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► **To cite this version:**

Marcel Fang, Pierre-Alexandre Bliman, Denis Efimov, Rosane Ushirobira. Nonlinear adaptive observers for SIS system with primary infections. 2024. hal-04726068

**HAL Id: hal-04726068**

**<https://inria.hal.science/hal-04726068v1>**

Preprint submitted on 8 Oct 2024

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# Nonlinear adaptive observers for SIS system with primary infections

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## Abstract

Observation and identification are crucially important for the practical use of compartmental models in epidemiology. Typically, the state and parameters of the epidemic model are evaluated based on the number of infected individuals (prevalence) or the newly infected cases per time unit (incidence). However, when reinfections are possible, additional data such as the number of primary infections are retrievable. In this note, we investigate whether measuring the number of primary infections can improve estimation. To do so, we present several nonlinear adaptive observers for a simple infection model with waning immunity and subsequent reinfections. We then prove the practical asymptotic stability of the estimation errors using the Lyapunov function method. Finally, we illustrate the efficacy of the observers via simulations.

*Key words:* Adaptive estimation; SIS model; Lyapunov method; Mathematical epidemiology

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## 1 Introduction

Compartmental models are popular and well-studied tools for modeling epidemic processes. Researchers in mathematical epidemiology are usually interested in analyzing and simulating these epidemic processes. However, practical studies of epidemics require knowledge about the existence of reliable parameter estimates. On the other hand, the state estimation of epidemiological systems is crucial for the surveillance and control of epidemics. Therefore, the inverse problem of estimating states and parameters remains essential for understanding and supervising the dynamics of epidemics.

Although identification and observation are well-studied issues in many fields, this is still not the case in mathematical epidemiology, where only a small amount of literature is available. Among them, most are only interested in the parameter identification of epidemic models, even though state estimation is equally significant for understanding and monitoring epidemic processes. Nevertheless, an interested reader may refer, for example, to [12, 5] for recent works about observer design of generic epidemiological model, but also [1, 3] for observer design with set-based approach. The typical difficulty or feature of estimation/identification problems for epidemiological models is the lack of observ-

ability/identifiability in the vicinity of the equilibria, which attracts system behavior.

In the present paper, we focus on the joint estimation of the state and parameters of a susceptible-infected-susceptible (SIS) epidemic model counting primary infections. The usual approach for estimating states and parameters is based on the measurement of prevalence (the number of infected individuals) or incidence (the number of newly infected cases), see [4, 17]. However, in case of an epidemic with reinfections, other data of interest may be available, such as the number of primary infections. Our main interest here is to see if the additional measurement of primary infections, a data rarely considered but available, could improve the observation and identification of epidemiological systems with reinfections. The model has been analyzed in [7, 6], where, in particular, the algebraic observability and identifiability were derived under the condition that the total number of infected as well as the number of primary infected are available. Moreover, a class of non-linear observers for the system is presented in [8] where the stability is deduced using both copositive and Lur'e-type Lyapunov functions and techniques based on linear matrix inequalities (LMI). In a third paper [9], we explored the adaptive estimation of the SIS model in the case where some parameters are unknown and derived an adaptive observer that relies on output differential algebraic relations. However, this method requires an estimation of the output derivatives, which may be hardly retrievable.

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\* The financial support of French National Research Agency (ANR) is acknowledged (ANR-23-CE48-0004).

The present paper proposes to overcome this issue with a new method for designing an adaptive observer independent of output derivatives. In Section 2, we recall the studied SIS model and its identifiability and observability properties before transforming the model into a suitable form for adaptive observer design. Some stability notions from the literature are recalled in Section 3, which are used in the subsequent analysis. After that, the adaptive observers and the related results are presented in sections 4 and 5 for cases where the entire or a portion of the infected and primary infected are measured, respectively.

### 1.1 Notation

- The sets of real and non-negative real numbers are denoted by  $\mathbb{R}$  and  $\mathbb{R}_+$ , respectively.
- The Euclidean norm for a vector  $x \in \mathbb{R}^n$  is denoted by  $|x|$ .
- For a Lebesgue measurable function of time  $d: \mathbb{K} \rightarrow \mathbb{R}^m$  define the  $L_\infty$ -norm  $\|d\|_\infty = \text{ess sup}_{t \in \mathbb{K}} |d(t)|$ . We further denote the space of functions  $d$  with  $\|d\|_\infty < +\infty$  as  $L_\infty(\mathbb{K})$ , the dimension  $m$  is omitted for simplicity and depends on the context.
- The definitions of classes of comparison functions  $\mathcal{K}$ ,  $\mathcal{K}_\infty$ , and  $\mathcal{KL}$  are standard and can be found in [16].

## 2 The model

### 2.1 A SIS model with primary infections

The studied SIS model at first introduced in [7, 8] has the following state-space representation:

$$\dot{S} = \mu - \beta SI - \mu S + \gamma I, \quad \dot{I} = \beta SI - (\mu + \gamma)I, \quad (1a)$$

$$\dot{S}_1 = \mu - \beta S_1 I - \mu S_1, \quad \dot{I}_1 = \beta S_1 I - (\mu + \gamma)I_1, \quad (1b)$$

$$y = \alpha I, \quad y_1 = \alpha I_1, \quad (1c)$$

where the four state variables  $(S(t), I(t), S_1(t), I_1(t)) \in [0, 1]^4$ ,  $t \in \mathbb{R}_+$ , represent the proportions of susceptible, infected, never infected susceptible and primary infected individuals, respectively; the coefficients  $\mu, \beta, \gamma > 0$  are the natural mortality rate, the contact rate, and the recovery rate; the output  $y \in [0, 1]^2$  provides the measurement of a given portion  $\alpha \in (0, 1]$  of the infected individuals and of the primary infected individuals. We assume that this portion, which corresponds to the infected individuals detected by the Public health system (e.g., the symptomatic cases), is identical for the primary and global infected populations. The formula (1a) shows that the total population is constant; it has been normalized to 1:  $S + I \equiv 1$  so that the system (1) may be reduced to a 3-dimensional model.

In the mathematical epidemiology community, the first stage (1a) of the system (1) is known as the classical SIS model with vital dynamics ([10]). The novelty of system (1) is the addition of equations (1b) modelling the dynamics of primary infections in a cascade connection to (1a). This

operation may be extended to obtain an infinite-dimensional system that also provides the number of infections already come through by the members of the population. One refers to [7, 6] for an analysis of the resulting infinite system in the more complex SEIRS model setting.

The *basic reproduction number* of this system is

$$\mathcal{R}_0 := \frac{\beta}{\mu + \gamma},$$

which is assumed to be greater than 1 so that convergence to the (unique, positive, globally asymptotically stable) endemic equilibrium occurs in the case when the disease is present.

**Theorem 1** *Assume that  $(S(0), S_1(0), I_1(0)) \in [0, 1]^3$ ,  $S(0) + I(0) = 1$ ,  $I(0) > 0$  and*

$$\mathcal{R}_0 > 1.$$

*System (1) has a unique endemic equilibrium*

$$x_{\mathbf{E}} := (\mathcal{R}_0^{-1}, 1 - \mathcal{R}_0^{-1}, \frac{\mu}{\beta - \gamma}, \frac{\beta - \gamma - \mu}{\beta - \gamma} \frac{\mu}{\gamma + \mu}),$$

*which is globally asymptotically stable.*

The proof of global stability of the endemic equilibrium  $(\mathcal{R}_0^{-1}, 1 - \mathcal{R}_0^{-1})$  for equations (1a) is contained in [11]. The rest may be deduced by taking advantage of the cascade structure and using arguments similar to [6].

