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On hyperexponential stability and stabilization [★]

Konstantin Zimenko ^a, Denis Efimov ^b, Andrey Polyakov ^b

^a*Faculty of Control Systems and Robotics, ITMO University, 49 Kronverkskiy av., 197101 Saint Petersburg, Russia*

^b*Inria, Univ. Lille, CNRS, UMR 9189 - CRISTAL, F-59000 Lille, France*

Abstract

Theorems on hyperexponential stability are proposed for both explicitly and implicitly defined Lyapunov functions. It is shown that finite-time and fixed-time controls based on the use of implicitly defined Lyapunov function provide hyperexponential stability under sampled-time realization. A rated hyperexponential control is proposed for linear systems. Preliminary numerical experiments show that the proposed control law is less sensitive to measurement noise and has better performance under input delays compared to finite-time analogs.

1 Introduction

Accelerated convergence rates are widely demanded nowadays for control and estimation algorithms in many applications [1], [6], [7], [12], [15], [18]. The most popular examples are finite- and fixed-time decays [1], [12], [18], which also possess other useful features. However, the discretization of these kinds of systems (or implementation in digital devices) is complicated (see, e.g., [10], [21]), and frequently it is difficult to recover all advantageous properties of the designed control or estimation algorithms in computer realization. Fortunately, there exist other types of accelerated convergence.

This paper focuses on hyperexponential, but not finite/fixed-time, stable systems. We call a stable system to be hyperexponential if it converges faster than any linear system, i.e., faster than any exponential (see, e.g., [8], [13], [15], [22], [23]). Such systems may reach the origin asymptotically, hence, they have a slower rate of convergence than finite/fixed-time stable systems. In its turn, being locally Lipschitz continuous at the origin, hyperexponential systems provide better robustness with respect to measurement noise and delays [5], [6], [11], [23] compared to finite/fixed-time stable systems. In this paper theorems on hyperexponential stability are proposed for systems described by autonomous

ordinary differential equations with the use of both explicitly and implicitly defined Lyapunov functions. Utilizing these results it is shown that finite/fixed-time controls of linear systems, which are synthesized with the use of implicit Lyapunov function (ILF) [18], [19] provide hyperexponential stability under sampled-time realization. An ILF-based and rated hyperexponential control is also proposed in the paper for linear systems. The preliminary results of this paper are discussed at [24]. The key differences are as follows: the proofs of all claims are given; the definition of hyperexponential stability has been modified; new hyperexponential stability analysis approach based on piecewise continuous Lyapunov functions is proposed (Theorem 2); the sufficient conditions for the ILF method are relaxed (Corollary 1); a new time-invariant control for linear systems with hyperexponential rate is proposed (Theorem 5); new numerical simulations demonstrating efficiency of the proposed control design schemes are considered.

Although hyperexponential convergence is weaker than finite-time one, according to simulation results it is difficult to distinguish between these properties (especially in the presence of disturbances). In turn, the proposed control demonstrates better robustness against measurement noise and delays compared to the finite-time counterpart. This study highlights the significance of these findings for practical applications and outlines the path for future research.

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Email addresses: konstantin.zimenko@itmo.ru (Konstantin Zimenko), denis.efimov@inria.fr (Denis Efimov), andrey.polyakov@inria.fr (Andrey Polyakov).

Notation: \mathbb{R} is the set of real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$; \mathbb{N} is the set of natural numbers; the symbol $\mathbb{1}, m$ is used to denote a sequence of integers $1, \dots, m$; the order relation $P > 0$ (< 0 ; ≥ 0 ; ≤ 0) for $P \in \mathbb{R}^{n \times n}$ means that P is symmetric and positive (negative) definite (semidef-

inite); the minimal and maximal eigenvalues of a symmetric matrix P are denoted by $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$, respectively; $\text{diag}\{\lambda_i\}_{i=1}^n$ is a diagonal matrix with elements λ_i ; a continuous function $\sigma : \mathbb{R}_+ \cup \{0\} \rightarrow \mathbb{R}_+ \cup \{0\}$ belongs to class \mathcal{K} if it is strictly increasing and $\sigma(0) = 0$; it belongs to class \mathcal{K}_∞ if it is also unbounded.

2 Preliminaries

The Implicit Lyapunov Function method allows one to determine the stability of a system without presenting the Lyapunov function in an explicit form (e.g., the Lyapunov function can be introduced as a solution of some algebraic equation, and it is not necessary to solve this equation for stability analysis). Consider the system

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \geq 0, \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a nonlinear vector field, $f(0) = 0$. The next theorem presents the ILF method.

Theorem 1 [18] *Let there exist a continuous function $Q : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying the conditions*

*C1) Q is continuously differentiable on $\mathbb{R}_+ \times \mathbb{R}^n \setminus \{0\}$;
C2) there exists $V : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}_+$ such that $Q(V(x), x) = 0$;*

*C3) for $\Omega = \{(V, x) \in \mathbb{R}_+ \times \mathbb{R}^n : Q(V, x) = 0\}$
 $\lim_{\substack{x \rightarrow 0 \\ (V, x) \in \Omega}} V = 0^+, \lim_{\substack{V \rightarrow 0^+ \\ (V, x) \in \Omega}} \|x\| = 0, \lim_{\substack{\|x\| \rightarrow \infty \\ (V, x) \in \Omega}} V = +\infty$;*

C4) the inequality $\frac{\partial Q(V, x)}{\partial V} < 0$ holds for all $V \in \mathbb{R}_+$ and $\forall x \in \mathbb{R}^n \setminus \{0\}$.

If $\frac{\partial Q(V, x)}{\partial x} z < 0$ for all $(V, x) \in \Omega$, then the origin of (1) is globally asymptotically stable.

Remark 1 Theorem 1 represents the well-known ILF result on asymptotic stability, where the conditions C1), C2) and C4) guarantee existence and uniqueness of a positive definite function $V : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}_+$ such that $Q(V(x), x) = 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$; C3) implies that $V(x) \rightarrow 0$ as $x \rightarrow 0$ ($V(x) \rightarrow +\infty$ as $x \rightarrow \infty$) and it can be continuously prolonged at the origin by $V(0) = 0$; the condition $\frac{\partial Q(V, x)}{\partial x} f(x) < 0$ ensures $\dot{V}(x) < 0$.

3 Stability definitions

Over the past several decades, increased interest has been paid to finite-time and fixed-time stability (see, [1], [12], [14], [18], [25] and references therein).

Let D be an open neighborhood of the origin in \mathbb{R}^n .

Definition 1 [1], [12], [14] *The origin of (1) is said to be finite-time stable if it is asymptotically stable and for any $x_0 \in D$ any solution $\Phi(t, x_0)$ of the system (1) reaches the origin at some finite time moment, i.e. $\Phi(t, x_0) =$*

$0 \forall t \geq T(x_0)$ and $\Phi(t, x_0) \neq 0 \forall t \in [0, T(x_0))$, $x_0 \neq 0$, where $T : \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{0\}$, $T(0) = 0$ is the settling-time function. The origin is said to be fixed-time stable if it is finite-time stable and the settling-time function $T(x_0)$ is bounded, i.e., $\exists T_{\max} > 0 : T(x_0) \leq T_{\max}, \forall x_0 \in D$. If $D = \mathbb{R}^n$, then the corresponding stability properties are called global.

Definition 2 [15] *A set $\mathcal{M} \subset \mathbb{R}^n$ is said to be globally fixed-time attractive if any solution $\Phi(t, x_0)$ of (1) reaches \mathcal{M} in some finite time moment $t = T(x_0) < T_{\max}$, $T_{\max} \in \mathbb{R}_+$ and remains there for all $t \geq T(x_0)$, where $T : \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{0\}$. The origin is said to be nearly fixed-time stable if it is globally Lyapunov stable and any neighborhood of the origin is fixed-time attractive.*

The interest in finite/fixed-time stability is due to existence of many control applications with time constraints, where a high precision is needed in the presence of significant perturbations (e.g., rejection of non-Lipschitz disturbances). An extension of Theorem 1 for finite/fixed-time stability analysis is proposed in [18]. Another type of fast convergence (faster than exponential), is hyperexponential stability (see, e.g., [2], [8], [13], [15]).

