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► **To cite this version:**

| Alexis Anagnostakis. Pricing and hedging for a sticky diffusion. 2024. hal-04756051

**HAL Id: hal-04756051**

**<https://inria.hal.science/hal-04756051v1>**

Preprint submitted on 28 Oct 2024

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# Pricing and hedging for a sticky diffusion

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## Abstract

We introduce a financial market model featuring a risky asset whose price follows a sticky geometric Brownian motion and a riskless asset that grows with a constant interest rate  $r \in \mathbb{R}$ . We prove that this model satisfies No Arbitrage (NA) and No Free Lunch with Vanishing Risk (NFLVR) only when  $r = 0$ . Under this condition, we derive the corresponding arbitrage-free pricing equation, assess replicability and representation of the replication strategy. We then show that all locally bounded replicable payoffs for the standard Black–Scholes model are also replicable for the sticky model. Last, we evaluate via numerical experiments the impact of hedging in discrete time and of misrepresenting price stickiness.

## 1. Introduction

The seminal work of Black and Scholes [12] stands as the founding result in the theory of derivatives valuation and forms the basis of most pricing models in mathematical finance. The model assumes that the price of the risky asset is a geometric Brownian motion and that the interest rate is constant. Under these conditions, it proposes a riskless replication (or hedging) strategy of the payoff of a European contingent claim through a self-financing portfolio composed of the underlying risky asset and the non-risky asset. In accordance with the arbitrage-free pricing principle, the claim's price is equal to the cost of the replication strategy. This allows effective market risk exposure management and fair pricing of financial products.

Most continuous models in mathematical finance extend the Black-Scholes framework to capture additional features of stock and derivatives price dynamics. The assumptions in the Black-Scholes model, such as constant volatility, drift, and interest rate, are unrealistic when applied to both historical and implied price behavior. Several models have been developed to address these limitations. Local volatility models [27, 28] and stochastic volatility models [34, 35, 36, 47, 49] account for the stochastic nature of volatility and provide a better fit for the implied volatility structure. Models incorporating jumps (e.g. [9])

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*Date:* October 28, 2024

*Keywords and phrases:* sticky geometric Brownian motion, no-arbitrage condition, derivatives pricing, arbitrage-free valuation, hedging time-granularity, model mismatch

*Mathematics Subject Classification 2020:* Primary 91G20, 91G30; Secondary 60J60

and stochastic interest rates (e.g. [7]) offer further refinements to market dynamics. In addition, skew-threshold dynamics and reflecting-threshold dynamics have been explored (see [19, 24, 45, 17] and references therein). Rough volatility models have been considered to better fit statistical observations [10, 33]. Models addressing liquidity and transaction costs have been proposed [1, 41], lifting additional assumptions of the Black-Scholes model.

An effect that is observed on price series is the presence of thresholds where the dynamic seems to linger or even spend a positive amount of time there. This is observed in the case of corporate takeovers, where the takeover price is considered to be fair by the market. This is mentioned in the Introduction of [8] as motivation for his work on the description of stochastic differential equations with a sticky point. An example of this occurrence is given in [21] on Hansen Technologies Limited between June 7, 2021 and September 6, 2021, where a takeover offer was made and then canceled. Softer psychological barriers are also observed in other markets, especially in commodities (see for example gold and silver).

In this paper, we consider a variant of the Black-Scholes model where the price-dynamic  $S$  of the risky asset is a sticky geometric Brownian motion. The process  $S$  is a diffusion that behaves like a geometric Brownian motion away for certain price threshold  $\zeta > 0$ , where it spends a positive amount time upon contact. We refer to this model as the *sticky Black-Scholes model*. We establish that there is *No Arbitrage* (NA) and *No Free Lunch with Vanishing Risk* (NFLVR) in the model if and only if the interest rate  $r$  is zero. In this case, we derive the arbitrage-free pricing equation, assess the existence of a replication strategy and provide an explicit representation of it in terms of the solution of the pricing problem. We then prove additional properties on the prices on claims and replicability. In particular, that the prices of claims with convex payoffs are decreasing and the class of locally bounded replicable payoffs is increasing with respect to the stickiness parameter. This yields that, for this model, one can consider all locally bounded payoff functions that can be replicated in the standard Black-Scholes model. Last, we conduct numerical evaluations of discrete-time hedging and analyze the hedging error incurred when ignoring or misrepresenting price stickiness (model mismatch) as realized volatility or local volatility in a smooth model.

The proof of sufficient and necessary condition for (NFLVR) is done by constructing an arbitrage strategy in the case  $r \neq 0$  and by finding an (ELMM) in the case  $r = 0$ . The construction of the arbitrage strategy is based on the proof of [26, Theorem 12.3.5] and relies on the presence of a local time term in the discounted price process that cannot be offset by the quadratic variation of the martingale part. This technique has been previously used to demonstrate the existence of arbitrage in [17], for a diffusion with a reflecting price threshold, and in [45], for a diffusion with skew returns. We note that, unlike these models, in a sticky price model, the local time term appears only for the discounted price process when  $r \neq 0$ .

In our analysis, we show that the sticky Black-Scholes model is an example of a model where there is no uniqueness of *Equivalent Local Martingale Measure* (ELMM) for the discounted price process and no uniqueness of payoff replication strategy. These peculiar effects arise from the ability to trade in the infinitesimal range of  $\mathbb{1}_{S_t=\zeta} dt$ , the times when the price process is located at the sticky threshold. Due to the stickiness, this set has positive Lebesgue measure. The arbitrage strategies in the case  $r \neq 0$  are necessarily

partially or fully supported on the set  $\mathbb{1}_{S_t=\zeta} dt$ . In the case  $r = 0$ , the (ELMM) and the payoff replication strategy are equivalence classes of processes that coincide over the range of  $\mathbb{1}_{S_t \neq \zeta} dt$ .

First discovered by Feller [30], sticky diffusions are Markov processes with continuous sample paths that spend positive amount of time at some threshold(s). They are characterized by the presence of atom(s) in their speed measures [8, 13] and can be expressed as time-delays of non-sticky diffusions (see [2, 29, 46]). The time-delay is proportional to the local time of the diffusion at the sticky threshold. This leads in some cases to path-space representations similar to classical SDEs for absolutely continuous diffusions [29, 46] (see also [4, Section 4]).

In finance, sticky diffusions have been studied to model other phenomena like near zero short rate dynamics [42, 51] and winning streak dynamics [31]. For other applications, see [14, 18, 23, 50, 52].

The paper is organized as follows. In Section 2, we define the sticky Black-Scholes model. In Section 3, we show that the sticky Black-Scholes model satisfies (NFLVR) iff  $r = 0$ . Under this condition, we identify an equivalent local martingale measure for the price process. In Section 4 we first derive the arbitrage-free pricing equation for the case  $r = 0$  and prove the existence of the associated payoff replication strategy. Section 5 explores monotonicity properties in both prices and the classes of replicable payoffs. In Section 6, we numerically evaluate properties of discrete-time hedging and assess the impact of model mismatch. Finally, Section 7 provides concluding remarks.

**Acknowledgments.** The author expresses sincere gratitude to David Criens for his valuable feedback and suggestions on the initial preprint. Special thanks go to Sara Mazzonetto and Denis Villemonais for their insightful discussions and helpful references during the early stages of this work. The author also appreciates the constructive comments from the Editor-in-Chief and the two anonymous referees.

## 2. The model

The standard Black-Scholes model makes the following assumptions:

- (i) infinite liquidity, divisibility of assets, and borrowing/lending capacity,
- (ii) frictionless market (no transaction costs, no borrowing or lending rate),
- (iii) possibility of continuous-time trading,
- (iv) constant interest rate  $r$ ,
- (v) geometric Brownian motion price dynamics.

In the sticky Black-Scholes model, assumption (v) is modified such that the price process of the risky asset  $S$  follows a sticky geometric Brownian motion. In particular, the prices of risky and non-risky assets  $(S, S^0)$  solve the following system:

$$\begin{cases} dS_t = \mathbb{1}_{S_t \neq \zeta} S_t (\mu dt + \sigma dB_t), \\ \mathbb{1}_{S_t = \zeta} dt = \rho dL_t^\zeta(S), \\ dS_t^0 = rS_t^0 dt, \end{cases} \quad (1)$$

where  $\mu, r \in \mathbb{R}$ ,  $\sigma, \zeta > 0$ ,  $\rho > 0$ , where  $B$  is a standard Brownian motion, and where  $(L_t^y(S); t \geq 0, y > 0)$  is the local time field of  $S$ .

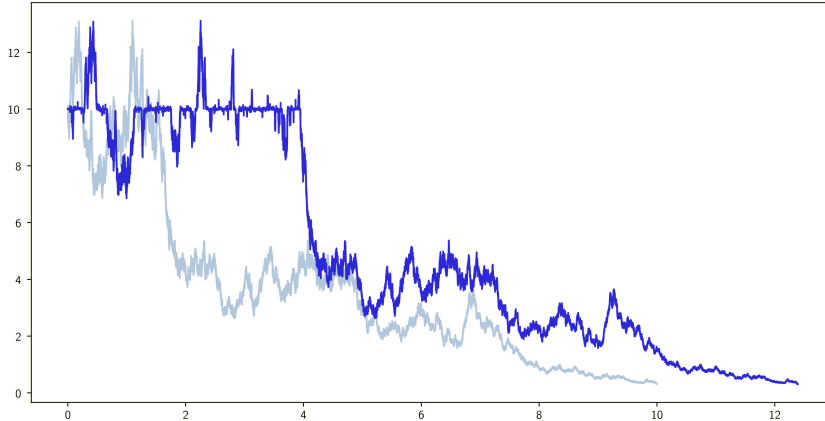


Figure 1: Simulated sample paths of the sticky (dark blue) geometric Brownian motion of parameters  $(\mu = 0, \sigma = 0.25, \rho = 1, \zeta = 10)$  and the non-sticky (light blue) geometric Brownian motion of parameters  $(\mu = 0, \sigma = 0.25)$ . The approximation method used for the simulation is from [6].

The process  $S$  defined this way has the following properties:

1. It is a semimartingale with the explicit Doob-Meyer decomposition:

$$S_t = \mu \int_0^t S_s \mathbb{1}_{S_s \neq \zeta} ds + \sigma \int_0^t S_s \mathbb{1}_{S_s \neq \zeta} dB_s.$$

2. It is the regular diffusion on  $(0, \infty)$  defined through scale and speed  $(s, m)$  (see [4, Section 4]), where  $s, m$  are defined for all  $x > 0$  by

$$s'(x) = e^{-\frac{2\mu}{\sigma^2} \int^x \frac{1}{\zeta} d\zeta}, \quad m(dx) = \frac{1}{s'(x)} \frac{1}{\sigma^2 x^2} dx + \frac{\rho}{s'(\zeta)} \delta_\zeta(dx). \quad (2)$$

We note that for equation (4.2) in [4] to be correct, the singular part of the expression should be divided by  $s'(0)$ . The author thanks David Criens and his student, Sebastian Hahn, for identifying and communicating this correction.

