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Robust Finite-time Stabilization of Linear Systems with Limited State Quantization

Yu Zhou ^a, Andrey Polyakov ^a, Gang Zheng ^a

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Abstract

This paper investigates the robust asymptotic stabilization of a linear time-invariant (LTI) system by static feedback with a static state quantization. It is shown that the controllable LTI system can be stabilized to zero in a finite time by means of nonlinear feedback with a quantizer having a limited (finite) number of values (quantization seeds) even when all parameters of the controller and the quantizer are time-invariant. The control design is based on generalized homogeneity. A homogeneous spherical quantizer is introduced. The static homogeneous feedback is shown to be a local (or global) finite-time stabilizer for any controllable linear system (depending on the system matrix). The tuning rules for both the quantizer and the feedback law are obtained in the form of Linear Matrix Inequalities (LMIs). The closed-loop system is proven to be robust with respect to some bounded matched and vanishing mismatched perturbations. Theoretical results are supported by numerical simulations.

Key words: static quantizer; limited data; homogeneous control; linear plant.

1 Introduction

Quantization is an important process of networked control system design aimed at the reduction of data transmission [Jiang and Teng-Fei \(2013\)](#), [Wang et al. \(2017\)](#). Quantization involves partitioning the state space into discrete and disjoint subsets known as quantization cells. Within each quantization cell, all states are represented by just one quantization seed (an element of the cell). In network control, a number of the quantization seed is transmitted and stored instead of the exact value of the state vector belonging to the corresponding quantization cell. Quantizers (quantization operators) can be classified into static (as in [Elia and Mitter \(2001\)](#)) and dynamic (see [Brockett and Liberzon \(2000\)](#)). A static quantizer has time-invariant quantization cells and seeds, whereas the quantization cells and seeds of a dynamic quantizer can be adjusted as time evolves.

The logarithmic quantizer proposed in [Elia and Mitter \(2001\)](#) is an example of the static one. The control design and analysis of linear control systems with logarithmic quantizer is based on the quadratic Lyapunov functions. The logarithmic quantizer follows the intuitive idea that

the farther state is from the origin, the less precise control action and knowledge about the state are required. Such a quantizer has an infinite (but countable) number of quantization levels/seeds. A commonly employed dynamic quantizer [Brockett and Liberzon \(2000\)](#) adjusts the quantization precision depending on the state or time. A finite static quantizer refers to a time-invariant quantizer having a finite number of quantization seeds on the whole state space.

A global asymptotic stabilizer designed for a “non-quantized” system generally fails to provide asymptotic stability in the case of a finite static quantization [Bullo and Liberzon \(2006\)](#). Existing studies have addressed asymptotic stabilization only through logarithmic or dynamic quantizers (see also e.g., [Fu and Xie \(2005\)](#), [Gu and Qiu \(2014\)](#), [Wang \(2021\)](#)). The former requires infinite number of quantization levels, while the latter needs a transmission of time-varying quantizer parameters for encoding/decoding over the network (e.g., [Fu and Xie \(2009\)](#), [Liu et al. \(2015\)](#), [Zheng et al. \(2018\)](#), [Wang \(2022\)](#)). A linear feedback with finite static state quantizer can only ensure just a practical stability of LTI system (see, e.g., [Delchamps \(1990\)](#), [Bullo and Liberzon \(2006\)](#), [Corradini and Orlando \(2008\)](#), [Ferrante et al. \(2015\)](#), [Wang \(2018\)](#)). This limitation is due to the fact that a linear feedback cannot reject a nonvanishing quantization errors. Motivated by the above observations, this paper is aimed at a non-

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linear feedback control design for asymptotic (in fact, finite-time) stabilization of LTI system with a finite static quantization of the states. The so-called generalized homogeneity is utilized for this purpose.

Homogeneity is a symmetry with respect to a group of transformations having topological characterization of a dilation [Kawski \(1991\)](#). Homogeneous systems are widely studied in control systems theory (e.g., [Zubov \(1958\)](#), [Rosier \(1992\)](#), [Grüne \(2000\)](#), [Bhat and Bernstein \(2005\)](#), [Andrieu et al. \(2008\)](#)) due to its various important properties, including the existence of a homogeneous Lyapunov function, finite/fixed-time convergence depending on a homogeneity degree, and the equivalence between local and global stability. There are few results on homogeneous controllers with state quantization for sliding mode [Yan et al. \(2019\)](#), [Li et al. \(2017\)](#) and linear systems [Zhou et al. \(2023a\)](#). However, all previous studies have focused on logarithmic quantizers or dynamic quantizers. The homogeneity allows a system analysis and control design to be reduced to a compact set (e.g., a sphere) [Bhat and Bernstein \(2005\)](#). Since, for compactly supported states, a finite quantization can provide any desired precision, then a compactly designed regulator may become a global stabilizer of the system due to homogeneity.

In our previous work on homogeneous stabilization using a logarithmic quantizer [Zhou et al. \(2023b\)](#), [Zhou et al. \(2023a\)](#), we demonstrated the feasibility of finite/fixed-time stabilization of an LTI plant with states quantization, which, however, requires an infinite number of quantization levels/seeds. Under the limited quantization, the previous approach could only achieve a practical stabilization.

In this paper, a special finite static homogeneous spherical quantizer is developed and combined with homogeneous feedback [Polyakov \(2019\)](#) to design a finite-time stabilizer for LTI systems. Unlike existing quantizers (uniform, logarithmic, or dynamic) which have bounded quantization cells, each quantization cell of the proposed homogeneous spherical quantizer is a homogeneous cone (unbounded set), but all quantization seeds lie on a unit sphere. Quantizers for the unit sphere are well studied in the literature [Haimovich et al. \(2006\)](#), [Bullo and Liberzon \(2006\)](#), [Matschkal and Huber \(2009\)](#), [Gu and Qiu \(2014\)](#), [Wang \(2018\)](#), [Wang \(2021\)](#). We show that their proper application in the theory of generalized homogeneous systems provides a novel solution for the stabilization of any controllable LTI system with limited data. Depending on the system matrix, local (or global) stabilization can be guaranteed. We also demonstrate that the proposed quantized stabilizer rejects some classes of additive non-vanishing perturbations.

This paper is organized as follows. In Section 2, we formulate the problem. Section 3 provides a brief introduc-

tion to homogeneous control systems analysis and design. In Section 4, we present the main results, which include the design of a homogeneous spherical quantizer as well as homogeneous quantized feedback. Finally, Section 5 presents the simulation results.

