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# Edges' Riemannian energy analysis for synchronization of multi-agent nonlinear systems over undirected weighted graphs

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## Abstract

In this note we investigate the problem of global exponential synchronization of multi-agent systems described by nonlinear input affine dynamics. We consider the case of networks described by undirected connected graphs possibly without leader. We present a set of sufficient conditions based on a Riemannian metric approach in order to design a state-feedback distributed control law. Then, we study the convergence properties of the overall network. By exploiting the properties of the edge Laplacian we construct a Lyapunov function that allows to conclude global exponential synchronization of the overall network.

*Keywords:* Synchronization, multi-agent systems, edge Laplacian, nonlinear systems

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## 1. Introduction

The synchronization problem of multi-agents systems has received increasing amounts of attention in the last decades. In this note, we focus on the case of homogeneous networks in which each agent is described by continuous-time nonlinear dynamics. The main objective consists in designing a distributed feedback law achieving synchronization of the network. Such a feedback law must respect two constraints: each agent can use only the information of its neighbours (i.e. the agents to which is communicating with), according to the given communication graph; the feedback law is zero on the synchronization manifold (i.e. when all the agents have the same behaviour). Building up on the seminal works in the linear context [1], notable results have been made for nonlinear systems, see [2–8] to cite a few. Existing works can be organized from two different point of view: the graph properties (e. g. directed or undirected, weighted or unweighted, with or without leader) or from the type of dynamics of each individual agent (e.g., linear or non-linear and which type of nonlinearities). However, while for linear dynamics, the synchronization problem can be solved for many types of graph (see, e.g. [9, 10]), for nonlinear dynamics a comprehensive result is still missing in the literature. To the best of our knowledge, the more general results in terms of graph and agent's dynamics are given by [11] (directed

strongly connected graphs) and [3] (directed weakly connected graphs). However, both results consider dynamics which are nonlinear in the state but linear in the input. Furthermore, quadratic Lyapunov functions are employed. Now, by re-organizing existing results from the contraction point of view [12–16], it can be noted that most of these works focused on the use of Euclidean metrics (or equivalently, quadratic Lyapunov functions) or  $L_1/L_\infty$  metrics (corresponding to linear Lyapunov functions). As a result, this severely restricts the individual agents' dynamics to be linear with respect to the input, with no coupling between the control input and the state [2–6, 11].

The first objective of this work is therefore to relax such a constrain by considering generic input-affine nonlinear systems. To this end we follow the recent results [7, 8, 13] where a Riemannian metric analysis is considered [12, 13]. Such an analysis (also denoted in literature as control contraction metric [13]) provide a set of sufficient conditions for the design of an infinite-gain margin law [17] which is essential to solve the synchronization problem even for the linear case see [9, 10]. The main limitation of [7, 8] is however that only undirected unweighted leader-connected graphs are considered. Therefore, the second objective of this work is to extend such an analysis to more generic graph connections. In particular, we focus on the case of undirected weighted graphs without the presence of a network leader. We highlight that the study of leaderless network is well motivated by several applications, such as electrical networks and grid-forming power-converters [18, 19]. For this, we need to modify the Lyapunov analysis developed in [7, 8], which is based on the use of  $N - 1$  error dynamics ( $N$  being the number of agents in the network) with respect to the leader and the properties of the Laplacian matrix to conclude overall synchronization. Such an

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approach, however, systematically fails when considering complex graphs which are not diagonalizable (see, e.g. [3]). Furthermore, the quadratic Lyapunov functions proposed in [3, 11] cannot be directly employed because of absence of a strong linear stable component in the agent's dynamics.

To overcome such a difficulty, we revisit the graph theory and the properties of the incidence matrix connecting the Laplacian and the edge Laplacian in [20, Chapter 2]. Such an edge Laplacian has been used, for instance, in [2, 5, 21–23] to solve the synchronization problem in the linear or Euclidean metric scenario. In particular, this approach focuses on the edges of the graph and a undirected network structure. The main idea consists in considering a number of error dynamics which is equivalent to the number of edges and not to the number of agents (e.g.,  $N-1$  in [3, 8], or  $N$  in [10]). To this end we use the properties of the edge Laplacian to create a specific edge matrix, and we use the control law in [7, Theorem 1]. Then, we propose a new Lyapunov function which corresponds to the Riemannian edge's energy. We remark that such a Lyapunov equation can be seen as an extension (to the Riemannian and weighted cases) of the Lyapunov function proposed in [2]. Its derivative along the flow of solutions is always negative and certify the overall network synchronization.