### 2.2 Identifiability and observability of the SIS model

The algebraic identifiability and observability of the SIS model (1) was investigated in [7], [6]. In particular, it is shown that the system with states  $S, I$  and output  $y = \alpha I$  is neither identifiable nor observable in case where  $\alpha, \beta, \gamma$  are unknown (the mortality rate  $\mu$  can be assumed to be available, since it can be easily estimated separately). However, when the additional states  $S_1, I_1$  and the output  $y_1 = \alpha I_1$  are included, the system is observable and identifiable.

**Theorem 2** *Assume that  $\beta, \gamma, \alpha$  are unknown, and the measurement  $y$  is available with  $\dot{y} \neq 0$ . Then system (1a) with states  $(S, I)$  and output  $y$  is neither observable nor identifiable. Additionally, if  $y_1$  is also available, then system (1) is both observable and identifiable.*

The proof of Theorem 1 is included in Appendix A. Theorem 2 underlines the interest of measuring the primary infections from the point of view of identification and observation of reinfection models, which is the main motivation of the present paper.

### 2.3 State-space representation for adaptive observer design

For the design of the adaptive observer, we introduce the change of variables:

$$z = S + I, \quad s_1 = \ln(S_1), \quad y = \alpha I, \quad y_1 = \alpha I_1.$$

Therefore, (1) may be rewritten in the equivalent form:

$$\begin{aligned} \dot{z} &= \mu - \mu z, \\ \dot{s}_1 &= -\mu - \frac{\beta}{\alpha} y + \mu e^{-s_1}, \\ \dot{y} &= \beta y - \frac{\beta}{\alpha} y^2 - y(\mu + \gamma), \\ \dot{y}_1 &= -\mu y_1 - \gamma y_1 + e^{s_1} \beta y. \end{aligned} \quad (2)$$

As the total normalized population  $z$  is constant and equal to 1, the dynamics of the state  $z$  can be disregarded in the rest of the paper.

### 3 Stability notions

For the stability analysis of non-linear observers, the theory of input-to-state stability (ISS) is a popular framework. In particular, as our objective is to reduce some specific estimation errors of the observer, we will introduce the closely related notions of input-to-output stability (IOS), considering only the components of estimation errors that should be minimized as outputs. We recall in this section the definition of IOS and its characterization by Lyapunov functions (see [15], [16]). To this end, consider the system

$$\dot{x}(t) = f(x(t), u(t)), \quad y(t) = h(x(t)), \quad (3)$$

with state  $x(t) \in \mathbb{R}^n$ , input  $u \in L_\infty(\mathbb{R}_+)$  and output  $y(t) \in \mathbb{R}^p$ . The functions  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ ,  $h : \mathbb{R}^n \rightarrow \mathbb{R}^p$  are assumed to be continuously differentiable and the system (3) is assumed to be forward complete, i.e., for every initial condition  $x(0)$  and every input signal  $u$ , the corresponding solution is defined for all  $t \geq 0$ .

**Definition 3** *The system (3) is called input-to-output stable (IOS) if there exist functions  $\kappa \in \mathcal{KL}$  and  $\lambda \in \mathcal{K}_\infty$  such that*

$$|y(t)| \leq \kappa(|x(0)|, t) + \lambda(\|u\|_\infty),$$

for all  $t \geq 0$ ,  $x(0) \in \mathbb{R}^n$  and  $u \in L_\infty(\mathbb{R}_+)$ .

The following definitions and result provide a characterization of an IOS system by a Lyapunov-like function:

**Definition 4** *A smooth function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  is called an IOS-Lyapunov function of (3) if there exist functions  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ ,  $\chi \in \mathcal{K}$  and  $\alpha_3 \in \mathcal{KL}$ , such that*

$$\alpha_1(|h(x)|) \leq V(x) \leq \alpha_2(|x|), \quad \forall x \in \mathbb{R}^n$$

and for all  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}^m$  it holds:

$$V(x) \geq \chi(\|u\|) \implies \nabla V(x) \cdot f(x, u) < -\alpha_3(V(x), |x|).$$

**Definition 5** *The system (3) is uniformly bounded input bounded state stable (UBIBS) if there exists  $\sigma \in \mathcal{K}$  such that the following estimate holds for all  $x(0) \in \mathbb{R}^n$ ,  $u \in L_\infty(\mathbb{R}_+)$  and  $t \geq 0$ :*

$$|x(t)| \leq \sigma(\max\{|x(0)|, \|u\|_\infty\}).$$

**Theorem 6 ([16])** *Assume the system (3) is UBIBS. The system is IOS if and only if it admits an IOS-Lyapunov function.*

### 4 Adaptive observation with complete measurements

In this section, we design an adaptive observer for the system (2) where the entire infected and primary infected populations are measured at each time. Hence,  $\alpha$  is assumed to be equal to 1, and the equations of (2) are simplified:

$$\begin{aligned} \dot{s}_1 &= -\mu - \beta y + \mu e^{-s_1}, \\ \dot{y} &= \beta y(1 - y) - y(\mu + \gamma), \\ \dot{y}_1 &= -\mu y_1 - \gamma y_1 + e^{s_1} \beta y. \end{aligned} \quad (4)$$

As  $\mu$  is known and  $y$  and  $y_1$  are outputs, our goal is to estimate the remaining state  $s_1$  and coefficients  $\beta, \gamma$ . Our observer design strategy has two steps. First, we estimate the unmeasured state component  $s_1$  based on auxiliary dummy identification of the uncertain parameters. Second, assuming that  $s_1$  is properly reconstructed, the estimates of the parameters are derived.

#### 4.1 Observer for $s_1$

We define the estimates  $\hat{s}_1, \hat{\beta}, \hat{\gamma}$  of  $s_1, \beta, \gamma$  as follows:

$$\begin{aligned} \hat{s}_1 &= \xi_{\hat{s}_1} + k_{\hat{s}_1} y_1, \\ \hat{\beta} &= \xi_{\hat{\beta}} + k_{\hat{\beta}} \ln(y), \quad \hat{\gamma} = \xi_{\hat{\gamma}} - k_{\hat{\gamma}} \ln(y), \end{aligned} \quad (5)$$

where  $k_{\hat{s}_1}, k_{\hat{\beta}}, k_{\hat{\gamma}} > 0$  are tuning parameters and  $\xi_{\hat{s}_1}, \xi_{\hat{\beta}}, \xi_{\hat{\gamma}}$  are auxiliary variables whose adaptive laws will be defined below in (6). The idea of mixing the states of the observer and the observed system (5) in the parameter estimate is similar to the Immersion & Invariance approach ([2]). To clarify the method, we will first introduce the approximations  $\hat{y}, \hat{y}_1$ , which imitate the dynamics of  $y, y_1$ :

$$\begin{aligned} \dot{\hat{y}} &= \hat{\beta} y(1 - y) - y(\mu + \hat{\gamma}), \\ \dot{\hat{y}}_1 &= e^{\hat{s}_1} \hat{\beta} y - (\hat{\gamma} + \mu) y_1. \end{aligned}$$