Definition 3 *The system (1) is said to be hyperexponentially stable at the origin if for any $r > 0$ there exist $\beta \in \mathcal{K}_\infty$ and $t' \geq 0$ such that $\|\Phi(t, x_0)\| \leq \beta(\|x_0\|)e^{-rt}$, $\forall t > t'$, for any $x_0 \in D$.*

In other words, the hyperexponential stability assumes increasing of convergence rates as $\Phi(t, x_0) \rightarrow 0$. In order to provide quantitative index to characterize hyperexponential convergence rates, the rated hyperexponential stability is presented in [20]. For $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_r)^\top \in \mathbb{R}_+^{r+1}$ and $r \in \mathbb{N}$ let us introduce the function of nested exponentials $\rho_{r, \alpha} : \mathbb{R} \rightarrow \mathbb{R}_+$ by the following recursive formula

$$\begin{aligned} \rho_{0, \alpha}(z) &= \alpha_0 z, \\ \rho_{i, \alpha}(z) &= \alpha_i (e^{\rho_{i-1, \alpha}(z)} - 1), \quad i = \overline{1, r}. \end{aligned} \quad (2)$$

Definition 4 *The origin of the system (1) is said to be hyperexponentially stable of degree $r \in \mathbb{N}$ if $\exists \beta \in \mathcal{K}_\infty$ and $\exists \alpha \in \mathbb{R}_+^{r+1}$ such that*

$$\|\Phi(t, x_0)\| \leq \beta(\|x_0\|)e^{-\rho_{r, \alpha}(t)} \quad (3)$$

holds for all $t \geq 0$ and $x_0 \in D$. If $D = \mathbb{R}^n$ then the origin of the system (1) is said to be globally hyperexponentially stable of degree r .

Notice that for the case $r = 0$, the expression (3) corresponds to exponential stability definition (and for any $r > 0$, (3) implies an exponential convergence). All finite-time and fixed-time stable systems are also hyperexponentially stable due to finite/fixed-time stability being

obtained through an "infinite eigenvalue assignment" for the system at the origin.

Hyperexponential stability (stabilization) is often the subject of research in the analysis of systems with delays (see, e.g., [16], [20]). Also, the problems of hyperexponential stability analysis and control via time-varying feedback are considered in [8], [22], [23]. In this paper, the hyperexponential stability is investigated with the use of both, explicit and implicit, Lyapunov function methods, and a time-invariant nonlinear hyperexponential control is proposed for linear systems.

Definition 4 implies hyperexponential convergence both in the vicinity and outside the origin. By analogy with nearly fixed-time (fast convergence outside the origin) and finite-time stability (fast convergence in the vicinity of the origin), one can give definitions of the stability of systems with rated hyperexponential convergence:

Definition 5 *The system (1) is said to be*

- *hyperexponentially stable of degree $r \in \mathbb{N}$ at infinity, if it is globally asymptotically stable, and (3) is satisfied for all $x_0 \in \mathbb{R}^n : \|x_0\| \geq 1$;*
- *hyperexponentially stable of degree $r \in \mathbb{N}$ at origin, if it is globally asymptotically stable, and (3) is satisfied for all $x_0 \in \mathbb{R}^n : \|x_0\| \leq 1$.*

An example of hyperexponential control at origin for linear systems is presented in [24, Theorem 5]. Note that Definition 3 does not provide an estimate of convergence rates outside the origin's vicinity (it assumes only increasing of convergence rates as $\Phi(t, x_0) \rightarrow 0$). In this sense, Definition 3 can be considered as an example of unrated hyperexponential stability at origin. Also note that for $D \neq \mathbb{R}^n$ Definition 4 is always satisfied in the case of hyperexponential stability of degree r at origin. Indeed, since any globally asymptotically stable system converges to the unit ball in a finite time, then there exist such β defined on D as in Definition 4.

4 Hyperexponential stability analysis using Lyapunov function method

The Lyapunov function method [9] is the main tool for stability analysis of nonlinear systems. The following theorem presents a sufficient condition for hyperexponential stability.

Theorem 2 *Let $\{D_i\}_{i=1}^{\infty} : D = D_1 \supset D_2 \supset \dots \supset D_i \supset \dots \supset \{0\}$ is a sequence of open sets. Suppose there exist positive definite functions $V_i : D_i \setminus D_{i+1} \rightarrow \mathbb{R}$, $i \in \mathbb{N}$ differentiable on $D_i \setminus D_{i+1}$, for which ∂D_i is a level set $\partial D_i = \{x \in \mathbb{R}^n : V_i(x) = v_i, v_i \in \mathbb{R}_+\}$. If $\exists a, C \in \mathbb{R}_+$*

and $\underline{k}_i, \bar{k}_i \in \mathbb{R}_+ : \frac{\max_i \bar{k}_i}{\min_i \underline{k}_i} \leq C$ such that

$$\begin{aligned} \underline{k}_i \|x\|^a &\leq V_i(x) \leq \bar{k}_i \|x\|^a, & \forall x \in D_i \setminus D_{i+1}, \\ \dot{V}_i(x) &\leq -c_i V_i(x), & \forall x \in D_i \setminus D_{i+1}, \\ V_{i+1}(x) &\leq V_i(x), & \forall x \in \partial D_{i+1}, \end{aligned} \quad (4)$$

where $c_{i+1} > c_i \in \mathbb{R}_+$ for any $i \in \mathbb{N}$ and $c_i \rightarrow +\infty$ as $i \rightarrow +\infty$, then the origin of system (1) is hyperexponentially stable. If $D = \mathbb{R}^n$, then the system (1) admits this property globally.

Proof. Let $x_0 \in D_j$, $j \in \mathbb{N}$. The proof is straightforward due to the condition (4) implies that for any $j \leq i \in \mathbb{N}$ the set D_i is strictly positively invariant and the solution $\Phi(t, x_0)$ reaches the nested set D_{i+1} in a finite time with corresponding exponential rates:

$$\begin{aligned} \|\Phi(t, x_0)\| &\leq \sqrt[a]{\frac{V_i(\Phi(t_i, x_0))}{\underline{k}_i}} e^{-\frac{c_i}{a}(t-t_i)} \leq \sqrt[a]{\frac{V_j(x_0)}{\underline{k}_i}} e^{-\frac{c_i}{a}(t-t_i)} \\ &\leq \sqrt[a]{\frac{\bar{k}_j}{\underline{k}_i}} \|x_0\| e^{-\frac{c_i}{a}(t-t_i)} \leq \sqrt[a]{C} \|x_0\| e^{-\frac{c_i}{a}(t-t_i)}, \end{aligned}$$

where t_i is the time when the trajectory entered in D_i ; the condition $c_i \rightarrow +\infty$ as $i \rightarrow +\infty$ implies increasing of the rates as solution reaches the nested set and $D_i \rightarrow \{0\}$ as $i \rightarrow \infty$ (i.e., the origin is attractive). ■

4.1 ILF-based finite-time control under sampled-time realization

Similarly to Theorem 2, other approaches for hyperexponential stability analysis can be used, based on piecewise continuous Lyapunov functions. The following application shows that sampled-time realization of ILF-based finite-time control from [18] provides hyperexponential stability.

Example 1 Consider the system

$$\dot{x} = Ax + Bu, \quad (5)$$

where $x \in \mathbb{R}^n$ is a state, $u \in \mathbb{R}$ is a control input, $A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & 1 \\ 0 & \dots & \dots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{n \times n}$, $B = \begin{bmatrix} 0 \\ \dots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{n \times 1}$. Introduce the ILF function $Q(V, x) = x^T D(V^{-1}) P D(V^{-1}) x - 1$, where $D(\lambda) = \begin{bmatrix} \lambda^{q_1} & 0 & \dots & 0 \\ 0 & \lambda^{q_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \lambda^{q_n} \end{bmatrix}$ for some $\lambda \in \mathbb{R}_+$, $q_i = 1 + (n - i)\mu$, $i = \overline{1, n}$, $\mu \in (0, 1]$ and $0 < P \in \mathbb{R}^{n \times n}$. Denote $H = \text{diag}\{q_i\}_{i=1}^n$. According to [18, Theorem 6] the control

$$u(V(x), x) = V^{1-\mu}(x) K D(V^{-1}(x)) x \quad (6)$$

stabilizes the system (5) in a finite time, where $V : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}_+$ such that $Q(V(x), x) = 0$,

$V(0) = 0$, $K = YX^{-1}$, and $X = P^{-1} \in \mathbb{R}^{n \times n}$, $Y \in \mathbb{R}^{1 \times n}$ satisfy for some $\alpha \in \mathbb{R}_+$ the system

$$\begin{cases} AX + XA^T + BY + Y^T B^T + \alpha X \leq 0, \\ XH + HX > 0, \quad X > 0. \end{cases}$$