## 2. Preliminaries and Problem Statement

A communication graph is described by a triplet  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ . With  $\mathcal{V} = \{v_1, v_2, \dots, v_N\}$  a set of  $N \in \mathbb{N}$  nodes (the  $N$  agents),  $\mathcal{E} = \{(v_{k_1}, v_{l_1}), \dots, (v_{k_m}, v_{l_m})\} \subset \mathcal{V} \times \mathcal{V}$  a set of edges that models the interconnection between the nodes. For an undirected graph, if  $(v_k, v_l) \in \mathcal{E}$ , then  $(v_l, v_k) \in \mathcal{E}$ . We denote with  $\mathcal{N}_i$  the set of in-neighbors of node  $i$ , i.e. the set  $\mathcal{N}_i := \{j \in \{1, \dots, N\} | e_{ji} \in \mathcal{E}\}$ . Furthermore,  $A = (a_{ij}) \in \mathbb{R}^{N \times N}$  is the adjacency matrix defined by the weight  $a_{kl} \geq 0$  of each edge from node  $k$  to node  $l$ . Notice that  $a_{kl} = a_{lk}$  for an undirected graph. We denote by  $L = (\ell_{ij}) \in \mathbb{R}^{N \times N}$  the Laplacian matrix of the graph, defined as  $\ell_{ij} = -a_{ij}$  for  $i \neq j$ , and  $\ell_{ij} = \sum_{k=1}^N a_{ik}$  for  $i = j$ . For an undirected connected graph with  $N$  nodes,  $L$  is a positive semi-definite matrix and its spectrum is known and ordered as:

$$0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_M, \text{ with } M = \text{rank}(L) + 1$$

Given an undirected graph with  $N$  nodes  $(v_1, \dots, v_N)$  and  $Q$  edges  $(\varepsilon_1, \dots, \varepsilon_Q)$ , by giving an arbitrary orientation for each edge of the graph, we denote by  $E = (e_{ij}) \in \mathbb{R}^{N \times Q}$  the incidence matrix defined as

$$e_{ij} = \begin{cases} -1 & \text{if } v_i \text{ is the initial node of edge } \varepsilon_j \\ +1 & \text{if } v_i \text{ is the terminal node of edge } \varepsilon_j \\ 0 & \text{otherwise} \end{cases}$$

Then, we denote  $E_l$  the matrix with the positive term of  $E$  and  $E_k$  the one with the absolute value of the negative term of  $E$ , such as we have  $E = E_l - E_k$ . We denote by  $w_i = a_{kl} > 0$ , for each edge  $\varepsilon_i = (v_k, v_l)$ , the weight of

the edge. Thus, we introduce the weighting matrix  $W$  defined as  $W = \text{diag}(w_1, \dots, w_Q)$ . Notice that the Laplacian is directly related to the incidence matrix and the weighting matrix as  $L = EWE^\top$ . Then, we introduce an edge variant of the Laplacian matrix, that is the edge Laplacian  $L_e \in \mathbb{R}^{Q \times Q}$  which is defined as

$$L_e = E^\top EW. \quad (1)$$

By definition, the non-zero eigenvalues of the Laplacian matrix  $L$  are eigenvalues of the edge Laplacian  $L_e$ .

