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► **To cite this version:**

Zhaonan Dong, Alexandre Ern. *hp*-error analysis of mixed-order hybrid high-order methods for elliptic problems on simplicial meshes. 2024. hal-04720237

**HAL Id: hal-04720237**

**<https://inria.hal.science/hal-04720237v1>**

Preprint submitted on 3 Oct 2024

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# $hp$ -error analysis of mixed-order hybrid high-order methods for elliptic problems on simplicial meshes

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October 3, 2024

## Abstract

We present both  $hp$ -a priori and  $hp$ -a posteriori error analysis of a mixed-order hybrid high-order (HHO) method to approximate second-order elliptic problems on simplicial meshes. Our main result on the  $hp$ -a priori error analysis is a  $\frac{1}{2}$ -order  $p$ -suboptimal error estimate. This result is, to our knowledge, the first of this kind for hybrid nonconforming methods and matches the state-of-the-art for other nonconforming methods as discontinuous Galerkin methods. Our second main result is a residual-based  $hp$ -a posteriori upper error bound, comprising residual, normal flux jump, tangential jump, and stabilization estimators (plus data oscillation terms). The first three terms are  $p$ -optimal and only the latter is  $\frac{1}{2}$ -order  $p$ -suboptimal. This result is, to our knowledge, the first  $hp$ -a posteriori error estimate for HHO methods. A novel approach based on the partition-of-unity provided by hat basis functions and on local Helmholtz decompositions on vertex stars is devised to estimate the nonconformity error. Finally, we establish local lower error bounds. Remarkably, the normal flux jump estimator is only  $\frac{1}{2}$ -order  $p$ -suboptimal, as it can be bounded by the stabilization owing to the local conservation property of HHO methods. Numerical examples illustrate the theory.

## 1 Introduction

Second-order elliptic PDEs are widely used in the modeling of diffusion phenomena. In the present work, we consider the following model problem:

$$\begin{cases} -\nabla \cdot (A \nabla u) = f & \text{in } \Omega, \\ u = g_D & \text{on } \Gamma_D, \\ (A \nabla u) \cdot \mathbf{n}_\Omega = g_N & \text{on } \Gamma_N, \end{cases} \quad (1)$$

where the domain  $\Omega$  is an open bounded, polytopal, Lipschitz set in  $\mathbb{R}^d$ ,  $d \in \{2, 3\}$ , with boundary  $\partial\Omega$  and unit outward normal  $\mathbf{n}_\Omega$ . The boundary  $\partial\Omega$  is split into two disjoint parts  $\Gamma_D$  and  $\Gamma_N$  with  $|\Gamma_D| > 0$ . In addition, the load  $f \in L^2(\Omega)$ ,  $g_D$  is a restriction to  $\Gamma_D$  of a function in  $H^{\frac{1}{2}}(\partial\Omega)$ ,  $g_N \in L^2(\Gamma_N)$  and  $A$  is a piecewise scalar-valued diffusion coefficient such that  $0 < A_\Omega^p \leq A(\mathbf{x}) \leq A_\Omega^\sharp$  for a.e.  $\mathbf{x} \in \Omega$ .

The hybrid high-order (HHO) method was introduced in [19] for linear diffusion and in [18] for locking-free linear elasticity. As shown in [16], the HHO method is closely related to hybridizable discontinuous Galerkin (HDG) and weak Galerkin (WG) methods. These links have been leveraged, e.g., in [21, 26] to devise a unified convergence analysis for the biharmonic problem and the acoustic wave equation. We also refer the reader to [38] and [15] for links to the nonconforming virtual element method (ncVEM), and to [13] for links to multiscale hybrid-mixed (MHM) methods. HHO methods are formulated in terms of broken cell and face polynomial spaces. The equal-order HHO method corresponds to cell and face unknowns having the same degree  $k \geq 0$ . Instead, the mixed-order HHO method corresponds to cell unknowns having degree  $(k + 1)$  and face unknowns having degree  $k \geq 0$ . Considering cell unknowns of degree  $(k - 1)$  with  $k \geq 1$  is also possible. One salient advantage of the mixed-order setting with cell

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unknowns of degree  $(k+1)$  is that  $h$ -optimal convergence can be achieved by using the simple Lehrenfeld–Schöberl (LS) HDG stabilization [37], in contrast with the more sophisticated HHO stabilization needed in the other settings.

The goal of the present work is to derive  $hp$ -a priori and  $hp$ -a posteriori error estimates for the mixed-order HHO method on simplicial meshes. Owing to the links highlighted above, the present results extend to HDG and WG methods in the same setting. Although HHO (and HDG, WG) methods can deal with polytopal meshes, we focus here on simplicial meshes because some of the  $hp$ -analysis tools we are going to invoke are only available on such meshes (and on tensor-product meshes as well). Our main result concerning the  $hp$ -a priori error analysis is Theorem 4.4, where we derive a  $\frac{1}{2}$ -order  $p$ -suboptimal and  $h$ -optimal error estimate. The only  $hp$ -a priori HHO error estimate we are aware of is derived in [1] for the equal-order HHO method and leads to a 1-order  $p$ -suboptimal error bound. Here, we use the same  $hp$ -scaling of the stabilization bilinear form, but the simpler form of the LS stabilization allows us to prove an error estimate with a tighter scaling in the polynomial degree. Notice in passing that  $\frac{1}{2}$ -order  $p$ -suboptimality corresponds to the state of the art for classical discontinuous Galerkin (dG) methods [33, 8], and was also obtained in [22] for a hybrid dG method applied to Stokes flow (recall that, when using polynomials of degree  $k$  on the mesh faces, a hybrid dG method typically leads to  $O(h^k)$  decay rates in the  $H^1$ -norm, whereas HHO methods lead to  $O(h^{k+1})$  decay rates).

The second main contribution of the present work is to derive a residual-based  $hp$ -a posteriori error estimate for the mixed-order HHO method in dimensions  $d \in \{2, 3\}$ . Our main result concerning the (global) upper error bound is Theorem 5.7, where all the terms in the upper bound are  $hp$ -optimal except one term which is  $\frac{1}{2}$ -order  $p$ -suboptimal. To the best of our knowledge, this is the first such estimate for HHO methods, whereas  $h$ -a posteriori error estimates for HHO methods were derived previously in [20, 5, 10], focusing on either the equal-order HHO setting or a stabilization-free variant of the method.

The main challenge in deriving an upper error bound for nonconforming methods is to estimate the nonconformity error, which essentially measures by how much the discrete solution departs from  $H^1$ . A first possibility is to invoke a nodal-averaging operator mapping to  $H^1$ , as done, e.g., in [34, 27, 2, 28] for dG methods. However,  $p$ -optimal approximation results for nodal-averaging operators are so far available only on tensor-product [6] and triangular [32], whereas the best bound available on tetrahedral meshes is  $p$ -suboptimal by one order [22, Lemma 7.6]. An alternative to using a nodal-averaging operator is to invoke a (global) Helmholtz decomposition on the nonconformity error, as in [17, 9, 11] for Crouzeix–Raviart finite elements and in [4, 7] for dG methods. Here, we adopt this technique, but we introduce a novel idea in that we additionally use the partition of unity provided by the hat basis functions to invoke a *local* Helmholtz decomposition on each vertex star (the subdomain covered by the mesh cells sharing the vertex). The benefit is that each vertex star is simply connected, and recent results on the stability constant in the local Helmholtz decomposition are available [30]. Instead, the stability constant for a global Helmholtz decomposition in a domain with  $N$  holes grows unfavorably with  $N$  [5].

We also address the efficiency of our a posteriori error estimate by establishing local lower error bounds. Our main result is Theorem 5.9 which leads to  $\frac{3}{2}$ -order  $p$ -suboptimality (only the tangential jump estimator leads to such suboptimality). Numerical experiments, though, indicate only  $\frac{1}{2}$ -order suboptimality in  $p$ . Our proof of the lower error bound uses bubble function techniques inspired from [41], but we introduce a novel argument in the proof in that we invoke the local conservation property of the HHO method to improve the efficiency result on the normal flux jump from  $\frac{3}{2}$ -order to  $\frac{1}{2}$ -order  $p$ -suboptimality. Another interesting numerical observation is that the normal flux jump is not the dominant component of the a posteriori error estimate for HHO methods, in contrast to the situation classically encountered with conforming finite elements [12].

The rest of this work is organised as follows. We present the weak formulation of the model problem together with the discrete setting in Section 2. In Section 3, we introduce the HHO method, and in Section 4, we derive the  $hp$ -a priori error estimate. In Section 5, we present the residual-based  $hp$ -a posteriori error analysis, leading to a (global) upper error bound and (local) lower error bounds. Numerical experiments are presented in Section 6 to illustrate the theory. Finally, in Section 7, we collect several (technical) proofs related to the  $hp$ -a posteriori error analysis.

## 2 Weak form and discrete setting

In this section, we introduce some basic notation, the weak formulation of the model problem, and the discrete setting to formulate and analyze the HHO discretization.

### 2.1 Basic notation and weak formulation

We use standard notation for the Lebesgue and Sobolev spaces and, in particular, for the fractional-order Sobolev spaces, we consider the Sobolev–Slobodeckij seminorm based on the double integral. For an open, bounded, Lipschitz set  $S$  in  $\mathbb{R}^d$ ,  $d \in \{1, 2, 3\}$ , we denote by  $(v, w)_S$  the  $L^2(S)$ -inner product, and we employ the same notation when  $v$  and  $w$  are vector- or matrix-valued fields. We denote by  $\nabla w$  the (weak) gradient of  $w$ . We use boldface notation to denote vectors in  $\mathbb{R}^d$ , as well as  $\mathbb{R}^d$ -valued fields and functional spaces composed of such fields.

Setting  $H_{g,D}^1(\Omega) := \{v \in H^1(\Omega) \mid v|_{\Gamma_D} = g_D\}$ , the weak formulation of (1) is as follows: Find  $u \in H_{g,D}^1(\Omega)$ , such that

$$(A\nabla u, \nabla v)_\Omega = (f, v)_\Omega + (g_N, v)_{\Gamma_N}, \quad (2)$$

for all  $v \in H_{0,D}^1(\Omega)$ . The well-posedness of (2) follows from the Lax–Milgram Lemma, see, e.g., [25, Proposition 31.21].

### 2.2 Mesh

Let  $\mathcal{T}_h$  be a simplicial mesh that covers the domain  $\Omega$  exactly and is compatible with the boundary partition as well as the domain partition on which  $A$  is piecewise constant. A generic mesh cell is denoted by  $K \in \mathcal{T}_h$ , its diameter by  $h_K$ , and its unit outward normal by  $\mathbf{n}_K$ . We set  $A_K := A|_K$ . We let  $\text{st}(K)$  denote the collection of cells  $\hat{K} \in \mathcal{T}_h$  sharing at least one vertex with the cell  $K$  ( $\text{st}(K)$  is often called cell star). Similarly,  $\text{es}(K)$  denotes the collection of cells  $\hat{K} \in \mathcal{T}_h$  sharing at least one vertex with  $\text{st}(K)$  ( $\text{es}(K)$  is often called extended cell star). For all  $k \geq 0$ ,  $\mathbb{P}^k(K)$  denotes the space of  $d$ -variate polynomials on  $K$  of degree at most  $k$ , and  $\Pi_K^k$  denotes the  $L^2$ -orthogonal projection onto  $\mathbb{P}^k(K)$ . Moreover,  $\mathbb{P}^k(\mathcal{T}_h) := \{v \in L^2(\Omega; \mathbb{R}) \mid v|_K \in \mathbb{P}^k(K)\}$  denotes the broken polynomial space of order  $k$  on the mesh  $\mathcal{T}_h$  (classically considered in dG methods).

The mesh faces are collected in the set  $\mathcal{F}_h$ , which is split as  $\mathcal{F}_h = \mathcal{F}_h^i \cup \mathcal{F}_h^b$ , where  $\mathcal{F}_h^i$  is the collection of interfaces (shared by two distinct mesh cells) and  $\mathcal{F}_h^b$  the collection of boundary faces. Moreover, we split  $\mathcal{F}_h^b$  into the Dirichlet subset,  $\mathcal{F}_h^D$ , and the Neumann subset,  $\mathcal{F}_h^N$ . Let  $\mathbf{n}_F$  denote the unit normal vector orienting the mesh face  $F \in \mathcal{F}_h$ . For all  $F \in \mathcal{F}_h^i$ , the direction of  $\mathbf{n}_F$  is arbitrary, but fixed, whereas we set  $\mathbf{n}_F := \mathbf{n}_\Omega|_F$  for all  $F \in \mathcal{F}_h^b$ . For every mesh cell  $K \in \mathcal{T}_h$ , the partition of its boundary  $\partial K$  is defined as  $\partial K = \partial K^i \cup \partial K^D \cup \partial K^N$  with obvious notation, and the mesh faces composing  $\partial K$  are collected in the set  $\mathcal{F}_{\partial K}$ , which is partitioned as  $\mathcal{F}_{\partial K} = \mathcal{F}_{\partial K^i} \cup \mathcal{F}_{\partial K^D} \cup \mathcal{F}_{\partial K^N}$  with obvious notation. For all  $F \in \mathcal{F}_h$ ,  $\Pi_F^k$  denotes the  $L^2$ -orthogonal projection onto  $\mathbb{P}^k(F)$ .

The set of mesh vertices is denoted by  $\mathcal{V}_h$  and is decomposed into the subset of interior vertices,  $\mathcal{V}_h^i$ , and the subset of boundary vertices,  $\mathcal{V}_h^b$ . For all  $\mathbf{a} \in \mathcal{V}_h$ ,  $\mathcal{T}_\mathbf{a}$  denotes the collection of mesh cells which share  $\mathbf{a}$  and  $\omega_\mathbf{a}$  the corresponding open subdomain (often called vertex star). In addition, we define  $\mathcal{F}_\mathbf{a}$  as the collection of faces in  $\mathcal{F}_h$  which share  $\mathbf{a}$ .

For all  $s > \frac{1}{2}$ , we define the broken Sobolev spaces  $H^s(\mathcal{T}_h; \mathbb{R}^q) := \{v \in L^2(\Omega; \mathbb{R}^q) \mid v|_K \in H^s(K; \mathbb{R}^q), \forall K \in \mathcal{T}_h\}$ ,  $q \in \{1, d\}$ . We define the jump  $[[w]]_F$  of any function  $w \in H^s(\mathcal{T}_h; \mathbb{R}^q)$  across any mesh interface  $F = \partial K_1 \cap \partial K_2 \in \mathcal{F}_h^i$  as  $[[w]]_F := w|_{K_1}|_F - w|_{K_2}|_F$ , where  $\mathbf{n}_F$  points from  $K_1$  to  $K_2$ . For any boundary face  $F = \partial K \cap \partial \Omega \in \mathcal{F}_h^b$ , we set  $[[w]]_F := w|_F$ . For all  $K \in \mathcal{T}_h$ , we define  $[[w]]_{\partial K}|_F := [[w]]_F$  for all  $F \in \mathcal{F}_{\partial K}$ . Finally, we define the broken gradient  $\nabla_{\mathcal{T}_h}$  as the gradient operator acting cellwise on  $H^1(\mathcal{T}_h; \mathbb{R}^q)$ .

### 2.3 $hp$ -analysis tools

Let us briefly review the main  $hp$ -analysis tools used in this work. We use the symbol  $C$  (sometimes with a subscript) to denote any positive generic constant whose value can change at each occurrence as long as it is independent of the mesh size  $h$  and the polynomial degree  $k$ . The value of  $C$  can depend on the mesh shape-regularity and the space dimension  $d$ .

