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# Finite/fixed-time Stabilization of Linear Systems with State Quantization

Yu Zhou, Andrey Polyakov, and Gang Zheng,

**Abstract**—This paper develops a homogeneity-based approach to finite/fixed-time stabilization of a linear time-invariant (LTI) system with quantized measurements. A sufficient condition for finite/fixed-time stabilization of multi-input LTI system under state quantization is derived. It is shown that a homogeneous state feedback with a novel logarithmic-like quantizer design can guarantee finite/fixed-time stability of the closed-loop system, provided that the quantization is sufficiently dense. Theoretical results are supported with numerical simulations.

**Index Terms**—homogeneous system, finite/fixed-time stabilization, quantization

## I. INTRODUCTION

Quantization and sampling are two main processes of digitization of a continuous signal. Considering a continuous signal as a function of time, the *digitization of the time domain* is known as sampling [1], while the *digitization of the co-domain* is referred to as quantization [2]. Both methods are crucial to network control systems with a limited data transmission rate. In the last two decades, finite-time and fixed-time state feedback stabilization (see Definition 2 for more details on finite/fixed-time stability) have attracted much attention due to their advantages such as controlled settling time, faster convergence, and better robustness (see, e.g., [3], [4], [5], [6]). It raises a question: *how to stabilize a system in finite/fixed time by a digitized control law?*

Continuous-time homogeneous systems with nonzero degrees are typical examples of dynamical systems with finite/fixed-time convergence rates [3], [7], [8]. Continuous-time homogeneous systems have many useful properties for control systems design, e.g., homogeneous Lyapunov functions, equivalence between local and global stability, tuning the finite/fixed-time convergence using homogeneity degree, small overshoots, etc. (see, e.g., [9], [10], [11], [3], [8]). Various homogeneous feedback control methods have been developed for continuous-time systems in the last two decades (see, e.g., [12], [13], [7], [14], [15], [16]). However, digitization may destroy the advantages of homogeneous controllers discovered in continuous-time cases. Even worse, arbitrarily small sampling period can result in a finite-time blowup of a fixed-time stable

continuous-time system, as demonstrated in [17]. Indeed, time sampling of the control signal generated by a homogeneous finite-time controller may invoke the so-called numerical chattering (see, e.g. [18], [19]). The problem of preservation of the finite/fixed-time convergence rate under the sampled-in-time implementation of homogeneous controllers has been solved recently in [20] using the concept of the so-called consistent discretization, while the problem of finite/fixed-time stabilization of multiple-input system under state quantization still remains open.

Mathematically, a quantizer can be defined as a discrete-valued mapping. Quantizers can be categorized as static (time-invariant) or dynamic (time-varying) based on whether their parameters can be adjusted over time. The former utilizes static parameters and commonly requires infinite quantization levels, while the latter has finite quantization values but requires time-varying parameters that depend on the current system state.

The logarithmic quantizer proposed in [21] and the dynamic “zooming” quantizer introduced in [22] initiate two approaches for control design with quantization. Researchers have extensively explored both static (e.g., [23], [24], [25], [26]) and dynamic (e.g., [27], [28], [29]) quantized control for LTI systems. Some attention also has been drawn to various related issues, such as mismatched quantization (see, e.g., [30], [31], [32]) and the reduction of sample rate and data size through event-triggering and dynamic quantization approaches (see, e.g., [33], [34], [35]). However, existing studies primarily focus on asymptotic stabilization and do not consider finite/fixed-time stabilization of LTI systems. In this note, we focus on the finite/fixed-time stabilization of an LTI system using nonlinear feedback with a static quantizer.

Nonlinear control systems with logarithmic quantizer are studied in [36], [37], [38], [39]. However, to the best of the author’s knowledge, the following question is still open: *is it possible to achieve finite/fixed-time stabilization of a multiple-input LTI system by means of a feedback law with a state quantization?* In this paper, we positively answer this question and design a homogeneity-based feedback preserving finite/fixed-time stability of a continuous-time LTI system in the case of logarithmic-like quantization.

In [23], an asymptotic stabilization of an LTI system by linear feedback with a logarithmic quantizer is addressed by treating a quantization error as sector-bounded uncertainty. This sector-bound approach is common for quadratic stability analysis of linear and quasi-linear control systems with state quantization (e.g., [26], [18], [40], [28], [29], [37]). In the case of a general nonlinear control system, the sector-bounded uncertainty may lead to instability. However, any

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asymptotically stable continuous-time homogeneous system is topologically equivalent to a quadratically stable system [15]. This motivates us to explore a similar sector-bound approach for homogeneous stabilization with quantization.

The key novel contributions of this paper are as follows. Firstly, a sufficient condition for finite/fixed-time stabilization of an LTI system by generalized homogeneous feedback with state quantization is established. In particular, a homogeneous sector-bound condition on quantization error is derived. It covers the conventional linear sector-bound condition as a particular case. Secondly, we introduce a novel quantizer design, which employs a decomposition of dilation and logarithmic quantization of transformed variables to ensure that the quantization error satisfies both homogeneous sector-bound and linear sector-bound conditions simultaneously. Finally, by leveraging the sector-bound properties of the proposed quantizer, we develop a global finite/fixed-time stabilizer with state quantization for a controllable LTI system.

A preliminary version of the paper has been presented at the Conference on Decision and Control 2023 [41], where only the controlled integrator chain has been studied.

The rest of the paper is organized as follows. The next section formulates the problem to be studied in the paper. In Section III, we briefly introduce the analysis of homogeneous control systems and homogeneous state feedback control design for linear systems. Section IV presents the main results, such as a general stability condition of the homogeneous control system with quantized state measurements and finite/fixed-time stabilizer design with logarithmic-like state quantization. Finally, Section V presents numerical simulation results and Section VI provides some concluding remarks.

### Notations

$\mathbb{R}$  is the set of real numbers,  $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ ;  $\mathbb{N}$  is set of all natural numbers without zero;  $\mathbf{0}$  denotes the zero vector from  $\mathbb{R}^n$ ;  $\text{diag}\{\lambda_i\}_{i=1}^n$  is the diagonal matrix with elements  $\lambda_i$ ;  $P \succ 0$  ( $\prec 0$ ,  $\succeq 0$ ,  $\preceq 0$ ) for  $P \in \mathbb{R}^{n \times n}$  means that the matrix  $P$  is symmetric and positive (negative) definite (semi-definite);  $\lambda_{\min}(P)$  and  $\lambda_{\max}(P)$  represent the minimum and maximum eigenvalues of a matrix  $P = P^\top$ ; for  $P \succeq 0$  the square root of  $P$  is a matrix  $M = P^{\frac{1}{2}}$  such that  $M^2 = P$ .  $|x|$ ,  $x \in \mathbb{R}$ , is the absolute value of a scalar.  $\|x\| = \sqrt{x^\top P x}$  is the weighted Euclidean norm of  $x \in \mathbb{R}^n$ , where  $P \succ 0$ ;  $\|x\|_2 = \sqrt{x^\top x}$  is the canonical Euclidean norm of  $x \in \mathbb{R}^n$ ;  $\mathcal{K}$  is the set of continuous strictly increasing functions  $\sigma : \mathbb{R}_+ \mapsto \mathbb{R}_+$  such that  $\sigma(0) = 0$ ;  $I_n$  denotes the identity matrix in  $\mathbb{R}^n$ .  $\mathbf{S} := \{x \in \mathbb{R}^n : \|x\| = 1\}$  denotes the unit sphere.

## II. PROBLEM STATEMENT

Consider the linear system given by:

$$\dot{x} = Ax + Bu, \quad t > 0, \quad x(0) = x_0, \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the system state,  $u(t) \in \mathbb{R}^m$  is the control,  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times m}$  are system matrices. The pair  $(A, B)$  is assumed to be controllable. All state measurements are subject to a quantization and the quantized state vector is available for feedback control.

Mathematically, the quantization involves partitioning the state space into disjoint sets. We define a quantization of  $\mathbb{R}^n$  following ideas of the Voronoi diagram [42].

Let  $\mathcal{E}$  be a subset of  $\mathbb{R}^n$  and  $\Xi$  be a subset of  $\mathbb{N}$ . Let  $\mathcal{D}_i \subseteq \mathcal{E}$  with  $i \in \Xi$  be non-empty disjoint subsets covering  $\mathcal{E}$ :  $\mathcal{D}_i \cap \mathcal{D}_j = \emptyset$  if  $i \neq j$  and  $\bigcup_{i \in \Xi} \mathcal{D}_i = \mathcal{E}$ . Let  $\mathcal{Q} = \{q_i\}_{i \in \Xi}$  such that  $q_i \in \mathcal{D}_i$  for any  $i \in \Xi$ .