Notations

\mathbb{R} : real numbers; \mathbb{R}_+ : non-negative real numbers; \mathbb{N} : natural numbers excluding zero; $\mathbf{0}$: zero element of a space; $P \succ 0$ ($\prec 0, \succeq 0, \preceq 0$): positive (negative) definite (or semidefinite) symmetric matrix P ; $\lambda_{\min}(P)$, $\lambda_{\max}(P)$: smallest and largest eigenvalues of matrix P ; $P^{\frac{1}{2}}$: square root matrix (i.e. $(P^{\frac{1}{2}})^2 = P$); $|x| = \sqrt{x^\top x}$: the conventional Euclidean norm for $x \in \mathbb{R}^n$; $\|x\| = \sqrt{x^\top P x}$: the weighted Euclidean norm of $x \in \mathbb{R}^n$ (with $P \succ 0$ to be defined); I_n : the identity $n \times n$ matrix; $S^{n-1}(1)$: unit sphere $S^{n-1}(1) = \{x \in \mathbb{R}^n : \|x\| = 1\}$; $B(r, x_0)$: open ball with radius r at center x_0 ; $B(r, x_0) = \{x \in \mathbb{R}^n : \|x - x_0\| < r\}$; $|Q|$: cardinality of a set Q ; $\lfloor x \rfloor = \max\{m \in \mathbb{Z} \mid m \leq x\}$: the floor operator for $x \in \mathbb{R}$.

2 Problem formulation

In this paper, we consider the system:

$$\dot{x} = Ax + Bu + g(t, x), \quad t > 0, \quad x(0) = x_0, \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^m$ is the control input, $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are known system matrices, $g : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is an unknown piecewise continuous function satisfying certain restrictions (see below). The pair (A, B) is controllable.

Let \mathcal{E} be a subset of \mathbb{R}^n and Ξ be a subset of \mathbb{N} . Let $\mathcal{D}_i \subseteq \mathcal{E}$ with $i \in \Xi$ be non-empty disjoint subsets covering \mathcal{E} : $\mathcal{D}_i \cap \mathcal{D}_j = \emptyset$ if $i \neq j$ and $\bigcup_{i \in \Xi} \mathcal{D}_i = \mathcal{E}$. Let $\mathcal{Q} = \{q_i\}_{i \in \Xi}$ such that $q_i \in \mathcal{D}_i$ for any $i \in \Xi$. A function $\mathbf{q} : \mathcal{E} \mapsto \mathcal{Q}$ is said to be a quantization function (*quantizer*) on the set \mathcal{E} if $\mathbf{q}(x) = q_i$ for all $x \in \mathcal{D}_i$ and all $i \in \Xi$.

The set \mathcal{Q} represents a collection of all quantization values. Each disjoint set \mathcal{D}_i is called by a quantization cell, and the vector q_i is the quantization seed of \mathcal{D}_i . In this paper, we assume that the whole state vector is available but it is quantized such that the set \mathcal{Q} is finite and time-invariant. This case corresponds to the so-called *finite static quantizer*. We refer the reader to [Brockett and Liberzon \(2000\)](#), [Bullo and Liberzon \(2006\)](#) for more details about state quantization.

The feedback control with quantization is given by:

$$u = \tilde{u}(\mathbf{q}(x)), \quad (2)$$

where $\tilde{u} : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a static feedback law, $\mathbf{q}(x)$ is a quantized state. The closed-loop system has discontinuous right-hand side. Its solutions are understood in the sense of Filippov (see [Filippov \(2013\)](#)).

The aim is to address the problem of finite-time stabilization by a static feedback with *finite static quantizer*. In other words, we need to design a quantizer $\mathbf{q} : \mathcal{E} = \mathbb{R}^n \mapsto \mathcal{Q} \subset \mathbb{R}^n, |\mathcal{Q}| < +\infty$ and a feedback law $\tilde{u} : \mathbb{R}^n \mapsto \mathbb{R}^n$ such that \mathbf{q} is time-invariant and the closed-loop system (1)-(2) is globally or locally uniformly finite-time stable¹. More specifically, we consider the problem of designing a finite static quantizer for a given finite-time stabilizer \tilde{u} . We are also interested in finding the size of the data, which has to be transmitted through the network, in order to use the feedback $\tilde{u}(\mathbf{q}(x))$. The size is determined by the number of bits M such that $|\mathcal{Q}| \leq \log_2 M$. We show that the considered problems can be solved for the so-called generalized homogeneous stabilizers (see, e.g., Polyakov et al. (2016), Polyakov (2019)).

3 Preliminaries

3.1 Homogeneity and homogeneous function/system

Homogeneity refers to a class of dilation symmetries, which have been shown to possess several useful properties for control design and analysis Zubov (1958), Khomeynuk (1961), Kawski (1991), Rosier (1992), Bhat and Bernstein (2005).

Definition 1 (Kawski (1991)) A mapping $\mathbf{d}(s) : \mathbb{R}^n \mapsto \mathbb{R}^n, s \in \mathbb{R}$ is said to be a dilation in \mathbb{R}^n if

- $\mathbf{d}(0) = I_n, \mathbf{d}(s+t) = \mathbf{d}(s)\mathbf{d}(t) = \mathbf{d}(t)\mathbf{d}(s), \forall s, t \in \mathbb{R}$;
- $\lim_{s \rightarrow -\infty} \|\mathbf{d}(s)x\| = 0$ and $\lim_{s \rightarrow \infty} \|\mathbf{d}(s)x\| = \infty, \forall x \neq \mathbf{0}$.

The dilation \mathbf{d} is continuous (linear) if the mapping $s \mapsto \mathbf{d}(s)$ is continuous (resp., linear). Any linear continuous dilation in \mathbb{R}^n is given by

$$\mathbf{d}(s) = e^{sG_{\mathbf{d}}} := \sum_{i=0}^{\infty} \frac{s^i G_{\mathbf{d}}^i}{i!}, \quad (3)$$

where an anti-Hurwitz matrix $G_{\mathbf{d}}$ is known as the *generator* of the dilation. For $G_{\mathbf{d}} = I_n$ we have the standard dilation $e^s I_n$, while a diagonal $G_{\mathbf{d}}$ corresponds to the weighted dilation (see Zubov (1958), Rosier (1992)).

Definition 2 (Polyakov (2019)) A dilation \mathbf{d} is strictly monotone with respect to a norm $\|\cdot\|$ in \mathbb{R}^n if $\exists \beta > 0 : \|\mathbf{d}(s)\| \leq e^{\beta s}, \forall s \leq 0$.

¹ The system (1) is said to be globally (locally) uniformly

- *Lyapunov stable* if $\exists \alpha \in \mathcal{K}$ such that $\|x(t)\| \leq \alpha(\|x_0\|), \forall t \geq 0, \forall x_0 \in \mathbb{R}^n$ (resp., $\in \Omega$ - a neighbourhood of $\mathbf{0}$);
- *finite-time stable* (see, e.g., Roxin (1966), Bhat and Bernstein (2000), Orlov (2004)) if it is globally (resp., locally) uniformly Lyapunov stable and there exists a locally bounded function $T : \mathbb{R}^n \mapsto \mathbb{R}_+$ such that $\|x(t)\| = 0, \forall t \geq T(x_0), \forall x_0 \in \mathbb{R}^n$ (resp., $\in \Omega$).

This paper deals only with the linear continuous dilation given by (3). The subsequent condition is a direct result of the quadratic Lyapunov function theorem for linear systems.