**Lemma 2.1** (Discrete trace inequality). *The following holds for all  $v \in \mathbb{P}^k(K)$ , all  $K \in \mathcal{T}_h$ , and all  $k \geq 0$ ,*

$$\|v\|_{\partial K} \leq C \frac{(k+1)}{h_K^{\frac{1}{2}}} \|v\|_K, \quad (3)$$

*Proof.* A proof can be found in [44] (with a slightly sharper dependency on  $k$ ).  $\square$

**Lemma 2.2** (Local  $L^2$ -orthogonal projection). *The following holds for  $v \in H^1(K)$ , all  $K \in \mathcal{T}_h$ , and all  $k \geq 0$ ,*

$$\|v - \Pi_K^k(v)\|_{\partial K} \leq C \left( \frac{h_K}{k+1} \right)^{\frac{1}{2}} |v|_{H^1(K)}. \quad (4)$$

*Proof.* A proof can be found in [14, 42].  $\square$

**Lemma 2.3** (Local Babuška–Suri operator). *There exists a positive constant  $C_{\text{BS}}$  such that, for all  $k \geq 1$  and all  $K \in \mathcal{T}_h$ , there exists an operator  $\mathcal{I}_{\text{BS},K}^k : L^2(K) \rightarrow \mathbb{P}^k(K)$ , called Babuška–Suri approximation operator, such that, for all  $r \in \{0, \dots, k\}$ , all  $m \in \{0, \dots, r\}$ , and all  $v \in H^r(K)$ ,*

$$|v - \mathcal{I}_{\text{BS},K}^k(v)|_{H^m(K)} \leq C_{\text{BS}} \left( \frac{h_K}{k} \right)^{r-m} \|v\|_{H^r(K)}. \quad (5)$$

*Proof.* A proof can be found in [3].  $\square$

**Lemma 2.4** (Global  $hp$ -Karkulik–Melenk operator). *There exists a constant  $C_{\text{KM}}$  such that, for all  $k \geq 1$ , there exists an operator  $I_{\text{KM}}^k : H_{0,\text{D}}^1(\Omega) \rightarrow \mathbb{P}^k(\mathcal{T}_h) \cap H_{0,\text{D}}^1(\Omega)$ , called Karkulik–Melenk interpolation operator, such that, for all  $v \in H_{0,\text{D}}^1(\Omega)$  and all  $K \in \mathcal{T}_h$ ,*

$$\left( \frac{k}{h_K} \right)^2 \|v - I_{\text{KM}}^k(v)\|_K^2 + \left( \frac{k}{h_K} \right) \|v - I_{\text{KM}}^k(v)\|_{\partial K}^2 + \|\nabla I_{\text{KM}}^k(v)\|_K^2 \leq C_{\text{KM}} \|v\|_{H^1(\text{st}(K))}^2. \quad (6)$$

*Proof.* A proof can be found in [40, 35].  $\square$

**Corollary 2.5** (Modified  $hp$ -Karkulik–Melenk operator). *There exists a constant  $C_{\text{mKM}}$  such that, for all  $k \geq 1$ , there exists an operator  $I_{\text{mKM}}^k : H_{0,\text{D}}^1(\Omega) \rightarrow \mathbb{P}^k(\mathcal{T}_h) \cap H_{0,\text{D}}^1(\Omega)$ , called modified Karkulik–Melenk interpolation operator, such that, for all  $v \in H_{0,\text{D}}^1(\Omega)$  and all  $K \in \mathcal{T}_h$ ,*

$$\left( \frac{k}{h_K} \right)^2 \|v - I_{\text{mKM}}^k(v)\|_K^2 + \left( \frac{k}{h_K} \right) \|v - I_{\text{mKM}}^k(v)\|_{\partial K}^2 + \|\nabla I_{\text{mKM}}^k(v)\|_K^2 \leq C_{\text{mKM}} \|\nabla v\|_{\text{es}(K)}^2. \quad (7)$$

*Proof.* The idea is to set, for all  $v \in H_{0,\text{D}}^1(\Omega)$ ,

$$I_{\text{mKM}}^k(v) := I_{\text{av},\text{D}}^1(v) + I_{\text{KM}}^k(v - I_{\text{av},\text{D}}^1(v)),$$

where the (first-order) nodal-averaging operator  $I_{\text{av},\text{D}}^1 : H_{0,\text{D}}^1(\Omega) \rightarrow \mathbb{P}^1(\mathcal{T}_h) \cap H_{0,\text{D}}^1(\Omega)$  is devised, e.g., in [23] when  $\Gamma_{\text{D}} = \partial\Omega$  and in [39] when  $\Gamma_{\text{D}}$  is a proper subset of  $\partial\Omega$ . This operator satisfies, for all  $v \in H_{0,\text{D}}^1(\Omega)$ ,

$$\left( \frac{1}{h_K} \right)^2 \|v - I_{\text{av},\text{D}}^1(v)\|_K^2 + \left( \frac{1}{h_K} \right) \|v - I_{\text{av},\text{D}}^1(v)\|_{\partial K}^2 + \|\nabla I_{\text{av},\text{D}}^1(v)\|_K^2 \leq C \|\nabla v\|_{\text{st}(K)}^2. \quad (8)$$

We can now prove (7). Using the approximation results (6) and (8) and the mesh shape-regularity, we infer that

$$\begin{aligned} \left( \frac{k}{h_K} \right)^2 \|v - I_{\text{mKM}}^k(v)\|_K^2 &= \left( \frac{k}{h_K} \right)^2 \|(v - I_{\text{av},\text{D}}^1(v)) - I_{\text{KM}}^k(v - I_{\text{av},\text{D}}^1(v))\|_K^2 \\ &\leq C \|v - I_{\text{av},\text{D}}^1(v)\|_{H^1(\text{st}(K))}^2 \leq C \sum_{\widehat{K} \in \text{est}(K)} \|\nabla v\|_{\text{st}(\widehat{K})}^2 \leq C \|\nabla v\|_{\text{es}(K)}^2. \end{aligned}$$

This proves the bound on the first term on the left-hand side of (7), and the other two terms can be bounded in a similar way.  $\square$

**Remark 2.6** (Applications). *The Babuška–Suri approximation operator is used in to establish our hp-a priori error estimate. Instead, the modified Karkulik–Melenk interpolation operator is used in the hp-a posteriori error analysis to establish the upper error bound. The advantage of the modified Karkulik–Melenk interpolation operator with respect to the original one is to invoke only the  $H^1$ -seminorm on the right-hand side of (7) (compare with (6)).*

### 3 HHO method

Let  $k \geq 0$  be the polynomial degree. We focus on the mixed-order HHO method where, for all  $K \in \mathcal{T}_h$ , the local HHO space is

$$\widehat{V}_K^k := \mathbb{P}^{k+1}(K) \times \mathbb{P}^k(\mathcal{F}_{\partial K}), \quad \mathbb{P}^k(\mathcal{F}_{\partial K}) := \bigtimes_{F \in \mathcal{F}_h} \mathbb{P}^k(F). \quad (9)$$

A generic element in  $\widehat{V}_K^k$  is denoted by  $\widehat{v}_K := (v_K, v_{\partial K})$  with  $v_K \in \mathbb{P}^{k+1}(K)$  and  $v_{\partial K} \in \mathbb{P}^k(\mathcal{F}_{\partial K})$ . The first component of the pair  $\widehat{v}_K$  aims at representing the solution inside the mesh cell and the second its trace at the cell boundary.

#### 3.1 Reconstruction and stabilization

The HHO method is formulated locally by means of a reconstruction and a stabilization operator. The local reconstruction operator  $R_K^{k+1} : \widehat{V}_K^k \rightarrow \mathbb{P}^{k+1}(K)$  is such that, for all  $\widehat{v}_K := (v_K, v_{\partial K}) \in \widehat{V}_K^k$ ,  $R_K^{k+1}(\widehat{v}_K) \in \mathbb{P}^{k+1}(K)$  is determined by solving the following well-posed problem:

$$(\nabla R_K^{k+1}(\widehat{v}_K), \nabla w)_K := (\nabla v_K, \nabla w)_K - (v_K - v_{\partial K}, \nabla w \cdot \mathbf{n}_K)_{\partial K}, \quad (10)$$

for all  $w \in \mathbb{P}^{k+1}(K)/\mathbb{R}$  and  $(R_K^{k+1}(\widehat{v}_K), 1)_K = (v_K, 1)_K$ . (Notice that (10) actually holds for all  $w \in \mathbb{P}^{k+1}(K)$ . Integration by parts gives

$$(\nabla R_K^{k+1}(\widehat{v}_K), \nabla w)_K = -(v_K, \Delta w)_K + (v_{\partial K}, \nabla w \cdot \mathbf{n}_K)_{\partial K}. \quad (11)$$

The local stabilization bilinear form  $S_{\partial K}$  is defined such that, for all  $(\widehat{v}_K, \widehat{w}_K) \in \widehat{V}_K^k \times \widehat{V}_K^k$ ,

$$S_{\partial K}(\widehat{v}_K, \widehat{w}_K) := \frac{(k+1)^2}{h_K} (\Pi_{\partial K}^k(v_{\partial K} - v_K), \Pi_{\partial K}^k(w_{\partial K} - w_K))_{\partial K}, \quad (12)$$

where  $\Pi_{\partial K}^k$  denotes the  $L^2$ -orthogonal projection onto  $\mathbb{P}^k(\mathcal{F}_{\partial K})$ . The reconstruction and stabilization operators are combined together to build the local bilinear form  $a_K$  on  $\widehat{V}_K^k \times \widehat{V}_K^k$  such that, for all  $K \in \mathcal{T}_h$ ,

$$a_K(\widehat{v}_K, \widehat{w}_K) := A_K(\nabla R_K^{k+1}(\widehat{v}_K), \nabla R_K^{k+1}(\widehat{w}_K))_K + A_K S_{\partial K}(\widehat{v}_K, \widehat{w}_K). \quad (13)$$

**Lemma 3.1** (Useful property). *The following holds for all  $\widehat{v}_K \in \widehat{V}_K^k$  and all  $K \in \mathcal{T}_h$ :*

$$\|\nabla(R_K^{k+1}(\widehat{v}_K) - v_K)\|_K^2 \leq C S_{\partial K}(\widehat{v}_K, \widehat{v}_K). \quad (14)$$

*Proof.* Using the definition (10) of the reconstruction operator with  $w := R_K^{k+1}(\widehat{v}_K) - v_K \in \mathbb{P}^{k+1}(K)$  gives

$$\|\nabla(R_K^{k+1}(\widehat{v}_K) - v_K)\|_K^2 := -(v_K - v_{\partial K}, \nabla(R_K^{k+1}(\widehat{v}_K) - v_K) \cdot \mathbf{n}_K)_{\partial K}.$$

Since  $\nabla(R_K^{k+1}(\widehat{v}_K) - v_K) \cdot \mathbf{n}_K \in \mathbb{P}^k(\mathcal{F}_{\partial K})$ , using the discrete trace inequality (3), the definition (12) of the stabilization, and the Cauchy–Schwarz inequality implies that

$$\begin{aligned} \|\nabla(R_K^{k+1}(\widehat{v}_K) - v_K)\|_K^2 &= -(\Pi_{\partial K}^k(v_K - v_{\partial K}), \nabla(R_K^{k+1}(\widehat{v}_K) - v_K) \cdot \mathbf{n}_K)_{\partial K} \\ &\leq C S_{\partial K}(\widehat{v}_K, \widehat{v}_K)^{\frac{1}{2}} \|\nabla(R_K^{k+1}(\widehat{v}_K) - v_K)\|_K. \end{aligned}$$

This concludes the proof of (14).  $\square$

### 3.2 Global discrete problem

We define the global HHO space as

$$\widehat{V}_h^k := \mathbb{P}^{k+1}(\mathcal{T}_h) \times \mathbb{P}^k(\mathcal{F}_h), \quad \mathbb{P}^{k+1}(\mathcal{T}_h) := \bigtimes_{K \in \mathcal{T}_h} \mathbb{P}^{k+1}(K), \quad \mathbb{P}^k(\mathcal{F}_h) := \bigtimes_{F \in \mathcal{F}_h} \mathbb{P}^k(F). \quad (15)$$

A generic element in  $\widehat{V}_h^k$  is denoted by  $\widehat{v}_h := (v_{\mathcal{T}_h}, v_{\mathcal{F}_h})$  with  $v_{\mathcal{T}_h} := (v_K)_{K \in \mathcal{T}_h}$  and  $v_{\mathcal{F}_h} := (v_F)_{F \in \mathcal{F}_h}$ . For all  $K \in \mathcal{T}_h$ , the local components of  $\widehat{v}_h$  are collected in the pair  $\widehat{v}_K := (v_K, v_{\partial K}) \in \widehat{V}_K^k$  with  $v_{\partial K}|_F := v_F$  for all  $F \in \mathcal{F}_{\partial K}$ . Similarly, let  $R_{\mathcal{T}_h}^{k+1}(\widehat{u}_h) \in \mathbb{P}^{k+1}(\mathcal{T}_h)$  be such that  $R_{\mathcal{T}_h}^{k+1}(\widehat{u}_h)|_K = R_K^{k+1}(\widehat{u}_K)$  for all  $K \in \mathcal{T}_h$ . To deal with the Dirichlet boundary condition, we define the subspaces

$$\widehat{V}_{h0,D}^k := \{\widehat{v}_h \in \widehat{V}_h^k \mid v_F = 0, \forall F \in \mathcal{F}_h^D\}, \quad (16a)$$

$$\widehat{V}_{hg,D}^k := \{\widehat{v}_h \in \widehat{V}_h^k \mid v_F = \Pi_F^k(g_D|_F), \forall F \in \mathcal{F}_h^D\}. \quad (16b)$$

The discrete HHO problem is as follows: Find  $\widehat{u}_h \in \widehat{V}_{hg,D}^k$  such that

$$a_h(\widehat{u}_h, \widehat{w}_h) = \ell_h(\widehat{w}_h), \quad \forall \widehat{w}_h \in \widehat{V}_{h0,D}^k, \quad (17)$$

where the global discrete bilinear form  $a_h$  and the global linear form  $\ell_h$  are assembled cellwise as

$$a_h(\widehat{v}_h, \widehat{w}_h) := \sum_{K \in \mathcal{T}_h} a_K(\widehat{v}_K, \widehat{w}_K), \quad \ell_h(\widehat{w}_h) := \sum_{K \in \mathcal{T}_h} \{(f, w_K)_K + (g_N, w_{\partial K})_{\partial K^N}\}. \quad (18)$$

It is well-known that the discrete problem (17) is amenable to static condensation, i.e., the cell unknowns can be eliminated locally in every mesh cell, leading to a global problem where the only remaining unknowns are those attached to the mesh faces, i.e., those in  $\mathbb{P}^k(\mathcal{F}_h \setminus \mathcal{F}_h^D)$ .