If we consider the errors  $\rho_1 := y - \hat{y}$  and  $\rho_2 := y_1 - \hat{y}_1$ , the dynamics are given by

$$\begin{aligned} \dot{\rho}_1 &= \theta_{\hat{\beta}} y - \theta_{\hat{\beta}} y^2 - \theta_{\hat{\gamma}} y, \\ \dot{\rho}_2 &= -\theta_{\hat{\gamma}} y_1 + (e^{s_1} - e^{\hat{s}_1}) \hat{\beta} y + e^{s_1} \theta_{\hat{\beta}} y, \end{aligned}$$

where  $\theta_{\hat{\beta}} := \beta - \hat{\beta}$ ,  $\theta_{\hat{\gamma}} := \gamma - \hat{\gamma}$  are parameters estimation errors. Thus, from the derivatives of  $\rho_1$  and  $\rho_2$  we obtain terms depending on the errors  $\theta_{\hat{\beta}}$ ,  $\theta_{\hat{\gamma}}$ ,  $e^{s_1} - e^{\hat{s}_1}$  which should be put into the adaptive law of  $\hat{s}_1$ ,  $\hat{\beta}$ ,  $\hat{\gamma}$  as negative feedbacks. This was implemented in the previous paper [9] under the assumption that the output derivatives  $\dot{y}$ ,  $\dot{y}_1$  are estimated using differentiators and put directly into the adaptive law. However, estimating derivatives may be unreliable in the noisy setting with low sampling of measurements, so we construct instead the estimates in the form (5) to remove the necessity of estimating  $\dot{y}$ ,  $\dot{y}_1$ . Therefore, if we define the dynamics of  $\xi_{\hat{s}_1}$ ,  $\xi_{\hat{\beta}}$ , and  $\xi_{\hat{\gamma}}$  as:

$$\begin{aligned}\dot{\xi}_{\hat{s}_1} &= -\mu - \hat{\beta}y + \mu e^{-\hat{s}_1} - k_{\hat{s}_1}(e^{\hat{s}_1}\hat{\beta}y - (\hat{\gamma} + \mu)y_1), \\ \dot{\xi}_{\hat{\beta}} &= -k_{\hat{\beta}}(\hat{\beta}(1-y) - \hat{\gamma} - \mu), \\ \dot{\xi}_{\hat{\gamma}} &= k_{\hat{\gamma}}(\hat{\beta}(1-y) - \hat{\gamma} - \mu),\end{aligned}\quad (6)$$

the dynamics of the errors  $\theta_{\hat{\beta}}$ ,  $\theta_{\hat{\gamma}}$  and  $\theta_{\hat{s}_1} := s_1 - \hat{s}_1$  become:

$$\begin{aligned}\dot{\theta}_{\hat{s}_1} &= \mu(e^{-s_1} - e^{-\hat{s}_1}) - y\theta_{\hat{\beta}} - k_{\hat{s}_1}[\hat{\beta}y(e^{s_1} - e^{\hat{s}_1}) \\ &\quad + e^{s_1}y\theta_{\hat{\beta}} - y_1\theta_{\hat{\gamma}}],\end{aligned}\quad (7a)$$

$$\dot{\theta}_{\hat{\beta}} = -k_{\hat{\beta}}[(1-y)\theta_{\hat{\beta}} - \theta_{\hat{\gamma}}],\quad (7b)$$

$$\dot{\theta}_{\hat{\gamma}} = k_{\hat{\gamma}}[(1-y)\theta_{\hat{\beta}} - \theta_{\hat{\gamma}}].\quad (7c)$$

The stability of the errors is checked in the following subsection.

#### 4.2 IOS stability through Lyapunov function

Consider first a weighted parameter error norm:

$$V_1 := |\theta_{\hat{\beta}}| + \frac{k_{\hat{\beta}}}{k_{\hat{\gamma}}}|\theta_{\hat{\gamma}}|.$$

Along the trajectories of the system (2)-(6), the derivative of  $V_1$  is

$$\begin{aligned}\dot{V}_1 &= -k_{\hat{\beta}}[|\theta_{\hat{\beta}}|(1-y) - \text{sgn}(\theta_{\hat{\gamma}})\theta_{\hat{\beta}}(1-y) + |\theta_{\hat{\gamma}}| \\ &\quad - \text{sgn}(\theta_{\hat{\beta}})\theta_{\hat{\gamma}}] \leq 0.\end{aligned}$$

Therefore, the observer has a non-increasing parameters error norm. Consider also,

$$V_2 := |\theta_{\hat{s}_1}|,$$

then

$$\begin{aligned}\dot{V}_2 &= -\mu|e^{-s_1} - e^{-\hat{s}_1}| - y\text{sgn}(\theta_{\hat{s}_1})\theta_{\hat{\beta}} \\ &\quad - k_{\hat{s}_1}\text{sgn}(\theta_{\hat{s}_1})[\hat{\beta}y(e^{s_1} - e^{\hat{s}_1}) + e^{s_1}y\theta_{\hat{\beta}} - y_1\theta_{\hat{\gamma}}] \\ &\leq -\mu|e^{-s_1} - e^{-\hat{s}_1}| - k_{\hat{s}_1}\hat{\beta}y|e^{s_1} - e^{\hat{s}_1}| + k_{\hat{s}_1}|\theta_{\hat{\gamma}}|y_1 \\ &\quad + (1 + k_{\hat{s}_1}e^{s_1})|\theta_{\hat{\beta}}|y \\ &\leq -\mu|e^{-s_1} - e^{-\hat{s}_1}| - k_{\hat{s}_1}\hat{\beta}y|e^{s_1} - e^{\hat{s}_1}| \\ &\quad + (1 + k_{\hat{s}_1}e^{s_1})V_1y + k_{\hat{s}_1}k_{\hat{\gamma}}k_{\hat{\beta}}^{-1}V_1y_1.\end{aligned}\quad (8)$$

Clearly, under the additional assumption that  $\hat{\beta}$  has a positive lower bound,  $V_2$  is an IOS-Lyapunov function from the input  $V_1 \leq V_1(0)$  (measuring the errors of the initial parametric guess) to the output  $\theta_{\hat{s}_1}$ . Moreover, we notice that the subsystem (7b)-(7c) is a cooperative system, i.e., the Jacobian has non-negative off-diagonal entries, it is then component-wise order-preserving, and in particular the errors  $\theta_{\hat{\beta}}$ ,  $\theta_{\hat{\gamma}}$  remains non-positive if the initial conditions  $\theta_{\hat{\beta}}(0)$ ,  $\theta_{\hat{\gamma}}(0)$  are non-positive, i.e.,  $\hat{\beta}(0) \geq \beta$ ,  $\hat{\gamma}(0) \geq \gamma$ . In this case,  $\hat{\beta} \geq \beta > 0$ , and the assumption above is satisfied.

Finally, the UBIBS property of (4)-(7) may be easily derived from the global stability property of the system (2) in addition to the fact that  $V_1$  is non-increasing, and  $V_2$  also has a negative derivative outside a region containing 0, which size depends on  $V_1(0)$ .

Therefore, we proved the following result:

**Theorem 7** For any  $k_{\hat{s}_1}, k_{\hat{\beta}}, k_{\hat{\gamma}} > 0$  and  $\hat{\beta}(0) > \beta$ ,  $\hat{\gamma}(0) > \gamma$ , the system (4) coupled with the adaptive observer (5), (6) is input-to-output stable with input  $V_1(0)$  (measure of the initial parametric estimation gap) and output  $\theta_{\hat{s}_1}$ .