Proposition 1 A linear continuous dilation \mathbf{d} in \mathbb{R}^n is strictly monotone with respect to the norm $\|z\| = \sqrt{z^T P z}$ if and only if the following linear matrix inequality holds

$$P \succ 0, PG_{\mathbf{d}} + G_{\mathbf{d}}^T P \succ 0, \quad (4)$$

where $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$ is the generator of dilation \mathbf{d} .

Inspired by Kawski (1991), the homogeneous function and vector field are defined as follows.

Definition 3 (Kawski (1991)) A vector field $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ (a function $h : \mathbb{R}^n \mapsto \mathbb{R}$) is said to be \mathbf{d} -homogeneous of degree $\mu \in \mathbb{R}$ if

$$\begin{aligned} f(\mathbf{d}(s)x) &= e^{\mu s} \mathbf{d}(s)f(x), \quad \forall x \in \mathbb{R}^n, \quad \forall s \in \mathbb{R}, \\ (\text{resp.}, h(\mathbf{d}(s)x) &= e^{\mu s} h(x), \quad \forall x \in \mathbb{R}^n, \quad \forall s \in \mathbb{R}), \end{aligned}$$

where \mathbf{d} is a continuous dilation in \mathbb{R}^n .

The linear continuous dilation \mathbf{d} introduces an alternative topology in \mathbb{R}^n by the so-called ‘‘homogeneous norm’’ Kawski (1995), Grune (2000) providing related geometric objects (e.g., sphere).

Definition 4 (Polyakov (2020)) The set

$$S_{\mathbf{d}}^{n-1}(r) = \{x \in \mathbb{R}^n : \|\mathbf{d}(-\ln r)x\| = 1\},$$

is called a \mathbf{d} -homogeneous sphere of the radius $r > 0$ with the center at $\mathbf{0}$.

The unit homogeneous sphere $S_{\mathbf{d}}^{n-1}(1)$ coincides with the unit sphere $S^{n-1}(1)$ in Euclidean space and

$$S_{\mathbf{d}}^{n-1}(r) = \mathbf{d}(-\ln r)S_{\mathbf{d}}^{n-1}(1) = \mathbf{d}(-\ln r)S^{n-1}(1). \quad (5)$$

The above equation means that any homogeneous sphere can be obtained through scaling the unit sphere.

Definition 5 (Polyakov (2019)) Let a continuous linear dilation \mathbf{d} be strictly monotone with respect to a norm $\|\cdot\|$ in \mathbb{R}^n . The function $\|\cdot\|_{\mathbf{d}} : \mathbb{R}^n \mapsto \mathbb{R}_+$ defined as $\|\mathbf{0}\|_{\mathbf{d}} = 0$ and

$$\|x\|_{\mathbf{d}} = e^s, \text{ where } s \in \mathbb{R} : \|\mathbf{d}(-s)x\| = 1, \quad x \neq \mathbf{0},$$

is called the canonical homogeneous norm in \mathbb{R}^n .

The canonical homogeneous norm is a continuous function defined implicitly and its derivative has the form (see Polyakov (2020))

$$\frac{\partial \|x\|_{\mathbf{d}}}{\partial x} = \|x\|_{\mathbf{d}} \frac{x^T \mathbf{d}^T(-\ln \|x\|_{\mathbf{d}}) P \mathbf{d}(-\ln \|x\|_{\mathbf{d}})}{x^T \mathbf{d}^T(-\ln \|x\|_{\mathbf{d}}) P G_{\mathbf{d}} \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) x}, \quad (6)$$

provided that $\|x\|_{\mathbf{d}}$ is induced by the weighted Euclidean norm $\|x\| = \sqrt{x^\top P x}$ with P satisfying (4).

The homogeneity of a linear vector field is characterized as follows (Polyakov (2019), Zimenko et al. (2020)).

Lemma 1 *Let \mathbf{d} be a linear continuous dilation. The following claims are equivalent:*

- the linear vector field $x \mapsto Ax$ with $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ is \mathbf{d} -homogeneous of degree $\mu \neq 0$;
- there exists a generator $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$ of linear dilation \mathbf{d} such that
$$AG_{\mathbf{d}} = (\mu I + G_{\mathbf{d}})A; \quad (7)$$
- the matrix A is nilpotent.

An asymptotically stable homogeneous system exhibits finite-time stability in the case of negative degree Bhat and Bernstein (2005). The following homogeneous stabilizer for an LTI system has been developed in the series of works Polyakov (2019), Zimenko et al. (2020) and Polyakov (2020).

Theorem 1 *A controllable disturbance-free linear system (1) (i.e., $g(t, x) = \mathbf{0}$) with the following control law*

$$u(x) = K_0 x + \|x\|_{\mathbf{d}}^{1+\mu} K \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) x, \quad \mu \in [-1, 0), \quad (8)$$

is globally uniformly finite-time stable, where $K = YX^{-1}$, $G_{\mathbf{d}} = I_n + \mu G_0$, $K_0 = Y_0(G_0 - I_n)^{-1}$, the pair $G_0 \in \mathbb{R}^{n \times n}$, $Y_0 \in \mathbb{R}^{m \times n}$ is a solution of the matrix equation:

$$AG_0 + BY_0 = G_0 A + A, \quad G_0 B = \mathbf{0}. \quad (9)$$

and the pair $X \in \mathbb{R}^n$ and $Y \in \mathbb{R}^{m \times n}$ is a solution of the linear algebraic system:

$$\begin{cases} XA_0^\top + A_0X + Y^\top B^\top + BY + \rho(XG_{\mathbf{d}}^\top + G_{\mathbf{d}}X) = \mathbf{0}, \\ XG_{\mathbf{d}}^\top + G_{\mathbf{d}}X \succ 0, \quad X \succ 0. \end{cases} \quad (10)$$

Based on the above theorem, we would like to emphasize the following facts:

- If the pair $\{A, B\}$ is controllable then the linear algebraic systems (9) and (10) always have solutions (see Polyakov et al. (2016)).
- Canonical homogeneous norm $\|x\|_{\mathbf{d}}$ is a Lyapunov function for the closed system (1), (8) and

$$\frac{\partial \|x\|_{\mathbf{d}}}{\partial x} (Ax + Bu(x)) = -\rho \|x\|_{\mathbf{d}}^{1+\mu}. \quad (11)$$

- The linear feedback $K_0 x$ homogenizes the system and guarantees that the matrix $A_0 = A + BK_0$ is nilpotent. Notice that we may have $K_0 = \mathbf{0}$ in (8) only if the equation (9) has a solution with $Y_0 = \mathbf{0}$. By Lemma 1, this is possible only if the matrix A is nilpotent.
- The closed-loop system with the control (8) is \mathbf{d} -homogeneous of degree μ .

For $\mu = -1$, solutions of the closed-loop system (2) are understood in the sense of Filippov. Let us recall the Lyapunov theorem for differential equations with discontinuous right-hand sides.