Denote an arbitrary sequence of time instances $\{t_i\}_{i=1}^{+\infty}$ such that $0 = t_0 < t_1 < t_2 < \dots$, $\lim_{i \rightarrow +\infty} t_i = +\infty$. In [18, Corollary 8] it is shown that sampled-time realization of the control (6) (i.e. $u = u(V_i, x)$ on each time interval $[t_i, t_{i+1})$, where $V_i \in \mathbb{R}_+$ such that $Q(V_i, x(t_i)) = 0$) provides asymptotic stabilization of the closed loop system independent of the length of the sampling interval. To show that sampled-time realization actually ensures *hyperexponential stability* of the closed loop system, define the set of nested ellipsoids $D_i := \{x \in \mathbb{R}^n : x^T P_i x \leq 1\}$, where $P_i = D(V_i^{-1}) P D(V_i^{-1})$ and the corresponding set of quadratic functions $\tilde{V}_i(x) = x^T P_i x$. For any $i \in \mathbb{N}$ on $D_i \setminus D_{i+1}$ by [18, Corollary 7, Corollary 8] we have $\dot{\tilde{V}}_i = 2x^T P_i (Ax + Bu(V_i, x)) \leq -\alpha V_i^{-\mu} \tilde{V}_i$ and $V(x(t)) < V(x(t_i)) \leq V(x(0))$, $\forall t \in (t_i, t_{i+1}]$, where $V(x)$ is such that $Q(V(x), x) = 0$. Moreover $\tilde{V}_i(x(t_i)) = 1$, then we obtain $\tilde{V}_i(t) \leq e^{-\alpha V_i^{-\mu}(t-t_i)}$. Taking into account that

$$\begin{aligned} \tilde{V}_i &\geq \lambda_{\min}(P) \min\{V_i^{-2}, V_i^{-2-2(n-1)\mu}\} \|x\|^2 \\ &\geq \lambda_{\min}(P) \min\{V_0^{-2}, V_0^{-2-2(n-1)\mu}\} \|x\|^2, \\ V_0 &\leq \begin{cases} \sqrt{\lambda_{\max}(P)} \|x_0\| & \text{if } V_0 \geq 1 \\ (\lambda_{\max}(P) \|x_0\|^2)^{1/(2+2(n-1)\mu)} & \text{if } V_0 \leq 1 \end{cases} \end{aligned}$$

for $V_0 \in \mathbb{R}_+$ such that $Q(V_0, x_0) = 0$, we obtain

$$\begin{aligned} \|\Phi(t, x_0)\| &\leq \frac{1}{\sqrt{\lambda_{\min}(P)}} e^{-0.5\alpha V_i^{-\mu}(t-t_i)} \\ &\times \max\left\{(\sqrt{\lambda_{\max}(P)} \|x_0\|)^{1+(n-1)\mu}, (\sqrt{\lambda_{\max}(P)} \|x_0\|)^{\frac{1}{1+(n-1)\mu}}\right\} \end{aligned}$$

for $t \in [t_i, t_{i+1})$ that by Definition 3 implies hyperexponential stability since $V_i \rightarrow 0$ as $i \rightarrow \infty$.

The same outcomes can be obtained for [18], [19], [25], where finite/fixed-time controls are proposed for linear controllable MIMO systems of the form (5).

Proposition 1 *Let $\{t_i\}_{i=0}^{\infty}$ be a strictly increasing sequence of arbitrary time instants, $0 = t_0 < t_1 < t_2 < \dots$ such that $\lim_{i \rightarrow \infty} t_i = +\infty$. Let for (5) all conditions proposed in [18] (or in [19], [25]) for ILF-based finite/fixed-time control $u(V, x)$ design be satisfied. Then sampled-time realization $u(t) = u(V_i, x(t))$ for $t \in [t_i, t_{i+1})$, where $V_i \in \mathbb{R}_+$ such that $Q(V_i, x(t_i)) = 0$ provides hyperexponential stability of the closed loop system.*

Proof. The proof follows exactly the same arguments as given in Example 1 and ILF properties. ■

4.2 Explicit Lyapunov function method

Since the hyperexponential stability needs increasing of exponential rates as $\Phi(t, x_0) \rightarrow 0$, then the following

result can be obtained:

Theorem 3 *Let $\beta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be positive nondecreasing function such that $\beta(s) \rightarrow \infty$ as $s \rightarrow \infty$. Suppose there exists a positive definite C^1 function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ defined on an open neighborhood of the origin $D \subset \mathbb{R}^n$, such that $\theta_1(\|x\|) \leq V(x) \leq \theta_2(\|x\|)$ for all $x \in D$ with some $\theta_1, \theta_2 \in \mathcal{K}_\infty$, and the condition $\dot{V}(x) \leq -\beta(V^{-1}(x))V(x)$ is true for the system (1) with $x \in D \setminus \{0\}$. Then the origin of system (1) is hyperexponentially stable. If $D = \mathbb{R}^n$, then the system (1) admits this property globally.*

Proof. Since the function β is positive and nondecreasing we have that $\dot{V}(x) \leq -C_0 V(x)$, $C_0 = \beta(V(x_0)^{-1})$ for all $x \in D \setminus \{0\}$, i.e., the system (1) is exponentially stable. Let $x(t)$ be a trajectory of the closed-loop system (1). Then, due to $\beta(s) \rightarrow \infty$ as $s \rightarrow \infty$, there exists a sequence of time instances $\{t_i\}_{i=0}^{+\infty}$: $0 = t_0 < t_1 < t_2 < \dots$, $\lim_{i \rightarrow +\infty} t_i = +\infty$ such that $\beta(V(x(t_0))^{-1}) < \beta(V(x(t_1))^{-1}) < \dots < \beta(V(x(t_i))^{-1}) < \dots$ and $\dot{V}(\Phi(t, x_0)) \leq -C_i V(\Phi(t, x_0))$, $C_i = \beta(V(x(t_i))^{-1})$, $\forall t \geq t_i$, that implies hyperexponential stability since $\theta_1(\|x\|) \leq V(x) \leq \theta_2(\|x\|)$ and $C_i \rightarrow +\infty$ as $t_i \rightarrow \infty$. ■

4.3 ILF method

Let us introduce the functions σ_i , $i = \overline{1, r}$ by the following recursive formula

$$\begin{aligned} \sigma_1(s) &= -\ln s + \alpha_r, \\ \sigma_i(s) &= \ln\left(\frac{1}{\alpha_{r-i+2}} \sigma_{i-1}(s)\right) + \alpha_{r-i+1}, \quad i = \overline{2, r} \end{aligned} \quad (7)$$

for $\alpha_i \in \mathbb{R}_+$ and $\{s \in \mathbb{R}_+ : \sigma_i(s) > 0, i = \overline{1, r}\}$. Obviously $\sigma_i(1) = \alpha_{r-i+1}$. Note that $\sigma_i(s) > 0$ for $s \leq 1$ and $\sigma_i(s) \rightarrow +\infty$ as $s \rightarrow 0$.

Definition 6 [20] *The function $q : \mathbb{R}_+^2 \rightarrow \mathbb{R}$, $(\sigma, s) \rightarrow q(\sigma, s)$ is said to be of the class \mathcal{IK}_∞ iff 1) q is continuous on \mathbb{R}_+^2 ; 2) for any $s \in \mathbb{R}_+$ there exists $\sigma \in \mathbb{R}_+$ such that $q(\sigma, s) = 0$; 3) for any fixed $s \in \mathbb{R}_+$ the function $q(\cdot, s)$ is strictly decreasing on \mathbb{R}_+ ; 4) for any fixed $\sigma \in \mathbb{R}_+$ the function $q(\sigma, \cdot)$ is strictly increasing on \mathbb{R}_+ ; 5) $\lim_{\substack{s \rightarrow 0^+ \\ (\sigma, s) \in \Gamma}} \sigma = 0$, $\lim_{\substack{\sigma \rightarrow 0^+ \\ (\sigma, s) \in \Gamma}} s = 0$, $\lim_{\substack{s \rightarrow +\infty \\ (\sigma, s) \in \Gamma}} \sigma = +\infty$, where $\Gamma = \{(\sigma, s) \in \mathbb{R}_+^2 : q(\sigma, s) = 0\}$.*

The next theorem presents the ILF method for *rated hyperexponential stability*.

Theorem 4 *Let there exist two functions Q_1 and Q_2 satisfying the conditions C1)-C4) of Theorem 1 for $V \in (0, 1]$, and $V \geq 1$, correspondingly. Let C5) $Q_1(1, x) = Q_2(1, x)$ for all $x \in \mathbb{R}^n \setminus \{0\}$; C6) there exist $\alpha_1 = (\alpha_{01}, \alpha_{11}, \dots, \alpha_{r1})^T \in \mathbb{R}_+^{r+1}$, $r \in \mathbb{N}$*

and corresponding functions σ_{i1} , $i = \overline{1, r}$, such that the inequality

$\frac{\partial Q_1(V, x)}{\partial x} f(x) \leq \alpha_{01} V \prod_{i=1}^r \sigma_{i1}(V) \frac{\partial Q_1(V, x)}{\partial V}$,
holds for all $V \in (0, 1]$ and $x \in \mathbb{R}^n \setminus \{0\}$ satisfying the equation $Q_1(V(x), x) = 0$;

C7) there exist $\alpha_2 = (\alpha_{02}, \alpha_{12}, \dots, \alpha_{r2})^\top \in \mathbb{R}_+^{r+1}$ and corresponding functions σ_{i2} , $i = \overline{1, r}$ such that the inequality

$\frac{\partial Q_2(V, x)}{\partial x} f(x) \leq \alpha_{02} V \prod_{i=1}^r \sigma_{i2} \left(\frac{1}{V}\right) \frac{\partial Q_2(V, x)}{\partial V}$,
holds for all $V \geq 1$ and $x \in \mathbb{R}^n \setminus \{0\}$ satisfying the equation $Q_2(V(x), x) = 0$;

C8) there exist $q_1 \in \mathcal{IK}_\infty$, $c \in \mathbb{R}_+$ such that

$$q_1(V, \|x\|) \leq Q_i(V, x), \quad q_1(cs, s) \geq 0,$$

holds for all $x \in \mathbb{R}^n \setminus \{0\}$ and $V : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}_+$ satisfying the equation $Q_1(V(x), x) = 0$ for $V \leq 1$ and $Q_2(V(x), x) = 0$ for $V \geq 1$, then the origin of the system (1) is globally hyperexponentially stable with degree r .