An important property of the HHO method we exploit in the a posteriori error analysis is local conservation. For all  $K \in \mathcal{T}_h$  and all  $F \in \mathcal{F}_{\partial K}$ , we define the flux

$$\phi_{K,F}(\widehat{u}_K) := -A_K \nabla R_K^{k+1}(\widehat{u}_K) \cdot \mathbf{n}_K|_F + A_K \frac{(k+1)^2}{h_K} \Pi_F^k(u_K|_F - u_F) \in \mathbb{P}^k(F). \quad (19)$$

Then, the following holds true [16]: (i) At every interface  $F = \partial K_1 \cap \partial K_2 \in \mathcal{F}_h^i$ , we have

$$\phi_{K_1,F}(\widehat{u}_{K_1}) + \phi_{K_2,F}(\widehat{u}_{K_2}) = 0; \quad (20a)$$

(ii) At every Neumann boundary face  $F = \partial K \cap \Gamma_N \in \mathcal{F}_h^N$ , we have

$$\phi_{K,F}(\widehat{u}_K) + \Pi_F^k(g_N|_F) = 0. \quad (20b)$$

### 3.3 Stability and well-posedness

We equip the local HHO space  $\widehat{V}_K^k$  with the  $H^1$ -like seminorm such that, for all  $\widehat{v}_K \in \widehat{V}_K^k$ ,

$$|\widehat{v}_K|_{\widehat{V}_K^k}^2 := \|\nabla v_K\|_K^2 + \frac{(k+1)^2}{h_K} \|\Pi_{\partial K}^k(v_{\partial K} - v_K)\|_{\partial K}^2. \quad (21)$$

**Lemma 3.2** (Local stability and boundedness). *There is a real number  $\alpha > 0$ , depending only on the mesh shape-regularity and the space dimension  $d$ , such that, for all  $\widehat{v}_K \in \widehat{V}_K^k$  and all  $K \in \mathcal{T}_h$ ,*

$$\alpha |\widehat{v}_K|_{\widehat{V}_K^k}^2 \leq \|\nabla R_K^{k+1}(\widehat{v}_K)\|_K^2 + S_{\partial K}(\widehat{v}_K, \widehat{v}_K) \leq \alpha^{-1} |\widehat{v}_K|_{\widehat{V}_K^k}^2. \quad (22)$$

*Proof.* The proof proceeds as in [16] using the discrete trace inequality (3).  $\square$

We equip the space  $\widehat{V}_{h0,D}^k$  with the norm

$$\|\widehat{v}_h\|_{\text{HHO}} := \sum_{K \in \mathcal{T}_h} A_K |\widehat{v}_K|_{\widehat{V}_K^k}^2, \quad \forall \widehat{v}_h \in \widehat{V}_{h0,D}^k. \quad (23)$$

**Corollary 3.3** (Coercivity and well-posedness). *The discrete bilinear form  $a_h$  is coercive on  $\widehat{V}_{h0,D}^k$ , and the discrete problem (17) is well-posed.*

*Proof.* Summing (22) multiplied by  $A_K$  over all the mesh cells shows the following coercivity and continuity properties:

$$\alpha \|\widehat{v}_h\|_{\text{HHO}}^2 \leq a_h(\widehat{v}_h, \widehat{v}_h) \leq \alpha^{-1} \|\widehat{v}_h\|_{\text{HHO}}^2, \quad \forall \widehat{v}_h \in \widehat{V}_{h0,D}^k. \quad (24)$$

The well-posedness of (17) with homogeneous Dirichlet boundary conditions then follows from the Lax–Milgram lemma. For inhomogeneous Dirichlet boundary conditions, the proof follows by introducing a lifting, say  $\widehat{g}_h \in \widehat{V}_{hg,D}^k$ , such that  $g_F = \Pi_F^k(g_D)$ , for all  $F \in \mathcal{F}_h^D$ .  $\square$

## 4 $hp$ -a priori error estimate

In this section, we establish our  $hp$ -a priori error estimate.

### 4.1 Approximation

For all  $K \in \mathcal{T}_h$ , we define the local reduction operator  $\widehat{\mathcal{I}}_K^k : H^1(K) \rightarrow \widehat{V}_K^k$  such that, for all  $v \in H^1(K)$ ,

$$\widehat{\mathcal{I}}_K^k(v) := (\Pi_K^{k+1}(v), \Pi_{\partial K}^k(v|_{\partial K})) \in \widehat{V}_K^k. \quad (25)$$

Moreover, the elliptic projection  $\mathcal{E}_K^{k+1} : H^1(K) \rightarrow \mathbb{P}^{k+1}(K)$  is defined such that

$$\begin{aligned} (\nabla(\mathcal{E}_K^{k+1}(v) - v), \nabla w)_K &= 0, \quad \forall w \in \mathbb{P}^{k+1}(K)/\mathbb{R}, \\ (\mathcal{E}_K^{k+1}(v) - v, 1)_K &= 0. \end{aligned} \quad (26)$$

One readily verifies by proceeding as in [19, Lemma 3] that

$$R_K^{k+1} \circ \widehat{\mathcal{I}}_K^k = \mathcal{E}_K^{k+1}. \quad (27)$$

**Lemma 4.1** (Bound on stabilization). *The following holds for all  $K \in \mathcal{T}_h$  and all  $v \in H^1(K)$ :*

$$S_{\partial K}(\widehat{\mathcal{I}}_K^k(v), \widehat{\mathcal{I}}_K^k(v)) \leq C(k+1) \|\nabla(v - \mathcal{I}_{\text{BS},K}^{k+1}(v))\|_K^2. \quad (28)$$

*Proof.* Recalling the definition (12) of  $S_{\partial K}$ , the definition (25) of  $\widehat{\mathcal{I}}_K^k$  and since  $\Pi_{\partial K}^k \circ \Pi_{\partial K}^k = \Pi_{\partial K}^k$ , we have

$$S_{\partial K}(\widehat{\mathcal{I}}_K^k(v), \widehat{\mathcal{I}}_K^k(v)) = \frac{(k+1)^2}{h_K} \|\Pi_{\partial K}^k(v - \Pi_K^{k+1}(v))\|_{\partial K}^2 \leq \frac{(k+1)^2}{h_K} \|v - \Pi_K^{k+1}(v)\|_{\partial K}^2,$$

where we used the  $L^2(\partial K)$ -stability of  $\Pi_{\partial K}^k$ . Then, we invoke the approximation result on the  $L^2$ -orthogonal projection, see (4), and that  $\Pi_K^{k+1} \circ \mathcal{I}_{\text{BS},K}^{k+1} = \mathcal{I}_{\text{BS},K}^{k+1}$ , giving

$$\begin{aligned} \frac{(k+1)^2}{h_K} \|v - \Pi_K^{k+1}(v)\|_{\partial K}^2 &= \frac{(k+1)^2}{h_K} \|v - \mathcal{I}_{\text{BS},K}^{k+1}(v) - \Pi_K^{k+1}(v - \mathcal{I}_{\text{BS},K}^{k+1}(v))\|_{\partial K}^2 \\ &\leq C(k+1) \|\nabla(v - \mathcal{I}_{\text{BS},K}^{k+1}(v))\|_K^2. \end{aligned}$$

Combining the above two bounds proves (28).  $\square$

For all  $K \in \mathcal{T}_h$  and all  $v \in H^{1+s}(K)$ ,  $s > \frac{1}{2}$ , we consider the following norm:

$$\|v\|_{\sharp,K}^2 := \|\nabla v\|_K^2 + \frac{h_K}{k+1} \|\nabla v\|_{\partial K}^2. \quad (29)$$

**Lemma 4.2** (Approximation). *The following holds for all  $K \in \mathcal{T}_h$  and all  $v \in H^{1+s}(K)$ ,  $s > \frac{1}{2}$ :*

$$\|v - \mathcal{E}_K^{k+1}(v)\|_{\sharp,K} \leq C(k+1)^{\frac{1}{2}} \|v - \mathcal{I}_{\text{BS},K}^{k+1}(v)\|_{\sharp,K}. \quad (30)$$



*Proof.* Using the triangle inequality, we have

$$\|v - \mathcal{E}_K^{k+1}(v)\|_{\sharp, K} \leq \|v - \mathcal{I}_{\text{BS}, K}^{k+1}(v)\|_{\sharp, K} + \|\mathcal{I}_{\text{BS}, K}^{k+1}(v) - \mathcal{E}_K^{k+1}(v)\|_{\sharp, K},$$

so that we only need to bound the second term on the right-hand side. Owing to the discrete trace inequality (3), we infer that

$$\|\mathcal{E}_K^{k+1}(v) - \mathcal{I}_{\text{BS}, K}^{k+1}(v)\|_{\sharp, K} \leq C(k+1)^{\frac{1}{2}} \|\nabla(\mathcal{E}_K^{k+1}(v) - \mathcal{I}_{\text{BS}, K}^{k+1}(v))\|_K,$$

and it remains to bound  $\|\nabla(\mathcal{E}_K^{k+1}(v) - \mathcal{I}_{\text{BS}, K}^{k+1}(v))\|_K$ . Since  $\mathcal{E}_K^{k+1} \circ \mathcal{I}_{\text{BS}, K}^{k+1} = \mathcal{I}_{\text{BS}, K}^{k+1}$ , we infer that

$$\|\nabla(\mathcal{E}_K^{k+1}(v) - \mathcal{I}_{\text{BS}, K}^{k+1}(v))\|_K = \|\nabla(\mathcal{E}_K^{k+1}(v - \mathcal{I}_{\text{BS}, K}^{k+1}(v)))\|_K \leq \|\nabla(v - \mathcal{I}_{\text{BS}, K}^{k+1}(v))\|_K,$$

where the last bound follows from the stability of the elliptic projection. Combining the above bounds completes the proof.  $\square$

## 4.2 Consistency

Let  $u \in H_{g, \text{D}}^1(\Omega)$  be the exact solution to (2). We define the consistency error  $\delta_h \in (\widehat{V}_{h0, \text{D}}^k)'$  such that

$$\langle \delta_h, \widehat{w}_h \rangle := \ell_h(\widehat{w}_h) - a_h(\widehat{\mathcal{I}}_h^k(u), \widehat{w}_h), \quad \forall \widehat{w}_h \in \widehat{V}_{h0, \text{D}}^k, \quad (31)$$

where  $\langle \cdot, \cdot \rangle$  denotes the duality pairing between  $(\widehat{V}_{h0, \text{D}}^k)'$  and  $\widehat{V}_{h0, \text{D}}^k$ , and where the global reduction operator  $\widehat{\mathcal{I}}_h^k : H^1(\Omega) \rightarrow \widehat{V}_h^k$  is defined such that, for all  $v \in H^1(\Omega)$ ,

$$\widehat{\mathcal{I}}_h^k(v) := ((\Pi_K^{k+1}(v|_K))_{K \in \mathcal{T}_h}, (\Pi_F^k(v|_F))_{F \in \mathcal{F}_h}) \in \widehat{V}_h^k, \quad (32)$$

observing that  $v$  is single-valued on every  $F \in \mathcal{F}_h^i$ . Notice that the local components of  $\widehat{\mathcal{I}}_h^k(v)$  attached to  $K$  and its faces are  $\widehat{\mathcal{I}}_K^k(v|_K)$  for all  $K \in \mathcal{T}_h$ .

**Lemma 4.3** (Consistency). *Assume that  $u \in H^{1+s}(\Omega)$  with  $s > \frac{1}{2}$ . The following holds true:*

$$\|\delta_h\|_{\text{HHO}'} := \sup_{\widehat{w}_h \in \widehat{V}_{h0, \text{D}}^k} \frac{|\langle \delta_h, \widehat{w}_h \rangle|}{\|\widehat{w}_h\|_{\text{HHO}}} \leq C \left( \sum_{K \in \mathcal{T}_h} A_K (k+1) \|u - \mathcal{I}_{\text{BS}, K}^{k+1}(u)\|_{\sharp, K}^2 \right)^{\frac{1}{2}}. \quad (33)$$

*Proof.* Let  $\widehat{w}_h \in \widehat{V}_{h0, \text{D}}^k$ . Using the definition of  $\ell_h$  in (18), the PDE and the boundary conditions satisfied by the exact solution  $u$ , and integrating by parts cellwise, we infer that

$$\ell_h(\widehat{w}_h) = \sum_{K \in \mathcal{T}_h} A_K \left\{ (\nabla u, \nabla w_K)_K - (\nabla u \cdot \mathbf{n}_K, w_K)_{\partial K} + (\nabla u \cdot \mathbf{n}_K, w_{\partial K})_{\partial K^{\text{N}}} \right\}.$$

The assumption  $u \in H^{1+s}(\Omega)$  with  $s > \frac{1}{2}$  implies that  $(\nabla u \cdot \mathbf{n}_K)|_{\partial K}$  is meaningful in  $L^2(\partial K)$  and single-valued at every mesh interface. Moreover, since  $w_{\partial K}$  is single-valued on  $\partial K^i$  and vanishes on  $\partial K^{\text{D}}$ , we infer that

$$\ell_h(\widehat{w}_h) = \sum_{K \in \mathcal{T}_h} A_K \left\{ (\nabla u, \nabla w_K)_K - (\nabla u \cdot \mathbf{n}_K, w_K - w_{\partial K})_{\partial K} \right\}.$$

Since  $a_h$  is assembled cellwise and the local components of  $\widehat{\mathcal{I}}_h^k(u)$  are  $\widehat{\mathcal{I}}_K^k(u|_K)$  for all  $K \in \mathcal{T}_h$ , we infer that  $a_h(\widehat{\mathcal{I}}_h^k(u), \widehat{w}_h) = \sum_{K \in \mathcal{T}_h} a_K(\widehat{\mathcal{I}}_K^k(u|_K), \widehat{w}_K)$ . Using the definition (13) of  $a_K$ , the definition (10) of  $R_K^{k+1}(\widehat{w}_K)$  and the identity (27) leads to

$$\begin{aligned} a_h(\widehat{\mathcal{I}}_h^k(u), \widehat{w}_h) &= \sum_{K \in \mathcal{T}_h} A_K \left\{ (\nabla \mathcal{E}_K^{k+1}(u|_K), \nabla w_K)_K - (\nabla \mathcal{E}_K^{k+1}(u|_K) \cdot \mathbf{n}_K, w_K - w_{\partial K})_{\partial K} \right. \\ &\quad \left. + S_{\partial K}(\widehat{\mathcal{I}}_K^k(u|_K), \widehat{w}_K) \right\}. \end{aligned}$$

Defining the function  $\eta$  cellwise as  $\eta|_K := u|_K - \mathcal{E}_K^{k+1}(u|_K)$  for all  $K \in \mathcal{T}_h$ , we infer that

$$\langle \delta_h, \widehat{w}_h \rangle = \sum_{K \in \mathcal{T}_h} A_K \left\{ -(\nabla \eta \cdot \mathbf{n}_K, w_K - w_{\partial K})_{\partial K} - S_{\partial K}(\widehat{\mathcal{I}}_K^k(u|_K), \widehat{w}_K) \right\}, \quad (34)$$

where we used that  $(\nabla \eta, \nabla w_K)_K = 0$ . Let us denote by  $\mathcal{T}_{1,K}$  the first term and by  $\mathcal{T}_{2,K}$  the second term inside braces on the right-hand side of (34). We bound  $\mathcal{T}_{1,K}$  by the Cauchy–Schwarz inequality, the triangle inequality, the trivial bound  $\|w_K - \Pi_{\partial K}^k(w_K)\|_{\partial K} \leq \|w_K - \Pi_K^k(w_K)\|_{\partial K}$ , and the approximation result (4) on the  $L^2$ -projection. Recalling the definition (29) of the  $\|\cdot\|_{\sharp,K}$ -norm, this yields

$$\begin{aligned} |\mathcal{T}_{1,K}| &\leq \|\nabla \eta\|_{\partial K} (\|\Pi_{\partial K}^k(w_K) - w_{\partial K}\|_{\partial K} + \|w_K - \Pi_K^k(w_K)\|_{\partial K}) \\ &\leq \left(\frac{h_K}{k+1}\right)^{\frac{1}{2}} \|\nabla \eta\|_{\partial K} \left(\frac{k+1}{h_K}\right)^{\frac{1}{2}} (\|\Pi_{\partial K}^k(w_K) - w_{\partial K}\|_{\partial K} + \|w_K - \Pi_K^k(w_K)\|_{\partial K}) \\ &\leq \|\eta\|_{\sharp,K} \left( \left(\frac{k+1}{h_K}\right)^{\frac{1}{2}} \|\Pi_{\partial K}^k(w_K) - w_{\partial K}\|_{\partial K} + C \|\nabla w_K\|_K \right) \leq C \|\eta\|_{\sharp,K} |\widehat{w}_K|_{\widehat{V}_K^k}. \end{aligned}$$

Moreover, owing to (28) and the upper bound in (22), we have

$$|\mathcal{T}_{2,K}| \leq S_{\partial K}(\widehat{\mathcal{I}}_K^k(u|_K), \widehat{\mathcal{I}}_K^k(u|_K))^{\frac{1}{2}} S_{\partial K}(\widehat{w}_K, \widehat{w}_K)^{\frac{1}{2}} \leq C(k+1)^{\frac{1}{2}} \|\nabla(u - \mathcal{I}_{\text{BS},K}^{k+1}(u))\|_K |\widehat{w}_K|_{\widehat{V}_K^k}.$$

Altogether, this implies that

$$|\langle \delta_h, \widehat{w}_h \rangle| \leq C \left( \sum_{K \in \mathcal{T}_h} A_K \left\{ \|\eta\|_{\sharp,K}^2 + (k+1) \|\nabla(u - \mathcal{I}_{\text{BS},K}^{k+1}(u))\|_K^2 \right\} \right)^{\frac{1}{2}} \|\widehat{w}_h\|_{\text{HHO}}.$$

Invoking Lemma 4.2 to bound  $\|\eta\|_{\sharp,K}$  completes the proof.  $\square$

### 4.3 Error estimate

We are now ready to establish our main result concerning the  $hp$ -a priori error analysis. The estimate is  $\frac{1}{2}$ -order  $p$ -suboptimal.