We consider a homogeneous network of  $N \geq 2$  agents connected via a weighted undirected graph, in which the dynamics of each agent is described by the following non-linear ode

$$\dot{x}_i = f(x_i) + g(x_i)u_i, \quad i \in \{1, \dots, N\}, \quad (2)$$

where  $x_i \in \mathbb{R}^n$  is the state of the node  $i$ ,  $u_i \in \mathbb{R}$  is the control action of node  $i$ , and  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $g: \mathbb{R}^n \rightarrow \mathbb{R}^n$  are  $C^1$  functions which are the same for each agent. The main goal of this article is to design a distributed diffusive coupling of the form

$$u_i = \sum_{j \in \mathcal{N}_i} a_{ij} (\varphi(x_j) - \varphi(x_i))$$

in order to guarantee global convergence of the distance between any two agents of the network, i.e.

$$\sum \|x_i(t) - x_j(t)\| \leq k \exp(-\mu t) \sum \|x_i(0) - x_j(0)\| \quad (3)$$

for any initial condition  $x_i(0) \in \mathbb{R}^n$ ,  $i = \{1, \dots, N\}$ , and for some  $k, \mu > 0$ . It's important to stress that for an agent  $i$ , the desired control law has to be designed taking into account only the state information of the agent itself and of the set of agents which are neighbours, namely  $j \in \mathcal{N}_i$ . Furthermore,  $u_i = 0$  whenever state agreement is achieved, i.e. when  $x_i = x_j$  for any  $j \in \mathcal{N}_i$ .

## 3. Properties of the edge Laplacian matrix

For an undirected graph we discussed that we can give an arbitrary orientation for each edge of the graph. In this paper, we take a specific orientation for each edge to simplify the notation. In particular, each edge between agents  $k$  and  $l$  (with  $k < l$ ) is noted  $(k, l)$ . Then, we order the arcs of the form  $(k, l)$  by increasing  $k$  then by increasing  $l$  for equal  $k$ . We then obtain the set of arcs of the network with  $Q$  edges that we note  $\Sigma = \{(k_1, l_1), (k_2, l_2), \dots, (k_Q, l_Q)\}$ . Thus, we defined the matrix  $E$  with this orientation and we obtain the edge Laplacian  $L_e$  with (1). Then, the following property can be established. Note that we don't require the graph to be connected.

**Lemma 1.** *Consider an undirected graph (not necessarily connected) with  $N$  agents and  $Q$  edges. Let  $W$ , resp.  $E$ ,*

resp.  $L = EWE^\top$ , resp.  $L_e = E^\top EW$ , be the weight matrix, the incidence, Laplacian and edge Laplacian matrix. Then there exists a matrix  $\Upsilon \in \mathbb{R}^{Q \times Q}$  verifying

$$E^\top L = \Upsilon E^\top, \quad (4)$$

$$W\Upsilon + \Upsilon^\top W \succ 0. \quad (5)$$

**Proof.** First of all, assume that  $\ker(E) \neq \emptyset$  and assume its dimension is  $\ell$ . Let  $(v_1, \dots, v_\ell)$  in  $\mathbb{R}^Q$  be an orthonormal basis of  $\ker(E)$ . Let

$$\Upsilon = E^\top EW + \mu \sum_{i=1}^{\ell} v_i v_i^\top$$

where  $\mu$  is a real number that will be selected later on. Note that

$$\begin{aligned} \Upsilon E^\top &= E^\top EWE^\top + \mu \sum_{i=1}^{\ell} v_i v_i^\top E^\top \\ &= E^\top L + \mu \sum_{i=1}^{\ell} v_i (Ev_i)^\top = E^\top L. \end{aligned}$$

Moreover,  $W\Upsilon + \Upsilon^\top W = Q_1 + \mu Q_2$  where

$$Q_1 = 2WE^\top EW, \quad Q_2 = W \sum_{i=1}^{\ell} v_i v_i^\top + \sum_{i=1}^{\ell} v_i v_i^\top W.$$

Let  $w$  in  $\mathbb{R}^Q$  be such that  $w^\top Q_1 w = 0$  and  $w \neq 0$ . Hence  $EWw = 0$  and this implies  $Ww = \sum_{j=1}^{\ell} a_j v_j$  for  $\ell$  real numbers  $a_i$  not all identically zero. Consequently, since  $W$  is diagonal,

$$\begin{aligned} w^\top Q_2 w &= 2(Ww)^\top \sum_{i=1}^{\ell} v_i v_i^\top W^{-1} Ww, \\ &= 2 \left( \sum_{j=1}^{\ell} a_j v_j^\top \right) \sum_{i=1}^{\ell} v_i v_i^\top W^{-1} \left( \sum_{j=1}^{\ell} a_j v_j \right), \\ &= 2 \left( \sum_{j=1}^{\ell} a_j v_j^\top \right) W^{-1} \left( \sum_{j=1}^{\ell} a_j v_j \right) > 0, \end{aligned}$$