3. It spends a positive amount of time at  $\zeta$  upon contact. Since  $S$  is a regular diffusion on  $(0, \infty)$ , this follows from the change of scale [43, Proposition VII.3.5], [43, Exercise VI.1.14] and the Markov property. This can be observed in the simulated path of the sticky geometric Brownian motion shown in Figure 1.
4. It has an alternative path-wise representation akin to that of the geometric Brownian motion. Indeed, from the Itô formula of [4, Lemma 4.5], the process  $(X, X^0) =$

$(\log S, \log S^0)$  solves:

$$\begin{cases} dX_t = \left( \mu - \frac{\sigma^2}{2} \right) \mathbb{1}_{X_t \neq \log \zeta} dt + \sigma \mathbb{1}_{X_t \neq \log \zeta} dB_t, \\ \mathbb{1}_{X_t = \log \zeta} dt = \frac{\rho}{\zeta} dL_t^{\log \zeta}(X), \\ dX_t^0 = r dt. \end{cases}$$

Consequently, we derive the following expressions for the price dynamics:

$$\begin{cases} S_t = S_0 \exp \left( \left( \mu - \frac{\sigma^2}{2} \right) \int_0^t \mathbb{1}_{S_t \neq \zeta} dt + \sigma \int_0^t \mathbb{1}_{S_t \neq \zeta} dB_t \right), \\ \mathbb{1}_{S_t = \zeta} dt = \rho dL_t^\zeta(S), \\ S_t^0 = S_0^0 \exp(rt). \end{cases} \quad (3)$$

**Example 2.1** (standard Black–Scholes model). If  $\rho = 0$ , then  $S$  is a geometric Brownian motion and the model  $(S, S^0)$  is the standard Black–Scholes model. Indeed, in this case,  $(s, m)$  defined in (2) are the scale function and speed measure of the geometric Brownian motion (see [13, p.132]). This can also be seen from (1), where if  $\rho = 0$ , the process  $S$  has a geometric Brownian dynamic away from  $\zeta$  and the time it spends time at  $\zeta$  is of Lebesgue measure 0.

**Remark 2.2.** The process  $(S, S^0)$  is a 2-dimensional sticky diffusion that, when  $r \neq 0$ , cannot be reduced to a time-change. The stickiness affecting the dynamic of the risky asset's price  $S$  slows down its motion on  $\zeta$ , while the log-value of the zero bond  $\log(S^0)$  steadily increases with rate  $r$ . This characteristic results in the violation the no-arbitrage hypothesis for the model when  $r \neq 0$ .

### 3. (No) Arbitrage

In this section we study the model in terms of arbitrage. The notion of arbitrage of interest is the free lunch with vanishing risk and the absence of it is called no free lunch with vanishing risk or (NFLVR). The reason we focus on (NFLVR) is that it is the key notion related to the existence of an equivalent local martingale measure (see [26, Theorem 12.1.2]) and the induced arbitrage-free pricing framework.

First, we recall all the necessary notions related to arbitrage along with some useful results. Then, we adapt these for the sticky Black-Scholes model. In particular, we prove that the model satisfies (NFLVR) iff  $r = 0$ . In this case, we find an equivalent local martingale measure for the discounted price process  $\tilde{S} = S/S^0$ .

#### 3.1. Preliminary notions

We start by introducing several concepts from the arbitrage theory for semimartingales. Let  $X$  be a semimartingale defined on the probability space  $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ , where the filtration  $(\mathcal{F}_t)_{t \geq 0}$  is assumed both right-continuous and complete. This means that

1. for all  $t \geq 0$ ,  $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ ,
2.  $\mathcal{F}_0$  contains all P-negligible events.

**Standing Assumption 3.1.** These assumptions on  $(\mathcal{F}_t)_{t \geq 0}$  will apply to all filtrations considered throughout this paper.

We may now define the following objects.

1. A *trading strategy* is a predictable process  $\theta$  with respect to  $(\mathcal{F}_t)_{t \geq 0}$ , i.e., for all stopping times  $\tau$ ,  $\theta_\tau$  is  $\mathcal{F}_{\tau-}$ -measurable.
2. An *elementary trading strategy* is a trading strategy of the form:

$$\theta_t = \sum_{i=0}^n \theta^{(i)} \mathbb{1}_{t \in (\tau_i, \tau_{i+1}]}$$

with  $n \in \mathbb{N}$ ,  $(\tau_i)_{i \leq n}$  a family of stopping times, and  $(\theta^{(i)})_i$  a family of random variables such that, for all  $i$ ,  $\theta^{(i)}$  is  $\mathcal{F}_{\tau_i}$ -measurable. Such strategies are also called simple trading strategies.

3. An *M-admissible trading strategy*, for some  $M \geq 0$ , is an adapted predictable process  $\theta$  with finite variation such that for all  $t \geq 0$ ,  $\int_0^t \theta_s dX_s > -M$  almost surely. An *admissible strategy* is a strategy that is *M-admissible* for some  $M \geq 0$ . We denote the set of all *M-admissible* strategies with  $\mathcal{A}_M$ .
4. A probability measure  $\mathbb{Q}$  is called an *Equivalent Local Martingale Measure* (or (ELMM)) for  $X$  if  $\mathbb{Q} \sim \mathbb{P}$  and  $X$  is a local martingale on  $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$ .

We now introduce three notions of arbitrage. For more on these, see [25, 40]. We note that arbitrage notions in a diffusion market with non-trivial interest rates should be expressed in terms of the discounted price process. Otherwise, for the case  $r \in \mathbb{R}$ , depending on the sign of  $r$ , longing or shorting the riskless asset results in an arbitrage strategy.

**Definition 3.2** ((No) Arbitrage). A trading strategy  $\theta$  is an *Arbitrage Opportunity* for  $X$  on  $[0, T]$  if  $\theta \in \mathcal{A}_0$  and  $\mathbb{P}(\int_0^T \theta_s dX_s > 0) > 0$ . A market is *arbitrage-free* (or satisfies (NA)) if, for all  $\theta \in \mathcal{A}_0$ , the probability  $\mathbb{P}(\int_0^T \theta_s dX_s > 0) = 0$ .

**Definition 3.3** ((No) Free Lunch with Vanishing Risk). A sequence  $(\theta^n)_n$  of trading strategies is said to realize a *Free Lunch with Vanishing Risk* if for each  $n$ ,  $\theta^n$  is  $\epsilon_n$ -admissible and there exists some  $M > 0$  such that,

$$\lim_{n \rightarrow \infty} \epsilon_n = 0 \quad \text{and} \quad \forall n \in \mathbb{N} : \quad \mathbb{P} \left( \int_0^t \theta_s^n dX_s > M \right) > 0.$$

A market satisfies No Free Lunch with Vanishing Risk (or satisfies (NFLVR)) if there is no sequence of strategies that is free lunch with vanishing risk.

**Definition 3.4** ((No) Unbounded Profit with Bounded Risk). A sequence  $(\theta^n)_n$  is said to realize an *Unbounded Profit with Bounded Risk* if their payoff is uniformly bounded in  $(n, t)$  from below but unbounded in probability from above, i.e. there exists some  $M > 0$  such that  $(\theta^n)_n \subset \mathcal{A}_M$  and the payoff sequence is unbounded in probability, i.e.

$$\forall M' > 0, \exists n \in \mathbb{N}, \exists t > 0 : \mathbb{P} \left( \int_0^t \theta_s^n dX_s > M' \right) > 0.$$

A market satisfies No Unbounded Profit with Bounded Risk (or satisfies (NUPBR)) if there is no sequence of strategies that is an unbounded profit with bounded risk.

The reason why we are interested in (NFLVR) is the following result which qualifies it as a sufficient and necessary condition for the existence of an (ELMM). The latter implies an arbitrage-free pricing framework.

**Theorem 3.5** (Theorem 12.1.2 of [26]). *Let  $X$  be a locally bounded semimartingale defined on the probability space  $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ . The following are equivalent:*

1. *there exists an (ELMM) for  $X$ ,*
2.  *$X$  satisfies (NFLVR),*
3.  *$X$  satisfies (NA) and (NUPBR).*

We observe that an arbitrage strategy also realizes a free lunch with vanishing risk. This confirms that (NFLVR) implies (NA). Hence, the following result provides a necessary condition for the existence of an (ELMM).

**Theorem 3.6** (one-dimensional version of Theorem 12.3.5 of [26]). *Let  $X$  be a locally bounded semimartingale that satisfies (NA). Then, if  $X = M + A$  is the Doob-Meyer decomposition of  $X$ , then we have that  $dA = h d\langle M \rangle$ , where  $h$  is a predictable process.*

## 3.2. The case of the sticky Black-Scholes model

We now study the sticky Black-Scholes model in terms of (NA) and (NFLVR). We first prove that (NFLVR) holds when  $r = 0$ , by finding an (ELMM) for the discounted price  $\tilde{S} = S/S^0$ . We then prove that (NA), and hence (NFLVR), does not hold when  $r \neq 0$ , by establishing an arbitrage strategy.

We first elaborate on the dynamic of the discounted risky asset. From (1) and the Itô formula (see [43, Theorem IV.3.3]),

$$d\tilde{S}_t = d(e^{-rt} S_t) = e^{-rt} dS_t - r e^{-rt} S_t dt = \left( \mu \mathbb{1}_{S_t \neq \zeta} - r \right) \tilde{S}_t dt + \sigma \tilde{S}_t \mathbb{1}_{S_t \neq \zeta} dB_t. \quad (4)$$

This qualifies  $\tilde{S}$  as a semimartingale.

**Proposition 3.7.** *We consider the sticky Black-Scholes model  $(S, S^0)$ , as defined in Section 2, defined on the probability space  $\mathcal{P}_x = (\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}_x)$  such that  $\mathbb{P}_x$ -a.s.,  $S_0 = x$ . If  $r = 0$ , then  $S = \tilde{S}$  and the following hold.*



(i) The measure  $\mathbb{Q}_x$  defined by  $d\mathbb{Q}_x = \mathcal{E}(\theta)_T d\mathbb{P}_x$ , where

$$\mathcal{E}(\theta)_T = \exp\left(\int_0^T \theta_s dB_s - \frac{1}{2} \int_0^T \theta_s^2 ds\right), \quad \text{and} \quad \theta_t = \frac{\mu}{\sigma}, \quad t \geq 0, \quad (5)$$

is an (ELMM) for  $S$ .

(ii) The process  $S$  solves

$$\begin{cases} dS_t = \sigma S_t \mathbb{1}_{S_t \neq \zeta} d\tilde{B}_t, \\ \mathbb{1}_{S_t = \zeta} dt = \rho dL_t^\zeta(S), \end{cases} \quad (6)$$

where  $\tilde{B} = (B_t - (\mu/\sigma)t)_{t \geq 0}$  is a  $\mathbb{Q}_x$ -standard Brownian motion.

(iii) The process  $S$  is a martingale under  $\mathbb{Q}_x$ .

*Proof.* Since  $r = 0$ , we have that  $\tilde{S} = S$ . From the Girsanov theorem of [4, Lemma 4.4],  $S$  solves (6) under  $\mathbb{Q}_x$ . This proves (i) and (ii).

The process  $S$  is a local martingale under  $\mathbb{Q}_x$  as the stochastic integral of a predictable process with respect to a standard Brownian motion. From [43, Ch.IV, Corollary 1.25], to prove that  $S$  is a martingale, it suffices to show that  $E^{\mathbb{Q}_x}(\langle S \rangle_t) < \infty$ . Indeed, from (3), we have

$$\begin{aligned} E^{\mathbb{Q}_x}(\langle S \rangle_t) &= E^{\mathbb{Q}_x}\left(\int_0^t \sigma^2 S_s^2 \mathbb{1}_{S_s \neq \zeta} ds\right) \leq \sigma^2 \int_0^t E^{\mathbb{Q}_x}(S_s^2) ds \\ &= \sigma^2 S_0^2 \int_0^t E^{\mathbb{Q}_x}\left(\exp\left(-\sigma^2 \int_0^s \mathbb{1}_{S_u \neq \zeta'} du + 2\sigma \int_0^s \mathbb{1}_{S_u \neq \zeta'} dB_u\right)\right) ds, \end{aligned}$$

where from [43, Corollary VIII.1.16], the quantity under the expectancy is a martingale. Hence for all  $t \geq 0$ , we have that  $E^{\mathbb{Q}_x}(\langle S \rangle_t) \leq \sigma^2 S_0^2 t$ , which is finite. This proves (iii), which completes the proof.  $\square$

**Theorem 3.8.** *The sticky Black-Scholes model satisfies (NA) and (NFLVR) if and only if  $r = 0$ .*

*Proof.* We recall that we work under the Standing Assumption 3.1. From Theorem 3.5 and Proposition 3.7 if  $r = 0$  both (NA) and (NFLVR) are satisfied. From Theorem 3.5, it thus suffices to prove the existence of an arbitrage strategy when  $r \neq 0$ . Inspired by the proof of [26, Theorem 12.3.5], we will construct an arbitrage strategy.