**Definition 1.** A function  $q : \mathcal{E} \mapsto \mathcal{Q}$  is said to be a quantization function (quantizer) on the set  $\mathcal{E}$  if  $q(x) = q_i$  for all  $x \in \mathcal{D}_i$  and all  $i \in \Xi$ .

The set  $\mathcal{Q}$  represents a collection of all quantization values. Each disjoint set  $\mathcal{D}_i$  is called as a quantization cell, but the vector  $q_i$  is the quantization seed of  $\mathcal{D}_i$  which represents all states within the quantization cell in order to reduce transmission data (by sending just the number  $i \in \Xi$  instead of the whole state vector  $x \in \mathcal{D}_i$ ). In this paper, we consider quantization functions defined on  $\mathbb{R}^n$ , i.e.,  $q : \mathbb{R}^n \rightarrow \mathcal{Q}$ .

The control system with quantized state measurements, denoted as:

$$\dot{x} = Ax + Bu(q(x)), \quad (2)$$

where  $u(\cdot)$  is a state feedback law which uses the quantized measurements  $q(x)$  of the state  $x$ . Such a closed-loop system is discontinuous. Its solutions are understood in the sense of Filippov [43].

The objective of this paper is to design a controller ensuring global finite/fixed-time stabilization of the system (2) with a state quantizer  $q$ .

## III. PRELIMINARIES

In this note, we focus on finite/fixed-time stability. Several definitions of stability are listed and recalled, as follows:

**Definition 2.** [44], [6], [45] The origin of the system  $\dot{x}(t) = f(t, x(t))$ ,  $t \geq t_0$ ,  $x(t_0) = x_0$ , is said to be

- globally uniformly **Lyapunov stable**, if  $\exists \alpha \in \mathcal{K}$  such that  $\|x(t)\| \leq \alpha(\|x_0\|)$ ,  $\forall t > t_0$ ,  $\forall t_0 \in \mathbb{R}$ ;
- globally uniformly **nearly fixed-time stable**, if it is globally Lyapunov stable and  $\forall r > 0$ ,  $\exists T_r > 0$  such that  $\|x(t)\| \leq r$ ,  $\forall t \geq t_0 + T_r$ ,  $\forall x_0 \in \mathbb{R}^n$ ,  $\forall t_0 \in \mathbb{R}$ ;
- globally uniformly **finite-time stable**, if it is globally Lyapunov stable and  $\forall x_0 \in \mathbb{R}^n$ ,  $\exists T_{x_0} > 0$  such that  $x(t) = 0$ ,  $\forall t \geq t_0 + T_{x_0}$ ,  $\forall t_0 \in \mathbb{R}$ ;
- globally uniformly **fixed-time stable**, if it is globally Lyapunov stable and  $\exists T_{\max} > 0$  such that  $x(t) = 0$ ,  $\forall t \geq t_0 + T_{\max}$ ,  $\forall x_0 \in \mathbb{R}^n$ ,  $\forall t_0 \in \mathbb{R}$ .

Notice that, the nearly fixed-time stable system reaches any neighborhood of the origin within a fixed time, whereas the so-called *practical fixed-time stability* requires the existence of just one such neighborhood.

Homogeneity refers to a class of dilation symmetries, which have been shown to possess several useful properties for control design and analysis [9], [46], [47], [10], [3]. In finite-dimensional systems, linear dilation is widely used.

**Definition 3.** [47] A mapping  $\mathbf{d}(s) : \mathbb{R}^n \mapsto \mathbb{R}^n$ ,  $s \in \mathbb{R}$  is said to be a dilation in  $\mathbb{R}^n$  if

- $\mathbf{d}(0) = I_n$ ,  $\mathbf{d}(s+t) = \mathbf{d}(s)\mathbf{d}(t) = \mathbf{d}(t)\mathbf{d}(s)$ ,  $\forall s, t \in \mathbb{R}$ ;
- $\lim_{s \rightarrow -\infty} \|\mathbf{d}(s)x\| = 0$  and  $\lim_{s \rightarrow +\infty} \|\mathbf{d}(s)x\| = +\infty$ .

The dilation  $\mathbf{d}$  is continuous if the mapping  $s \mapsto \mathbf{d}(s)$  is continuous. A dilation is linear [15] if the mapping  $\mathbf{d}(s)$  is linear for each  $s \in \mathbb{R}$ . Any linear continuous dilation in  $\mathbb{R}^n$  is given by [48]:  $\mathbf{d}(s) = e^{sG_{\mathbf{d}}} := \sum_{i=0}^{\infty} \frac{s^i G_{\mathbf{d}}^i}{i!}$ . Since  $\frac{d}{ds} e^{sG_{\mathbf{d}}} = e^{sG_{\mathbf{d}}} G_{\mathbf{d}} = G_{\mathbf{d}} e^{sG_{\mathbf{d}}}$ , the derivative of  $\mathbf{d}$  at  $s = 0$  is an anti-Hurwitz matrix  $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$  (i.e.,  $-G_{\mathbf{d}}$  is Hurwitz) called the generator of a dilation.

**Definition 4.** [15] A dilation  $\mathbf{d}$  is strictly monotone with respect to a norm  $\|\cdot\|$  in  $\mathbb{R}^n$  if  $\exists \beta > 0 : \|\mathbf{d}(s)\| \leq e^{\beta s}$ ,  $\forall s \leq 0$ .

By default, below, we deal with linear continuous dilation. The following result is the straightforward consequence of the quadratic Lyapunov function theorem for linear systems.

**Proposition 1.** [15] A linear continuous dilation  $\mathbf{d}$  in  $\mathbb{R}^n$  is strictly monotone with respect to the norm  $\|z\| = \sqrt{z^T P z}$  if and only if the following linear matrix inequality holds

$$P \succ 0, PG_{\mathbf{d}} + G_{\mathbf{d}}^T P \succ 0, \quad (3)$$

where  $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$  is the generator of the linear continuous dilation  $\mathbf{d}$  in  $\mathbb{R}^n$ .

The linear continuous dilation  $\mathbf{d}$  induces an alternative topology in  $\mathbb{R}^n$  via a ‘‘homogeneous norm’’ [49], [50].

**Definition 5.** [15] The function  $\|\cdot\|_{\mathbf{d}} : \mathbb{R}^n \mapsto \mathbb{R}_+$  defined as  $\|\mathbf{0}\|_{\mathbf{d}} = 0$  and  $\|x\|_{\mathbf{d}} = e^s$ , where  $s \in \mathbb{R} : \|\mathbf{d}(-s)x\| = 1$ ,  $x \neq \mathbf{0}$ , is called the canonical homogeneous norm in  $\mathbb{R}^n$ , where  $\mathbf{d}$  is a continuous monotone linear dilation on  $\mathbb{R}^n$ .

The monotonicity of the dilation is required to guarantee that the functional  $\|\cdot\|_{\mathbf{d}}$  is single-valued and continuous. Although the canonical homogeneous norm does not satisfy the triangle inequality in  $\mathbb{R}^n$ , it is a norm in a special Euclidean space [8] being homeomorphic to  $\mathbb{R}^n$ .

**Proposition 2.** [15], [8, Corollary 6.4] Let  $\mathbf{d}$  be a strictly monotone linear continuous dilation on  $\mathbb{R}^n$ . Then

$$\begin{cases} e^{\eta s} \leq \|\mathbf{d}(s)\| \leq \|\mathbf{d}(s)\| \leq e^{\bar{\eta} s}, & s \geq 0, \\ e^{\bar{\eta} s} \leq \|\mathbf{d}(s)\| \leq \|\mathbf{d}(s)\| \leq e^{\eta s}, & s \leq 0, \end{cases} \quad \forall s \in \mathbb{R}, \quad (4)$$

$$\begin{cases} \|x\|_{\mathbf{d}}^{\eta} \leq \|x\| \leq \|x\|_{\mathbf{d}}^{\bar{\eta}}, & \|x\| \geq 1, \\ \|x\|_{\mathbf{d}}^{\bar{\eta}} \leq \|x\| \leq \|x\|_{\mathbf{d}}^{\eta}, & \|x\| \leq 1, \end{cases} \quad \forall x \in \mathbb{R}^n, \quad (5)$$

where  $\|\mathbf{d}(s)\| = \inf_{u \in S} \|\mathbf{d}(s)u\| = \inf_{u \neq 0} \frac{\|\mathbf{d}(s)u\|}{\|u\|}$ ,

$$\bar{\eta} = \frac{1}{2} \lambda_{\max}(P^{\frac{1}{2}} G_{\mathbf{d}} P^{-\frac{1}{2}} + P^{-\frac{1}{2}} G_{\mathbf{d}}^T P^{\frac{1}{2}}) > 0,$$

$$\eta = \frac{1}{2} \lambda_{\min}(P^{\frac{1}{2}} G_{\mathbf{d}} P^{-\frac{1}{2}} + P^{-\frac{1}{2}} G_{\mathbf{d}}^T P^{\frac{1}{2}}) > 0.$$

The following lemma establishes a connection between Euclidean and canonical homogeneous norms offering a valuable tool for the analysis of quantizers.