Proof. Let two functions V_1 and V_2 be defined by the equations $Q_1(V_1, x) = 0$ and $Q_2(V_2, x) = 0$. Consider the sets $\Sigma_1 = \{x \in \mathbb{R}^n : V_1(x) > 1\}$, $\Sigma_2 = \{x \in \mathbb{R}^n : V_2(x) > 1\}$ and prove that $\Sigma_1 = \Sigma_2$. Suppose the contrary, i.e. $\exists z \in \mathbb{R}^n$ such that $z \in \Sigma_1$ and $z \notin \Sigma_2$. On the one hand, $Q_1(V_1, z) = 0$ implies $V_1 > 1$ and $Q_1(1, z) > Q_1(V_1, z) = 0$ due to condition C4). On the other hand, $Q_2(V_2, z) = 0$ implies $V_2 \leq 1$ and $Q_2(1, z) \leq Q_2(V_2, z) = 0$. The contradiction follows from C5). In the same way one can show that $z \in \Sigma_2$ implies $z \in \Sigma_1$. Thus, $\Sigma_1 = \Sigma_2$. Therefore, due to C5) and C4) the function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by the equality

$$V(x) = \begin{cases} V_1(x) & \text{for } V_1(x) < 1, \\ V_2(x) & \text{for } V_2(x) > 1, \\ 1 & \text{for } V_1(x) = V_2(x) = 1, \end{cases}$$

is positive definite, continuous in \mathbb{R}^n and continuously differentiable for $x \notin \{0\} \cup \{x \in \mathbb{R}^n : V(x) = 1\}$. Note that the conditions C1)-C4) guarantee existence, uniqueness and radial unboundedness of the function V (see Remark 1). The function V is Lipschitz continuous outside the origin and has the Clarke's gradient [3] $\nabla_C V(x) = \xi \nabla V_1(x) + (1 - \xi) \nabla V_2(x)$ for $x \in \mathbb{R}^n$, where $\xi = 1$ for $0 < V_1(x) < 1$, $\xi = 0$ for $V_2(x) > 1$, $\xi \in [0, 1]$ for $V_1(x) = V_2(x) = 1$ and ∇V_i is the gradient of the function V_i , $i = 1, 2$. Hence, due to conditions C6) and C7), by means of Implicit Function Theorem [4] the inequality

$$\frac{\partial V(\Phi(t, x_0))}{\partial t} \leq \begin{cases} -\alpha_{01} V(\Phi(t, x_0)) \prod_{i=1}^r \sigma_{i1}(V(\Phi(t, x_0))) & \text{for } V(\Phi(t, x_0)) < 1, \\ -\alpha_{02} V(\Phi(t, x_0)) \prod_{i=1}^r \sigma_{i2} \left(\frac{1}{V(\Phi(t, x_0))}\right) & \text{for } V(\Phi(t, x_0)) > 1, \\ -\min\{\alpha_{01} \prod_{i=1}^r \alpha_{i1}, \alpha_{02} \prod_{i=1}^r \alpha_{i2}\} & \text{for } V(\Phi(t, x_0)) = 1 \end{cases} \quad (8)$$

holds for almost all t such that $\Phi(t, x_0) \neq 0$.

The following sections **I** and **II** of the proof show that the scalar system (8), representing the Lyapunov function dynamics, converges hyperexponentially near zero and away from zero, respectively. Finally, section **III** demonstrates that the system (1) converges hyperexponentially if the Lyapunov function has this property under condition C8).

I. Let $\alpha \in \mathbb{R}_+^{r+1}$ and consider the scalar differential equation

$$\dot{y}(t) = -\alpha_0 y(t) \prod_{i=1}^r \sigma_i \left(\frac{y(t)}{y_0}\right), \quad y(0) = y_0 \in \mathbb{R}_+. \quad (9)$$

The solution of (9) is $y(t) = y_0 e^{-\rho_{r, \alpha}(t)}$. Indeed, due to

$$\begin{aligned} \rho_{r, \alpha}(t) &= \ln \left(\frac{y_0}{y(t)}\right), \\ \rho_{i-1, \alpha}(t) &= \ln \left(\frac{1}{\alpha_i} \rho_{i, \alpha}(t) + 1\right), \\ \frac{\partial \rho_{i, \alpha}(t)}{\partial t} &= \alpha_i e^{\rho_{i-1, \alpha}(t)} \frac{\partial \rho_{i-1, \alpha}(t)}{\partial t} \\ &= (\rho_{i, \alpha}(t) + \alpha_i) \frac{\partial \rho_{i-1, \alpha}(t)}{\partial t}, \\ \frac{\partial \rho_{0, \alpha}(t)}{\partial t} &= \alpha_0, \end{aligned}$$

we have

$$\begin{aligned} \frac{\partial y_0 e^{-\rho_{r, \alpha}(t)}}{\partial t} &= -y(t) \frac{\partial \rho_{r, \alpha}(t)}{\partial t} = -y(t) \sigma_1 \left(\frac{y(t)}{y_0}\right) \frac{\partial \rho_{r-1, \alpha}(t)}{\partial t} \\ &= -y(t) \sigma_1 \left(\frac{y(t)}{y_0}\right) \sigma_2 \left(\frac{y(t)}{y_0}\right) \frac{\partial \rho_{r-2, \alpha}(t)}{\partial t} \\ &= \dots = -\alpha_0 y(t) \prod_{i=1}^r \sigma_i \left(\frac{y(t)}{y_0}\right). \end{aligned} \quad (10)$$

Thus, the system (9) is globally hyperexponentially stable with degree r and convergence rate α . Returning to (10) we have

$$\begin{aligned} \frac{\partial y(t)}{\partial t} &= -\alpha_0 y(t) \prod_{i=1}^r \sigma_i \left(\frac{y(t)}{y_0}\right) \\ &= -\alpha_0 y(t) \prod_{i=1}^r \sigma_i (e^{-\rho_{r, \alpha}(t)}) < 0 \quad \forall t \geq 0, \end{aligned}$$

i.e., $y(t)$ is strictly decreasing function of t and $y(t) \leq y_0$.

Taking into account that $\prod_{i=1}^r \sigma_i \left(\frac{y(t)}{y_0}\right) \leq \prod_{i=1}^r \sigma_i (y(t))$ for $y_0 \leq 1$, then by the comparison lemma we have that the system in the form $\frac{\partial y(t)}{\partial t} \leq -\alpha_{01} y(t) \prod_{i=1}^r \sigma_{i1}(y(t))$ for $y_0 \leq 1$ converges hyperexponentially with degree r and convergence rate α_1 .

II. Let $\bar{\alpha}_0 \in \mathbb{R}_+$, $\bar{\alpha}_i = 1$ for $i = \overline{1, r}$ and define the corresponding functions $\bar{\sigma}_i$ according to (7) (i.e., $\bar{\sigma}_1(s) = -\ln s + 1$, $\sigma_i(s) = \ln(\sigma_{i-1}(s)) + 1$ for $i = \overline{2, r}$). Let us consider the scalar equation $\dot{y}(t) = -\bar{\alpha}_0 y(t) \prod_{i=1}^r \bar{\sigma}_i \left(\frac{1}{y(t)}\right)$, $y(0) = y_0 \in \mathbb{R}_+$. Since $\int y^{-1} \prod_{i=1}^r \bar{\sigma}_i^{-1} \left(\frac{1}{y}\right) dy = \ln \left(\bar{\sigma}_r \left(\frac{1}{y}\right)\right) + C$, $C = \text{const}$

one can show $\ln\left(\bar{\sigma}_r\left(\frac{1}{y}\right)\right) = -\bar{\alpha}_0 t + \ln\left(\bar{\sigma}_r\left(\frac{1}{y_0}\right)\right)$ or $y(t) = \exp\left(-1 + \exp\left(\dots + \exp\left(-1 + e^{-1+e^{-\bar{\alpha}_0 t} \bar{\sigma}_r\left(\frac{1}{y_0}\right)}\right)\dots\right)\right)$.