#### 4.3 Estimating $\beta$ and $\gamma$ from the measure of $s_1$

For the previous observer, the simulations in Section 6 below show excellent accuracy for estimating  $s_1$  at the disease outbreak. This leads us to construct a simple asymptotic estimator of  $\beta$ ,  $\gamma$  assuming that  $s_1$  is an additional accessible quantity (for implementation,  $s_1$  will be replaced by its estimated value  $\hat{s}_1$  from (5)):

$$\dot{\check{\beta}} = \xi_{\check{\beta}} - k_{\check{\beta}}s_1, \quad \dot{\check{\gamma}} = \xi_{\check{\gamma}} - k_{\check{\gamma}}\ln(y),\quad (9)$$

where  $\xi_{\check{\beta}}$ ,  $\xi_{\check{\gamma}}$  are auxiliary variables whose dynamics are defined by

$$\begin{aligned}\dot{\xi}_{\check{\beta}} &= k_{\check{\beta}}(\mu(e^{-s_1} - 1) - \check{\beta}y), \\ \dot{\xi}_{\check{\gamma}} &= k_{\check{\gamma}}(\check{\beta}(1-y) - \check{\gamma} - \mu).\end{aligned}\quad (10)$$

Consider now the errors

$$\theta_{\check{\beta}} = \beta - \check{\beta}, \quad \theta_{\check{\gamma}} = \gamma - \check{\gamma},$$

whose dynamics can be written as follows:

$$\begin{aligned}\dot{\theta}_{\hat{\beta}} &= -k_{\hat{\beta}}y\theta_{\hat{\beta}}, \\ \dot{\theta}_{\hat{\gamma}} &= -k_{\hat{\gamma}}(\theta_{\hat{\gamma}} - \theta_{\hat{\beta}}(1-y)).\end{aligned}\quad (11)$$

Taking now the candidate Lyapunov function

$$V_3 := |\theta_{\hat{\beta}}| + b|\theta_{\hat{\gamma}}|,$$

we obtain the estimate:

$$\dot{V}_3 \leq -(k_{\hat{\beta}} - b)|\theta_{\hat{\beta}}|y - bk_{\hat{\gamma}}|\theta_{\hat{\gamma}}|,$$

taking  $b < k_{\hat{\beta}}$ , this proves that the estimator (9), (10) converges exponentially (the output  $y$  is separated from zero approaching the endemic equilibrium).

**Theorem 8** *Let  $k_{\hat{\beta}}, k_{\hat{\gamma}} > 0$ . The adaptive observer (9), (10) converges exponentially to  $\beta, \gamma$ .*

In particular, if we take the imperfect estimation  $\hat{s}_1$  instead of  $s_1$ , the estimators are IOS-stable with input  $\theta_{\hat{s}_1}$  and outputs  $\theta_{\hat{\beta}}$  and  $\theta_{\hat{\gamma}}$ . Then cascade connection of two IOS systems is again IOS (see [14]).

**Corollary 9** *Let  $k_{\hat{s}_1}, k_{\hat{\beta}}, k_{\hat{\gamma}}, k_{\hat{\beta}}, k_{\hat{\gamma}} > 0$ . The cascade connection of systems (4)-(5)-(6)-(9)-(10) is input-to-output stable with input  $V_1(0)$  and output  $\theta_{\hat{s}_1}, \theta_{\hat{\beta}}, \theta_{\hat{\gamma}}$ .*

## 5 Adaptive observation with unknown measured proportion

In this section, we examine the case where an unknown proportion  $\alpha$  of infected and primary infected is measured. Denoting  $\alpha^{-1} = \zeta$ , we rewrite again the system (2) excluding the dynamics of  $z$ .

$$\begin{aligned}\dot{s}_1 &= -\mu - \beta\zeta y + \mu e^{-s_1}, \\ \dot{y} &= \beta y - \beta\zeta y^2 - (\gamma + \mu)y, \\ \dot{y}_1 &= -(\gamma + \mu)y_1 + e^{s_1}\beta y.\end{aligned}\quad (12)$$

The additional unknown parameter  $\alpha$  governing the outputs  $y, y_1$  renders the estimation problem more delicate and nonlinearly parameterized. For the adaptive observer design, we will adapt the same two-step strategy employed for the case with complete measurements in section 4.

### 5.1 Adaptive observer for $s_1$

As in subsection 4.1, the main objective in this subsection is to derive an adaptive observer that, in particular, delivers an accurate estimation of  $s_1$ .

We introduce the estimates  $\hat{s}_1, \hat{\gamma}, \hat{\zeta}$  in the form:

$$\begin{aligned}\hat{s}_1 &= \xi_{\hat{s}_1} + k_{\hat{s}_1}y_1, \quad \hat{\gamma} = \xi_{\hat{\gamma}} - k_{\hat{\gamma}}\ln y, \\ \hat{\zeta} &= \xi_{\hat{\zeta}} - k_{\hat{\zeta}}\ln y.\end{aligned}\quad (13)$$

Additionally, in light of the inequality upper bounding  $\dot{V}_2$  in (8), it would be needed for the proof of stability to have  $\hat{\beta} > \beta$  for a  $\beta > 0$ . However, we could not recover an observer with similar cooperative property governing the error dynamics due to the additional parameter to estimate. Thus, following [13], we introduce a projection in the adaptive law instead, to guarantee that the estimate  $\hat{\beta}$  remains inside a specific interval  $(\underline{\beta}, \bar{\beta})$  with  $\underline{\beta}, \bar{\beta} > 0$  determined beforehand. The observer writes:

$$\dot{\xi}_{\hat{s}_1} = \mu e^{-\hat{s}_1} - \hat{\beta}\hat{\zeta}y - \mu - k_{\hat{s}_1}(e^{\hat{s}_1}\hat{\beta}y - (\hat{\gamma} + \mu)y_1), \quad (14a)$$

$$\dot{\xi}_{\hat{\gamma}} = k_{\hat{\gamma}}(\hat{\beta} - \mu - \hat{\gamma} - \hat{\beta}\hat{\zeta}y), \quad (14b)$$

$$\dot{\xi}_{\hat{\zeta}} = k_{\hat{\zeta}}(\hat{\beta} - \mu - \hat{\gamma} - \hat{\beta}\hat{\zeta}y), \quad (14c)$$

$$\begin{aligned}\dot{\hat{\beta}} &= - \left( 1 - \frac{\varepsilon - (\bar{\beta} - \hat{\beta})}{\varepsilon} \mathbb{1}_{h \leq 0} \mathbb{1}_{\hat{\beta} \geq \bar{\beta} - \varepsilon} \right. \\ &\quad \left. - \frac{\varepsilon - (\hat{\beta} - \underline{\beta})}{\varepsilon} \mathbb{1}_{h \geq 0} \mathbb{1}_{\hat{\beta} \leq \underline{\beta} + \varepsilon} \right) h, \end{aligned}\quad (14d)$$

$$h = f(\hat{\beta}, \hat{\gamma}, \hat{\zeta}, \hat{s}_1, y, y_1), \quad (14e)$$

where  $\varepsilon > 0$ ,  $\mathbb{1}_c$  is the condition  $c$  indicator and  $f$  is a generic  $C^1$  function. For instance, an example of  $f$  is in Section 6 for simulation purpose.

Due to the projection algorithm,  $\hat{\beta}$  stays inside the admissible interval  $(\underline{\beta}, \bar{\beta})$ , and our goal in this section is to estimate  $s_1$ . The signal  $f$  plays then an auxiliary role.