**Lemma 1.** Let  $G_{\mathbf{d}}$  be a diagonalizable generator of a continuous linear dilation  $\mathbf{d}$  and  $G_{\mathbf{d}} = \Gamma \Lambda \Gamma^{-1}$ , where  $\Lambda$  is a diagonal matrix. For any  $z, y \in \mathbb{R}^n$ ,  $0 < \varepsilon < 1$ , there exists  $\delta > 0$  such that

$$|\xi_{z_i}| \leq \delta |\xi_{y_i}|, \forall i = 1 \dots n \quad \Rightarrow \quad \|z\|_{\mathbf{d}} \leq \varepsilon \|y\|_{\mathbf{d}},$$

where  $\xi_z = \Gamma^{-1}z = [\xi_{z,1}, \xi_{z,2}, \dots, \xi_{z,n}]^T$ ,  $\xi_y = \Gamma^{-1}y = [\xi_{y,1}, \xi_{y,2}, \dots, \xi_{y,n}]^T$ , the canonical homogeneous norm  $\|\cdot\|_{\mathbf{d}}$  is induced by the norm  $\|x\| = \sqrt{x^T P x}$ ,  $x \in \mathbb{R}^n$  with a symmetric matrix  $P \in \mathbb{R}^{n \times n}$  satisfying (3).

*Proof.* Notice that  $|\xi_{z,i}| \leq \delta |\xi_{y,i}|$  indicates that  $\sum_{i=1}^n \xi_{z,i}^2 \leq \delta^2 \sum_{i=1}^n \xi_{y,i}^2$ . Besides, since  $G_{\mathbf{d}}$  is anti-Hurwitz, we have

$$\sum_{i=1}^n e^{-2\lambda_i s} \xi_{z,i}^2 \leq \delta^2 \sum_{i=1}^n e^{-2\lambda_i s} \xi_{y,i}^2, \quad \forall s \in \mathbb{R}, \lambda_i > 0,$$

and we note  $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \succ 0$ . The above inequality is equivalent to

$$\|e^{-\Lambda s} \Gamma^{-1} z\|_2 \leq \delta \|e^{-\Lambda s} \Gamma^{-1} y\|_2, \quad s \in \mathbb{R}.$$

By Definition 5 of the canonical homogeneous norm, we have

$$1 = \|\Gamma e^{-\Lambda s_y} \Gamma^{-1} y\|_2^2 \geq \lambda_{\min}(\Gamma^T P \Gamma) \|e^{-\Lambda s_y} \Gamma^{-1} y\|_2^2,$$

where  $s_y = \ln \|y\|_{\mathbf{d}}$ . The latter inequality indicates that

$$\|e^{-\Lambda s_y} \Gamma^{-1} z\|_2 \leq \delta \lambda_{\min}^{-1/2}(\Gamma^T P \Gamma).$$

Then, since  $G_{\mathbf{d}}$  is anti-Hurwitz and  $0 < \varepsilon < 1$ , we can derive

$$\begin{aligned} \|e^{-\Lambda \ln \varepsilon} e^{-\Lambda s_y} \Gamma^{-1} z\|_2 &\leq \frac{\delta \lambda_{\max}(e^{-\Lambda \ln \varepsilon})}{\lambda_{\min}^{1/2}(\Gamma^T P \Gamma)} \\ \Rightarrow \|\Gamma e^{-\Lambda \ln \varepsilon} e^{-\Lambda s_y} \Gamma^{-1} z\| &\leq \frac{\delta \lambda_{\max}(e^{-\Lambda \ln \varepsilon}) \lambda_{\max}^{1/2}(\Gamma^T P \Gamma)}{\lambda_{\min}^{1/2}(\Gamma^T P \Gamma)} \quad (6) \\ \Rightarrow \|\mathbf{d}(-\ln \varepsilon \|y\|) z\| &\leq \delta \lambda_{\max}(e^{-\Lambda \ln \varepsilon}) \frac{\lambda_{\max}^{1/2}(\Gamma^T P \Gamma)}{\lambda_{\min}^{1/2}(\Gamma^T P \Gamma)}. \end{aligned}$$

The latter inequality indicates that for a given  $0 < \varepsilon < 1$ , there exists a sufficiently small  $\delta > 0$  such that

$$\|\mathbf{d}(-\ln \varepsilon \|y\|) z\| = \|\mathbf{d}(\ln \frac{\|z\|_{\mathbf{d}}}{\varepsilon \|y\|_{\mathbf{d}}}) \mathbf{d}(-\ln \|z\|_{\mathbf{d}}) z\| \leq 1.$$

Taking into account  $\|\mathbf{d}(-\ln \|z\|_{\mathbf{d}}) z\| = 1$ , we obtain

$$1 \geq \|\mathbf{d}(-\ln \varepsilon \|y\|_{\mathbf{d}}) z\| \geq \left[ \mathbf{d} \left( \ln \frac{\|z\|_{\mathbf{d}}}{\varepsilon \|y\|_{\mathbf{d}}} \right) \right].$$

Hence, using (4), we conclude  $\ln \frac{\|z\|_{\mathbf{d}}}{\varepsilon \|y\|_{\mathbf{d}}} \leq 0$ . The proof is complete.  $\square$

The homogeneous function and vector field are defined following the paper [47].

**Definition 6.** A vector field  $f : \mathbb{R}^n \mapsto \mathbb{R}^n$  (resp., a function  $h : \mathbb{R}^n \mapsto \mathbb{R}$ ) is said to be  $\mathbf{d}$ -homogeneous of degree  $\mu \in \mathbb{R}$  if

$$f(\mathbf{d}(s)) = e^{\mu s} \mathbf{d}(s) f(x), \quad (\text{resp.}, h(\mathbf{d}(s)) = e^{\mu s} h(x)),$$

$\forall x \in \mathbb{R}^n, \forall s \in \mathbb{R}$ , where  $\mathbf{d}$  is a linear continuous dilation.

For a linear vector field  $Ax$ , its homogeneity is revealed in [15] and [16]. The properties are listed as follows:

**Lemma 2.** [15], [16] Let  $\mathbf{d}$  be a linear continuous dilation in  $\mathbb{R}^n$ . The linear vector field  $f(x) = Ax$  with  $x \in \mathbb{R}^n$ ,  $A \in \mathbb{R}^{n \times n}$  is  $\mathbf{d}$ -homogeneous of degree  $\mu \in \mathbb{R}$  if and only if

$$AG_{\mathbf{d}} = (\mu I + G_{\mathbf{d}})A, \quad (7)$$

where  $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$  is a generator of  $\mathbf{d}$ . Moreover, the following two claims are equivalent:

- There exists a dilation  $\mathbf{d}(s) = e^{sG_{\mathbf{d}}}$  in  $\mathbb{R}^n$  such that the vector field  $x \mapsto Ax$  is  $\mathbf{d}$ -homogeneous of degree  $\mu \neq 0$ .
- $A$  is nilpotent.

Any asymptotically stable homogeneous system with a non-zero degree exhibits finite-time stability (for negative degree)

or nearly fixed-time stability (for positive degree) [51], [3]. A state feedback homogeneous stabilization of LTI systems has been proposed in [52], [16], and [8].