We consider the trading strategy  $H$  defined by

$$H_t = \text{sgn}(-r)e^{rt} \mathbb{1}_{S_t = \zeta}, \quad t > 0. \quad (7)$$

We now show that for all  $t \geq 0$ ,  $\int_0^t H_s d\tilde{S}_s \geq 0$ , and that there exists some  $t > 0$  such that  $P(\int_0^t H_s d\tilde{S}_s > 0) > 0$ . From (1), (4) and (7),

$$\begin{aligned} \int_0^t H_s d\tilde{S}_s &= \int_0^t H_s d\tilde{S}_s = \int_0^t H_s \mathbb{1}_{S_s = \zeta} d\tilde{S}_s \\ &= -r\zeta \int_0^t H_s e^{-rs} \mathbb{1}_{S_s = \zeta} ds = |r|\zeta \int_0^t \mathbb{1}_{S_s = \zeta} ds = |r|\rho\zeta L_t^0(S). \end{aligned}$$

We observe that for all  $t \geq 0$ ,  $\int_0^t H_s d\tilde{S}_s \geq 0$ , i.e.  $H \in \mathcal{A}_0$ . From [39, Corollary 29.18] and [16, Theorem 1.1],  $\mathbb{P}_x \left( \int_0^t H_s d\tilde{S}_s > 0 \right) = \mathbb{P}_x(L_t^0(S) > 0) = \mathbb{P}_x(\inf\{s > 0 : S_s = \zeta\} < t) > 0$ . These qualify  $H$  as an arbitrage strategy. This completes the proof.  $\square$

**Example 3.9** (standard Black–Scholes ( $\rho = 0$ ), continuation of Example 2.1). In the standard Black–Scholes model the discounted price process has dynamic

$$d\tilde{S}_t = (\mu - r)\tilde{S}_t dt + \sigma\tilde{S}_t dB_t.$$

An (ELMM) for  $\tilde{S}$  is the measure  $\mathbb{Q}_x$  defined by

$$d\mathbb{Q}_x = \mathcal{E}(\theta)_T d\mathbb{P}_x, \quad \text{where} \quad \mathcal{E}(\theta)_T = \exp\left(\int_0^T \frac{\mu - r}{\sigma} dB_s - \frac{1}{2} \int_0^T \left(\frac{\mu - r}{\sigma}\right)^2 ds\right).$$

Hence, (NFLVR) is satisfied.

We note also that in the standard Black–Scholes model, if  $A + M$  is the Doob–Meyer decomposition of the discounted price–process  $\tilde{S}$ , we have that  $dA_t \ll d\langle M \rangle_t$ . Indeed,

$$\forall t \geq 0, \quad \begin{cases} A_t = \int_0^t (\mu - r)\tilde{S}_s ds, \\ M_t = \int_0^t \sigma\tilde{S}_s dB_s. \end{cases}$$

**Remark 3.10.** From Proposition 3.7, we note that the price process under the (ELMM)  $\mathbb{Q}_x$  is a driftless sticky geometric Brownian motion, sticky at  $\zeta > 0$ , of stickiness parameter  $\rho$ . Therefore, under  $\mathbb{Q}_x$  the process  $S$  is also a diffusion on  $(0, \infty)$ . This also implies that stickiness persists under the (ELMM)  $\mathbb{Q}_x$ .

**Remark 3.11.** In the sticky Black–Scholes model ( $\rho > 0$ ), there is no uniqueness of (ELMM) for  $S$ . Indeed, we observe that taking any  $\theta$  in (5) such that

$$\forall t \geq 0 : \int_0^t \theta_s \mathbb{1}_{S_s \neq \zeta} ds = \int_0^t \frac{\mu}{\sigma} \mathbb{1}_{S_s \neq \zeta} ds,$$

also results in an equivalent local martingale measure  $\mathbb{Q}^\theta$  for  $S$ . In this case, the process  $S$  solves

$$\begin{cases} dS_t = \sigma S_t \mathbb{1}_{S_t \neq \zeta} dB_t^\theta, \\ \mathbb{1}_{S_t = \zeta} dt = \rho dL_t^\zeta(S), \end{cases}$$

where  $\tilde{B}^\theta = (B_t - \int_0^t \theta_s ds)_{t \geq 0}$  is a  $\mathbb{Q}^\theta$ -Brownian motion

**Remark 3.12.** The arbitrage strategy constructed in the proof of Theorem 3.8 is highly erratic: it is supported on  $\{t \in [0, T] : S_t = \zeta\}$ . It is therefore not a simple arbitrage (an arbitrage that is an elementary strategy). In the forthcoming paper [5], it is proven that there is no simple arbitrage strategy in this model for any  $r \in \mathbb{R}$ . This leads to two key implications: First, arbitrage strategies are practically unattainable as they require an infinite number of transactions. Second, by Theorem 3.5, the presence of such an unattainable arbitrage precludes the existence of an (ELMM) for  $S$  and of an arbitrage–free pricing framework.

## 4. Pricing and hedging

This section is dedicated to the pricing and hedging problem in the sticky Black–Scholes model. We first prove the pricing equation and deduce pricing and hedging expressions. These require the existence of a unique classical solution to the pricing problem. Next, we apply the martingale representation theorem to prove, under an integrability condition, the existence of a replication strategy and the representation of the claim’s price as the expectancy of the payoff under the (ELMM). In our analysis, we observe another distinctive feature of the sticky model: the non-uniqueness of the replication strategy.

These derivations are only valid for the case  $r = 0$ . When  $r \neq 0$ , the (NFLVR) condition is not met and, as per Theorem 3.5, there is no (ELMM) for the discounted price process. In this case, this pricing framework does not apply.

We recall the concept of arbitrage pricing principle.

**Definition 4.1** (see [38], Section 2.1.9). The *Arbitrage pricing principle* (or (APP)) refers to the concept that the price of a contingent claim is determined in such a way that no party involved can achieve an arbitrage.

For the remainder of this section, we consider the sticky Black-Scholes model defined in Section 2 with  $r = 0$ , defined on the probability space  $\mathcal{P}_x = (\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}_x)$  so that  $S_0 = x$ ,  $\mathbb{P}_x$ -almost surely. Let  $\mathbb{Q}_x$  be the (ELMM) for  $S$  derived in Proposition 3.7.

We also consider the notation of Markovian families used in [32] for the diffusion  $S$  under the equivalent local martingale measure  $\mathbb{Q}_x$ . Let  $(s_{\mathbb{Q}}, m_{\mathbb{Q}})$  be the scale and speed of  $S$  under  $\mathbb{Q}_x$ . We note with  $(Q_x^{S, \mathbb{Q}}, x \in J)$  the canonical diffusion, of scale and speed  $(s_{\mathbb{Q}}, m_{\mathbb{Q}})$ , on the path-space  $C([0, \infty), \mathbb{R})$  and  $Y$  be the coordinate process. Hence, for all almost surely finite stopping time  $\tau$ , we have

$$Q_{X_\tau}^{S, \mathbb{Q}}(h(Y_t)) = \mathbb{E}^{\mathbb{Q}_x}(h(X_{\tau+t}) | \mathcal{F}_\tau).$$

### 4.1. Pricing equation

We consider the generic pricing problem

$$\begin{cases} v_t = -\frac{\sigma^2 x^2}{2} v_{xx}, & \forall (t, x) \in [0, T) \times (0, \infty), \\ \frac{\sigma^2 \zeta^2}{2} v_{xx}(t, \zeta-) = \frac{\sigma^2 \zeta^2}{2} v_{xx}(t, \zeta+) = \frac{1}{2\rho} (v_x(t, \zeta+) - v_x(t, \zeta-)), & \forall t \in [0, T), \\ v(T, x) = h(x), & \forall x \in (0, \infty), \end{cases} \quad (8)$$

We now define the space of classical solutions of (8). This is justified from the forthcoming Remark 4.5.

For  $I$  an interval of  $[0, \infty)$  and  $L$  an interval of  $(0, \infty)$ , we consider the spaces  $C^L(I)$ ,  $C^{1,L}(I \times L)$ , defined by

$$C^L(L) := \left\{ f \in C(L) \cap C^2(L) : \frac{\sigma^2 \zeta^2}{2} f''(\zeta-) = \frac{\sigma^2 \zeta^2}{2} f''(\zeta+) = \frac{1}{2\rho} (f'(\zeta+) - f'(\zeta-)) \right\},$$

and

$$C^{1,L}(I \times L) := \left\{ u \in C(I \times L) : \begin{array}{l} \forall x \in L, u(\cdot, x) \in C^1(I); \\ \forall t \in I, u(t, \cdot) \in C^L(L) \end{array} \right\},$$

where all considered derivatives are right-derivatives.

We can now define the notion of classical solution of (8).

**Definition 4.2.** A function  $v$  is said to be a classical solution of (8) if  $v \in C^{1,L}([0, T] \times (0, \infty))$  and if it solves (8).

We note that the space  $C^L((0, \infty))$  is comprised of functions that are differences of convex functions. Indeed, if  $f \in C^L((0, \infty))$ , the function  $\phi$  defined by

$$\phi(x) = f(x) - \frac{\rho(\zeta\sigma)^2}{2} f''(\zeta)(x - \zeta)_+, \quad x \in (0, \infty),$$

is  $C^2((0, \infty))$ . It suffices to observe that the regularity of  $\phi$  ensures it is difference of convex functions and that  $[x \rightarrow (x - \zeta)_+]$  is convex. This justifies the usage of the Itô-Tanaka formula (see [43, Theorem VI.1.5]) on any  $f \in C^L((0, \infty))$ .