Taking into account $e^{\varepsilon^{-a} \ln(\varepsilon b) - 1} \leq b e^{1-e^a}$ for $\forall b \geq 1$, $a \in \mathbb{R}_+ : b e^{1-e^a} \geq 1$, for $y_0 \geq 1$ we have

$$\begin{aligned} y(t) &= \exp\left(-1 + \exp\left(\dots + \exp\left(-1 + e^{-1+e^{-\bar{\alpha}_0 t} \bar{\sigma}_r\left(\frac{1}{y_0}\right)}\right)\dots\right)\right) \\ &= \exp\left(-1 + \exp\left(\dots + \exp\left(-1 + e^{-1+e^{-\bar{\alpha}_0 t} \ln\left(e^{\bar{\sigma}_{r-1}\left(\frac{1}{y_0}\right)}\right)}\right)\dots\right)\right) \\ &\leq \exp\left(-1 + \exp\left(\dots + \exp\left(-1 + \bar{\sigma}_{r-1}\left(\frac{1}{y_0}\right) e^{-\rho_{1,\bar{\alpha}}(t)}\right)\dots\right)\right) \\ &= \exp\left(-1 + \exp\left(\dots + \exp\left(-1 + \ln\left(e^{\bar{\sigma}_{r-2}\left(\frac{1}{y_0}\right)}\right) e^{-\rho_{1,\bar{\alpha}}(t)}\right)\dots\right)\right) \\ &\leq \exp\left(-1 + \exp\left(\dots + \bar{\sigma}_{r-2}\left(\frac{1}{y_0}\right) e^{-\rho_{2,\bar{\alpha}}(t)}\right)\dots\right) \\ &\leq \dots \leq \exp\left(-1 + \bar{\sigma}_1\left(\frac{1}{y_0}\right) e^{-\rho_{r-1,\bar{\alpha}}(t)}\right) \\ &= \exp\left(-1 + \ln(\varepsilon y_0) e^{-\rho_{r-1,\bar{\alpha}}(t)}\right) \leq y_0 e^{-\rho_{r,\bar{\alpha}}(t)} \end{aligned} \quad (11)$$

for $t \leq T_1$, where $T_1 = \bar{\alpha}^{-1} \ln\left(\bar{\sigma}_r\left(\frac{1}{y_0}\right)\right)$ and $y(T_1) \leq 1$. Taking into account that $\sigma_{12}(s) \geq \min\{1, \alpha_{r2}\} \bar{\sigma}_1(s)$, $\sigma_{22}(s) \geq \min\{1, \alpha_{(r-1)2} + \min\{1, \alpha_{r2}^{-1}\}\} \bar{\sigma}_2(s)$ and so on, then by the comparison lemma for the system in the form $\frac{\partial y(t)}{\partial t} \leq -\alpha_{02} y(t) \prod_{i=1}^r \sigma_{i2}\left(\frac{1}{y(t)}\right)$, $y_0 \geq 1$, we have that $y(t) \leq y_0 e^{-\rho_{r,\bar{\alpha}}(t)}$ for some $\bar{\alpha}_0 \in \mathbb{R}_+$, and for $t \leq T_1$, where $T_1 = \bar{\alpha}^{-1} \ln\left(\bar{\sigma}_r\left(\frac{1}{y_0}\right)\right)$, $y(T_1) \leq 1$.

III. Combining **I** and **II** we have that the system (8) is hyperexponentially stable with degree r and convergence rates $\tilde{\alpha}_i = \min\{\alpha_{i1}, \bar{\alpha}\}$ for some $\bar{\alpha}_0 \in \mathbb{R}_+$, i.e., $V(t) \leq V(x_0) e^{-\rho_{r,\bar{\alpha}}(t)}$. Finally, according to [20] the conditions C3), C8) imply that there exists $\theta_1, \theta_2 \in \mathcal{K}_\infty$ such that $c\|x\| \leq \theta_1(\|x\|) \leq V(x) \leq \theta_2(\|x\|)$. Thus, we have $\|\Phi(t, x_0)\| \leq c^{-1} \theta_2(\|x_0\|) e^{-\rho_{r,\bar{\alpha}}(t)}$. ■

Note that due to the condition C3) the function V is radially unbounded and it can be continuously extended to the origin by $V(0) = 0$.

Define $\sigma_0(s) = s^{-1}$. The condition C8) in Theorem 4 may be relaxed as it is shown in the following corollary.

Corollary 1 *Let there exist two functions Q_1 and Q_2 that satisfy the conditions C1)-C4) of Theorem 1, the conditions C5)-C7) of Theorem 4 with $r \in \mathbb{N}$ and C9) for some $k_1, k_2, \eta \in \mathbb{R}_+$ and $\mathbb{N} \ni i < r$, $r \geq 2$ the inequalities*

$$\begin{aligned} k_1 \|x\|^\eta &\leq \sigma_i^{-1}(V(x)), \quad \text{for } V(x) \in (0, 1] \\ k_2 \|x\|^\eta &\leq \sigma_i(V^{-1}(x)), \quad \text{for } V(x) \geq 1 \end{aligned} \quad (12)$$

hold for $x \in \mathbb{R}^n \setminus \{0\}$ and $V(x)$ satisfying the equation $Q_1(V(x), x) = 0$ for $V(x) \leq 1$ and $Q_2(V(x), x) = 0$ for $V(x) > 1$,

then the origin of the system (1) is globally hyperexponentially stable with degree $r - i$.

Proof. Without loss of generality, let $\alpha_{i1} = \alpha_{i2} = 1$, $i = 1, r$ and $\alpha = \alpha_1 = \alpha_2$. The conditions C1)-C7) guarantee Lyapunov stability of the system. Since V is positive definite and continuous function then $\exists \theta_2 \in \mathcal{K}_\infty$ such that $V(x) \leq \theta_2(\|x\|)$. Moreover, according to the proof of Theorem 4 the conditions C1)-C7) provide $V(\Phi(t, x_0)) \leq V(x_0) e^{-\rho_{r,\alpha}(t)}$ that, taking into account (2), implies

$$\frac{1}{\sigma_i\left(\frac{V(\Phi(t, x_0))}{V(x_0)}\right)} \leq e^{-\rho_{r-i,\alpha}(t)}. \quad (13)$$

The subsequent proof sections **I-III** analyze three possible cases: $V(x) \leq 1 \leq V(x_0)$, $V(x) \leq V(x_0) \leq 1$, and $1 \leq V(x) \leq V(x_0)$ respectively, demonstrating for each case that (12), (13) guarantee the hyperexponential convergence of the state vector with degree $r - i$.

I. Let $V(x) \leq 1 \leq V(x_0)$ (note that in this case $V(x_0) - 1 \geq \ln V(x_0) \geq 0$). Let in (12) $i = 1$. The inequality (12) implies

$$\begin{aligned} k_1 \|x\|^\eta &\leq \frac{1}{1 - \ln V(x)} \leq \frac{V(x_0)}{V(x_0) - \ln V(x)} \\ &= \frac{V(x_0)}{(V(x_0) - 1) - \ln V(x) + 1} \leq \frac{V(x_0)}{\ln(V(x_0)/V(x)) + 1}. \end{aligned} \quad (14)$$

Then, for $t \in \mathbb{R} : V(\Phi(t, x_0)) \leq 1$, (12), (13) and (14) imply

$$\begin{aligned} \|\Phi(t, x_0)\| &\leq \left(\frac{V(x_0)}{-k_1 \ln\left(\frac{V(\Phi(t, x_0))}{V(x_0)}\right) + k_1} \right)^{\frac{1}{\eta}} \\ &\leq \left(\frac{V(x_0)}{k_1} e^{-\rho_{r-1,\alpha}(t)} \right)^{\frac{1}{\eta}} \leq \left(\frac{1}{k_1} \theta_2(\|x_0\|) \right)^{\frac{1}{\eta}} e^{-\frac{1}{\eta} \rho_{r-1,\alpha}(t)} \end{aligned}$$

Let us consider the case $i \geq 2$ in (12). The inequality (12) implies

$$\begin{aligned} k_1 \|x\|^\eta &\leq \frac{1}{\ln(\sigma_{i-1}(V(x))) + 1} \leq \frac{V(x_0)}{\ln(\sigma_{i-1}(V(x))) + V(x_0)} \\ &\leq \frac{V(x_0)}{\ln(\sigma_{i-1}(V(x))) + \ln(V(x_0)) + 1} \leq \frac{V(x_0)}{\ln(V(x_0) \sigma_{i-1}(V(x))) + 1}. \end{aligned} \quad (15)$$