**Theorem 1.** [8], [16] *If the linear equation*

$$AG_0 + BY_0 = G_0A + A, \quad G_0B = \mathbf{0}. \quad (8)$$

*has a solution  $G_0 \in \mathbb{R}^{n \times n}$  and  $Y_0 \in \mathbb{R}^{m \times n}$  such that  $G_0 - I_n$  is invertible and for any  $\mu \in [-1, 1/n]$  the matrix  $G_{\mathbf{d}} = I_n + \mu G_0$  is anti-Hurwitz, then the linear system (1) with the control*

$$u(x) = K_0x + \|x\|_{\mathbf{d}}^{1+\mu} K_{\mathbf{d}} (-\ln \|x\|_{\mathbf{d}}) x, \quad K = YX^{-1} \quad (9)$$

*is  $\mathbf{d}$ -homogeneous of degree  $\mu$  and globally asymptotically stable provided that  $K_0 = Y_0(G_0 - I_n)^{-1}$ ,  $A_0 = A + BK_0$  is nilpotent,  $X \in \mathbb{R}^{n \times n}$  and  $Y \in \mathbb{R}^{m \times n}$  satisfy the following algebraic system*

$$\begin{cases} XA_0^{\top} + A_0X + Y^{\top}B^{\top} + BY + \rho(XG_{\mathbf{d}}^{\top} + G_{\mathbf{d}}X) = \mathbf{0}, \\ XG_{\mathbf{d}}^{\top} + G_{\mathbf{d}}X \succ 0, \quad X \succ 0, \end{cases} \quad (10)$$

*and the canonical homogeneous norm  $\|\cdot\|_{\mathbf{d}}$  is induced by the norm  $\|x\| = \sqrt{x^{\top}X^{-1}x}$ . Moreover, the closed-loop system is: globally uniformly finite-time stable for  $\mu < 0$ ; globally uniformly exponentially stable for  $\mu = 0$ ; globally uniformly nearly fixed-time stable for  $\mu > 0$ . Moreover,  $\|\cdot\|_{\mathbf{d}}$  is a Lyapunov function of the system:*

$$\frac{d}{dt}\|x\|_{\mathbf{d}} = -\rho\|x\|_{\mathbf{d}}^{1+\mu}. \quad (11)$$

If the pair  $\{A, B\}$  is controllable, then any solution of the system (8) satisfies conditions of the latter theorem for  $\mu \leq 1/\underline{k}$ , where  $\underline{k}$  is a minimal number such that  $\text{rank}[B, AB, \dots, A^{\underline{k}-1}B] = n$ . The equation (8) determines the matrices  $G_{\mathbf{d}}$  and  $K_0$  such that  $A_0 := A + BK_0$  is  $\mathbf{d}$ -homogeneous of degree  $\mu$  (i.e., (7) holds). The feasibility of (10) has been proven in [52], [16]. Notice also that the matrix  $G_0$  (and respectively,  $G_{\mathbf{d}}$ ) in Theorem 1 can always be selected diagonalizable [8].

#### IV. HOMOGENEOUS CONTROL WITH STATE QUANTIZATION

##### A. Stability of homogeneous control system with quantized state measurements

Due to nonlinearity, the design of finite/fixed-time controllers with quantized state measurements is a non-trivial task. To tackle this problem we propose the following two-step procedure:

*Step 1:* First, we design a continuous-time closed-loop system with finite/fixed-time stabilizing feedback. Next, we replace the continuous states with quantized states in the feedback law and derive some sufficient finite/fixed-time stability conditions of the closed-loop system with quantization.

*Step 2:* Leveraging the derived quantization conditions, we design a quantization function using a logarithmic quantizer such that finite/fixed-time stabilization is preserved.

The continuous-time homogeneous finite/fixed-time stabilizer (9) is assumed to be designed for an LTI system by means of Theorem 1. Our goal now is to derive a sufficient condition for the quantization function to preserve the finite/fixed

stability of the closed-loop system in the case of quantized state measurements. The homogeneous controller (9) contains two terms. The linear one  $K_0x$  is aimed at homogenization of  $A_0 = A + BK_0$  with non-zero degree (see Lemma 2), while the nonlinear term stabilizes the closed-loop homogeneous system. If  $A$  is a nilpotent matrix, then  $K_0$  may be equal to zero. Below we study both cases  $K_0 \neq \mathbf{0}$  and  $K_0 = \mathbf{0}$ , since, in the latter case, the control law also becomes a homogeneous function, allowing the restrictions on the quantizer to be relaxed.

**Theorem 2.** *Let all parameters of the homogeneous feedback (9) be designed as in Theorem 1, and define the quantized feedback as:*

$$u(\mathbf{q}(x)) = K_0\mathbf{q}(x) + \|\mathbf{q}(x)\|_{\mathbf{d}}^{1+\mu} K_{\mathbf{d}} (-\ln \|\mathbf{q}(x)\|_{\mathbf{d}}) \mathbf{q}(x), \quad (12)$$

*where  $\mathbf{q}: \mathbb{R}^n \mapsto \mathcal{Q}$  is a quantization function. If  $K_0 = \mathbf{0}$  and, for a sufficiently small  $\epsilon > 0$ , the quantization error satisfies*

$$\|\mathbf{q}(x) - x\|_{\mathbf{d}} \leq \epsilon \|x\|_{\mathbf{d}}, \quad \forall x \in \mathbb{R}^n, \quad (13)$$

*then the system (2) with (12) is*

- globally uniformly finite-time stable for  $-1 < \mu < 0$ ;
- globally uniformly nearly fixed-time stable for  $\mu > 0$ .

*If  $K_0 \neq \mathbf{0}$  and the quantization error additionally satisfies*

$$\|\mathbf{q}(x) - x\| \leq \kappa \|x\|, \quad \forall x \in \mathbb{R}^n, \quad (14)$$

*with a sufficiently small  $\kappa > 0$ , then the system (2) with (12) is*

- locally uniformly finite-time stable for  $-1 < \mu < \min\{\underline{\eta} - 1, 0\}$ ;
- globally uniformly practically fixed-time stable for  $\mu > \max\{\bar{\eta} - 1, 0\}$ ,

*where  $0 < \underline{\eta} \leq \bar{\eta}$  are defined in Proposition 2.*

*Proof.* The inequality (13) implies that  $\mathbf{q}(\mathbf{0}) = \mathbf{0}$  and  $x = \mathbf{0}$  is the equilibrium of the system. Since  $\|x\|_{\mathbf{d}}$  serves as a Lyapunov function for the quantization-free system (1), let us calculate its derivative with respect to time:

$$\begin{aligned} \frac{d}{dt}\|x\|_{\mathbf{d}} &= \|x\|_{\mathbf{d}} \frac{x^{\top} \mathbf{d}^{\top} (-s_x) P \mathbf{d} (-s_x)}{x^{\top} \mathbf{d}^{\top} (-s_x) P G_{\mathbf{d}} \mathbf{d} (-s_x) x} (Ax + Bu(\mathbf{q}_x)) \\ &= \frac{\|x\|_{\mathbf{d}} x^{\top} \mathbf{d}^{\top} (-s_x) P \mathbf{d} (-s_x) (Ax + B(u(x) + u(\mathbf{q}_x) - u(x)))}{x^{\top} \mathbf{d}^{\top} (-s_x) P G_{\mathbf{d}} \mathbf{d} (-s_x) x}, \end{aligned} \quad (15)$$

where  $s_x = \ln \|x\|_{\mathbf{d}}$  and  $\mathbf{q}_x = \mathbf{q}(x)$ . The equation (8) leads to  $A_0 \mathbf{d}(s) = e^{\mu s} \mathbf{d}(s) A_0$ ,  $\mathbf{d}(s) B = e^s B$ ,  $\forall s \in \mathbb{R}$  and

$$\frac{d}{dt}\|x\|_{\mathbf{d}} = -\rho\|x\|_{\mathbf{d}}^{1+\mu} + \frac{x^{\top} \mathbf{d}^{\top} (-s_x) P B (u(\mathbf{q}_x) - u(x))}{x^{\top} \mathbf{d}^{\top} (-s_x) P G_{\mathbf{d}} \mathbf{d} (-s_x) x}. \quad (16)$$

The control  $u(x)$  consists of two parts: the linear part  $K_0x$  and the homogeneous part  $\tilde{u}(x) = \|x\|_{\mathbf{d}}^{1+\mu} K_{\mathbf{d}} (-\ln \|x\|_{\mathbf{d}}) x$ . Therefore, we derive

$$\begin{aligned} \frac{d}{dt}\|x\|_{\mathbf{d}} &= -\rho\|x\|_{\mathbf{d}}^{1+\mu} + \frac{x^{\top} \mathbf{d}^{\top} (-s_x) P B K_0 (\mathbf{q}_x - x)}{x^{\top} \mathbf{d}^{\top} (-s_x) P G_{\mathbf{d}} \mathbf{d} (-s_x) x} \\ &\quad + \frac{x^{\top} \mathbf{d}^{\top} (-s_x) P B (\tilde{u}(\mathbf{q}_x) - \tilde{u}(x))}{x^{\top} \mathbf{d}^{\top} (-s_x) P G_{\mathbf{d}} \mathbf{d} (-s_x) x}. \end{aligned} \quad (17)$$

By definition of canonical homogeneous norm we have  $\|\mathbf{d}(-s_x)x\| = 1$ . The Cauchy–Schwarz inequality yields that

$$\frac{d\|x\|_{\mathbf{d}}}{dt} \leq -\rho\|x\|_{\mathbf{d}}^{1+\mu} + c_1 \|\mathbf{q}_x - x\| + \frac{x^{\top} \mathbf{d}^{\top} (-s_x) P B (\tilde{u}(\mathbf{q}_x) - \tilde{u}(x))}{x^{\top} \mathbf{d}^{\top} (-s_x) P G_{\mathbf{d}} \mathbf{d} (-s_x) x}, \quad (18)$$

where  $c_1 = \frac{2\sqrt{\lambda_{\max}(P^{-1/2}K_0^{\top}B^{\top}PBK_0P^{-1/2})}}{\lambda_{\min}(P^{1/2}G_{\mathbf{d}}P^{-1/2} + P^{-1/2}G_{\mathbf{d}}^{\top}P^{1/2})}$ .