**Theorem 4.3.** *Let  $h$  be a measurable function such that (8) has a unique classical solution  $v$ . Then, the following hold.*

- (i) *The (APP) price at  $t \in [0, T]$  of the contingent claim with payoff  $h(S_T)$  is  $v(t, S_t)$ .*
- (ii) *A replication strategy for  $h(S_T)$  is  $\delta = (v_x(t, S_t); t \in [0, T])$ .*

*Proof.* Let  $v$  be the unique classical solution of (8). Since for all  $t \in [0, T]$ ,  $v(t, \cdot) \in C^L((0, \infty))$ , from the Itô-Tanaka formula,

$$\begin{aligned} dv(t, S_t) &= v_t(t, S_t) dt + v_x(t, S_t) dS_t \\ &\quad + \frac{1}{2} v_{xx}(t, S_t) d\langle S \rangle_t + \frac{1}{2} (v_x(t, \zeta+) - v_x(t, \zeta-)) dL_t^\zeta(S). \end{aligned}$$

From (6),

$$d\langle S \rangle_t = \sigma^2 S_t^2 \mathbb{1}_{S_t \neq \zeta} dt, \quad \mathbb{1}_{S_t = \zeta} dt = \rho dL_t^\zeta(S).$$

This, combined with (8), yields

$$\frac{1}{2} (v_x(t, \zeta+) - v_x(t, \zeta-)) dL_t^\zeta(S) = \frac{\rho\sigma^2\zeta^2}{4} v_{xx}(t, \zeta) dL_t^\zeta(S) = -v_t(t, \zeta) \mathbb{1}_{S_t = \zeta} dt.$$

Consequently,

$$\begin{aligned} dv(t, S_t) &= v_t(t, S_t) \mathbb{1}_{S_t \neq \zeta} dt + v_x(t, S_t) dS_t + \frac{\sigma^2 S_t^2}{2} v_{xx}(t, S_t) \mathbb{1}_{S_t \neq \zeta} dt \\ &\quad + v_t(t, S_t) \mathbb{1}_{S_t = \zeta} dt + \frac{1}{2} (v_x(t, \zeta+) - v_x(t, \zeta-)) dL_t^\zeta(S) \\ &= v_x(t, S_t) dS_t + \left( v_t(t, S_t) + \frac{\sigma^2 S_t^2}{2} v_{xx}(t, S_t) \right) \mathbb{1}_{S_t \neq \zeta} dt. \end{aligned}$$

Integrating in time yields

$$h(S_T) = v(t, S_t) + \int_t^T v_x(t, S_s) dS_s.$$

This proves the two assertions. □

**Remark 4.4.** The replication strategy  $\delta = (v_x(t, S_t); t \in [0, T])$  in the sticky Black-Scholes model is typically discontinuous at the sticky threshold  $\zeta$ . Indeed, from (8), if  $v_{xx}(t, \zeta) \neq 0$ , then  $v_x(t, \zeta+) - v_x(t, \zeta-) \neq 0$ .

For our next remark, we recall some notions. The semi-group of a diffusion is the family of operators  $(P_t)_{t \geq 0}$  defined by

$$P_t f(x) := \mathbb{E}(f(S_t) | S_0 = x), \quad \forall f \in C_b((0, \infty)), t \geq 0, x \in (0, \infty).$$

We note that  $S$  is a Feller process (see [20, Theorem 1]), i.e. it is strong Markov and its semi-group  $(P_t)_{t \geq 0}$  is internal on  $C_0((0, \infty))$  and continuous at 0, i.e.

1.  $\forall f \in C_0((0, \infty))$ :  $P_t f \in C_0((0, \infty))$ ,
2.  $\forall f \in C_0((0, \infty))$ ,  $x \in (0, \infty)$ :  $\lim_{t \rightarrow 0} P_t f(x) = f(x)$ .

Its semi-group is therefore generated by the following operator

$$L f = \frac{(x\sigma)^2}{2} f'', \quad \text{dom}_{C_0}(L) = C_0((0, \infty)) \cap C^L((0, \infty)).$$

**Remark 4.5.** We note that if  $h \in \text{dom}_{C_0}(L)$ , from the Kolmogorov equations (see [43, Proposition VII.1.2]), the problem (8) has a unique classical solution  $v \in C^{1,L}([0, T] \times (0, \infty))$ , that admits the following representation

$$\forall (t, x) \in [0, T] \times (0, \infty), \quad v(t, x) = Q_x^{S, Q}(h(Y_{T-t})).$$

## 4.2. Representation

In Remark 4.5, we observed that if  $h \in \text{dom}_{C_0}(L)$ , then the pricing equation has a unique classical solution, the price function has a representation in terms of an expectancy and there exists a replication strategy. In this section we prove that the two latter assertions are part of a more general principle, which is consequence of the martingale representation property [22, Theorem 2.7].

Our motivation is that the condition  $h \in \text{dom}_{C_0}(L)$  is quite restrictive. Indeed, classical claims like Calls and Puts have payoff functions that are neither in  $C_0((0, \infty))$  nor in  $C^L((0, \infty))$ . This holds true even for the standard Black-Scholes model, where  $\text{dom}_{C_0}(L) = C_0((0, \infty)) \cap C^2((0, \infty))$ . Nevertheless, as depicted in Section 6.1, we obtain satisfactory numerical results for a Call payoff in the sticky and the non-sticky model.

**Theorem 4.6.** For all  $h \in \mathbf{H}(S_T) := \{g \text{ measurable} : g(S_T) \in L^1(\Omega, \mathcal{F}_T, \mathbb{Q}_x)\}$ ,

- (i) there exists a trading strategy  $\delta$  in  $L_{\text{loc}}^2(S)$  (for a definition of  $L_{\text{loc}}^2(S)$ , see [37, §I.4.39]), such that for all  $t \in [0, T]$ ,

$$\begin{cases} \mathbb{E}^{\mathbb{Q}_x}(h(S_T) | \mathcal{F}_t) = \mathbb{E}^{\mathbb{Q}_x}(h(S_T)) + \int_0^t \delta_s dS_s, \\ h(S_T) = \mathbb{E}^{\mathbb{Q}_x}(h(S_T) | \mathcal{F}_t) + \int_t^T \delta_s dS_s, \end{cases} \quad (9)$$

(ii) and for all  $t \in [0, T]$ , the (APP) price of the claim of terminal payoff  $h(S_T)$  at  $t$  is  $V_t := \mathbb{E}^{\mathbb{Q}_x}(h(S_T) | \mathcal{F}_t) = Q_{S_t}^{S, \mathbb{Q}}(h(Y_{T-t}))$ .

*Proof.* Regarding (i), since  $h(S_T) \in L^1(\Omega, \mathcal{F}_T, \mathbb{Q}_x)$ , the process  $M$  defined as follows is a uniformly integrable martingale

$$M_t := \mathbb{E}^{\mathbb{Q}_x}(h(S_T) | \mathcal{F}_t), \quad t \in [0, T].$$

From the Markov property of  $S$  under  $\mathbb{Q}_x$  (see Remark 3.10), we have that  $\mathbb{E}^{\mathbb{Q}}(h(S_T) | \mathcal{F}_t) = \mathbb{E}^{\mathbb{Q}}(h(S_T) | \sigma(S_t))$ . Hence,  $M$  is adapted to the canonical filtration of  $S$ . Since  $S$  is a local martingale, from [22, Theorem 2.7], there exists a predictable process  $\delta \in L_{\text{loc}}^2(S)$  such that

$$M_t = M_0 + \int_0^t \delta_s dS_s.$$

This proves the first equation of (i) Regarding the second equation, we consider (9) for different values of  $t$ . Subtracting them results in

$$h(S_T) = \mathbb{E}^{\mathbb{Q}_x}(h(S_T) | \mathcal{F}_{t_0}) - \int_{t_0}^T \delta_s dS_s.$$

Regarding (ii), from the (APP) (see Definition 4.1) and the payoff replication relations 9, the price of the claim at  $t$  is necessarily  $\mathbb{E}^{\mathbb{Q}_x}(h(S_T) | \mathcal{F}_t)$ . Any other value would result in an arbitrage opportunity for one of the two parties. This finishes the proof.  $\square$

**Remark 4.7.** We note that in the sticky Black–Scholes model ( $\rho > 0$ ) we have no uniqueness of the payoff replication strategy. Indeed, let  $\delta$  be a payoff replication strategy and  $\delta'$  a strategy such that

$$\delta \neq \delta' \quad \text{and} \quad \forall t \geq 0 : \delta_t \mathbb{1}_{S_t \neq \zeta} = \delta'_t \mathbb{1}_{S_t \neq \zeta}.$$

Then, we have

$$\mathbb{E}^{\mathbb{Q}_x}(h(S_T) | \mathcal{F}_t) - \mathbb{E}^{\mathbb{Q}_x}(h(S_T)) = \int_0^t \delta_s dS_s = \int_0^t \delta'_s dS_s.$$

**Corollary 4.8.** Let  $h \in \mathcal{H}(S_T)$  such that (8) has a unique classical solution  $v$ . Then, from the (APP),

$$\forall (t, x) \in [0, T] \times (0, \infty), \quad v(t, x) = Q_x^{S, \mathbb{Q}}(h(Y_{T-t})),$$

and if  $\delta$  is a replication strategy for  $h(S_T)$ , then

$$\forall t \in [0, T], \quad v_x(t, S_t) \mathbb{1}_{S_t \neq \delta} = \delta_t \mathbb{1}_{S_t \neq \delta},$$

and  $(v_x(t, S_t); t \in [0, T]) \in L_{\text{loc}}^2(S)$ .

**Example 4.9** (standard Black–Scholes ( $\rho = 0$ ), continuation of Examples 2.1, 3.9). For the standard Black–Scholes model ( $\rho = 0$ ), the risky asset and discounted risky asset price dynamics  $S, \tilde{S}$  under  $\mathbb{Q}_x$  are

$$dS_t = S_t(r dt + \sigma dB_t), \quad d\tilde{S}_t = \sigma \tilde{S}_t dB_t.$$

The pricing equation is (see [48, Theorems 6.4.3 and 7.3.1])

$$\begin{cases} v_t(t, x) + rxv_x(t, x) + \frac{\sigma^2 x^2}{2} v_{xx}(t, x) = rv(t, x), & \forall (t, x) \in (0, T) \times (0, \infty), \\ v(T, x) = h(x), \end{cases} \quad (10)$$

If  $h \in \mathbf{H}(S_T)$ , there exists a replication strategy in  $L^2$  and the price at  $t \in [0, T]$  of the contingent claim of payoff at maturity  $h(S_T)$  is given by

$$V_t := \mathbb{E}^{\mathbb{Q}_x} (e^{-r(T-t)} h(S_T) | \mathcal{F}_t).$$

If (10) has a unique classical solution  $v$ , the (APP) price of the contingent claim at  $t \in [0, T]$  is  $V_t = v(t, S_t)$ , and the replication strategy is the process  $\delta$  defined by (see [48, Remark 7.3.3])

$$\delta_t := v_x(t, S_t), \quad t \in [0, T].$$

## 5. Price monotonicity and payoff inclusion

In this section we prove price monotonicity, and inclusion properties for a class of replicable payoffs, with respect to the stickiness parameter.

We first prove that higher stickiness leads to lower prices for European derivative with convex payoff functions. We deduce the respective inclusion on the subset of convex positive replicable payoffs. Second, we generalize the latter and prove that within the class of locally bounded ( $L_{\text{loc}}^\infty$ ) payoffs, the same inclusion holds. The proof relies on fine properties of the transition kernel. These inclusions justify the usage of classical payoffs in the sticky Black-Scholes model.

We first define the following objects. For each  $\rho \geq 0$ , let  $S^{(\rho)}$  be the solution of (1) and  $\mathbf{H}(S_T^{(\rho)})$  be the class of replicable payoff functions on  $S_T^{(\rho)}$ , given by

$$\mathbf{H}(S_T^{(\rho)}) := \{h \text{ measurable} : h(S_T^{(\rho)}) \in L^1(\Omega, \mathcal{F}_T, \mathbb{Q}_x)\}.$$

**Proposition 5.1** (price monotonicity). *Let  $\text{Opt}_h(S^{(\rho)}, x, T)$  be the (APP) price at  $t = 0$  of the contingent claim on the underlying asset  $S^{(\rho)}$  with payoff function  $h$ , maturity  $T > 0$  and spot price  $S_0^{(\rho)} = x$ , i.e.  $\text{Opt}_h(S^{(\rho)}, x, T) = \mathbb{E}^{\mathbb{Q}_x} (h(S_T^{(\rho)}))$ . If  $h$  is convex, then*

(i) *for all  $x, T > 0$ , the application  $[\rho \rightarrow \text{Opt}_h(S^{(\rho)}, x, T)]$  is decreasing,*

(ii) *for all  $x > 0$ ,  $\rho \geq 0$ , the application  $[T \rightarrow \text{Opt}_h(S^{(\rho)}, x, T)]$  is increasing,*

The proof relies on the following result, which allows a sticky diffusion to be expressed as a time change of a non-sticky diffusion. Although this result can be inferred from the proof of [3, Proposition 2.1.17], we provide a more detailed proof in Appendix A.1.