where the last inequality is obtained since  $W$  is composed of positive element. In conclusion, we have the implication

$$(w^\top Q_1 w = 0, w \neq 0) \Rightarrow w^\top Q_2 w > 0.$$

Hence, with Finsler's lemma, there exists  $\mu$  such that  $W\Upsilon + \Upsilon^\top W \succ 0$  concluding the proof.  $\square$

#### 4. Main Result

In this section we consider a network represented by a graph  $\mathcal{G}$  and satisfying the next assumption of  $N$  identical agents described by the nonlinear dynamics (2).

**Assumption 1.** The graph  $\mathcal{G}$  is undirected and connected.

The objective is to design a distributed control law solving the synchronization problem as in (3). We recall the definition of Lie derivative of  $C^1$  tensor  $P : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  along a  $C^1$  vector field  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  defined as

$$\mathcal{L}_f P(x) = \mathfrak{d}_f P(x) + P(x) \frac{\partial f}{\partial x}(x) + \frac{\partial f}{\partial x}(x)^\top P(x)$$

where  $\mathfrak{d}_f P(x) = \lim_{h \rightarrow 0} \frac{P(x+hf(x)) - P(x)}{h}$  for all  $x \in \mathbb{R}^n$ . Then, following [7, 8], we introduce the following set of assumptions.

**Assumption 2.** There exists a  $C^1$  function  $P : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  taking symmetric positive definite values and real numbers  $\mu, \rho, \underline{p}, \bar{p} > 0$  satisfying for all  $x \in \mathbb{R}^n$

$$\begin{aligned} \mathcal{L}_f P(x) - \rho P(x) g(x) g(x)^\top P(x) &\preceq -2\mu P(x), \\ \underline{p} I_n &\preceq P(x) \preceq \bar{p} I_n. \end{aligned} \quad (6)$$

**Assumption 3.** The following Killing vector property and integrability conditions are verified

$$\mathcal{L}_g P(x) = 0, \quad \frac{\partial \alpha}{\partial x}(x) = g(x)^\top P(x), \quad \forall x \in \mathbb{R}^n, \quad (7)$$

for some  $C^2$  function  $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$ .

As detailed in [17], Assumptions 2 and 3 are sufficient to establish the existence of an infinite-gain margin feedback law making contractive a system of the form (2). Further comments are given at the end of the section. We can state now the following theorem.

**Theorem 1.** Consider a network described by  $N$  agents (2) and suppose that Assumptions 1-3 hold. Assume moreover, that the functions  $f, \alpha$  are globally Lipschitz and  $g$  is globally bounded. Let  $\beta^* = \frac{\rho \bar{w}}{2\underline{\lambda}}$  with  $\rho$  satisfying (6),  $\bar{w}$  being the maximal eigenvalue of the matrix  $W$ , and  $\underline{\lambda}$  the minimal eigenvalue of the matrix  $W\Upsilon$  with  $\Upsilon$  satisfying (4). Then, for any  $\beta \geq \beta^*$ , the distributed feedback

$$u_i = -\beta \sum_{j=1}^N \ell_{ij} \alpha(x_j) = \beta \sum_{j \in \mathcal{N}_i} a_{ij} (\alpha(x_j) - \alpha(x_i)) \quad (8)$$

with  $\alpha$  satisfying (7), solve the global synchronization problem for the network (2). More precisely, (3) holds with  $\mu$  given by Assumption 2.