Since  $B\tilde{u}(\mathbf{d}(s)z) = e^{(1+\mu)s}B\tilde{u}(z), \forall s \in \mathbb{R}, \forall z \in \mathbb{R}^n$ ,

$$\begin{aligned} \frac{d}{dt}\|x\|_{\mathbf{d}} \leq & -\rho\|x\|_{\mathbf{d}}^{1+\mu} + c_1\|q_x - x\| \\ & + c_2\|x\|_{\mathbf{d}}^{1+\mu}\|\tilde{u}(\mathbf{d}(-s_x)q_x) - \tilde{u}(\mathbf{d}(-s_x)x)\|, \end{aligned} \quad (19)$$

where  $c_2 = \frac{2\sqrt{\lambda_{\max}(P^{-1/2}B^{\top}PBP^{-1/2})}}{\lambda_{\min}(P^{1/2}G_{\mathbf{d}}P^{-1/2} + P^{-1/2}G_{\mathbf{d}}^{\top}P^{1/2})}$ .

Taking into account the monotonicity of the dilation  $\mathbf{d}$  (see Definition 4), the condition (13) implies that

$$\begin{aligned} e^{-\ln \epsilon} e^{-s_x}\|q_x - x\|_{\mathbf{d}} \leq 1 & \Rightarrow \|\mathbf{d}(-s_x - \ln \epsilon)(q_x - x)\|_{\mathbf{d}} \leq 1 \\ & \Rightarrow \|\mathbf{d}(-\ln \epsilon)\mathbf{d}(-s_x)(q_x - x)\| \leq 1 \\ & \Rightarrow \lfloor \mathbf{d}(-\ln \epsilon) \rfloor \cdot \|\mathbf{d}(-s_x)(q_x - x)\| \leq 1. \end{aligned} \quad (20)$$

Due to (4), there exists  $\alpha \in \mathcal{K}$  such that  $\frac{1}{\lfloor \mathbf{d}(-\ln \epsilon) \rfloor} \leq \alpha(\epsilon)$ , leading to

$$\|\mathbf{d}(-s_x)q_x\| = \|\mathbf{d}(-s_x)x + \mathbf{d}(-s_x)(q_x - x)\| \leq 1 + \alpha(\epsilon).$$

Since  $\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x$  lies on the unit sphere for any  $x \neq 0$ , then for any  $0 < \epsilon < \infty$ , there exists a compact set, which contains the origin, such that  $\mathbf{d}(-\ln \|x\|_{\mathbf{d}})q_x$  and  $\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x$  always belong to this compact set for all  $x \in \mathbb{R}^n \setminus \{0\}$ . Since  $\tilde{u}$  is continuous on  $\mathbb{R}^n$ , it is uniformly continuous on a compact set. Therefore, there exists a class- $\mathcal{K}$  function  $\gamma$ , such that

$$\|\tilde{u}(\mathbf{d}(-s_x)q_x) - \tilde{u}(\mathbf{d}(-s_x)x)\| \leq \gamma(\|\mathbf{d}(-s_x)(q_x - x)\|). \quad (21)$$

Then, using the above inequality and (20), we derive

$$\frac{d}{dt}\|x\|_{\mathbf{d}} \leq -(\rho - \bar{\gamma}(\epsilon))\|x\|_{\mathbf{d}}^{1+\mu} + c_1\|q_x - x\|, \quad (22)$$

where  $\bar{\gamma} = c_2\gamma \circ \alpha \in \mathcal{K}$ . For  $K_0 = 0 \Rightarrow c_1 = 0$ , we have

$$\frac{d}{dt}\|x\|_{\mathbf{d}} \leq -(\rho - \bar{\gamma}(\epsilon))\|x\|_{\mathbf{d}}^{1+\mu}, \quad (23)$$

where  $\bar{\gamma}^{-1}$  is the inverse function of  $\bar{\gamma}$ . If  $0 < \epsilon < \bar{\gamma}^{-1}(\rho)$ , then the system is globally finite-time stable for  $\mu < 0$ ,  $x(t) = 0, \forall t \geq \frac{\|x(0)\|_{\mathbf{d}}^{\mu}}{-\mu(\rho - \bar{\gamma}(\epsilon))}$ ; exponentially stable for  $\mu = 0$  and nearly fixed-time stable for  $\mu > 0$ ,  $\|x(t)\|_{\mathbf{d}} \leq r, \forall t \geq \frac{1}{(\rho - \bar{\gamma}(\epsilon))\mu r^{\mu}}, \forall r \in \mathbb{R}_+$ .

If  $K_0 \neq 0$  (i.e.,  $c_1 \neq 0$ ), let us consider the relation between the Euclidean norm and the canonical homogeneous norm given in Proposition 2. It yields that

$$\begin{cases} \|q - x\| \leq \kappa\|x\|_{\mathbf{d}}^{\underline{\eta}}, \|x\|_{\mathbf{d}} \leq 1, \\ \|q - x\| \leq \kappa\|x\|_{\mathbf{d}}^{\bar{\eta}}, \|x\|_{\mathbf{d}} \geq 1. \end{cases} \quad (24)$$

Using the above estimate and (22), we derive that

$$\frac{d\|x\|_{\mathbf{d}}}{dt} \leq \begin{cases} -\left[\rho - \bar{\gamma}(\epsilon) - c_1\kappa\|x\|_{\mathbf{d}}^{\underline{\eta}-(1+\mu)}\right]\|x\|_{\mathbf{d}}^{1+\mu}, \|x\|_{\mathbf{d}} \leq 1, \\ -\left[\rho - \bar{\gamma}(\epsilon) - c_1\kappa\|x\|_{\mathbf{d}}^{\bar{\eta}-(1+\mu)}\right]\|x\|_{\mathbf{d}}^{1+\mu}, \|x\|_{\mathbf{d}} \geq 1, \end{cases} \quad (25)$$

for some  $\kappa > 0$ . Let  $0 < \epsilon < \bar{\gamma}^{-1}(\rho)$ , and we have the following two cases:

- 1) If  $\mu > \max\{\bar{\eta}-1, \underline{\eta}-1\} = \bar{\eta}-1$  we have  $\|x\|_{\mathbf{d}}^{\bar{\eta}-(1+\mu)} \rightarrow 0$  and  $\|x\|_{\mathbf{d}}^{\underline{\eta}-(1+\mu)} \rightarrow 0$  as  $\|x\|_{\mathbf{d}} \rightarrow \infty$ . Thus, the system (2) with (12) is globally practically fixed-time stable due to  $\mu > 0$ .
- 2) If  $\mu \leq \min\{\bar{\eta}-1, \underline{\eta}-1\} = \underline{\eta}-1$  then  $\|x\|_{\mathbf{d}}^{\bar{\eta}-(1+\mu)} \rightarrow 0$  and  $\|x\|_{\mathbf{d}}^{\underline{\eta}-(1+\mu)} \rightarrow 0$  as  $\|x\|_{\mathbf{d}} \rightarrow 0$ . Hence, the system (2) with (12) is locally finite-time stable if  $\mu < 0$ .

The proof is complete.  $\square$

A larger gain  $\rho$  and a denser quantization (i.e., smaller  $\epsilon$ ) imply faster convergence. For  $K_0 \neq 0$ , the region of attraction (resp., the attractive set) of locally finite-time (resp., globally practically fixed-time) stable system (2), (12) with  $\mu < 0$  (resp.  $\mu > 0$ ) can be tuned arbitrarily large (resp., small) by a selection of small enough  $\epsilon > 0$  and  $\kappa > 0$ .