**Theorem 4.4** (*hp*-a priori error estimate). *Let  $u$  be the weak solution to (2), and let  $\widehat{u}_h$  be the discrete solution to (17). Assume that  $u \in H^{1+s}(\Omega)$  with  $s > \frac{1}{2}$ . The following holds:*

$$\sum_{K \in \mathcal{T}_h} A_K \left\{ \|\nabla(u - R_K^{k+1}(\widehat{u}_K))\|_K^2 + \|\nabla(u - u_K)\|_K^2 + S_{\partial K}(\widehat{u}_K, \widehat{u}_K) \right\} \leq C \sum_{K \in \mathcal{T}_h} A_K (k+1) \|u - \mathcal{I}_{\text{BS},K}^{k+1}(u)\|_{\sharp,K}^2. \quad (35a)$$

Moreover, if  $u \in H^{l+1}(\mathcal{T}_h; \mathbb{R})$ , with  $l \in \{1, \dots, k+1\}$ , we have

$$\sum_{K \in \mathcal{T}_h} A_K \left\{ \|\nabla(u - R_K^{k+1}(\widehat{u}_K))\|_K^2 + \|\nabla(u - u_K)\|_K^2 + S_{\partial K}(\widehat{u}_K, \widehat{u}_K) \right\} \leq C \sum_{K \in \mathcal{T}_h} A_K (k+1) \left(\frac{h_K}{k+1}\right)^{2l} \|u\|_{H^{l+1}(K)}^2. \quad (35b)$$

*Proof.* (i) Proof of (35a). We set  $\widehat{e}_h := \widehat{\mathcal{I}}_h^k(u) - \widehat{u}_h$  and observe that  $\widehat{e}_h \in \widehat{V}_{h,0,D}^k$ . Moreover, since  $a_h(\widehat{e}_h, \widehat{e}_h) = -\langle \delta_h, \widehat{e}_h \rangle$ , the coercivity property (24) implies that

$$\alpha \|\widehat{e}_h\|_{\text{HHO}}^2 \leq a_h(\widehat{e}_h, \widehat{e}_h) = -\langle \delta_h, \widehat{e}_h \rangle \leq \|\delta_h\|_{\text{HHO}'} \|\widehat{e}_h\|_{\text{HHO}},$$

so that  $\|\widehat{e}_h\|_{\text{HHO}} \leq \frac{1}{\alpha} \|\delta_h\|_{\text{HHO}'}$ . Since  $\sum_{K \in \mathcal{T}_h} A_K \left\{ \|\nabla R_K^{k+1}(\widehat{e}_K)\|_K^2 + S_{\partial K}(\widehat{e}_K, \widehat{e}_K) \right\} \leq \alpha^{-1} \|\widehat{e}_h\|_{\text{HHO}}^2$  follows from (24), we infer from Lemma 4.3 that

$$\sum_{K \in \mathcal{T}_h} A_K \left\{ \|\nabla R_K^{k+1}(\widehat{e}_K)\|_K^2 + S_{\partial K}(\widehat{e}_K, \widehat{e}_K) \right\} \leq C \sum_{K \in \mathcal{T}_h} A_K (k+1) \|u - \mathcal{I}_{\text{BS},K}^{k+1}(u)\|_{\sharp,K}^2.$$

Since  $u - R_K^{k+1}(\widehat{u}_K) = (u - \mathcal{E}_K^{k+1}(u)) + R_K^{k+1}(\widehat{e}_K)$  and  $\widehat{u}_K = \widehat{\mathcal{I}}_K^k(u) - \widehat{e}_K$ , the triangle inequality combined with Lemma 4.1, Lemma 4.2, and the above bound proves that

$$\sum_{K \in \mathcal{T}_h} A_K \left\{ \|\nabla(u - R_K^{k+1}(\widehat{u}_K))\|_K^2 + S_{\partial K}(\widehat{u}_K, \widehat{u}_K) \right\} \leq C \sum_{K \in \mathcal{T}_h} A_K (k+1) \|u - \mathcal{I}_{\text{BS},K}^{k+1}(u)\|_{\sharp,K}^2.$$

Moreover, using  $\|\nabla(u - u_K)\|_K^2 \leq 2(\|\nabla(u - R_K^{k+1}(\hat{u}_K))\|_K^2 + \|\nabla(u_K - R_K^{k+1}(\hat{u}_K))\|_K^2)$  and (14), we infer that

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} A_K \|\nabla(u - u_K)\|_K^2 &\leq 2 \sum_{K \in \mathcal{T}_h} A_K \left\{ \|\nabla(u - R_K^{k+1}(\hat{u}_K))\|_K^2 + CS_{\partial K}(\hat{u}_K, \hat{u}_K) \right\} \\ &\leq C \sum_{K \in \mathcal{T}_h} A_K (k+1) \|u - \mathcal{I}_{\text{BS},K}^{k+1}(u)\|_{\sharp,K}^2. \end{aligned}$$

This completes the proof of (35a).

(ii) The proof of (35b) follows by invoking on all  $K \in \mathcal{T}_h$  the approximation properties of  $\mathcal{I}_{\text{BS},K}^{k+1}$  (see Lemma 2.3), together with a multiplicative trace inequality (see, e.g., [24, Sec. 12.3.2]) to estimate the term involving the normal derivative in  $\|u - \mathcal{I}_{\text{BS},K}^{k+1}(u)\|_{\sharp,K}$ .  $\square$

## 5 $hp$ -a posteriori error analysis

In this section, we perform the  $hp$ -a posteriori error analysis of the above HHO discretization. We establish both (global) upper and (local) lower error bounds. Since we are interested in the  $hp$ -a posteriori error analysis, we only consider in what follows the case  $k \geq 1$ . For the case  $k = 0$ , we refer the reader to Remark 7.1. We shall also assume that the Dirichlet datum satisfies  $g_D \in H^{\frac{1}{2}}(\partial\Omega)$ . For all  $K \in \mathcal{T}_h$ , we define the following data oscillation terms:

$$O_K(f) := A_K^{-\frac{1}{2}} \left( \frac{h_K}{k+1} \right) \|f - \Pi_K^{k+1}(f)\|_K, \quad (36a)$$

$$O_K(g_N) := A_K^{-\frac{1}{2}} \left( \frac{h_K}{k+1} \right)^{\frac{1}{2}} \|g_N - \Pi_{\partial K}^k(g_N)\|_{\partial K^N}, \quad (36b)$$

$$O_K(g_D) := A_K^{\frac{1}{2}} \left( \frac{h_K}{k+1} \right)^{\frac{1}{2}} \|\nabla(g_D - \Pi_{\partial K}^{k+1}(g_D)) \times \mathbf{n}_\Omega\|_{\partial K^D}. \quad (36c)$$

It is useful to define some contrast factors related to the diffusion coefficient  $A$ . For all  $K \in \mathcal{T}_h$  and all  $\mathbf{a} \in \mathcal{V}_h$ , we set

$$\chi_K(A) := A_K \max_{\widehat{K} \in \text{es}(K)} A_{\widehat{K}}^{-1}, \quad \chi_{\mathbf{a}}(A) := A_{\mathbf{a}}^{\sharp} (A_{\mathbf{a}}^{\flat})^{-1}, \quad (37)$$

with  $A_{\mathbf{a}}^{\sharp} := \max_{K \in \mathcal{T}_{\mathbf{a}}} A_K$  and  $A_{\mathbf{a}}^{\flat} := \min_{K \in \mathcal{T}_{\mathbf{a}}} A_K$ . Moreover, it is convenient to set

$$A_F^{\flat} := \min(A_{K_1}, A_{K_2}), \quad \forall F = \partial K_1 \cap \partial K_2 \in \mathcal{F}_h^i, \quad (38a)$$

$$A_F^{\flat} := A_K, \quad \forall F = \partial K \cap \partial\Omega \in \mathcal{F}_h^b. \quad (38b)$$

For all  $K \in \mathcal{T}_h$ , we also define  $A_{\partial K}^{\flat}|_F := A_F^{\flat}$  for all  $F \in \mathcal{F}_{\partial K}$ .

### 5.1 Global upper error bound

We decompose the error into two components as follows:

$$e := u - u_{\mathcal{T}_h} = (u - u_c) + (u_c - u_{\mathcal{T}_h}) =: e_c + e_d, \quad u_c \in H_{g,D}^1(\Omega). \quad (39)$$

The function  $u_c$  is constructed from  $\hat{u}_h$  as detailed in Section 5.1.2. The precise definition of  $u_c$  is irrelevant for bounding  $e_c$  (we only use that  $e_c \in H_{0,D}^1(\Omega)$ ), and is only relevant for bounding  $e_d$ . We call  $e_c$  the conforming error and  $e_d$  the nonconformity error.

#### 5.1.1 Bound on conforming error $e_c$

In this section, we derive two bounds on the conforming error  $e_c$ . The first bound avoids the normal flux jump (classically considered in the context of finite elements) and is  $\frac{1}{2}$ -order  $p$ -suboptimal. The second bound includes the normal flux jump and is  $p$ -optimal. The first bound is, however, interesting in its own right. We will also see that, in the context of the lower error bound, the normal flux jump leads to  $\frac{1}{2}$ -order  $p$ -suboptimality anyway. The proofs of the following two results is postponed to Section 7.

**Lemma 5.1** (Conforming error ( $p$ -suboptimal bound)). *The following holds:*

$$\begin{aligned} \|A^{\frac{1}{2}}\nabla e_c\|_{\Omega} &\leq C_{c,1} \left\{ \sum_{K \in \mathcal{T}_h} \left\{ A_K^{-1} \left( \frac{h_K}{k+1} \right)^2 \|\Pi_K^{k+1}(f) + A_K \Delta R_K^{k+1}(\hat{u}_K)\|_K^2 + A_K(k+1) S_{\partial K}(\hat{u}_K, \hat{u}_K) \right. \right. \\ &\quad \left. \left. + O_K(f)^2 + O_K(g_N)^2 \right\} \right\}^{\frac{1}{2}} + \|A^{\frac{1}{2}}\nabla_{\mathcal{T}_h} e_d\|_{\Omega}, \end{aligned} \quad (40)$$

where the constant  $C_{c,1}$  depends on the mesh shape-regularity and on  $\max_{K \in \mathcal{T}_h} \chi_K(A)^{\frac{1}{2}}$ , but is independent of  $h$  and  $k$ .

**Lemma 5.2** (Conforming error ( $p$ -optimal bound)). *The following holds:*

$$\begin{aligned} \|A^{\frac{1}{2}}\nabla e_c\|_{\Omega} &\leq C_{c,2} \left\{ \sum_{K \in \mathcal{T}_h} \left\{ A_K^{-1} \left( \frac{h_K}{k+1} \right)^2 \|\Pi_K^{k+1}(f) + A_K \Delta R_K^{k+1}(\hat{u}_K)\|_K^2 + A_K S_{\partial K}(\hat{u}_K, \hat{u}_K) \right. \right. \\ &\quad \left. \left. + A_K^{-1} \left( \frac{h_K}{k+1} \right) \left( \|\llbracket A \nabla R_{\mathcal{T}_h}^{k+1}(\hat{u}_h) \rrbracket_{\partial K} \cdot \mathbf{n}_K\|_{\partial K}^2 + \|A_K \nabla R_K^{k+1}(\hat{u}_K) \cdot \mathbf{n}_K - \Pi_{\partial K}^k(g_N)\|_{\partial K^N}^2 \right) \right. \right. \\ &\quad \left. \left. + O_K(f)^2 + O_K(g_N)^2 \right\} \right\}^{\frac{1}{2}} + \|A^{\frac{1}{2}}\nabla_{\mathcal{T}_h} e_d\|_{\Omega}, \end{aligned} \quad (41)$$

where the constant  $C_{c,2}$  depends on the mesh shape-regularity and on  $\max_{K \in \mathcal{T}_h} \chi_K(A)^{\frac{1}{2}}$ , but is independent of  $h$  and  $k$ .

