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Characterizing Signalling: Connections between Causal Inference and Space-time Geometry

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Causality is pivotal to our understanding of the world, presenting itself in different forms: information-theoretic and relativistic, the former linked to the flow of information, and the latter to the structure of space-time. Leveraging a framework introduced in PRA, 106, 032204 (2022), which formally connects these two notions in general physical theories, we study their interplay. The framework defines information-theoretic causality through a causal modelling approach, which enables the inference of causal connections through agents' interventions and correlations. First, we improve the characterization of information-theoretic signalling as defined through so-called affects relations. Specifically, we provide conditions for identifying redundancies in different parts of such a relation, introducing techniques for causal inference in unfaithful causal models (where the observable data does not “faithfully” reflect the causal dependences). In particular, this demonstrates the possibility of causal inference using the absence of signalling between certain nodes. Second, we define an order-theoretic property called conicality, showing that it is satisfied for light cones in Minkowski space-times with $d > 1$ spatial dimensions but violated for $d = 1$. Finally, we study the embedding of information-theoretic causal models in space-time without violating relativistic principles such as no superluminal signalling (NSS). In general, we observe that constraints imposed by NSS in a space-time and those imposed by purely information-theoretic causal inference behave differently. We then prove a correspondence between conical space-times and faithful causal models: in both cases, there emerges a parallel between these two types of constraints. This indicates a connection between informational and geometric notions of causality, and offers new insights and tools for studying the relations between the principles of NSS and no causal loops in different space-time geometries and theories of information processing.

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1 Introduction

A cornerstone of scientific inquiry is the search for causal explanations for our observations of the world around us. The field of causal modelling and causal inference provides rigorous mathematical frameworks for connecting observable data with cause-effect relations. Originating in classical statistics [1, 2] where all causal relata are classical random variables, its versatility has led to applications across classical data driven disciplines [2, 3, 4, 5, 6, 7]. More recently, the approach has seen generalizations within quantum information theory, enabling successful causal explanations of fundamental correlations and phenomena in the quantum world [8, 9]. Causal models enable an operational definition of causality, in terms of the flow of information, discernible to agents’ interventions, which can be formulated without reference to space or time.

However, in other crucial disciplines such as relativistic physics, causality is intrinsically linked to the geometry of space and time. Moreover the two notions must exhibit a compatible interplay as space-time structure does constrain the flow of information in physical processes through principles of relativistic causality, such as the impossibility of signalling outside the future light cone. Recognizing the imperative to bridge this gap, recent works involving one of us [10, 11] have developed formalisms to formally link these informational and spatio-temporal notions of causality in quantum theory as well as more general (possibly non-classical) theories.

The work [10] develops a causal modelling formalism applicable to a general class of physical theories, and relates it to space-time structure through graph-theoretic compatibility conditions. This enables a theory-independent study of the interplay of the two causality notions. The concept of (*higher-order*) *affects relations* was introduced to capture the general possibilities for agents to signal to each other through interventions, and we will thus refer to the approach of [10] as the *affects framework* in the rest of this paper. The scope of the affects framework is quite broad, encompassing scenarios with cyclic and fine-tuned causal influences and latent non-classical causes. In particular, fine-tuning refers to the possibility of carefully tuned causal mechanisms that wash out certain correlations and signalling possibilities, making the observed data not “faithful” to the underlying causal dependencies. As a consequence, causation does not imply signalling in the presence of fine-tuning. Such fine-tuning is crucial in practical scenarios, such as in the security of cryptographic protocols.

By embedding a causal model in space-time, the principle of no superluminal signalling (NSS) corresponds to the compatibility between the higher-order affects relations and the light cone structure of the space-time. The affects framework led to the surprising finding that causal loops can be embedded in 1+1-Minkowski space-time without leading to superluminal signalling, even when the existence of the loop can be operationally verified through interventions [12]. Whether such loops are possible in higher-dimensional Minkowski space-time, as well as the necessary and sufficient conditions for ruling out such causal loops in a physical theory remain important open questions. More generally, the affects framework provides a platform to study what kind of properties are common or distinct between informational and spatio-temporal notions of causality in different theories, which is relatively little explored. Further investigating this aspect can lead to useful insights for understanding practical information processing scenarios in space-time as well as how space-time structure might emerge from informational structures, at a more fundamental level.

Addressing such questions requires a careful characterization of (a) the operational properties of higher-order affects relations and (b) order-theoretic properties relating to space-time geometry, and their interplay. This also calls for effective tools for handling fine-tuning, as several relativistic principles and information-theoretic notions become inequivalent when allowing for fine-tuning [10, 12, 13]. In the present work, we further develop such characterizations and their applications for causal inference, and identify an interesting correspondence between the aspects (a) and (b). This provides useful techniques for the aforementioned open questions on causal loops and for probing the interface of informational and spatio-temporal causation in non-classical theories.

2 Summary of contributions

We outline the main contributions of this work, while aiming for this summary to be self-contained even for those without prior knowledge of the affects framework [10] (which is reviewed in Section 3). An (higher-order) affects relation, denoted as $X \vDash Y \mid \text{do}(Z)$, carries three arguments, which are disjoint sets

of random variables, and it captures that an agent who intervenes on X can signal to an agent who can observe data on Y and is given information about interventions performed on Z .¹ Depending on whether Z is empty or not, we have a zeroth-order or a higher-order relation. Here, X and Z are interventional arguments while Y is a purely observational argument. While affects relations refer to classical variables (such as measurement settings and outcomes), they can generally arise from operations performed on non-classical systems of any underlying theory. In the following, we will refer to a random variable (which goes into the argument of an affects relation) as a node, as such variables form the nodes or vertices of the directed graphs representing information-theoretic causal structures.

Characterizing affects relations and applications to causal inference Generally, an affects relation $X \vDash Y \mid \text{do}(Z)$ can carry some redundancies and may be operationally equivalent to a “reduced” affects relation on sets of smaller cardinality. Identifying such redundancies is important both for information-theoretic causal inference, and for understanding precisely how relativistic causality principles in space-time constrain information processing protocols. In [10], only the reducibility of affects relations in the first interventional argument X was defined. In Section 4.1, we extend the concept of reducibility to all the different arguments of the affects relation while discussing its operational significance in identifying redundancies in these arguments.

In Section 4.2, we introduce the concept of *clustering* and relate it to reducibility. An affects relation $X \vDash Y \mid \text{do}(Z)$ is clustered in an argument if the affects relation no longer holds when replacing the set of nodes in that argument by a strict subset of it. We prove that clustering (in any argument) is a signature of fine-tuning and thereby distinguish between at least three different types of operationally detectable fine-tuning, i.e., the fine-tuning of underlying causal mechanisms which can be detected from the affects relations. Such possibilities for information transfer between sets of systems that is not detectable within subsets of those systems, also have applications in cryptography [14, 15, 16], quantum error correction [17, 18] and distributed information processing protocols in space-time [19, 20, 21]. Relations between irreducibility, clustering and fine-tuning are illustrated in Figure 3.

The causal inference implications of these concepts are then presented in Section 4.3. Typical causal inference results and algorithms assume no fine-tuning due to complications for causal inference that arise in the presence of fine-tuning (see [2, 22] for examples). Our results show that in certain fine-tuned models, the absence of signalling between some of the nodes can be employed for causal inference. More generally, our work contributes to open questions regarding causal inference in presence of fine-tuning, as raised in Section IX.e of [10].

Order-theoretic properties of space-time Modelling space-time structure as a partially ordered set, an order-theoretic property of the causal structure of space-time is introduced in Section 5, namely, *conicality*. We then show that $d+1$ -dimensional Minkowski space-times with $d > 1$ are conical space-times, while this is not the case for 1+1-dimensional Minkowski space-time. Conicality captures the requirement that the joint future region $f(L)$ (intersection of future light cones) of a set L of space-time points uniquely determines the location of all points in L that contribute non-trivially to $f(L)$.² In Minkowski space-time with $d = 1$, we can have two distinct pairs of space-like separated points $L = \{p, q\}$ and $L' = \{p', q'\}$ that have the same joint future, $f(L) = f(L')$ and conicality is thus violated (Figure 6). For $d > 1$ however, our result implies that this cannot happen.

We discuss further order-theoretic properties that distinguish Minkowski space-time for $d = 1$ and $d > 1$ spatial dimensions in Appendix B. Intuitively, these distinctions are related to the fact that in $d = 1$ the joint future of any two points has the same geometry as the light cone of an individual point (the unique earliest point in this joint future), but the geometries of these regions will differ for $d > 1$ as there is no longer a unique earliest point in the joint future of any two points.

Correspondence between causal inference and space-time geometry In Section 6.1, we study the properties of affects relations and causal models, which can be embedded in a space-time compatibly, i.e., without violating no superluminal signalling (NSS). We find that an affects relation $X \vDash Y \mid \text{do}(Z)$ irreducible in the first and third arguments implies that all nodes in X and in Z are causes of some node in

¹Technically, this corresponds to an unconditional (higher-order) affects relation. The concept of conditional higher-order affects relations is also introduced in [10] by including a fourth argument $X \vDash Y \mid \text{do}(Z), W$ which captures an additional post-selection on W (without interventions), i.e. the agent receiving the signal is also given information about this post-selection. We focus on the unconditional case in the main text and generalise many of the results to conditional relations in the appendix.

²For example, if $L = \{p, q\}$ with $p \prec q$, then $f(L) = f(q)$ and p does not contribute non-trivially to $f(L)$.

Y i.e., the set Y of nodes is ordered later than each node in X and Z relative to the *information-theoretic causal order*. However, we observe that when embedding these nodes in space-time, imposing that the affects relation does not lead to superluminal signalling does not generally imply an analogous ordering of nodes relative to the *light cone structure of the space-time*: the set Y of nodes can generally be jointly accessible outside the future light cone of some nodes in X and Z . That is, the relativistic principle of NSS and purely informational principles of causal inference generally impose different ordering constraints on the relevant operational events (here, the arguments of an affects relation).

We then prove a correspondence between (1) affects relations that are *not clustered in the third argument* but embedded compatibly in an arbitrary space-time and (2) arbitrary affects relations but embedded compatibly in a *conical space-time*. Specifically, we show that in both these cases, a clear ordering emerges between interventional arguments X and Z and the observational argument Y of an irreducible affects relation $X \models Y \mid \text{do}(Z)$, with the former ordered before the latter relative to both the information-theoretic and spatio-temporal causal orders. This correspondence is illustrated and summarised in Table 1.

As clustering implies fine-tuning, this suggests links between faithful causal models and conical space-times. This result also sheds light on the relation between fine-tuning in the causal model and a notion of fine-tuning in the space-time embedding suggested in [10], which plays an important role in the possibility of embeddable causal loops in 1+1-Minkowski space-time [12]. It also highlights that in conical space-times, one can often reduce statements regarding NSS for higher-order affects relations to equivalent statements in terms of the much simpler zeroth-order relations. Further, another type of correspondence between affects relations without clustering in the first argument and conical space-times is discussed in Appendix F.

We conclude by discussing the implications of our results and future outlook in Section 7, which may be of interest on the one hand for disciplines beyond physics that apply causal inference and on the other for relativistic physics.

3 Review of the affects framework

In this section, we give a brief introduction into the affects framework introduced in previous work [10] involving one of us, which formalizes the notion of signalling under minimal and theory-independent assumptions. This generality makes it applicable to scenarios with non-classical or cyclic causal influences as well as scenarios where causal influences may be fine-tuned so that they wash out certain observable correlations or possibilities for signalling. Here we review the concepts through specific examples, and refer the reader to Appendix A for technical definitions of the concepts mentioned here, but not defined in full generality.

3.1 Theory-independent causal models

The affects framework formulates information-theoretic causality independently of a notion of space and time, by building on the *causal modelling* approach. Here, a causal structure is a directed graph \mathcal{G} where each node can either be *observed* or *unobserved*. Observed nodes correspond to classical random variables or RVs (such as settings and outcomes of measurements) while unobserved nodes can be associated with systems of any physical theory (classical, quantum or post-quantum). The directed edges \rightsquigarrow are understood as direct causal influences between the systems involved.

Causal models are then formulated in terms of a directed graph \mathcal{G} as above over a set of nodes N , together with a probability distribution $P_{\mathcal{G}}$ over a set of RVs, which are in 1-to-1 correspondence to the observed nodes $N_{\text{obs}} \subseteq N$, where $P_{\mathcal{G}}$ satisfies a linking property (the *d-separation property*) relative to \mathcal{G} . Going forward, we will therefore refer to these RVs as observed RVs and drop the distinction between them and their associated nodes.

Notation 1 *When considering sets of nodes or RVs $N_1, N_2 \subset N$ as well as individual nodes $X \in N$, we will generally denote $X \cong \{X\}$ and $N_1 N_2 = N_1 \cup N_2$.*

The idea behind this linking property is that graph separation relations between sets of observed nodes in \mathcal{G} should imply conditional independences between corresponding variables in the observed distribution $P_{\mathcal{G}}$. The graph separation criterion used for this purpose is called *d-separation* (directed

separation), which is a standard measure used in the classical [2, 23, 24], quantum [8, 9, 22, 25] and post-quantum [26] causal modelling literature. If two sets of nodes N_1 and N_2 in \mathcal{G} are d-separated given a third set N_3 , potentially empty, we will write $(N_1 \perp^d N_2 | N_3)_{\mathcal{G}}$. This entails the idea that certain kinds of paths between N_1 and N_2 are blocked by N_3 . Conditional independence is defined as follows.

Definition 3.1 (Conditional Independence) *Let \mathcal{G} be a causal structure associated with a probability distribution $P_{\mathcal{G}}$. For three disjoint sets of RVs A, B, C with A and B non-empty, we say that A and B are conditionally independent given C , denoted as $(A \perp\!\!\!\perp B | C)_{\mathcal{G}}$, if the marginal of $P_{\mathcal{G}}$ on these variables satisfies $P_{\mathcal{G}}(AB | C) = P_{\mathcal{G}}(A | C)P_{\mathcal{G}}(B | C)$.*

Then the *d-separation property* entails that whenever X_1, X_2 and X_3 are disjoint sets of RVs associated with observed nodes of \mathcal{G} , then

$$(X_1 \perp^d X_2 | X_3)_{\mathcal{G}} \implies (X_1 \perp\!\!\!\perp X_2 | X_3)_{\mathcal{G}} \quad (1)$$

In particular, when $X_3 = \emptyset$, we write $(X_1 \perp^d X_2)_{\mathcal{G}} \implies (X_1 \perp\!\!\!\perp X_2)_{\mathcal{G}}$. When X_1 and X_2 are individual RVs, $(X_1 \perp^d X_2)_{\mathcal{G}}$ is equivalent to saying that neither of them is a cause of the other (i.e. there are no directed paths $X_1 \rightsquigarrow \dots \rightsquigarrow X_2$ or vice versa, connecting them in \mathcal{G}) and they share no common ancestor (no common node C which has directed paths to both) in the graph. This encapsulates the Reichenbach principle of common cause [27], which fundamentally asserts that correlations between events must have some underlying causal explanation (in terms of one event being a cause of another, or common causes).

Intuitively, the causal structure captures the flow of information through the network of nodes. How this flow of information is modelled specifically – the *causal mechanisms* – is dependent on the respective theory. For example, in classical probabilistic theories, for each node, the mechanisms are provided by the conditional distributions $P(N | \text{par}(N))$. These can be thought of as classical channels or stochastic maps, and in deterministic theories, we would have functions for each node instead, $f_N : \text{par}(N) \mapsto N$. Analogously, in quantum causal models, we can refer to conditional density operators $\rho_{N | \text{par}(N)}$ to characterize the causal mechanisms, these are representations of quantum channels [8, 25].

Commonly, the literature adopts a *bottom-up* approach to causal modelling (e.g., [2, 8, 9, 22, 26]) which starts with assumptions on the causal mechanisms, deriving conditions (like d-separation) on the observed distributions. These often focus on faithful (not fine-tuned) and acyclic causal models³. These approaches have proven useful for characterising the nature of causation and correlations in specific theories, like quantum theory. On the other hand, the affects framework [10] pursues a *top-down* approach that applies to any causal mechanisms which satisfy the d-separation property on a level of observed correlations. While it assumes the existence of a causal structure \mathcal{G} (not necessarily acyclic), it does not assume a specific meaning to the causal arrows \rightsquigarrow , and can be applied to rather general notions of causality in different theories. This makes it well-suited for deriving general results on causal inference and signalling or generic impossibility results for a wide class of theories.

Further, the concept of fine-tuning in a causal model will be important in this work as we do not assume its absence. A causal model is considered *unfaithful* or *fine-tuned* if the converse implication of Equation (1) is not satisfied for some disjoint sets of observed RVs X_1, X_2 and X_3 :

$$(X_1 \perp^d X_2 | X_3)_{\mathcal{G}} \not\Leftarrow (X_1 \perp\!\!\!\perp X_2 | X_3)_{\mathcal{G}} \quad (2)$$

This captures the idea that independences in the distribution faithfully reflect the connectivity of the causal graph. This can fail in causal models where the underlying causal mechanisms are fine-tuned to hide certain causal connections from being detectable through probabilistic dependences. Usually, this property is only evaluated in the original causal model (associated with \mathcal{G}). Within this work, we extend this notion to also account for interventions (described in the next section), which alter the graph \mathcal{G} in a specific manner.

3.2 Interventions and (higher-order) affects relations

Generally, the same observed probability distribution can admit several different causal explanations. This fundamentally arises from the fact that correlations are symmetric, while causation, as represented

³See [25, 28] for recent developments in bottom-up approaches for cyclic causal models.

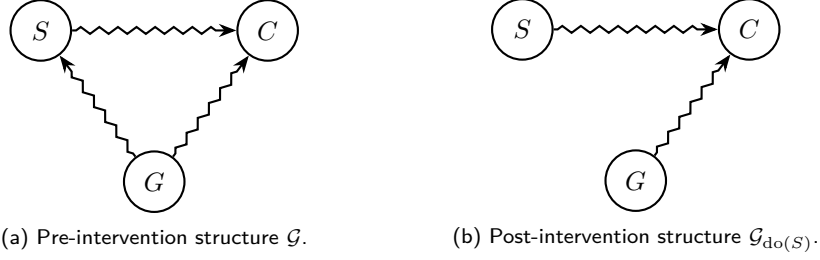


Figure 1: Starting from the original (pre-intervention) causal structure, an intervention on the node S is performed by removing all its incoming edges, yielding the post-intervention causal structure $\mathcal{G}_{\text{do}(S)}$. For the associated distributions, an intervention then corresponds to fixing S to fix a certain value $S = s$ in the post-intervention causal model.

in the causal structure, is an asymmetric, directed relation. For instance, the existence of correlation between the prevalence of smoking (modelled as a variable S) and the incidence of cancer (modelled as a variable C) in a population does not necessarily allow to infer that smoking is a cause of cancer, because it is possible to have a common factor G (such as a genetic predisposition) that influences both an individual’s likelihood to (become addicted to) smoking and the onset of cancer, and could thereby explain the same correlations.

Therefore, to deduce causal relations, we need to supplement the correlations, collected through *passive observation* of the variables involved, with *free interventions*, which also actively control the variables. Indeed, such interventions form the basis of controlled trials, for instance where one wishes to deduce if a drug causes recovery from a disease.

In Figure 1, we illustrate how this idea is captured within causal models. In our example, to infer whether S is a cause of C , one performs an intervention on S , denoted as $\text{do}(S)$. Such an intervention would correspond to actively forcing people to start or quit smoking, which may be practically infeasible due to ethical reasons, but can be considered in theory. This amounts to removing all incoming arrows of S , and fixing the RV to a particular value. This captures the basic assumption of such experiments, that the intervention choice is independent of prior causes of S , such as the common genetic factor G . The result is a post-intervention causal model, built from the post-intervention causal structure $\mathcal{G}_{\text{do}(S)}$ and the associated post-intervention distribution $P_{\mathcal{G}_{\text{do}(S)}}$. Then, by comparing if the post-intervention distribution $P_{\mathcal{G}_{\text{do}(S)}}(C|S)$ is distinct from the original distribution $P_{\mathcal{G}}(C)$, we can infer whether or not S is a cause of C . We would like to highlight that $P_{\mathcal{G}_{\text{do}(S)}}$ is related to but can generally not be fully inferred from the pre-intervention distribution $P_{\mathcal{G}}$ alone [2, 10].

This notion of interventions is independent of the theory under consideration and its respective causal mechanisms. Operationally, this allows to capture a notion of *signalling*: If an operation performed at node X , encoded in the choice of intervention $\text{do}(X)$, yields an observably distinct distribution at another node Y , we can understand this as X signalling to Y . More generally, we can consider an additional set of RVs Z and ask whether X can signal to Y given that some interventions have been performed on Z . This idea is formalised through *affects relations*, where Z being trivial or non-trivial delineates 0^{th} -order and higher-order affects relations.

Definition 3.2 ((Unconditional Higher-Order) Affects Relations) Consider a causal model over a set of S observed nodes, associated with a causal structure \mathcal{G} . For pairwise disjoint subsets $X, Y, Z \subset S$, with X, Y non-empty, we say

$$X \text{ affects } Y \text{ given } \text{do}(Z), \quad (3)$$

which we alternatively denote as

$$X \vDash Y \mid \text{do}(Z), \quad (4)$$

if there exist values x of X and z of Z such that

$$P_{\mathcal{G}_{\text{do}(XZ)}}(Y|X = x, Z = z) \neq P_{\mathcal{G}_{\text{do}(Z)}}(Y|Z = z) \quad (5)$$

If $Z \neq \emptyset$, we have a higher-order (HO) affects relation. More specifically, it is also called a $|Z|^{\text{th}}$ -order affects relation.

We understand affects relations as the *formalization of signalling* to be used in the remainder of this work. We will contrast this definition with notions from previous literature in Section 3.3, exploring possibilities for signalling beyond correlations.

Note that for $X := S$, $Y := C$ and $Z = \emptyset$, this definition recovers the intuition for signalling explained for the particular example of smoking and cancer.

Even more generally, one can consider conditional affects relations $X \vDash Y \mid \text{do}(Z), W$, which are conditioned on a set of RVs W that are not intervened upon. A definition of these will be provided in Appendix E. However, in [10] it is shown that each such conditional affects relation implies an unconditional affects relation $X \vDash YW$ (see also Lemma E.2), and we will therefore disregard conditional affects relations and only work with unconditional relations for the bulk of this work.

The next lemma highlights how affects relations can be used to infer information about the causal structure, and uses the following definition of *cause*.

Definition 3.3 (Cause) *We say that a set X of nodes is a cause of a set Y of nodes in a causal model if $\exists e_X \in X$ and $e_Y \in Y$ with a directed path $e_X \rightsquigarrow \dots \rightsquigarrow e_Y$ between them in the graph \mathcal{G} representing the causal structure of the model.*

Lemma 3.4 *Let S be a set of RVs in a causal model and $X, Y, Z \subset S$ disjoint. Then $X \vDash Y \mid \text{do}(Z) \implies X$ is a cause of Y .*

We have seen how a given causal model on a graph \mathcal{G} can be associated with a family of post-intervention causal models associated with the derived graphs $\mathcal{G}_{\text{do}(S)}$. Accordingly, we can extend the definition of fine-tuning from Equation (2) and in this work, we will say that a causal model is fine-tuned if the non-implication of Equation (2) holds in any post-intervention causal model obtained from the original one. Further details are found in Appendix A.

3.3 Affects relations capture signalling beyond correlations

In the quantum information community, the notion of signalling is most commonly discussed in the context of protocols where the measurement settings of parties are freely chosen and modelled as parentless variables. A parentless variable, say X (a setting) is said to signal to some outcome variable, say A if there exist distinct values x and x' of X and a value a of A such that $P(A = a \mid X = x) \neq P(A = a \mid X = x')$ (which we denote as $P(A \mid X = x) \neq P(A \mid X = x')$). This condition encompasses precisely what it means for A to be correlated with X , i.e., for parentless variables (or sets of parentless variables) X , correlation with another variable (or a set of variables) A is equivalent to signalling.

Based on this, we consider the case where (some of) the variables in X have non-trivial parents. In this case, by means of interventions on X , a natural definition of signalling one might consider is the following.

$$\exists x, x' : P_{\mathcal{G}_{\text{do}(X)}}(Y \mid X = x) \neq P_{\mathcal{G}_{\text{do}(X)}}(Y \mid X = x') \quad (6)$$

This is equivalent to saying that X and Y are correlated in the post-intervention causal model on $\mathcal{G}_{\text{do}(X)}$. Concepts such as average causal effect used in operationally identifying causation in classical and non-classical causal models follow a similar definition [2, 29]. It is shown in [10] (Lemma IV.2) that this implies $X \vDash Y$, but the converse is not true (Example IV.1). This illustrates that affects relations strictly capture more general ways of signalling than the alternative definition for signalling given in Equation (6) informed by the previous literature.

This generality of affects relations has a clear operational significance: the alternative definition exclusively captures the possibility of signalling by means of two distinct interventions on X , while affects relations additionally capture the additional possibility of signalling through the very fact that some non-trivial intervention was performed as opposed to passive observation (even when the exact choice of intervention does not matter).

These two cases can only diverge when X is not parentless. In this case, we can have

$$\forall x \text{ of } X : P_{\mathcal{G}_{\text{do}(X)}}(Y \mid X = x) = P_{\mathcal{G}_{\text{do}(X)}}(Y) \neq P_{\mathcal{G}}(Y), \quad (7)$$

yielding $X \vDash Y$ although X and Y are uncorrelated in $\mathcal{G}_{\text{do}(X)}$. The same is true for higher-order affects relations, as illustrated in Example 4 where we will have $X \vDash Y \mid \text{do}(Z)$ even though $P_{\mathcal{G}_{\text{do}(XZ)}}(Y \mid XZ) = P_{\mathcal{G}_{\text{do}(XZ)}}(Y \mid Z)$.

4 Characterization of signalling in the affects framework

As discussed in the previous section, affects relations formalize general possibilities for signalling in a theory-independent manner, while accounting for signalling via interventions on variables that have parents in the causal structure, and capturing operational possibilities for signalling that are not encoded in correlations alone.

However, affects relations can still contain certain redundancies. In this section, we address this issue by providing an operationally motivated notion of irreducibility for each of the four arguments of an affects relation, generalizing on the notion for the first argument introduced in [10]. Irreducibility of an affects relation in a given argument will capture the absence of any reduced affects relation (where that argument is replaced by a strict subset thereof) that is operationally equivalent to the original relation. We also introduce a related notion called clustering for each argument of an affects relations, which formalizes the absence of such reduced affects relations altogether. For instance, $X_1 X_2 \vDash Y$ but $X_1 \not\vDash Y$, $X_2 \not\vDash Y$ is an example of clustering in the first argument.

Having defined these new and operationally motivated concepts, we link irreducibility and clustering, showing the application of these concepts for detecting causal fine-tuning using affects relations, and we conclude by exploring implications of these properties for causal inference. In doing so, we will employ the relations between conditional dependencies and affects relations encapsulated in the following lemma.

Lemma 4.1 $X \not\vDash Y | \text{do}(Z) \implies (X \perp\!\!\!\perp Y | Z)_{\mathcal{G}_{\text{do}}(XZ)}$.

Going forward, we will focus on the case of unconditional affects relations, and accordingly, on the first three arguments (X , Y and Z), as these admit the most useful intuition. This is due the fact that (by Lemma E.2) for each conditional affects relation $X \vDash Y | \text{do}(Z), W$, we can infer an unconditional affects relation $X \vDash YW | \text{do}(Z)$, which is equivalent in terms of causal inference and will allow the application of the concepts of irreducibility and clustering in the second argument to the full set YW . However, most results of this section fully generalize when applied to conditional affects relations, as explicitly carried out in Appendix E. In the same appendix, a detailed treatment of the properties of clustering and irreducibility in the additional fourth argument of such conditional affects relations is also provided.

4.1 Reducibility of affects relations in different arguments

In this section, we will define multiple concepts relating to the absence of certain affects relations and the presence of certain others. It is to be understood that these concepts are always defined relative to some given set \mathcal{A} of affects relations. If a causal model is specified, then \mathcal{A} is the set of all affects relations in the model, otherwise the set \mathcal{A} must be explicitly specified when applying these concepts.

Definition 4.2 (Reducibility in the first argument [10]) *We say that an affects relation $X \vDash Y | \text{do}(Z)$ is reducible in the first argument (or Red_1) if there exists a non-empty subset $s_X \subsetneq X$ such that $s_X \vDash Y | \text{do}(\tilde{s}_X Z)$, where $\tilde{s}_X := X \setminus s_X$. Otherwise, we say that $X \vDash Y | \text{do}(Z)$ is irreducible in the first argument and denote it as Irred_1 .*

Using this definition, the following lemma is proven in [10].