Theorem 2 (Filippov (2013), page 152) *Consider the system $\dot{x} = f(t, x)$, $t > 0$, where x is the state vector and f is a piece-wise continuous vector field. Suppose there exists a positive definite radially unbounded function $V : C(\mathbb{R}^n, \mathbb{R}_+) \cap C^1(\mathbb{R}^n \setminus \{\mathbf{0}\}, \mathbb{R}_+)$:*

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(t, x) < 0,$$

almost everywhere in $\mathbb{R} \times \mathbb{R}^n$. Then, the system is globally asymptotically stable provided that its solutions are understood in the sense of Filippov.

The above result means that the stability analysis of the discontinuous ODE can be done in the conventional way (using the right-hand side of the ODE) provided that Lyapunov function V is continuously differentiable away from the origin and solutions are understood in the sense of Filippov. A Lipschitz continuous function V cannot be utilized in the same way (see, e.g., Utkin (2013)) despite its differentiability almost everywhere.

4 Main results

4.1 Generalized homogeneous spherical quantizer

One of the key features of homogeneous systems is the completeness of their global definition based on local data. Indeed, if a homogeneous vector field is defined on a sphere then it can be expanded to the whole $\mathbb{R}^n \setminus \{\mathbf{0}\}$ using the dilation (see Definition 4). This inspires the spherical quantization for homogeneous systems.

It is known that any non-zero element of \mathbb{R}^n can be projected to the unit sphere using the uniform dilation. The corresponding projector $\pi : \mathbb{R}^n \setminus \{\mathbf{0}\} \mapsto S^{n-1}(1)$ is defined as

$$\pi(x) = \frac{x}{\|x\|} = \mathbf{d}(-\ln \|x\|)x,$$

where $\mathbf{d}(s) = e^{sI_n}$ is a uniform dilation. A similar projection can be constructed using any strictly monotone continuous dilation \mathbf{d} (see Polyakov (2020)). Indeed, for a given strictly monotone dilation $\mathbf{d}(s) = e^{G_{\mathbf{d}} s}$, $s \in \mathbb{R}$, the canonical homogeneous norm $\|x\|_{\mathbf{d}}$ is continuous on \mathbb{R}^n and Lipschitz continuous on $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Moreover, the canonical homogeneous norm introduces a \mathbf{d} -homogeneous projector $\pi_{\mathbf{d}} : \mathbb{R}^n \setminus \{\mathbf{0}\} \mapsto S^{n-1}(1)$,

$$\pi_{\mathbf{d}}(x) = \mathbf{d}(-\ln \|x\|_{\mathbf{d}})x. \quad (12)$$

since, by definition of the canonical homogeneous norm we have $\|\pi_{\mathbf{d}}(x)\| = 1$. Notice that $\pi_{\mathbf{d}}(\mathbf{d}(s)x) = \pi_{\mathbf{d}}(x)$, $\forall s \in \mathbb{R}, \forall x \neq \mathbf{0}$.

Definition 6 Let a linear continuous dilation \mathbf{d} be strictly monotone with respect to the weighted Euclidean norm $\|x\| = \sqrt{x^\top P x}$, $x \in \mathbb{R}^n$, and $\mathcal{Q}_s \subset S^{n-1}(1)$ be a discrete set. A quantizer $\mathbf{q}_{\pi_{\mathbf{d}}} : \mathbb{R}^n \mapsto \mathcal{Q}_s$ given by

$$\mathbf{q}_{\pi_{\mathbf{d}}}(x) = \begin{cases} \mathbf{q}_s(\pi_{\mathbf{d}}(x)), & x \neq \mathbf{0}, \\ \mathbf{0}, & x = \mathbf{0}, \end{cases} \quad (13)$$

is called \mathbf{d} -homogeneous spherical quantizer, where $\mathbf{q}_s : S^{n-1}(1) \mapsto \mathcal{Q}_s \subset S^{n-1}(1)$ is a discrete map, $|\mathcal{Q}_s| < +\infty$.

The \mathbf{d} -homogeneous spherical quantizer is a finite static quantizer being invariant with respect to linear dilation \mathbf{d} as follows

$$\mathbf{q}_{\pi_{\mathbf{d}}}(\mathbf{d}(s)x) = \mathbf{q}_{\pi_{\mathbf{d}}}(x), \quad \forall s \in \mathbb{R}, \quad \forall x \in \mathbb{R}^n. \quad (14)$$

The homogeneous spherical quantization consists of two steps: first, state $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ is projected to the unit sphere using the homogeneous projector $\pi_{\mathbf{d}}$, and next, the discrete map \mathbf{q}_s quantizes the projected state. Since the value $\pi_{\mathbf{d}}(x)$ belongs to the unit sphere, the following proposition is straightforward due to the compactness of the sphere.

Proposition 2 For a \mathbf{d} -homogeneous spherical quantizer $\mathbf{q}_{\pi_{\mathbf{d}}} : \mathbb{R}^n \mapsto \mathcal{Q}_s \subset S^{n-1}(1)$, any given $\delta > 0$, there exists a set \mathcal{Q}_s such that $|\mathcal{Q}_s| = N < +\infty$ and

$$\|\mathbf{q}_{\pi_{\mathbf{d}}}(z) - z\| \leq \delta, \quad \forall z \in S^{n-1}(1).$$

The proposed homogeneous spherical quantizer divides the space into disjoint homogeneous cones² being quantization cells. All such cells are unbounded, but quantization seeds lie on the unit sphere. The form of a quantization cell depends on the dilation (i.e., on the matrix $G_{\mathbf{d}}$). However, since the canonical homogeneous norm has the same unit sphere as the weighted Euclidean norm (i.e., $\|x\|_{\mathbf{d}} = 1 \Leftrightarrow \|x\| = 1$), the quantization cells of the homogeneous spherical quantizer $\mathbf{q}_{\pi_{\mathbf{d}}}(x)$ have the same intersection with the unit sphere (see Fig. 1).

Remark 1 The proposed homogeneous sphere quantizer has quantization seeds on the unit sphere similar to the widely used controversial sphere quantizer (see, e.g., Haimovich et al. (2006), Bullo and Liberzon (2006), Matschkal and Huber (2009), Gu and Qiu (2014), Wang (2018), Wang (2021)), but proposed quantizer uses completely different quantization cells due to the homogeneous projection.

For the proposed sphere quantizer, the corresponding practical realization issues will be discussed in Section 4.3.

² A nonempty set $\mathcal{H} \subseteq \mathbb{R}^n$ is said to be a \mathbf{d} -homogeneous cone in \mathbb{R}^n (Polyakov (2020)) if \mathcal{H} is invariant with respect to the dilation \mathbf{d} , i.e., $\mathbf{d}(s)z \in \mathcal{H}$ for $z \in \mathcal{H}$ and $s \in \mathbb{R}$.

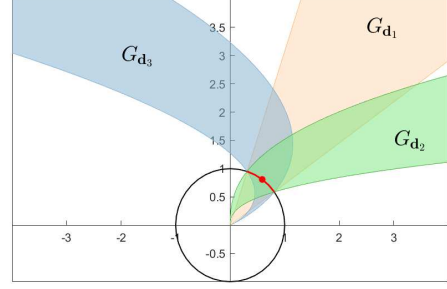


Fig. 1. The quantization cell of homogeneous spherical quantizer $\mathbf{q}_{\pi_{\mathbf{d}}}(x) = \mathbf{q}_s(\pi_{\mathbf{d}}(x))$ dependently of the dilation: $G_{\mathbf{d}_1} = I_2$, $G_{\mathbf{d}_2} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $G_{\mathbf{d}_3} = \begin{bmatrix} 1.5 & -1.2 \\ 0 & 1 \end{bmatrix}$.