Denoting the errors

$$\theta_{\hat{s}_1} := s_1 - \hat{s}_1, \quad \theta_1 := \beta - \hat{\beta}, \quad \theta_{\hat{\gamma}} := \gamma - \hat{\gamma}, \quad \theta_{\hat{\zeta}} = \zeta - \hat{\zeta},$$

then,

$$\begin{aligned}\dot{\theta}_{\hat{s}_1} &= -k_{\hat{s}_1}(e^{s_1}y\theta_1 + \hat{\beta}y(e^{s_1} - e^{\hat{s}_1}) - y_1\theta_{\hat{\gamma}}) \\ &\quad + \mu(e^{-s_1} - e^{-\hat{s}_1}) - \hat{\beta}y\theta_{\hat{\zeta}} - \zeta y\theta_1,\end{aligned}$$

$$\dot{\theta}_{\hat{\gamma}} = k_{\hat{\gamma}}(\theta_1 - \hat{\beta}y\theta_{\hat{\zeta}} - \zeta y\theta_1 - \theta_{\hat{\gamma}}),$$

$$\dot{\theta}_{\hat{\zeta}} = k_{\hat{\zeta}}(\theta_1 - \hat{\beta}y\theta_{\hat{\zeta}} - \zeta y\theta_1 - \theta_{\hat{\zeta}}).$$

### 5.2 IOS stability through Lyapunov function

To study the stability, we introduce the quantity  $\theta_{\hat{\beta}}^{\max} = \bar{\beta} - \underline{\beta}$  and the candidate Lyapunov function

$$V := |\theta_{\hat{s}_1}| + b|\theta_{\hat{\gamma}}| + c|\theta_{\hat{\zeta}}|.$$

Assume that  $\beta \in (\underline{\beta}, \bar{\beta})$ , the derivative of  $V$  along trajectories is

$$\begin{aligned}
\dot{V} &= -\mu|e^{-s_1} - e^{-\hat{s}_1}| - \hat{\beta}y \operatorname{sgn}(\theta_{\hat{s}_1})\theta_{\hat{\zeta}} - \zeta y \operatorname{sgn}(\theta_{\hat{s}_1})\theta_1 \\
&\quad - k_{\hat{s}_1} \operatorname{sgn}(\theta_{\hat{s}_1})(-y_1\theta_{\hat{\gamma}} + \hat{\beta}y(e^{s_1} - e^{\hat{s}_1}) + e^{s_1}y\theta_1) \\
&\quad - b|\theta_{\hat{\gamma}}| + k_{\hat{\gamma}} \operatorname{sgn}(\theta_{\hat{\gamma}})(\theta_1 - \hat{\beta}y\theta_{\hat{\zeta}} - \zeta y\theta_1) - c|\theta_{\hat{\zeta}}| \\
&\quad + k_{\hat{\zeta}} \operatorname{sgn}(\theta_{\hat{\zeta}})(\theta_1 - \hat{\beta}y\theta_{\hat{\zeta}} - \zeta y\theta_1) \\
&\leq -\mu|e^{-s_1} - e^{-\hat{s}_1}| - k_{\hat{s}_1}|\hat{\beta}y|e^{s_1} - e^{\hat{s}_1}| + |\theta_{\hat{\zeta}}|\hat{\beta}y \\
&\quad + \zeta y|\theta_1| + k_{\hat{s}_1}|\theta_{\hat{\gamma}}|y_1 + k_{\hat{s}_1}e^{s_1}|\theta_1|y - bk_{\hat{\gamma}}|\theta_{\hat{\gamma}}| \\
&\quad + bk_{\hat{\gamma}}(|\theta_1|(1 - \zeta y) + |\theta_{\hat{\zeta}}|\hat{\beta}y) - ck_{\hat{\zeta}}|\theta_{\hat{\zeta}}|\hat{\beta}y \\
&\quad + ck_{\hat{\zeta}}(|\theta_1|(1 - \zeta y) + |\theta_{\hat{\gamma}}|y) \\
&\leq -\mu|e^{-s_1} - e^{-\hat{s}_1}| - k_{\hat{s}_1}|\hat{\beta}y|e^{s_1} - e^{\hat{s}_1}| - ck_{\hat{\zeta}}|\theta_{\hat{\zeta}}|\hat{\beta}y \\
&\quad + [\zeta y + k_{\hat{s}_1}e^{s_1}y + (bk_{\hat{\gamma}} + ck_{\hat{\zeta}})(1 - \zeta y)]|\theta_1| \\
&\quad + (k_{\hat{s}_1}y_1 + ck_{\hat{\zeta}}y)|\theta_{\hat{\gamma}}| + (1 + bk_{\hat{\gamma}})\hat{\beta}y|\theta_{\hat{\zeta}}| \\
&\leq -\mu|e^{-s_1} - e^{-\hat{s}_1}| - \hat{\beta}yk_{\hat{s}_1}|e^{s_1} - e^{\hat{s}_1}| \\
&\quad - (bk_{\hat{\gamma}} - k_{\hat{s}_1}y_1 - ck_{\hat{\zeta}}y)|\theta_{\hat{\gamma}}| - (ck_{\hat{\zeta}} - 1 - bk_{\hat{\gamma}})|\theta_{\hat{\zeta}}|\hat{\beta}y \\
&\quad + [\zeta y + k_{\hat{s}_1}e^{s_1}y + (bk_{\hat{\gamma}} + ck_{\hat{\zeta}})(1 - \zeta y)]\theta_{\hat{\beta}}^{\max}.
\end{aligned}$$

Thus  $V$  is an IOS-Lyapunov function with input  $\theta_{\hat{\beta}}^{\max}$  and outputs  $\theta_{\hat{s}_1}, \theta_{\hat{\gamma}}, \theta_{\hat{\zeta}}$  if

$$bk_{\hat{\gamma}} > k_{\hat{s}_1}y_1 + ck_{\hat{\zeta}}y, \quad ck_{\hat{\zeta}} > 1 + bk_{\hat{\gamma}},$$

These inequalities are equivalent to

$$\frac{ck_{\hat{\zeta}} - 1}{k_{\hat{\gamma}}} > b > \frac{k_{\hat{s}_1}y_1 + ck_{\hat{\zeta}}y}{k_{\hat{\gamma}}}.$$

In order to guarantee that there exists indeed  $b$  such that the previous inequality is fulfilled, it is necessary and sufficient to select  $c$  sufficiently large to have

$$ck_{\hat{\zeta}}(1 - y) > 1 + k_{\hat{s}_1}y_1.$$

This is possible because, for any trajectory, there exists  $\varepsilon_1 > 0$  such that  $1 - y = 1 - \alpha I > \varepsilon_1$  since the derivative  $\dot{I}$  of the SIS system (1a) is strictly negative while  $I$  is close to 1. Moreover, the IOS stability is also deduced from the IOS-Lyapunov property of  $V$  and the fact that  $\hat{\beta} \in (\underline{\beta}, \bar{\beta})$ . Therefore, the following result is proved:

**Theorem 10** *Let  $f$  be a  $C^1$  function,  $k_{\hat{s}_1}, k_{\hat{\gamma}}, k_{\hat{\zeta}} > 0$  and  $\beta, \hat{\beta}(0) \in (\underline{\beta}, \bar{\beta})$ . The system (12)-(14) is input-to-output stable, with input  $\theta_{\hat{\beta}}^{\max}$  and output  $\theta_{\hat{s}_1}, \theta_{\hat{\gamma}}, \theta_{\hat{\zeta}}$ .*

### 5.3 Estimating $\beta, \gamma$ and $\alpha$ from the measure of $s_1$

The simulations provided below in section 6 show that the observer (13)-(14) offers again good accuracy for estimating

$s_1$  at the outbreak of the disease. Therefore, we explore once more if a reliable estimation of the unknown parameters can be recovered assuming the state  $s_1$  is measured (again, for implementation, the estimate of  $s_1$  given in (13) should be used). We introduce first the notation  $\psi = \frac{\beta}{\alpha}$  and rewrite (12) as

$$\begin{aligned}
\dot{s}_1 &= -\mu - \psi y + \mu e^{-s_1}, \\
\dot{y} &= \beta y - \psi y^2 - (\gamma + \mu)y, \\
\dot{y}_1 &= e^{s_1}\beta y - (\gamma + \mu)y_1.
\end{aligned}$$