Lemma 4.3 *If $X \vDash Y | \text{do}(Z)$ is a Red_1 affects relation, then there exists $\tilde{s}_X \subsetneq X$ such that $\tilde{s}_X \vDash Y | \text{do}(Z)$ holds.*

Operational motivation for definition Red_1 captures the idea that the original affects relation $X \vDash Y | \text{do}(Z)$ and the reduced affects relation $\tilde{s}_X \vDash Y | \text{do}(Z)$ (for $\tilde{s}_X \subsetneq X$) carry the same information. Writing out these affects relations, we have

$$\begin{aligned} P_{\mathcal{G}_{\text{do}}(XZ)}(Y|XZ) &\neq P_{\mathcal{G}_{\text{do}}(Z)}(Y|Z) \\ P_{\mathcal{G}_{\text{do}}(\tilde{s}_X Z)}(Y|\tilde{s}_X Z) &\neq P_{\mathcal{G}_{\text{do}}(Z)}(Y|Z) \end{aligned} \tag{8}$$

Notice that the right hand side of the two expressions are the same, and Red_1 requires the left hand sides to be identical (which is equivalent to saying that $s_X \vDash Y | \text{do}(\tilde{s}_X Z)$). Once this is imposed, the two affects relations carry the same information, and are expressed by equivalent expressions.

Example 1 (*Motivating example for Red₁*) Consider the simple causal model where X_1 is a cause of Y with $Y = X_1$, while X_2 is a additional causally disconnected node. Here we would expect $X_1X_2 \models Y$ to be reducible to $X_1 \models Y$ as X_2 is clearly redundant. Indeed this is the case using the above definition, as we have $X_2 \not\models Y | \text{do}(X_1)$ in this example.

We apply a similar logic to define reducibility in the remaining arguments. We start with the third argument as its operational motivation is closer to Red₁, since both the first and third arguments correspond to nodes on which active interventions have been performed.

Definition 4.4 (Reducibility in the third argument) We say that an affects relation $X \models Y | \text{do}(Z)$ is reducible in the third argument (or Red₃) if there exists a non-empty subset $s_Z \subseteq Z$ such that both the following conditions hold, where $\tilde{s}_Z := Z \setminus s_Z$

- $s_Z \not\models Y | \text{do}(X\tilde{s}_Z)$
- $s_Z \not\models Y | \text{do}(\tilde{s}_Z)$

Otherwise, we say that $X \models Y | \text{do}(Z)$ is irreducible in the third argument and denote it as Irred₃.

Lemma 4.5 If $X \models Y | \text{do}(Z)$ is a Red₃ affects relation, then there exists $\tilde{s}_Z \subsetneq Z$ such that $X \models Y | \text{do}(\tilde{s}_Z)$ holds.

Operational motivation for the definition As with the case of Red₁, Red₃ captures that the original affects relation $X \models Y | \text{do}(Z)$ and the reduced (in the third argument) affects relation $X \models Y | \text{do}(\tilde{s}_Z)$, for $\tilde{s}_Z \subsetneq Z$ carry the same information, while ensuring that it recovers the expected notion of redundancy in simple test examples. Noting that the original and reduced relations are equivalent to the following two conditions respectively,

$$\begin{aligned} P_{\mathcal{G}_{\text{do}(XZ)}}(Y|XZ) &\neq P_{\mathcal{G}_{\text{do}(Z)}}(Y|Z), \\ P_{\mathcal{G}_{\text{do}(X\tilde{s}_Z)}}(Y|X\tilde{s}_Z) &\neq P_{\mathcal{G}_{\text{do}(\tilde{s}_Z)}}(Y|\tilde{s}_Z). \end{aligned} \tag{9}$$

We observe that the two conditions in Definition 4.4 ensure that the left-hand sides and right-hand sides of the expressions for the original and reduced affects relations are identical. Specifically, this corresponds to $s_Z \not\models Y | \text{do}(\tilde{s}_Z)$ and $s_Z \not\models Y | \text{do}(\tilde{s}_ZX)$. Unlike the case of Red₁, where the right-hand sides of the expressions for the original and reduced affects relations are equal by default, here we require two conditions because this equality does not hold by default in Red₃. Consequently, an additional level of subtlety arises. In the above, we equate the two left-hand sides and the two right-hand sides of the relevant expressions. This corresponds to equating the respective pre- and post-intervention distributions of the two affects relations. One might consider an alternative approach, equating the left-hand side of one expression with the right-hand side of another, to derive a different condition for reducibility in the third argument, which would also imply the equivalence of the expressions in Equation (9). However, in doing so, we would disregard the information specifying which distribution is pre-/post-intervention, even though it is operationally accessible via the set of nodes intervened upon. Therefore, a more natural definition of operational equivalence of the original and reduced affects relation is the former one which account for this additional, operationally accessible information. We can further confirm this argument with a simple test example for ruling out the alternative definition.

Example 2 (*Motivating example for Red₃*) Let X , Y , and Z be individual random variables where X is a cause of Y and Z is a causally disconnected node. In this scenario, we have $X \models Y | \text{do}(Z)$ and $X \models Y$ in general, but we would expect the former to be reducible to the latter since Z is entirely redundant. This is indeed the case by Definition 4.4 because $Z \not\models Y$ and $Z \not\models Y | \text{do}(X)$. However, the alternative definition mentioned above, which mixes the pre- and post-intervention distributions, would incorrectly classify $X \models Y | \text{do}(Z)$ as irreducible despite Z 's redundancy.⁴

⁴One could nonetheless consider a definition of reducibility based on the logical OR of the two conditions discussed, which are two ways by which the expressions in Equation (9) can be equivalent. This alternative would classify a smaller subset of affects relations as irreducible compared to Definition 4.4, thus limiting the generality of our main results. The current results, which use Definition 4.4 to explore the consequences of Irred₃ affects relations, are more general and would imply the same for this alternative definition.

The following examples both illustrate affects relations $X \vDash Y|\text{do}(Z)$ which are Irred_3 , but show that the two conditions of Definition 4.4 can be independently violated.⁵

Example 3 (Violating the first Red_3 condition) Consider a one-time pad over binary variables where $X \rightsquigarrow Y$ and $Z \rightsquigarrow Y$, with X and Z uniformly distributed and $Y = X \oplus Z$. Here, we have $X \vDash Y|\text{do}(Z)$, $Z \not\vDash Y$ and $Z \vDash Y|\text{do}(X)$. Z is a singleton and hence the only possible non-empty subset s_Z here is Z itself. Thus $X \vDash Y|\text{do}(Z)$ is an Irred_3 affects relation, it violates the the first condition of Definition 4.4 for $s_Z = Z$ (since $Z \vDash Y|\text{do}(X)$). However it satisfies the second condition for the same s_Z , (since $Z \not\vDash Y$).

Example 4 (Violating the second Red_3 condition) This is identical to Example IV.4 from [10] and illustrated in Figure 2. Consider the causal structure \mathcal{G} of Figure 2a with all nodes being binary variables, and the causal model where W is uniformly distributed, $Y = X \oplus Z \oplus W$, $Z = X$ and $X = W$. This gives us $Y = W$ in \mathcal{G} with $P_{\mathcal{G}}(Y) = P_{\mathcal{G}}(W)$ being uniform. Consider an intervention on X associated with the post-intervention graph $\mathcal{G}_{\text{do}(X)}$ of Figure 2b. Here, we no longer have $X = W$ (but the remaining functional dependences hold). We still have $Y = W$ (since $Z = X$) and this tells us that $P_{\mathcal{G}_{\text{do}(X)}}(Y|X)$ is also uniform, independently of the value of X . Under interventions on Z , we obtain the graph $\mathcal{G}_{\text{do}(Z)}$ of Figure 2c where $Z = X$ no longer holds. We obtain $Y = Z$ in this graph since $X = W$ which implies that $P_{\mathcal{G}_{\text{do}(Z)}}(Y|Z)$ is deterministic. Since $Y = X \oplus Z \oplus W$ still holds, this tells us that $Z \vDash Y$. Finally, under joint intervention on X and Z , we obtain the graph $\mathcal{G}_{\text{do}(XZ)}$ of Figure 2d where we neither have $Z = X$ nor $X = W$. It is easy to see that $P_{\mathcal{G}_{\text{do}(XZ)}}(Y|XZ)$ is then uniform, independent of the values of X and Z , since W is uniform. Since $P_{\mathcal{G}_{\text{do}(XZ)}}(Y|XZ)$ differs from $P_{\mathcal{G}_{\text{do}(Z)}}(Y|Z)$ but not from $P_{\mathcal{G}_{\text{do}(X)}}(Y|X)$, we have $X \vDash Y|\text{do}(Z)$ and $Z \not\vDash Y|\text{do}(X)$. Again, as in Example 3, the only possible non-empty subset s_Z of Z is Z itself, and the affects relation $X \vDash Y|\text{do}(Z)$ of this example is also Irred_3 , but in this case, it satisfies the first condition (since $Z \not\vDash Y|\text{do}(X)$) but violates the second condition (since $Z \vDash Y$) of Definition 4.4.

Clearly Z is non-redundant in the affects relation $X \vDash Y|\text{do}(Z)$ in both of the above examples, and these examples independently motivate the relevance of both conditions of Definition 4.4.

We conclude by introducing the notion of affects relations that are reducible in their second argument. In analogy to the fact that higher-order affects relations are required for defining Red_1 even for a 0^{th} -order relation $X \vDash Y$ [10], we require conditional affects relations (Definition E.1) to define Red_2 for unconditional relations. Moreover, as the second argument of an affects relations emerges in the main argument of the associated probability distributions (as opposed to other arguments which appear as the conditionals), we additionally gain a condition on conditional independence (denoted by $\perp\!\!\!\perp$, cf. Definition 3.1). We define Red_2 below and subsequently motivate the definition further.

Definition 4.6 (Reducibility in the second argument) We say that an affects relation $X \vDash Y|\text{do}(Z)$ is reducible in the second argument (or Red_2) if there exists a non-empty subset $s_Y \subsetneq Y$ such that both the following conditions hold, where $\tilde{s}_Y := Y \setminus s_Y$.

- $X \not\vDash s_Y|\text{do}(Z)$, \tilde{s}_Y (cf. Definition E.1)
- $(s_Y \perp\!\!\!\perp \tilde{s}_Y|XZ)_{\mathcal{G}_{\text{do}(XZ)}}$ or $(s_Y \perp\!\!\!\perp \tilde{s}_Y|Z)_{\mathcal{G}_{\text{do}(Z)}}$

Otherwise, we say that $X \vDash Y|\text{do}(Z)$ is irreducible in the second argument and denote it as Irred_2 .

Lemma 4.7 If $X \vDash Y|\text{do}(Z)$ is a Red_2 affects relation, then there exists $\tilde{s}_Y \subsetneq Y$ such that $X \vDash \tilde{s}_Y|\text{do}(Z)$ holds.

Operational motivation for the definition As before, the Red_2 property of an affects relation $X \vDash Y|\text{do}(Z)$ captures that it encodes the same information as an affects relation $X \vDash \tilde{s}_Y|\text{do}(Z)$ for a strictly smaller second argument $\tilde{s}_Y \subsetneq Y$. For example, consider a causal model over X , Y_1 and Y_2 where $X \rightsquigarrow Y_1$ is the only edge and we have $Y_1 = X$. Then for any distribution over Y_2 and X , we have $X \vDash Y_1$ and $X \vDash Y_1 Y_2$. However, we know that Y_2 is entirely superficial in this example, and would expect that

⁵For general causal models, that is. For faithful causal models, we demonstrate in Appendix D that the first condition implies the second one.

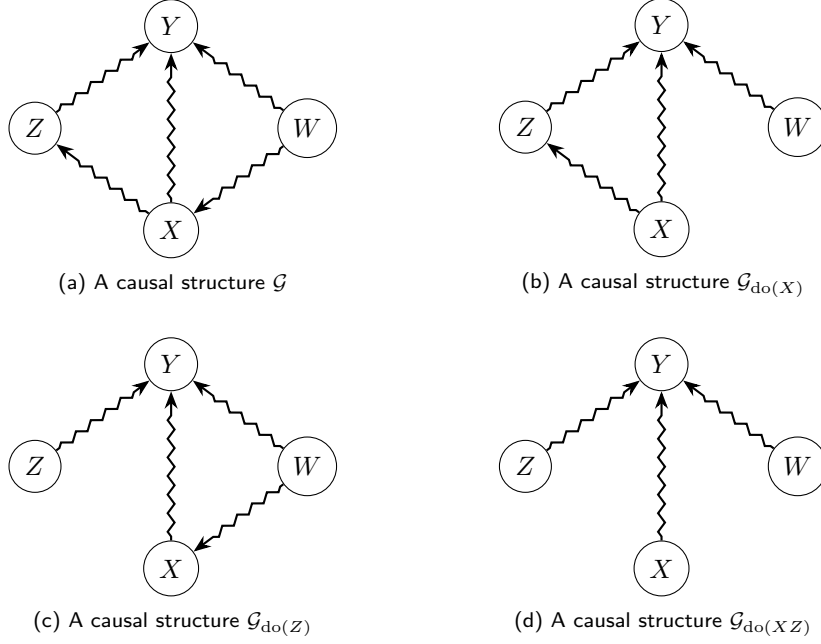


Figure 2: Pre and post-intervention causal structures for Example 4.

the latter affects relation should be reducible to the former one. Expressing the two relations explicitly, we have

$$\begin{aligned} P_{\mathcal{G}_{\text{do}(X)}}(Y_1|X) &\neq P_{\mathcal{G}}(Y_1), \\ P_{\mathcal{G}_{\text{do}(X)}}(Y_1Y_2|X) &\neq P_{\mathcal{G}}(Y_1Y_2) \end{aligned} \quad (10)$$

Notice however that in this case, it does not make sense to simply equate the left and right hand sides of the expressions for the original and reduced affects relations, as we did for Red_1 and Red_3 . Even in our simple example, $P_{\mathcal{G}_{\text{do}(X)}}(Y_1|X) \neq P_{\mathcal{G}_{\text{do}(X)}}(Y_1Y_2|X)$ and $P_{\mathcal{G}}(Y_1Y_2) \neq P_{\mathcal{G}}(Y_1)$. Therefore, the motivation for the definition of Irred_2 is more complex than for the other cases. Moreover, Irred_2 will not be as central to our results as Irred_1 and Irred_3 . Therefore, we conclude with a final example and defer further details regarding the rationale and examples for Irred_2 to Appendix C.

Example 5 (*Motivating example for Red_2*) The concept of jamming was first introduced in [30] as a new form of post-quantum correlation in the context of a tripartite Bell scenario. Suppose that Y is the (freely chosen) setting of a party Bob while A and C are outcomes of the remaining two parties Alice and Charlie. Jamming correlations allow for Y to jointly signal to A and C (i.e., $P(AC|Y) \neq P(AC)$), without signalling individually i.e., $P(A|Y) = P(A)$, $P(C|Y) = P(C)$.⁶

In [12, 13, 31] explicit causal models for such jamming scenarios have been proposed and analyzed in the affects framework. In the simplest case using the three variables mentioned here, the following causal model exhibits jamming, where the signalling is now expressed through affects relations. We have $Y \rightsquigarrow A$, $\Lambda \rightsquigarrow A$ and $\Lambda \rightsquigarrow C$ with $P(\Lambda)$ and $P(Y)$ being uniform, $A = \Lambda \oplus Y$ and $C = \Lambda$. Here, $Y \vDash AC$ but $Y \not\vDash A$ and $Y \not\vDash C$. However both $Y \vDash A | C$ and $Y \vDash C | A$ hold. Therefore, $Y \vDash AC$ is Irred_2 .

4.2 Clustering of affects relations: detecting and classifying fine-tuning

So far we have focused on the reducibility and irreducibility of affects relations, and seen that an affects relation $X \vDash Y|\text{do}(Z)$ which is reducible in any of the arguments X , Y or Z implies a reduced affects relation of the same form where the corresponding argument is replaced by a strict subset of it. Conversely,

⁶The motivation for this is that when we embed the tripartite scenario in spacetime such that the three measurements are spacelike separated and Y 's future contains the joint future of A and C , then the joint signalling from Y to AC can only be verified within Y 's future. Therefore, it cannot lead to superluminal communication since Y does not individually signal to the spacelike variables.

how does the irreducibility of $X \models Y | \text{do}(Z)$ in a given argument relate to the existence/non-existence of affects relations of the same form involving strict subsets of that argument? To study this, we introduce a related property to irreducibility, which also applies to different arguments and which we call clustering. It captures the property of an affects relation to hold without the presence of any reduced affects relations (involving strict subsets of a given argument).

Definition 4.8 (Clustering in the first argument) *An affects relation $X \models Y | \text{do}(Z)$ is called clustered in the first argument (denoted Clus_1) if $|X| \geq 2$ and there exists no $s_X \subsetneq X$ such that $s_X \models Y | \text{do}(Z)$.*

Definition 4.9 (Clustering in the second argument) *An affects relation $X \models Y | \text{do}(Z)$ is called clustered in the second argument (denoted Clus_2) if $|Y| \geq 2$ and there exists no $s_Y \subsetneq Y$ such that $X \models s_Y | \text{do}(Z)$.*

Definition 4.10 (Clustering in the third argument) *An affects relation $X \models Y | \text{do}(Z)$ is called clustered in the third argument (denoted Clus_3) if $|Z| \geq 1$ and there exists no $s_Z \subsetneq Z$ such that $X \models Y | \text{do}(s_Z)$.*

We note that the third argument is treated slightly differently from the first two arguments, because for any affects relation the latter must necessarily be non-empty while the former need not. The following theorem then shows that clustered affects relations constitute a special case of irreducibility.

Theorem 4.11 [Clustering implies irreducibility] *For any affects relation $X \models Y | \text{do}(Z)$, $\text{Clus}_i \Rightarrow \text{Irred}_i$ for all $i \in \{1, 2, 3\}$.*

While clustering implies irreducibility, the following examples show that the converse is not true. The conclusions of these examples hold for all choices of distributions over the parentless nodes, and we therefore do not specify them.

Example 6 ($\text{Irred}_1 \not\Rightarrow \text{Clus}_1$) *Consider a causal model over the nodes X_1, X_2 and Y with edges $X_1 \rightsquigarrow Y$ and $X_2 \rightsquigarrow Y$ with $Y = X_1 \cdot X_2$, where \cdot represents binary AND. Let X_1 and X_2 be non-deterministically distributed. We then have $X_1 \models Y$ and $X_1 X_2 \models Y$, which implies that $X_1 X_2 \models Y$ does not satisfy the Clus_1 property. However $X_1 X_2 \models Y$ is an Irred_1 affects relation since $X_1 \models Y | \text{do}(X_2)$ and $X_2 \models Y | \text{do}(X_1)$ both hold.*

Example 7 ($\text{Irred}_2 \not\Rightarrow \text{Clus}_2$) *Consider a causal model over the nodes X, Y_1 and Y_2 which are associated binary variables, with edges $X \rightsquigarrow Y_1 \rightsquigarrow Y_2$ and $X \rightsquigarrow Y_2$ with $Y_1 = X$ and $Y_2 = Y_1 \cdot X$, where \cdot represents binary AND. Then we have $X \models Y_1$ which implies that $X \models Y_1 Y_2$ does not satisfy the Clus_2 property. The only non-empty and strict subsets of $\{Y_1, Y_2\}$ are $\{Y_1\}$ and $\{Y_2\}$, but we have $X \models Y_1$ and $X \models Y_2$. Additionally, we observe that both $(Y_1 \not\models Y_2)_{\mathcal{G}}$ and $(Y_1 \not\models Y_2 | X)_{\mathcal{G}_{\text{do}(X)}}$, which means that the second condition of Definition 4.6 is always violated. Therefore $X \models Y_1 Y_2$ is an Irred_2 affects relation.*

Example 8 ($\text{Irred}_3 \not\Rightarrow \text{Clus}_3$) *Consider the causal model of Example 6. We have $X_1 \models Y$ and $X_1 \models Y | \text{do}(X_2)$ which implies that $X_1 \models Y | \text{do}(X_2)$ does not satisfy the Clus_3 property. However, $X_1 \models Y | \text{do}(X_2)$ does satisfy Irred_3 since $X_2 \models Y | \text{do}(X_1)$ and $X_2 \models Y$ both hold, which violate both conditions of Definition 4.4 for $s_{X_2} = X_2$ (which is the only possible non-empty $s_{X_2} \subseteq X_2$ in this case).*

In Appendix D, we will point out some further strong implications of affects relations being clustered. These will not be relevant for our main points on the fine-tuning of causal models, but will be used in some proofs later on. The next connection we show is between clustering and fine-tuning.

Theorem 4.12 [Clustering implies fine-tuning] *Any affects relation $X \models Y | \text{do}(Z)$ that satisfies Clus_i for any $i \in \{1, 2, 3\}$ necessarily arises from a fine-tuned causal model.*

Therefore, together with the aforementioned examples this theorem demonstrates that in contrast to clustering, irreducibility applies to both faithful and fine-tuned models. While we show that all other statements do, we will see in Appendix E.2 that this theorem does not fully generalize to the case of conditional affects relations (which include the additional argument W). There, we prove the generalization under an additional assumption (that X is not a cause of W in the underlying model) and conjecture that the full generalization holds.

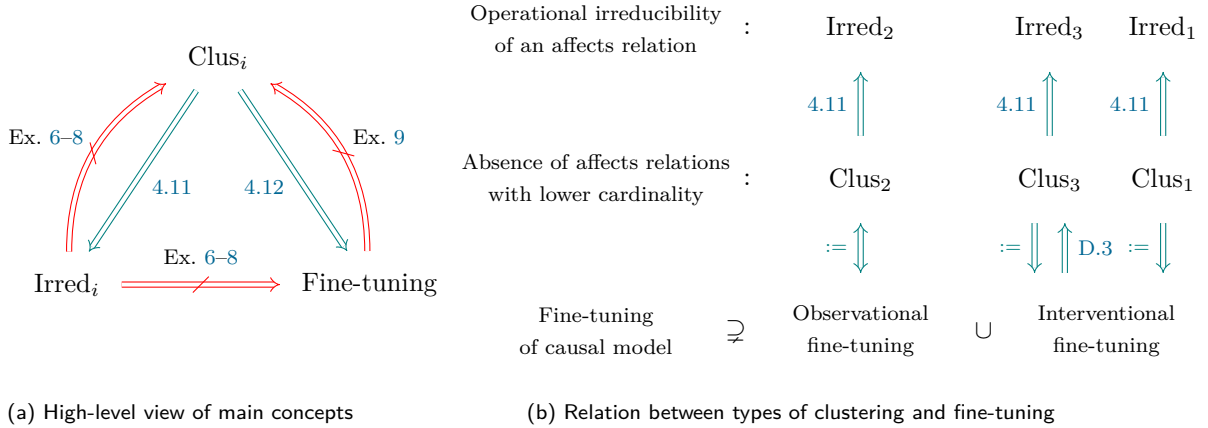


Figure 3: Overview of causal inference concepts and relationships given by our results. Generally, an affects relation $X \vDash Y \mid \text{do}(Z)$ may contain elements in X, Y, Z which are operationally redundant. While irreducibility (Irred_i) signifies that this is not the case (per i^{th} argument), clustering (Clus_i) additionally certifies that it is impossible to recover signalling by considering a subset of variables (per i^{th} argument). The latter is a signature of fine-tuning, which we furthermore use to propose a distinction into different types of fine-tuning. In (b), for every one-way implication shown, inequivalence holds. Most of the implications between clustering and the specific types of fine-tuning are by definition, as indicated by the annotation $:=$.

The connection between clustering and fine-tuning established above allow us to differentiate between three independent types of fine-tuning. For instance, we will demonstrate a distinction between *observational* fine-tuning, as indicated by Clus_2 , and *interventional* fine-tuning, as indicated by Clus_1 and Clus_3 , which could potentially be relevant to characterize information processing tasks.

If it is necessary to intervene on multiple nodes to demonstrate the presence of signalling, we encounter *interventional* fine-tuning, as encoded by Clus_1 or Clus_3 . In Appendix D we show that we can determine the presence of interventional fine-tuning entirely by the presence of affects relations with Clus_3 .

Regarding *observational* fine-tuning, we may consider the jamming causal model [12, 13, 31] presented in Example 5 where Y jams the correlations between A and C , yielding $Y \vDash AC$. This model exhibits clustering (only) in the second argument as it has $Y \not\vDash A$ and $Y \not\vDash C$.⁷ Therefore, in this case we need to observe multiple nodes in the pre- and post-intervention distribution to register the presence of signalling. In [31], it is shown that any causal model where the jamming variable Y (in [31], this variable is called B) is parentless, and which exhibits jamming must necessarily be fine-tuned, independently of the theory describing hidden common causes. Our Theorem 4.12 generalizes this previous result, since jamming is characterized by a Clus_2 relation (signalling to a set of RVs without signalling to a subset). Moreover, it also applies to situations beyond Bell-type scenarios⁸ where Y is not a freely chosen parentless variable, and to affects relations clustered in any argument. It provides a way to identify the fine-tuning of possible hidden causal parameters in general physical theories, solely from the observable affects relations.

Here, we note that the previous work considered only the original causal model in the definition of fine-tuning, while we also consider post-intervention causal models obtained from the original one. This means that potentially, a larger set of causal models are regarded as fine-tuned according to our definition as compared to the previous work, since one might in-principle have fine-tuning at the level of a post-intervention causal model even when the original pre-intervention model is faithful. Whether this is possible, however, remains an open question. See Definition A.6 for further details on fine-tuning.

Conversely, examples with all three types of clustering also exist: The one-time pad causal model given in Example 3 involves all three types of clustering. $XZ \vDash Y$ is a Clus_1 relation (since $X \not\vDash Y$ and $Z \not\vDash Y$), we also have $X \vDash YZ$ which is Clus_2 (since $X \not\vDash Y$ and $X \not\vDash Z$) and $X \vDash Y \mid \text{do}(Z)$ is Clus_3 (since $X \not\vDash Y$). Nonetheless, not every type of fine-tuning can be related to clustering in one way or another.

⁷Since this is the only affects relation, and it emanates from a single RV while having no third argument, it can be neither Clus_1 nor Clus_3 .

⁸Note that jamming was originally formulated in a Bell-type scenario, see Example 5.

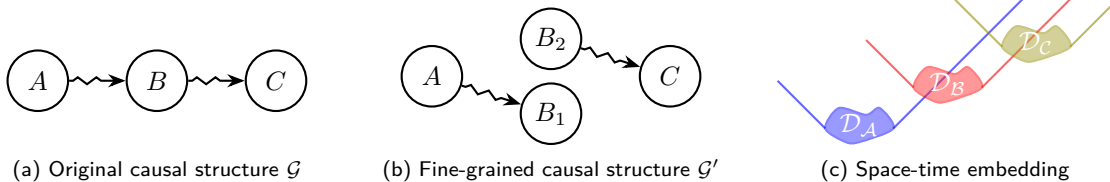


Figure 4: Example for a fine-graining of a causal structure, as given in Example 9, as well as its space-time embedding that corresponds to the configuration of Sorkin’s problem [32]. In contrast to our approach within Section 5, just for this illustration, we consider an embedding that associates each RV, say X , with a space-time region \mathcal{D}_X which has the relativistic future $\bar{\mathcal{F}}_{\text{reg}}(\mathcal{X}) := \{p \in \mathcal{T} \mid \exists q \in \mathcal{D}_X, q \preceq p\}$. We can order regions \mathcal{D}_X and \mathcal{D}_Y by considering whether the future of \mathcal{D}_X has an overlap with \mathcal{D}_Y [11]. However, this will not be a partial order, as it lacks transitivity: Even though we have $\bar{\mathcal{F}}_{\text{reg}}(\mathcal{A}) \cap \mathcal{D}_B \neq \emptyset$ and $\bar{\mathcal{F}}_{\text{reg}}(\mathcal{B}) \cap \mathcal{D}_C \neq \emptyset$, we still have $\bar{\mathcal{F}}_{\text{reg}}(\mathcal{A}) \cap \mathcal{D}_C = \emptyset$. Note that in the spatio-temporal case as well, analogous to the information-theoretic fine-graining, the “node” corresponding to the region \mathcal{D}_B can be fine-grained into two regions $\mathcal{D}_B = \mathcal{D}_{B_1} \cup \mathcal{D}_{B_2}$, where \mathcal{D}_{B_1} includes all points in \mathcal{D}_B that lie the future of \mathcal{D}_A and \mathcal{D}_{B_2} includes the remaining points in \mathcal{D}_B , such that $\bar{\mathcal{F}}_{\text{reg}}(\mathcal{B}_1) \cap \mathcal{D}_C = \emptyset$.

Example 9 (*Fine-tuning does not imply Clustering.*) Consider a causal model over $S = \{A, B, C\}$ associated with the causal structure of Figure 4a, where $A, C \in \{0, 1\}$ and $B \in \{00, 01, 10, 11\}$. The model is specified by an arbitrary exogenous distribution $P(A)$ and the conditional distributions $P(B|A)$, $P(C|B)$, where $P(B = i0|A = i) = P(B = i1|A = i) = \frac{1}{2}$ for $i \in \{0, 1\}$ (and remaining entries 0), and $P(C = j|B = 0j) = P(C = j|B = 1j) = 1$ for $j \in \{0, 1\}$ (and remaining entries 0). Notice that B can be fine-grained into two nodes $B = B_1 \times B_2$ with $B_1, B_2 \in \{0, 1\}$. Then the given causal model can be equivalently expressed as a fine-grained model on the causal structure of Figure 4b with $B_1 = A$ and $C = B_2$, with B_2 uniformly distributed and A distributed as $P(A)$. Then, relative to the coarse-grained scenario of Figure 4a, we have $A \not\perp^d C$, but $A \perp C$, and hence a fine-tuned causal model. Accordingly, we have $A \models B$ and $B \models C$, but $A \not\models C$ (and $A \not\models C \mid \text{do}(B)$). By doing so, we use a similar setup to [11], where they fine-grain causal structures to unravel cyclic causal structures into acyclic ones. Note that here, the fine-graining removes the fine-tuning of the model. Furthermore, this example bears some resemblance with the setup of the Sorkin problem [32, 33] in algebraic QFT, where signalling can not be transitive, as otherwise superluminal signalling would be possible. In this scenario, RVs are associated to regions rather than points of space-time, with access to the entire region generally required to extract the entire information associated with a RV.