4.2 Finite-time stabilization with homogeneous spherical quantization

The dilation symmetry of a closed-loop system may not be preserved in the presence of quantization. For example, the scalar closed-loop system

$$\dot{x} = ax + u(x), \quad a \in \mathbb{R}, \quad u(x) = -|x|^{0.5} \text{sign}(x) - ax,$$

is standard homogeneous of degree -0.5 . The open-loop system is standard homogeneous of degree 1, but the feedback $u(x)$ with $a \neq 0$ is not homogeneous. That is why the closed-loop system with the quantized state feedback $\dot{x} = ax + u(\mathbf{q}(x))$ is not homogeneous even if the quantizer $\mathbf{q}(x)$ has: $\mathbf{q}(e^s x) = \mathbf{q}(x), \forall s, x \in \mathbb{R}$. Based on this observation, we first study the case, when the \mathbf{d} -homogeneous spherical quantization does not destroy the dilation symmetry of the closed-loop system. For a MIMO LTI system, this is possible if the matrix A is nilpotent (see Lemma 1). In this case, the equation (9) may have a solution $Y_0 = K_0 = \mathbf{0}$, so the feedback (8) contains only the nonlinear term:

$$u_{\text{hom}}(x) = \|x\|_{\mathbf{d}}^{1+\mu} K \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) x.$$

Taking into account $\|\pi_{\mathbf{d}}(x)\|_{\mathbf{d}} = 1$, we derive:

$$\begin{aligned} u_{\text{hom}}(\pi_{\mathbf{d}}(x)) &= \|\pi_{\mathbf{d}}(x)\|_{\mathbf{d}}^{1+\mu} K \mathbf{d} (-\ln \|\pi_{\mathbf{d}}(x)\|_{\mathbf{d}}) \pi_{\mathbf{d}}(x) \\ &= K \pi_{\mathbf{d}}(x), \end{aligned}$$

where $\pi_{\mathbf{d}}$ is the \mathbf{d} -homogeneous projector (see above). The homogeneous control becomes a static feedback of the projected state. Following this projection strategy, and considering nonvanishing quantization error, homogeneous feedback with a \mathbf{d} -homogeneous spherical quantizer is constructed. For the disturbance-free system, stability is investigated as follows.

Theorem 3 Let the parameters $G_0 \in \mathbb{R}^{n \times n}$, $K_0 \in \mathbb{R}^{m \times n}$, $G_{\mathbf{d}} = I_n - G_0$ and $A_0 = A + BK_0$ be defined as in Theorem 1. Let $X \in \mathbb{R}^{n \times n}$ and $Y \in \mathbb{R}^{m \times n}$ be a solution of the LMI

$$XG_{\mathbf{d}}^{\top} + G_{\mathbf{d}}X \succ 0, \quad X \succ 0, \quad (15a)$$

$$\begin{bmatrix} XA_0^{\top} + A_0X + Y^{\top}B^{\top} + BY + \delta^2\tau X & BY \\ Y^{\top}B^{\top} & -\tau X \end{bmatrix} \prec 0, \quad (15b)$$

for some $\delta > 0$ and some $\tau > 0$. Let the canonical homogeneous norm $\|\cdot\|_{\mathbf{d}}$ be induced by norm $\|x\| = \sqrt{x^{\top}X^{-1}x}$. Let $\mathbf{q}_{\pi_{\mathbf{d}}} : \mathbb{R}^n \mapsto \mathcal{Q}_s \subset S^{n-1}(1)$ be a \mathbf{d} -homogeneous spherical quantizer such that

$$\|\mathbf{q}_{\pi_{\mathbf{d}}}(z) - z\| \leq \delta, \quad \forall z \in S^{n-1}(1).$$

Then, the closed-loop system (1) with

$$u = K\mathbf{q}_{\pi_{\mathbf{d}}}(x), \quad K = YX^{-1}, \quad (16)$$

and with $g = \mathbf{0}$ is locally uniformly finite-time stable. Moreover, if $K_0 = \mathbf{0}$, it is globally uniformly finite-time stable.

Proof: If $K_0 = \mathbf{0}$ then $A_0 = A$, according to (9), we have:

$$A_0G_{\mathbf{d}} = (-I_n + G_{\mathbf{d}})A_0, \quad G_{\mathbf{d}}B = B$$

where $G_{\mathbf{d}} = I_n - G_0$. Considering that $\mathbf{d}(s) = e^{G_{\mathbf{d}}s} = \sum_{i=0}^{\infty} \frac{s^i G_{\mathbf{d}}^i}{i!}$, we derive

$$e^{-s}\mathbf{d}(s)A_0 = A_0\mathbf{d}(s), \quad \mathbf{d}(s)B = e^s B, \quad s \in \mathbb{R}. \quad (17)$$

Then, we conclude that the open-loop system $\dot{x} = A_0x$ is \mathbf{d} -homogeneous of degree -1 . Let us show in the following that the closed-loop system

$$\dot{x} = f(x) := A_0x + BK\mathbf{q}_{\pi_{\mathbf{d}}}(x)$$

is \mathbf{d} -homogeneous too. For this, since the dilation $\mathbf{d}(s)x = e^{G_{\mathbf{d}}s}x$, we have

$$f(\mathbf{d}(s)x) = A_0\mathbf{d}(s)x + BK\mathbf{q}_{\pi_{\mathbf{d}}}(\mathbf{d}(s)x), \quad \forall s \in \mathbb{R}.$$

Using the identity (17) and the dilation invariance of $\mathbf{q}_{\pi_{\mathbf{d}}}$ (see the formula (14)), we derive

$$\begin{aligned} f(\mathbf{d}(s)x) &= e^{-s}\mathbf{d}(s)A_0x + e^{-s}\mathbf{d}(s)BK\mathbf{q}_{\pi_{\mathbf{d}}}(x) \\ &= e^{-s}\mathbf{d}(s)f(x). \end{aligned} \quad (18)$$

Therefore, the closed-loop system is \mathbf{d} -homogeneous of degree -1 .