Note that $V(x_0) \sigma_1(V(x)) = V(x_0) (-\ln V(x) + 1) \geq -\ln V(x) + \ln V(x_0) + 1 = \sigma_1\left(\frac{V(x)}{V(x_0)}\right)$, and

$$\begin{aligned} V(x_0) \sigma_2(V(x)) &= V(x_0) (\ln(\sigma_1(V(x))) + 1) \\ &\geq \ln(\sigma_1(V(x))) + V(x_0) \geq \ln(V(x_0) \sigma_1(V(x))) + 1 \\ &\geq \ln\left(\sigma_1\left(\frac{V(x)}{V(x_0)}\right)\right) + 1 = \sigma_2\left(\frac{V(x)}{V(x_0)}\right). \end{aligned}$$

Then by mathematical induction we have

$$V(x_0) \sigma_{i-1}(V(x)) \geq \sigma_{i-1}\left(\frac{V(x)}{V(x_0)}\right). \quad (16)$$

For $t \in \mathbb{R} : V(\Phi(t, x_0)) \leq 1$, (12), (13), (15), (16) imply

$$\begin{aligned} \|\Phi(t, x_0)\| &\leq \left(\frac{V(x_0)}{k_1 \left[\ln(\sigma_{i-1} \left(\frac{V(\Phi(t, x_0))}{V(x_0)} \right)) + 1 \right]} \right)^{\frac{1}{\eta}} \\ &= \left(\frac{V(x_0)}{k_1 \sigma_i \left(\frac{V(\Phi(t, x_0))}{V(x_0)} \right)} \right)^{\frac{1}{\eta}} \leq \left(\frac{V(x_0)}{k_1} e^{-\rho_{r-i, \alpha}(t)} \right)^{\frac{1}{\eta}} \\ &\leq \left(\frac{\theta_2(\|x_0\|)}{k_1} \right)^{\frac{1}{\eta}} e^{-\frac{1}{\eta} \rho_{r-i, \alpha}(t)}. \end{aligned}$$

II. Let $V(x) \leq V(x_0) \leq 1$. Taking into account that σ_i is strictly decreasing function, the inequality (12) implies

$$k_1 \|x\|^\eta \leq \sigma_i^{-1}(V(x)) \leq \sigma_i^{-1} \left(\frac{V(x)}{V(x_0)} \right). \quad (17)$$

On the other hand we have

$$k_1 \|x\|^\eta \leq \sigma_i^{-1}(V(x)) \leq \sigma_i^{-1}(V(x_0)) \leq \sigma_i^{-1}(\theta_2(\|x_0\|)), \quad (18)$$

$\sigma_i^{-1}(\theta_2(\cdot)) \in \mathcal{K}_\infty$. Then (12), (13), (17) and (18) provide

$$\begin{aligned} \|\Phi(t, x_0)\| &\leq \left(k_1^2 \sigma_i \left(\frac{V(\Phi(t, x_0))}{V(x_0)} \right) \sigma_i(\theta_2(\|x_0\|)) \right)^{-\frac{1}{2\eta}} \\ &\leq \left(k_1^2 \sigma_i(\theta_2(\|x_0\|)) \right)^{-\frac{1}{2\eta}} e^{-\frac{1}{2\eta} \rho_{r-i, \alpha}(t)}. \end{aligned}$$

III. Consider the case $V(x_0) \geq V(x) \geq 1$. According to (11) one can obtain

$$\sigma_i \left(\frac{1}{V(\Phi(t, x_0))} \right) \leq \sigma_i \left(\frac{1}{V(x_0)} \right) e^{-\rho_{r-i, \alpha}(t)} \quad (19)$$

for $t \in \mathbb{R}_+ : V(\Phi(t, x_0)) \geq 1$. Then (12) implies $\|\Phi(t, x_0)\| \leq k_2^{-1} \sigma_i \left(\frac{1}{\theta(\|x_0\|)} \right) e^{-\rho_{r-i, \alpha}(t)}$ for $t \in \mathbb{R}_+ : V(\Phi(t, x_0)) \geq 1$.

Thus, generalizing the considered cases **I-III**, the system is globally hyperexponentially stable with degree $r-i$. ■

5 Hyperexponential control

Consider the system (5). Let us introduce the ILF functions

$$\begin{aligned} Q_i(V, x) &= \sigma_{r-1}^{2(3-2j)} (V^{3-2j}) x^T D_j (\sigma_r^{3-2j} (V^{3-2j})) P \\ &\quad \times D_j (\sigma_r^{3-2j} (V^{3-2j})) x - 1, \quad j = \overline{1, 2}, \end{aligned} \quad (20)$$

where $V \in \mathbb{R}_+$, $r \in \mathbb{N}$ (if $r = 1$ then we define $\sigma_{r-1}(s) = s^{-1}$), $D_1(\lambda) = \text{diag}\{\lambda^{q_i}\}_{i=1}^n$, $q_i = 1 + (n-i)$, $D_2(\lambda) = \text{diag}\{\lambda^{p_i}\}_{i=1}^n$, $p_i = 1 + (i-1)$ for $\lambda \in \mathbb{R}_+$ and $0 < P \in \mathbb{R}^{n \times n}$. Denote $H_1 = \text{diag}\{q_i\}_{i=1}^n$, $H_2 = \text{diag}\{p_i\}_{i=1}^n$. The following theorem is on rated hyperexponential control design for the linear system (5).

Theorem 5 Let the system of matrix inequalities

$$\begin{aligned} AX + XA^T + BY + Y^T B^T + \gamma_j (2X + XH_j + H_j X) &\leq 0, \\ XH_j + H_j X > 0, \quad X > 0, \quad j = \overline{1, 2} \end{aligned} \quad (21)$$

be feasible for some $\gamma_j \in \mathbb{R}_+$, $j = \overline{1, 2}$, and $X \in \mathbb{R}^{n \times n}$, $Y \in \mathbb{R}^{1 \times n}$. Let

$$u(V(x), x) = \begin{cases} KD_1(\sigma_r(V(x))) x & \text{for } x^T P x < 1, \\ KD_2(\sigma_r^{-1}(V^{-1}(x))) x & \text{for } x^T P x \geq 1, \end{cases} \quad (22)$$

where $\alpha_i = 1$, $i = \overline{1, r}$; $K = YX^{-1}$, $X = P^{-1} \in \mathbb{R}^{n \times n}$, $Y \in \mathbb{R}^{1 \times n}$ and $V(x)$ is such that $Q_1(V(x), x) = 0$ for $x^T P x \leq 1$; $V(x)$ is such that $Q_2(V(x), x) = 0$ for $x^T P x > 1$. Then the closed-loop system (5), (22) is hyperexponentially stable with degree 1.

Proof. The following section **I** of the proof demonstrates that the ILF function (20) satisfies conditions *C1*)-*C7*), respectively, i.e., that the implicitly defined Lyapunov function converges hyperexponentially with degree r . Section **II** shows, based on Corollary 1, that the state vector of the system (5), (22) converges hyperexponentially with degree 1.

I. The function $Q_j(V, x)$, $j = \overline{1, 2}$ satisfies the conditions *C1*)-*C4*) of Theorem 1 for $V \in (0, e^{-\rho_{r-2}(e^{-1}-1)})$ with $j = 1$, $r \geq 2$, and $V > e^{\rho_{r-2}(e^{-1}-1)}$ with $j = 1$, $r \geq 2$ ($V \in (0, e)$ with $j = 1$, $r = 1$, and $V > e^{-1}$ with $j = 2$, $r = 1$). Indeed, it is continuously differentiable for all $V \in \mathbb{R}_+$ and $\forall x \in \mathbb{R}^n$ (i.e., the condition *C1*) hold). Note that $\sigma_i^{3-2j}(V^{3-2j})$, $i = \overline{0, r}$ is positive and strictly decreasing for $V \in (0, e^{-\rho_{r-2}(e^{-1}-1)})$ with $j = 1$, $r \geq 2$, and $V > e^{\rho_{r-2}(e^{-1}-1)}$ with $j = 1$, $r \geq 2$ ($V \in (0, e)$ with $j = 1$, $r = 1$, and $V > e^{-1}$ with $j = 2$, $r = 1$). Then, taking into account $\lim_{V \rightarrow 0^+} \sigma_i(V) = +\infty$, $\lim_{V \rightarrow +\infty} \sigma_i^{-1}(V^{-1}) = 0^+$, $\sigma_i(1) = 1$, the chain of inequalities

$$\begin{aligned} &\sigma_{r-1}^{2(3-2j)} (V^{3-2j}) \lambda_{\min}(P) \|x\|^2 \\ &\quad \times \min \left\{ \sigma_r^{2n(3-2j)} (V^{3-2j}), \sigma_r^{2(3-2j)} (V^{3-2j}) \right\} \\ &\leq Q_1(V, x) + 1 \leq \sigma_{r-1}^{2(3-2j)} (V^{3-2j}) \lambda_{\max}(P) \|x\|^2 \\ &\quad \times \max \left\{ \sigma_r^{2n(3-2j)} (V^{3-2j}), \sigma_r^{2(3-2j)} (V^{3-2j}) \right\} \end{aligned}$$