### 5.1.2 Bound on nonconformity error $e_d$

We start by defining the function  $u_c \in H_{g,D}^1(\Omega)$  introduced in (39). To this purpose, in the spriti of [29], we solve local minimization problems in  $H^1(\omega_{\mathbf{a}})$  with suitable boundary conditions for every mesh vertex  $\mathbf{a} \in \mathcal{V}_h$  (recall that  $\omega_{\mathbf{a}}$  denotes the star associated with the vertex  $\mathbf{a}$ ). Let  $\psi_{\mathbf{a}}$  be the hat basis function equal to 1 at  $\mathbf{a}$  and having a support in the vertex star  $\omega_{\mathbf{a}}$ . Recall that the hat basis functions satisfy the following partition-of-unity property:

$$\sum_{\mathbf{a} \in \mathcal{V}_h} \psi_{\mathbf{a}} = 1. \quad (42)$$

**Definition 5.3** (Patchwise and global potential reconstruction). *For all  $\mathbf{a} \in \mathcal{V}_h$ , let  $u_c^{\mathbf{a}} \in H_{g,D}^1(\omega_{\mathbf{a}})$  be the solution of the following well-posed problem:*

$$(A \nabla u_c^{\mathbf{a}}, \nabla v_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (A \nabla_{\mathcal{T}_h}(\psi_{\mathbf{a}} u_h), \nabla v_{\mathbf{a}})_{\omega_{\mathbf{a}}}, \quad \forall v_{\mathbf{a}} \in H_{0,D}^1(\omega_{\mathbf{a}}), \quad (43)$$

with

$$H_{g,D}^1(\omega_{\mathbf{a}}) := \{v \in H^1(\omega_{\mathbf{a}}) \mid v|_{\partial\omega_{\mathbf{a}} \cap \Gamma_D} = \psi_{\mathbf{a}} g_D \text{ and } v|_{\partial\omega_{\mathbf{a}} \cap \Omega} = 0\}. \quad (44)$$

An equivalent definition is

$$u_c^{\mathbf{a}} := \arg \min_{\rho_{\mathbf{a}} \in H_{g,D}^1(\omega_{\mathbf{a}})} \|A^{\frac{1}{2}}(\nabla \rho_{\mathbf{a}} - \nabla_{\mathcal{T}_h}(\psi_{\mathbf{a}} u_{\mathcal{T}_h}))\|_{\omega_{\mathbf{a}}}. \quad (45)$$

Then, extending  $u_c^{\mathbf{a}}$  by zero to  $\Omega$ , we set

$$u_c := \sum_{\mathbf{a} \in \mathcal{V}_h} u_c^{\mathbf{a}}. \quad (46)$$

Notice that we indeed have  $u_c \in H_{g,D}^1(\Omega)$  as required in (39); this follows from the partition-of-unity property (42) and the definition (44) of  $H_{g,D}^1(\omega_{\mathbf{a}})$ . The proof of the following result is postponed to Section 7.

**Lemma 5.4** (Local nonconformity error). *Let  $u_c^{\mathbf{a}}$  solve (43) and set  $e_d^{\mathbf{a}} := u_c^{\mathbf{a}} - \psi_{\mathbf{a}} u_{\mathcal{T}_h}$  for all  $\mathbf{a} \in \mathcal{V}_h$ . The following holds:*

$$\begin{aligned} \|A^{\frac{1}{2}} \nabla_{\mathcal{T}_h} e_d^{\mathbf{a}}\|_{\omega_{\mathbf{a}}} &\leq C_d \left\{ \sum_{K \in \mathcal{T}_{\mathbf{a}}} \left\{ A_K S_{\partial K}(\hat{u}_K, \hat{u}_K) + O_K(g_D)^2 \right\} + \sum_{F \in \mathcal{F}_{\mathbf{a}} \cap \mathcal{F}_h^i} A_F^b \left( \frac{h_F}{k+1} \right) \|\llbracket \nabla u_{\mathcal{T}_h} \rrbracket_F \times \mathbf{n}_F\|_F^2 \right. \\ &\quad \left. + \sum_{F \in \mathcal{F}_{\mathbf{a}} \cap \mathcal{F}_h^D} A_F^b \left( \frac{h_F}{k+1} \right) \|\nabla(u_{\mathcal{T}_h} - \Pi_{\partial K}^{k+1}(g_D)) \times \mathbf{n}_{\Omega}\|_F^2 \right\}^{\frac{1}{2}}, \end{aligned} \quad (47)$$

where the constant  $C_d^{\mathbf{a}}$  depends on the mesh shape-regularity and on  $\chi_{\mathbf{a}}(A)^{\frac{1}{2}}$ , but is independent of  $h$  and  $k$ .

**Corollary 5.5** (Global bound on nonconformity error). *The following holds:*

$$\begin{aligned} \|A^{\frac{1}{2}} \nabla_{\mathcal{T}_h} e_d\|_{\Omega} &\leq C_d \left\{ \sum_{K \in \mathcal{T}_h} \left\{ A_K S_{\partial K}(\hat{u}_K, \hat{u}_K) + O_K(g_D)^2 + A_{\partial K}^b \left( \frac{h_K}{k+1} \right) \|\llbracket \nabla u_{\mathcal{T}_h} \rrbracket_{\partial K} \times \mathbf{n}_K\|_{\partial K^i}^2 \right. \right. \\ &\quad \left. \left. + A_K \left( \frac{h_K}{k+1} \right) \|\nabla(u_K - \Pi_{\partial K}^{k+1}(g_D)) \times \mathbf{n}_{\Omega}\|_{\partial K^D}^2 \right\} \right\}^{\frac{1}{2}}, \end{aligned} \quad (48)$$

where the constant  $C_d$  depends on the mesh shape-regularity and on  $\max_{\mathbf{a} \in \mathcal{V}_h} \chi_{\mathbf{a}}(A)^{\frac{1}{2}}$ , but is independent of  $h$  and  $k$ .

*Proof.* Using that  $u_c|_K = \sum_{\mathbf{a} \in \mathcal{V}_K} u_c^{\mathbf{a}}|_K$  for all  $K \in \mathcal{T}_h$ , where  $\mathcal{V}_K$  is the set of vertices of  $K$ , the partition-of-unity property (42), and the triangle inequality, we infer that

$$\begin{aligned} \|A^{\frac{1}{2}} \nabla_{\mathcal{T}_h} e_d\|_{\Omega}^2 &= \sum_{K \in \mathcal{T}_h} A_K \|\nabla(u_c - u_K)\|_K^2 = \sum_{K \in \mathcal{T}_h} A_K \left\| \sum_{\mathbf{a} \in \mathcal{V}_K} \nabla(u_c^{\mathbf{a}} - \psi_{\mathbf{a}} u_K) \right\|_K^2 \\ &\leq \sum_{K \in \mathcal{T}_h} \sum_{\mathbf{a} \in \mathcal{V}_K} (d+1) A_K \|\nabla(u_c^{\mathbf{a}} - \psi_{\mathbf{a}} u_K)\|_K^2 = \sum_{\mathbf{a} \in \mathcal{V}_h} (d+1) \|A^{\frac{1}{2}} \nabla_{\mathcal{T}_h} e_d^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}^2. \end{aligned}$$

Invoking Lemma 5.4 and using that  $h_F \leq h_K$  for all  $F \in \partial K$  and all  $K \in \mathcal{T}_h$  completes the proof.  $\square$

**Remark 5.6** (Constant  $C_d$ ). *The constant  $C_d$  in (48) does not depend on the topology of  $\Omega$ . The reason is that we do not invoke a global Helmholtz decomposition in  $\Omega$ , but instead invoke a local decomposition in each vertex star  $\omega_{\mathbf{a}}$  (see Section 7.3 for further insight).*

### 5.1.3 Main result

For all  $K \in \mathcal{T}_h$ , we define the following error indicators:

$$\eta_{K,\text{res}} := A_K^{-\frac{1}{2}} \left( \frac{h_K}{k+1} \right) \|\Pi_K^{k+1}(f) + A_K \Delta R_K^{k+1}(\hat{u}_K)\|_K, \quad (49a)$$

$$\eta_{K,\text{sta}} := A_K^{\frac{1}{2}} S_{\partial K}(\hat{u}_K, \hat{u}_K)^{\frac{1}{2}}, \quad (49b)$$

$$\eta_{K,\text{tan}} := (A_{\partial K}^b)^{\frac{1}{2}} \left( \frac{h_K}{k+1} \right)^{\frac{1}{2}} \left\{ \|\llbracket \nabla u_{\mathcal{T}_h} \rrbracket_{\partial K} \times \mathbf{n}_K\|_{\partial K^i} + \|\nabla(u_K - \Pi_{\partial K}^{k+1}(g_D)) \times \mathbf{n}_{\Omega}\|_{\partial K^D} \right\}, \quad (49c)$$

$$\eta_{K,\text{nor}} := A_K^{-\frac{1}{2}} \left( \frac{h_K}{k+1} \right)^{\frac{1}{2}} \left\{ \|\llbracket A \nabla R_{\mathcal{T}_h}^{k+1}(\hat{u}_h) \rrbracket_{\partial K} \cdot \mathbf{n}_K\|_{\partial K^i} + \|A_K \nabla R_K^{k+1}(\hat{u}_K) \cdot \mathbf{n}_{\Omega} - \Pi_{\partial K}^k(g_N)\|_{\partial K^N} \right\}, \quad (49d)$$

$$O_{K,\text{dat}} := O_K(f) + O_K(g_N) + O_K(g_D), \quad (49e)$$

where the three data oscillation terms on the right-hand side of (49e) are defined in (36).

**Theorem 5.7** (*hp*-error upper bound). *Let  $u$  be the weak solution to (2), and let  $\hat{u}_h$  be the discrete solution to (17). The following holds:*

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} \left\{ \|A^{\frac{1}{2}} \nabla e\|_K^2 + A_K S_{\partial K}(\hat{u}_K, \hat{u}_K) \right\} &\leq C_u \left\{ \sum_{K \in \mathcal{T}_h} \left\{ \eta_{K,\text{res}}^2 + \eta_{K,\text{tan}}^2 + \eta_{K,\text{sta}}^2 + O_{K,\text{dat}}^2 \right\} \right. \\ &\quad \left. + \min \left( \sum_{K \in \mathcal{T}_h} k \eta_{K,\text{sta}}^2, \sum_{K \in \mathcal{T}_h} \eta_{K,\text{nor}}^2 \right) \right\}, \end{aligned} \quad (50)$$

where  $C_u$  depends on the mesh shape-regularity and on  $\max(\max_{K \in \mathcal{T}_h} \chi_K(A), \max_{\mathbf{a} \in \mathcal{V}_h} \chi_{\mathbf{a}}(A))^{\frac{1}{2}}$ , but is independent of  $h$  and  $k$ .

*Proof.* Combining the bounds from Lemma 5.1 and Lemma 5.2 gives

$$\|A^{\frac{1}{2}} \nabla e_c\|_{\Omega} \leq C \left\{ \sum_{K \in \mathcal{T}_h} \left\{ \eta_{K,\text{res}}^2 + \eta_{K,\text{sta}}^2 + O_K(f)^2 + O_K(g_N)^2 \right\} + \min \left( \sum_{K \in \mathcal{T}_h} k \eta_{K,\text{sta}}^2, \sum_{K \in \mathcal{T}_h} \eta_{K,\text{nor}}^2 \right) \right\}^{\frac{1}{2}},$$

where the constant  $C$  depends on  $\max_{K \in \mathcal{T}_h} \chi_K(A)^{\frac{1}{2}}$ . Moreover, we can rewrite the bound of Corollary 5.5 as follows:

$$\|A^{\frac{1}{2}} \nabla_{\mathcal{T}_h} e_d\|_{\Omega} \leq C \left\{ \sum_{K \in \mathcal{T}_h} \left\{ \eta_{K,\text{sta}}^2 + \eta_{K,\text{tan}}^2 + O_K(g_D)^2 \right\} \right\}^{\frac{1}{2}},$$

where the constant  $C$  depends on  $\max_{\mathbf{a} \in \mathcal{V}_h} \chi_{\mathbf{a}}(A)^{\frac{1}{2}}$ . Combining the above two bounds proves that  $\sum_{K \in \mathcal{T}_h} \|A^{\frac{1}{2}} \nabla e\|_K^2$  is bounded by the right-hand side of (50). Since  $\sum_{K \in \mathcal{T}_h} A_K S_{\partial K}(\hat{u}_K, \hat{u}_K) = \sum_{K \in \mathcal{T}_h} \eta_{K,\text{sta}}^2$ , the proof is complete.  $\square$

**Remark 5.8** (Estimator without normal flux jump). *Notice that (50) implies that*

$$\sum_{K \in \mathcal{T}_h} \left\{ \|A^{\frac{1}{2}} \nabla e\|_K^2 + A_K S_{\partial K}(\hat{u}_K, \hat{u}_K) \right\} \leq C_u \sum_{K \in \mathcal{T}_h} \left\{ \eta_{K,\text{res}}^2 + \eta_{K,\text{tan}}^2 + (k+1) \eta_{K,\text{sta}}^2 + O_K^2 \right\}.$$

*This upper bound does not contain the normal flux jump, which is often the dominant component of the error estimator for  $H^1$ -conforming FEM [12]. The price to pay is a  $\frac{1}{2}$ -order  $p$ -suboptimality for the stabilization term. Our numerical experiments confirm that the normal flux jump term does not dominate the total error estimator.*

## 5.2 Local lower error bound

In this section, we establish a local lower error bound. Specifically, we bound the local error indicators  $\eta_{K,\text{res}}$ ,  $\eta_{K,\text{nor}}$ , and  $\eta_{K,\text{tan}}$ , for all  $K \in \mathcal{T}_h$ , in terms of the error  $e = u - u_{\mathcal{T}_h}$  in (a neighborhood of)  $K$  and the data oscillation indicators defined in (36). We do not bound the local error indicator  $\eta_{K,\text{sta}}$  since it is present on both sides of the upper error bound (50). This is classical in a posteriori error estimates for nonconforming methods.

**Theorem 5.9.** *The following holds for all  $K \in \mathcal{T}_h$ :*

$$\eta_{K,\text{res}} \leq C_1(k+1) (\|A^{\frac{1}{2}} \nabla e\|_K + O_K(f)), \quad (51a)$$

$$\eta_{K,\text{nor}} \leq C_1(k+1)^{\frac{1}{2}} \left\{ \sum_{\mathcal{K} \in \omega_K} A_{\mathcal{K}} S_{\partial \mathcal{K}}(\hat{u}_{\mathcal{K}}, \hat{u}_{\mathcal{K}}) \right\}^{\frac{1}{2}}, \quad (51b)$$

$$\eta_{K,\text{tan}} \leq C_1(k+1)^{\frac{3}{2}} \left\{ \sum_{\mathcal{K} \in \omega_K} \|A^{\frac{1}{2}} \nabla e\|_{\mathcal{K}}^2 \right\}^{\frac{1}{2}}, \quad (51c)$$

where  $\omega_K$  collects all the mesh cells (including  $K$ ) sharing at least an interface with  $K$ , and the constant  $C_1$  depends on the mesh shape-regularity and on the diffusion contrast factor  $\chi'_K(A) := A_K^{-1} \max_{K' \in \omega_K} A_{K'}$ , but is independent of  $h$  and  $k$ .

**Remark 5.10** (Bound on normal flux jump). *We observe that the upper bound on  $\eta_{K,\text{nor}}$  has only a  $\frac{1}{2}$ -order  $p$ -suboptimality. This rather sharp result is achieved by exploiting the local conservation property of the HHO method, and is in contrast with the upper bound that could be obtained using bubble function techniques and which would feature a  $\frac{3}{2}$ -order  $p$ -suboptimality (details not shown for brevity).*

## 6 Numerical examples

In this section, we present numerical examples to illustrate our theoretical results.