Calculating the time derivative of the Lyapunov function candidate $\|x\|_{\mathbf{d}}$ along trajectories of the system (2) with (16), by means of the formula (6), we derive

$$\frac{d\|x\|_{\mathbf{d}}}{dt} = \|x\|_{\mathbf{d}} \frac{2x^{\top}\mathbf{d}^{\top}(-s_x)P\mathbf{d}(-s_x)[A_0x + BK\mathbf{q}_{\pi_{\mathbf{d}}}(x)]}{x^{\top}\mathbf{d}^{\top}(-s_x)(G_{\mathbf{d}}^{\top}P + PG_{\mathbf{d}})\mathbf{d}(-s_x)x}, \quad (19)$$

where $s_x = \ln \|x\|_{\mathbf{d}}$. Using the \mathbf{d} -homogeneity of closed-loop system (18) we obtain

$$\frac{d\|x\|_{\mathbf{d}}}{dt} = \frac{2x^{\top}\mathbf{d}^{\top}(-s_x)P[A_0\mathbf{d}(-s_x)x + BK\mathbf{q}_{\pi_{\mathbf{d}}}(x)]}{x^{\top}\mathbf{d}^{\top}(-s_x)(G_{\mathbf{d}}^{\top}P + PG_{\mathbf{d}})\mathbf{d}(-s_x)x}. \quad (20)$$

Taking the quantization error $\sigma = \mathbf{q}_{\pi_{\mathbf{d}}}(\pi_{\mathbf{d}}(x)) - \pi_{\mathbf{d}}(x)$ into account, and using the dilation invariance of $\mathbf{q}_{\pi_{\mathbf{d}}}$, we obtain

$$\frac{d\|x\|_{\mathbf{d}}}{dt} = \frac{[\pi_{\mathbf{d}}(\sigma)]^{\top} \begin{bmatrix} (A_0+BK)^{\top}P + P(A_0+BK) & PBK \\ K^{\top}B^{\top}P & \mathbf{0} \end{bmatrix} [\pi_{\mathbf{d}}(\sigma)]}{\pi_{\mathbf{d}}^{\top}(x)(G_{\mathbf{d}}^{\top}P + PG_{\mathbf{d}})\pi_{\mathbf{d}}(x)}. \quad (21)$$

Since $\|\sigma\| \leq \delta$ and $\|\pi_{\mathbf{d}}(x)\| = 1$, it holds

$$\|\sigma\| \leq \delta \|\pi_{\mathbf{d}}(x)\| \Leftrightarrow [\pi_{\mathbf{d}}(\sigma)]^{\top} \begin{bmatrix} -\delta^2 P & \mathbf{0} \\ \mathbf{0} & P \end{bmatrix} [\pi_{\mathbf{d}}(\sigma)] \leq 0. \quad (22)$$

Then, according to S-procedure (see, for example, [Boyd and Vandenberghe \(2004\)](#)), the norm $\|x\|_{\mathbf{d}}$ is Lyapunov function of system (1) and (16) if and only if there exists a positive constant $\tau > 0$ such that

$$W := \begin{bmatrix} A_0^{\top}P + PA_0 + K^{\top}B^{\top}P + BK^{\top}P + \delta^2\tau P & PBK \\ K^{\top}B^{\top}P & -\tau P \end{bmatrix} \prec 0. \quad (23)$$

This matrix inequality is equivalent to (15b) for $X = P^{-1}$ and $K = YX^{-1}$.

Since $W \prec 0$, $P \succ 0$ and $G_{\mathbf{d}}^{\top}P + PG_{\mathbf{d}} \succ 0$ then there exists $\rho > 0$: $W \leq -\rho \begin{bmatrix} G_{\mathbf{d}}^{\top}P + PG_{\mathbf{d}} & \mathbf{0} \\ \mathbf{0} & P \end{bmatrix}$. Hence, taking into account $\pi_{\mathbf{d}}^{\top}(x)P\pi_{\mathbf{d}}(x) = 1$ we derive $\frac{d\|x\|_{\mathbf{d}}}{dt} < -\rho$ almost everywhere on \mathbb{R}^n .

Applying Theorem 2, we conclude the global asymptotic stability of the system. Additionally, the negative homogeneity degree of the asymptotically stable closed-loop system implies finite-time stability (see e.g., [Nakamura et al. \(2002\)](#)).

For $K_0 \neq \mathbf{0}$, the closed-loop system is not \mathbf{d} -homogeneous. Instead, it can be regarded as \mathbf{d} -homogeneous system with a linear term, which is not \mathbf{d} -homogeneous:

$$\dot{x} = A_0x + BK\mathbf{q}_{\pi_{\mathbf{d}}}(x) - BK_0x, \quad (24)$$

where $A_0 = A + BK_0$ is nilpotent. Repeating the above considerations, we have A_0x and $A_0x + BK\mathbf{q}_{\pi_{\mathbf{d}}}(x)$ are \mathbf{d} -homogeneous of degree -1 , and we derive

$$\frac{d\|x\|_{\mathbf{d}}}{dt} \leq -\rho - \frac{x^{\top}\mathbf{d}^{\top}(-s_x)PBK_0x}{x^{\top}\mathbf{d}^{\top}(-s_x)PG_{\mathbf{d}}\mathbf{d}(-s_x)x} \leq -\rho + \frac{\|BK_0x\|}{\beta},$$

where $\beta = \frac{1}{2}\lambda_{\min} \left(P^{\frac{1}{2}}G_{\mathbf{d}}P^{-\frac{1}{2}} + P^{-\frac{1}{2}}G_{\mathbf{d}}^{\top}P^{\frac{1}{2}} \right) > 0$ and the identity $x^{\top}\mathbf{d}^{\top}(-\ln \|x\|_{\mathbf{d}})P\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x = 1$ is utilized in the derivations. Hence, we obtain the local finite-time stability. \square

The feasibility of LMIs (10) has been proven in Polyakov et al. (2016). Taking $\tau = \frac{1}{\delta}$ we conclude the feasibility of (15), at least, for small δ .

The above theorem shows that the feedback (16) with the \mathbf{d} -homogeneous spherical quantizer $\mathbf{q}_{\pi_{\mathbf{d}}}$ is a global finite-time stabilizer for the unperturbed linear controllable plant (1) if $K_0 = \mathbf{0}$. As mentioned above (see discussions after Theorem 1), the matrix A is nilpotent in this case. The simplest example of such a globally finite-time stabilizable system with $g = \mathbf{0}$ is the controlled integrator chain:

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Indeed, in this case, the solution of the algebraic equation (9) is

$$Y_0 = \mathbf{0}, \quad G_0 = \begin{bmatrix} -n+1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -n+2 & 0 & \dots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \dots & -1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

So, $K_0 = \mathbf{0}$ and, by Theorem 3, the controlled integrator chain is globally finite-time stabilizable using the quantized homogeneous feedback (16).

Corollary 1 *Let all conditions of Theorem 3 hold and $g : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ be a measurable locally bounded function. If there exists sufficiently small $\kappa > 0$, such that, for $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, the following inequality holds*

$$\|x\|_{\mathbf{d}} \frac{x^{\top} \mathbf{d}^{\top} (-\ln \|x\|_{\mathbf{d}}) P \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) g(t, x)}{x^{\top} \mathbf{d}^{\top} (-\ln \|x\|_{\mathbf{d}}) P G_{\mathbf{d}} \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) x} \leq \kappa, \quad \forall t \geq 0, \quad (25)$$

almost everywhere on \mathbb{R}^n , then the perturbed closed-loop system (1) and (16) is globally (locally) uniformly finite-time stable for $K_0 = \mathbf{0}$ (resp., $K_0 \neq \mathbf{0}$).