This yields the aforementioned third type of fine-tuning, which is not captured by interventional or observational clustering of affects relations. This type of fine-tuning is also detectable from the affects relations, as it is certified through their non-transitivity.⁹ Here, this is due to the causal model being fine-grainable. Do clustering and fine-grainability cover all operationally detectable fine-tunings? We leave these questions for future work.

4.3 Causal inference using irreducibility and clustering

Using irreducibility, one can obtain stronger causal inference results than given in Lemma 3.4. For Irred_1 the following lemma was shown in [10]. Here we use Definition 3.3 of *cause*.

Lemma 4.13 For any disjoint sets X, Y and Z of observed nodes $X \models Y \mid \text{do}(Z)$ is $\text{Irred}_1 \Rightarrow$ each $e_X \in X$ is a cause of at least one element $e_Y \in Y$.

Analogously, we obtain the following results for Irred_3 .

Lemma 4.14 For any disjoint sets X, Y and Z of observed nodes, $X \models Y \mid \text{do}(Z)$ is $\text{Irred}_3 \Rightarrow$ each $e_Z \in Z$ is a cause of at least one element $e_Y \in Y$.

Therefore, Irred_1 and Irred_3 complement each other, allowing to perform causal inference for individual nodes from all nodes that are intervened upon. For affects relations which are irreducible in both arguments we can summarize this to a single expression.

⁹Specifically, the non-transitivity of affects relations in our example certifies fine-tuning because $A \models B$ and $B \models C$ tells us that A is a cause of C (Corollary 4.15), which gives $A \not\perp^d C$. However, $A \not\models C$ here implies $A \perp C$ (Lemma 4.1), which implies fine-tuning (Equation (2)).

Corollary 4.15 For any disjoint sets X, Y and Z of observed nodes, if $X \vDash Y | \text{do}(Z)$ satisfies Irred_1 and Irred_3 then $\forall e_{XZ} \in XZ, \exists e_Y \in Y$ such that e_{XZ} is a cause of e_Y .

Combining this and Theorem 4.11 we obtain the following corollaries.

Corollary 4.16 For any disjoint sets X, Y and Z of observed nodes, $X \vDash Y | \text{do}(Z)$ is Clus_1 implies that each $e_X \in X$ is a cause of at least one element $e_Y \in Y$.

Corollary 4.17 For any disjoint sets X, Y and Z of observed nodes, $X \vDash Y | \text{do}(Z)$ is Clus_3 implies that each $e_Z \in Z$ is a cause of at least one element $e_Y \in Y$.

As clustering properties necessitate the absence of certain affects relations and also indicate fine-tuning (Theorem 4.12), this yields the interesting result that in certain fine-tuned causal models, we can use the *absence* of signalling between nodes for successful causal inference.

Furthermore, this raises the question whether we can use Irred_2 in a similar fashion to pinpoint individual nodes $e_Y \in Y$ to be caused by an affects relation $X \vDash Y | \text{do}(Z)$. However, the jamming scenario given in Example 5 shows that this is not the case: There, $Y \vDash AC$ and is Irred_2 (and Clus_2), yet Y is not a cause of A . However, A and C share a common cause distinct from Y . Therefore, for an Irred_2 affects relation $X \vDash Y | \text{do}(Z)$, while we cannot infer that every element of Y is an effect of some element of X , we can infer some d -connections between these elements, as shown below.

Lemma 4.18 For any disjoint sets X, Y and Z of observed nodes, $X \vDash Y | \text{do}(Z)$ is Irred_2 implies that both $(s_Y \not\perp^d \tilde{s}_Y | Z)_{\mathcal{G}_{\text{do}(Z)}}$ and $(X \not\perp^d s_Y | Z \tilde{s}_Y)_{\mathcal{G}_{\text{do}(X(\tilde{s}_Y)Z)}}$ for any partition of Y into subsets s_Y, \tilde{s}_Y . Here, $X(W) := X \setminus (X \cap \text{anc}(W))$ for $\text{anc}(W)$ denoting the set of all ancestors of W in the original graph \mathcal{G} .

5 Space-time as a partial order

To model the causal properties of space-time in this framework, we aim to be as general as possible. Therefore, we will model the causal structure of space-time as a partially ordered set (poset) \mathcal{T} of its points, as suggested by [34]. Thereby, we match the fundamental approach taken in [10], albeit analyzing the properties of the respective poset in more detail.

Modelling space-time as a poset allows us to study its causal structure in order-theoretic terms, while setting aside the additional mathematical structure which is usually assumed when studying space-time, such as the differential manifold structure and symmetries. However, Malament's theorem [35] shows that most of this structure, including topological properties, can be captured using partial order relations alone. The partial order relations between space-time points $a, b \in \mathcal{T}$ can be one of the following:

$$a = b, \quad a < b, \quad a > b, \quad a \not\prec b. \quad (11)$$

These correspond respectively to a and b being identical, a being in the causal past/future of b (and therefore time-like or light-like relative to each other), or causally unordered (and therefore space-like with regard to each other).

By allowing to order events in space-time in a transitive way, this approach is generic enough to model causality for an arbitrary space-time manifold without closed time-like curves (CTC), an example being Minkowski space-time depicted in Figure 5. Allowing for CTCs would allow for orderings of the form $b < a < b$, which would be incompatible with a partial order. Such space-times would be modelled as a *preorder* instead. However, this generality is not of interest here as it allows the causal future and the causal past of a point to be non-disjoint, and would thereby trivially allow the embedding of arbitrary causal models, as we will appreciate later on.

Further, it allows to study discrete generalizations of space-time [36], since we make no further assumptions on the properties of \mathcal{T} . Here, the most notable representative of the latter group are *causal sets* (causets) [37, 38], which are locally finite in addition to being posets.

The following definition captures the notion of causal future, matching literature conventions of the field of general relativity. There, it is commonly used to study causal structures on space-time manifolds [39].

Definition 5.1 (Causal Future and Causal Past) Let \mathcal{T} be a poset and $x \in \mathcal{T}$. Then the causal future of x is given by $J^+(a) := \{b \in \mathcal{T} : b \succeq a\}$. Dually, the causal past of x is given by $J^-(a) := \{b \in \mathcal{T} : b \preceq a\}$.

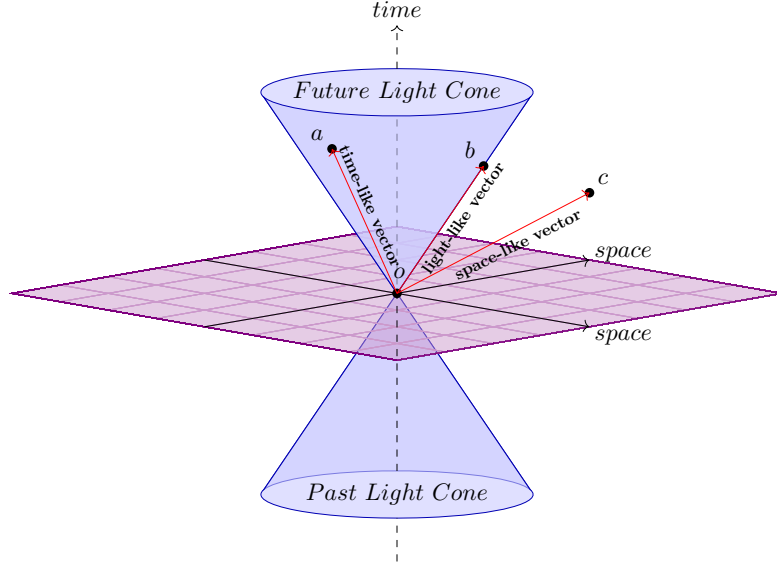


Figure 5: A light cone in 2+1-Minkowski space-time. For each point o , there exist time-like separated points a , in this case with $a \succ o$, space-like separated points c with $c \not\prec o$, which are unordered with regard to o , and light-like separated points, which are again ordered with respect to o : Here, $b \succ o$. Therefore, relative to o , there exist a region of space-like separated points and two regions of time-like separated points, which are separated by a surface of codimension 1 of points which are separated in a light-like way. Adapted from picture published by SandyG and Nick at [TeX Stack Exchange](#) and licensed under [CC BY-SA 4.0](#).

Of course, this definition applies equally well to posets \mathcal{T} which do not form a manifold. Informally, we will nonetheless generally refer to the causal future as a light cone.

While there is a wide variety of well-studied properties a poset may implement, they will not be of vital importance for the arguments of this work. We give a brief formal overview of posets and some of their properties in Appendix B, where we will also relate them to Minkowski space-time.

5.1 An order-theoretic property of physical space-times: conicality

In this section, we introduce the order-theoretic property of conicality, which, to the best of our knowledge, has not been studied so far. This property is in particular satisfied by higher-dimensional Minkowski space-time, as we will show. As a prerequisite to express conicality, we introduce a notion of *spanning elements* for subsets of the poset. For each set of points, these denote the subset that is contributing to the shape of their joint causal future.

Definition 5.2 (Spanning Points) *Let \mathcal{T} be a poset and $L \subset \mathcal{T}$ be finite. Then, the set of spanning elements, denoted by $\text{span}(L)$, is given by the union of all sets $L' \subseteq L$ that satisfy*

$$f(L') = f(L) \quad \wedge \quad \nexists L'' \subsetneq L': f(L'') = f(L) \quad (12)$$

where $f(L) := \bigcap_{x \in L} J^+(x)$.

Hence it forms the union of all possible minimal subsets of L which share the same joint future light cone as L . In particular, $\text{span}(L)$ is a subset of the “latest” elements of L in the poset, given by

$$\text{late}(L) := \{x \in L \mid \nexists y \in L : y \succ x\}. \quad (13)$$

Generally, and in particular, for Minkowski space-time in any dimension, both sets are not identical, as is illustrated in Figure 6 for the example of three equidistant points $a, b, c \in \mathcal{T}$, located on a space-like line next to each other.¹⁰

$$J^+(a) \cap J^+(b) \cap J^+(c) = J^+(a) \cap J^+(c). \quad (14)$$

¹⁰Equidistance is assumed to ensure $J^+(b) \subseteq J^+(a) \cap J^+(c)$. Alternatively, positioning b earlier than the space-like line connecting a and c , yet space-like to the individual points, relaxes the equidistance requirement.

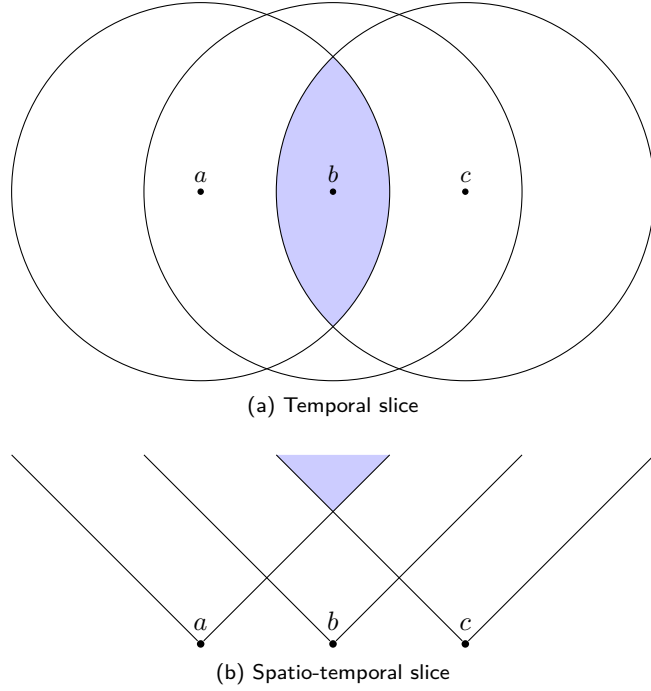


Figure 6: Light cones originating from three points located on a space-like line in Minkowski space-time with $d = 2$ spatial dimensions. Even though $a, b, c \in \mathcal{T}$ are unordered with regard to one another and therefore $\{a, b, c\} = \text{late}(\{a, b, c\})$, the point b does not contribute to the shape of their joint future. We will denote this fact as $b \notin \text{span}(\{a, b, c\}) = \{a, c\}$. This property holds even if we move b earlier in time. However, it would fail if we move it later in time, as the joint future of a and c would no longer be contained in the future of b in that case (for $d = 2$ spatial dimensions) [31].

Therefore, in this case, $\text{span}(\{a, b, c\}) = \{a, c\} \neq \text{late}(\{a, b, c\}) = \{a, b, c\}$. If $|L| \leq 2$ however, both sets are indeed always identical:

$$\text{span}(L) = \text{late}(L) \quad \forall L \subset \mathcal{T} : |L| \leq 2. \quad (15)$$

Definition 5.3 (Conicality) Let \mathcal{T} be a poset. We say that \mathcal{T} is a conical poset if for any two finite subsets $L_i, L_j \subset \mathcal{T}$,

$$\text{span}(L_i) = \text{span}(L_j) \quad \implies \quad f(L_i) = f(L_j) \quad (16)$$

holds.

Hence, in conical posets, it is possible to recover the origin of each light cone contributing to the shape of the joint future from that shape alone. We now consider whether this property is satisfied for a relevant physical models for space-time.

Lemma 5.4 For $d \geq 2$, $d+1$ -Minkowski space-time is a conical poset.

Figure 7 illustrates the difference of the geometric properties of Minkowski space-time in 1+1 dimensions, where it does not satisfy conicality, in contrast to the higher-dimensional case. Finally, we will conjecture that this property generalizes to other space-times than Minkowski. Therefore, all results we show which require conicality would generalize to this class of space-times (if the conjecture holds true).

Conjecture 5.5 Any $d+1$ -dimensional space-time manifold with $d \geq 2$, time orientation and no CTCs which is homotopic to Minkowski space-time is conical.

Here, we refer to *closed time-like curves as used in general relativity*, which we have ruled out by choosing a partial order model for space-time in Section 5. This is distinct from *causal loops in an information-theoretic causal model*. Even when embedding a causal model in partially ordered space-times and forbidding signalling outside the future light cone, the arrows of the causal model need not

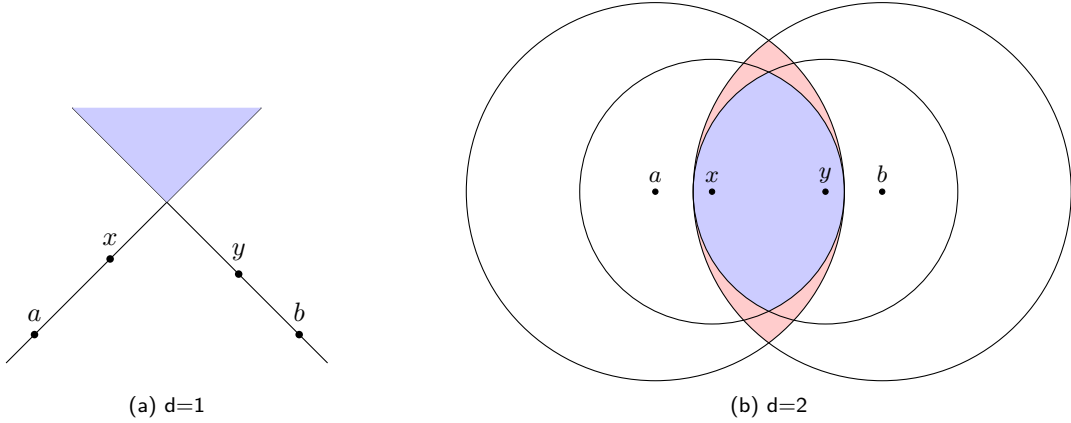


Figure 7: Schematic representation of light cones in Minkowski space-time for d spatial dimensions. (a) For $d = 1$ Minkowski space-time does not show conicality. This is because $L_1 := \{a, b\}$ and $L_2 := \{x, y\}$ are distinct sets where both satisfy $L_i = \text{span}(L_i)$ but $f(L_1) = J^+(a) \cap J^+(b) = J^+(x) \cap J^+(y) = f(L_2)$. (b) For $d = 2$, Minkowski space-time shows conicality (cf. Definition 5.3). As can be seen from the figure which represents one particular time slice, the joint futures are distinct between L_1 and L_2 .

align with the direction of time [12]. Thus, we will not refer to cyclic (information-theoretic) causal structures as featuring CTCs, but refer to them as *causal loops*.

The conjecture is based on the observation that conicality only refers to qualitative features of the light cone structure of $d+1$ -Minkowski space-time, which is locally upheld by arbitrary space-time manifolds. However, it does not necessarily carry over if the space-time is finite, is not simply-connected (e.g. has singularities or light cones interfering with themselves), as light cones can change their fundamental geometry in such situations. Further, we presume it plausible that this conjecture may require additional technical conditions on the space-time, as for example given by global hyperbolicity [40]. Moreover, there can be finite posets (not necessarily coming from an underlying manifold structure) which satisfy the conicality property. Further research is needed to determine and characterize the full space of posets satisfying these properties.

5.2 Ordered random variables

To gain a link between the set of RVs S of a causal model as outlined in Section 3.1, and a space-time given by a poset \mathcal{T} , in which the respective physical experiments are ultimately performed, the concept of a space-time embedding \mathcal{E} of a causal model was introduced in [10]. Each observed random variable in the model will be embedded into a single location in space-time. In doing so, we will introduce the formal framework to describe an embedding independent of any desired compatibility conditions with the causal model, which will follow in Section 6. While reviewing [10], we will however perform some simplifications and also highlight certain properties of sets of ORVs more explicitly.

Definition 5.6 (Ordered Random Variable (ORV)) An ORV is defined as the pair $\mathcal{X} := (X, O(X))$, where X is a random variable (RV) and $O(X) \in \mathcal{T}$ its assigned location in the partial order \mathcal{T} . Here, O is referred to as an Ordering. Given a set of RVs S , the associated set of ORVs is denoted by \mathcal{S} .

We then have the following definition of a space-time embedding, adapted from [10].

Definition 5.7 (Embedding) An embedding \mathcal{E} specifies an ordering O on a set of RVs S in \mathcal{T} , yielding a set of ORVs $\mathcal{S} = (S, O(S))$.

$$\mathcal{E} : S \mapsto \mathcal{S} := \{(X, O(X)) | X \in S\} \quad (17)$$

If O is injective, its embedding is considered non-degenerate ($O(X) \neq O(Y)$ for all distinct $X, Y \in S$).

Notation 2 We will carry over the notion of a partial order from \mathcal{T} to \mathcal{S} . Hence, $O(X) \prec O(Y)$ will be expressed as $\mathcal{X} \prec \mathcal{Y}$. Furthermore, we will often denote $O(\mathcal{X}) := O(X)$.

Notation 3 Going forward, we will also use the notation $O(S) \equiv O(S) := \{O(X) : X \in S\}$ for sets S of RVs. Thereby, we obtain a set of ORV locations.

Notation 4 From now on, let the (sets of) ORVs $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{S}_i, e_{\mathcal{X}}$ etc. always be associated to the (sets of) RVs $A, B, C, X, Y, Z, S_i, e_X$ etc. with respect to some embedding \mathcal{E} , and vice versa.

Accordingly, we carry over the notion of causal future $J^+(x)$, introduced for points $x \in \mathcal{T}$ within Definition 5.1, to ORVs.

Definition 5.8 (Future) The (inclusive) future of an ORV \mathcal{X} are defined as

$$\bar{\mathcal{F}}(\mathcal{X}) := \{a \in \mathcal{T} : a \succeq O(\mathcal{X})\}.^{11} \quad (18)$$

Connecting this to the standard notion of causal future introduced in Definition 5.1, we can equivalently write $\bar{\mathcal{F}}(\mathcal{X}) := J^+(O(\mathcal{X}))$.

Having assigned a location to each RV in S to obtain an ORV, this induces a notion of joint future for sets of ORVs [10]. We explicitly distinguish the future of a single ORV from the joint future of a set of more than one ORVs, by referring to the latter as the *support future*.

Definition 5.9 (Support Future) Let \mathcal{X} be a subset of a set of ORVs \mathcal{S} . We define

$$\bar{\mathcal{F}}_s(\mathcal{X}) \equiv \bar{\mathcal{F}}_s \left(\bigcup_{\mathcal{X}_i \in \mathcal{X}} \mathcal{X}_i \right) := \bigcap_{\mathcal{X}_i \in \mathcal{X}} \bar{\mathcal{F}}(\mathcal{X}_i) \quad (19)$$

and call it the support future.

Hence, future and support future coincide for individual ORVs $e_{\mathcal{X}} \in \mathcal{S}$, i.e. $\bar{\mathcal{F}}_s(e_{\mathcal{X}}) = \bar{\mathcal{F}}(e_{\mathcal{X}})$. Note further that associating the joint future of a set of locations $L \subset \mathcal{T}$ to be the *intersection* of the futures of the individual locations is not the usual convention in general relativity. There, the causal future of a set of points is usually understood as the *union* $J^+[L] = \bigcup_{x \in L} J^+(x)$ [39, 40, 41]. The convention adopted here (based on [10, 12]) is relevant for tightly characterizing the principle of no signalling outside the future light cone, where the relevant aspect to consider is where can a set of variables be jointly accessed (see Section 6.1).

Remark 5.10 By this definition, $\bar{\mathcal{F}}_s(\emptyset) = \mathcal{T}$. This follows as for subsets of \mathcal{T} , intersection with \mathcal{T} is the identity operation. Specifically, for any set \mathcal{X} of ORVs, $\bar{\mathcal{F}}_s(\mathcal{X}) = \bar{\mathcal{F}}_s(\mathcal{X} \cup \emptyset) = \bar{\mathcal{F}}_s(\mathcal{X}) \cap \bar{\mathcal{F}}_s(\emptyset)$.

We conclude by carrying over the notions of spanning elements and conicality, which were originally formulated for sets of points in a poset \mathcal{T} , to sets of RVs embedded in a poset (ORVs). In doing so, we gain additional complexity due to multiple ORVs $\mathcal{X}, \mathcal{Y} \in \mathcal{S}$ potentially sharing the same location $O(\mathcal{X}) = O(\mathcal{Y})$ in \mathcal{T} .

Definition 5.11 (Spanning Elements for ORVs) For a set of ORVs \mathcal{X} , we define its set of spanning elements, denoted by $\text{span}(\mathcal{X})$, as the union of all sets $s_{\mathcal{X}} \subseteq \mathcal{X}$ which satisfy

$$\bar{\mathcal{F}}_s(s_{\mathcal{X}}) = \bar{\mathcal{F}}_s(\mathcal{X}) \quad \wedge \quad \nexists t_{\mathcal{X}} \subsetneq s_{\mathcal{X}} : \bar{\mathcal{F}}_s(t_{\mathcal{X}}) = \bar{\mathcal{F}}_s(\mathcal{X}). \quad (20)$$

Indeed, we obtain:

Lemma 5.12 Let \mathcal{T} be a conical poset and $L \subset \mathcal{T}$ be finite. Let \mathcal{X} be a set of ORVs on this poset. Then, the knowledge of $\bar{\mathcal{F}}_s(\mathcal{X})$ implies the locations $O(\mathcal{X}_i)$ for all its spanning elements $\mathcal{X}_i \in \text{span}(\mathcal{X})$.

From here, we can derive a more technical equivalent formulation of conicality that we will refer to as *location symmetry*, as is detailed in Appendix G.2. This formulation will be a vital ingredient in the proofs of our main results in Section 6.2. In the particularly simple case of $\mathcal{X} \subset \mathcal{S}, \mathcal{Y}_1, \mathcal{Y}_2 \in \mathcal{S}$, it can be written as

$$\bar{\mathcal{F}}_s(\mathcal{X}\mathcal{Y}_1) = \bar{\mathcal{F}}_s(\mathcal{X}\mathcal{Y}_2) \quad \implies \quad \bar{\mathcal{F}}_s(\mathcal{X}) \subseteq \bar{\mathcal{F}}_s(\mathcal{Y}_1\mathcal{Y}_2) \vee O(\mathcal{Y}_1) = O(\mathcal{Y}_2). \quad (21)$$

¹¹Additionally, [10] introduced a notion of *exclusive* future, given by $\mathcal{F}(\mathcal{X}) := \{a \in \mathcal{T} : a \succ O(\mathcal{X})\}$. We will not use it in this work.

Generally, location symmetry captures the intuition that if two non-disjoint sets of ORVs (with \mathcal{X} being their overlap) share the same joint future, this is either due to the joint future of the overlapping ORVs \mathcal{X} being contained in joint futures of ORVs \mathcal{Y}_1 and \mathcal{Y}_2 outside the overlap, or due to the latter ORVs being embedded at the same location (i.e., the embedding is degenerate).

Remark 5.13 We note that in [10], a distinct notion of accessible regions $\mathcal{R}_X \subseteq \mathcal{T}$ for each RV $X \in S$ was also introduced, which is the space-time region where the information associated with the respective RV is accessible. Accordingly, a priori [10] considers this region to be independent of the location or the future of the ORV. This allows to distinguish the information-theoretic aspects of accessibility from the partial order of \mathcal{T} . Nonetheless, when imposing compatibility which captures relativistic principles in the space-time, $\mathcal{R}_X = \bar{\mathcal{F}}(\mathcal{X})$ is imposed, identifying the accessible regions with the causal future a posteriori. Thus \mathcal{R}_X is then determined by the location $O(X)$ of the RV. Operationally, this identification allows for the broadcasting of classical information everywhere in the future light cone. For ease of presentation, we have dropped the distinction between the accessible region and the spatio-temporal future in this work (thereby not needing to introduce the former separately).

6 Compatibility of affects relations with space-time

In this section, we will review compatibility conditions for embeddings of RVs into a space-time in the presence of affects relations as introduced in [10]. Thereby, we relate the information-theoretic concept of signalling, as encoded into the affects relations, with relativistic causality, as given by the space-time poset \mathcal{T} . Subsequently we will present the main correspondence between conical space-times and faithful causal models mentioned in the abstract and introduction.

6.1 Reviewing compatibility

While demanding all causation to go into the future seems to be a natural assumption, it is actually not necessary to prevent superluminal *signalling*, since causation and signalling are inequivalent concepts (owing to the possibility of fine-tuning). The latter is sufficient to disallow agents from being able to transmit information into the spatio-temporal past.

Operationally, transmitting information, or rather, signals, is encoded (in the current framework) into affects relations between the observed RVs. More precisely, $X \vDash Y \mid \text{do}(Z)$ captures that an agent (Alice) who intervenes on X can signal to another agent (Bob) who can observe Y and is given information on interventions performed on Z . If these RVs are embedded in space-time, Bob needs to jointly access Y and Z (which can be done in $\bar{\mathcal{F}}_s(\mathcal{Y}\mathcal{Z}) \subseteq \bar{\mathcal{F}}_s(\mathcal{Y})$) to receive the signal from Alice, who has access to X (accessible in $\bar{\mathcal{F}}_s(\mathcal{X})$). To avoid superluminal signalling, we therefore need Bob's variables to be jointly accessible in the future of Alice's interventions only.

Moreover, as motivated in [10] we only need to consider Irred₁ affects relations to ensure that the conditions are not unnecessarily strong. For instance, the reducible relation $X_1 X_2 \vDash Y$ may arise in a model where $X_1 \rightsquigarrow Y$ and X_2 is a dummy variable with no incoming or outgoing arrows (no causes or effects). In that case $\mathcal{Y} \succ \mathcal{X}_2$ is not implied by any relativistic causality principle, and only the constraint implied by the equivalent irreducible relation $X_1 \vDash Y$ is relevant for no superluminal signalling, imposing $\mathcal{Y} \succ \mathcal{X}_1$. As motivated in Section 4.1, irreducibility serves to eliminate such redundancies.

This can be formalized to the following condition proposed in [10]. Here we provide a simplified version, with regard to the embedding as well as in the restriction to unconditional affects relations.

Definition 6.1 (compat) Let S be a set of ORVs from a set of RVs S and a poset \mathcal{T} with an embedding \mathcal{E} . Then a set of unconditional affects relations \mathcal{A} is said to be compatible with \mathcal{E} (or satisfies **compat**) if the following condition holds:

- Let $X, Y \subset S$ be disjoint non-empty sets of RVs, $Z \subset S$ another disjoint set of RVs, potentially empty. If $(X \vDash Y \mid \text{do}(Z)) \in \mathcal{A}$ and is Irred₁, then $\bar{\mathcal{F}}_s(\mathcal{Y}\mathcal{Z}) = \bar{\mathcal{F}}_s(\mathcal{Y}) \cap \bar{\mathcal{F}}_s(\mathcal{Z}) \subseteq \bar{\mathcal{F}}_s(\mathcal{X})$.