**Lemma 5.2.** *Let  $X$  be a regular diffusion on  $J$ , an open interval of  $\mathbb{R}$ , on natural scale of speed measure  $m$  so that  $\rho := m(\{\zeta\}) \in (0, \infty)$ . We suppose  $X$  to be defined on the probability space  $\mathcal{P}_x = (\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}_x)$ . There exists a regular diffusion  $Z$  on  $J$ , on natural scale, with speed measure  $m_Z = m - \rho \delta_\zeta$ , defined on an extension of  $\mathcal{P}_x$ , such that*

$$\forall t \geq 0 : \quad X_t = Z_{\gamma_\rho(t)}, \quad A_\rho(t) = t + \rho L_t^\zeta(Z),$$

where  $\gamma_\rho$  is the right-inverse of  $A_\rho$ .

*Proof of Proposition 5.1.* Let  $S^{(0)}$  be defined on the probability space  $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}_x)$  such that  $S^{(0)} = x$ ,  $\mathbb{P}_x$ -almost surely. Let  $\mathbb{Q}_x$  be the (ELMM) for  $S^{(0)}$  defined in Proposition 3.7. From Theorem 4.6(ii),

$$\text{Opt}_h(S^{(0)}, x, T) = \mathbb{E}^{\mathbb{Q}_x}(h(S_T^{(0)})).$$

For all  $\rho \geq 0$ , let  $A_\rho$  be the random time change defined by  $A_\rho(t) = t + \rho L_t^\zeta(S^{(0)})$  and let  $\gamma_\rho$  be its right-inverse. We observe that

1. if  $\rho' < \rho$ , then  $0 \leq \gamma_\rho(T) \leq \gamma_{\rho'}(T) \leq T$ ,
2. from Lemma 5.2,  $(S_{\gamma_\rho(t)}^{(0)})_{t \geq 0} = (S_t^{(\rho)})_{t \geq 0}$  in law,
3.  $\gamma_\rho^{-1}, \gamma_\rho$  are stopping times,
4. since  $S^{(0)}$  is a  $\mathbb{Q}_x$ -martingale (Proposition 3.7(iii)),  $(h(S_t^{(0)}))_{t \geq 0}$  is a  $\mathbb{Q}_x$ -submartingale.

Thus, from the stopping theorem, if  $\rho' < \rho$ ,

$$\begin{aligned} \text{Opt}_h(S^{(\rho')}, x, T) &= \mathbb{E}^{\mathbb{Q}_x}(h(S_T^{(\rho')})) = \mathbb{E}^{\mathbb{Q}_x}(h(S_{\gamma_{\rho'}(T)}^{(0)})) \\ &\leq \mathbb{E}^{\mathbb{Q}_x}(h(S_{\gamma_\rho(T)}^{(0)})) = \mathbb{E}_x^{\mathbb{Q}}(h(S_T^{(\rho)})) = \text{Opt}_h(S^{(\rho)}, x, T). \end{aligned}$$

This proves (i). With similar arguments, if  $0 \leq T \leq T'$ ,

$$\text{Opt}_h(S^{(\rho)}, x, T) = \mathbb{E}^{\mathbb{Q}_x}(h(S_T^{(\rho)})) \leq \mathbb{E}^{\mathbb{Q}_x}(h(S_{T'}^{(\rho)})) = \text{Opt}_h(S^{(\rho)}, x, T').$$

This proves (ii), which completes the proof.  $\square$

**Remark 5.3.** Lemma 5.2 yields that if  $C^+$  is the cone of positive convex functions, then

$$\mathbf{H}(S_T^{(\rho)}) \cap C^+ \subset \mathbf{H}(S_T^{(\rho')}) \cap C^+, \quad \forall 0 \leq \rho < \rho'.$$

In particular, this holds for Call and Put payoff functions.

The next proposition goes beyond this observation.

**Proposition 5.4.** *Let  $L_{\text{loc}}^\infty((0, \infty))$  be the family of locally bounded functions on  $(0, \infty)$ , i.e. the functions that are bounded on all compacts of  $(0, \infty)$ . For all  $0 \leq \rho < \rho'$ , we have*

$$\mathbf{H}(S_T^{(\rho)}) \cap L_{\text{loc}}^\infty((0, \infty)) \subset \mathbf{H}(S_T^{(\rho')}) \cap L_{\text{loc}}^\infty((0, \infty)).$$

For this, we use the following lemma. For a proof, we refer the reader to Appendix A.2.

**Lemma 5.5.** *Let  $[(t, x, y) \rightarrow p_0(t, x, y)]$  and  $m_0(dy) = \dot{m}_0(y) dy$  be the probability transition kernel and speed measure of the geometric Brownian motion. For all  $x > 0$  and  $T > 0$ , there exists a compact  $K_{x,T}$  such that on  $(t, y) \in [0, T] \times K_{x,T}^c$ ,  $[t \rightarrow p_0(t, x, y)\dot{m}_0(y)]$  is increasing.*



*Proof of Proposition 5.4.* For all  $\rho \geq 0$ , let  $m_\rho$  be the speed measure, and  $[(t, x, y) \rightarrow p_\rho(t, x, y)]$  the probability transition kernel, of the process  $S^{(\rho)}$ . Let  $\gamma_\rho$  be the right-inverse of the time-change  $A_\rho(t) = t + \rho L_t^{(0)}(S)$ ,  $t \geq 0$ . This yields in particular that for all  $t \geq 0$ ,  $\gamma_\rho(t) \leq t$ .

From Lemma 5.2, since  $S^{(\rho)} = (S_{\gamma_\rho(t)}^{(0)})_{t \geq 0}$  in law, for all positive measurable function  $f$ ,

$$\mathbb{E}^{\mathbb{Q}_x} (f(S_T^{(\rho)})) = \mathbb{E}^{\mathbb{Q}_x} (f(S_{\gamma_\rho(T)}^{(0)})) = \mathbb{E}^{\mathbb{Q}_x} \left( \int_{(0, \infty)} f(y) p_0(\gamma_\rho(T), x, y) \dot{m}_0(y) dy \right). \quad (11)$$

Let  $K_{x,T}$  be some compact set of  $(0, \infty)$  inferred from Lemma 5.5. Hence, from (11),

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}_x} (|h(S_T^{(\rho)})|) &= \mathbb{E}^{\mathbb{Q}_x} (\mathbb{1}_{S_T^{(\rho)} \in K_{x,T}} |h(S_T^{(\rho)})|) + \mathbb{E}^{\mathbb{Q}_x} (\mathbb{1}_{S_T^{(\rho)} \in K_{x,T}^c} |h(S_T^{(\rho)})|) \\ &\leq \|\mathbb{1}_{K_{x,T}} h\|_\infty + \mathbb{E}^{\mathbb{Q}_x} (\mathbb{1}_{S_{\gamma_\rho(T)}^{(0)} \in K_{x,T}^c} |h(S_{\gamma_\rho(T)}^{(0)})|) \\ &\leq \|\mathbb{1}_{K_{x,T}} h\|_\infty + \mathbb{E}^{\mathbb{Q}_x} \left( \int_0^\infty \mathbb{1}_{y \in K_{x,T}^c} |h(y)| p_0(\gamma_\rho(T), x, y) \dot{m}_0(y) dy \right). \end{aligned}$$

Let  $0 \leq \rho < \rho'$ . We recall that for all  $T \geq 0$  and  $t \in [0, T]$ ,  $0 \leq \gamma_{\rho'}(t) \leq \gamma_\rho(t) \leq T$ . Hence, from the definition of  $K_{x,T}$ ,

$$\mathbb{E}^{\mathbb{Q}_x} \left( \int_{K_{x,T}^c} |h(y)| p_0(\gamma_{\rho'}(T), x, y) \dot{m}_0(y) dy \right) \leq \mathbb{E}^{\mathbb{Q}_x} \left( \int_{K_{x,T}^c} |h(y)| p_0(\gamma_\rho(T), x, y) \dot{m}_0(y) dy \right).$$

Since  $h$  is locally bounded,  $\|\mathbb{1}_{K_{x,T}} h\|_\infty < \infty$ . These prove the inclusion  $\mathbf{H}(S_T^{(\rho)}) \cap L_{\text{loc}}^\infty((0, \infty)) \subset \mathbf{H}(S_T^{(\rho')}) \cap L_{\text{loc}}^\infty((0, \infty))$ . This completes the proof.  $\square$

**Remark 5.6.** Proposition 5.4 ensures that the  $L^1$ -condition of Theorem 4.6 is met for all locally bounded payoff functions that are replicable in the standard Black-Scholes model, i.e. for all  $h \in \mathbf{H}(S_T^{(0)}) \cap L_{\text{loc}}^\infty((0, \infty))$ .

## 6. Numerical Experiments

This section is dedicated to numerical experiments aimed at evaluating practical aspects of hedging a contingent claim on an asset with a sticky price dynamic. We investigate two key properties. The first property involves the hedging time granularity of the sticky Black-Scholes model, referred to as “the impact on replication when the agent hedges in discrete time.” The second property is model mismatch, or “the impact on replication when an agent ignores or misrepresents price stickiness.”

In the rest of this section we consider the assets dynamic (1) with  $r = 0$ ,  $\sigma = 0.25$  and various degrees of stickiness  $\rho \in \{0, 1, 2\}$  at the threshold  $\zeta = 10$ . We recall that a stickiness value of  $\rho = 0$  corresponds to the standard Black-Scholes dynamic. The claim of interest will be a Call option with strike  $K$  that delivers  $(S_T - K)_+$  at maturity  $T = 10$ .

The approximation scheme we use to simulate sample paths of the sticky geometric Brownian motion is the Space-Time Markov Chain Approximation or STMCA of one-dimensional diffusions [6].