implies that for any $x \in \mathbb{R}^n \setminus \{0\}$ there exist $V^- \in \mathbb{R}_+$ and $V^+ \in \mathbb{R}_+$: $Q_j(V^-, x) < 0 < Q_j(V^+, x)$. Moreover, if $Q_j(V, x) = 0$ then the same chain of inequalities gives

$$\begin{aligned} &\frac{\sigma_{r-1}^{-2(3-2j)} (V^{3-2j})}{\lambda_{\max}(P) \max \left\{ \sigma_r^{2n(3-2j)} (V^{3-2j}), \sigma_r^{2(3-2j)} (V^{3-2j}) \right\}} \leq \|x\|^2 \\ &\leq \frac{\sigma_{r-1}^{-2(3-2j)} (V^{3-2j})}{\lambda_{\min}(P) \min \left\{ \sigma_r^{2n(3-2j)} (V^{3-2j}), \sigma_r^{2(3-2j)} (V^{3-2j}) \right\}} \end{aligned}$$

implying that *C2*), *C3*) hold.

Note that according to (7) we have

$$\frac{\partial \sigma_r(s)}{\partial s} = -s^{-1} \alpha_2 \alpha_3 \cdots \alpha_r \prod_{i=1}^{r-1} \sigma_i^{-1}(s) = -s^{-1} \prod_{i=1}^{r-1} \sigma_i^{-1}(s)$$

for $r \geq 2$. Taking this into account, for $r \in \mathbb{N}$ we get

$$\begin{aligned} \frac{\partial Q_j(V,x)}{\partial V} &= -\frac{1}{V} \sigma_{r-1}^{2(3-2j)} (V^{3-2j}) \sigma_r (V^{3-2j}) \prod_{i=1}^r \sigma_i^{-1}(V^{3-2j}) \\ &\quad \times x^T D_j (\sigma_r^{3-2j} (V^{3-2j})) [2P + \sigma_r^{-1}(V^{3-2j}) (PH_j + H_j P)] \\ &\quad \times D_j (\sigma_r^{3-2j} (V^{3-2j})) x. \end{aligned}$$

Then, the system (21) and $P := X^{-1}$ imply $H_j P + PH_j > 0$ and $\frac{\partial Q_1}{\partial V} < 0$ for all $V \in \mathbb{R}_+$ and $x \in \mathbb{R}^n \setminus \{0\}$, and the condition C_4) of Theorem 1 also holds.

Now let us check that the condition C_6) holds for all $V \in (0, 1]$ and $x \in \mathbb{R}^n \setminus \{0\}$. Taking into account (21) with $P := X^{-1}$, $K = YP$, we have

$$\begin{aligned} \frac{\partial Q_1(V,x)}{\partial x} \dot{x} &= 2\sigma_{r-1}^2(V) x^T D_1(\sigma_r(V)) P D_1(\sigma_r(V)) (Ax + Bu) \\ &= \sigma_{r-1}^2(V) \sigma_r(V) x^T D_1(\sigma_r(V)) \\ &\quad \times (PA + A^T P + PBK + K^T B^T P) D_1(\sigma_r(V)) x \\ &\leq -\gamma_1 \sigma_{r-1}^2(V) \sigma_r(V) x^T D_1(\sigma_r(V)) \\ &\quad \times (2P + PH_1 + H_1 P) D_1(\sigma_r(V)) x \\ &\leq \gamma_1 V \prod_{i=1}^r \sigma_i(V) \frac{\partial Q_1(V,x)}{\partial V} \end{aligned}$$

i.e., C_6) holds and $V(\Phi(t, x_0))$ converges hyperexponentially with degree r for $t \in \mathbb{R} : V(\Phi(t, x_0)) \leq 1$.

The condition C_7) also hold for all $V \geq 1$ and $x \in \mathbb{R}^n \setminus \{0\}$ due to

$$\begin{aligned} \frac{\partial Q_2(V,x)}{\partial x} \dot{x} &= 2\sigma_{r-1}^{-2}(V^{-1}) x^T D_2(\sigma_r^{-1}(V^{-1})) P D_2(\sigma_r^{-1}(V^{-1})) \\ &\quad \times (Ax + Bu) \\ &= \sigma_{r-1}^{-2}(V) \sigma_r(V^{-1}) x^T D_2(\sigma_r^{-1}(V^{-1})) \\ &\quad \times (PA + A^T P + PBK + K^T B^T P) D_2(\sigma_r^{-1}(V^{-1})) x \\ &\leq -\gamma_2 \sigma_{r-1}^{-2}(V^{-1}) \sigma_r(V^{-1}) x^T D_2(\sigma_r^{-1}(V^{-1})) \\ &\quad \times (2P + PH_2 + H_2 P) D_2(\sigma_r^{-1}(V^{-1})) x \\ &\leq \gamma_2 V \prod_{i=1}^r \sigma_i(V^{-1}) \frac{\partial Q_2(V,x)}{\partial V}. \end{aligned}$$

Also note, that due to $\alpha_i = 1$, $i = \overline{1, r}$ we have $Q_1(1, x) = Q_2(1, x) = x^T P x - 1$, i.e., the condition C_5) is satisfied.

II. From (5) for $V \leq 1$, $j = 1$ we have

$$\|x\| \leq \frac{1}{\sqrt{\lambda_{\min}(P) \sigma_{r-1}(V) \sigma_r(V)}} \leq \frac{1}{\sqrt{\lambda_{\min}(P) \sigma_{r-1}(V)}},$$

i.e., C_9) from Corollary 1 is satisfied for $V \leq 1$. Analogously, taking into account (5), (19), for $V \geq 1$, $j = 2$

we have

$$\begin{aligned} \|x(t)\| &\leq \frac{\sigma_{r-1}(V^{-1}) \sigma_r^n(V^{-1})}{\sigma_{r-1}(V^{-1}) \sqrt{\lambda_{\min}(P)}} \\ &\leq \frac{\sigma_{r-1}(V^{-1}) \sigma_r^n(V^{-1}(x_0))}{\sqrt{\lambda_{\min}(P)}} \\ &\leq \frac{(\lambda_{\max}(P) \|x_0\|^2)^{0.25n} \sigma_{r-1}(V^{-1})}{\sqrt{\lambda_{\min}(P)}} \\ &\leq \frac{(\lambda_{\max}(P) \|x_0\|^2)^{0.25n} \sigma_{r-1}(V^{-1}(x_0))}{\sqrt{\lambda_{\min}(P)}} e^{-\rho_{1,\alpha}(t)} \\ &\leq \frac{(\lambda_{\max}(P) \|x_0\|^2)^{0.25n+0.5}}{\sqrt{\lambda_{\min}(P)}} e^{-\rho_{1,\alpha}(t)}. \end{aligned}$$

Thus, the solution converges hyperexponentially with degree 1. \blacksquare

Note that due to $\sigma_r(V) = 1$ on the set $\{x \in \mathbb{R}^n : x^T P x = 1\}$, the control function $u = u(V, x)$ is continuous in $\mathbb{R}_+ \times \mathbb{R}^n$.

In comparison with the paper [22], the proposed control (22) is *time-invariant*, which brings a number of advantages in the application and further development of the results. Let us make some additional remarks and extensions about presented control scheme:

- Theorem 5 can be easily extended to linear MIMO systems using block decomposition like in [19].
- The function V is continuous, positive definite and radial unbounded on $\mathbb{R}^n \setminus \{0\}$. In addition, since $\lim_{x \rightarrow 0} V = 0^+$ the function V can be continuously prolonged to the origin by setting $V(0) = 0$.
- In order to calculate the function $V(x)$ implicitly defined by (20) the bisection method may be utilized (see, e.g., [17]):

Algorithm (the case $x^T P x \leq 1$)

INITIALIZATION: $V_0 = 1$; $a = V_{\min}$; $b = 1$;

STEP :

If $\sigma_{r-1}^2(b) x_i^T D_1(\sigma_r(b)) P D_1(\sigma_r(b)) x_i > 1$ then

$a = b$; $b = 2b$;

elseif $\sigma_{r-1}^2(a) x_i^T D_1(\sigma_r(a)) P D_1(\sigma_r(a)) x_i < 1$ then

$b = a$; $a = \max\{\frac{a}{2}, V_{\min}\}$;

else

$c = \frac{a+b}{2}$

If $\sigma_{r-1}^2(c) x_i^T D_1(\sigma_r(c)) P D_1(\sigma_r(c)) x_i < 1$ then

$b = c$;

else $a = \max\{V_{\min}, c\}$;

endif;

endif; $V_i = b$;

If **STEP** is applied recurrently many times to the same vector x_i then it allows to localize the unique positive root of the equation $\sigma_{r-1}^2(V) x^T D_1(\sigma_r(V)) P D_1(\sigma_r(V)) x - 1 = 0$. Similar algorithm can be applied for the case $x^T P x > 1$.