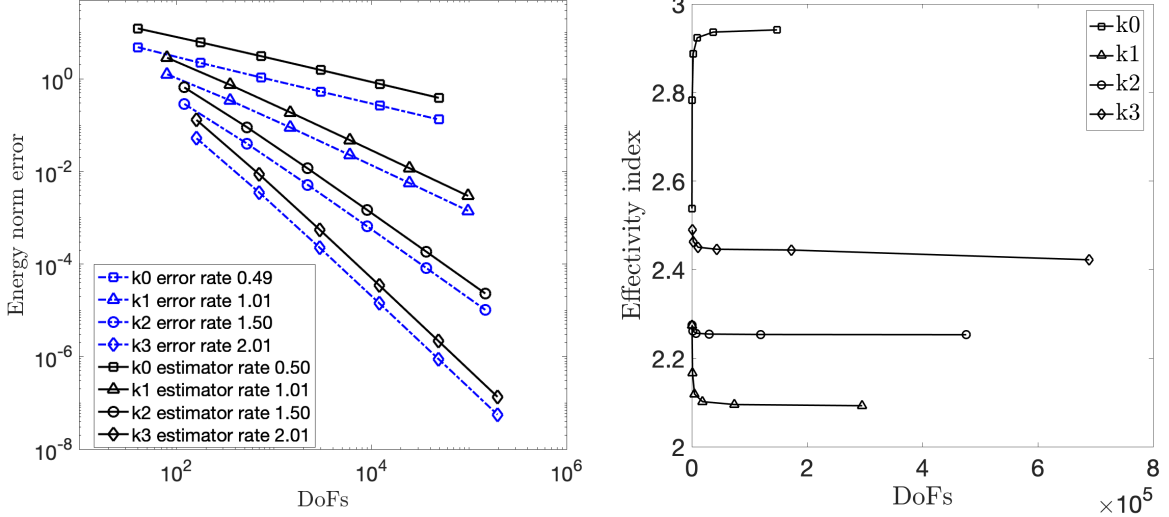


Figure 1: Example 1. Energy error and a posteriori error estimator for  $k \in \{0, 1, 2, 3\}$  as a function of DoFs (left) and effectivity index as a function of DoFs

### 6.1 Example 1: Convergence rates for smooth solution

We select  $f$ ,  $A := I_{2 \times 2}$  and Dirichlet boundary conditions on the unit square  $\Omega := (-1, 1)^2$ , so that the exact solution is

$$u(x, y) := \sin(\pi x) \sin(\pi y). \quad (52)$$

We employ the polynomial degrees  $k \in \{0, \dots, 3\}$  and a sequence of successively refined triangular meshes consisting of  $\{32, 128, 512, 2048, 8192\}$  cells.

Let us first verify the convergence rates obtained with the HHO methods with  $k \in \{0, \dots, 3\}$ . We measure the error in the energy norm  $\sum_{K \in \mathcal{T}_h} A_K \{ \|\nabla(u - u_K)\|_K^2 + S_{\partial K}(\hat{u}_K, \hat{u}_K) \}$ . The energy error and the a posteriori estimator on the right-hand side of (50) (with constant  $C_u$  set to one) are reported in Figure 1. The rates are computed as a function of DoFs, which denotes the total number of globally coupled discrete unknowns (that is, the total number of face unknowns except those located on the boundary faces). We observe that the energy error and the a posteriori estimator both converge at the optimal rate  $\mathcal{O}(\text{DoFs}^{-\frac{(k+1)}{2}})$ . The convergence rate of the energy error is optimal in view of the result of Theorem 4.4. Moreover, the effectivity index, defined as the ratio of the a posteriori estimator to the energy error, remains well behaved as a function of DoFs. The effectivity index takes values between 2 and 2.8 for  $k \geq 1$ , whereas the effectivity index is almost 3 for  $k = 0$ .

As the results in Theorem 5.7 (upper error bound) and 5.9 (lower error bound) differ by an algebraic rate in the polynomial degree  $k$ , we investigate the dependence of the effectivity index on  $k$ . In Figure 2, we report the effectivity index as a function of the polynomial degree  $k \in \{1, \dots, 9\}$  on a mesh consisting of 128 triangular cells. We observe an algebraic rate of  $p^{\frac{1}{2}}$ , which matches the statement in Theorem 5.7.

Finally, we compare the relative contributions of the various terms composing the a posteriori error estimator (all in percentage). Setting  $\eta_X := \{ \sum_{K \in \mathcal{T}} \eta_{K,X}^2 \}^{\frac{1}{2}}$  for  $X \in \{\text{res}, \text{sta}, \text{nor}, \text{tan}\}$ , we report in Table 1 the relative contribution of the residual estimator  $\eta_{\text{res}}$ , the stabilization estimator  $\eta_{\text{sta}}$ , the normal flux jump estimator  $\eta_{\text{nor}}$ , and the tangential flux jump estimator  $\eta_{\text{tan}}$ , for polynomial degrees  $k \in \{0, 1, 2\}$ . For  $k \in \{1, 2\}$ , the residual estimator dominates the total estimator (by more than 60%), followed by  $\eta_{\text{nor}}$  (about 20%),  $\eta_{\text{tan}}$  (about 9%), and  $\eta_{\text{sta}}$  (about 8%). For  $k = 0$ , the residual dominates the total estimator (by more than 50%), followed by  $\eta_{\text{tan}}$  (25%),  $\eta_{\text{nor}}$  (15%), and  $\eta_{\text{sta}}$  (6%).

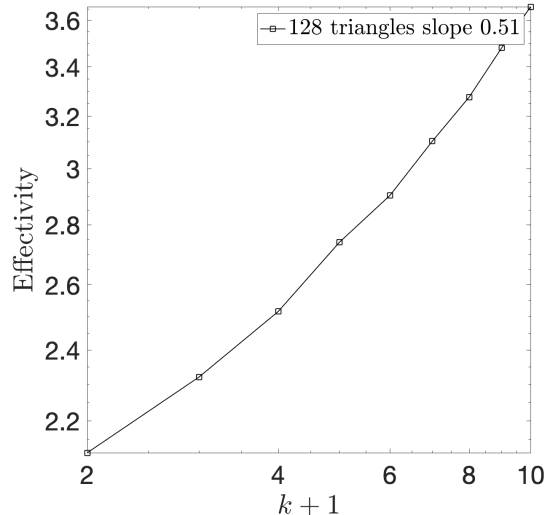


Figure 2: Example 1. Effectivity index with  $k \in \{1, \dots, 9\}$  on a mesh composed of 128 cells.

	$k = 0$				$k = 1$				$k = 2$			
# cell	$\eta_{\text{res}}$	$\eta_{\text{sta}}$	$\eta_{\text{nor}}$	$\eta_{\text{tan}}$	$\eta_{\text{res}}$	$\eta_{\text{sta}}$	$\eta_{\text{nor}}$	$\eta_{\text{tan}}$	$\eta_{\text{res}}$	$\eta_{\text{sta}}$	$\eta_{\text{nor}}$	$\eta_{\text{tan}}$
128	54	6	15	25	64	9	19	8	66	8	19	7
512	54	6	15	25	62	9	22	7	65	8	20	7
2048	53	6	15	26	62	9	24	5	65	8	20	7
8192	53	6	15	26	61	9	25	5	65	8	20	7

Table 1: Example 1. Relative contribution of the various terms composing the a posteriori error estimator for  $k \in \{0, 1, 2\}$ .

## 6.2 Example 2: Adaptive algorithm for singular solution.

We select  $f$ ,  $A := I_{2 \times 2}$  and Dirichlet boundary conditions on the L-shaped domain  $\Omega := (-1, 1)^2 \setminus \{(0, 1) \times (-1, 0)\}$ , so that the exact solution is in polar coordinates

$$u = r^{\frac{2}{3}} \sin(2\theta/3). \quad (53)$$

We test an adaptive algorithm driven by the a posteriori error estimator from Section 5. The adaptive algorithm starts from a coarse mesh and uses the estimator on the right-hand side of (50) to mark mesh cells for refinement through a bulk-chasing criterion (also known as Dörfler's marking). The adaptive algorithm can be classically described as

$$\text{SOLVE} \longrightarrow \text{ESTIMATE} \longrightarrow \text{MARK} \longrightarrow \text{REFINE}.$$

We first test the convergence rate of the above adaptive algorithm with  $k \in \{0, 1, 2, 3\}$  and a bulk-chasing criterion set to 40%. The energy error and the a posteriori error estimator are reported in Figure 3 with convergence rates computed as a function of DoFs. We observe that the energy error and the a posteriori error estimator converge at the optimal rate  $\mathcal{O}(\text{DoFs}^{-\frac{(k+1)}{2}})$ . Moreover, we observe in the left panel of Figure 4 that the effectivity index remains well behaved as a function of DoFs and that it slightly increases with the polynomial degree  $k \geq 1$ , with values between 2 and 2.8. For  $k = 0$ , the effectivity index is almost 3. To gain further insight, we report in the right panel of Figure 4 the effectivity index as a function of the polynomial degree  $k \in \{1, \dots, 10\}$  on a mesh consisting of 96 triangular cells. We observe an algebraic rate of  $p^{\frac{1}{2}}$ , again in agreement with Theorem 5.7.

Finally, we report in Table 2 the relative contributions of the various terms composing the a posteriori error estimator (all in percentage) with polynomial degrees  $k \in \{0, 1, 2\}$ . For  $k = 1$ , the tangential jump



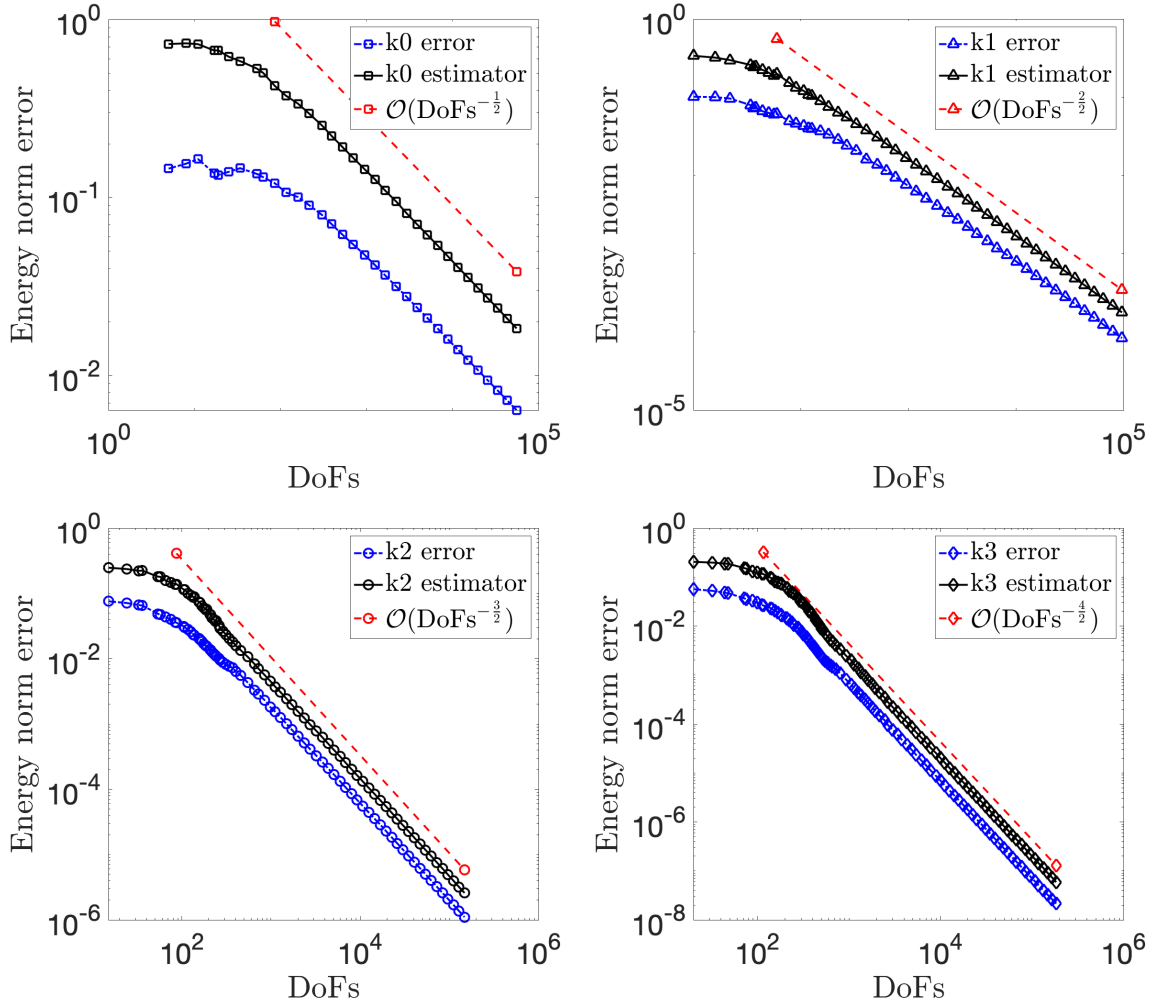


Figure 3: Example 2. Energy error and a posteriori error estimator as a function of DoFs for  $k \in \{0, 1, 2, 3\}$ .

estimator dominates the total estimator (by more than 64%), followed by  $\eta_{\text{nor}}$  (25%),  $\eta_{\text{sta}}$  (10%) and  $\eta_{\text{res}}$  (0%). For  $k = 2$ , the tangential jump estimator dominates the total estimator (by more than 33%), followed by  $\eta_{\text{nor}}$  and  $\eta_{\text{res}}$  (both at about 30%), and  $\eta_{\text{sta}}$  (9%). For  $k = 0$ , only the tangential jump estimator is nonzero. Clearly,  $\eta_{\text{res}} = 0$  since  $f = 0$  and  $u_K$  is affine for  $k = 0$ . Moreover, one can show that for  $f = 0$  and  $k = 0$ , the Crouzeix–Raviart FEM solution is the cellwise component of the HHO solution, while the facewise component is the mean-value of the trace of the cell components. This explains why  $\eta_{\text{sta}} = 0$  in Table 2, and consequently  $\eta_{\text{nor}} = 0$  owing to the local conservation property (20).

$k = 0$					$k = 1$					$k = 2$				
# cell	$\eta_{\text{res}}$	$\eta_{\text{sta}}$	$\eta_{\text{nor}}$	$\eta_{\text{tan}}$	# cell	$\eta_{\text{res}}$	$\eta_{\text{sta}}$	$\eta_{\text{nor}}$	$\eta_{\text{tan}}$	# cell	$\eta_{\text{res}}$	$\eta_{\text{sta}}$	$\eta_{\text{nor}}$	$\eta_{\text{tan}}$
116	0	0	0	100	118	0	10	32	58	172	28	9	29	34
1118	0	0	0	100	1207	0	10	26	64	1348	28	8	30	34
5948	0	0	0	100	6098	0	10	26	64	5856	27	8	32	33
22306	0	0	0	100	21762	0	10	25	65	21574	26	8	33	33

Table 2: Example 2. Relative contribution of the various terms composing the a posteriori error estimator  $k \in \{0, 1, 2\}$ .

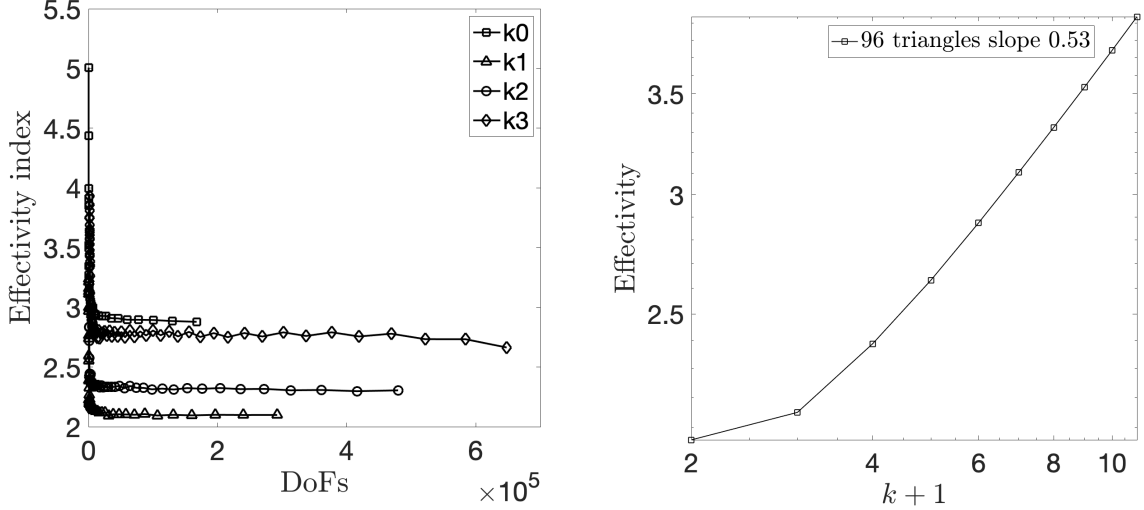


Figure 4: Example 2. Effectivity index as a function of DoFs for  $k \in \{0, 1, 2, 3\}$  (left). Effectivity index as a function of  $k \in \{1, \dots, 10\}$  on a mesh composed of 96 cells (right).