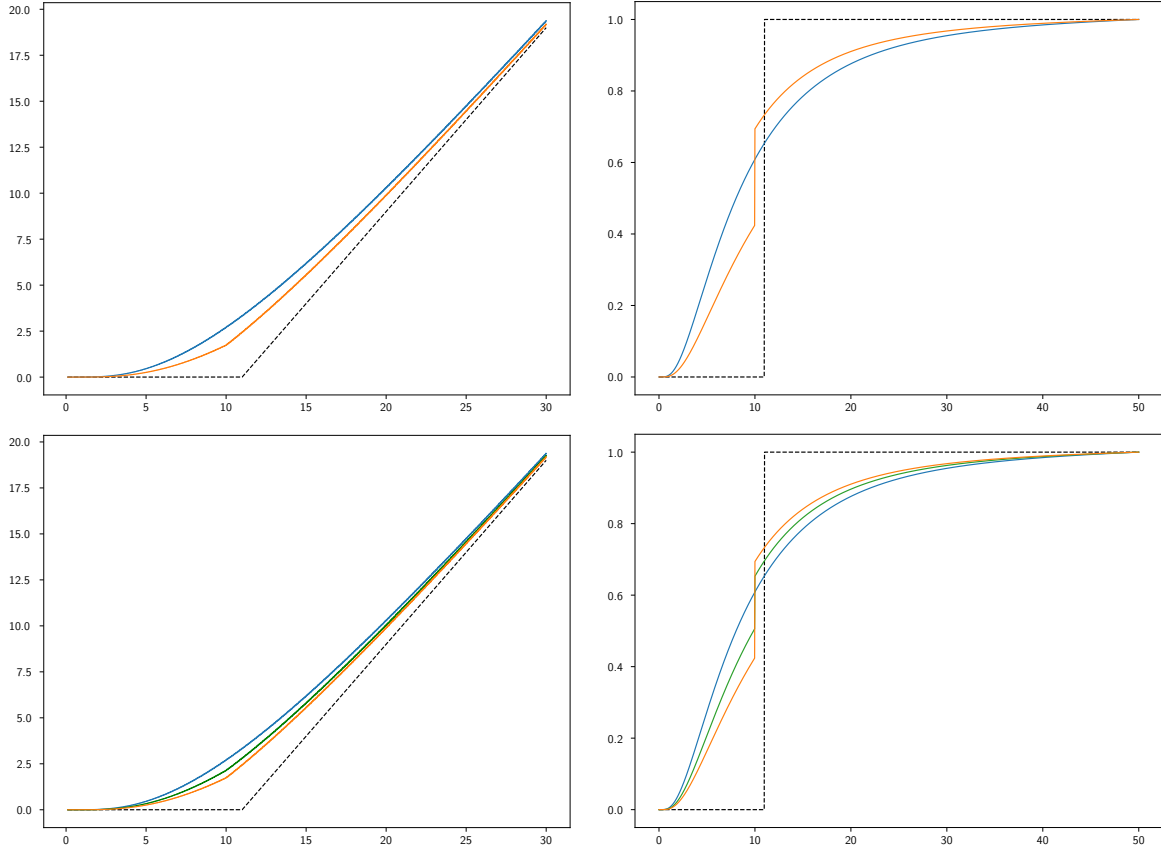


Figure 2: **Price and delta of the Black-Scholes and sticky Black-Scholes.** Finite-elements approximations of the price (left) and delta (right) curves of a Call option of strike  $K = 11$  and  $T = 10$  in the  $(r, \sigma) = (0, 0.25)$  standard Black-Scholes model (blue), the  $(r, \sigma, \rho, \zeta) = (0, 0.25, 1.0, 10.0)$  sticky Black-Scholes model (green) and the  $(r, \sigma, \rho, \zeta) = (0, 0.25, 2.0, 10.0)$  sticky Black-Scholes model (orange).

## 6.1. Hedging time-granularity

The message of the Black-Scholes model is as follows: It is possible to replicate the payoff of a European option using a self-financing portfolio composed of quantities of the risky and non-risky assets. In particular, the option seller can hedge their risk by holding a portfolio that, at each time  $t$ , comprises  $\delta_t = v_x(t, S_t)$  parts of the risky asset.

However, a practical challenge arises since achieving this requires continuous rebalancing of the portfolio composition over time. This, as mentioned in [11], proves impossible in real-world scenarios as it necessitates an infinite number of operations. One way to address this is to introduce a hedging window  $h > 0$ , resulting in hedging times  $(t_i)_i$  such that  $t_{i+1} - t_i = h$ . The agent can then rebalance the portfolio at each time  $t_i$  so that for all  $i$ , they hold  $v_x(t_i, S_{t_i})$  parts of the risky asset on  $[t_i, t_{i+1})$ . The resulting discretized version of the continuous-time hedging strategy  $\delta$  is the strategy  $\delta^{(N)}$  defined for all  $t$  by

$$\delta_t^{(N)} = \delta_{[Nt]/N},$$

where  $[x] = \sup\{k \in \mathbb{N} : k \leq x\}$ .

---

**Algorithm 1:** Delta Hedging of an Option

---

**Data:** Stock price time-series  $(S_t)_{t \geq 0}$ , Strike price  $K$ , Time to maturity  $T$ ,  
Volatility  $\sigma$ , Stickiness  $\rho$ , Hedging window  $\Delta t$

**Result:** Deltas and Portfolio Values over time

- 1 Compute or approximate the price function `u_price`( $x, K, T, \sigma, \rho$ );
  - 2 Compute or approximate the delta function `u_delta`( $x, K, T, \sigma, \rho$ );
  - 3 Initialize *cash* to `u_price`( $S_0, K, T, \sigma, \rho$ );
  - 4 Initialize *t* to 0;
  - 5 Initialize *old\_delta* to 0;
  - 6 **while**  $t < T$  **do**
  - 7     Update time:  $t \leftarrow t + \Delta t$ ;
  - 8     Retrieve current stock price  $S_t$ ;
  - 9     Calculate current option value:  $price \leftarrow \text{u\_price}(S_t, K, T - t, \sigma, \rho)$ ;
  - 10    Calculate current option delta:  $delta \leftarrow \text{u\_delta}(S_t, K, T - t, \sigma, \rho)$ ;
  - 11    Update cash position:  $cash \leftarrow cash - (delta - old\_delta) \times S_t$ ;
  - 12    Set *old\_delta*  $\leftarrow delta$ ;
  - 13    Update portfolio value:  $portfolio\_value \leftarrow cash + (delta \times S_t) - price$ ;
  - 14    Append *delta* to the list of deltas;
  - 15    Append *portfolio\_value* to the list of portfolio values;
  - 16 **return** *List of deltas*, *List of portfolio values*;
- 

The PnLs at  $t$  of the strategies  $\delta$  and  $\delta^{(N)}$  are

$$V_t(\delta) - V_0(\delta) = \int_0^t \delta_s dS_s, \quad V_t(\delta^{(N)}) - V_0(\delta) = \int_0^t \delta_s^{(N)} dS_s,$$

where from Theorem 4.6, the option premium is  $V_0(\delta) = \mathbb{E}^{\mathbb{Q}}(h(S_T))$  and  $V_t(\delta) = \mathbb{E}^{\mathbb{Q}}(h(S_T) | \mathcal{F}_t) = Q_{S_t}^{S, \mathbb{Q}}(h(Y_{T-t}))$ .

This discrete replication strategy, outlined in Algorithm 1, results in replication errors known as the *tracking error*, defined by

$$\varepsilon_t^{(N)} := V_t(\delta^{(N)}) - V_t(\delta).$$

In [11], the metric used to assess the replication quality is the mean squared tracking error, defined by

$$\mathbf{R}_T^{(N)} := \sqrt{\mathbb{E}^{\mathbb{Q}} [(\varepsilon_T^{(N)})^2]}.$$

Instead of this, we will quantify the replication properties of each model with the pair  $(\mathbb{E}^{\mathbb{Q}}[\varepsilon_t^{(N)}], (\text{Var}^{\mathbb{Q}}[\varepsilon_t^{(N)}])^{1/2})$ , where  $\mathbb{E}^{\mathbb{Q}}[\varepsilon_t^{(N)}]$  is the expected replication PnL and  $(\text{Var}^{\mathbb{Q}}[\varepsilon_t^{(N)}])^{1/2}$  is the replication PnL standard deviation. The reason is that  $\mathbf{R}_t^{(N)}$  is more adapted to the case where  $\mathbb{E}^{\mathbb{Q}}[\varepsilon_t^{(N)}] = 0$ . If an agent is unsure about the pricing/replication model he applies, he is likely to miscalculate the option premium and end up with non-zero PnL, making the metric  $\mathbf{R}_t^{(N)}$  irrelevant. This is the object of Section 6.2. Another reason

Stickiness $\rho = 0$				Stickiness $\rho = 1$			
N	premium	$\hat{\mu}_{MC}^{(N)}$	$\hat{\sigma}_{MC}^{(N)}$	N	premium	$\hat{\mu}_{MC}^{(N)}$	$\hat{\sigma}_{MC}^{(N)}$
2000	3.03923	-0.03438	0.06251	2000	2.43607	-0.02426	0.08783
1000	3.03923	-0.03481	0.08605	1000	2.43607	-0.01573	0.17748
250	3.03923	-0.03687	0.16781	250	2.43607	-0.01959	0.29477
100	3.03923	-0.04529	0.26832	100	2.43607	-0.02013	0.40177
10	3.03923	-0.00601	0.69078	10	2.43607	-0.00182	0.80982

Stickiness $\rho = 2$			
N	premium	$\hat{\mu}_{MC}^{(N)}$	$\hat{\sigma}_{MC}^{(N)}$
2000	1.9976	-0.01487	0.04602
1000	1.9976	-0.01502	0.18474
250	1.9976	-0.03455	0.38181
100	1.9976	-0.03731	0.51832
10	1.9976	0.01637	0.96878

Table 1: Premiums and Monte Carlo estimation of the expected replication error (1000 simulated trajectories) for a Call  $(K, T) = (10, 10)$  for various stickiness and hedging windows.  $N_{MC} = 1000$ .

for this choice is that the pair  $(\mathbb{E}^Q[\varepsilon_t^{(N)}], (\text{Var}^Q[\varepsilon_t^{(N)}])^{1/2})$  contains the information in  $\mathbf{R}_t^{(N)}$ . Indeed, the three quantities are linked with the relation:

$$\text{Var}^Q[\varepsilon_t^{(N)}] = (\mathbf{R}_t^{(N)})^2 - (\mathbb{E}^Q[\varepsilon_t^{(N)}])^2.$$

We assess the discrete-time hedging error of the sticky Black-Scholes model as follows. We sample trajectories of the sticky geometric Brownian motion. Then, we apply Algorithm 1 to hedge a call option for various hedging windows. Last, we compute the empiric mean and standard deviation of the PnL with

$$\hat{\mu}_{MC}^{(N)} = \frac{1}{N_{MC}} \sum_{i=1}^{N_{MC}} \varepsilon_T^{(N)}(\omega_i), \quad \hat{\sigma}_{MC}^{(N)} = \frac{1}{\sqrt{N_{MC}}} \sqrt{\sum_{i=1}^{N_{MC}} (\varepsilon_T^{(N)}(\omega_i) - \hat{\mu}_{MC}^{(N)})^2}.$$

We do not have closed form expressions for the replication strategy and the induced prices for the sticky Black-Scholes model. We thus use finite differences approximations of the solution of (8). Simulation results are summarized in Table 1.

From numerical simulations, we observe the following:

1. The replication errors decreases the higher the hedging frequency  $N$  is. Convergence seems to occur at the same rate as for the standard Black-Scholes model. In [11], it is proven that the rate of convergence for the Black-Scholes model is  $\mathcal{O}(N^{-1/2})$ .
2. Higher stickiness leads to higher tracking errors for coarse hedging time-windows. We attribute this behavior to the inherent discontinuity of the delta function at the sticky threshold. As depicted in Figure 2, higher stickiness at  $\zeta$  corresponds to more

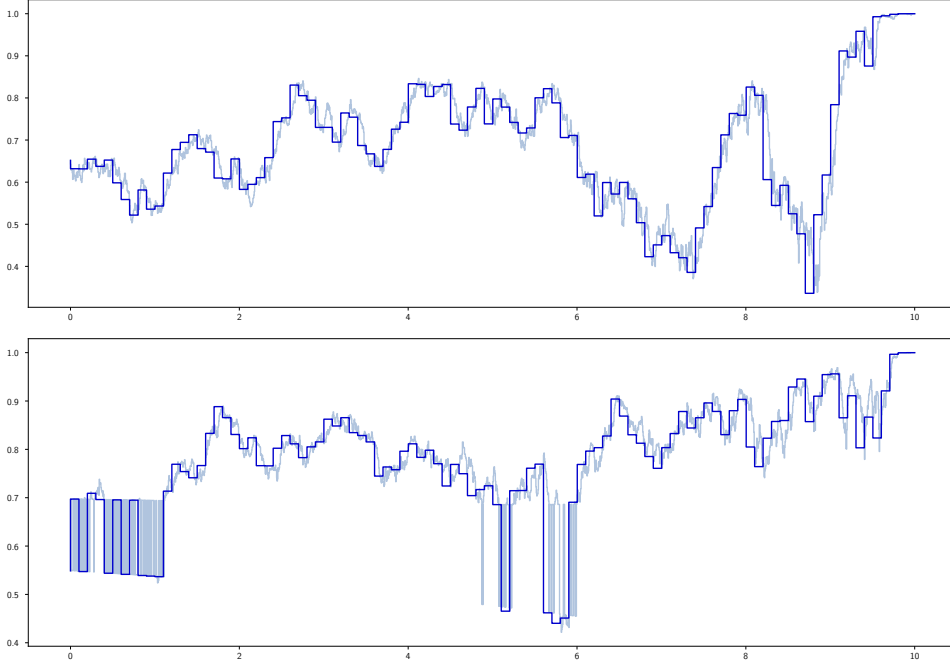


Figure 3: (top): Parts of risky asset in the payoff replication strategy for one sample path of  $S$  with  $\rho = 0$ , for (dark blue)  $N = 100$  and (light blue)  $N = 2000$ . (bottom): Same, but with  $\rho = 1$ .

substantial jumps in the delta, resulting in sudden shifts in portfolio composition whenever the price process  $S$  crosses the sticky threshold. Consequently, the discrete-time hedging portfolio manifests substantial deviations in portfolio composition from the ideal configuration whenever the price crosses the sticky threshold. This is illustrated in Figure 3, where spikes and oscillations in the portfolio composition for fine-grained hedging do not appear in coarser regimes.