This compatibility condition provides a necessary and sufficient condition to ensure that we have no superluminal signalling once the affects relations are embedded in the space-time [10]. Notice that the relevant condition for strictly capturing no superluminal signalling (NSS) is on the intersections and not

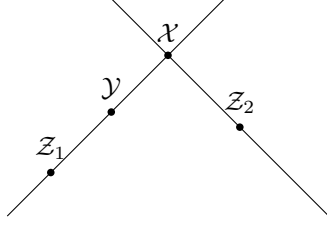


Figure 8: Sketch of a compatible non-degenerate embedding of Example 10 into 1+1-Minkowski space-time. In this embedding, all ORVs which are located on a light-like surface, and therefore on the boundary of the light cone of the respective earlier RVs. As this space-time is not conical, even for $X \# Y \mid \text{do}(Z)$ satisfying Irred_1 and Irred_3 , in accordance with Corollary 6.5. we may have $\bar{\mathcal{F}}(\mathcal{Y}) \not\subseteq \bar{\mathcal{F}}(\mathcal{X}) \cap \bar{\mathcal{F}}_s(\mathcal{Z})$.

union of the futures (the latter relates to prohibiting superluminal causation, which is inequivalent to NSS [13]). In particular, in the jamming scenario of Example 5, we have $Y \# AC$ (which is Irred_1) and NSS only requires the joint future of the ORVs \mathcal{A} and \mathcal{C} to be contained in the future of the ORV \mathcal{Y} and not that \mathcal{A} or \mathcal{C} is embedded in the future light cone of \mathcal{Y} , since $Y \# A$ and $Y \# C$.

Remark 6.2 For $X, Y, Z \in S$, consider an affects relation $X \# Y \mid \text{do}(Z)$. If $\bar{\mathcal{F}}_s(\mathcal{Y}\mathcal{Z}) = \emptyset$, there is no location in \mathcal{T} where both Y and Z are accessible. Hence, it is impossible to experimentally verify the respective affects relation, rendering it operationally meaningless. Similarly, if $\bar{\mathcal{F}}_s(\mathcal{X}) = \emptyset$, compatibility implies that $\bar{\mathcal{F}}_s(\mathcal{Y}\mathcal{Z}) = \emptyset$: It is impossible to signal from nowhere to somewhere. Therefore, we will usually disregard the case where either of these sets is empty.

A physical example for the former case would be for \mathcal{Y} and \mathcal{Z} located in two distinct classical black holes or outside their respective cosmological event horizons, as these would have no joint future.

6.2 A correspondence between conical space-times and causal models without clustering

Consider a higher-order affects relation $X \# Y \mid \text{do}(Z)$ between disjoint sets X, Y and Z of RVs. The Irred_1 property of such an affects relation enables us to infer that each element $e_X \in X$ is a cause of some element $e_Y \in Y$ [10], and as we have shown in Section 4.3, the Irred_3 property enables an analogous inference relative to the other interventional argument, Z , that all $e_Z \in Z$ are a cause of some $e_Y \in Y$. In other words, imposing Irred_1 and Irred_3 for the higher-order affects relation $X \# Y \mid \text{do}(Z)$ (or $Z \# Y \mid \text{do}(X)$) has the same implications for causal inference as imposing Irred_1 for the 0^{th} -order relation $XZ \# Y$. This indicates an interchangeability between X and Z for causal inference statements derived from higher-order affects relations irreducible in the first and third arguments. Does this interchangeability also carry forth to compatibility constraints imposed by higher-order affects relations in a space-time? The following example illustrates that the answer to this question is generally negative.

Example 10 (*Correspondence does not generally hold*) Let X and Y be two RVs while $Z := \{Z_1, Z_2\}$. Then $X \# Y \mid \text{do}(Z)$ is Irred_1 by construction. Suppose that it is also Irred_3 . Then by Definition 4.4, it follows that we must have (1) $Z_1 \# Y \mid \text{do}(Z_2)$ or $Z_1 \# Y \mid \text{do}(XZ_2)$, and (2) $Z_2 \# Y \mid \text{do}(Z_1)$ or $Z_2 \# Y \mid \text{do}(XZ_1)$. We resolve this condition by picking the first affects relation of (1) and the second of (2) i.e., $Z_1 \# Y \mid \text{do}(Z_2)$ and $Z_2 \# Y \mid \text{do}(XZ_1)$. Consider the space-time embedding of these RVs in 1+1-Minkowski space-time where $Z_1 \prec \mathcal{Y} \prec \mathcal{X} \succ Z_2$, as depicted in Figure 8. All three of the above affects relations are compatible with this embedding, as we have $\bar{\mathcal{F}}(\mathcal{Y}) \cap \bar{\mathcal{F}}_s(\mathcal{Z}) \subseteq \bar{\mathcal{F}}(\mathcal{X})$, $\bar{\mathcal{F}}(\mathcal{Y}) \cap \bar{\mathcal{F}}(\mathcal{Z}_2) \subseteq \bar{\mathcal{F}}(\mathcal{Z}_1)$ and $\bar{\mathcal{F}}(\mathcal{Y}) \cap \bar{\mathcal{F}}(\mathcal{X}) \cap \bar{\mathcal{F}}(\mathcal{Z}_1) \subseteq \bar{\mathcal{F}}(\mathcal{Z}_2)$. However, the compatibility condition implied by an additional Irred_1 , 0^{th} -order relation $XZ \# Y$ is violated as $\bar{\mathcal{F}}(\mathcal{Y}) \not\subseteq \bar{\mathcal{F}}(\mathcal{X}) \cap \bar{\mathcal{F}}_s(\mathcal{Z})$.¹²

¹²Further, from [10], we know that $X \# Y \mid \text{do}(Z)$ implies that $Z \# Y$ or $XZ \# Y$ must hold, which can be resolved by choosing $Z \# Y$. This may be reducible or irreducible, we take it to be reducible to $Z_1 \# Y$, which is by construction irreducible since Z_1 is a single RV. This imposes the additional compatibility condition $Z_1 \prec \mathcal{Y}$ which is already satisfied in the space-time embedding of our example. Similarly, we can resolve the implied affects relations of $Z_1 \# Y \mid \text{do}(Z_2)$ and $Z_2 \# Y \mid \text{do}(XZ_1)$ with the same affects relation $Z_1 \# Y$. Thus the compatibility of the affects relations of this example with the given embedding continues to hold when considering implied affects relations.

$$X \vDash Y_1 Y_2 \mid \text{do}(Z_1 Z_2) \text{ with Irred}_1$$

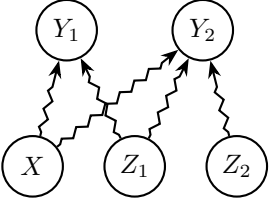
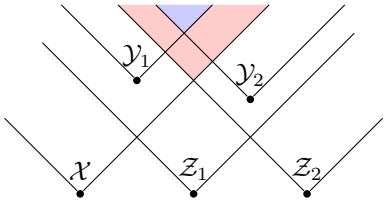
	$X, Z_1, Z_2 \rightsquigarrow Y_1 Y_2$ for Causal Inference?	$\bar{\mathcal{F}}_s(\mathcal{X} Z_1 Z_2) \supseteq \bar{\mathcal{F}}_s(\mathcal{Y}_1 \mathcal{Y}_2)$ for Spacetime Embedding? with no superluminal signalling
in general	\times	\times
w/ Irred ₃	\checkmark	\times
w/ Irred ₃ & no Clus ₃ in model	\checkmark	\checkmark
w/ Irred ₃ & conical spacetime	\checkmark	\checkmark
	 Example for compliant causal structure	 Example for compliant spacetime embedding

Table 1: The consequences of an affects relation $X \vDash Y_1 Y_2 \mid \text{do}(Z_1 Z_2)$ regarding causal inference and spacetime compatibility, given that various additional conditions apply. Here, X, Y_1, Y_2, Z_1, Z_2 represent individual RVs in the causal model. Therefore, these affects relation is Irred₁ by definition. The juxtaposition within the table demonstrates the correspondence between causal and spatiotemporal order, which is established under certain additional conditions only. The example is based on Example 10, albeit considering Y_1 and Y_2 rather than a single RV Y in its second argument.

This example is described in terms of affects relations (that can arise from some underlying, unknown causal model). It however remains an open question to find a causal model that generates exactly a given set of affects relations (such as those of this example) and no more. Further, the example is set in 1+1-Minkowski space-time, which is not a conical space-time.

Interestingly, as we show in the following theorem, in conical space-times, the answer to the aforementioned question is positive for any set of affects relations: Here, the interchangeability of X and Z does carry forth to compatibility. This results in a simplification of compatibility considerations, which can have useful applications for characterizing relativistic causality for information-processing protocols in conical space-times (such as our physical 3+1-Minkowski space-time), as further discussed in Section 7.

Theorem 6.3 *Let \mathcal{A} be a set of unconditional affects relations and $\mathcal{A}' \subseteq \mathcal{A}$ consist of all affects relations in \mathcal{A} that are both Irred₁ and Irred₃. Then for any non-degenerate embedding \mathcal{E} into a conical space-time \mathcal{T} satisfying *compat*, we have*

$$(X \vDash Y \mid \text{do}(Z)) \in \mathcal{A}' \quad \implies \quad \bar{\mathcal{F}}_s(\mathcal{Y}) \subseteq \bar{\mathcal{F}}_s(\mathcal{X}) \cap \bar{\mathcal{F}}_s(\mathcal{Z}). \quad (22)$$

However, imposing restrictions on the underlying space-time is not the only way to recover this resemblance between causal inference and space-time structure. Alternatively, we can restrict the allowed sets of affects relations, and by extension, the space of causal models giving rise to them. In particular, we can demand the absence of clustering in the third argument.

Theorem 6.4 *Let \mathcal{A} be a set of unconditional affects relations arising from a causal model not yielding any affects relations with Clus₃ and let $\mathcal{A}' \subseteq \mathcal{A}$ consist of all affects relations in \mathcal{A} that are both Irred₁ and Irred₃. Then compatibility of \mathcal{A} with an embedding \mathcal{E} in any space-time \mathcal{T} implies Equation (22).*

In particular, the above theorem holds for any faithful (or not fine-tuned) causal model since such models cannot have any clustered affects relations (Theorem 4.12). Accordingly, we can check that the set of affects relations considered in Example 10 does indeed admit affects relations clustered in the third argument, as $X \vDash Y \mid \text{do}(Z)$, yet $X \not\vDash Y$. This implies (by Theorem 4.12) that any causal model that can give rise to these affects (and non-affects) relations must be fine-tuned.

These results reveal a correspondence between properties of causal models, and the geometry of space-time, in particular between no clustering in the interventional arguments and conicality. This

parallel is rooted in the fact that both of these properties affirm the independent impact of individual elements in the interventional arguments of affects relation, albeit in different ways: either information-theoretically through non-clustered signalling or via non-degenerate compatibility through conicality. For each condition, we have an interchangeability between X and Z for higher-order affects relations irreducible in the first and the third argument, allowing us to treat them on an equal footing for the purpose of compatibility constraints (as we can generally do for causal inference, by Corollary 4.15). Effectively, this then tells us that in these scenarios, compatibility implies that the interventional data given by X and Z in a higher-order affects relation must be entirely in the “past” of the region where the observational data captured by Y is completely accessible, irrespective of whether “past” is defined relative to the relations \rightsquigarrow of the information-theoretic causal model or the relations \prec capturing the light-cone structure of the space-time. This main result is illustrated and summarized in Table 1.

In Appendix F, we introduce another compatibility condition named **compat-atomic**, which is a restriction of **compat** (Definition 6.1) to affects relations $X \vDash Y \mid \text{do}(Z)$ where X is a single RV. While this does not fully capture no superluminal signalling and is not equivalent to **compat** in general, we will show that **compat-atomic** and **compat** are equivalent under analogous restrictions as the above theorem: namely when restricting the space-time to be conical or the causal model to have no clustered relations of a certain type. This suggests another correspondence between causal inference and space-time geometry which also leads to notable simplifications.

6.3 Generalizing the correspondence to conditional affects relations

Using that any conditional affects relation $X \vDash Y \mid \text{do}(Z), W$ implies the presence of an unconditional affects relation $X \vDash YW \mid \text{do}(Z)$, it is possible to transform each set of affects relations to a set of unconditional affects relations with the same implications for causal inference (according to Section 4.3) and compatibility. This follows from Lemma E.2 and Corollary E.3. Therefore, all statements of the previous section also apply for conditional affects relations when replacing Y with YW accordingly. Together with Corollary 4.15, Theorem 6.3 and Theorem 6.4, this yields that if either the space-time or the causal model is not fine-tuned, an affects relation $X \vDash Y \mid \text{do}(Z), W$ with Irred_1 and Irred_3 is equivalent to an affects relation $XZ \vDash YW$ with regard to causal inference and compatibility¹³.

This allows for direct generalization of statements concerning the embeddability of sets of unconditional 0^{th} -order affects relations into space-time to sets of conditional higher-order affects relations. Conjoining the contrapositions of our two main theorems and generalizing to conditional affects relations, we obtain:

Corollary 6.5 *Let \mathcal{A} be a set of (conditional) affects relations (c.f. Definition E.1) and $\mathcal{A}' \subseteq \mathcal{A}$ consist of all affects relations in \mathcal{A} that are both Irred_1 and Irred_3 . Then the existence of a non-degenerate embedding \mathcal{E} in a space-time \mathcal{T} with*

$$(X \vDash Y \mid \text{do}(Z), W) \in \mathcal{A}' \quad \wedge \quad \bar{\mathcal{F}}_s(\mathcal{Y}) \cap \bar{\mathcal{F}}_s(\mathcal{W}) \not\subseteq \bar{\mathcal{F}}_s(\mathcal{X}) \cap \bar{\mathcal{F}}_s(\mathcal{Z}). \quad (23)$$

implies that (1) the space-time \mathcal{T} does not satisfy conicality and that (2) the affects relations \mathcal{A} exhibit interventional fine-tuning (Clus_3).

This shows that in a conditional higher-order affects relation, the interventional (1^{st} and 3^{rd}) and observational (2^{nd} and 4^{th}) arguments respectively exhibit interchangeability amongst each other when either restricting to conical space-times or affects relations without clustering.

To summarize, we have established that in cases (1) and (2) of the above corollary, an affects relation $X \vDash Y \mid \text{do}(Z), W$ being Irred_1 and Irred_3 give us the following causal inference and compatibility statements, for any $e_{XZ} \in XZ$:

$$\begin{aligned} \text{Causal inference: } & e_{XZ} \text{ is a cause of } YW \\ \text{Compatibility: } & \bar{\mathcal{F}}(e_{XZ}) \supseteq \bar{\mathcal{F}}_s(\mathcal{Y}\mathcal{W}) \end{aligned} \quad (24)$$

These exhibit an analogous order structure. The former statement follows from the results of Section 4.3 holding independently of restrictions (1) and (2), while the latter statement does not generally hold without these restrictions.

¹³For conditional affects relations, compatibility is spelled out in Corollary E.21.

7 Conclusions

The affects framework [10] provides a formal platform for investigating the interplay between information-theoretic and spatio-temporal causation in rather general scenarios. For addressing many of the open problems brought to light by this approach, it is beneficial to develop a tighter characterisation of the associated concepts and techniques, relating to causal modelling, space-time structures and the compatibility of the two. Within our work we have improved on these characterizations, generating new insights and tools for many of the open problems identified in [10].

For causal models, our contributions are three-fold. We generalized the concept of reducibility to identify redundancies in different arguments of an affects relation, which captures information-theoretic signalling through interventions in a causal model. Introducing the concept of clustering of affects relations, we characterized different types of operationally detectable fine-tunings depending on whether it involves *observational* or *interventional* arguments. Subsequently, we derived applications of these concepts for causal inference, showing that interestingly, the absence of certain affects relations can also be successfully employed for causal inference.

For space-time structure, we have introduced the order-theoretic property of *conicality* showing that it is satisfied in Minkowski space-times with $d > 1$ spatial dimensions and violated for $d = 1$. Using this, we have shown a correspondence between conical space-times and causal models without a form of clustering – both cases exhibit a parallel between constraints on the observed variables coming from compatibility with a space-time embedding and those coming from purely information-theoretic causal inference (although these two types of constraints behave differently in general). As clustering is linked to fine-tuning, this suggests a more general link between conical space-times and faithful causal models. Moreover, this correspondence reveals that in conical space-times, the principle of no superluminal signalling ensures a clear temporal ordering originating from an affects relation $\mathcal{X} \models \mathcal{Y} \mid \text{do}(\mathcal{Z}), \mathcal{W}$ irreducible in \mathcal{X} and \mathcal{Z} : between when the interventional arguments \mathcal{X} and \mathcal{Z} and the observational arguments \mathcal{Y} and \mathcal{W} are jointly accessible, with the former ordered before the latter.

While we have focused on foundational questions here, the relevance of fine-tuning in the security of cryptographic protocols (as highlighted in [10]) motivates potential applications of our results in more practical scenarios. Moreover, given the widespread use of causal modelling and inference in data driven disciplines, combined with the interest in order-theoretic properties of space-time in general relativity and quantum gravity approaches, this work may be of interest in broader communities beyond quantum information and physics.

7.1 Open questions

There is still much scope for future work. We discuss some interesting future directions that can be investigated by building on this work and the affects framework.

Causal models and inference

Here, we have focused on causal models defined using the d-separation property, which holds for all acyclic models [2, 26] and for a class of cyclic models [10, 11] in quantum and post-quantum theories, but can fail in certain cyclic causal models already in classical theories [28]. A natural open question is whether all the techniques used here generalize to the case when the d-separation condition is replaced by the σ -separation condition [42], which applies to an even wider variety of cyclic (and possibly continuous variable) classical causal models, or to p -separation, which holds in all finite-dimensional cyclic quantum causal models [43]. Further, we have focused on unconditional affects relations: While we generalized the concepts and some of the results to conditional relations, we leave the full generalization for future work, as discussed in Appendix E.

Another important open problem relates to the classification of fine-tuning, as initiated in Section 4.2 through the concept of clustering. As highlighted there, clustering in different arguments presents us with observational and interventional types of fine-tuning, but there can be additional forms of fine-tuning relating to the fine-grainability of the causal model (in the sense introduced in [11]). It remains to be explored whether there can exist other types of fine-tuning of a causal model, which are not explained by these types, and whether all such types can be operationally detected through its pattern of affects relations. Such classifications of fine-tuning would prove useful for developing robust algorithms for

causal discovery in the presence of fine-tuning, as existing ones usually assume faithfulness, owing to inherent challenges posed by fine-tuning for causal inference [1, 2].

A related question is whether it is possible to detect fine-tuning through interventions when it is impossible to detect it through observations alone. Formally, this relates to a question raised in Appendix A regarding equivalence of two definitions of fine-tuning: one which compares d-separation and conditional independence in the pre-intervention causal model alone and the other where all post-intervention models are also taken into account. Presently, it is unknown if there is a gap between these definitions.

The affects framework significantly abstracts the usual causal modelling approaches, showing that many known results (such as Pearl’s rules of do-calculus [2]) can be obtained from more minimal assumptions. However, proving the completeness of the causal inference results derived under these minimal assumptions remains an important open challenge. It would be interesting to prove this, even under certain restrictions on the causal models (possibly coming from the form of causal mechanisms assumed for the underlying theory). We aim to shed further light on this question within a follow-up work [44].

Space-time structure

There are interesting questions relating to the characterization of the order-theoretic properties of space-times without referring to differential geometry. In particular, can we characterize the set of space-times which satisfy *conicality*, both for pseudo-Riemannian manifolds and for discrete models of space-time? Are there further, yet to be uncovered, properties of light cones that are reflected within the associated partial order, which exhibit useful correspondences with causal inference concepts? Conversely, can we formulate notions such as space-time distance and curvature in terms of graph-theoretic properties studied in causal models, such as d-separation? Progress in this direction could provide valuable tools for programs that aim on the discretization of space-time, as is the case for causal set quantum gravity [37] or causal dynamic triangulations [45] as at least in the limit, certain properties of light cones need be recovered by physical space-times.

Compatibility of causal models and space-time

The possibility of operationally detectable causal loops in 1+1-Minkowski space-time without superluminal signalling was shown in [12]. In the technical language of the framework, these are cyclic causal loops in a causal model whose presence can be certified through the resulting affects relations (and hence called affects causal loops or ACLs), which can be compatibly and non-degenerately embedded in a partially ordered set. A crucial question in this regard has been whether such loops are possible in higher-dimensional Minkowski space-time.

The formal concept of conicality introduced here distinguishes between Minkowski space-time with $d = 1$ and $d > 1$ spatial dimensions, as the former does not satisfy it (and can embed such ACLs) while the latter does (and no such embeddings are known). Moreover, it is only possible to compatibly and non-degenerately embed such ACLs in a poset when the causal model is fine-tuned [10]. The correspondence between compatible embeddings in conical space-times and those of faithful (i.e., not fine-tuned) causal models found in Section 6.2, lends support to the conjecture that compatible and non-degenerate embeddings of ACLs in conical (and hence higher-dimensional Minkowski, Lemma 5.4) space-times are in fact impossible.

In this regard, it is worth noting that the structure of ACLs can be rather complicated, even more so when higher-order affects relations are involved and several distinct classes have been identified in [10]. Our results show that when studying compatible embeddings in conical space-times, for affects relations irreducible in the first and third arguments, higher-order affects relations can be replaced equivalently with 0^{th} -order ones. This provides a useful simplification for studying the compatible embeddings of general ACLs involving higher-order affects relations, in conical space-times. Building on the present work, we aim to prove the aforementioned conjecture about such ACLs in a follow-up work [44]. Although this is not the case for non-conical space-times [12], the conjecture being true would show that in conical space-times, the fundamental principle of no superluminal signalling is sufficient for ruling out all (operationally detectable) causal loops.

Applications to cryptography and quantum tasks in space-time

The concept of clustering captures signalling between sets of nodes without signalling between individual nodes. This is a property that is necessary for the security of cryptographic protocols such as the one-time pad and secret sharing schemes [14, 15, 16], where information is distributed over multiple systems and cannot be recovered from a subset thereof. This is also similar to the desired properties of quantum error correcting codes [17, 18] that are intimately linked to quantum information-processing tasks in space-time such as summoning [19, 20, 21], where achieving the task efficiently can require quantum information to be cleverly distributed over multiple space-time locations. Developing a causal modelling approach to such protocols, and investigating the applications of these concepts there, hence presents an intriguing inter-disciplinary avenue for future research.

In this regard, it is important to note that the affects framework and our work focus on signalling between classical variables obtained by interacting with (e.g., by measuring) non-classical systems, and it makes no assumptions about the non-classical causal mechanisms. This facilitates intrinsically device- and theory-independent statements. In quantum theory, we understand the causal mechanisms and can define signalling between quantum in/output systems of quantum channels and study their compatibility with space-time [11]. Developing the language of conditional higher-order affects relations further for the case of quantum theory, can enable us to study more refined notions of signalling in quantum circuits, where conditioning on additional interventions and measurements outcomes can may allow or forbid signalling between agents. Moreover, we focused on space-time embeddings where variables are well-localized at space-time points. More generally, systems may be embedded into space-time regions and quantum systems exchanged by parties can also be non-localized in space and in time, as captured by the framework of [11]. Investigating the aforementioned applications for quantum protocols in space-time will therefore benefit from combining and linking techniques from the frameworks of [10] and [11].

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A Further details on the causal modelling framework

In this section, we provide a formal definition for the technical concept of d-separation, which was described on an intuitive level in Section 3. This concept was originally introduced in the classical causal modelling literature (e.g., [2]) and has been applied to define a general class of causal models applicable to situations with non-classical, cyclic and fine-tuned causal influences, in the affects framework of [10]. We provide a more technical overview of these concepts here.

Definition A.1 (Blocked Paths) Let \mathcal{G} be a directed graph, where X and Y are distinct nodes and Z is a set of nodes not containing X and Y . An (undirected) path from X to Y is blocked by Z if the path contains A, B such that either $A \rightsquigarrow W \rightsquigarrow B$, $A \leftarrow W \rightsquigarrow B$ with $W \in Z$, or $A \rightsquigarrow V \leftarrow B$ with neither V nor any child of V in Z .

Definition A.2 (d-separation) Let \mathcal{G} be a directed graph, with X, Y, Z being disjoint subsets of nodes. X and Y are d-separated by Z in \mathcal{G} , denoted as $(X \perp^d Y | Z)_{\mathcal{G}}$, if every path from an element of X to an element of Y is blocked by Z . Otherwise, X is d-connected to Y given Z . If obvious from context, the index \mathcal{G} may be suppressed.

Using d-separation, the following minimal definition of a causal model was proposed in [10].

Definition A.3 (Causal model) A causal model over a set of observed random variables $\{X_1, \dots, X_n\}$ consists of a directed graph \mathcal{G} over them (possibly additionally involving classical/quantum/GPT unobserved systems) and a joint distribution $P_{\mathcal{G}}(X_1, \dots, X_n)$ that respects the d-separation property relative to \mathcal{G} .

Definition A.4 (d-separation property) Let $\{X_1, \dots, X_n\}$ be a set of random variables denoting the observed nodes of a directed graph \mathcal{G} , and $P(X_1, \dots, X_n)$ be a joint probability distribution over them. Then P is said to satisfy the d-separation property with respect to \mathcal{G} if for all disjoint subsets X, Y and Z of $\{X_1, \dots, X_n\}$,

$$X \perp^d Y | Z \implies X \perp\!\!\!\perp Y | Z \quad \text{i.e., } P(XY | Z) = P(X | Z)P(Y | Z). \quad (25)$$

We continue by stating formally how an intervention on a set of nodes I influences the causal model, i.e. how the post-intervention causal model relates to the pre-intervention model, following the formulation of [31]. This procedure, as developed within [10], is independent on causal mechanisms or other information on the underlying theory, but is based directly on Definition A.4 instead.

Definition A.5 (Post-intervention causal model) Consider a causal model on a graph \mathcal{G} specified by the graph together with a distribution $P_{\mathcal{G}}$ satisfying Definition A.4. A post-intervention causal model associated with interventions on a subset X of the observed nodes of \mathcal{G} is specified by a graph $\mathcal{G}_{\text{do}(X)}$ obtained from \mathcal{G} by removing all incoming directed edges to the set X , together with a distribution $P_{\mathcal{G}_{\text{do}(X)}}$ satisfying Definition A.4 relative to $\mathcal{G}_{\text{do}(X)}$. Further, consider a (potentially empty) set of observed nodes Y in \mathcal{G} and a disjoint set of nodes x exogenous in \mathcal{G} . Then, we require

$$P_{\mathcal{G}_{\text{do}(Y)}}(S | XY) = P_{\mathcal{G}_{\text{do}(XY)}}(S | XY) \quad (26)$$

where S is the set of the remaining observed nodes.

For all the results in this paper we only use the above-mentioned general definition of a post-intervention causal model. However, for specific classical examples, we will apply the usual causal modelling approach of Pearl [2] to describe the post-intervention scenario (which adheres with the minimal requirements of the above definition). Explicitly, a classical causal model on a graph \mathcal{G} is typically specified by providing a probability distribution $P(X)$ for every parentless node X together with a function $f_X : \text{par}(X) \mapsto X$ from the set of all parents of X to X . The post-intervention model associated with interventions on a set S of nodes is then obtained by considering the post-intervention graph $\mathcal{G}_{\text{do}(S)}$ as defined before (where the incoming arrows to S are removed), together with a causal model that is identical to the original model except that for every $X \in S$, we replace the dependence $X = f_X(\text{par}(X))$ with $X = x$ (denoting that X takes on a fixed value x).

Having defined post-interventional models, we now formalize what we mean by fine-tuning and faithfulness (the absence of fine-tuning). Faithfulness is related to the idea that conditional independences in the distribution imply a corresponding d-separation in the graph (intuitively, the probabilities are faithful to the graph structure). Usually this is defined only by considering the original pre-intervention model, but here we extend the definition by accounting for all post-intervention causal models.

Definition A.6 (Faithfulness and fine-tuning) A causal model over a directed graph \mathcal{G} is said to be faithful if for every post-intervention causal structure $\mathcal{G}_{\text{do}(I)}$ obtained from \mathcal{G} by intervening of a set I of observed nodes, and for any mutually disjoint subsets X, Y, Z of the observed nodes (possibly overlapping with I),

$$(X \perp\!\!\!\perp Y|Z)_{\mathcal{G}_{\text{do}(I)}} \implies (X \perp^d Y|Z)_{\mathcal{G}_{\text{do}(I)}}. \quad (27)$$

Otherwise, the causal model is said to be fine-tuned.