**Remark 1.** *The sector-bounded condition  $\|q(x) - x\| \leq \kappa\|x\|$  is sufficient for exponential stabilization of a linear control system with state quantization (see, e.g., [23], [53]). For  $\mathbf{d}$ -homogeneous LTI system with the  $\mathbf{d}$ -homogeneous quantized feedback, the finite/fixed-time stabilization condition is expressed in the same form but utilizing the canonical homogeneous norm (13). Furthermore, for  $G_{\mathbf{d}} = I_n$  (i.e.,  $\mu = 0$ ) we have  $\|x\|_{\mathbf{d}} = \|x\|$ . This corresponds to the linear case considered above. Below we call the inequality (13) as **the homogeneous sector-bounded condition**.*

At first sight, the condition for the quantizer in Theorem 2 is not easy to check since it is expressed in terms of the canonical homogeneous norm defined implicitly. However, in the next section, we present a quantizer design approach that leverages the well-known logarithmic quantizer and exploits the homogeneity to fulfill the required condition.

### B. Finite/fixed-time stabilization under logarithmic-like quantization

The objective of this section is to study the feasibility of quantizer design for the finite/fixed-time stabilization of LTI system with state quantization. Our design is based on the logarithmic quantizer. The distinguishing property of a logarithmic quantizer is that the quantization error decreases as the states approach the origin. This allows the convergence/stability property of the original system to be preserved. Let us briefly recall the corresponding definitions.

**Logarithmic quantizer** is described by

$$q(\phi) = \begin{cases} \nu^i \zeta_0, & \frac{\nu^i \zeta_0}{1+\delta} < \phi \leq \frac{\nu^i \zeta_0}{1-\delta}, i=0, \pm 1, \pm 2, \dots, \\ 0, & \phi = 0, \\ -q(-\phi), & \phi < 0, \end{cases} \quad (26)$$

where  $q(\phi), \phi \in \mathbb{R}, \zeta_0 \in \mathbb{R}_+, \nu \in (0, 1)$  represents the quantization density and  $\delta = (1-\nu)/(1+\nu)$ . A small  $\nu$  (resp., large  $\delta$ ) implies a coarse quantization, but a large  $\nu$  (resp., small  $\delta$ ) means a dense quantization. For the logarithmic quantizer, the quantization error is sector-bounded [23]:

$$|q(\phi) - \phi| \leq \delta|\phi|, \delta \in (0, 1), \quad (27)$$

and vanishing as the state goes to the origin.

Directly using logarithmic quantizers may not generally guarantee the homogeneous sector-bounded condition (13) to be fulfilled. To validate the stability condition (13), a special relation between  $|\cdot|$  and  $\|\cdot\|_{\mathbf{d}}$  has been derived in Lemma 1. Based on the mentioned results, we design a novel quantizer  $q(x)$  that employs the transformation matrix  $\Gamma$  and logarithmic quantization, such that the conditions of Theorem 2 are fulfilled.

**Corollary 1.** Let  $G_{\mathbf{d}}$  be a diagonalizable generator of the linear continuous dilation  $\mathbf{d}$  in  $\mathbb{R}^n$ :

$$\exists \Gamma \in \mathbb{R}^{n \times n} : G_{\mathbf{d}} = \Gamma \Lambda \Gamma^{-1}, \quad \Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \succ 0,$$

and the canonical homogeneous norm  $\|x\|_{\mathbf{d}}$  be induced by the weighted Euclidean norm  $\|x\| = \sqrt{x^T P x}$ , where  $x \in \mathbb{R}^n$  and  $0 \prec P = P^T \in \mathbb{R}^{n \times n}$  satisfies (3). If  $q$  is a logarithmic quantization function with parameter  $\delta > 0$  and

$$\mathbf{q}(x) = \Gamma [q(\xi_{x,1}), q(\xi_{x,2}), \dots, q(\xi_{x,n})]^T, \quad (28)$$

where  $\xi_x = \Gamma^{-1}x = [\xi_{x,1}, \xi_{x,2}, \dots, \xi_{x,n}]^T$ ,  $x \in \mathbb{R}^n$ , then for any  $\epsilon > 0$  and any  $\kappa > 0$ , there exists  $\delta > 0$  such that the inequalities (13), (14) are fulfilled simultaneously.

*Proof.* On the one hand, from the formula (28), we have

$$\Gamma^{-1}(\mathbf{q}(x) - x) = [q(\xi_{x,1}) - \xi_{x,1}, q(\xi_{x,2}) - \xi_{x,2}, \dots, q(\xi_{x,n}) - \xi_{x,n}]^T,$$

where  $\Gamma^{-1}x = [\xi_{x,1}, \xi_{x,2}, \dots, \xi_{x,n}]^T$ . Since  $G_{\mathbf{d}}$  is diagonalizable, according to the quantization error of the logarithmic quantizer (27), by Lemma 1, we conclude that for any given  $\epsilon > 0$ , there exists a sufficiently small  $\delta > 0$  such that the inequality  $|q(\xi_{x,i}) - \xi_{x,i}| \leq \delta |\xi_{x,i}|$ ,  $\forall i = 1, 2, \dots, n$  implies the inequality (13).

On the other hand, since  $\Gamma$  is invertible,  $\Gamma^{-T} \Gamma^{-1}$  is positive definite, the inequality (27) also implies that

$$\begin{aligned} \|\mathbf{q}(x) - x\|_2 &= \|\Gamma ([q(\xi_{x,1}), q(\xi_{x,2}), \dots, q(\xi_{x,n})]^T - \xi_x)\|_2 \\ &\leq \lambda_{\max}^{1/2}(\Gamma^T \Gamma) \left( \sum_{i=1}^n |q(\xi_{x,i}) - \xi_{x,i}|^2 \right)^{1/2} \\ &\leq \lambda_{\max}^{1/2}(\Gamma^T \Gamma) \left( \sum_{i=1}^n |\delta \xi_{x,i}|^2 \right)^{1/2} \\ &\leq \delta \lambda_{\max}^{1/2}(\Gamma^T \Gamma) \lambda_{\max}^{1/2}(\Gamma^{-T} \Gamma^{-1}) \|x\|_2. \end{aligned} \quad (29)$$

This yields (14) with

$$\kappa = \delta \lambda_{\max}^{1/2}(\Gamma^T \Gamma) \lambda_{\max}^{1/2}(\Gamma^{-T} \Gamma^{-1}) \frac{\sqrt{\lambda_{\min}(P)}}{\sqrt{\lambda_{\max}(P)}}. \quad (30)$$

The proof is complete.  $\square$

The proven corollary immediately implies that all conditions of Theorem 2 are fulfilled for the constructed quantizer (28) with a sufficiently small  $\delta > 0$ , provided that the linear dilation  $\mathbf{d}$  has a diagonalizable generator. Therefore, the quantized control (12) is a global (local) stabilizer of the system (2).

**Remark 2.** For the controlled chain of integrators:  $A = \begin{pmatrix} 0 & I_{n-1} \\ 0 & 0 \end{pmatrix}$ ,  $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  the dilation  $\mathbf{d}$  has a diagonal generator, so  $\Gamma = I_n$  and all conditions of Theorem 2 hold even for the classical logarithmic quantizer:  $\mathbf{q}(x) = [q(x_1), \dots, q(x_n)]^T$ ,  $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ .

The global finite/fixed-time stabilizer can be designed by a combination of various homogeneous stabilizers with negative and positive degrees.