### 6.3 Example 3: Adaptive algorithm for Kellogg’s test case

Our last example is Kellogg’s test case [36], i.e., a diffusion problem on the square domain  $\Omega := (-1, 1)^2$  with a checkerboard pattern for the diffusion coefficient  $A$ , namely  $A := b$  for  $xy \geq 0$  and  $A := 1$  otherwise. The exact solution (for zero right-hand side and suitable Dirichlet boundary conditions) takes the form  $u := r^\alpha \phi(\theta)$  in polar coordinates, where the explicit expression for the function  $\phi$  can be found in [43, Section 5.3]. We select  $\alpha = 0.1$  and  $b = 161.4476387975881$ , so that  $u \in H^{1.1-\varepsilon}(\Omega)$  with  $\varepsilon > 0$  arbitrarily small.

We first test the convergence rate of the adaptive algorithm described in the previous section with  $k \in \{0, 1, 2, 3\}$  and a bulk-chasing criterion set to 10%. The energy error and the a posteriori error estimator are reported in Figure 5 with convergence rates reported as a function of DoFs. We observe that the energy error and the a posteriori error estimator converge at the optimal rate  $\mathcal{O}(\text{DoFs}^{-\frac{(k+1)}{2}})$ . Moreover, we observe in Figure 6 that the effectivity index remains well behaved as a function of DoFs and that it slightly increases with the polynomial degree  $k \geq 1$ , with values between 1.9 and 2.6. For  $k = 0$ , the effectivity index is almost 2.8.

Finally, we report in Table 3 the relative contributions of the various terms composing the a posteriori error estimator (all in percentage) with polynomial degrees  $k \in \{0, 1, 2\}$ . For  $k = 1$ , the tangential jump estimator dominates the total estimator (by more than 63%), followed by  $\eta_{\text{nor}}$  (27%),  $\eta_{\text{sta}}$  (10%), and  $\eta_{\text{res}}$  (0%). For  $k = 2$ , the tangential jump estimator dominates the total estimator on the coarsest mesh, whereas the contributions of the tangential jump and normal flux jump estimators are equilibrated on the other meshes. For  $k = 0$ , only the tangential jump residual is nonzero, for the same reasons as discussed in the previous example.

$k = 0$					$k = 1$					$k = 2$				
# cell	$\eta_{\text{res}}$	$\eta_{\text{sta}}$	$\eta_{\text{nor}}$	$\eta_{\text{tan}}$	# cell	$\eta_{\text{res}}$	$\eta_{\text{sta}}$	$\eta_{\text{nor}}$	$\eta_{\text{tan}}$	# cell	$\eta_{\text{res}}$	$\eta_{\text{sta}}$	$\eta_{\text{nor}}$	$\eta_{\text{tan}}$
122	0	0	0	100	124	0	7	8	85	142	22	9	11	58
1386	0	0	0	100	1376	0	8	33	59	1326	29	9	30	32
5962	0	0	0	100	5864	0	9	28	63	5486	28	8	32	32
15642	0	0	0	100	15342	0	10	27	63	14894	28	8	32	32

Table 3: Example 3. The relative contribution of different estimator components to the total error estimator for  $k \in \{0, 1, 2\}$ .

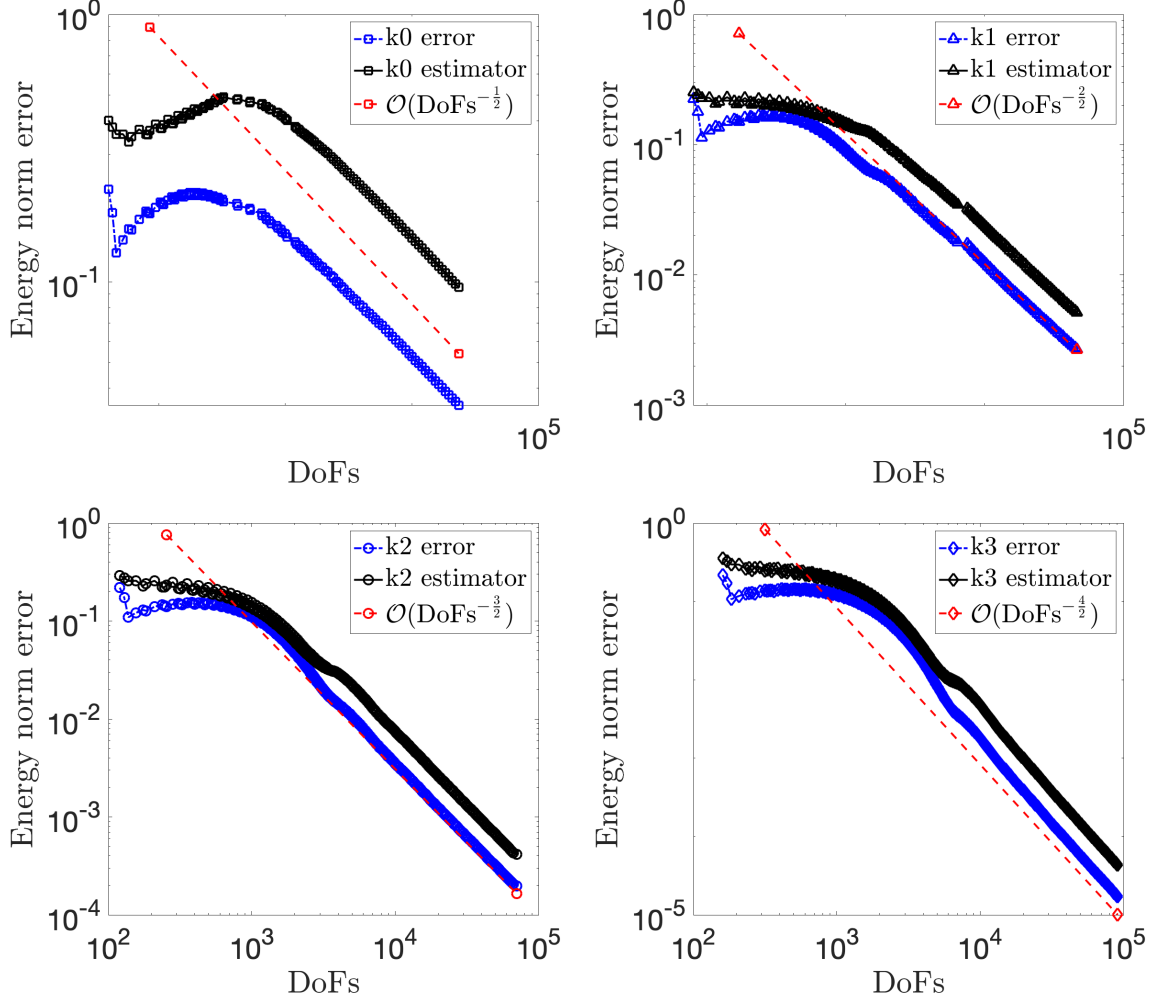


Figure 5: Example 3. Energy error and a posteriori error estimator as a function of DoFs for  $k \in \{0, 1, 2, 3\}$ .

## 7 Proofs

This section collects the proofs of all the preparatory results stated in the previous section, namely Lemma 5.1, Lemma 5.2, Lemma 5.4, and Theorem 5.9.

### 7.1 Proof of Lemma 5.1

The proof is split into two steps.

(i) Since  $e_c \in H_{0,D}^1(\Omega)$ , using (39) leads to

$$\|A^{\frac{1}{2}} \nabla e_c\|_{\Omega}^2 = (A \nabla_{\mathcal{T}_h} e, \nabla e_c)_{\Omega} - (A \nabla_{\mathcal{T}_h} e_d, \nabla e_c)_{\Omega}. \quad (54)$$

For the second term, the Cauchy–Schwarz inequality gives

$$|(A \nabla_{\mathcal{T}_h} e_d, \nabla e_c)_{\Omega}| \leq \|A^{\frac{1}{2}} \nabla e_c\|_{\Omega} \|A^{\frac{1}{2}} \nabla_{\mathcal{T}_h} e_d\|_{\Omega}. \quad (55)$$

Next, we focus on bounding  $(A \nabla_{\mathcal{T}_h} e, \nabla e_c)_{\Omega}$ . Using the weak form of the PDE (2), adding and subtracting the term  $(A_K \nabla R_K^{k+1}(\hat{u}_K), \nabla e_c)_K$  for all  $K \in \mathcal{T}_h$ , and using the discrete problem (18) with some test

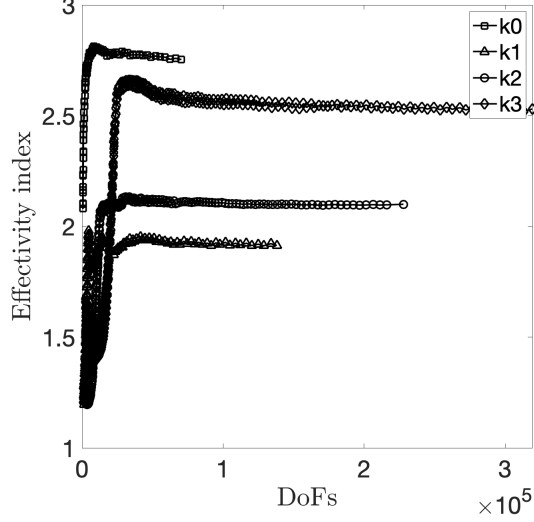


Figure 6: Example 3. Effectivity index as a function of DoFs with  $k \in \{0, 1, 2, 3\}$ .

function  $\hat{w}_h \in \hat{V}_{h0}^k$ , we infer that

$$\begin{aligned} (A\nabla_{\mathcal{T}_h} e, \nabla e_c)_\Omega &= \sum_{K \in \mathcal{T}_h} \left\{ (f, e_c)_K + (g_N, e_c)_{\partial K^N} - (A_K \nabla R_K^{k+1}(\hat{u}_K), \nabla e_c)_K \right. \\ &\quad + (A_K \nabla (R_K^{k+1}(\hat{u}_K) - u_K), \nabla e_c)_K + (A_K \nabla R_K^{k+1}(\hat{u}_K), \nabla R_K^{k+1}(\hat{w}_K))_K \\ &\quad \left. + S_{\partial K}(\hat{u}_K, \hat{w}_K) - (f, w_K)_K - (g_N, w_{\partial K})_{\partial K^N} \right\}. \end{aligned}$$

Let us set  $\hat{w}_h := ((I_{\text{mKM}}^k(e_c)|_K)_{K \in \mathcal{T}_h}, (I_{\text{mKM}}^k(e_c)|_F)_{F \in \mathcal{F}_h})$ . This definition is meaningful since  $I_{\text{mKM}}^k(e_c)$  is single-valued at the mesh interfaces; moreover, since  $I_{\text{mKM}}^k(e_c)$  vanishes on the boundary faces in  $\mathcal{F}_h^D$ , we have  $\hat{w}_h \in \hat{V}_{h0}^k$ . We observe that, for all  $K \in \mathcal{T}_h$ , the definition (10) of the reconstruction operator gives  $R_K^{k+1}(\hat{w}_K) = w_K$ , and we also have  $S_{\partial K}(\hat{u}_K, \hat{w}_K) = 0$ . Hence, defining  $\hat{\eta}_K = (\eta_K, \eta_{\partial K}) := (e_c|_K - w_K, e_c|_{\partial K} - w_{\partial K})$  for all  $K \in \mathcal{T}_h$ , we infer that

$$\begin{aligned} (A\nabla_{\mathcal{T}_h} e, \nabla e_c)_\Omega &= \sum_{K \in \mathcal{T}_h} \left\{ (f, \eta_K)_K + (g_N, \eta_{\partial K})_{\partial K^N} - (A_K \nabla R_K^{k+1}(\hat{u}_K), \nabla \eta_K)_K \right. \\ &\quad \left. + (A_K \nabla (R_K^{k+1}(\hat{u}_K) - u_K), \nabla e_c)_K \right\}. \end{aligned}$$

Integrating by parts and using that  $\eta_{\partial K^D} = 0$  and that  $A_K$  is constant on  $K$  gives

$$\begin{aligned} (A\nabla_{\mathcal{T}_h} e, \nabla e_c)_\Omega &= \sum_{K \in \mathcal{T}_h} \left\{ (f + A_K \Delta R_K^{k+1}(\hat{u}_K), \eta_K)_K - (A_K \nabla R_K^{k+1}(\hat{u}_K) \cdot \mathbf{n}_K, \eta_{\partial K})_{\partial K^i} \right. \\ &\quad \left. + (g_N - A_K \nabla R_K^{k+1}(\hat{u}_K) \cdot \mathbf{n}_K, \eta_{\partial K})_{\partial K^N} + (A_K \nabla (R_K^{k+1}(\hat{u}_K) - u_K), \nabla e_c)_K \right\}. \quad (56) \end{aligned}$$

(ii) Since  $\eta_{\partial K}|_F = (e_c - I_{\text{mKM}}^k(e_c))|_F$  is single-valued on each interface  $F \in \mathcal{F}_{\partial K^i}$ , using the local conservation property (20) of the HHO method gives

$$\begin{aligned} (A\nabla_{\mathcal{T}_h} e, \nabla e_c)_\Omega &= \sum_{K \in \mathcal{T}_h} \left\{ (f + A_K \Delta R_K^{k+1}(\hat{u}_K), \eta_K)_K - A_K \frac{(k+1)^2}{h_K} (\Pi_{\partial K}^k(u_K - u_{\partial K}), \eta_{\partial K})_{\partial K^i} \right. \\ &\quad + (g_N - \Pi_{\partial K}^k(g_N), \eta_{\partial K})_{\partial K^N} - A_K \frac{(k+1)^2}{h_K} (\Pi_{\partial K}^k(u_K - u_{\partial K}), \eta_{\partial K})_{\partial K^N} \\ &\quad \left. + (A_K \nabla (R_K^{k+1}(\hat{u}_K) - u_K), \nabla e_c)_K \right\}. \end{aligned}$$