While we conjecture this to hold, to the best of our knowledge, it is generally unknown whether the above definition is equivalent to its restriction to the case $I = \emptyset$ i.e., whether fine-tuning in a post-intervention model (for some choice I of nodes being intervened on) always implies fine-tuning in the original (or pre-intervention) causal model. One way to prove this equivalence (which we leave for future work) would be to show that in any general (d-separation) causal model which is faithful by this restricted definition, an intervention such that $(X \not\perp\!\!\!\perp Y|Z)_{\mathcal{G}} \wedge (X \perp\!\!\!\perp Y|Z)_{\mathcal{G}_{\text{do}(I)}}$ always implies that $(X \perp^d Y|Z)_{\mathcal{G}_{\text{do}(I)}}$.¹⁴

We conclude by pointing out that while it does capture a variety of cyclic causal models, some functional equation models are known which do not satisfy d-separation [46]. To capture these scenarios, the generalized notion of σ -separation can be used [42]. However, σ -separation is not the most general property either, and there are causal models known to violate σ -separation as well. An upcoming work [43] proposes another graph separation property called p -separation that is valid for all fine-dimensional, possibly cyclic quantum causal models (and therefore for all finite and discrete variable classical causal models). The extension of our results to these alternative graph separation properties is an interesting avenue for future work.

B Properties of partially ordered sets

In this section, we will provide a formal introduction of posets. We will study properties posets may show as well as their relation to the properties of relevant physical examples for \mathcal{T} .

Definition B.1 (Partial Order) A (strict) partial order is a binary relation \prec on a set T which satisfies

- *Irreflexivity:* $a \not\prec a$.
- *Asymmetry:* $a \prec b \implies b \not\prec a$.
- *Transitivity:* $a \prec b$ and $b \prec c \implies a \prec c$.

for all $a, b, c \in T$.

Definition B.2 (Poset) A poset \mathcal{T} is given by a set T endowed with a partial order: $\mathcal{T} := (T, \prec)$.

We start by introducing a notion of immediate neighbors:

Definition B.3 Let \mathcal{T} be a poset and $x, y \in \mathcal{T}$. We say y covers x if $x \prec y$ and there is no $z \in \mathcal{T}$ such that $x \prec z \prec y$.

We can use this definition to give discrete posets a graphical representation in terms of Hasse diagrams [47], as illustrated in Figure 9.

In the remainder of this section, we review the definitions of join and meet and how they yield the notion of an (order) lattice. We will then identify how these generic notions relate to physical space-time.

Definition B.4 (Join) Let \mathcal{T} be a poset. For two elements $x, y \in \mathcal{T}$, their join $x \vee y$ is an element of \mathcal{T} such that:

- $x \preceq x \vee y$ and $y \preceq x \vee y$

¹⁴Note that for $(X \perp\!\!\!\perp Y|Z)_{\mathcal{G}}$, we have the d-separation condition $(X \perp^d Y|Z)_{\mathcal{G}}$, which follows from the restricted definition of fine-tuning, noting that $\mathcal{G} := \mathcal{G}_{\text{do}(I)}$ for $I = \emptyset$. And $(X \perp^d Y|Z)_{\mathcal{G}}$ implies $(X \perp^d Y|Z)_{\mathcal{G}_{\text{do}(I)}}$ for any choice of I since removing arrows cannot add d-connections.

- if $x \preceq a$ and $y \preceq a$, then $x \vee y \preceq a \forall a \in \mathcal{T}$

or equivalently: $\forall a \in \mathcal{T}, x \vee y \preceq a \iff x \preceq a \text{ and } y \preceq a$.

Definition B.5 (Meet) Let \mathcal{T} be a poset. For two elements $x, y \in \mathcal{T}$, their meet $x \wedge y$ is an element of \mathcal{T} such that:

- $x \succeq x \wedge y$ and $y \succeq x \wedge y$
- if $x \succeq a$ and $y \succeq a$, then $x \wedge y \succeq a \forall a \in \mathcal{T}$

or equivalently: $\forall a \in \mathcal{T}, x \wedge y \succeq a \iff x \succeq a \text{ and } y \succeq a$.

For each pair of elements in a poset, the existence of their join and meet is not guaranteed. However if they exist, they are respectively unique. Both fulfill the following properties, where \odot stands either for \wedge or \vee :

- Associativity: $(a \odot b) \odot c = a \odot (b \odot c)$
- Commutativity: $a \odot b = b \odot a$
- Idempotency: $a \odot a = a$

Generalizing the concept of the join, we will define the notion of the minimal elements of a poset. Analogously, we could also define the maximal elements for the meet. This, however, will not be required going forward, as we will focus on the causal future to model the availability of information.

Definition B.6 (Minimal Elements of a Poset) Let $M \subseteq \mathcal{T}$. Then $\min M := \{x \in M \mid \nexists y \in M : y \prec x\}$.

Lemma B.7 Let \mathcal{T} be a poset. If and only if two elements $x, y \in \mathcal{T}$ have a join $x \vee y$, then $\min\{a \in \mathcal{T} \mid x \preceq a \succeq y\} = \{x \vee y\}$.

Proof: This follows directly from the definition of the join. □

Definition B.8 (Semilattice) A poset \mathcal{T} is a join-semilattice if $\forall x, y \in \mathcal{T}$ the join $x \vee y$ exists. Dually, \mathcal{T} is a meet-semilattice if $\forall x, y \in \mathcal{T}$ the meet $x \wedge y$ exists.

Intuitively, it becomes apparent that precisely for a join-semilattice, we can associate *sets* of points in \mathcal{T} , or equivalently sets of ORVs, with a single location in space-time to encode their joint future.

Definition B.9 (Join- and Meet-free Poset) Let \mathcal{T} be a poset. We call \mathcal{T} a join-free poset if for all $x, y \in \mathcal{T}$, the existence of a join $x \vee y$ implies $x \vee y = x$ or $x \vee y = y$. Dually, \mathcal{T} is a meet-free poset if no meet $x \wedge y$ exists for analogous choices of x and y .

Hence, a join-free poset can be considered the “opposite” of a join-semilattice, since in the former, for a pair of points x, y , joins exist only for the trivial case of $x \prec y$ or $x \succ y$. All other posets are in-between these two extremes: Some non-trivial combinations of two points have a join there, while others do not.

Definition B.10 (Lattice) A lattice is a poset \mathcal{T} which is both a join- and a meet-semilattice.

An example of a lattice, in contrast to a poset which does not form a lattice, is depicted in Figure 9. The theory of (order) lattices constitutes a well-studied subbranch of mathematics, which can be approached both from an order-theoretic (as done here) and from an algebraic perspective. More precisely, instead of deriving join and meet from a partial order, one may axiomatically consider any two binary operations \wedge and \vee that are commutative and associative and satisfy the association law

$$a \vee (a \wedge b) = a \wedge (a \vee b) = a \quad \forall a, b \in \mathcal{T}. \quad (28)$$

It can be shown that satisfying these algebraic properties guarantees that both join and meet define the same partial order. Accordingly, a huge amount of further properties a lattice may fulfill are known [48]. Most of these have not proven directly relevant to this work and will therefore not be reviewed here.

We continue by making a connection of these more generic poset properties with the notion of conicality introduced in Section 5.1, and apply them to the particular case of Minkowski space-time.

Lemma B.11 Any conical poset (cf. Definition 5.3) is join-free.

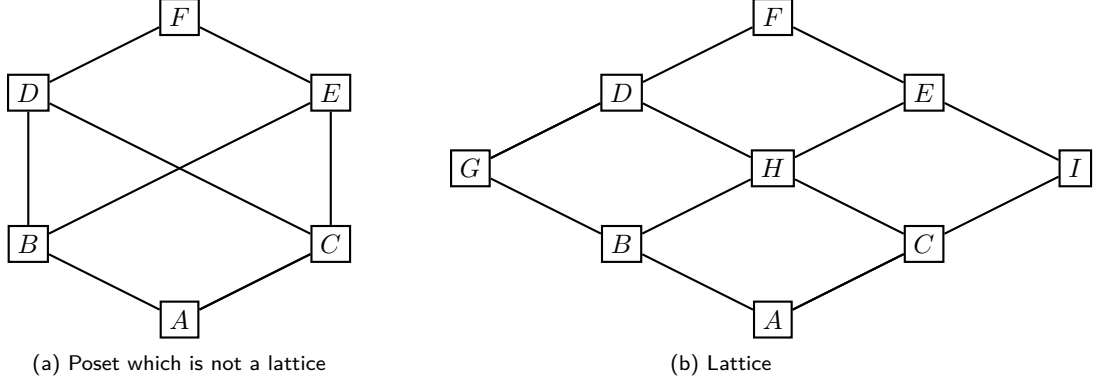


Figure 9: Finite posets can be depicted using *Hasse diagrams*. Here, two nodes A and B are connected and B is shown above A if B covers A (cf. Definition B.3). In (a), we see a poset which is not a lattice, since $B \prec D \succ C$ and $B \prec E \succ C$. Therefore, since $D \not\leq E$, B and C have no least upper bound. In (b), we see a poset which forms a lattice. Here, $H = B \vee C$.

Proof: We prove this via contraposition. Let \mathcal{T} be a conical poset with $a, b \in \mathcal{T}$. Assume that $a \not\leq b$ and that there exists a join $a \vee b \in \mathcal{T}$, which by definition is unique. Then, $f(\{a \vee b\}) = J^+(a \vee b) = J^+(a) \cap J^+(b) = f(\{a, b\})$. As $\text{span}(\{a, b\}) = \{a, b\}$ due to $a \not\leq b$, and $\text{span}(\{a \vee b\}) = a \vee b$, we have two distinct spanning sets which admit the same joint future, posing a contradiction to conicality. \square

Proposition B.12 (1+1-Minkowski space-time) *The light cone structure of 1+1-Minkowski space-time, having one spatial and one temporal dimension, forms a lattice of its points. The meet of two points is given by the latest point where their past light cones intersect, while their join is given by the earliest point where their future light cones intersect [49].*

Corollary B.13 (Higher-dim. Minkowski space-time) *In the case of d spatial dimensions, we speak of $d+1$ -Minkowski space-time. For $d \geq 2$, Minkowski space-time (as depicted in Figure 5) is a join- and meet-free poset.*

Proof: That Minkowski space-time is join-free follows from Lemma B.11, as it is conical. As Minkowski space-time is also invariant under time inversion, it is also meet-free. \square

This can also be seen from the fact that there are no unique minimal and maximal points, respectively. Specifically, the intersection of the future of two light cones is larger than any light cone that can be found within the intersection [49].

Nonetheless, causally closed subsets of Minkowski space-time actually assemble a so-called *complete orthomodular lattice* for any number of dimensions. The respective sets are sometimes referred to as causal diamonds. Interestingly, the same mathematical structure is also exhibited by the set of projectors in Hilbert space, as used in quantum mechanics [50][51, p. 109–115]. However, we are not aware of any deeper results which have originated from this observation.

C A formal motivation for reducibility in the second argument

In this section, we provide further details and examples on the definition of reducibility in the second argument, which in contrast to the first and third argument, is observational rather than interventional. We begin by repeating the definition below for convenience.

Definition 4.6 (Reducibility in the second argument) *We say that an affects relation $X \vDash Y | \text{do}(Z)$ is reducible in the second argument (or Red_2) if there exists a non-empty subset $s_Y \subsetneq Y$ such that both the following conditions hold, where $\tilde{s}_Y := Y \setminus s_Y$.*

- $X \not\equiv s_Y | \text{do}(Z), \tilde{s}_Y$ (cf. Definition E.1)
- $(s_Y \perp\!\!\!\perp \tilde{s}_Y | XZ)_{\mathcal{G}_{\text{do}(XZ)}}$ or $(s_Y \perp\!\!\!\perp \tilde{s}_Y | Z)_{\mathcal{G}_{\text{do}(Z)}}$

Otherwise, we say that $X \vDash Y | \text{do}(Z)$ is irreducible in the second argument and denote it as Irred_2 .

Operational motivation for the definition Consider again the example mentioned in the main text (Section 4.1), in relation to Red_2 , where we have a causal structure with $X \rightsquigarrow Y_1$ while Y_2 is a node with no in or outgoing arrows. Clearly $X \vDash Y_1 Y_2$ here must be reducible to $X \vDash Y_1$ here, according to any reasonable notion of Red_2 , as Y_2 is entirely redundant in this scenario. Operationally, we want the definition to capture that the original affects relation $X \vDash Y_1 Y_2$ and the reduced relation $X \vDash Y_1$ capture equivalent phenomena. As mentioned in the main text, in this case, this operational intuition cannot be captured mathematically by simply equating the left and right hand sides of the expressions for the two affects relations. Instead, the redundancy of Y_2 in $X \vDash Y_1 Y_2$ is captured through conditional independences.

- $(Y_1 \perp\!\!\!\perp Y_2)_{\mathcal{G}}$, i.e. $P_{\mathcal{G}}(Y_1 Y_2) = P_{\mathcal{G}}(Y_1)P_{\mathcal{G}}(Y_2)$
- $(Y_1 \perp\!\!\!\perp Y_2 | X)_{\mathcal{G}_{\text{do}(X)}}$, i.e. $P_{\mathcal{G}_{\text{do}(X)}}(Y_1 Y_2 | X) = P_{\mathcal{G}_{\text{do}(X)}}(Y_1 | X)P_{\mathcal{G}_{\text{do}(X)}}(Y_2 | X)$

While both the above independences are satisfied in our simple example, we now show that even if one of these independences is satisfied, along with $X \not\vDash Y_2 | Y_1$ (this is the first condition of Definition 4.6 applied to this example, which is also satisfied here), then the two affects relations $X \vDash Y_1 Y_2$ and $X \vDash Y_1$ are equivalent. Since satisfying one of these conditional independences is the second condition of Definition 4.6, for the case of this example, this would establish that under the defining conditions of Red_2 , the original and reduced affects relations are indeed equivalent as we require. More generally, for any general higher-order affects relation $X \vDash Y | \text{do}(Z)$, if the conditions of Definition 4.6 hold for some $s_Y \subsetneq Y$, then through similar arguments as we present below, it can be shown that the original affects relation is equivalent to the reduced relation $X \vDash \tilde{s}_Y | \text{do}(Z)$.

We start by assuming $X \not\vDash Y_2 | Y_1$ and the first conditional independence $(Y_1 \perp\!\!\!\perp Y_2)_{\mathcal{G}}$. Writing out the expressions for $X \not\vDash Y_2 | Y_1$ and noting that $(Y_1 \perp\!\!\!\perp Y_2)_{\mathcal{G}}$ implies $P_{\mathcal{G}}(Y_2 | Y_1) = P_{\mathcal{G}}(Y_2)$, we obtain

$$P_{\mathcal{G}_{\text{do}(X)}}(Y_2 | Y_1 X) = P_{\mathcal{G}}(Y_2). \quad (29)$$

Now, to show that $X \vDash Y_1 Y_2$ implies $X \vDash Y_1$ under this assumption, we write the expression for the former and use $(Y_1 \perp\!\!\!\perp Y_2)_{\mathcal{G}}$.

$$P_{\mathcal{G}_{\text{do}(X)}}(Y_1 Y_2 | X) \neq P_{\mathcal{G}}(Y_1 Y_2) = P_{\mathcal{G}}(Y_1)P_{\mathcal{G}}(Y_2). \quad (30)$$

Using $P_{\mathcal{G}_{\text{do}(X)}}(Y_1 Y_2 | X) = P_{\mathcal{G}_{\text{do}(X)}}(Y_2 | Y_1 X)P_{\mathcal{G}_{\text{do}(X)}}(Y_1 | X)$, we have

$$P_{\mathcal{G}_{\text{do}(X)}}(Y_2 | Y_1 X)P_{\mathcal{G}_{\text{do}(X)}}(Y_1 | X) \neq P_{\mathcal{G}}(Y_1)P_{\mathcal{G}}(Y_2). \quad (31)$$

Combining this with Equation (29) simplifies the condition to

$$P_{\mathcal{G}_{\text{do}(X)}}(Y_1 | X) \neq P_{\mathcal{G}}(Y_1), \quad (32)$$

which is equivalent to $X \vDash Y_1$. By running this proof in the other direction, it is easy to see that under the same assumptions, $X \vDash Y_1$ also implies $X \vDash Y_1 Y_2$, making them equivalent under these conditions. Similarly, it can also be checked that if we start by assuming $X \not\vDash Y_2 | Y_1$ and the second conditional independence, $(Y_1 \perp\!\!\!\perp Y_2 | X)_{\mathcal{G}_{\text{do}(X)}}$, we can analogously show the equivalence of $X \vDash Y_1 Y_2$ and $X \vDash Y_1$.

Therefore, we see that the two conditions of Definition 4.6 have allowed us to operationally identify the redundancy of Y_2 in this example just from the correlations and affects relations. Some examples are as follows, these examples also illustrate that all of the conditions of Definition 4.6 can be violated independently, resulting in irreducible affects relations.

Example 11 (*Violating the first Red_2 condition*) To illustrate that the first condition of Definition 4.6 can be violated independently, consider the causal structure of Figure 10a with the causal model: E_1 is non-uniform, E_2 is uniform, $X = E_1 \oplus E_2$, $Y_2 = X \oplus E_2$ and $Y_1 = X$. We notice that $P_{\mathcal{G}}(Y_1)$ is uniform (since $Y_1 = X$ but $X = E_1 \oplus E_2$ and E_2 is uniform) and $P_{\mathcal{G}}(Y_2) = P_{\mathcal{G}}(E_1)$ is non-uniform (since $Y_2 = X \oplus E_2$ and $X = E_1 \oplus E_2$ gives $Y_2 = E_1$). In $\mathcal{G}_{\text{do}(X)}$, we only have the dependences: $Y_2 = X \oplus E_2$ and $Y_1 = X$ and we can see that $P_{\mathcal{G}_{\text{do}(X)}}(Y_1 Y_2 | X)$ will be different from $P_{\mathcal{G}}(Y_1 Y_2)$ (in particular since Y_1 is fully determined by X) hence $X \vDash Y_1 Y_2$ holds. Moreover, $(Y_1 \perp\!\!\!\perp Y_2)_{\mathcal{G}}$. We also

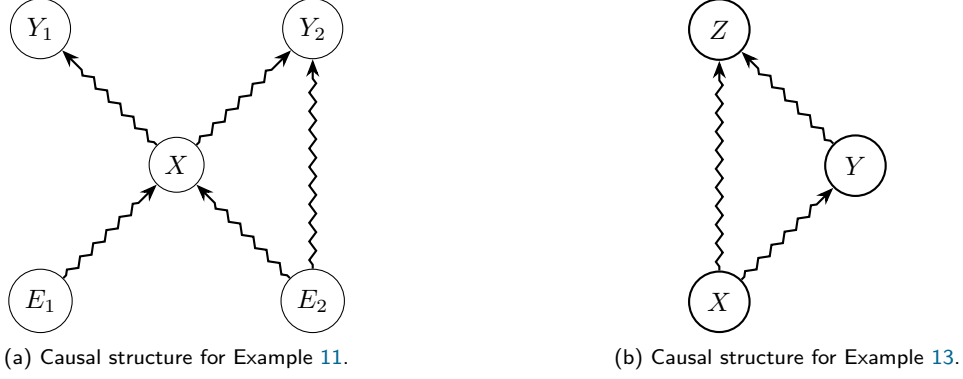


Figure 10: Causal structures for various examples.

see that $(Y_1 \perp\!\!\!\perp Y_2 | X)_{\mathcal{G}_{\text{do}(X)}}$ holds, as this follows from the d -separation $(Y_1 \perp^d Y_2 | X)_{\mathcal{G}_{\text{do}(X)}}$ in the causal structure. Noting that the only non-empty strict subsets s_Y in this case are $\{Y_1\}$ and $\{Y_2\}$, the second condition is satisfied. Finally, notice that $P_{\mathcal{G}_{\text{do}(X)}}(Y_1 | X)$ is deterministic and $P_{\mathcal{G}_{\text{do}(X)}}(Y_2 | X)$ is uniform (since $Y_2 = X \oplus E_2$ with X independent of E_2 and E_2 uniform in $\mathcal{G}_{\text{do}(X)}$). Therefore we have $X \vDash Y_1$ and $X \vDash Y_2$ which in particular imply that $X \vDash Y_1 | Y_2$ and $X \vDash Y_2 | Y_1$, as additional conditioning preserves an affects relation if Y_1 and Y_2 are conditionally independent both in the pre- and post-intervention causal structure. Therefore, the first condition is violated. In summary, $X \vDash Y_1 Y_2$ is Irred_2 in this example, it violates the first and satisfies the second two conditions of Definition 4.6.

Example 12 (Violating the second Red_2 condition) To illustrate that the second condition of Definition 4.6 can be violated independently, consider the causal structure $Y_1 \leftarrow X \leftarrow E \rightarrow Y_2$ and the causal model where $Y_1 = X$, $X = E$, $Y_2 = E$ and any distribution over E . Clearly $X \vDash Y_1$, $X \vDash Y_1 Y_2$. We also have $X \not\vDash Y_2$ and $X \not\vDash Y_2 | Y_1$ due to d -separation and therefore the first condition of Definition 4.6 is satisfied for the affects relation $X \vDash Y_1 Y_2$. Moreover $(Y_1 \perp\!\!\!\perp Y_2 | X)_{\mathcal{G}_{\text{do}(X)}}$ holds due to the d -separation $(Y_1 \perp^d Y_2 | X)_{\mathcal{G}_{\text{do}(X)}}$ which means that the first alternative second condition would also be satisfied. However, the second alternative is violated as we clearly have $(Y_1 \not\perp\!\!\!\perp Y_2)_{\mathcal{G}}$. Therefore, $X \vDash Y_1 Y_2$ is Red_2 .

D Further properties of clustering

In this section we present interesting properties of clustering in the first three arguments of an affects relation, which may be of independent interest, but are not pivotal for our main results. First, we show that the presence of an affects relation that is clustered in its first or third argument is incompatible with the presence of a wide variety of affects relations.

Lemma D.1 For any affects relation $X \vDash Y | \text{do}(Z)$ which is Clus_1 , $\tilde{s}_X^1 \not\vDash Y | \text{do}(\tilde{s}_X^2 Z)$ for all $\tilde{s}_X^1, \tilde{s}_X^2 \subsetneq X$ disjoint with $\tilde{s}_X^1 \tilde{s}_X^2 \neq X$.

Proof: Due to $X \vDash Y | \text{do}(Z)$ being Clus_1 , $\tilde{s}_X \not\vDash Y | \text{do}(Z)$ for all $\tilde{s}_X \subsetneq X$. Hence,

$$P_{\mathcal{G}_{\text{do}(Z)}}(Y | Z) = P_{\mathcal{G}_{\text{do}(\tilde{s}_X Z)}}(Y | \tilde{s}_X Z), \quad \forall \tilde{s}_X \subsetneq X \quad (33)$$

$$\implies P_{\mathcal{G}_{\text{do}(\tilde{s}_X^1 Z)}}(Y | \tilde{s}_X^1 Z) = P_{\mathcal{G}_{\text{do}(\tilde{s}_X^2 Z)}}(Y | \tilde{s}_X^2 Z), \quad \forall \tilde{s}_X^1, \tilde{s}_X^2 \subsetneq X \quad (34)$$

$$\implies P_{\mathcal{G}_{\text{do}(\tilde{s}_X Z)}}(Y | \tilde{s}_X^1 Z) = P_{\mathcal{G}_{\text{do}(\tilde{s}_X^1 \tilde{s}_X^2 Z)}}(Y | \tilde{s}_X^2 \tilde{s}_X^1 Z), \quad \forall \tilde{s}_X^2 \subseteq \tilde{s}_X^1 \tilde{s}_X^2 \subsetneq X \quad (35)$$

However, this line is equivalent to stating $\tilde{s}_X^1 \not\vDash Y | \text{do}(\tilde{s}_X^2 Z)$ for all $\tilde{s}_X^1, \tilde{s}_X^2$ disjoint with $\tilde{s}_X^1 \tilde{s}_X^2 \neq X$. \square

Lemma D.2 For any affects relation $X \vDash Y | \text{do}(Z)$ which is Clus_3 , $\tilde{s}_Z^1 \not\vDash Y | \text{do}(\tilde{s}_Z^2)$ for all $\tilde{s}_Z^1, \tilde{s}_Z^2 \subsetneq Z$ disjoint with $\tilde{s}_Z^1 \tilde{s}_Z^2 \neq Z$.

Proof: Due to $X \vDash Y | \text{do}(Z)$ being Clus_3 , $X \not\vDash Y | \text{do}(Xs_Z)$ for all $\tilde{s}_Z \subsetneq Z$. Hence,

$$P_{\mathcal{G}_{\text{do}(Z)}}(Y|XZ) = P_{\mathcal{G}_{\text{do}(\tilde{s}_Z X)}}(Y|X\tilde{s}_Z W), \quad \forall \tilde{s}_Z \subsetneq Z \quad (36)$$

$$\implies P_{\mathcal{G}_{\text{do}(\tilde{s}_X Z)}}(Y|X\tilde{s}_Z^1) = P_{\mathcal{G}_{\text{do}(\tilde{s}_Z X)}}(Y|X\tilde{s}_Z^2), \quad \forall \tilde{s}_Z^2, \tilde{s}_Z \subsetneq Z \quad (37)$$

$$P_{\mathcal{G}_{\text{do}(\tilde{s}_Z X)}}(Y|X\tilde{s}_Z^1) = P_{\mathcal{G}_{\text{do}(\tilde{s}_Z^1 \tilde{s}_Z^2 Z X)}}(Y|X\tilde{s}_Z^2 \tilde{s}_Z^1 X), \quad \forall \tilde{s}_Z^2 \subseteq \tilde{s}_Z^1 \tilde{s}_Z^2 \subsetneq Z \quad (38)$$

However, this line is equivalent to stating $\tilde{s}_Z^1 \not\vDash Y | \text{do}(\tilde{s}_Z^2 X)$ for all $\tilde{s}_Z^1, \tilde{s}_Z^2$ disjoint with $\tilde{s}_Z^1 \tilde{s}_Z^2 \neq Z$. \square

Second, we demonstrate that for a causal model to admit affects relations that are Clus_1 , it is necessary but not sufficient for the model to also admit affects relations that are Clus_3 .

Lemma D.3 *For any affects relation $X \vDash Y | \text{do}(Z)$ which is Clus_1 , $s_X \vDash Y | \text{do}(\tilde{s}_X Z)$ is Clus_3 for each partition $s_X \tilde{s}_X = X$.*

Proof: By Theorem 4.11, $X \vDash Y | \text{do}(Z)$ is Irred_1 . Hence, for all $s_X \tilde{s}_X = X$, we have $s_X \vDash Y | \text{do}(\tilde{s}_X Z)$. Due to Lemma D.1, we also have $s_X \not\vDash Y | \text{do}(\tilde{s}'_X Z)$, where $\tilde{s}'_X \subsetneq \tilde{s}_X$. This is equivalent for Clus_3 for $s_X \vDash Y | \text{do}(\tilde{s}_X Z)$. \square

By contrast, we can construct an example which has clustering only in the third argument.

Example 13 (*Clus₃ only*) *Consider a causal model over $S = \{X, Y, Z\}$, where X is exogenous and non-uniform, $Y = X$ and $Z = Y \oplus X$. The respective causal structure is shown in Figure 10b. Then pre-intervention, Z is deterministically 0. Clearly, this causal model is fine-tuned with regard to the value of Z : No matter which intervention we choose on X , this does not change, hence $X \not\vDash Z$, yet clearly $X \vDash Y$ and also $Y \vDash Z$, as Z is no longer uniform after an intervention. Accordingly, since we have only three observable nodes in this model, any clustered affects relation must have X in its first and Z in its second argument, while not having Y in its first and second argument. Hence, $X \vDash Z | \text{do}(Y)$, clustered in its third argument, is the only clustered affects relation in this example. Finally, we would like to point out that in this causal model, affects relations are not transitive: We have $X \vDash Y$ and $Y \vDash Z$, yet $X \not\vDash Z$.*

Finally, we demonstrate that in absence of clustering in a certain argument, a causal model must allow each affects relation to be reduced to minimal cardinality for the respective argument.

Lemma D.4 ($\neg \text{Clus}_1$) *Consider an affects relation $X \vDash Y | \text{do}(Z)$ in a causal model which does not yield any affects relation that are Clus_1 . Then there exists $e_X \in X$ such that $e_X \vDash Y | \text{do}(Z)$.*

Proof: As $X \vDash Y | \text{do}(Z)$ is not Clus_1 , there exists a subset $s_X \subsetneq X$ such that $s_X \vDash Y | \text{do}(Z)$. However, as this is again not Clus_1 , we can repeat this inductively until we reach $e_X \in Y$ with $e_X \vDash Y | \text{do}(Z)$. \square

Lemma D.5 ($\neg \text{Clus}_2$) *Consider an affects relation $X \vDash Y | \text{do}(Z)$ in a causal model which does not yield any affects relation that are Clus_2 . Then there exists $e_Y \in Y$ such that $X \vDash e_Y | \text{do}(Z)$.*

Proof: Directly analogous to $\neg \text{Clus}_1$, but for the second argument. \square

Lemma D.6 ($\neg \text{Clus}_3$) *Consider an affects relation $X \vDash Y | \text{do}(Z)$ in a causal model which does not yield any affects relation that are Clus_3 . Then there exists $e_X \in X$ such that $e_X \vDash Y$.*

Proof: As $X \vDash Y | \text{do}(Z)$ is not Clus_3 , there exists a subset $s_Z \subsetneq Z$ such that $X \vDash Y | \text{do}(s_Z)$. However, as this is again not Clus_3 , we can repeat this inductively until we reach the 0^{th} -order affects relation $X \vDash Y$. Additionally, by Lemma D.3, the absence of any affects relations with Clus_1 implies the same for Clus_3 . Therefore, we can use Lemma D.4 to deduce the claim. \square

Considering that by Theorem 4.12 the absence of clustering is necessary for faithfulness, we can summarize these results accordingly.