**Proof.** Let  $\Sigma = \{(k_1, l_1), (k_2, l_2), \dots, (k_Q, l_Q)\}$  be the set of arcs of the network as defined in Section 3. For a given initial condition  $(x_1(0), \dots, x_N(0))$  in  $\mathbb{R}^{Nn}$  let  $t \mapsto (x_1(t), \dots, x_N(t))$  be the solution of (2). This one is defined for all positive time since by assumption, the nonlinear system is globally Lipschitz. For any  $i \in \{1, \dots, Q\}$ , we

define the  $C^2$  function  $(s, t) \in [0, 1] \times \mathbb{R}_+ \mapsto \Gamma_i(s, t) \in \mathbb{R}^n$  solution of

$$\begin{aligned} \frac{\partial \Gamma_i}{\partial t}(s, t) &= f(\Gamma_i(s, t)) + g(\Gamma_i(s, t))\tilde{u}_i(s, t), \\ \tilde{u}_i(s, t) &= -\beta \left[ \sum_{j=1}^Q \Upsilon_{ij} \alpha(\Gamma_j(s, t)) + \sum_{l=1}^N \Omega_{il} \alpha(x_l(t)) \right], \end{aligned} \quad (9)$$

and with  $\Gamma_i(0, s) = (1-s)x_{k_i} + sx_{l_i}$ , where  $\Upsilon = (\Upsilon_{ij})$  is solution of (4) and  $\Omega = (\Omega_{il}) \in \mathbb{R}^{Q \times N}$  is defined by  $\Omega = \frac{1}{2}(|E^\top|L - \Upsilon|E^\top|)$ , with the notation  $|A| := (|a_{ij}|)$ , for any matrix  $A = (a_{ij})$ . The function  $\Gamma_i$  is defined as any  $C^2$  path between the points  $x_{k_i}$  and  $x_{l_i}$  satisfying the dynamics (9), consistently with the agent dynamics (2). Consider now the  $C^1$  function  $V : \mathbb{R}_+ \mapsto \mathbb{R}_+$  defined by

$$\begin{aligned} V(t) &:= \sum_{i=1}^Q w_i V_i(t), \\ V_i(t) &:= \int_0^1 \left( \frac{\partial \Gamma_i^\top}{\partial s}(s, t) P(\Gamma_i(s, t)) \frac{\partial \Gamma_i}{\partial s}(s, t) \right) ds, \end{aligned} \quad (11)$$

where  $P$  is such that (6) and (7) are verified. It corresponds to the sum of the energies  $V_i$  associated to the paths  $\Gamma_i$  between any two agents connected via an edge  $i \in \{1, \dots, Q\}$ . Note that using the second inequality in (6) and the fact that for all  $t \geq 0$ ,  $\Gamma_i(0, t) = x_{k_i}(t)$  and  $\Gamma_i(1, t) = x_{l_i}(t)$ , we obtain

$$V_i(t) \geq \underline{p} \int_0^1 \frac{\partial \Gamma_i^\top}{\partial s}(s, t) \frac{\partial \Gamma_i}{\partial s}(s, t) ds \geq \underline{p} \|x_{k_i}(t) - x_{l_i}(t)\|^2.$$

So for all  $t \geq 0$  it yields

$$\begin{aligned} V(t) &\geq \underline{p} \sum_{i=1}^Q w_i \|x_{k_i}(t) - x_{l_i}(t)\|^2 \\ &= \underline{p} \mathbf{x}(t)^\top (EW \otimes I_n) (E^\top \otimes I_n) \mathbf{x}(t) = \underline{p} \mathbf{x}^\top (L \otimes I_n) \mathbf{x}(t) \end{aligned}$$

with  $\mathbf{x} = (x_1^\top, \dots, x_N^\top)^\top$ . The graph being connected, it implies  $\ker(L) = \text{Vect}(\mathbf{1}_n)$ . Hence,  $\ker(L \otimes I_n) = \text{Vect}(\mathbf{1}_n \otimes I_n)$ . Consequently there exists a positive real number  $\underline{\alpha}$  such that,

$$V(t) \geq \underline{\alpha} \sum_{i,j \in \{1, \dots, N\}} \|x_i(t) - x_j(t)\|^2, \quad \forall t \geq 0. \quad (12)$$