The estimators are chosen in the form

$$\begin{aligned}
\check{\beta} &= \xi_{\check{\beta}} + \operatorname{sgn}(e^{s_1}\frac{y}{y_1} - 1)k_{\check{\beta}}(s_1 - \ln y + \ln y_1), \\
\check{\gamma} &= \xi_{\check{\gamma}} - k_{\check{\gamma}}y_1, \quad \check{\psi} = \xi_{\check{\psi}} - k_{\check{\psi}}s_1,
\end{aligned} \tag{15}$$

and the auxiliary variables  $\xi_{\check{\beta}}, \xi_{\check{\gamma}}, \xi_{\check{\psi}}$  are determined by the adaptive law:

$$\begin{aligned}
\dot{\xi}_{\check{\beta}} &= \operatorname{sgn}\left(\frac{e^{s_1}y}{y_1} - 1\right)k_{\check{\beta}}\left[\mu(1 - e^{-s_1}) + \check{\beta}\left(1 - \frac{e^{s_1}y}{y_1}\right)\right], \\
\dot{\xi}_{\check{\gamma}} &= k_{\check{\gamma}}[e^{s_1}\check{\beta}y - y_1(\mu + \check{\gamma})], \\
\dot{\xi}_{\check{\psi}} &= k_{\check{\psi}}(-\mu + \mu e^{-s_1} - \check{\psi}y).
\end{aligned} \tag{16}$$

Let the errors  $\theta_{\check{\beta}}, \theta_{\check{\gamma}}, \theta_{\check{\psi}}$  be

$$\theta_{\check{\beta}} := \beta - \check{\beta}, \quad \theta_{\check{\gamma}} = \gamma - \check{\gamma}, \quad \theta_{\check{\psi}} = \psi - \check{\psi}.$$

The errors follow the differential equations :

$$\begin{aligned}
\dot{\theta}_{\check{\beta}} &= -k_{\check{\beta}}|1 - e^{s_1}\frac{y}{y_1}|\theta_{\check{\beta}}, \\
\dot{\theta}_{\check{\gamma}} &= -k_{\check{\gamma}}(y_1\theta_{\check{\gamma}} - e^{s_1}y\theta_{\check{\beta}}), \\
\dot{\theta}_{\check{\psi}} &= -k_{\check{\psi}}y\theta_{\check{\psi}}.
\end{aligned}$$

It is then not difficult to deduce that, under the condition that  $1 - e^{s_1}\frac{y}{y_1} \neq 0$ , the estimation errors converge exponentially to zero. Moreover,  $e^{s_1}\frac{y}{y_1}$  converges to  $\mathcal{R}_0^{-1} < 1$ , hence the asymptotic convergence is guaranteed.

**Theorem 11** *Let  $k_{\check{\beta}}, k_{\check{\gamma}}, k_{\check{\psi}} > 0$ . The adaptive observer (15)-(16) provides estimates of  $\beta, \gamma, \psi$  with exponential convergence.*

Similarly to the section 4, the cascade connection is still IOS.

**Corollary 12** *Let  $f$  be a  $C^1$  function,  $k_{\hat{s}_1}, k_{\hat{\gamma}}, k_{\hat{\zeta}}, k_{\check{\beta}}, k_{\check{\gamma}}, k_{\check{\psi}} > 0$  and  $\beta, \hat{\beta}(0) \in (\underline{\beta}, \bar{\beta})$ . The cascade connection of systems (12)-(13)-(14)-(15)-(16) is input-to-output stable with input  $\theta_{\hat{\beta}}^{\max}$  and output  $\theta_{\hat{s}_1}, \theta_{\check{\beta}}, \theta_{\check{\gamma}}, \theta_{\check{\psi}}$ .*

## 6 Numerical simulations

### 6.1 Tuning of the observers

For the settings of the observer (5)-(6), the inequality regarding  $\dot{V}_2$  in (8) suggests taking  $k_{\hat{s}_1}$  and  $k_{\hat{\beta}}$  relatively large and  $k_{\hat{\gamma}}$  small in order to improve the practical estimation of  $\hat{s}_1$  (notice that usually the positive terms  $e^{s_1} = S_1$  and  $y_1 = \alpha I_1$  in the inequality converge to relatively low values, in fact to  $\frac{\mu}{\beta-\gamma}$  and  $\alpha \frac{\beta-\gamma-\mu}{\beta-\gamma} \frac{\mu}{\gamma+\mu}$  due to the effects of reinfections). On the other hand, the gains  $k_{\hat{\beta}}$ ,  $k_{\hat{\gamma}}$  of the asymptotic estimator (9)-(11) will be set sufficiently high to accelerate the convergence.

For the adaptive observer (13)-(14), we choose a large value for the parameter  $k_{\hat{s}_1}$  to have a reliable estimate of  $s_1$ . Moreover, the function  $f$  in (14e) is defined as

$$f(\hat{\beta}, \hat{\gamma}, \hat{\zeta}, \hat{s}_1, y, y_1) = k_1(\hat{\beta} - \mu - \hat{\gamma} - \hat{\beta}\hat{\zeta}y) + k_2(-(\hat{\gamma} + \mu)y_1 + e^{\hat{s}_1}\hat{\beta}y), \quad (17)$$

for some positive gains  $k_1$  and  $k_2$ . This is motivated by the fact that we expect  $\hat{\beta}$  to fulfill

$$\hat{\beta} - \mu - \hat{\gamma} - \hat{\beta}\hat{\zeta}y = 0, \quad e^{\hat{s}_1}\hat{\beta}y - (\hat{\gamma} + \mu)y_1 = 0,$$

because the true coefficients  $\beta, \gamma$  satisfy these equalities asymptotically. However, as stated in Theorem 10, the IOS stability is conserved for any  $C^1$  function  $f$ . For example, we may choose instead the function

$$f_1 = k_1\hat{\beta}, \quad (18)$$

which yields similar results numerically. The advantage of the function defined in (6.1) over (18) is to obtain an adaptive estimate  $\hat{\beta}$  which does not converge to boundary values  $\underline{\beta}, \bar{\beta}$ .

### 6.2 Case with $\alpha$ known

We consider in this subsection the simulation of the cascade connection of systems (4)-(5)-(6)-(9)-(10) with the settings:  $\mu^{-1} = 70$  years,  $\gamma = 12^{-1}$  days $^{-1}$ ,  $\beta = 0.2$  days $^{-1}$ ,  $k_{\hat{s}_1} = k_{\hat{\beta}} = k_{\hat{\gamma}} = 100$  days $^{-1}$ ,  $k_{\hat{\gamma}} = 10$  days $^{-1}$ ,  $k_{\hat{\gamma}} = 0.01$  days $^{-1}$ . The result indicates that the observer starts converging to an almost exact value of  $S_1$  during the outbreak (Fig. 1) before losing its precision slowly. Nevertheless, this initial estimation is sufficient to obtain accurate estimates  $\check{\beta}, \check{\gamma}$  of  $\beta, \gamma$  (Figs. 2 and 3). In Fig. 2, at the end of the estimation interval, when the system has approached the equilibrium, we can see the augmentation of the estimation error and a kind of unstable behavior, which is related to the loss of the previously mentioned identifiability close to the steady state.