Corollary D.7 *Consider an affects relation $X \vDash Y | \text{do}(Z)$ in a causal model without clustering. Then there exist $e_X \in X, e_Y \in Y$ such that $e_X \vDash e_Y$. In particular, this holds for any faithful causal model.*

Recollecting Example 9, this indicates that in absence of clustering, any fine-tuning detectable from affects relations alone is possible to deduce via non-transitive affects relations.

E Conditional higher-order affects relations

Within this section, we generalize results from the main text to the case of conditional affects relations, and provide some additional useful transformation rules which are not relevant to the main results of this work. To start off, we provide a formal definition of conditional affects relations.

Definition E.1 (Conditional (Higher-Order) Affects Relations [10]) *Consider a causal model over a set of S observed nodes, associated with a causal structure \mathcal{G} . For pairwise disjoint subsets $X, Y, Z, W \subset S$, with X, Y non-empty, we say*

$$X \text{ affects } Y \text{ given } \text{do}(Z), W, \quad (39)$$

which we alternatively denote as

$$X \vDash Y \mid \text{do}(Z), W, \quad (40)$$

if there exist values x of X , z of Z and w of W such that

$$P_{\mathcal{G}_{\text{do}(XZ)}}(Y \mid X = x, Z = z, W = w) \neq P_{\mathcal{G}_{\text{do}(Z)}}(Y \mid Z = z, W = w) \quad (41)$$

For $W \neq \emptyset$, we speak of a conditional affects relation, denoted by $X \vDash Y \mid W$. If $Z \neq \emptyset$, we have a higher-order (HO) affects relation, denoted by $X \vDash Y \mid \text{do}(Z)$. More specifically, it is also called a $|Z|^{\text{th}}$ -order affects relation. The trivial case of $W = Z = \emptyset$ is called an unconditional 0^{th} -order affects relation, denoted by $X \vDash Y$.

E.1 Properties

We begin by restating Lemma IV.8 of [10], which makes clear how conditional and unconditional affects relations relate with one another.

Lemma E.2 *For a causal model over a set S of RVs, where $X, Y, Z, W \subset S$ disjoint,*

1. $X \vDash Y \mid \text{do}(Z), W \implies X \vDash YW$.
2. $X \vDash Y \mid \text{do}(Z), W \text{ is Irred}_1 \implies X \vDash YW \text{ is Irred}_1$.
3. $X \vDash YW \mid \text{do}(Z) \iff X \vDash Y \mid \text{do}(Z), W \vee X \vDash W \mid \text{do}(Z)$.

By the same argument as for Lemma E.2.2., one can also derive:

Corollary E.3 *For a causal model over a set S of RVs, where $X, Y, Z, W \subset S$ disjoint, $X \vDash Y \mid \text{do}(Z), W \text{ is Irred}_3 \implies X \vDash YW \mid \text{do}(Z) \text{ is Irred}_3$.*

We continue with a general result about conditional affects relations that will be useful, this is essentially a generalization of Lemma 4.1 to the conditional case. The proof method is exactly analogous to that of Lemma 4.1, but we repeat it for completeness.

Lemma E.4 *For a causal model over a set S of RVs, where $X, Y, Z, W \subset S$ disjoint, $X \not\vDash Y \mid \text{do}(Z), W \implies (X \perp\!\!\!\perp Y \mid ZW)_{\mathcal{G}_{\text{do}(XZ)}}$.*

Lemma E.5 *For a causal model over a set S of RVs, where $X, Y, Z, W \subset S$ disjoint,*

$$(X \perp\!\!\!\perp^d Y \mid ZW)_{\mathcal{G}_{\text{do}(X(W)Z)}} \implies X \not\vDash Y \mid \{\text{do}(Z), W\},$$

with $X(W) := X \setminus (X \cap \text{anc}(W))$. where $\text{anc}(W)$ is the set of all ancestors of W in the original graph \mathcal{G} .

It is proven in Theorem IV.1 of [10] that Pearl's three rules of do-calculus [2] also hold in the general affects framework and can be derived from the d-separation property alone. The above lemma follows immediately from Pearl's third rule of do-calculus.

Notice that between these two statements, there is a subtle difference regarding the relation between X and W in the causal structure. Specifically, the requirement in Lemma E.5 is satisfied whenever $W = \emptyset$ and whenever X is not a cause of W . This impacts the generalization of some of the statements from the main text for unconditional affects relations ($W = \emptyset$) to conditional relations. For the case that

both X is not a cause of W (and thus, $X = X(W)$) and the causal model is faithful (hence, we have $(X \perp\!\!\!\perp Y|ZW)_{\mathcal{G}_{\text{do}}(XZ)} \iff (X \perp^d Y|ZW)_{\mathcal{G}_{\text{do}}(XZ)}$), then Lemma E.4 and Lemma E.5 taken together show that,

$$X \not\equiv Y|\{\text{do}(Z), W\} \iff (X \perp\!\!\!\perp Y|ZW)_{\mathcal{G}_{\text{do}}(XZ)} \iff (X \perp^d Y|ZW)_{\mathcal{G}_{\text{do}}(XZ)}, \quad (42)$$

while for unfaithful models, conditional independence (middle) need not imply absence of affects relations (left) or d-separation (right).

E.2 Generalizing reducibility and clustering

In this section, we present the generalization of the results of Section 4, originally presented for unconditional affects relations, to the case of conditional affects relations. While some of the results fully generalize, the others require an additional condition when introducing a non-trivial fourth argument $W \neq \emptyset$ of the conditional affects relation, owing to the condition appearing in Lemma E.5.

We begin by defining the reducibility concepts introduced in Section 4.1 for the first three arguments of unconditional affects relations $X \equiv Y | \text{do}(Z)$ to all four arguments of conditional affects relations $X \equiv Y | \text{do}(Z), W$. For the first three arguments, this is the same as Definition 4.2, Definition 4.6 and Definition 4.4 but with the inclusion of W , but we repeat them here for completeness.

Definition E.6 (Reducibility in the first argument [10]) *We say that an affects relation $X \equiv Y | \text{do}(Z), W$ is reducible in the first argument (or Red_1) if there exists a non-empty subset $s_X \subsetneq X$ such that $s_X \equiv Y | \text{do}(\tilde{s}_X Z), W$, where $\tilde{s}_X := X \setminus s_X$. Otherwise, we say that $X \equiv Y | \text{do}(Z), W$ is irreducible in the first argument and denote it as Irred_1 .*

Definition E.7 (Reducibility in the second argument) *We say that an affects relation $X \equiv Y | \text{do}(Z), W$ is reducible in the second argument (or Red_2) if there exists a non-empty subset $s_Y \subsetneq Y$ such that both the following conditions hold, where $\tilde{s}_Y := Y \setminus s_Y$.*

- $X \equiv s_Y | \text{do}(Z), \tilde{s}_Y W$
- $(s_Y \perp\!\!\!\perp \tilde{s}_Y | XZW)_{\mathcal{G}_{\text{do}}(XZ)}$ or $(s_Y \perp\!\!\!\perp \tilde{s}_Y | ZW)_{\mathcal{G}_{\text{do}}(Z)}$

Otherwise, we say that $X \equiv Y | \text{do}(Z), W$ is irreducible in the second argument and denote it as Irred_2 .

Definition E.8 (Reducibility in the third argument) *We say that an affects relation $X \equiv Y | \text{do}(Z), W$ is reducible in the third argument (or Red_3) if there exists a non-empty subset $s_Z \subseteq Z$ such that both the following conditions hold, where $\tilde{s}_Z := Z \setminus s_Z$*

- $s_Z \equiv Y | \text{do}(X \tilde{s}_Z), W$
- $s_Z \equiv Y | \text{do}(\tilde{s}_Z), W$

Otherwise, we say that $X \equiv Y | \text{do}(Z), W$ is irreducible in the third argument and denote it as Irred_3 .

Definition E.9 (Reducibility in the fourth argument) *We say that an affects relation $X \equiv Y | \text{do}(Z), W$ is reducible in the fourth argument (or Red_4) if there exists a non-empty subset $s_W \subseteq W$ such that all the following conditions hold, where $\tilde{s}_W := W \setminus s_W$.*

- $(Y \perp\!\!\!\perp s_W | XZ \tilde{s}_W)_{\mathcal{G}_{\text{do}}(XZ)}$
- $(Y \perp\!\!\!\perp s_W | Z \tilde{s}_W)_{\mathcal{G}_{\text{do}}(XZ)}$

Otherwise, we say that $X \equiv Y | \text{do}(Z), W$ is irreducible in the fourth argument and denote it as Irred_4 .

It is then easy to check that the following generalizations of Lemma 4.3, Lemma 4.7 and Lemma 4.5 for the first three arguments follow from the same proof method (the first of these is proven in [10]).

Lemma E.10 *If $X \equiv Y | \text{do}(Z), W$ is a Red_1 affects relation, then there exists $\tilde{s}_X \subsetneq X$ such that $\tilde{s}_X \equiv Y | \text{do}(Z), W$ holds.*

Lemma E.11 *If $X \models Y|\text{do}(Z), W$ is a Red_2 affects relation, then there exists $\tilde{s}_Y \subsetneq Y$ such that $X \models \tilde{s}_Y|\text{do}(Z), W$ holds.*

Lemma E.12 *If $X \models Y|\text{do}(Z), W$ is a Red_3 affects relation, then there exists $\tilde{s}_Z \subsetneq Z$ such that $X \models Y|\text{do}(\tilde{s}_Z), W$ holds.*

For the fourth argument, the following analogous lemma is proven in Appendix G.4.

Lemma E.13 *If $X \models Y|\text{do}(Z), W$ is a Red_4 affects relation, then there exists $\tilde{s}_W \subsetneq W$ such that $X \models Y|\text{do}(Z), \tilde{s}_W$ holds.*

We note that the operational motivations for $\text{Red}_1, \text{Red}_2$ and Red_3 given in Section 4.1 also extend to the conditional case (as can be easily checked). The operational motivation for Red_4 introduced in this appendix is discussed below.

Operational motivation for the definition of Red_4 The rationale in this case mirrors that of Red_3 , the two conditions of Definition E.9 demand equality between the left hand side and right hand side respectively of the expressions for the original affects relation $X \models Y|\text{do}(Z), W$ and the reduced affects relation $X \models Y|\text{do}(Z), \tilde{s}_W$. This ensures that the two affects relations are equivalent and carry the same information.

The following examples illustrate two instances of Irred_4 affects relations, showing that the two conditions of Definition E.9 can be independently violated.

Example 14 (Violating the first Red_4 condition) *To illustrate that the first condition of Definition E.9 can be independently violated, consider the causal structure $X \rightsquigarrow Y, X \rightsquigarrow W$ and $E \rightsquigarrow W$ with any distribution over $X, P(E)$ being non-uniform and non-deterministic, $W = X \oplus W, Y = X$. Since X is exogenous and $P_G(Y|XW) \neq P_G(Y|W)$ whenever W is non-deterministic, we have $X \models Y|W$. However Y is correlated with W since E is non-uniform and we have $(Y \not\perp W)_{\mathcal{G}_{\text{do}(X)}}$ which violates the second condition of Definition E.9 for the affects relation $X \models Y|W$ with $s_W = W$ (which is the only possibility for s_W in this case). However, $(Y \perp W|X)_{\mathcal{G}_{\text{do}(X)}}$ holds due to the d-separation property and we therefore satisfy the second condition for $X \models Y|W$ with $s_W = W$.*

Example 15 (Violating the second Red_4 condition) *To illustrate that the second condition of Definition E.9 can be independently violated, consider again a jamming causal model with $X \rightsquigarrow Y, \Lambda \rightsquigarrow Y$ and $\Lambda \rightsquigarrow W$, with Λ uniformly distributed, $Y = \Lambda \oplus X$ and $W = \Lambda$. We clearly have $X \models Y|W$ which will be our original affects relation, where $s_W = W$ is the only possible non-empty subset of the fourth argument. Since X is exogenous, we have $(Y \perp W)_{\mathcal{G}_{\text{do}(X)}}$ and $(Y \not\perp W|X)_{\mathcal{G}_{\text{do}(X)}}$ i.e., for the affects relation $X \models Y|W$, the first condition of Definition E.9 is satisfied but the second condition is violated.*

Analogously, we can generalize the definition of clustering in different arguments to conditional affects relations. The following three definitions are a direct generalization of Definition 4.8, Definition 4.9 and Definition 4.10, with an inclusion of W .

Definition E.14 (Clustering in the first argument) *An affects relation $X \models Y|\text{do}(Z), W$ is called clustered in the first argument (denoted Clus_1) if $|X| \geq 2$ and there exists no $s_X \subsetneq X$ such that $s_X \models Y|\text{do}(Z), W$.*

Definition E.15 (Clustering in the second argument) *An affects relation $X \models Y|\text{do}(Z), W$ is called clustered in the second argument (denoted Clus_2) if $|Y| \geq 2$ and there exists no $s_Y \subsetneq Y$ such that $X \models s_Y|\text{do}(Z), W$.*

Definition E.16 (Clustering in the third argument) *An affects relation $X \models Y|\text{do}(Z), W$ is called clustered in the third argument (denoted Clus_3) if $|Z| \geq 1$ and there exists no $s_Z \subsetneq Z$ such that $X \models Y|\text{do}(s_Z), W$.*

Definition E.17 (Clustering in the fourth argument) *An affects relation $X \models Y|\text{do}(Z), W$ is called clustered in the fourth argument (denoted Clus_4) if there exists no non-empty subset $s_W \subsetneq W$ such that $X \models Y|\text{do}(Z), s_W$.*

Notice that due to the requirement that s_W must be a non-empty and strict subset of W , the minimum cardinality of W needed for a conditional affects relation to be Clus_4 is 2. The reason for this is because the fourth argument captures post-selection without intervention, thus allowing W to act as a collider to X and Y . This means that in any faithful causal model where W is a collider between X and Y and X is not a cause of Y , we would generically have $X \vDash Y|W$ even though $X \not\vDash Y$ (in contrast, we have seen in Theorem 4.12 that the analogous situation $X \vDash Y|\text{do}(Z)$ and $X \not\vDash Y$ for the third argument is only possible in fine-tuned models). With the above definitions, we can now generalize Theorem 4.11 (relating clustering and irreducibility) to conditional affects relations and all four arguments, as shown below.

Theorem E.18 [*Clustering implies irreducibility*] For any affects relation $X \vDash Y|\text{do}(Z), W$, $\text{Clus}_i \Rightarrow \text{Irred}_i$ for all $i \in \{1, 2, 3, 4\}$.

Example 16 ($\text{Irred}_4 \not\Rightarrow \text{Clus}_4$) Consider the jamming-type causal model over binary variables, with $X \rightsquigarrow Y$, $\Lambda \rightsquigarrow Y$ and $\Lambda \rightsquigarrow W$, with $P(\Lambda)$ being neither uniformly distributed nor a deterministic distribution, $W = \Lambda$ and $Y = X \oplus \Lambda$. Here we have $X \vDash Y$ since Λ is non-uniform as well as $X \vDash Y|W$ since Λ is non-deterministic. Therefore $X \vDash Y|W$ does not satisfy the Clus_4 property. However since $Y \not\perp W|X$ (and X is exogenous to the same holds under interventions on X), $X \vDash Y|W$ is not reducible in the fourth argument and is an Irred_4 affects relation.

Theorem 4.12 in the main text proves that clustering in the first, second and third argument of an unconditional affects relation $X \vDash Y|\text{do}(Z)$ implies fine-tuning of the underlying causal model. Here we extend the result to conditional affects relations and clustering in the fourth argument, introduced in this section. Due to the restriction to the subset $X(W)$ of X in Lemma E.5, which is only relevant when considering non-trivial conditional affects relations with $W \neq \emptyset$, our result for the conditional case is not fully general and applies to cases where $X(W) = X$, or equivalently where X is not a cause of W .¹⁵ We conjecture that the theorem also holds more generally, without this restriction and leave a proof of this to future work, as it is not directly relevant for the main results of this paper.

Theorem E.19 [*Clustering implies fine-tuning (conditional version)*] Any affects relation $X \vDash Y|\text{do}(Z), W$ that satisfies Clus_i for $i \in \{1, 2, 3, 4\}$ and generated in a causal graph \mathcal{G} where X is not a cause of W , necessarily arises from a fine-tuned causal model on \mathcal{G} .

Finally, we provide an additional result relating Clus_2 and Clus_4 (clustering in observational arguments) which is analogous to Lemma D.3 which links Clus_1 and Clus_3 (clustering in the interventional arguments). However, due to the minimum cardinality for affects relations to be Clus_4 , it only holds in a more restricted case.

Corollary E.20 For any unconditional affects relation $X \vDash Y|\text{do}(Z)$ which is Clus_2 and has $|Y| \geq 3$, $X \vDash s_Y|\text{do}(Z)$, \tilde{s}_Y is Clus_4 for each partition $s_Y \tilde{s}_Y = Y$ with $|\tilde{s}_Y| \geq 2$.

The above result is only established for unconditional affects relations, and the full generalization of these statements to conditional relations remains open.

E.3 Compatibility for conditional affects relations

We provide the generalization of Definition 6.1 to conditional affects relations, as given in [10], yet simplified with regard to the embedding.

Corollary E.21 Let \mathcal{S} be a set of ORVs from a set of RVs S and a poset \mathcal{T} with an embedding \mathcal{E} . Then a set of affects relations \mathcal{A} is said to be compatible with \mathcal{E} (or satisfies **compat**) if the following condition holds:

- Let $X, Y \subset S$ be disjoint non-empty sets of RVs, $Z, W \subset S$ further disjoint sets of RVs, potentially empty. If $(X \vDash Y | \text{do}(Z), W) \in \mathcal{A}$ and is Irred_1 , then $\bar{\mathcal{F}}_s(\mathcal{Y}\mathcal{W}\mathcal{Z}) \subseteq \bar{\mathcal{F}}_s(\mathcal{X})$.

This generalization follows from the unconditional definition of compatibility by applying Lemma E.2.

¹⁵ X is a cause of W is equivalent to saying that there exists an element e_X that has a directed path to some element e_W of W , therefore X is not a cause of W is equivalent to $X \cap \text{anc}(W) = \emptyset$, which is equivalent to $X(W) = X$ for $X(W)$ as defined in Lemma E.5.

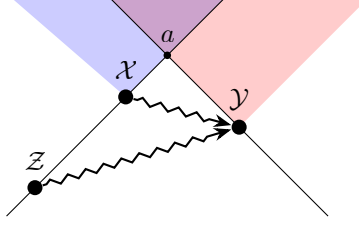


Figure 11: Combined representation of the causal structure of Example 17 and its space-time embedding into 1+1-Minkowski space-time satisfying **compat-atomic** and accordingly, $\bar{\mathcal{F}}_s(\mathcal{Y}\mathcal{Z}) = \bar{\mathcal{F}}_s(\mathcal{Y}\mathcal{X})$. Even though $XZ \vDash Y$, we have $\bar{\mathcal{F}}(\mathcal{Y}) \not\subseteq \bar{\mathcal{F}}_s(\mathcal{X}\mathcal{Z})$ and hence violate **compat**.

F Relaxing compatibility to atomic affects relations

As we have seen, the causal structure is a graph whose nodes are associated with individual random variables or physical systems (of some theory) while the signalling structure, which is captured by affects relations in our framework, is defined on sets of random variables. In other frameworks, which consider notions of signalling defined for quantum systems in a circuit, it has been shown that in unitary quantum circuits, signalling relations between individual in/output systems are sufficient to fully characterize all the signalling relations.¹⁶ This property of unitary circuits is referred to as atomicity [52]. In d-separation causal models, whenever there is an inequivalence between causation and signalling, this indicates fine-tuning [10]. The assumption of a unitary quantum circuit is closely related to the assumption of a faithful (not fine-tuned) causal model in the quantum case, as discussed in [9].

This motivates us to consider a similar property for affects relations (which capture signalling), and call an affects relation atomic if it originates from a single RV: $X \vDash Y \mid \text{do}(Z)$ with $|X| = 1$.¹⁷ This is particularly appealing as all such affects relations are Irred_1 by construction.

Then we ask, when does imposing compatibility only for atomic affects relations with a given space-time imply general compatibility for all irreducible affects relations. We find that this is not true in general but it is true when we either restrict to faithful causal models or to conical space-times, revealing yet another correspondence between these distinct causality concepts.

Definition F.1 (compat-atomic) *Let \mathcal{S} be a set of ORVs from a set of RVs S and a poset \mathcal{T} with an embedding \mathcal{E} . Then a set of unconditional affects relations \mathcal{A} satisfies **compat-atomic** with respect to \mathcal{E} , if it satisfies **compat** restricted to such affects relations whose first argument $X \subset S$ satisfies $|X| = 1$.*

While **compat** implies **compat-atomic**, both notions are not equivalent for general causal models and space-times as illustrated by the following example.

Example 17 (compat-atomic** $\not\Rightarrow$ **compat**)** *Consider a One-Time Pad of binary variables, where $X \rightsquigarrow Y$ and $Z \rightsquigarrow Y$, with a model satisfying $Y = X \oplus Z$. Then for any distribution on the parentless nodes X and Z , the affects relations with a singleton in their first argument ($X \vDash Y \mid \text{do}(Z)$ and $Z \vDash Y \mid \text{do}(X)$) imply $\bar{\mathcal{F}}_s(\mathcal{Y}\mathcal{Z}) \subseteq \bar{\mathcal{F}}(\mathcal{X})$ as well as $\bar{\mathcal{F}}_s(\mathcal{Y}\mathcal{X}) \subseteq \bar{\mathcal{F}}(\mathcal{Z})$ as compatibility conditions. Therefore **compat-atomic** yields $\bar{\mathcal{F}}_s(\mathcal{Y}\mathcal{Z}) = \bar{\mathcal{F}}_s(\mathcal{Y}\mathcal{X})$. For 1+1-Minkowski space-time, this is satisfied in an embedding where $Z \prec X \prec a \succ Y$ for $a \in \mathcal{T}$. By contrast, **compat** additionally demands $\bar{\mathcal{F}}(\mathcal{Y}) \subseteq \bar{\mathcal{F}}_s(\mathcal{X}\mathcal{Z})$, which is not satisfied in this embedding. Note however that the stronger condition imposed by **compat** is necessary for having no superluminal signalling: In the given embedding, an agent with access to Y can learn about the parity of X and Z outside the future light cone of any of these variables.*

We now show that analogous to the main result of Theorem 6.3, under almost identical restrictions on either the space-time or the causal model/affects relations, the desired implication does hold, hinting at a deeper correspondence between conical space-times and causal models without clustered affects

¹⁶This is not the case in non-unitary circuits. There, we can have signalling between sets of in/output systems without signalling between individual elements of the set (analogous to clustered affects relations we have encountered in Section 4.2).

¹⁷In contrast to the atomicity property on unitaries, this notion still allows the second argument of the affects relation to refer to sets of RVs. As this is not relevant for our main results, we leave an exploration of atomicity in different arguments to future work, focusing only on the first argument here.

relations. For this, recall that according to Definition 4.2, an affects relation $X \vDash Y \mid \text{do}(Z)$ to be Irred_1 is actually just a short-hand for a whole family of affects relations to hold:

$$s_X \vDash Y \mid \text{do}(Z\tilde{s}_X) \quad \forall s_X \subseteq X, \quad (43)$$

where $\tilde{s}_X := X \setminus s_X$. Therefore, we have a rather analog situation to Lemma G.7 concerning irreducibility in the third argument, and can follow the same idea for the proof.

Lemma F.2 *Let \mathcal{A} be a set of unconditional affects relations. Then for any non-degenerate embedding \mathcal{E} into a conical space-time \mathcal{T} ,*

$$\text{compat-atomic} \implies \text{compat}. \quad (44)$$

Proof: By definition, $X \vDash Y \mid \text{do}(Z)$ being Irred_1 implies $e_X \vDash Y \mid \text{do}(ZX \setminus e_X)$. **compat-atomic** then yields

$$\bar{\mathcal{F}}_s(\mathcal{Y}\mathcal{Z}) \cap \bar{\mathcal{F}}_s(\mathcal{X} \setminus e_X) \subseteq \bar{\mathcal{F}}(e_X) \quad \forall e_X \in \mathcal{X}. \quad (45)$$

With Lemma G.5 for $\mathcal{A} = \mathcal{Y}\mathcal{Z}$ and $\mathcal{B} = \mathcal{X}$ it follows that

$$\bar{\mathcal{F}}_s(\mathcal{Y}\mathcal{Z}\mathcal{X}) = \bar{\mathcal{F}}_s(\mathcal{Y}\mathcal{Z}\mathcal{X} \setminus e_X) \quad \forall e_X \in \mathcal{X}. \quad (46)$$

We proceed analogous to Lemma G.7, deriving that the embedding is either degenerate or

$$\bar{\mathcal{F}}_s(\mathcal{Z}\mathcal{Y}) \subseteq \bar{\mathcal{F}}_s(\mathcal{X}). \quad (47)$$

This is precisely **compat**, concluding the proof. \square

Lemma F.3 *Let \mathcal{A} be a set of unconditional affects relations which does not contain any affects relations that are Clus_1 . Then for any non-degenerate embedding \mathcal{E} of \mathcal{A} into a conical space-time \mathcal{T} ,*

$$\text{compat-atomic} \implies \text{compat}. \quad (48)$$

In particular, this holds for any faithful causal model.

Proof: Given that every affects relation $X \vDash Y \mid \text{do}(Z)$ in \mathcal{A}' is Irred_1 , this implies that for every non-empty $s_X \subseteq X$, we have $s_X \vDash Y \mid \text{do}(ZX \setminus s_X)$.

By Lemma D.4, the absence of Clus_1 relations in \mathcal{A} implies that there exists $e_X^1 \in X$ such that $e_X^1 \vDash Y \mid \text{do}(Z)$, and therefore, by **compat-atomic**, $\bar{\mathcal{F}}_s(\mathcal{Y}\mathcal{Z}) \subseteq \bar{\mathcal{F}}(e_X^1)$, which again implies $\bar{\mathcal{F}}_s(\mathcal{Y}\mathcal{Z}e_X^1) = \bar{\mathcal{F}}_s(\mathcal{Y}\mathcal{Z})$. However, due to Irred_1 , we also have $X \setminus e_X^1 \vDash Y \mid \text{do}(Ze_X^1)$. Invoking again the absence of Clus_1 relations and applying Lemma D.4, we know that there exists $e_X^2 \in X$ such that $e_X^2 \vDash Y \mid \text{do}(Ze_X^1)$, yielding $\bar{\mathcal{F}}_s(\mathcal{Y}\mathcal{Z}e_X^1) \subseteq \bar{\mathcal{F}}(e_X^2)$. Together by the earlier condition on $\bar{\mathcal{F}}(e_X^1)$, this yields $\bar{\mathcal{F}}_s(\mathcal{Y}\mathcal{Z}) \subseteq \bar{\mathcal{F}}(e_X^2)$.

As Irred_1 for the original affects relation implies the same for $X \setminus e_X^1 \vDash Y \mid \text{do}(Ze_X^1)$, we can now repeat this procedure recursively, yielding $\bar{\mathcal{F}}_s(\mathcal{Y}\mathcal{Z}) \subseteq \bar{\mathcal{F}}(e_X) \forall e_X \in \mathcal{X}$. Taken together, this is equivalent to $\bar{\mathcal{F}}_s(\mathcal{Y}\mathcal{Z}) \subseteq \bar{\mathcal{F}}_s(X)$, yielding the claim. \square

Using the absence of affects relations with Clus_2 or Clus_3 as a condition, similar results can be obtained for atomic compatibility in the second and third argument.

Analogous to Theorem 6.3 and Theorem 6.4, this demonstrates another useful implication which does not hold in general space-times and causal models, but does hold when either restricting to conical space-times or restricting to causal models without a type of clustering. A difference however is that Theorem 6.4 uses the (stronger) absence of Clus_3 while the above result uses only the absence of Clus_1 , indeed the former was related to Irred_3 affects relations while the latter is about Irred_1 affects relations. Moreover, while our main results imply an ability to reduce compatibility statements for higher-order affects relations to equivalent statements for 0^{th} -order relations, the results of this section enable a reduction of compatibility statements for non-atomic affects relations to equivalent statements for atomic relations. There is scope for exploring further relations and implications of these results, as well as other possible correspondences between conical space-times and causal models without clustering of a certain type. We leave this for future work.