On the other hand, the time-derivative of  $V$  satisfies,

$$\begin{aligned} \dot{V}(t) &= \sum_{i=1}^Q w_i \int_0^1 \left( \frac{\partial^2 \Gamma_i^\top}{\partial s \partial t}(s, t) P(\Gamma_i(s, t)) \frac{\partial \Gamma_i}{\partial s}(s, t) \right. \\ &\quad + \frac{\partial \Gamma_i^\top}{\partial s}(s, t) \mathfrak{D}_{f+g\tilde{u}_i} P(\Gamma_i) \frac{\partial \Gamma_i}{\partial s}(s, t) \\ &\quad \left. + \frac{\partial \Gamma_i^\top}{\partial s}(s, t) P(\Gamma_i(s, t)) \frac{\partial^2 \Gamma_i}{\partial s \partial t}(s, t) \right) ds. \end{aligned}$$

Using the definition of  $\Gamma$  in (9) and  $\tilde{u}$  in (10), the term  $\partial^2 \Gamma / \partial s \partial t$  is computed as

$$\begin{aligned} \frac{\partial^2 \Gamma_i}{\partial s \partial t}(s, t) &= \frac{\partial}{\partial s} [f(\Gamma_i(s, t)) + g(\Gamma_i(s, t))\tilde{u}_i] \\ &= \frac{\partial f(\Gamma_i(s, t))}{\partial x} \frac{\partial \Gamma_i}{\partial s}(s, t) + \frac{\partial g(\Gamma_i(s, t))}{\partial x} \frac{\partial \Gamma_i}{\partial s}(s, t) \tilde{u}_i \\ &\quad - \beta g(\Gamma_i(s, t)) \frac{\partial \alpha(x)}{\partial x} \left( \sum_{j=1}^Q \Upsilon_{ij} \frac{\partial \Gamma_j}{\partial s}(s, t) \right) \\ &= \left( \frac{\partial f(\Gamma_i(s, t))}{\partial x} + \frac{\partial g(\Gamma_i(s, t))}{\partial x} \tilde{u}_i \right) \frac{\partial \Gamma_i}{\partial s}(s, t) \\ &\quad - \beta g(\Gamma_i) g(\Gamma_j)^\top P(\Gamma_j) \left( \sum_{j=1}^Q \Upsilon_{ij} \frac{\partial \Gamma_j}{\partial s}(s, t) \right). \end{aligned}$$

Then, combining (7) with the previous expression in the derivative of  $V$  yields for all  $t \geq 0$

$$\begin{aligned} \dot{V}(t) &= \sum_{i=1}^Q w_i \int_0^1 \left( \frac{\partial \Gamma_i^\top}{\partial s} \left[ \mathcal{L}_f P(\Gamma_i) + \tilde{u}_i \mathcal{L}_g P(\Gamma_i) \right] \frac{\partial \Gamma_i}{\partial s} \right. \\ &\quad \left. - 2\beta \sum_{j=1}^Q \Upsilon_{ij} \text{He} \left[ \frac{\partial \Gamma_i^\top}{\partial s} P(\Gamma_i) g(\Gamma_i) g(\Gamma_j)^\top P(\Gamma_j) \frac{\partial \Gamma_j}{\partial s} \right] \right) ds, \end{aligned}$$

with the compact notation  $\text{He}(A) = \frac{A+A^\top}{2}$  for any element  $A$ . By introducing the notation  $\bar{\Lambda}(s, t) = (\Lambda_i^\top, \dots, \Lambda_Q^\top)^\top$  and  $\Lambda_i(s, t) = g(\Gamma_i(s, t))^\top P(\Gamma_i(s, t)) \frac{\partial \Gamma_i}{\partial s}(s, t)$ , we obtain

$$\begin{aligned} \dot{V}(t) &\leq -2\mu V(t) + k \sum_{i=1}^Q \int_0^1 w_i \Lambda_i^\top \Lambda_i ds \\ &\quad - \beta \sum_{i=1}^Q \sum_{j=1}^Q \int_0^1 w_i \Upsilon_{ij} [\Lambda_i^\top \Lambda_j + \Lambda_j^\top \Lambda_i] ds, \end{aligned}$$

with (6) and (7), and therefore

$$\begin{aligned} \dot{V}(t) &\leq -2\mu V(t) - \int_0^1 \bar{\Lambda}^\top [\beta(W\Upsilon + \Upsilon^\top W) - kW] \bar{\Lambda} ds \\ &\leq -2\mu V(t), \end{aligned}$$

for any  $\beta \geq \beta^*$ ,  $\beta^* = \frac{\rho \bar{w}}{2\lambda}$ . This shows exponential convergence of  $V$  to 0. With (12), it ensures the synchronization of the agents.  $\square$