Indeed, as shown in the proof of Theorem 3, there exists a positive constant ρ such that $\frac{\partial \|x\|_{\mathbf{d}}}{\partial x} (Ax + BK \mathbf{q}_{\pi_{\mathbf{d}}}(x)) \leq -\rho + \frac{\|BK_0 x\|}{\beta}$. Then for the perturbed system we have

$$\begin{aligned} \frac{d\|x\|_{\mathbf{d}}}{dt} &\leq -\rho + \frac{\|BK_0 x\|}{\beta} + \|x\|_{\mathbf{d}} \frac{x^{\top} \mathbf{d}^{\top} (-s_x) P \mathbf{d} (-s_x) g(t, x)}{x^{\top} \mathbf{d}^{\top} (-s_x) P G_{\mathbf{d}} \mathbf{d} (-s_x) x} \\ &\leq -(\rho - \kappa) + \frac{\|BK_0 x\|}{\beta}, \end{aligned}$$

almost everywhere on \mathbb{R}^n . Then, by Theorem 2, the closed-loop system is globally (resp., locally) finite-time stable for $K_0 = \mathbf{0}$ (resp., $K_0 \neq \mathbf{0}$). Notice that the inequality (25) is fulfilled, for example, if

$$g(t, x) = B\gamma(t, x) : \|B\gamma(t, x)\| \leq \beta\kappa, \quad \forall t \geq 0, \quad \forall x \in \mathbb{R}^n.$$

Indeed, in this case, due to $\mathbf{d}(s)B = e^s B$, we have

$$\|x\|_{\mathbf{d}} \frac{x^{\top} \mathbf{d}^{\top} (-\ln \|x\|_{\mathbf{d}}) P \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) g(t, x)}{x^{\top} \mathbf{d}^{\top} (-\ln \|x\|_{\mathbf{d}}) P G_{\mathbf{d}} \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) x} \leq \frac{\|B\gamma\|}{\beta} \leq \kappa. \quad (26)$$

4.3 A possible spherical quantizer design

To guarantee the feasibility of LMI (15), the parameter δ has to be small enough. This means that quantization on the unit sphere has to be sufficiently dense. However, in practice, the quantization number is often constrained by limited memory or limited data transmission (network bandwidth). For an implementation of the quantized feedback law designed in Theorem 3, an admissible number M of bits for encoding (numbering) the quantization seeds (i.e., $|\mathcal{Q}| \leq N = 2^M$) is given in practice. In this case, an explicit formula relating the quantization error δ with the number of bits is required for the spherical quantizer design. In this section, given an integer N and $P = X^{-1}$ satisfying LMIs (15), we design a quantizer $\mathbf{q}(x)$ to ensure $\|\mathbf{q}(x) - x\| \leq \delta_N$, where $\delta_N > 0$ is the quantization error for the given N . To provide an explicit formula for δ_N , we design the homogeneous spherical quantizer based on spherical coordinates.

In fact, any unit vector $z = [z_1, z_2, \dots, z_n]^{\top} \in S^{n-1}(1)$ can be represented using spherical coordinates: $\xi = [1, \varphi_1, \varphi_2, \dots, \varphi_{n-1}]^{\top}$, $0 \leq \varphi_1, \varphi_2, \dots, \varphi_{n-2} \leq \pi$, $0 \leq \varphi_{n-1} < 2\pi$. Let the map from spherical coordinates to Cartesian coordinates be denoted by $g_1 : \mathbb{R}_+ \times [0, \pi]^{n-1} \times [0, 2\pi) \mapsto S^{n-1}(1)$,

$$z = g_1(\xi), \quad z_1 = \cos \varphi_1, \quad z_n = \prod_{j=1}^{n-1} \sin \varphi_j, \quad z_k = \cos \varphi_k \prod_{j=1}^{k-1} \sin \varphi_j,$$

where $k = 2, 3, \dots, n-1$. Let the inverse map (from Cartesian coordinates to spherical coordinates) be denoted by $g_2 : S^{n-1}(1) \mapsto \mathbb{R}_+ \times [0, \pi]^{n-1} \times [0, 2\pi)$,

$$\xi = g_2(z), \quad \varphi_i = \text{atan2}(\sqrt{\sum_{j=i+1}^n x_j^2}, x_i), \quad i = 1, \dots, n-2,$$

$$\varphi_{n-1} = \text{atan2}(x_n, x_{n-1}),$$

where $\text{atan2}(\cdot, \cdot)$ is the so-called two-argument arctangent function.

Proposition 3 *Wang (2021) Let $z, y \in S^{n-1}(1)$ be two unit vectors with spherical coordinates $\xi_z = g_2(z)$ and $\xi_y = g_2(y)$, respectively. If for any $i \in 1, 2, \dots, n-1$, it holds that $|\varphi_{z,i} - \varphi_{y,i}| \leq \Delta$, $\Delta \in (0, \frac{\pi}{2})$ then*

$$|z - y| \leq \sqrt{2 - 2(2 \cos^{2(n-1)}(\frac{\Delta}{2}) - 1)}. \quad (27)$$

For a strictly monotone dilation \mathbf{d} with respect to $P \succ 0$, the \mathbf{d} -homogeneous spherical quantizer in \mathbb{R}^n is defined as follows:

$$\mathbf{q}_{\pi_{\mathbf{d}}}(x) = P^{-1/2} g_1([1, q(\varphi_{\pi_{\mathbf{d}},1}), \dots, q(\varphi_{\pi_{\mathbf{d}},n-1})]^{\top}), \quad (28)$$

where $[1, \varphi_{\pi_{\mathbf{d}},1}, \varphi_{\pi_{\mathbf{d}},2}, \dots, \varphi_{\pi_{\mathbf{d}},n-1}]^{\top} = g_2(P^{1/2} \pi_{\mathbf{d}}(x))$,

$$q(\varphi_{\pi_{\mathbf{d}},k}) = (\lfloor \frac{\varphi_{\pi_{\mathbf{d}},k}}{\Delta} \rfloor + \frac{1}{2}) \Delta, \quad \Delta \in (0, \frac{\pi}{2}),$$

$k=1, 2, \dots, n-1$.

The proposed quantizer uses the uniform quantization of the angles of the spherical coordinates with equal intervals.