Remark F.4 Due to Lemma G.6, any affects relation $X \vDash Y \mid \text{do}(Z)$ irreducible in its third argument implies that for any embedding satisfying **compat-atomic**, we have $\bar{\mathcal{F}}_s(\mathcal{Y}\mathcal{X}) \cap \bar{\mathcal{F}}_s(\mathcal{Z} \setminus e_Z) \subseteq \bar{\mathcal{F}}(e_Z)$. Conjoining this with Equation (45) (implied by **compat-atomic**), we obtain

$$\bar{\mathcal{F}}_s(\mathcal{Y}) \cap \bar{\mathcal{F}}_s(\mathcal{X}\mathcal{Z} \setminus e_{\mathcal{X}\mathcal{Z}}) \subseteq \bar{\mathcal{F}}(e_{\mathcal{X}\mathcal{Z}}) \quad \forall e_{\mathcal{X}\mathcal{Z}} \in \mathcal{X}\mathcal{Z}. \quad (49)$$

Since **compat** implies **compat-atomic**, the same follows from imposing **compat** for an Irred_3 affects relation. Hence, even without imposing the restrictions of Theorem 6.3 or Theorem 6.4, namely the absence of clustering or conical space-times, we obtain this weaker type of interchangeability of \mathcal{X} and \mathcal{Z} directly from **compat-atomic** (or **compat**). However, this is not strong enough to yield the correspondence of Equation (24) between causal inference statements and compatibility statements.

G Proofs of results

G.1 Proofs for Section 4

Lemma 4.1 $X \not\vDash Y \mid \text{do}(Z) \implies (X \perp\!\!\!\perp Y \mid Z)_{\mathcal{G}_{\text{do}}(XZ)}$.

Proof: Suppose by contradiction that $X \not\vDash Y \mid \text{do}(Z)$ (equivalently $P_{\mathcal{G}_{\text{do}}(XZ)}(Y \mid XZ) = P_{\mathcal{G}_{\text{do}}(Z)}(Y \mid Z)$) and $(X \perp\!\!\!\perp Y \mid Z)_{\mathcal{G}_{\text{do}}(XZ)}$. The latter is equivalent to saying that there exist distinct values z of Z and distinct values x and x' of X such that $P_{\mathcal{G}_{\text{do}}(XZ)}(Y \mid X = x, Z = z) \neq P_{\mathcal{G}_{\text{do}}(XZ)}(Y \mid X = x', Z = z)$. However, this implies that $P_{\mathcal{G}_{\text{do}}(XZ)}(Y \mid XZ)$ does have a non-trivial dependence on X and contradicts $P_{\mathcal{G}_{\text{do}}(XZ)}(Y \mid XZ) = P_{\mathcal{G}_{\text{do}}(Z)}(Y \mid Z)$ (which would imply that it is equal to an X -independent quantity). This proves the result. \square

Lemma 4.5 If $X \vDash Y \mid \text{do}(Z)$ is a Red_3 affects relation, then there exists $\tilde{s}_Z \subsetneq Z$ such that $X \vDash Y \mid \text{do}(\tilde{s}_Z)$ holds.

Proof: Suppose $X \vDash Y \mid \text{do}(Z)$ holds and is a Red_3 affects relation. Then writing out this affects relation along with the two non-affects relations implied by the reducibility (while recalling that $s_Z \cup \tilde{s}_Z = Z$), we have

$$\begin{aligned} P_{\mathcal{G}_{\text{do}}(XZ)}(Y \mid XZ) &\neq P_{\mathcal{G}_{\text{do}}(Z)}(Y \mid Z) \\ P_{\mathcal{G}_{\text{do}}(XZ)}(Y \mid XZ) &= P_{\mathcal{G}_{\text{do}}(X\tilde{s}_Z)}(Y \mid X\tilde{s}_Z) \\ P_{\mathcal{G}_{\text{do}}(Z)}(Y \mid Z) &= P_{\mathcal{G}_{\text{do}}(\tilde{s}_Z)}(Y \mid \tilde{s}_Z) \end{aligned} \quad (50)$$

Taken together, these imply that $P_{\mathcal{G}_{\text{do}}(X\tilde{s}_Z)}(Y \mid X\tilde{s}_Z) \neq P_{\mathcal{G}_{\text{do}}(\tilde{s}_Z)}(Y \mid \tilde{s}_Z)$ which is equivalent to $X \vDash Y \mid \text{do}(\tilde{s}_Z)$. \square

Lemma 4.7 If $X \vDash Y \mid \text{do}(Z)$ is a Red_2 affects relation, then there exists $\tilde{s}_Y \subsetneq Y$ such that $X \vDash \tilde{s}_Y \mid \text{do}(Z)$ holds.

Proof: Suppose $X \vDash Y \mid \text{do}(Z)$ holds and is a Red_2 affects relation. Then writing out this affects relation along with the three conditions implied by the reducibility (while recalling that $s_Y \cup \tilde{s}_Y = Y$), we have

$$\begin{aligned} P_{\mathcal{G}_{\text{do}}(XZ)}(Y \mid XZ) &\neq P_{\mathcal{G}_{\text{do}}(Z)}(Y \mid Z) \\ P_{\mathcal{G}_{\text{do}}(XZ)}(s_Y \mid XZ) &= P_{\mathcal{G}_{\text{do}}(Z)}(s_Y \mid Z) \\ P_{\mathcal{G}_{\text{do}}(XZ)}(Y \mid XZ) &= P_{\mathcal{G}_{\text{do}}(XZ)}(s_Y \mid XZ) P_{\mathcal{G}_{\text{do}}(XZ)}(\tilde{s}_Y \mid XZ) \\ P_{\mathcal{G}_{\text{do}}(Z)}(Y \mid Z) &= P_{\mathcal{G}_{\text{do}}(Z)}(s_Y \mid Z) P_{\mathcal{G}_{\text{do}}(Z)}(\tilde{s}_Y \mid Z). \end{aligned} \quad (51)$$

The third and fourth of the above expressions allow us to re-express the left and right hand sides of the first expression as a product of two terms. The second expression allows us to cancel out a term in the product, leaving us with $P_{\mathcal{G}_{\text{do}}(XZ)}(\tilde{s}_Y \mid XZ) \neq P_{\mathcal{G}_{\text{do}}(Z)}(\tilde{s}_Y \mid Z)$ which is equivalent to $X \vDash \tilde{s}_Y \mid \text{do}(Z)$. \square

Theorem 4.11 [Clustering implies irreducibility] For any affects relation $X \vDash Y \mid \text{do}(Z)$, $\text{Clus}_i \implies \text{Irred}_i$ for all $i \in \{1, 2, 3\}$.

Proof: **Clus₁ ⇒ Irred₁** Suppose that $X \models Y|\text{do}(Z)$ holds and satisfies Clus₁ but not Irred₁ i.e., it is Red₁. Then by Definition 4.2 and Definition 4.8 of these properties, we have the following requirements: $\tilde{s}_X \not\models Y|\text{do}(Z)$ for all $\tilde{s}_X \subsetneq X$ and $\exists \tilde{s}_X \subsetneq X$, with $\tilde{s}_X \cup s_X = X$ such that $s_X \not\models Y|\text{do}(\tilde{s}_X Z)$. Writing these out explicitly, we have:

$$\begin{aligned} P_{\mathcal{G}_{\text{do}(\tilde{s}_X Z)}}(Y|\tilde{s}_X Z) &= P_{\mathcal{G}_{\text{do}(Z)}}(Y|Z), \quad \forall \tilde{s}_X \subsetneq X, \\ P_{\mathcal{G}_{\text{do}(XZ)}}(Y|XZ) &= P_{\mathcal{G}_{\text{do}(\tilde{s}_X Z)}}(Y|\tilde{s}_X Z), \quad \exists \tilde{s}_X \subsetneq X, \end{aligned} \quad (52)$$

These imply $P_{\mathcal{G}_{\text{do}(XZ)}}(Y|XZ) = P_{\mathcal{G}_{\text{do}(Z)}}(Y|Z)$ which is equivalent to $X \not\models Y|\text{do}(Z)$. This contradicts our initial assumption that $X \models Y|\text{do}(Z)$ holds and therefore proves that Clus₁ implies Irred₁ for any affects relation $X \models Y|\text{do}(Z)$.

Clus₂ ⇒ Irred₂ We assume that $X \models Y|\text{do}(Z)$ holds, it is Clus₂ but not Irred₂ and derive a contradiction, which will establish that Clus₂ ⇒ Irred₂ for any affects relation $X \models Y|\text{do}(Z)$. Clus₂ implies the following

$$P_{\mathcal{G}_{\text{do}(XZ)}}(s_Y|XZ) = P_{\mathcal{G}_{\text{do}(Z)}}(s_Y|Z) \quad \forall s_Y \subsetneq Y. \quad (53)$$

On the other hand, Red₂ implies that $\exists s_Y \subsetneq Y$ such that $s_Y \not\models X|\text{do}(Z)$, \tilde{s}_Y . However, by Lemma E.2.3, both of these together imply $X \models \tilde{s}_Y|\text{do}(Z)$, which contradicts our assumption that $X \models Y|\text{do}(Z)$ is Clus₂.

Clus₃ ⇒ Irred₃ As before, we assume that $X \models Y|\text{do}(Z)$ holds and satisfies Clus₃ and Red₃. From Definition 4.10 and Definition 4.4 this implies the following conditions.

$$\begin{aligned} P_{\mathcal{G}_{\text{do}(X\tilde{s}_Z)}}(Y|X\tilde{s}_Z) &= P_{\mathcal{G}_{\text{do}(\tilde{s}_Z)}}(Y|\tilde{s}_Z), \quad \forall \tilde{s}_Z \subsetneq Z, \\ P_{\mathcal{G}_{\text{do}(XZ)}}(Y|XZ) &= P_{\mathcal{G}_{\text{do}(X\tilde{s}_Z)}}(Y|X\tilde{s}_Z), \quad \exists \tilde{s}_Z \subsetneq Z, \\ P_{\mathcal{G}_{\text{do}(Z)}}(Y|Z) &= P_{\mathcal{G}_{\text{do}(\tilde{s}_Z)}}(Y|\tilde{s}_Z), \quad \exists \tilde{s}_Z \subsetneq Z. \end{aligned} \quad (54)$$

Combining these, we obtain $P_{\mathcal{G}_{\text{do}(XZ)}}(Y|XZ) = P_{\mathcal{G}_{\text{do}(Z)}}(Y|Z)$ which is equivalent to $X \not\models Y|\text{do}(Z)$, which contradicts our initial assumption and therefore proves the claim. \square

Theorem 4.12 [Clustering implies fine-tuning] *Any affects relation $X \models Y|\text{do}(Z)$ that satisfies Clus_i for any $i \in \{1, 2, 3\}$ necessarily arises from a fine-tuned causal model.*

Proof: **Clus₁ ⇒ fine-tuning** Suppose that $X \models Y|\text{do}(Z)$ is a Clus₁ affects relation, which means that $|X| \geq 2$ and $s_X \not\models Y|\text{do}(Z)$ for all $s_X \subsetneq X$. From the proof of Lemma IV.3 of [10], it follows that $X \models Y|\text{do}(Z)$ implies that there exists $e_X \in X$ with a directed path from e_X to Y in $\mathcal{G}_{\text{do}(XZ)}$, which in turn implies $(X \not\perp^d Y|Z)_{\mathcal{G}_{\text{do}(XZ)}}$. Further, this tells us that there must exist a subset $s_X \subsetneq X$ such that $(s_X \not\perp^d Y|Z)_{\mathcal{G}_{\text{do}(XZ)}}$, which implies the same d-connection in the graph where we intervene only on this subset of X , i.e., $(s_X \not\perp^d Y|Z)_{\mathcal{G}_{\text{do}(s_X Z)}}$. This is because the graph $\mathcal{G}_{\text{do}(XZ)}$ will have the same nodes and less edges than $\mathcal{G}_{\text{do}(s_X Z)}$ for any $s_X \subsetneq X$ (since interventions cut off incoming edges), and d-connection cannot be lost by adding edges. Therefore we have established that $(s_X \not\perp^d Y|Z)_{\mathcal{G}_{\text{do}(s_X Z)}}$ holds for some $s_X \subsetneq X$. However Clus₁ implies that for all $s_X \subsetneq X$, $s_X \not\models Y|\text{do}(Z)$. This in turn implies that $(s_X \perp Y|Z)_{\mathcal{G}_{\text{do}(s_X Z)}}$ for all $s_X \subsetneq X$ (Lemma 4.1), which along with $(s_X \not\perp^d Y|Z)_{\mathcal{G}_{\text{do}(s_X Z)}}$ which we have previously established, implies that any underlying causal model giving rise to these affects relations must be fine-tuned.

Clus₂ ⇒ fine-tuning Suppose that $X \models Y|\text{do}(Z)$ is a Clus₂ affects relation, which means that $|Y| \geq 2$ and $X \not\models s_Y|\text{do}(Z)$ for all $s_Y \subsetneq Y$. As before, $X \models Y|\text{do}(Z)$ implies that $(X \not\perp^d Y|Z)_{\mathcal{G}_{\text{do}(XZ)}}$. By the definition of d-separation, this us that there must exist $e_Y \in Y$ such that $(X \not\perp^d e_Y|Z)_{\mathcal{G}_{\text{do}(XZ)}}$. However Clus₂ implies that for all $e_Y \in Y$, $X \not\models e_Y|\text{do}(Z)$ which gives $(X \perp e_Y|Z)_{\mathcal{G}_{\text{do}(XZ)}}$ for all $e_Y \in Y$ (Lemma 4.1). Then it is clear that there is at least one d-connection $(X \not\perp^d e_Y|Z)_{\mathcal{G}_{\text{do}(XZ)}}$ not matched by a corresponding conditional independence, thus making the model fine-tuned.

Clus₃ ⇒ fine-tuning Suppose that $X \models Y|\text{do}(Z)$ is a Clus₃ affects relation, which means that $|Z| \geq 1$ and $X \not\models Y|\text{do}(s_Z)$ for all $s_Z \subsetneq Z$. As in the proof for the previous case, we start with the fact that $X \models Y|\text{do}(Z)$ implies $(X \not\perp^d Y|Z)_{\mathcal{G}_{\text{do}(XZ)}}$. Since all nodes of the conditioning set Z are parentless in this d-connection, it follows that $(X \not\perp^d Y|s_Z)_{\mathcal{G}_{\text{do}(XZ)}}$ holds for subsets $s_Z \subsetneq Z$. This is because the only case where removing elements from the conditioning set could remove a d-connection is when those elements act as a collider (or descendent of a collider) on a sole unblocked path between X and Y ,

which is not possible if those elements are parentless. Now, observe that the only distinction between the graphs $\mathcal{G}_{\text{do}}(Xs_Z)$ for $s_Z \subsetneq Z$ and $\mathcal{G}_{\text{do}}(XZ)$ is that the former has a strictly larger set of edges (the incoming edges to nodes in $Z \setminus s_Z$ are present in the former and not in the latter). Since d-connection cannot be lost by adding edges, $(X \not\perp^d Y|s_Z)_{\mathcal{G}_{\text{do}}(XZ)}$ implies $(X \not\perp^d Y|s_Z)_{\mathcal{G}_{\text{do}}(Xs_Z)}$. We then use the fact that Clus_3 implies $X \not\perp Y|\text{do}(s_Z)$ for all $s_Z \subsetneq Z$, and consequently that $(X \perp Y|s_Z)_{\mathcal{G}_{\text{do}}(XZ)}$ for all $s_Z \subsetneq Z$ (Lemma 4.1), which when taken together with the d-connection $(X \not\perp^d Y|s_Z)_{\mathcal{G}_{\text{do}}(Xs_Z)}$ indicates that the causal model must be fine-tuned. \square

Lemma 4.14 *For any disjoint sets X, Y and Z of observed nodes, $X \perp Y|\text{do}(Z)$ is $\text{Irred}_3 \Rightarrow$ each $e_Z \in Z$ is a cause of at least one element $e_Y \in Y$.*

Proof: $X \perp Y|\text{do}(Z)$ satisfies Irred_3 implies in particular that for each $e_Z \in Z$, either $e_Z \perp Y|\text{do}(XZ \setminus e_Z)$ or $e_Z \perp Y|\text{do}(Z \setminus e_Z)$. As e_Z is a singleton, each of these implies by Lemma 4.13 that e_Z is a cause of at least one $e_Y \in Y$, which establishes the claim. \square

Lemma 4.18 *For any disjoint sets X, Y and Z of observed nodes, $X \perp Y|\text{do}(Z)$ is Irred_2 implies that both $(s_Y \not\perp^d \tilde{s}_Y|Z)_{\mathcal{G}_{\text{do}}(Z)}$ and $(X \not\perp^d s_Y|Z\tilde{s}_Y)_{\mathcal{G}_{\text{do}}(X(\tilde{s}_Y)Z)}$ for any partition of Y into subsets s_Y, \tilde{s}_Y . Here, $X(W) := X \setminus (X \cap \text{anc}(W))$ for $\text{anc}(W)$ denoting the set of all ancestors of W in the original graph \mathcal{G} .*

Proof: If $X \perp Y|\text{do}(Z)$ is Irred_2 , then $X \perp s_Y|\text{do}(Z), \tilde{s}_Y$ must hold and either $(s_Y \not\perp \tilde{s}_Y|XZ)_{\mathcal{G}_{\text{do}}(XZ)}$ or $(s_Y \not\perp \tilde{s}_Y|Z)_{\mathcal{G}_{\text{do}}(Z)}$ must hold, for each partition of $Y = s_Y \tilde{s}_Y$. Using the d-separation property, this implies that either $(s_Y \not\perp^d \tilde{s}_Y|XZ)_{\mathcal{G}_{\text{do}}(XZ)}$ or $(s_Y \not\perp^d \tilde{s}_Y|Z)_{\mathcal{G}_{\text{do}}(Z)}$ must hold. Noting that in $\mathcal{G}_{\text{do}}(XZ)$, X only consist of parentless nodes, it cannot act as a collider and therefore removing it from the conditioning set cannot remove d-connection. Moreover, in going from $\mathcal{G}_{\text{do}}(XZ)$ to $\mathcal{G}_{\text{do}}(XZ)$ we would introduce additional edges, which cannot remove d-connection. Thus in both cases we have $(s_Y \not\perp^d \tilde{s}_Y|Z)_{\mathcal{G}_{\text{do}}(Z)}$. The first condition, i.e. $X \perp s_Y|\text{do}(Z), \tilde{s}_Y$ further yields by Lemma E.5, $(X \not\perp^d s_Y|Z\tilde{s}_Y)_{\mathcal{G}_{\text{do}}(X(\tilde{s}_Y)Z)}$. \square

G.2 Proofs for Section 5

Lemma 5.4 *For $d \geq 2$, $d+1$ -Minkowski space-time is a conical poset.*

Proof: In $d+1$ -dim. Minkowski space-time \mathcal{T} , the boundary of the light cone of a space-time point $(\vec{x}_0, t_0) \in \mathcal{T}$ is given as follows, where $\vec{x}_0 = (x_0^1, x_0^2, \dots, x_0^d)$ are the d -dimensional spatial co-ordinates of the point (in some chosen co-ordinate system).

$$c^2(t - t_0)^2 = \sum_{1 < i \leq d} (x^i - x_0^i)^2. \quad (55)$$

Observe that for each time slice (fixed value of t), the above equation corresponds to a d -dimensional sphere with radius $c^2(t - t_0)^2$. Whenever $d \geq 2$, for a fixed t , given any finite portion of this spherical boundary, we can determine the radius of the sphere and therefore its center, which is associated with the spatial co-ordinate \vec{x}_0 . If we are considering the future light cone, this is non-empty only for $t > t_0$. Given a non-empty time slice of the future light cone of a space-time point $(\vec{x}_0, t_0) \in \mathcal{T}$, we can consider light rays emanating from each point on the boundary of the sphere and extending towards the negative t direction, these light rays would necessarily intersect at a unique time t_0 which allows us to fully determine the original point (\vec{x}_0, t_0) , starting from any portion of the future light cone boundary on a given non-empty time slice.

Now let L be a finite subset of points in $d+1$ -dim. Minkowski space-time and consider the joint future $f(L) = \bigcap_{x \in L} J^+(x)$ of all the points in L . The boundary of this joint future is compiled from pieces of the light cone boundaries of points in L . In particular, the boundary of $f(L)$ associated with a non-empty time slice is compiled from spherical pieces which are portions of light cone boundaries of points in L , associated with that time slice (for $d = 2$, these are circular arcs). We can show that each $x \in \text{span}(L)$ contributes a unique, finite portion to the boundary of the joint future region $f(L)$ at every non-empty time slice. For this, recall that by definition, if $x \in \text{span}(L)$, then $x \in L' \subseteq L$ such that $f(L') = f(L)$ and there is no $L'' \subsetneq L'$ satisfying $f(L'') = f(L)$. If we assume that x doesn't contribute a finite piece to the boundary of $f(L)$, then this yields a contradiction as it would mean that for $L'' := L' \setminus x$, we have

$f(L'') = f(L)$. To show uniqueness, suppose $x \in \text{span}(L)$ contributes a finite portion to the boundary of $f(L)$ at a time slice t . No other $x' \in \mathcal{T}$ can contribute the same portion of the boundary of $f(L)$, due to the argument presented in the first paragraph of this proof: each such portion allows us to uniquely reconstruct the point x as it is also a portion of the future light cone of x .

Thus, the distinct spherical portions that comprise the boundary of $f(L)$ can each be mapped back uniquely to a distinct spanning element of L . Since L is arbitrary here, this implies that $f(L)$ is invertible on all sets L with $L = \text{span}(L)$, which proves the conicality of Minkowski space-times with 2 or more spatial dimensions. \square

Lemma 5.12 *Let \mathcal{T} be a conical poset and $L \subset \mathcal{T}$ be finite. Let \mathcal{X} be a set of ORVs on this poset. Then, the knowledge of $\bar{\mathcal{F}}_s(\mathcal{X})$ implies the locations $O(\mathcal{X}_i)$ for all its spanning elements $\mathcal{X}_i \in \text{span}(\mathcal{X})$.*

Proof: Recalling that $O(\mathcal{X}) = \{O(\mathcal{X}_i) | \mathcal{X}_i \in \mathcal{X}\}$ and $f(L) := \bigcap_{x \in L} J^+(x)$, then by definition, for any $\mathcal{X} \subset \mathcal{S}$,

$$\bar{\mathcal{F}}_s(\mathcal{X}) \equiv \bigcap_{\mathcal{X}_i \in \mathcal{X}} J^+(O(\mathcal{X}_i)) = f(O(\mathcal{X})) = f(O(\text{span}(\mathcal{X}))). \quad (56)$$

If the poset is conical, by Definition 5.3, $f(\text{span}(O(\mathcal{X})))$ is injective. Hence, it remains to show that $\text{span}(O(\mathcal{X})) \stackrel{!}{=} O(\text{span}(\mathcal{X}))$. To this purpose, we differentiate between the case that the embedding is degenerate or not. If the embedding is non-degenerate, we have a bijection between \mathcal{X} and $L_{\mathcal{X}}$, which immediately yields $\text{span}(O(\mathcal{X})) = O(\text{span}(\mathcal{X}))$. Otherwise, there are multiple ORVs $u_{\mathcal{X}} \in \mathcal{X}$ sharing the same location. In this case, it is easy to see that either all of them or none of them belong to $\text{span}(\mathcal{X})$.¹⁸ Therefore, we can group every set of ORVs embedded degenerately at the same location, into a single ORV embedded at the same location without changing the span. Thus we have reduced a degenerate problem to a non-degenerate one such that $\text{span}(O(\mathcal{X})) = O(\text{span}(\mathcal{X}))$ holds in the former if and only if it holds in the latter. Since we have established that this always holds in the non-degenerate case, this proves the claim. \square

As outlined in the main text, our primary results rely on showing that conicality of a poset is equivalent to another poset property called location symmetry. We proceed by motivating this property. Suppose we have a family $\{S_i\}_{i \in I}$ where each element is a set of locations $S_i \subset \mathcal{T}$ all of which share the same span, let us denote this by K . Then we can write $S_i = KL_i$ (where KL_i is short for the union of the sets), making clear that the locations in K must be shared among the sets and here the different L_i may also share locations. Plugging S_i for L_i and K for L_j into Equation (16), which holds in conical space-times, we obtain that the the joint futures of these sets must be identical in such space-times,

$$f(K) = f(S_i) \subseteq f(S_i \setminus K) = f(L_i), \forall i \in I \quad (57)$$

where $f(L) := \bigcap_{x \in L} J^+(x)$. Here, the subset relation follows from the fact that removing sets from an intersection always yields a (potentially improper) superset of the original intersection. As this holds for all $i \in I$, we can replace $f(L_i)$ with $\bigcap_{i \in I} f(L_i)$. This yields that in conical space-times where $\text{span}(KL_i) = K$ for all $i \in I$ (or equivalently $\bigcap_{i \in I} \text{span}(KL_i) = K$),

$$f(KL_i) = f(KL_k) \quad \forall i, k \in I \quad \implies \quad f(K) \subseteq \bigcap_{i \in I} f(L_i) \quad (58)$$

More generally, we can consider sets $S_i = KL_i$ where K is not the common span of the sets. Then $\bigcap_{i \in I} \text{span}(KL_i) \neq K$. This implies that there must be further shared locations in $\text{span}(KL_i) \setminus K$ for each i , that are commonly shared between all the L_i 's, and using conicality, it can be shown that for all i , $\text{span}(KL_i) \setminus K$ must be identical in this case (as we do in the proofs below). Based on this motivation, we have the following definition of location symmetry which captures the two alternative conditions discussed above.

¹⁸If at least one $u_{\mathcal{X}} \ni e_{\mathcal{X}} \in \text{span}(\mathcal{X})$, for each allowed choice of $s_{\mathcal{X}}$ (as according to the definition), $|s_{\mathcal{X}} \cap u_{\mathcal{X}}| = 1$. However, due to considering the union of such sets we then regain the entirety of $u_{\mathcal{X}} \subset \bigcup s_{\mathcal{X}}$.

Definition G.1 (Location Symmetry) Let \mathcal{T} be a poset and I be an index set. Let $K, L_i \subset \mathcal{T}$ discrete with $i \in I$. Then \mathcal{T} satisfies location symmetry if

$$\begin{aligned} f(K) \cap f(L_i) &= f(K) \cap f(L_k) \quad \forall i, k \in I \\ \implies f(K) &\subseteq \bigcap_{i \in I} f(L_i) \quad \vee \quad \exists! M \subset \mathcal{T} : \text{span}(KL_i) \setminus K = M \neq \emptyset \quad \forall i \in I \end{aligned} \quad (59)$$

where

$$f(L) := \bigcap_{x \in L} J^+(x)$$

If K or L_i is empty, we understand their intersection as being equal to the full poset \mathcal{T} .

Alternatively, we can also phrase location symmetry in terms of a set of ORVs \mathcal{S} on this poset.

Definition G.2 (Location Symmetry for ORVs) Let \mathcal{T} be a poset and I be an index set. Let $\mathcal{X}, \mathcal{Y}_i \subset \mathcal{S}$ with $i \in I$, not necessarily disjoint or non-empty. Then \mathcal{T} satisfies location symmetry

$$\begin{aligned} \bar{\mathcal{F}}_s(\mathcal{X}\mathcal{Y}_i) &= \bar{\mathcal{F}}_s(\mathcal{X}\mathcal{Y}_k) \quad \forall i, k \in I \\ \implies \bar{\mathcal{F}}_s(\mathcal{X}) &\subseteq \bar{\mathcal{F}}_s(\bigcup_{i \in I} \mathcal{Y}_i) \quad \vee \quad \exists! M \subset \mathcal{T} : O(\text{span}(\mathcal{X}\mathcal{Y}_k)) \setminus O(\mathcal{X}) = M \neq \emptyset \quad \forall k \in I \end{aligned} \quad (60)$$

Especially in the respective second alternative, these definitions takes care to play well with the respective sets / their locations potentially not being disjoint, as this will be of importance for considering degenerate embeddings of ORVs.

Lemma G.3 \mathcal{T} is a conical poset if and only if it satisfies location symmetry.

Proof: For this proof, we will refer to ORVs directly, as that is more compact and matches the notation we will use for the proofs going forward. All notation matches Definition G.2.