Example 2 Consider the system (5) for $n = 3$. Let us choose the same P , K as in Example 1 that satisfy (21) with $X = P^{-1}$, $Y = KX$. The results of simulation for $r = 1$ are shown in Fig. 1, Fig. 2, which

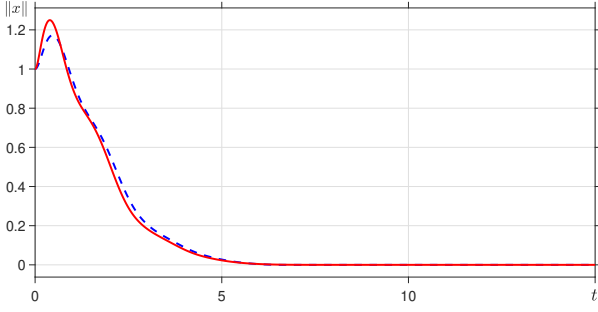


Fig. 1. Simulation plot of $\|x\|$ for the hyperexponential (red line) and finite-time (blue line) control with $x_0 = [1 \ 0 \ 0]^T$

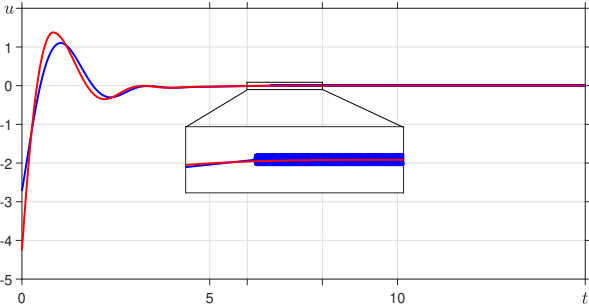


Fig. 2. Simulation plot of hyperexponential (red line) and finite-time (blue line) control signal

demonstrate plots of the state norm $\|x\|$ and control inputs for the hyperexponential control (22) (red line) and finite-time control (6) with $\mu = 0.5$ (blue line). Fig. 3 demonstrates plots of $\|x\|$ for the hyperexponential and finite-time controls with measurement noise $v(t) = 0.01(\text{rnd}(n) + \sin(\omega t))$, where $\text{rnd}(n)$ denotes uniformly distributed in the interval $[0, 1]^n$ random numbers, $\omega = [11, 12, 13]$. Fig. 4 shows plots of the state norm $\|x\|$ in the presence of delay $\tau = 0.05$ in the control channel. It is easy to see that the control (22) has better robustness properties than the finite-time control (6) and it is free of numerical chattering (see Fig. 2). The root mean square errors for the final time interval $[6, 15]$ are 0.048 (0.003) for the hyperexponential control and 0.067 (0.04) for the finite-time control for the noised (delayed) case illustrating less sensitivity to measurement noise and input delays. A detailed study of the presented control algorithms on robustness analysis with respect to disturbances, uncertainties, delays and extension of these results on a wider class of systems goes beyond the scope of the paper providing the subjects for a future research.

In order to further accelerate convergence outside the vicinity of the origin one can combine a nearly fixed-time control with hyperexponential one.

Define $\bar{Q}_2(V, x) = x^T \bar{D}(V^{-1}) P \bar{D}(V^{-1}) x - 1$, where $\bar{D}(\lambda) = \text{diag}\{\lambda^{p_i}\}_{i=1}^n$, $p_i = 1 + (i-1)\nu$, $\nu \in \mathbb{R}_+$, $\lambda \in \mathbb{R}_+$ and $0 < P \in \mathbb{R}^{n \times n}$. Denote $\bar{H} = \text{diag}\{p_i\}_{i=1}^n$.

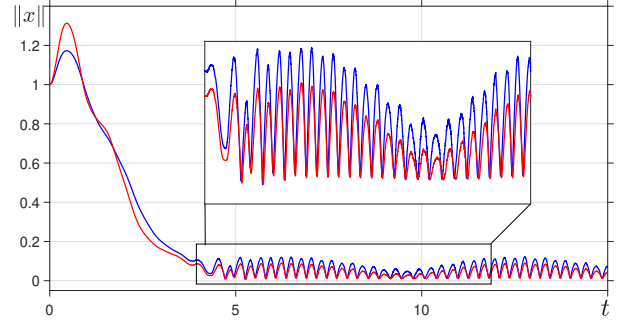


Fig. 3. Simulation plot of $\|x\|$ for the hyperexponential (red line) and finite-time (blue line) control with measurement noise and $x_0 = [1 \ 0 \ 0]^T$

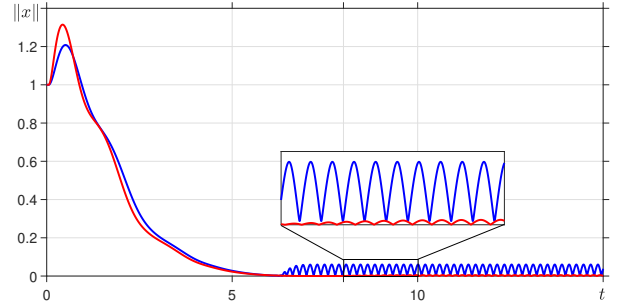


Fig. 4. Simulation plot of $\|x\|$ for the hyperexponential (red line) and finite-time (blue line) control in the presence of delay $\tau = 0.05$ in the control channel for $x_0 = [0.5 \ 0 \ 0]^T$

Corollary 2 Let the system of matrix inequalities (21) and $X\bar{H} + \bar{H}X > 0$ be feasible for some $\gamma \in \mathbb{R}_+$ and $X \in \mathbb{R}^{n \times n}$, $Y \in \mathbb{R}^{1 \times n}$. Let

$$u(V(x), x) = \begin{cases} KD(\sigma_r(V(x)))x & \text{for } x^T Px < 1, \\ V^{1+\nu} K \bar{D}(V^{-1}(x))x & \text{for } x^T Px \geq 1, \end{cases} \quad (23)$$

where $K = YX^{-1}$, $X = P^{-1} \in \mathbb{R}^{n \times n}$, $Y \in \mathbb{R}^{1 \times n}$ and $V : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}_+$ such that $Q_1(V(x), x) = 0$ for $x^T Px < 1$, and $Q_2(V(x), x) = 0$ for $x^T Px \geq 1$. Then the set $\{x \in \mathbb{R}^n : x^T Px \leq 1\}$ is fixed-time attractive and the closed-loop system (5), (23) is globally hyperexponentially stable with degree 1.

Proof. The proof follows exactly the same arguments as one of Theorem 5 and [18, Theorem 9]. ■

Using the same arguments as in Proposition 1 it is easy to show that under sampled-time realization the ILF-based control (22) preserves hyperexponential stability:

Corollary 3 Let $\{t_i\}_{i=0}^\infty$ be a strictly increasing sequence of arbitrary time instants, $0 = t_0 < t_1 < t_2 < \dots$ such that $\lim_{i \rightarrow \infty} t_i = +\infty$. Let for (5) all conditions of Theorem 5 (Corollary 2) be satisfied. Then sampled-time realization $u(t) = u(V_i, x(t))$ for $t \in [t_i, t_{i+1})$, where $V_i \in \mathbb{R}_+$ such that $Q_1(V_i, x(t_i)) = 0$ for $x^T Px \leq 1$, and

$Q_2(V_i, x(t_i)) = 0$ for $x^T Px > 1$, provides hyperexponential stability of the closed-loop system.

Therefore, the hyperexponentially stabilizing controls preserve their convergence rates under sampled-and-hold discretization, which is not the case for the finite/fixed-time stabilizers.

6 CONCLUSIONS

In the paper sufficient Lyapunov characterizations of hyperexponential stability are presented for the system (1) using explicit and implicit Lyapunov function approaches. Firstly, it is shown that ILF-based finite/fixed-time control methods provide hyperexponential stability under sampled-time realization. Next, the hyperexponential control (22) was proposed for the linear system (5). The preliminary numeric experiments indicate that this control is less sensitive with respect to noises than its finite-time analog. In addition, according to simulations the hyperexponential control demonstrates better performance in the presence of delays being free of numerical chattering, and shows negligible differences in convergence rates. Tuning of control parameters is presented in the form of linear matrix inequalities. The performance of the proposed control is illustrated through simulations. The presented results open a lot of topics for future research. For example, development of ILF-based hyperexponential controls and observers for nonlinear and MIMO systems, or robustness analysis for the presented controls with respect to disturbances, uncertainties, delays, etc.

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