**Remark.** Note that in the proof of Theorem 1, the function  $\tilde{u}_i$  in (10) is designed by the equations verified by  $\Gamma_i$  for  $s = 0$  and  $s = 1$  for all  $i \in \{1, \dots, Q\}$ . Indeed these equations yield  $\tilde{u}_i(0, t) = u_{k_i}(t)$  and  $\tilde{u}_i(1, t) = u_{l_i}(t)$ . And with (8), it leads to

$$\begin{aligned} \tilde{u}_i(0, t) &= \sum_{m=1}^N (\Upsilon E_k^\top + \Omega)_{i,m} \alpha(x_m) \\ u_{k_i}(t) &= \sum_{m=1}^N \ell_{k_i, m} \alpha(x_m) = \sum_{m=1}^N (E_k^\top L)_{i,m} \alpha(x_m) \end{aligned}$$

so that  $\tilde{u}_i(0, t) = u_{k_i}(t) \Leftrightarrow E_k^\top L = \Upsilon E_k^\top + \Omega$  and with the same equation for  $s = 1$ , we obtain  $\tilde{u}_i(1, t) = u_i(t) \Leftrightarrow E_l^\top L = \Upsilon E_l^\top + \Omega$ . Subtracting the last two equations we recover (4).

**Remark.** We highlight that the controller (8) is a diffusive coupling feedback which compares the difference between the  $i$ -th agent state and its  $j$ -th neighbours. As already remarked in [7, 8, 17], in the case of linear dynamics of the form

$$\dot{x}_i = Ax_i + Bu_i,$$

Assumptions 2-3 boil down to the stabilizability of the pair  $A, B$ , and the matrix function  $P$  is the solution to the Algebraic Riccati Inequality

$$PA + A^\top P - \rho PBB^\top P \preceq -2\mu P.$$

As a consequence, the function  $\alpha$  can be selected as  $\alpha(x) = Kx$ , with  $K = B^\top P$ , and the distributed feedback law (8) simply reads  $u_i = -\beta \sum_{j=1}^N \ell_{ij} B^\top P x_j$ , thus recovering standard existing solution for the linear context [1, 9, 10]. In this linear case, the Lyapunov function certifying global exponential synchronization is given by

$$V = e^\top (W \otimes P) e = \mathbf{x}^\top (EWE^\top \otimes P) \mathbf{x}$$

where  $e := (E^\top \otimes I_n)x \in \mathbb{R}^{Qn}$  with  $\mathbf{x} = (x_1, \dots, x_N)$  being the vector of the state of the  $N$  agents.

## 5. Illustration

The conditions of Assumption 2 and 3 are based on contraction theory. As a consequence, although they are very hard to verify analytically, they provide a feedback design which is robust by construction. For this reasons, many recent works studied how to approximate contraction-based controllers (also denoted in literature as control contraction matrices [13]). See, for instance, [15]. In particular, as already shown in [8], the integrability and Killing vector conditions can be relaxed at the price of obtaining a feedback design which allows to achieve only semi-global state synchronization (where the domain of attraction has to be intended with respect to a asymptotic behaviour of the overall network).

Building up on similar robustness properties, we illustrate in this section that we can learn the feedback  $\alpha$  of Theorem 1 and yet achieving asymptotic synchronization of a complex undirected network. To this end, we use the Deep Neural Networks (DNN) based algorithm for metric estimation presented in [8, Section VII-B] and we apply to a network of  $N$  identical (chaotic) Lorenz attractors. Each agent  $i \in \{1, \dots, N\}$  is described by the following dynamics

$$\begin{cases} \dot{x}_{i,1} = a(x_{i,2} - x_{i,1}) + u_i \\ \dot{x}_{i,1} = x_{i,1}(b - x_{i,3}) - x_{i,2} + (2 + \sin(x_{i,1}))u_i \\ \dot{x}_{i,3} = x_{i,1}x_{i,2} - cx_{i,3} \end{cases}$$

$$E = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Figure 1: Incidence matrix  $E$ .