“ \implies ”: Here we assume location symmetry and prove conicality. We only require the former for the special case where $\mathcal{X} = \emptyset$ while \mathcal{Y}_1 and \mathcal{Y}_2 are any disjoint sets of ORVs with $\bar{\mathcal{F}}_s(\mathcal{Y}_1) = \bar{\mathcal{F}}_s(\mathcal{Y}_2)$. Then, applying Definition 5.9 to empty sets of ORVs, we recover $\bar{\mathcal{F}}_s(\mathcal{X}) = \mathcal{T}$. Moreover, the pre-condition $\bar{\mathcal{F}}_s(\mathcal{X}\mathcal{Y}_1) = \bar{\mathcal{F}}_s(\mathcal{X}\mathcal{Y}_2)$ for applying location symmetry is satisfied as it is equivalent to $\bar{\mathcal{F}}_s(\mathcal{Y}_1) = \bar{\mathcal{F}}_s(\mathcal{Y}_2)$. Therefore, location symmetry implies that

$$\mathcal{T} = \bar{\mathcal{F}}_s(\mathcal{Y}_1\mathcal{Y}_2) \vee O(\text{span}(\mathcal{X}\mathcal{Y}_1)) \setminus O(\mathcal{X}) = O(\text{span}(\mathcal{X}\mathcal{Y}_2)) \setminus O(\mathcal{X}). \quad (61)$$

Here, the second alternative immediately reduces to $O(\text{span}(\mathcal{Y}_1)) = O(\text{span}(\mathcal{Y}_2))$, due to \mathcal{X} being empty. The first is an equality as $\bar{\mathcal{F}}_s(\mathcal{Y}_1\mathcal{Y}_2) \subseteq \mathcal{T}$ trivially holds. Moreover, notice that the first case $\mathcal{T} = \bar{\mathcal{F}}_s(\mathcal{Y}_1\mathcal{Y}_2)$ is equivalent to $\mathcal{T} = \bar{\mathcal{F}}_s(\mathcal{Y}_1) = \bar{\mathcal{F}}_s(\mathcal{Y}_2)$, and this can only be satisfied if both sets of ORVs are embedded into the minimal element of the poset (which may not exist). If the minimal element exists, we therefore have $O(\mathcal{Y}_1) = O(\mathcal{Y}_2)$, which in particular implies $O(\text{span}(\mathcal{Y}_1)) = O(\text{span}(\mathcal{Y}_2))$. In other words, for both alternatives of location symmetry we get that for any disjoint sets of ORVs \mathcal{Y}_1 and \mathcal{Y}_2 , $\bar{\mathcal{F}}(\mathcal{Y}_1) = \bar{\mathcal{F}}(\mathcal{Y}_2) \implies O(\text{span}(\mathcal{Y}_1)) = O(\text{span}(\mathcal{Y}_2))$, reproducing the definition of conicality (Equation (16)).

“ \Leftarrow ”: Let $\mathcal{X}, \mathcal{Y}_i \subset \mathcal{S}$ with $i \in I$ such that

$$\bar{\mathcal{F}}_s(\mathcal{X}\mathcal{Y}_i) = \bar{\mathcal{F}}_s(\mathcal{X}\mathcal{Y}_k) \quad \forall i, k \in I. \quad (62)$$

We now show that in any conical space-time, this implies one of the two conditions in the second line of Equation (60), thus proving location symmetry.

Applying the property Equation (16) of conical space-times to Equation (62), we see that for any choice of $i \in I$, we can deduce the value of $O(\mathcal{A})$ for all $\mathcal{A} \in \text{span}(\mathcal{X}\mathcal{Y}_i)$ for ORVs embedded in any conical poset. This implies for the set of respective locations to be identical.

$$O(\text{span}(\mathcal{X}\mathcal{Y}_i)) = O(\text{span}(\mathcal{X}\mathcal{Y}_k)) \quad \forall i, k \in I. \quad (63)$$

We now consider two cases and show that these yield respectively the two alternatives for location symmetry, thus concluding the proof.

1. **Case 1** ($\exists i : O(\mathcal{X}) = O(\text{span}(\mathcal{X}\mathcal{Y}_i))$) : In this case, we can deduce that the same holds for all \mathcal{Y}_k with $k \in I$, due to Equation (62). Hence,

$$\bar{\mathcal{F}}_s(\mathcal{X}) = \bar{\mathcal{F}}_s(\mathcal{X}\mathcal{Y}_i) \quad (64)$$

$$\implies \bar{\mathcal{F}}_s(\mathcal{X}) \subseteq \bar{\mathcal{F}}_s(\mathcal{Y}_i) \quad \forall i \in I \quad (65)$$

$$\iff \bar{\mathcal{F}}_s(\mathcal{X}) \subseteq \bigcap_{i \in I} \bar{\mathcal{F}}_s(\mathcal{Y}_i) = \bar{\mathcal{F}}_s\left(\bigcup_{i \in I} \mathcal{Y}_i\right), \quad (66)$$

yielding the first alternative of Equation (60).

2. **Case 2** ($\forall i : O(\mathcal{X}) \neq O(\text{span}(\mathcal{X}\mathcal{Y}_i))$) : This implies that $O(\text{span}(\mathcal{X}\mathcal{Y}_i)) \setminus O(\mathcal{X}) \neq \emptyset$. We have already established in Equation (63) that in conical space-times where Equation (62) is satisfied, $O(\text{span}(\mathcal{X}\mathcal{Y}_k))$ must be identical for all $i \in I$, this implies the same holds for $O(\text{span}(\mathcal{X}\mathcal{Y}_i)) \setminus O(\mathcal{X})$, and yields the second alternative of Equation (60). □

We conclude by providing a weaker, but significantly shorter version of the second alternative that will be useful for clarity in some later proofs.

Lemma G.4 *Let $M \subseteq \mathcal{T}$ non-empty. Let $I, \mathcal{X}, \mathcal{Y}_k$ as in Definition G.2. Then*

$$O(\text{span}(\mathcal{X}\mathcal{Y}_k)) \setminus O(\text{span}(\mathcal{X})) = M \quad \forall k \in I \quad \implies \quad \bigcap_k O(\mathcal{Y}_k) \neq \emptyset. \quad (67)$$

Proof: First, we see that $\text{span}(\mathcal{X}\mathcal{Y}_i) \setminus \mathcal{X} \subseteq \mathcal{Y}_i$. As this carries over to their respective locations, we obtain

$$\emptyset \neq M = O(\text{span}(\mathcal{X}\mathcal{Y}_k)) \setminus O(\text{span}(\mathcal{X})) \quad \forall k \in I \quad (68)$$

$$\iff \emptyset \neq M = O(\text{span}(\mathcal{X}\mathcal{Y}_k)) \cap O(\mathcal{Y}_k) \setminus O(\text{span}(\mathcal{X})) \quad \forall k \in I \quad (69)$$

$$\implies \emptyset \neq M \subseteq O(\text{span}(\mathcal{X}\mathcal{Y}_k)) \cap O(\mathcal{Y}_k) \quad \forall k \in I \quad (70)$$

$$\implies \emptyset \neq M \subseteq \bigcap_k O(\text{span}(\mathcal{X}\mathcal{Y}_k)) \cap O(\mathcal{Y}_k) \quad (71)$$

$$\implies \emptyset \neq \bigcap_k O(\mathcal{Y}_k), \quad (72)$$

where the last implication holds as intersecting larger sets always yields a (potentially improper) superset. □

G.3 Proofs for Section 6

To prove Theorem 6.3, we require some auxiliary lemmas, which we first prove before stating and proving the main theorem.

Lemma G.5 *Let $\mathcal{A}, \mathcal{B} \subset \mathcal{S}$. Then*

$$\bar{\mathcal{F}}_s(\mathcal{A}) \cap \bar{\mathcal{F}}_s(\mathcal{B} \setminus e_{\mathcal{B}}) \subseteq \bar{\mathcal{F}}(e_{\mathcal{B}}) \quad \forall e_{\mathcal{B}} \in \mathcal{B} \quad \implies \quad \bar{\mathcal{F}}_s(\mathcal{A}\mathcal{B}) = \bar{\mathcal{F}}_s(\mathcal{A}\mathcal{B} \setminus e_{\mathcal{B}}) \quad \forall e_{\mathcal{B}} \in \mathcal{B}. \quad (73)$$

Proof: Let $e_{\mathcal{B}}^i := e_{\mathcal{B}}$. We can transform

$$\bar{\mathcal{F}}_s(\mathcal{A}) \cap \bar{\mathcal{F}}_s(\mathcal{B} \setminus e_{\mathcal{B}}^i) \subseteq \bar{\mathcal{F}}(e_{\mathcal{B}}^i) \quad (74)$$

$$\implies \bar{\mathcal{F}}_s(\mathcal{A}) \cap \bar{\mathcal{F}}_s(\mathcal{B} \setminus e_{\mathcal{B}}^i) \subseteq \bar{\mathcal{F}}_s(\mathcal{A}) \cap \bar{\mathcal{F}}(e_{\mathcal{B}}^i) \quad \forall e_{\mathcal{B}}^i \in \mathcal{B} \quad (75)$$

$$\implies \bar{\mathcal{F}}_s(\mathcal{A}) \cap \bar{\mathcal{F}}_s(\mathcal{B} \setminus e_{\mathcal{B}}^i) \subseteq \bar{\mathcal{F}}_s(\mathcal{A}) \cap \bar{\mathcal{F}}(e_{\mathcal{B}}^i) \cap \bar{\mathcal{F}}_s(\mathcal{B} \setminus \{e_{\mathcal{B}}^i, e_{\mathcal{B}}^j\}) \quad \forall e_{\mathcal{B}}^i, e_{\mathcal{B}}^j \in \mathcal{B} \quad (76)$$

$$\iff \bar{\mathcal{F}}_s(\mathcal{A}) \cap \bar{\mathcal{F}}_s(\mathcal{B} \setminus e_{\mathcal{B}}^i) \subseteq \bar{\mathcal{F}}_s(\mathcal{A}) \cap \bar{\mathcal{F}}_s(\mathcal{B} \setminus e_{\mathcal{B}}^j) \quad \forall e_{\mathcal{B}}^i, e_{\mathcal{B}}^j \in \mathcal{B}. \quad (77)$$

Combining these for all e_B , we get

$$\bar{\mathcal{F}}_s(\mathcal{A}) \cap \bar{\mathcal{F}}_s(\mathcal{B} \setminus e_B^i) = \bar{\mathcal{F}}_s(\mathcal{A}) \cap \bar{\mathcal{F}}_s(\mathcal{B} \setminus e_B^j) \quad \forall e_B^i, e_B^j \in \mathcal{B} \quad (78)$$

$$\implies \bar{\mathcal{F}}_s(\mathcal{A}\mathcal{B}) = \bar{\mathcal{F}}_s(\mathcal{A}\mathcal{B} \setminus e_B) \quad \forall e_B \in \mathcal{B}. \quad (79)$$

This is precisely the claim. \square

Lemma G.6 *Let \mathcal{A} be a set of unconditional affects relations and $\mathcal{A}' \subseteq \mathcal{A}$ consist of all affects relations in \mathcal{A} that are Irred₃. Then for any non-degenerate embedding \mathcal{E} into an arbitrary space-time \mathcal{T} satisfying **compat** we have*

$$(X \vDash Y \mid \text{do}(Z)) \in \mathcal{A}' \implies \bar{\mathcal{F}}_s(\mathcal{Y}\mathcal{X}) \cap \bar{\mathcal{F}}_s(\mathcal{Z} \setminus e_Z) \subseteq \bar{\mathcal{F}}(e_Z) \quad \forall e_Z \in \mathcal{Z}. \quad (80)$$

Proof: As $X \vDash Y \mid \text{do}(Z)$ satisfies Irred₃, we have for $\tilde{Z} := Z \setminus e_Z$

$$\forall e_Z \in \mathcal{Z} : \quad e_Z \vDash Y \mid \text{do}(\tilde{Z}) \vee e_Z \vDash Y \mid \text{do}(\tilde{Z}X), \quad (81)$$

either of which is Irred₁, if it holds. Therefore, by **compat**, we get

$$\bar{\mathcal{F}}_s(\mathcal{Y}) \cap \bar{\mathcal{F}}_s(\tilde{Z}) \subseteq \bar{\mathcal{F}}(e_Z) \vee \bar{\mathcal{F}}_s(\mathcal{Y}) \cap \bar{\mathcal{F}}_s(\mathcal{X}\tilde{Z}) \subseteq \bar{\mathcal{F}}(e_Z) \quad (82)$$

$$\implies \bar{\mathcal{F}}_s(\mathcal{Y}\mathcal{X}) \cap \bar{\mathcal{F}}_s(\tilde{Z}) \subseteq \bar{\mathcal{F}}(e_Z) \quad (83)$$

for any e_Z . \square

With these three auxiliary lemmas in place, we can continue to prove the theorem.

Lemma G.7 *Let \mathcal{A} be a set of unconditional affects relations and $\mathcal{A}' \subseteq \mathcal{A}$ consist of all affects relations in \mathcal{A} that are Irred₃. Then for any non-degenerate embedding \mathcal{E} into a conical space-time \mathcal{T} satisfying **compat**, we have*

$$(X \vDash Y \mid \text{do}(Z)) \in \mathcal{A}' \implies \bar{\mathcal{F}}_s(\mathcal{Y}) \cap \bar{\mathcal{F}}_s(\mathcal{X}) \subseteq \bar{\mathcal{F}}_s(\mathcal{Z}). \quad (84)$$

Proof: By conjoining Lemma G.6 with Lemma G.5 for $\mathcal{A} = \mathcal{Y}\mathcal{X}$ and $\mathcal{B} = \mathcal{Z}$, for any affects relation $X \vDash Y \mid \text{do}(Z)$ irreducible in the third argument, we get

$$\bar{\mathcal{F}}_s(\mathcal{X}\mathcal{Y}\mathcal{Z}) = \bar{\mathcal{F}}_s(\mathcal{X}\mathcal{Y}\mathcal{Z} \setminus e_Z) \quad \forall e_Z \in \mathcal{Z}. \quad (85)$$

Due to conicality, Lemma G.3 implies that \mathcal{T} satisfies location symmetry. By using this property on the chain of equalities (setting \mathcal{X} in Equation (60) to $\mathcal{X}\mathcal{Y}$, and \mathcal{Y}_i to $\mathcal{Z}_i = \mathcal{Z} \setminus e_Z^i$) as well as the transformation of Lemma G.4, we arrive at

$$\bar{\mathcal{F}}_s(\mathcal{X}\mathcal{Y}) \subseteq \bar{\mathcal{F}}_s(\mathcal{Z}) \vee \bigcap_i O(\mathcal{Z}_i) \neq \emptyset, \quad (86)$$

where $\mathcal{Z}_i := \mathcal{Z} \setminus e_i$.

For the second alternative, beware that the different \mathcal{Z}_i are not disjoint. Nonetheless, $\bigcap_i \mathcal{Z}_i = \emptyset$. Therefore, multiple ORVs must share the same location. As this corresponds to a degenerate embedding, we arrive at the claim. \square

Theorem 6.3 *Let \mathcal{A} be a set of unconditional affects relations and $\mathcal{A}' \subseteq \mathcal{A}$ consist of all affects relations in \mathcal{A} that are both Irred₁ and Irred₃. Then for any non-degenerate embedding \mathcal{E} into a conical space-time \mathcal{T} satisfying **compat**, we have*

$$(X \vDash Y \mid \text{do}(Z)) \in \mathcal{A}' \implies \bar{\mathcal{F}}_s(\mathcal{Y}) \subseteq \bar{\mathcal{F}}_s(\mathcal{X}) \cap \bar{\mathcal{F}}_s(\mathcal{Z}). \quad (22)$$

Proof: Due to Lemma G.7, **compat** implies the following for an Irred₁ and Irred₃ affects relation

$$(X \vDash Y \mid \text{do}(Z)) \in \mathcal{A}' \implies \bar{\mathcal{F}}_s(\mathcal{Y}) \cap \bar{\mathcal{F}}_s(\mathcal{X}) \subseteq \bar{\mathcal{F}}_s(\mathcal{Z}). \quad (87)$$

Conjoining this with **compat** for this affects relation (which implies $\bar{\mathcal{F}}_s(\mathcal{Y}) \cap \bar{\mathcal{F}}_s(\mathcal{Z}) \subseteq \bar{\mathcal{F}}_s(\mathcal{X})$) and using elementary set theory yields

$$\bar{\mathcal{F}}_s(\mathcal{Y}) \cap \bar{\mathcal{F}}_s(\mathcal{Z}) \subseteq \bar{\mathcal{F}}_s(\mathcal{X}) \cap \bar{\mathcal{F}}_s(\mathcal{Y}) \wedge \bar{\mathcal{F}}_s(\mathcal{Y}) \cap \bar{\mathcal{F}}_s(\mathcal{X}) \subseteq \bar{\mathcal{F}}_s(\mathcal{Z}) \cap \bar{\mathcal{F}}_s(\mathcal{Y}) \quad (88)$$

$$\implies \bar{\mathcal{F}}_s(\mathcal{Y}) \cap \bar{\mathcal{F}}_s(\mathcal{X}) = \bar{\mathcal{F}}_s(\mathcal{Y}) \cap \bar{\mathcal{F}}_s(\mathcal{Z}) \quad (89)$$

Due to conicality, Lemma G.3 implies that \mathcal{T} satisfies location symmetry. Using location symmetry, we arrive at

$$\bar{\mathcal{F}}_s(\mathcal{Y}) \subseteq \bar{\mathcal{F}}_s(\mathcal{X}\mathcal{Z}) \vee O(\mathcal{X}) \cap O(\mathcal{Z}) \neq \emptyset. \quad (90)$$

As the second option would imply the embedding to be degenerate, the claim, corresponding to the first option, follows. \square

Theorem 6.4 *Let \mathcal{A} be a set of unconditional affects relations arising from a causal model not yielding any affects relations with Clus₃ and let $\mathcal{A}' \subseteq \mathcal{A}$ consist of all affects relations in \mathcal{A} that are both Irred₁ and Irred₃. Then compatibility of \mathcal{A} with an embedding \mathcal{E} in any space-time \mathcal{T} implies Equation (22).*

Proof: Every affects relation $X \vDash Y \mid \text{do}(Z)$ in \mathcal{A}' is Irred₁ and Irred₃. By Lemma D.6, due to the absence of affects relations with Clus₃, there exists $e_X^1 \in X$ such that $e_X^1 \vDash Y$. Therefore, for any compatible embedding \mathcal{E} , we have $\bar{\mathcal{F}}_s(\mathcal{Y}) \subseteq \bar{\mathcal{F}}(e_X^1)$. However, due to $X \vDash Y \mid \text{do}(Z)$ being Irred₁, we also have $X \setminus e_X^1 \vDash Y \mid \text{do}(Ze_X^1)$, which again by Lemma D.6 implies that there is $e_X^2 \in X \setminus e_X^1$ such that $e_X^2 \vDash Y$. Repeating this argument recursively, we find that any compatible embedding \mathcal{E} of such affects relations in a space-time will satisfy $\bar{\mathcal{F}}_s(\mathcal{Y}) \subseteq \bar{\mathcal{F}}(e_{\mathcal{X}}) \forall e_{\mathcal{X}} \in \mathcal{X}$, or in short, $\bar{\mathcal{F}}_s(\mathcal{Y}) \subseteq \bar{\mathcal{F}}_s(\mathcal{X})$.

Next we use the fact that $X \vDash Y \mid \text{do}(Z)$ is Irred₃, which gives $s_Z \vDash Y \mid \text{do}(\bar{s}_Z X)$ or $s_Z \vDash Y \mid \text{do}(\bar{s}_Z)$ for all choices of $s_Z \subsetneq Z$. Applying Lemma D.6, both these alternatives yield $\exists e_Z^1 \in Z$ s.t. $e_Z^1 \vDash Y$ and $\bar{\mathcal{F}}_s(\mathcal{Y}) \subseteq \bar{\mathcal{F}}(e_Z^1)$. Moreover, $X \vDash Y \mid \text{do}(Z)$ being Irred₃ implies in particular that $Z \setminus e_Z^1 \vDash Y \mid \text{do}(e_Z^1 X)$ or $Z \setminus e_Z^1 \vDash Y \mid \text{do}(e_Z^1)$, both of which imply by Lemma D.6 that $\exists e_Z^2 \in Z \setminus e_Z^1$ s.t. $e_Z^2 \vDash Y$. Repeating this procedure recursively, we obtain that for any compatible embedding \mathcal{E} of these affects relations, we have $\bar{\mathcal{F}}_s(\mathcal{Y}) \subseteq \bar{\mathcal{F}}(e_{\mathcal{Z}}) \forall e_{\mathcal{Z}} \in \mathcal{Z}$, or in short, $\bar{\mathcal{F}}_s(\mathcal{Y}) \subseteq \bar{\mathcal{F}}_s(\mathcal{Z})$.

Combining both results, we recover the claim. \square

G.4 Proofs for Appendix E

Lemma E.4 *For a causal model over a set S of RVs, where $X, Y, Z, W \subset S$ disjoint, $X \not\vDash Y \mid \text{do}(Z), W \implies (X \perp\!\!\!\perp Y \mid ZW)_{\mathcal{G}_{\text{do}(XZ)}}$.*

Proof: Suppose by contradiction that $X \not\vDash Y \mid \{\text{do}(Z), W\}$ (which amounts to $P_{\mathcal{G}_{\text{do}(XZ)}}(Y \mid XZW) = P_{\mathcal{G}_{\text{do}(Z)}}(Y \mid ZW)$) and $(X \perp\!\!\!\perp Y \mid ZW)_{\mathcal{G}_{\text{do}(XZ)}}$. The latter is equivalent to saying that there exist values z of Z , w of W and distinct values x and x' of X such that $P_{\mathcal{G}_{\text{do}(XZ)}}(Y \mid X = x, Z = z, W = w) \neq P_{\mathcal{G}_{\text{do}(XZ)}}(Y \mid X = x', Z = z, W = w)$. However, this implies that $P_{\mathcal{G}_{\text{do}(XZ)}}(Y \mid XZ)$ does have a non-trivial dependence on X and contradicts $P_{\mathcal{G}_{\text{do}(XZ)}}(Y \mid XZW) = P_{\mathcal{G}_{\text{do}(Z)}}(Y \mid ZW)$ (which would imply that it is equal to an X -independent quantity). This proves the result. \square

Lemma E.13 *If $X \vDash Y \mid \text{do}(Z), W$ is a Red₄ affects relation, then there exists $\tilde{s}_W \subsetneq W$ such that $X \vDash Y \mid \text{do}(Z), \tilde{s}_W$ holds.*

Proof: Suppose $X \vDash Y \mid \text{do}(Z), W$ holds and is a Red₄ affects relation. Then writing out this affects relation along with the two conditions implied by the reducibility (while recalling that $s_W \cup \tilde{s}_W = W$), we have

$$\begin{aligned} P_{\mathcal{G}_{\text{do}(XZ)}}(Y \mid XZW) &\neq P_{\mathcal{G}_{\text{do}(Z)}}(Y \mid ZW) \\ P_{\mathcal{G}_{\text{do}(XZ)}}(Y s_W \mid XZ \tilde{s}_W) &= P_{\mathcal{G}_{\text{do}(XZ)}}(Y \mid XZ \tilde{s}_W) P_{\mathcal{G}_{\text{do}(XZ)}}(s_W \mid XZ \tilde{s}_W) \\ P_{\mathcal{G}_{\text{do}(XZ)}}(Y s_W \mid Z \tilde{s}_W) &= P_{\mathcal{G}_{\text{do}(XZ)}}(Y \mid Z \tilde{s}_W) P_{\mathcal{G}_{\text{do}(XZ)}}(s_W \mid Z \tilde{s}_W). \end{aligned} \quad (91)$$

The last two equalities are equivalent to the following.

$$\begin{aligned} P_{\mathcal{G}_{\text{do}(XZ)}}(Y|XZW) &= P_{\mathcal{G}_{\text{do}(XZ)}}(Y|XZ\tilde{s}_W) \\ P_{\mathcal{G}_{\text{do}(XZ)}}(Y|ZW) &= P_{\mathcal{G}_{\text{do}(XZ)}}(Y|Z\tilde{s}_W) \end{aligned} \quad (92)$$

Plugging these back into the first expression of Equation (91) we obtain $P_{\mathcal{G}_{\text{do}(XZ)}}(Y|XZ\tilde{s}_W) \neq P_{\mathcal{G}_{\text{do}(Z)}}(Y|Z\tilde{s}_W)$, which is equivalent to $X \not\equiv Y|\text{do}(Z), \tilde{s}_W$. \square

Theorem E.18 [Clustering implies irreducibility] *For any affects relation $X \equiv Y|\text{do}(Z), W$, $\text{Clus}_i \Rightarrow \text{Irred}_i$ for all $i \in \{1, 2, 3, 4\}$.*

Proof: For $i \in \{1, 2, 3\}$, the proofs are entirely analogous to that of Theorem 4.11 and will not be repeated. We carry out the proof for $i = 4$ below.

Again, start by assuming that $X \equiv Y|\text{do}(Z), W$ holds and satisfies Clus_4 and Red_4 . From Definition E.17 and Definition E.9 this implies the following conditions (see also Equation (92)).

$$\begin{aligned} P_{\mathcal{G}_{\text{do}(XZ)}}(Y|XZ\tilde{s}_W) &= P_{\mathcal{G}_{\text{do}(Z)}}(Y|Z\tilde{s}_W), \quad \forall \tilde{s}_W \subsetneq W, \\ P_{\mathcal{G}_{\text{do}(XZ)}}(Y|XZW) &= P_{\mathcal{G}_{\text{do}(XZ)}}(Y|XZ\tilde{s}_W) \quad \exists \tilde{s}_W \subseteq W, \\ P_{\mathcal{G}_{\text{do}(XZ)}}(Y|ZW) &= P_{\mathcal{G}_{\text{do}(XZ)}}(Y|Z\tilde{s}_W) \quad \exists \tilde{s}_W \subseteq W. \end{aligned} \quad (93)$$

These imply that $P_{\mathcal{G}_{\text{do}(XZ)}}(Y|XZW) = P_{\mathcal{G}_{\text{do}(Z)}}(Y|ZW)$, which is equivalent to $X \not\equiv Y|\text{do}(Z), W$. This contradicts our initial assumption that $X \equiv Y|\text{do}(Z), W$ and establishes the result. \square

Theorem E.19 [Clustering implies fine-tuning (conditional version)] *Any affects relation $X \equiv Y|\text{do}(Z), W$ that satisfies Clus_i for $i \in \{1, 2, 3, 4\}$ and generated in a causal graph \mathcal{G} where X is not a cause of W , necessarily arises from a fine-tuned causal model on \mathcal{G} .*

Proof: The proofs for $i \in \{1, 2, 3\}$ are entirely analogous as those of Theorem 4.12 whenever we are given that X is not a cause of W . This is because the main step of the proof is to infer from $X \equiv Y|\text{do}(Z)$ that $(X \not\perp^d Y|Z)_{\mathcal{G}_{\text{do}(XZ)}}$, and the analogous inference holds for conditional affects relations when X is not a cause of W , due to Lemma E.5 i.e., $X \equiv Y|\{\text{do}(Z), W\}$ implies $(X \not\perp^d Y|ZW)_{\mathcal{G}_{\text{do}(XZ)}}$ whenever X is not a cause of W . We therefore do not repeat the proof.

The proof for $i = 4$ is also analogous, but we repeat it for completeness. Suppose that $X \equiv Y|\text{do}(Z), W$ holds and is a Clus_4 affects relation. This implies in particular that for all $e_W \in W$, $X \not\equiv Y|\text{do}(Z), e_W$. Writing this out, we have

$$\begin{aligned} P_{\mathcal{G}_{\text{do}(XZ)}}(Y|XZW) &\neq P_{\mathcal{G}_{\text{do}(Z)}}(Y|ZW) \\ P_{\mathcal{G}_{\text{do}(XZ)}}(Y|XZe_W) &= P_{\mathcal{G}_{\text{do}(Z)}}(Y|Ze_W), \quad \forall e_W \in W. \end{aligned} \quad (94)$$

Further, using Lemma E.4, the non-affects relations $X \not\equiv Y|\text{do}(Z), e_W$ imply the conditional independences $(X \perp Y|Z, e_W)_{\mathcal{G}_{\text{do}(XZ)}}$. Now consider the corresponding d-separation for each $e_W \in W$, we can either have $(X \perp^d Y|Z, e_W)_{\mathcal{G}_{\text{do}(XZ)}}$ or $(X \not\perp^d Y|Z, e_W)_{\mathcal{G}_{\text{do}(XZ)}}$. In the latter case we could have a d-connection with a corresponding conditional independence, which would make the model fine-tuned. So we consider the case where $(X \perp^d Y|Z, e_W)_{\mathcal{G}_{\text{do}(XZ)}}$ holds for all $e_W \in W$ (for all other cases the above argument establishes the result about fine-tuning). However, by assumption that X is not a cause of W , we have $X(W) = W$ and we know from Lemma E.5 that $(X \perp^d Y|Z, e_W)_{\mathcal{G}_{\text{do}(XZ)}}$ implies $X \not\equiv Y|\{\text{do}(Z), W\}$, which contradicts our initial assumption and established the result. \square

Corollary E.20 *For any unconditional affects relation $X \equiv Y|\text{do}(Z)$ which is Clus_2 and has $|Y| \geq 3$, $X \equiv s_Y|\text{do}(Z), \tilde{s}_Y$ is Clus_4 for each partition $s_Y \tilde{s}_Y = Y$ with $|\tilde{s}_Y| \geq 2$.*

Proof: Any $X \equiv Y|\text{do}(Z)$ which is Clus_2 , by Lemma E.2.3 is equivalent to $X \equiv s_Y|\text{do}(Z), \tilde{s}_Y$ for each partition $s_Y \tilde{s}_Y = Y$.

If for the case of $|\tilde{s}_Y| \geq 2$ – which is only possible for $|Y| \geq 2$ – this affects relation were not Clus_4 , there would exist $X \equiv s_Y|\text{do}(Z), \tilde{s}'_Y$ with $\tilde{s}'_Y \subsetneq \tilde{s}_Y$, implying $X \equiv s_Y \tilde{s}'_Y|\text{do}(Z)$ by Lemma E.2.1. This is in contradiction to the original affects relation being Clus_2 and proves the claim. \square