### 6.3 Case with $\alpha$ unknown

The simulation of the cascade connection of systems (12)-(13)-(14)-(15)-(16) is computed with the same coefficients  $\beta, \gamma, \mu$  as above, additionally we set  $\alpha = 0.8$ ,  $k_{\hat{s}_1} = k_{\hat{\gamma}} = 100$  days $^{-1}$ ,  $k_{\hat{\gamma}} = k_{\hat{\beta}} = k_{\hat{\gamma}} = 10$  days $^{-1}$ ,  $k_{\hat{\zeta}} = k_1 = k_2 = 1$  days $^{-1}$ ,  $\varepsilon = 0.1$  days $^{-1}$  and  $(\underline{\beta}, \bar{\beta}) = (0.01, 2)$  days $^{-1}$ . For better estimation of  $s_1$ , fixing  $k_{\hat{s}_1}$  and  $k_{\hat{\gamma}}$  sufficiently large is necessary. Moreover,  $k_{\hat{\beta}}$  can take theoretically arbitrarily large values. However, in this case, the estimation would have significant noise. In the present settings, Fig. 4 shows that, even for a large confidence interval of  $\beta$ , the observer for unknown  $\alpha$  exhibits the same behavior as above, delivering an accurate estimate of  $s_1$  at the outbreak of the epidemics, however at a slower pace. This estimate is then sufficient for deriving a rather precise estimation of the remaining parameters (figures 5-7).

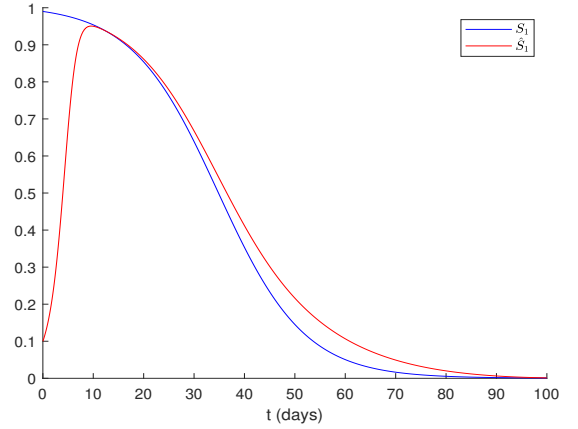


Fig. 1. Estimation of  $S_1$ , case with  $\alpha$  known

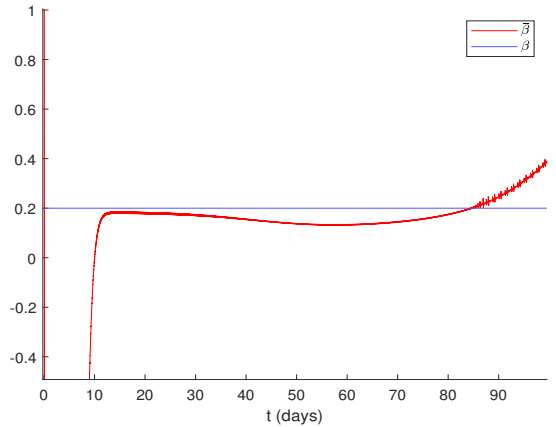


Fig. 2. Estimation of  $\beta$ , case with  $\alpha$  known

## 7 Conclusion

This paper considered the design of an adaptive observer for an SIS model that includes compartments of primary in-



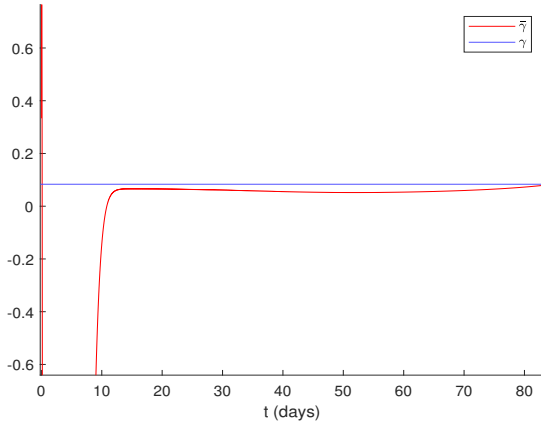


Fig. 3. Estimation of  $\gamma$ , case with  $\alpha$  known

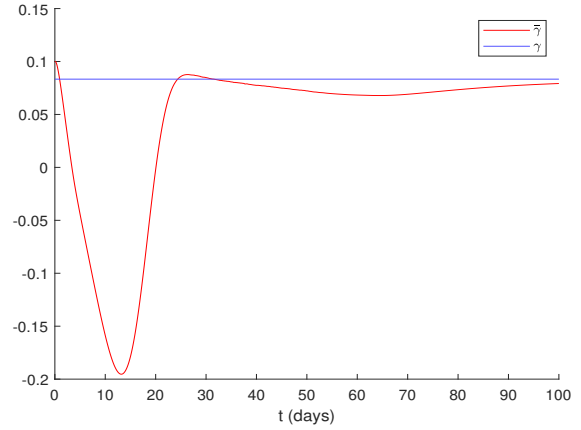


Fig. 6. Estimation of  $\gamma$ , case with  $\alpha$  unknown

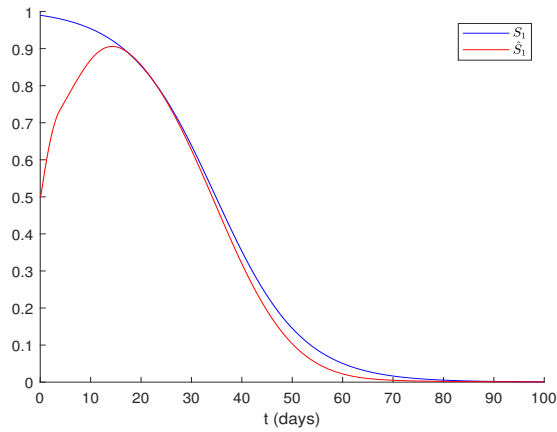


Fig. 4. Estimation of  $S_1$ , case with  $\alpha$  unknown

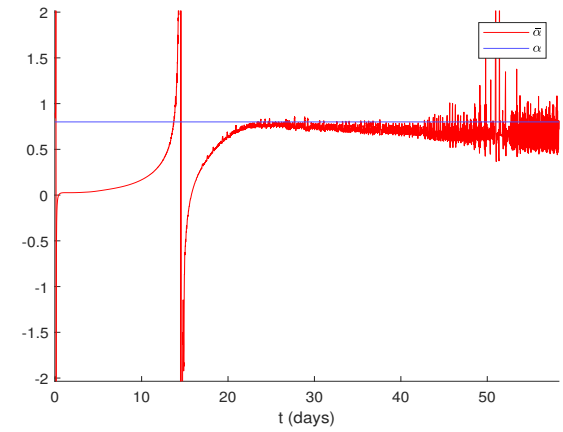


Fig. 7. Estimation of  $\alpha$ , case with  $\alpha$  unknown

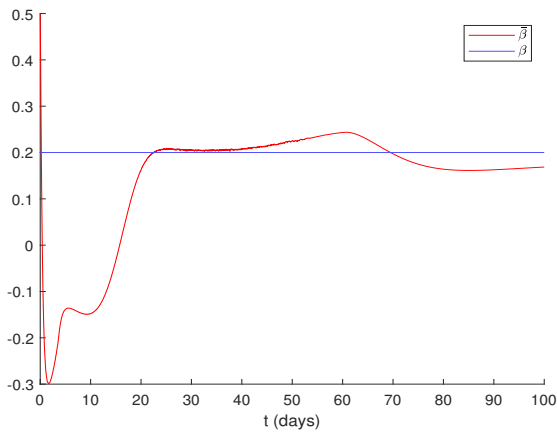


Fig. 5. Estimation of  $\beta$ , case with  $\alpha$  unknown

fectured individuals. The nonlinearity of the model and the measurements that depend on unknown parameters make this problem difficult. A cascade of adaptive observers was proposed to estimate all unmeasured state components and

unknown parameters. The stability of the observers was investigated under the input-to-output stability framework. The simulation results indicated that the proposed observers can effectively solve the problem. Moreover, the results illustrated the pertinence of measuring primary infections to identify and observe reinfection models.