## 6.2. Model mismatch: constant volatility

Now, let's assume that an agent intends to hedge a contingent claim on an underlying asset with a sticky geometric Brownian motion price dynamic with parameters  $(\sigma, \rho, \zeta)$ . The agent considers the following models:

**Model 1:** The standard Black-Scholes model with volatility  $\sigma$ .

**Model 2:** The standard Black-Scholes model with volatility  $\Sigma_n(X)$ , where for some  $T' > 0$ ,  $\Sigma_n(X)$  is the estimator of the realized volatility on  $[-T', 0)$  of the current sample path. The estimator is defined by

$$\Sigma_n(X) = \frac{1}{\sqrt{T'}} \sqrt{\sum_{i=1}^{\lfloor nT' \rfloor} (\log S_{-\frac{i}{n}} - \log S_{-\frac{i+1}{n}})^2}.$$

Model 1						Model 2					
$\rho$	prem.	(APP)	$\epsilon_{\text{MP}}$	$\hat{\mu}_{\text{MC}}$	$\hat{\sigma}_{\text{MC}}^{(N)}$	$\rho$	prem.	(APP)	$\epsilon_{\text{MP}}$	$\hat{\mu}_{\text{MC}}$	$\hat{\sigma}_{\text{MC}}^{(N)}$
0	3.073	3.039	0.034	-0.0018	0.057	0	3.076	3.039	0.037	0.0016	0.077
1	3.073	2.436	0.637	0.6224	0.515	1	2.670	2.436	0.234	0.2172	0.508
2	3.073	1.997	1.076	1.0551	0.727	2	2.359	1.997	0.362	0.3238	0.735

Model 3					
$\rho$	prem.	(APP)	$\epsilon_{\text{MP}}$	$\hat{\mu}_{\text{MC}}$	$\hat{\sigma}_{\text{MC}}^{(N)}$
0	3.039	3.039	0	-0.0343	0.062
1	2.436	2.436	0	-0.0242	0.087
2	1.997	1.997	0	-0.0014	0.046

Table 2: Theoretical premiums in each model, (APP) prices, mean PnLs and tracking errors for a Call  $(K, T) = (10, 10)$  on an assets with price-processes  $\sigma = 0.25$  sticky at  $\zeta = 10$  for various degrees of stickiness  $\rho$  in each model. The hedging frequency is  $N = 2000$  and the number of simulated sample paths of each process is  $N_{\text{MC}} = 1000$ .

**Model 3:** The sticky Black-Scholes model with parameters  $(\sigma, \rho, \zeta)$ .

According to Theorem 4.6, the replication cost for the claim with payoff  $h(S_T)$  is  $E^{\mathbb{Q}}[h(S_T)]$ . This matches the (APP) price or premium in Model 3, i.e.,

$$\text{(APP) price} = \text{(Premium in Model 3)} = E^{\mathbb{Q}}[h(S_T)].$$

Therefore, the premium is miscalculated in Models 1 and 2. To address the premium miscalculation in each model, we introduce the mispricing error, defined as

$$\epsilon_{\text{MP}} := \text{(Premium in Model } i) - \text{(APP) price.}$$

This error compensates for the error made by the agent when using an incorrect model to compute premiums for contingent claims. It is to be compared with the empiric mean PnL of the replication portfolios  $\hat{\mu}_{\text{MC}}$ . The discrepancy between values between Model 1 and Model 3 when  $\rho = 0$  is due to error incurred from the finite elements approximations of quantities in Model 3.

We observe the following:

1. We observe in Models 1 that the mispricing error  $\epsilon_{\text{MP}}$  is close to the empiric mean PnL of the replication portfolios  $\hat{\mu}_{\text{MC}}$ . This is due to two facts. First, the agent ultimately holds  $h(S_T)$  at maturity. Second, given that the price dynamics are martingales on  $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$ , from (9) the self-financing portfolio is also a martingale. Hence, the expected PnL of any self-financing portfolio is 0 and the expected PnL of the option seller is

$$\text{Premium} - E^{\mathbb{Q}}[h(S_T)],$$

which equals 0 only when pricing with the correct model, Model 3.

2. Among Models 1, 2 and 3, it is Model 3 that exhibits the lowest hedging error, indicating superior replication properties (see Table 2).

### 6.3. Model mismatch: smooth SDE

We now suppose the agent misrepresents price stickiness as local volatility in a smooth SDE model.

It is possible to approximate in law diffusions with sticky features by smooth diffusions. Let  $X$  be the diffusion on  $(0, \infty)$ , on natural scale, of speed measure

$$m(dx) = \frac{2}{\sigma^2 x^2} dx + \rho \delta_\zeta(dx), \quad x > 0.$$

It corresponds to the law of  $S$  under  $Q$ , see (6). We can define an approximating sequence  $(X^n)_n$  of  $S$  in law as follows. For all  $n$ : let  $X^n$  solve  $dX_t^n = \sigma_n(X_t^n) dB_t^n$  with  $B^n$  a standard Brownian motion and  $\sigma_n$  a smooth function, such that

$$\frac{2}{\sigma_n^2(x)} dx \xrightarrow[n \rightarrow \infty]{\text{weakly}} m(dx).$$

From [15],  $X^n \rightarrow X$  in law.

We make two key observations. First, the prices in the smooth models converge to the prices in the sticky model. Indeed, let  $v$  be the price function and  $(Q_x^X)_x$  the Markovian family of the sticky model  $X$ , and, for all  $n \in \mathbb{N}$ , let  $v^{(n)}$  be the price function and  $(Q_x^{X^n})_x$  the Markovian family of the smooth model  $X^n$ . The convergence in law implies that

$$\lim_{n \rightarrow \infty} v^{(n)}(t, x) = \lim_{n \rightarrow \infty} Q_x^{X^n}(h(Y_{T-t})) = Q_x^X(h(Y_{T-t})) = v(t, x),$$

where  $Y$  is the coordinate process. Second, while the deltas in the smooth models are continuous, the delta of the sticky model is discontinuous (see Remark 4.4). Indeed, due to smoothness of coefficient  $\sigma_n$ , the function  $x \rightarrow v^{(n)}(t, x)$  is also smooth, for all  $t \in [0, T]$  and  $n \in \mathbb{N}$ .

Stickiness $\rho = 0$				Stickiness $\rho = 1$			
N	premium	$\hat{\mu}_{MC}$	$\hat{\sigma}_{MC}^{(N)}$	N	premium	$\hat{\mu}_{MC}$	$\hat{\sigma}_{MC}^{(N)}$
2000	3.03923	-0.03438	0.06251	2000	2.43607	-0.02311	0.17189
1000	3.03923	-0.03481	0.08605	1000	2.43607	-0.01909	0.19802
250	3.03923	-0.03687	0.16781	250	2.43607	-0.02165	0.29681
100	3.03923	-0.04529	0.26832	100	2.43607	-0.02050	0.38760
10	3.03923	-0.00601	0.69078	10	2.43607	-0.01681	0.78004

Stickiness $\rho = 2$			
N	premium	$\hat{\mu}_{MC}$	$\hat{\sigma}_{MC}^{(N)}$
2000	1.9976	-0.00126	0.24402
1000	1.9976	-0.00939	0.28132
250	1.9976	-0.02760	0.40503
100	1.9976	-0.02075	0.52987
10	1.9976	0.00303	0.94837

Table 3: Same as Table 1 but for the locally linearly interpolated price curve around  $\zeta$ .

In Section 6.1, we approximated the delta function using an interpolation of the finite-difference scheme that preserved the discontinuity. In this section, we use an approximation that does not preserve the discontinuity. Thus, the delta for the two approximations differ only locally around the threshold of stickiness. We observe that the second method produces high tracking error (see Table 3, to compare with Table 1). This indicates that hedging a sticky diffusion using a smooth model results in some significant irreducible tracking error. The reason is that a smooth function of the spot price cannot replicate the discontinuity of the true sticky hedging strategy.

## 7. Conclusion

In conclusion, we have demonstrated that the sticky Black-Scholes model satisfies the No Free Lunch with Vanishing Risk (NFLVR) and is consistent with arbitrage-free pricing theory only when the interest rate  $r = 0$ . Under this condition, we proved the existence of a replication strategy and that in this model there is no uniqueness in both the replication strategy and the equivalent local martingale measure (ELMM). Moreover, we established that any locally bounded payoff that can be replicated in the standard Black-Scholes model can also be replicated in the sticky model.

Our numerical experiments indicate that the mean squared tracking error of discrete-time hedging converges at a rate of  $\mathcal{O}(N^{1/2})$ , to 0 as  $N$  diverges to  $\infty$ , matching the rate in the standard Black-Scholes model. We also observed that ignoring or misrepresenting stickiness results in significant irreducible tracking errors, emphasizing the importance of using hedging strategies that accurately account for sticky price features.

This work leaves several topics for future investigation. First, exploring whether the model with  $r \neq 0$  remains arbitrage-free under proportional or fixed transaction costs. Second, identifying the class of payoffs for which the pricing problem stated in Theorem 4.3 admits a unique classical solution. Lastly, determining the rate of convergence of the mean squared tracking error that arises from discrete-time hedging.

## A. Proof-assisting results

### A.1. Proof of Lemma 5.2

For reader's convenience, we recall the statement of Lemma 5.2.

**Lemma.** *Let  $X$  be a regular diffusion on  $J$ , an open interval of  $\mathbb{R}$ , on natural scale of speed measure  $m$  so that  $\rho := m(\{\zeta\}) \in (0, \infty)$ . We suppose  $X$  to be defined on the probability space  $\mathcal{P}_x = (\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}_x)$ . There exists a regular diffusion  $Z$  on  $J$ , on natural scale, with speed measure  $m_Z = m - \rho \delta_\zeta$ , defined on an extension of  $\mathcal{P}_x$ , such that*

$$\forall t \geq 0 : \quad X_t = Z_{\gamma_\rho(t)}, \quad A_\rho(t) = t + \rho L_t^\zeta(Z),$$

where  $\gamma_\rho$  is the right-inverse of  $A_\rho$ .