**Corollary 2.** Let  $G_0$ ,  $Y_0$ , and  $K_0 = Y_0(G_0 - I_n)^{-1}$ ,  $A_0 = A + BK_0$  be defined as in Theorem 1. Let  $G_0$  be diagonalizable

$$\exists \Gamma \in \mathbb{R}^{n \times n} : G_0 = \Gamma \Lambda_0 \Gamma^{-1}, \quad \Lambda_0 = \text{diag}(\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_n) \succ 0,$$

and the logarithmic quantizer  $\mathbf{q}$  be defined by the formula (28) with a parameter  $\delta > 0$ . Suppose that the feedback control with quantized state is defined as follows:

$$u = \begin{cases} K_0 \mathbf{q}(x) + \|\mathbf{q}(x)\|_{\mathbf{d}_1}^{1+\mu_1} K_{\mathbf{d}_1} (-\ln \|\mathbf{q}(x)\|_{\mathbf{d}_1}) \mathbf{q}(x) & \text{if } \|\mathbf{q}(x)\| \geq 1, \\ K_0 \mathbf{q}(x) + \|\mathbf{q}(x)\|_{\mathbf{d}_2}^{1+\mu_2} K_{\mathbf{d}_2} (-\ln \|\mathbf{q}(x)\|_{\mathbf{d}_2}) \mathbf{q}(x) & \text{if } \|\mathbf{q}(x)\| < 1, \end{cases} \quad (31)$$

where  $K = YX^{-1}$ ,  $\mathbf{d}_i(\cdot)$ ,  $i = 1, 2$  are generated by  $G_{\mathbf{d}_i} = G_0 + \mu_i I_n$ , with  $0 \leq \mu_1 \leq 1/n$  and  $-1 \leq \mu_2 < 0$ ,  $\|\cdot\|_{\mathbf{d}_i}$  are induced by the norm  $\|x\| = \sqrt{x^T P x}$ ,  $P = X^{-1}$ . If  $X \in \mathbb{R}^{n \times n}$  and  $Y \in \mathbb{R}^{m \times n}$  satisfy the following algebraic system:

$$\begin{cases} X A_0^T + A_0 X + Y^T B^T + B Y \prec 0, & X \succ 0, \\ 2(1 + \mu_1)X \succeq X G_{\mathbf{d}_1}^T + G_{\mathbf{d}_1} X \succ 0, \\ X G_{\mathbf{d}_2}^T + G_{\mathbf{d}_2} X \succeq 2(1 + \mu_2)X. \end{cases} \quad (32)$$

Then for sufficiently small  $0 < \delta < 1$ , the system (2), (31) is:

- globally uniformly finite-time stable for  $\mu_1 = 0$ ;
- globally uniformly fixed-time stable for  $\mu_1 > 0$ .

*Proof.* Let us consider the following Lyapunov function candidate:

$$V = \begin{cases} \|x\|_{\mathbf{d}_1}, & \|x\| \geq 1, \\ \|x\|_{\mathbf{d}_2}, & \|x\| \leq 1. \end{cases} \quad (33)$$

Notice that  $V \in C^{\mathbb{R}^n} \cap C^1(\mathbb{R}^n \setminus \mathbf{S} \setminus \{\mathbf{0}\})$ . For sufficiently small  $\delta \in (0, 1)$ , we have

$$(1 - \kappa) \|x\| \leq \|\mathbf{q}(x)\| \leq (1 + \kappa) \|x\|, \quad (34)$$

where  $\kappa = \kappa(\delta)$  has the form as in (30). Hence

$$\frac{1}{(1-\kappa)} \|x\| \Rightarrow 1 \leq \|\mathbf{q}(x)\|, \quad \|x\| \leq \frac{1}{(1+\kappa)} \|x\| \Rightarrow \|\mathbf{q}(x)\| \leq 1. \quad (35)$$

Let  $\Omega_1 := \{x \in \mathbb{R}^n : \frac{1}{(1-\kappa)} \leq \|x\|\}$ ,  $\Omega_2 := \{x \in \mathbb{R}^n : \|x\| \leq \frac{1}{(1+\kappa)}\}$ ,  $\Omega_3 := \{x \in \mathbb{R}^n : \frac{1}{(1+\kappa)} \leq \|x\| \leq \frac{1}{(1-\kappa)}\}$ . By Corollary 1, for a sufficiently small  $0 < \delta < 1$ , the quantizer  $\mathbf{q}(x)$  satisfies the conditions (13) and (14). Thus, according to (25), the derivative of  $V$  admits the estimate

$$\dot{V} \leq \begin{cases} -[\rho_1 - \bar{\gamma}_1(\epsilon) - c_{1,\mu_1} \kappa \|x\|_{\mathbf{d}_1}^{\bar{\eta}_1 - (1+\mu_1)}] \|x\|_{\mathbf{d}_1}^{1+\mu_1}, & x \in \Omega_1, \\ -[\rho_2 - \bar{\gamma}_2(\epsilon) - c_{1,\mu_2} \kappa \|x\|_{\mathbf{d}_2}^{\underline{\eta}_2 - (1+\mu_2)}] \|x\|_{\mathbf{d}_2}^{1+\mu_2}, & x \in \Omega_2, \end{cases} \quad (36)$$

where  $c_{1,\mu_i} = \frac{2\lambda_{\max}^{1/2}(P^{-1/2} K_0^T B^T P B K_0 P^{-1/2})}{\lambda_{\min}(X A_0^T + A_0 X + Y^T B^T + B Y)}$ ,  $\rho_i = \frac{2\lambda_{\min}(P^{1/2} G_{\mathbf{d}_i} P^{-1/2} + P^{-1/2} G_{\mathbf{d}_i}^T P^{1/2})}{h_i}$ ,  $h_i = \lambda_{\min}(P^{1/2} G_{\mathbf{d}_i} P^{-1/2} + P^{-1/2} G_{\mathbf{d}_i}^T P^{1/2})$ ,  $i = 1, 2$ . The functions  $\bar{\gamma}_1, \bar{\gamma}_2 \in \mathcal{K}$  are obtained from (20), (21) with  $(\mathbf{d}_i, \mu_i)$ ,  $i = 1, 2$ .

The LMIs defined in (32) guarantee  $\bar{\eta}_1 - 1 \leq \mu_1$  and  $\mu_2 \leq \underline{\eta}_2 - 1$ , where  $\bar{\eta}_i = \frac{1}{2} \lambda_{\max}(X^{-\frac{1}{2}} G_{\mathbf{d}_i} X^{\frac{1}{2}} + X^{\frac{1}{2}} G_{\mathbf{d}_i}^T X^{-\frac{1}{2}})$ ,  $\underline{\eta}_i = \frac{1}{2} \lambda_{\min}(X^{-\frac{1}{2}} G_{\mathbf{d}_i} X^{\frac{1}{2}} + X^{\frac{1}{2}} G_{\mathbf{d}_i}^T X^{-\frac{1}{2}})$ ,  $i = 1, 2$ . Then, we have  $\|x\|_{\mathbf{d}_1}^{\bar{\eta}_1 - (1+\mu_1)}$  monotonically decreasing to zero as  $\|x\|_{\mathbf{d}_1}$  increases to infinity, and  $\|x\|_{\mathbf{d}_2}^{\underline{\eta}_2 - (1+\mu_2)}$  monotonically decreasing to zero as  $\|x\|_{\mathbf{d}_2}$  decreases to zero. Thus, we conclude that for a sufficiently small  $\delta$  we have

$$\dot{V} < 0, \quad x \in \Omega_1 \cup \Omega_2.$$

On the other hand, for  $\frac{1}{1+\kappa} \leq \|x\| \leq \frac{1}{1-\kappa}$ , the quantization error satisfies  $\|\mathbf{q}_x - x\| \leq \frac{\kappa}{1-\kappa}$ . Using (22), for a sufficiently small  $\delta > 0$  we derive

$$\dot{V} \leq \max_{i=1,2} \left\{ (\bar{\gamma}_i(\epsilon) - \rho_i) \|x\|_{\mathbf{d}_i}^{1+\mu_i} + \frac{c_{1,\mu_i} \kappa}{1-\kappa} \right\} < 0, \quad x \in \Omega_3. \quad (37)$$

Therefore,  $V$  is a global Lyapunov function and the system is globally asymptotically stable. According to (36), (37), there exist  $\bar{\rho}_1 > 0$  and  $\bar{\rho}_2 > 0$ , such that

$$\dot{V} = \begin{cases} -\bar{\rho}_1 \|x\|_{\mathbf{d}_1}^{1+\mu_1}, & \|x\| \geq 1, \\ -\bar{\rho}_2 \|x\|_{\mathbf{d}_2}^{1+\mu_2}, & \|x\| \leq 1. \end{cases} \quad (38)$$

Since  $\mu_2 < 0$ , by Theorem 2, it is finite-time stable if  $\mu_1 = 0$  and fixed-time stable if  $\mu_1 > 0$ . The proof is complete.  $\square$

To address the robustness in the sense of input-to-state stability (ISS) [54], we consider the closed-loop system with additive perturbations  $g \in L^\infty$ :

$$\dot{x} = Ax + Bu(q(x)) + g(t), \quad t > 0, \quad x(0) = x_0. \quad (39)$$

**Corollary 3.** *Let all parameters of homogeneous quantization feedback (31) be designed as in Corollary 1. Then, the system (39) with (31) is input-to-state stable provided that  $\mu_2 > -1$ .*