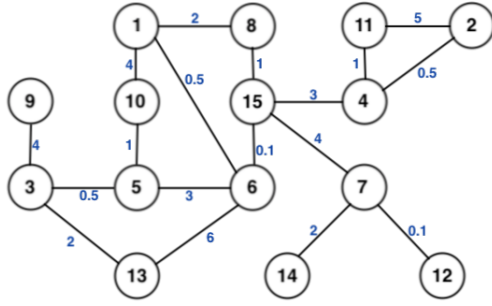
where  $x_i = (x_{i,1}, x_{i,2}, x_{i,3}) \in \mathbb{R}^3$  is the state of agent  $i$ ,  $u_i \in \mathbb{R}$  the control input, and the parameters  $a, b, c$  are selected as  $a = 10$ ,  $b = \frac{8}{3}$ , and  $c = 28$  in order to guarantee chaotic behaviour. Note that with such a choice of control vector field  $g$ , the system is not feedback linearizable and conditions of Assumption 2 and 3 cannot be easily verified due to the presence of both polynomial and trigonometric functions. A DDN is used to learn an approximated solution on a sufficiently large region following [8, Section VI]. In the illustration, we considered an undirected and connected network of  $N = 15$  agents. The network is illustrated in Figure 2a, where the numbers over each arcs are the weight of each connections. For this example we have the incidence matrix depicted in Figure 1 with a matrix of the weight of each nodes  $W = \text{diag}(0.5, 2, 4, 0.5, 5, 0.5, 4, 2, 1, 3, 3, 1, 6, 0.1, 0.1, 2, 4, 1)$ .

Figure 2b show that if the initial conditions are not too far, all the agents of the network synchronize among them with the distributed control law (8) derived in Theorem 1.

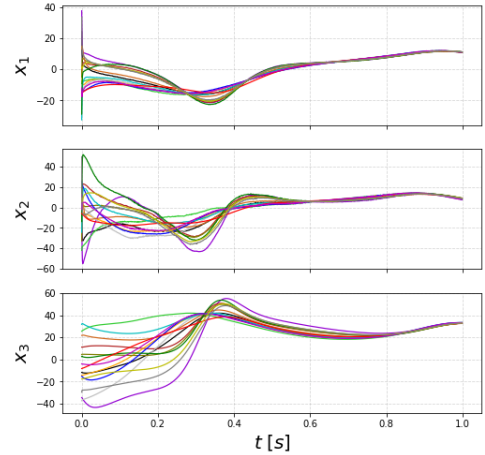
## 6. Conclusion

Based on a new technical result on the properties of the Edge Laplacian matrix of a graph, we have been able to prove that the distributed feedback design proposed in [7, 8] is able to solve the problem of global state synchronization of a network of input-affine nonlinear agents connected via a weighted undirected graph. The proof is based on the construction of a new Lyapunov function which is formally equivalent to the Edge's Riemannian energy. It is interesting to note that the derivative of the proposed Lyapunov function is always negative regardless the connectivity properties of the graph. However, it defines a suitable Lyapunov function for the network synchronization only when the graph is connected.

We remark that similarly to [8], the Killing and integrability conditions of Assumption 3 can be relaxed at the price of guaranteeing only semi-global synchronization. In turns, such a relaxation allows the use of an approximation of the feedback law  $\alpha$  which is generically hard to obtain in analytic form. Such an approximation can be computed via numerical methods such as neural networks



(a) Connected network with 15 nodes. The numbers above arrows denote the weights of the edge between any two agents.



(b) Evolution of the states  $(x_1, x_2, x_3)$  of agents  $a_i, i = 1, \dots, N$ .

Figure 2: Synchronization of Lorentz's oscillators

(e.g. [8, 15]) while guaranteeing asymptotic synchronization.

Finally, we conjecture that the proposed feedback law can apply also to the more general case of weighted directed graphs in the analysis. As a future work, we would like to certify this conjecture by a suitable Lyapunov analysis (which may include also the use of colour theory as in [4]).

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