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## A Proof of Theorem 2

**Measuring only  $y$ .** Consider first the use of the measurement  $y$ . Because  $S + I \equiv 1$ , the two formulas in (1a) provide indeed the same equation, namely:  $\dot{I} = (\beta - (\mu + \gamma) - \beta I)I$ . Therefore,

$$\dot{y} = \left( \beta - (\mu + \gamma) - \frac{\beta}{\alpha} y \right) y. \quad (\text{A.1})$$

By differentiation one gets that

$$\frac{d}{dt} \left( \frac{\dot{y}}{y} \right) = -\frac{\beta}{\alpha} \dot{y},$$

and one may express the two quantities  $\frac{\beta}{\alpha}$  and  $\mu + \gamma$ , as

$$\frac{\beta}{\alpha} = -\frac{1}{\dot{y}} \frac{d}{dt} \left( \frac{\dot{y}}{y} \right) = -\frac{1}{\dot{y}} \frac{d^2}{dt^2} (\ln y), \quad (\text{A.2a})$$

$$\begin{aligned} \beta - \gamma &= \frac{\dot{y}}{y} + \mu + \frac{\beta}{\alpha} y = \frac{\dot{y}}{y} + \mu - \frac{y}{\dot{y}} \frac{d^2}{dt^2} (\ln y) \\ &= \mu + \frac{d}{dt} (\ln y) - \frac{d}{dt} \left( \ln \left| \frac{d}{dt} (\ln y) \right| \right). \end{aligned} \quad (\text{A.2b})$$

These two quantities are thus identifiable, but this is not sufficient to obtain each of the three coefficients  $\alpha, \beta, \gamma$ . On the other hand, it is clear that nothing more may be learned when measuring only  $y$ , which fulfils equation (A.1). Therefore, the system (1a) is *not* identifiable over  $\alpha, \beta, \gamma$ .

Also, notice that it is not possible to determine  $I = \frac{1}{\alpha} y$ , otherwise  $\alpha$  would be identifiable, and all other parameters too. Thus the system is not observable.

### • Measuring $y$ and $y_1$ .

We include now the equations (1b) and exploit the knowledge of the supplementary measured output  $y_1$ . From the definition of  $y_1$  and the second formula in (1b) one deduces, putting  $w := \beta S_1$ , that

$$\dot{y}_1 = (\beta S_1) y - (\mu + \gamma) y_1 = w y - (\mu + \gamma) y_1, \quad (\text{A.3})$$

and thus:

$$\ddot{y}_1 = \dot{w} y + w \dot{y} - (\mu + \gamma) \dot{y}_1. \quad (\text{A.4})$$

On the other hand, due the first equation in (1b) and replacing  $\frac{\beta}{\alpha}$  by its value obtained from (A.2a) yields

$$\begin{aligned} \dot{w} &= \beta \dot{S}_1 = \beta (\mu - \beta S_1 I - \mu S_1) = \beta \mu - \frac{\beta}{\alpha} w y - \mu w \\ &= \beta \mu + \left( \frac{d}{dt} \left( \ln \left| \frac{d}{dt} (\ln y) \right| \right) - \mu \right) w. \end{aligned} \quad (\text{A.5})$$

Inserting in (A.4) the value of  $w$  extracted from (A.5), we get

$$\ddot{y}_1 = \left( \beta\mu + \left( \frac{d}{dt} \left( \ln \left| \frac{d}{dt} (\ln y) \right| \right) - \mu \right) w \right) y + w\dot{y} - (\mu + \gamma)\dot{y}_1,$$

and gathering the terms in  $w$  leads to the following equivalent form:

$$\ddot{y}_1 = \beta\mu y + \left( \dot{y} + y \frac{d}{dt} \left( \ln \left| \frac{d}{dt} (\ln y) \right| \right) - \mu y \right) w - (\mu + \gamma)\dot{y}_1. \quad (\text{A.6})$$

For clarity, let us write (A.3) and (A.6) under matrix form:

$$\begin{pmatrix} -y_1 & y \\ -\dot{y}_1 & \dot{y} + y \frac{d}{dt} \left( \ln \left| \frac{d}{dt} (\ln y) \right| \right) - \mu y \end{pmatrix} \begin{pmatrix} \mu + \gamma \\ w \end{pmatrix} = \begin{pmatrix} \dot{y}_1 \\ \ddot{y}_1 - \beta\mu y \end{pmatrix}.$$

One deduces by partial inversion of the matrix that

$$\mu + \gamma = \frac{\begin{vmatrix} \dot{y}_1 & y \\ \ddot{y}_1 - \beta\mu y & \dot{y} + y \left[ \frac{d}{dt} \left( \ln \left| \frac{d}{dt} (\ln y) \right| \right) - \mu \right] \end{vmatrix}}{\begin{vmatrix} -y_1 & y \\ -\dot{y}_1 & \dot{y} + y \left[ \frac{d}{dt} \left( \ln \left| \frac{d}{dt} (\ln y) \right| \right) - \mu \right] \end{vmatrix}} \quad (\text{A.7})$$

on the other hand, one has, by (A.2b), an alternative expression of  $\mu + \gamma$ , namely

$$\mu + \gamma = \beta - \frac{d}{dt} (\ln y) + \frac{d}{dt} \left( \ln \left| \frac{d}{dt} (\ln y) \right| \right). \quad (\text{A.8})$$

Achieving elimination of  $\mu + \gamma$  between (A.7) and (A.8) yields

$$\begin{aligned} & \beta - \frac{d}{dt} (\ln y) + \frac{d}{dt} \left( \ln \left| \frac{d}{dt} (\ln y) \right| \right) \\ &= \frac{\begin{vmatrix} \dot{y}_1 & y \\ \ddot{y}_1 - \beta\mu y & \dot{y} + y \frac{d}{dt} \left( \ln \left| \frac{d}{dt} (\ln y) \right| \right) - \mu y \end{vmatrix}}{\begin{vmatrix} -y_1 & y \\ -\dot{y}_1 & \dot{y} + y \frac{d}{dt} \left( \ln \left| \frac{d}{dt} (\ln y) \right| \right) - \mu y \end{vmatrix}}. \end{aligned}$$

Now, one checks that the previous identity is indeed *affine* in  $\beta$ . Factorizing all terms in  $\beta$ , it may be written

$$\begin{aligned} & \left( - \left( \dot{y} + y \frac{d}{dt} \left( \ln \left| \frac{d}{dt} (\ln y) \right| \right) - \mu y \right) y_1 + y\dot{y}_1 - \mu y^2 \right) \beta \\ &= \Phi(y, \dot{y}, \ddot{y}, y_1, \dot{y}_1, \ddot{y}_1), \end{aligned}$$

where the function  $\Phi$  depends upon the two outputs  $y$  and  $y_1$  and their derivatives up to the second one. In particular,  $\Phi$  does not contain any occurrence of the unknown coefficients  $\alpha, \beta, \gamma$ . One deduces that  $\beta$  may be identified, and thus also  $\alpha, \gamma$  with the help of (A.2). Therefore, in these conditions, system (1) is identifiable.

Once these parameters have been identified, one has  $I = \frac{1}{\alpha}y$  and  $S = 1 - I$ , while on the other hand,  $I_1 = \frac{1}{\alpha}y_1$  and  $S_1 = \frac{1}{\beta}w$ , where (see (A.3))  $w$  is given by

$$w = \frac{y_1}{y} \left( \frac{d}{dt} (\ln y_1) + \mu + \gamma \right).$$

The system (1) is thus observable.