Corollary 2 *Let the \mathbf{d} -homogeneous spherical quantizer in \mathbb{R}^n be defined by the formula (28) with $\Delta \in (0, \frac{\pi}{2})$ given by $\Delta = \frac{\pi}{\lfloor (N/2)^{1/(n-1)} \rfloor}$. Then, the set of quantization seeds admits the estimate $|\mathcal{Q}_s| \leq N$ and the quantization error of (28) is bounded by $\|\mathbf{q}_{\pi_{\mathbf{d}}}(\pi_{\mathbf{d}}(x)) - \pi_{\mathbf{d}}(x)\| \leq \delta_N$,*

$$\delta_N := 2 \left[1 - \cos^{2(n-1)} \left(\frac{\pi}{2 \lfloor (N/2)^{1/(n-1)} \rfloor} \right) \right]^{\frac{1}{2}}.$$

Proof: Indeed, according to Proposition 3, we have

$$\begin{aligned} & \left| g_1([1, q(\varphi_{\pi_{\mathbf{d},1}}), \dots, q(\varphi_{\pi_{\mathbf{d},n-1}})]^\top) - P^{\frac{1}{2}} \pi_{\mathbf{d}}(x) \right| \\ & \leq 2 \left[1 - \cos^{2(n-1)} \left(\frac{\Delta}{2} \right) \right]^{\frac{1}{2}}. \end{aligned}$$

Then, $\|\mathbf{q}_{\pi_{\mathbf{d}}}(x) - \pi_{\mathbf{d}}(x)\| \leq 2 \left[1 - \cos^{2(n-1)} \left(\frac{\Delta}{2} \right) \right]^{\frac{1}{2}}$. Thus, for each angle $\varphi_{\pi_{\mathbf{d},i}}$, $i = 1, 2, \dots, n-2$, the interval $[0, \pi/2]$ can be divided into $\lfloor (\frac{N}{2})^{\frac{1}{n-1}} \rfloor$ non-overlapping intervals. Additionally, $\varphi_{\pi_{\mathbf{d},n-1}}$ is divided into $2 \lfloor (\frac{N}{2})^{\frac{1}{n-1}} \rfloor$ non-overlapping intervals. The number of quantization seeds is given by $|\mathcal{Q}_s| = 2 \lfloor (N/2)^{\frac{1}{n-1}} \rfloor^{n-1} \leq N$. \square

Geometrically, the uniform quantizer and logarithmic quantizer partition the space into disjoint rectilinear quantization cells. In contrast, the proposed homogeneous spherical quantizer divides the space into disjoint homogeneous cones as quantization cells, where all cells are unbounded, and the quantization seeds lie on a sphere. An illustrative example for a two-dimensional space is presented in Fig. 2.

5 Numerical example

For numerical validation, we consider the linear system with $A = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 1.5 \end{bmatrix}$. The perturbation is considered as a function $g(x, t) = B(0.2 \sin(t))$. The initial state is $x_0 = [2, 4, 1]^\top$. Since A is nilpotent, by solving equation (9), we obtain $K_0 = \mathbf{0}$, $G_{\mathbf{d}} = \begin{bmatrix} 3 & -0.75 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. By solving the (15) with $\delta = 0.4$, $\tau = \frac{1}{8}$, we obtain

$$P = \begin{bmatrix} 0.0053 & 0.0037 & 0.0185 \\ 0.0037 & 0.0212 & 0.0381 \\ 0.0185 & 0.0381 & 0.2522 \end{bmatrix}, \quad K = -[0.1327 \ 0.4089 \ 1.7270].$$

Following Lemma 2, we calculate the numbers of bits $M = 8$ required to encode the quantized states:

$$\|\mathbf{q}_{\pi_{\mathbf{d}}}(\pi_{\mathbf{d}}(x)) - \pi_{\mathbf{d}}(x)\| \leq \delta_N = 0.3896 < \delta = 0.4.$$

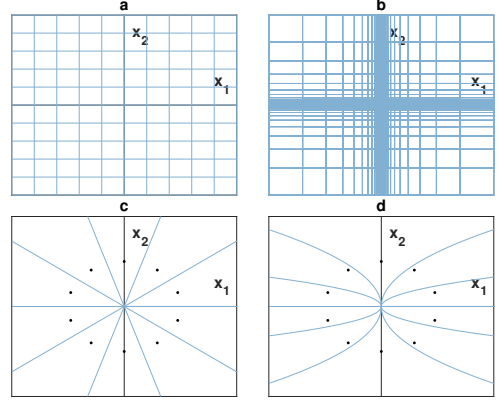


Fig. 2. Illustration of quantization in \mathbb{R}^2 . (a) Uniform quantizer per axis. (b) Logarithmic quantizer per axis. (c) Homogeneous spherical quantizer. ($G_{\mathbf{d}} = I_2$). (d) Homogeneous spherical quantizer ($G_{\mathbf{d}} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$). The dots in (c) and (d) represent quantization seeds on the sphere.

The homogeneous spherical quantizer is designed by the formula (28). The simulation results are presented in Figs. 3, 4, and 5. Fig. 3 illustrates the quantization algorithm with quantization seeds positioned on the weighted sphere. The evolution of the state of the closed-loop system is shown in Fig. 4. It validates the finite-time convergence of the state to the origin (up to discretization error) under perturbation. Fig. 5 presents the quantization values during stabilization. The chattering phenomenon in Fig. 5 results from the discontinuity in the projection $\pi_{\mathbf{d}}(x)$ at $x = \mathbf{0}$.

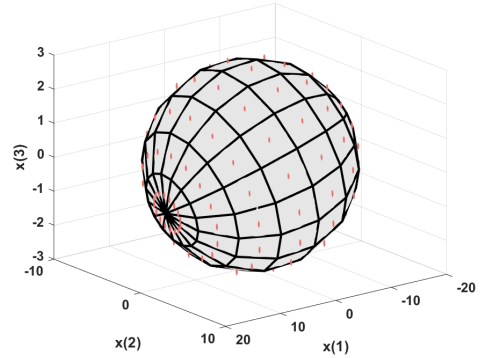


Fig. 3. Quantization seeds (red markers) with $N = 512$ are placed on the unit sphere $x^\top P x = 1$. The lines represent the intersections of quantization cells with the weighted sphere.

6 Conclusion

This paper proposes a finite-time stabilizer for an LTI system with a finite static quantization of the system state. The design is based on the generalized homogeneity. The quantizer is designed as a composition of a homogeneous projector to the unit sphere with a spherical quantizer. In this case, the feedback law only uses discrete values on the unit sphere as quantization seeds of

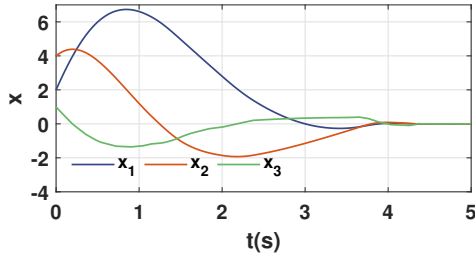


Fig. 4. The system state under finite static quantization feedback.

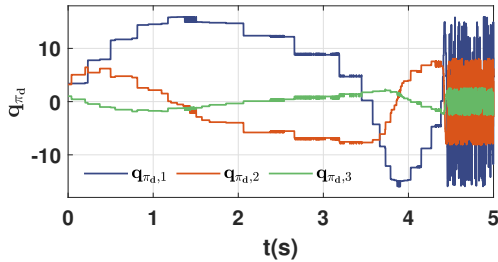


Fig. 5. The quantization value $q_{\pi_d}(x)$.

the state from \mathbb{R}^n . The design under limited information is addressed by solving parametric linear matrix inequalities. The chattering of the control input appears as the system state reaches the origin due to discontinuity. Future work may focus on the optimal partition of the unit sphere and chattering attenuation.

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