*Proof.* Consider the Lyapunov function (33), according to (37), for  $\|x\| \geq 1$ , we have

$$\begin{aligned} \dot{V} &\leq -\bar{\rho}_1 \|x\|_{\mathbf{d}_1}^{1+\mu_1} + \frac{\|x\|_{\mathbf{d}_1} x^\top \mathbf{d}_1^\top (-\ln \|x\|_{\mathbf{d}_1}) P \mathbf{d}_1 (-\ln \|x\|_{\mathbf{d}_1}) g}{x^\top \mathbf{d}_1^\top (-\ln \|x\|_{\mathbf{d}_1}) P G_{\mathbf{d}_1} \mathbf{d}_1 (-\ln \|x\|_{\mathbf{d}_1}) x} \\ &\leq -(\bar{\rho}_1 - \|\tilde{\mathbf{d}}_1(-\ln \|x\|_{\mathbf{d}_1}) \frac{g}{h_1}\|) \|x\|_{\mathbf{d}_1}^{1+\mu_1}, \end{aligned} \quad (40)$$

where  $\tilde{\mathbf{d}}_1(s) = e^{(G_{\mathbf{d}_1} + \mu_1 I)s}$ ,  $\forall s \in \mathbb{R}$  and  $h_1 > 0$  is some constant. By the definition of canonical homogeneous norm, for  $\forall s = \ln \|x\|_{\mathbf{d}_1} \in \mathbb{R}$ ,  $\|\frac{\tilde{\mathbf{d}}_1(-s)g}{0.5h_1\bar{\rho}_1}\| \leq 1 \Leftrightarrow \|\frac{g}{0.5h_1\bar{\rho}_1}\| \|\tilde{\mathbf{d}}_1\| \leq \|x\|_{\mathbf{d}_1}$ . Then, by monotonicity of dilation, we have  $\dot{V} \leq -\frac{\bar{\rho}_1}{2} \|x\|_{\mathbf{d}_1}^{1+\mu_1}$ , provided that  $V = \|x\|_{\mathbf{d}_1} \geq \|\frac{2g}{\bar{\rho}_1 h_1}\|_{\tilde{\mathbf{d}}_1}$ .

For  $\|x\| \leq 1$ , following the same procedure, we can conclude that  $\dot{V} \leq -\frac{\bar{\rho}_2}{2} \|x\|_{\mathbf{d}_2}^{1+\mu_2}$ , provided that  $V = \|x\|_{\mathbf{d}_2} \geq \|\frac{2g}{\bar{\rho}_2 h_2}\|_{\tilde{\mathbf{d}}_2}$ , where  $h_2 > 0$  is some constant and  $\tilde{\mathbf{d}}_2(s) = e^{(G_{\mathbf{d}_2} + \mu_2 I)s}$ ,  $\forall s \in \mathbb{R}$  is a linear continuous dilation in  $\mathbb{R}^n$  if  $\mu_2 > -1$ . For the system (39) with (31), we have  $\dot{V} \leq -\sigma_1(V)$  if  $V \geq \sigma_2(\|g\|)$ , where  $\sigma_1, \sigma_2 \in \mathcal{K}$ . This means that  $V$  is an ISS Lyapunov function [54].  $\square$

The perturbation  $g$  can also model the sampling-induced error as in [19], [55].

## V. NUMERICAL SIMULATION

To validate the proposed approach on numerical simulations, we consider an LTI plant with  $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & 2 \\ 2.5 & 3 & 0 \end{bmatrix}$ ,  $B = \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ 1 & 2 \end{bmatrix}$ . We select  $\mu_1 = 0.3$ ,  $\mu_2 = -0.6$ , then the generators  $G_{\mathbf{d}_i} = \mu_i G_0 + I_3$  and linear feedback are obtained by solving equation (8):  $G_0 = \begin{bmatrix} -0.6351 & -0.4763 & 0.9526 \\ 1.0332 & 0.7749 & -1.5498 \\ 0.7599 & 0.5699 & -1.1398 \end{bmatrix}$ ,  $K_0 = \begin{bmatrix} 1.4206 & 1.5829 & -3.3246 \\ -0.9155 & -0.3918 & -0.0474 \end{bmatrix}$ . The control parameters are obtained by solving the LMIs (32):  $K = \begin{bmatrix} -1.0414 & -2.5677 & 1.0562 \\ 0.0217 & 0.6953 & -0.2595 \end{bmatrix}$ ,  $P = \begin{bmatrix} 1.6720 & 0.4360 & -0.7185 \\ 0.4360 & 1.3316 & -0.6291 \\ -0.7185 & -0.6291 & 2.2893 \end{bmatrix}$ .

For  $\mu = 0$ , the homogeneous feedback (9) becomes the linear control:  $u_l(x) := \tilde{K}x$ ,  $\tilde{K} = K_0 + K$ . Thus, to illustrate the proposed quantizer design and control performance, we make a comparison between two controllers:  $u(q(x))$  - the homogeneous feedback (31) with proposed quantizer  $q$  given by (31);  $u_l(q_l(x)) := \tilde{K}q_l(x)$  - linear feedback with the conventional component-wise logarithmic quantizer, where  $q_l(x) = [q(x_1), q(x_2), q(x_3)]^\top$ , where  $q$  is a scalar logarithmic quantizer. In all cases, the parameter  $\nu$  of the logarithmic quantizer in  $q$  and  $q_l$  is set to 0.7.

Simulation results, depicted in Fig. 1, demonstrate that both the homogeneous feedback  $u(q(x))$  and the linear feedback  $u_l(q_l(x))$  can stabilize the system to the origin, but the homogeneous approach achieves faster convergence. Furthermore,

as shown in Fig. 2, under perturbation, the homogeneous feedback  $u(q(x))$  drives the system to a smaller neighborhood of the origin than the linear feedback  $u_l(q_l(x))$ . This illustrates the better robustness of the homogeneous control law.

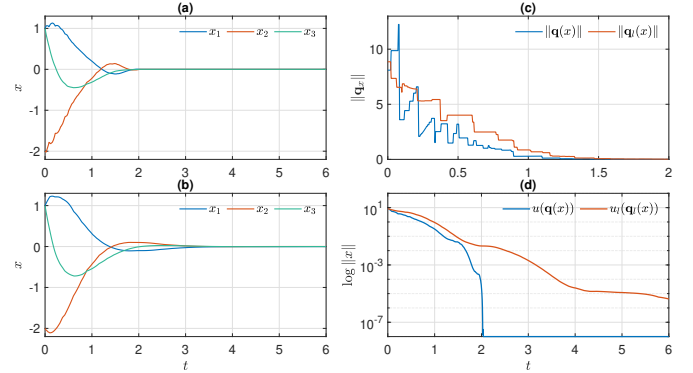


Fig. 1. The system trajectory with the homogeneous feedback  $u(q(x))$  (the subfigure (a)), the linear feedback  $u_l(q_l(x))$  (the subfigure (b)), and norm of quantized state (the subfigure (c)). The subfigure (d) depicts the evolution of  $\|x\|$  under different controllers in the logarithmic scale.

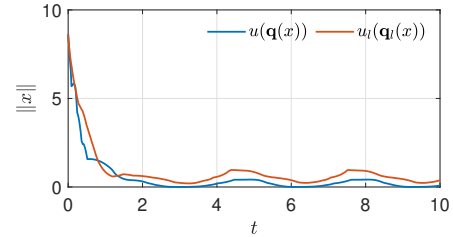


Fig. 2. The states of the system with the homogeneous feedback  $u(q(x))$  and the linear feedback  $u_l(q_l(x))$  in the case of the additive perturbation  $g(t) = (0.3 \sin(2t) + 0.2)[1, 2, 1]^\top$ .

## VI. CONCLUSION

This paper addressed the problem of finite/fixed-time stabilization of LTI systems under quantized state measurements. It is discovered that the stability and convergence rates of the LTI systems with a homogeneous control can be preserved at least locally under a logarithmic-like quantization. Global finite/fixed-time stability can be guaranteed for generalized homogeneous linear plants (i.e., for LTI systems with nilpotent matrices). For a general LTI plant, a global finite/fixed-time stabilization with quantized measurements can be achieved using a locally homogeneous control. The theoretical results are supported by numerical simulations. By providing valuable insights into the behavior of homogeneous systems with quantization effects, these results contribute to a better understanding and applicability of finite/fixed-time controllers under quantized state measurements. Future research may be focused on exploring new kinds of quantizers inspired by homogeneity.

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