*Proof.* From [44, Theorem V.47.1], there exists a standard Brownian motion  $W$ , defined on an extension of  $\mathcal{P}_x$ , such that

$$\forall t \geq 0 : \quad X_t = W_{\gamma_X(t)} \quad A_X(t) = \int_J L_t^y(W) m(dy),$$

where  $\gamma_X$  is the right-inverse of  $A_X$ . We consider the process  $Z$  defined by

$$\forall t \geq 0 : \quad Z_t = W_{\gamma_Z(t)} \quad A_Z(t) = \int_J L_t^y(W) m_Z(dy),$$

where  $\gamma_Z$  is the right-inverse of  $A_Z$ . From [44, Section V.47, Remark (ii)],  $Z$  is a diffusion on  $J$  on natural scale with speed measure  $m_Z$ . Thus,  $X = (Z_{A_Z \circ \gamma_X(t)})_{t \geq 0}$ . From [43, Proposition IV.1.12] and the proof of [44, Theorem V.47.1], for the case of a regular diffusion with unattainable boundaries, we have that  $\gamma_Z(\cdot) = \langle Z \rangle$  and that  $\langle Z \rangle$  is almost surely strictly increasing. Thus, the time-changes  $\gamma_Z, A_Z, \gamma_X, A_X$  are all almost surely strictly increasing and hence invertible, i.e.  $A_Z = \gamma_Z^{-1}$ ,  $A_X = \gamma_X^{-1}$ . This with [43, Exercise VI.1.27] yield that

$$\gamma_X^{-1}(t) = \int_J L_t^y(W) m(dy) = \int_J L_t^y(W) m_Z(dy) + L_t^\zeta(W) m(\{\zeta\}) = \gamma_Z^{-1}(t) + m(\{\zeta\}) L_{\gamma_Z^{-1}(t)}^\zeta(Z)$$

and that

$$(\gamma_Z^{-1} \circ \gamma_X)^{-1}(t) = \gamma_X^{-1} \circ \gamma_Z = t + m(\{\zeta\}) L_t^\zeta(Z) = A_\rho(t).$$

This completes the proof.  $\square$

## A.2. Proof of Lemma 5.5

For reader's convenience, we recall the statement of Lemma 5.5.

**Lemma.** *Let  $[(t, x, y) \rightarrow p_0(t, x, y)]$  and  $m_0(dy) = \dot{m}_0(y) dy$  be the probability transition kernel and speed measure of the geometric Brownian motion. For all  $x > 0$  and  $T > 0$ , there exists a compact  $K_{x,T}$  of  $(0, \infty)$  such that on  $(t, y) \in [0, T] \times K_{x,T}^c$ , the function  $[t \rightarrow p_0(t, x, y) \dot{m}_0(y)]$  is increasing.*

*Proof.* From the expressions for the probability transition kernel and speed measure of the geometric Brownian motion (see [13, Appendix 1.20]):

$$\dot{m}(y) p_0(t, x, y) = \frac{1}{y \sigma \sqrt{2\pi t}} \exp \left( - \frac{(\log y - \log x - (\mu - \frac{1}{2}\sigma^2)t)^2}{2\sigma^2 t} \right).$$

Taking the derivative in time yields

$$\begin{aligned} \partial_t (\dot{m}(y) p_0(t, x, y)) &= \left( \frac{(\log y - \log x - (\mu - \frac{1}{2}\sigma^2)t)^2}{2\sigma^2 t^2} \right. \\ &\quad \left. + 2 \frac{(\log y - \log x - (\mu - \frac{1}{2}\sigma^2)t)}{2\sigma^2 t} \left( \mu - \frac{1}{2}\sigma^2 \right) t - \frac{1}{2t} \right) \dot{m}(y) p_0(t, x, y). \end{aligned}$$

We observe that for all  $t, x > 0$ , as  $y \rightarrow 0$  and  $y \rightarrow \infty$ , the quantity  $\partial_t p_0(t, x, y)$  becomes positive. This completes the proof.  $\square$

## References

- [1] R. Almgren and N. Chriss. Optimal execution of portfolio transactions. *Journal of Risk*, 3:5–40, 2001.
- [2] M. Amir. Sticky Brownian motion as the strong limit of a sequence of random walks. *Stochastic Process. Appl.*, 39(2):221–237, 1991.
- [3] A. Anagnostakis. *Path-wise study of singular diffusions*. PhD thesis, Université de Lorraine (Nancy), 2022.
- [4] A. Anagnostakis. Functional convergence to the local time of a sticky diffusion. *Electronic Journal of Probability*, 28(88):1–26, 2023.
- [5] A. Anagnostakis, D. Criens, and M. Urusov. On weak notions for no arbitrage in a general diffusion market with non-zero interest rate.
- [6] A. Anagnostakis, A. Lejay, and D. Villemonais. General diffusion processes as limit of time-space Markov chains. *The Annals of Applied Probability*, 33(5):3620 – 3651, 2023.
- [7] G. Bakshi, C. Cao, and Z. Chen. Empirical performance of alternative option pricing models. *The Journal of finance*, 52(5):2003–2049, 1997.
- [8] R. F. Bass. A stochastic differential equation with a sticky point. *Electron. J. Probab.*, 19:no. 32, 22, 2014.
- [9] D. S. Bates. Jumps and stochastic volatility: Exchange rate processes implicit in deutsche mark options. *The Review of Financial Studies*, 9(1):69–107, 1996.
- [10] C. Bayer, P. Friz, and J. Gatheral. Pricing under rough volatility. *Quantitative Finance*, 16(6):887–904, 2016.
- [11] D. Bertsimas, L. Kogan, and A. W. Lo. When is time continuous? In *Quantitative analysis in financial markets. Collected papers of the New York University Mathematical Finance Seminar. Vol. II*, pages 71–102. Singapore: World Scientific, 2001.
- [12] F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654, 1973.
- [13] A. N. Borodin and P. Salminen. *Handbook of Brownian motion—facts and formulae*. Probability and its Applications. Birkhäuser Verlag, Basel, 1996.
- [14] N. Bou-Rabee and M. C. Holmes-Cerfon. Sticky Brownian motion and its numerical solution. *SIAM Rev.*, 62(1):164–195, 2020.
- [15] J. Brooks and R. Chacon. Weak convergence of diffusions, their speed measures and time changes. *Advances in Mathematics*, 46(2):200–216, 1982.
- [16] C. Bruggeman and J. Ruf. A one-dimensional diffusion hits points fast. *Electron. Common. Probab.*, 21(22):1–7, 2016.
- [17] D. Buckner, K. Dowd, and H. Hulley. Arbitrage problems with reflected geometric brownian motion. *Finance and Stochastics*, 28(1):1–26, 2024.

- [18] A. Calsina and J. Z. Farkas. Steady states in a structured epidemic model with Wentzell boundary condition. *J. Evol. Equ.*, 12(3):495–512, 2012.
- [19] T. R. A. Corns and S. E. Satchell. Skew brownian motion and pricing european options. *The European Journal of Finance*, 13(6):523–544, 2007.
- [20] D. Criens. On the Feller-Dynkin and the martingale property of one-dimensional diffusions. *Electron. Commun. Probab.*, 28:15, 2023. Id/No 20.
- [21] D. Criens and M. Urusov. Separating times for one-dimensional diffusions. *arXiv preprint arXiv:2211.06042*, 2022.
- [22] D. Criens and M. Urusov. On the representation property for 1d general diffusion semi-martingales, 2024.
- [23] M. J. Davies and A. Truman. Brownian motion with a sticky boundary and point sources in quantum mechanics. volume 20, pages 173–193. 1994. Lagrange geometry, Finsler spaces and noise applied in biology and physics.
- [24] M. Decamps, A. De Schepper, and M. Goovaerts. Applications of  $\delta$ -function perturbation to the pricing of derivative securities. *Physica A: Statistical Mechanics and its Applications*, 342(3-4):677–692, 2004.
- [25] F. Delbaen and W. Schachermayer. A general version of the fundamental theorem of asset pricing. *Math. Ann.*, 300:463–520, 1994.
- [26] F. Delbaen and W. Schachermayer. *The mathematics of arbitrage*. Springer Finance. Berlin: Springer, 2006.
- [27] B. Dupire. Pricing and hedging with smiles. In *Mathematics of derivative securities. Forewords are given by R. C. Merton and M. F. Atiyah*, pages 103–111. Cambridge: Cambridge Univ. Press, 1997.
- [28] B. Dupire et al. Pricing with a smile. *Risk*, 7(1):18–20, 1994.
- [29] H.-J. Engelbert and G. Peskir. Stochastic differential equations for sticky Brownian motion. *Stochastics*, 86(6):993–1021, 2014.
- [30] W. Feller. The parabolic differential equations and the associated semigroups of transformation. *Ann. Math. (2)*, 55:468–519, 1952.
- [31] R. Feng, P. Jiang, and H. Volkmer. Modeling financial market movement with winning streaks: Sticky maximum process. *ERN: Asset Pricing Models (Topic)*, 2020.
- [32] D. Freedman. *Brownian motion and diffusion*. Springer-Verlag, New York-Berlin, second edition, 1983.
- [33] J. Gatheral, T. Jaisson, and M. Rosenbaum. Volatility is rough. *Quantitative finance*, 18(6):933–949, 2018.
- [34] P. S. Hagan, D. Kumar, A. S. Lesniewski, and D. E. Woodward. Managing smile risk. *The Best of Wilmott*, 1:249–296, 2002.

- [35] S. L. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Rev. Financ. Stud.*, 6(2):327–343, 1993.
- [36] J. Hull and A. White. The pricing of options on assets with stochastic volatilities. *The journal of finance*, 42(2):281–300, 1987.
- [37] J. Jacod and A. N. Shiryaev. *Limit theorems for stochastic processes.*, volume 288 of *Grundlehren Math. Wiss.* Berlin: Springer, 2nd ed. edition, 2003.
- [38] Y. M. Kabanov and M. Safarian. *Markets with transaction costs. Mathematical theory.* Springer Finance. Berlin: Springer, 2009.
- [39] O. Kallenberg. *Foundations of modern probability.* Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002.
- [40] I. Karatzas and C. Kardaras. The numéraire portfolio in semimartingale financial markets. *Finance Stoch.*, 11:447–493, 2007.
- [41] H. E. Leland. Option pricing and replication with transactions costs. *The journal of finance*, 40(5):1283–1301, 1985.
- [42] Y. Nie and V. Linetsky. Sticky reflecting ornstein-uhlenbeck diffusions and the vasicek interest rate model with the sticky zero lower bound. *Stochastic Models*, 0(0):1–19, 2019.
- [43] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.
- [44] L. C. G. Rogers and D. Williams. *Diffusions, Markov processes, and martingales. Vol. 2.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2000. Itô calculus, Reprint of the second (1994) edition.
- [45] D. Rossello. Arbitrage in skew Brownian motion models. *Insur. Math. Econ.*, 50(1):50–56, 2012.
- [46] M. Salins and K. Spiliopoulos. Markov processes with spatial delay: path space characterization, occupation time and properties. *Stoch. Dyn.*, 17(6):21, 2017. Id/No 1750042.
- [47] L. O. Scott. Option pricing when the variance changes randomly: Theory, estimation, and an application. *Journal of Financial and Quantitative analysis*, 22(4):419–438, 1987.
- [48] S. E. Shreve. *Stochastic calculus for finance. II: Continuous-time models.* Springer Finance. New York, NY: Springer, 2004.
- [49] E. M. Stein and J. C. Stein. Stock price distributions with stochastic volatility: an analytic approach. *The review of financial studies*, 4(4):727–752, 1991.
- [50] G. Stell. Sticky spheres and related systems. *Journal of Statistical Physics*, 63:1203–1221, 1991.
- [51] H. Zhang and Y. Tian. Hitting time problems of sticky brownian motion and their applications in optimal stopping and bond pricing. *Methodology and Computing in Applied Probability*, 24, 06 2022.
- [52] J. Y. Zhu. Optimal contracts with shirking. *Rev. Econ. Stud.*, 80(2):812–839, 2013.