Invoking the Cauchy–Schwarz inequality leads to

$$\begin{aligned}
|(A\nabla_{\mathcal{T}_h} e, \nabla e_c)_\Omega| &\leq \sum_{K \in \mathcal{T}_h} \left\{ \left( A_K^{-\frac{1}{2}} \frac{h_K}{k+1} \right) \|f + A_K \Delta R_K^{k+1}(\hat{u}_K)\|_K \left( A_K^{\frac{1}{2}} \frac{k+1}{h_K} \right) \|\eta_K\|_K \right. \\
&\quad + \left( A_K \frac{(k+1)^3}{h_K} \right)^{\frac{1}{2}} \|\Pi_{\partial K}^k(u_K - u_{\partial K})\|_{\partial K^i} \left( A_K \frac{k+1}{h_K} \right)^{\frac{1}{2}} \|\eta_{\partial K}\|_{\partial K^i} \\
&\quad + \left( A_K^{-1} \frac{h_K}{k+1} \right)^{\frac{1}{2}} \|g_N - \Pi_{\partial K}^k(g_N)\|_{\partial K^N} \left( A_K \frac{k+1}{h_K} \right)^{\frac{1}{2}} \|\eta_{\partial K}\|_{\partial K^N} \\
&\quad + \left( A_K \frac{(k+1)^3}{h_K} \right)^{\frac{1}{2}} \|\Pi_{\partial K}^k(u_K - u_{\partial K})\|_{\partial K^N} \left( A_K \frac{k+1}{h_K} \right)^{\frac{1}{2}} \|\eta_{\partial K}\|_{\partial K^N} \\
&\quad \left. + A_K^{\frac{1}{2}} \|\nabla(R_K^{k+1}(\hat{u}_K) - u_K)\|_K A_K^{\frac{1}{2}} \|\nabla e_c\|_K \right\}.
\end{aligned}$$

Using the approximation result (7), we infer that, for all  $K \in \mathcal{T}_h$ ,

$$A_K^{\frac{1}{2}} \left\{ \left( \frac{k+1}{h_K} \right) \|\eta_K\|_K + \left( \frac{k+1}{h_K} \right)^{\frac{1}{2}} \|\eta_{\partial K}\|_{\partial K} \right\} \leq C A_K^{\frac{1}{2}} \|\nabla e_c\|_{\text{es}(K)} \leq C \chi_K(A) \|A^{\frac{1}{2}} \nabla e_c\|_{\text{es}(K)}. \quad (57)$$

Using the above bounds, the bound (14) on  $\|\nabla(R_K^{k+1}(\hat{u}_K) - u_K)\|_K$ , the mesh shape-regularity and the triangle inequality for the residual term gives

$$\begin{aligned}
|(A\nabla_{\mathcal{T}_h} e, \nabla e_c)_\Omega| &\leq C \max_{K \in \mathcal{T}_h} \chi_K(A) \left\{ \sum_{K \in \mathcal{T}_h} \left\{ A_K^{-1} \left( \frac{h_K}{k+1} \right)^2 \|\Pi_K^{k+1}(f) + A_K \Delta R_K^{k+1}(\hat{u}_K)\|_K^2 \right. \right. \\
&\quad \left. \left. + A_K(k+1)S_{\partial K}(\hat{u}_K, \hat{u}_K) + A_K S_{\partial K}(\hat{u}_K, \hat{u}_K) + O_K(f)^2 + O_K(g_N)^2 \right\} \right\}^{\frac{1}{2}} \|A^{\frac{1}{2}} \nabla e_c\|_\Omega.
\end{aligned}$$

Combining (54) and (55) with the above bound completes the proof.

## 7.2 Proof of Lemma 5.2

The starting point is the bound (56) obtained at the end of the first step of the proof of Lemma 5.1. However, we no longer invoke the local conservation property (20), but consider the jump of  $A\nabla R_{\mathcal{T}_h}^{k+1}(\hat{u}_h)$  across all the mesh interfaces and its value at all the Neumann boundary faces. Recalling that  $\eta_{\partial K}|_F = (e_c - I_{\text{mKM}}^k(e_c))|_F$  is single-valued on every interface  $F \in \mathcal{F}_{\partial K^i}$ , we infer from (56) that

$$\begin{aligned}
(A\nabla_{\mathcal{T}_h} e, \nabla e_c)_\Omega &= \sum_{K \in \mathcal{T}_h} \left\{ (f + A_K \Delta R_K^{k+1}(\hat{u}_K), \eta_K)_K + (A_K \nabla(R_K^{k+1}(\hat{u}_K) - u_K), \nabla e_c)_K \right. \\
&\quad - \sum_{F \in \mathcal{F}_{\partial K^i}} \frac{1}{2} (\llbracket A\nabla R_{\mathcal{T}_h}^{k+1}(\hat{u}_h) \rrbracket_F \cdot \mathbf{n}_F, \eta_{\partial K})_F \\
&\quad \left. - (A_K \nabla R_K^{k+1}(\hat{u}_K) \cdot \mathbf{n}_K - \Pi_{\partial K}^k(g_N), \eta_{\partial K})_{\partial K^N} + (g_N - \Pi_{\partial K}^k(g_N), \eta_{\partial K})_{\partial K^N} \right\}.
\end{aligned}$$

Invoking the Cauchy–Schwarz inequality leads to

$$\begin{aligned}
|(A\nabla_{\mathcal{T}_h} e, \nabla e_c)_\Omega| &\leq \sum_{K \in \mathcal{T}_h} \left\{ \left( A_K^{-\frac{1}{2}} \frac{h_K}{k+1} \right) \|f + A_K \Delta R_K^{k+1}(\hat{u}_K)\|_K \left( A_K^{\frac{1}{2}} \frac{k+1}{h_K} \right) \|\eta_K\|_K \right. \\
&\quad + A_K^{\frac{1}{2}} \|\nabla(R_K^{k+1}(\hat{u}_K) - u_K)\|_K \|A^{\frac{1}{2}} \nabla e_c\|_K \\
&\quad + \sum_{F \in \mathcal{F}_{\partial K^i}} \frac{1}{2} \left( A_K^{-1} \frac{h_K}{k+1} \right)^{\frac{1}{2}} \|\llbracket A\nabla R_{\mathcal{T}_h}^{k+1}(\hat{u}_h) \rrbracket_F \cdot \mathbf{n}_F\|_F \left( A_K \frac{k+1}{h_K} \right)^{\frac{1}{2}} \|\eta_{\partial K}\|_F \\
&\quad + \left( A_K^{-1} \frac{h_K}{k+1} \right)^{\frac{1}{2}} \|A_K \nabla R_K^{k+1}(\hat{u}_K) \cdot \mathbf{n}_\Omega - \Pi_{\partial K}^k(g_N)\|_{\partial K^N} \left( A_K \frac{k+1}{h_K} \right)^{\frac{1}{2}} \|\eta_{\partial K}\|_{\partial K^N} \\
&\quad \left. + \left( A_K^{-1} \frac{h_K}{k+1} \right)^{\frac{1}{2}} \|g_N - \Pi_{\partial K}^k(g_N)\|_{\partial K^N} \left( A_K \frac{k+1}{h_K} \right)^{\frac{1}{2}} \|\eta_{\partial K}\|_{\partial K^N} \right\}.
\end{aligned}$$

Using the Cauchy–Schwarz inequality, the bound (14) on  $\|\nabla(R_K^{k+1}(\hat{u}_K) - u_K)\|_K$ , the approximation result (57) on  $\eta$ , and the triangle inequality for the residual term, we obtain

$$\begin{aligned} |(A\nabla_{\mathcal{T}_h} e, \nabla e_c)_\Omega| &\leq C \max_{K \in \mathcal{T}_h} \chi_K(A) \left\{ \sum_{K \in \mathcal{T}_h} \left\{ A_K^{-1} \left( \frac{h_K}{k+1} \right)^2 \|\Pi_K^{k+1}(f) + A_K \Delta R_K^{k+1}(\hat{u}_K)\|_K^2 + A_K S_{\partial K}(\hat{u}_K, \hat{u}_K) \right. \right. \\ &\quad \left. \left. + A_K^{-1} \left( \frac{h_K}{k+1} \right) \left( \|[A\nabla R_{\mathcal{T}_h}^{k+1}(\hat{u}_h)]\|_{\partial K} \cdot \mathbf{n}_K\|_{\partial K}^2 + \|A_K \nabla R_K^{k+1}(\hat{u}_K) \cdot \mathbf{n}_\Omega - \Pi_{\partial K}^k(g_N)\|_{\partial K}^2 \right) \right. \right. \\ &\quad \left. \left. + O_K(f)^2 + O_K(g_N)^2 \right\} \right\}^{\frac{1}{2}} \|A^{\frac{1}{2}} \nabla e_c\|_\Omega. \end{aligned}$$

Combining (54) and (55) with the above bound completes the proof.

### 7.3 Proof of Lemma 5.4

To fix the ideas, we assume that  $d = 3$ ; all the results also hold for  $d = 2$ . First, we recall a result on the Helmholtz decomposition on simply connected domains [31, Theorem 3]. Let  $\omega$  be a bounded, simply connected, Lipschitz domain and consider a partition of its boundary  $\partial\omega$  into two disjoint parts  $\gamma_D$  and  $\gamma_N$ . Then, for all  $\mathbf{w} \in \mathbf{L}^2(\omega)$ , there exist

$$\xi \in H_{0,D}^1(\omega) := \{\zeta \in H^1(\omega) \mid \zeta|_{\gamma_D} = 0\}, \quad (58a)$$

$$\phi \in \mathbf{H}_{0,N}^1(\omega) := \{\psi \in \mathbf{H}^1(\omega) \mid \psi|_{\gamma_N} = \mathbf{0}\}, \quad (58b)$$

such that

$$A\mathbf{w} = A\nabla\xi + \mathbf{curl}\phi \quad \text{in } \omega. \quad (59)$$

Moreover, the following holds with a constant  $C_\omega$  only depending on  $\omega$  and its boundary partition:

$$\|A^{\frac{1}{2}}\mathbf{w}\|_\omega^2 = \|A^{\frac{1}{2}}\nabla\xi\|_\omega^2 + \|A^{-\frac{1}{2}}\mathbf{curl}\phi\|_\omega^2, \quad (60a)$$

$$\|\nabla\phi\|_\omega \leq C_\omega \|\mathbf{curl}\phi\|_\omega. \quad (60b)$$

For  $d = 2$ , one has  $C_\omega = 1$ . For  $d = 3$ , the constant  $C_\omega$  depends on  $\omega$  and is more delicate to estimate. In particular, the constant  $C_\omega$  grows with the number of holes in  $\omega$ . An explicit bound on  $C_\omega$  is derived in [30] when  $\omega$  is the union of overlapping star-shaped domains. This is the case, in particular, when  $\omega$  is a vertex star, which is the application setting we consider in what follows.

We apply the Helmholtz decomposition (59) to  $\mathbf{w} := \nabla_{\mathcal{T}_h} e_d^\alpha$  on the vertex star  $\omega := \omega_\alpha$  with the boundary partition such that  $\gamma_D := \partial\omega_\alpha \cap (\Omega \cup \Gamma_D)$  and  $\gamma_N := \partial\omega_\alpha \cap \Gamma_N$ . This gives  $\xi \in H_{0,D}^1(\omega_\alpha)$  and  $\phi \in \mathbf{H}_{0,N}^1(\omega_\alpha)$  such that  $A\nabla_{\mathcal{T}_h} e_d^\alpha = A\nabla\xi + \mathbf{curl}\phi$  in  $\omega_\alpha$ . Notice that  $H_{0,D}^1(\omega_\alpha)$  is indeed the functional space defined in (44) for  $g = 0$ . Notice that the stability constant  $C_\omega$  in (60) only depends on the mesh shape-regularity, since, in particular, there is only a finite number of possible partitions of the boundary of  $\omega_\alpha$  and this number is bounded in terms of the mesh shape-regularity.

Taking the  $L^2(\omega_\alpha)$ -inner product with  $\nabla_{\mathcal{T}_h} e_d^\alpha$  and observing that  $(A\nabla_{\mathcal{T}_h} e_d^\alpha, \nabla\xi)_{\omega_\alpha} = 0$  owing to the definition (43) of  $u_c^\alpha$ , we infer that

$$\|A^{\frac{1}{2}}\nabla_{\mathcal{T}_h} e_d^\alpha\|_{\omega_\alpha}^2 = (\nabla_{\mathcal{T}_h} e_d^\alpha, \mathbf{curl}\phi)_{\omega_\alpha}.$$

Let  $I_{\text{mKM}}^{k,\alpha}$  be the (componentwise) modified  $hp$ -Karkulik–Melenk interpolation operator on the vertex star  $\omega_\alpha$  preserving the homogeneous boundary condition on  $\gamma_N = \partial\omega_\alpha \cap \Gamma_N$ . Recall that  $k \geq 1$  by assumption so that this operator is indeed well defined. We have

$$\|A^{\frac{1}{2}}\nabla_{\mathcal{T}_h} e_d^\alpha\|_{\omega_\alpha}^2 = (\nabla_{\mathcal{T}_h} e_d^\alpha, \mathbf{curl}(\phi - I_{\text{mKM}}^{k,\alpha}(\phi)))_{\omega_\alpha} + (\nabla_{\mathcal{T}_h} e_d^\alpha, \mathbf{curl} I_{\text{mKM}}^{k,\alpha}(\phi))_{\omega_\alpha} =: T_1 + T_2,$$

and it remains to bound  $T_1$  and  $T_2$ .

(1) Bound on  $T_1$ . Since  $\mathbf{curl}\nabla u_c^\alpha = \mathbf{0}$ ,  $(\phi - I_{\text{mKM}}^{k,\alpha}(\phi))|_{\partial\omega_\alpha \cap \Gamma_N} = \mathbf{0}$ , and  $u_c^\alpha \in H_{g,D}^1(\omega_\alpha)$  (see (44)), integrating by parts the curl operator gives

$$(\nabla u_c^\alpha, \mathbf{curl}(\phi - I_{\text{mKM}}^{k,\alpha}(\phi)))_{\omega_\alpha} = (\phi - I_{\text{mKM}}^{k,\alpha}(\phi), \nabla(\psi_\alpha g_D) \times \mathbf{n}_\Omega)_{\partial\omega_\alpha \cap \Gamma_D}.$$

Similar arguments, together with the fact that  $\phi - I_{\text{mKM}}^{k,\mathbf{a}}(\phi)$  is single-valued across every interface  $F \in \mathcal{F}_{\mathbf{a}} \cap \mathcal{F}_h^i$ , give

$$\begin{aligned} (\nabla_{\mathcal{T}_h}(\psi_{\mathbf{a}} u_{\mathcal{T}_h}), \mathbf{curl}(\phi - I_{\text{mKM}}^{k,\mathbf{a}}(\phi)))_{\omega_{\mathbf{a}}} &= \sum_{F \in \mathcal{F}_{\mathbf{a}} \cap \mathcal{F}_h^i} (\phi - I_{\text{mKM}}^{k,\mathbf{a}}(\phi), \llbracket \nabla_{\mathcal{T}_h}(\psi_{\mathbf{a}} u_{\mathcal{T}_h}) \rrbracket_F \times \mathbf{n}_F)_F \\ &\quad + \sum_{F \in \mathcal{F}_{\mathbf{a}} \cap \mathcal{F}_h^D} (\phi - I_{\text{mKM}}^{k,\mathbf{a}}(\phi), \nabla(\psi_{\mathbf{a}} u_{\mathcal{T}_h}) \times \mathbf{n}_{\Omega})_F. \end{aligned}$$

Putting the above two identities together and using that

$$\nabla_{\mathcal{T}_h} e_d^{\mathbf{a}} = \nabla u_c^{\mathbf{a}} - \nabla_{\mathcal{T}_h}(\psi_{\mathbf{a}} u_{\mathcal{T}_h}), \quad \nabla_{\mathcal{T}_h}(\psi_{\mathbf{a}} u_{\mathcal{T}_h}) = \psi_{\mathbf{a}} \nabla_{\mathcal{T}_h} u_{\mathcal{T}_h} + u_{\mathcal{T}_h} \nabla \psi_{\mathbf{a}},$$

and that  $(\nabla \psi_{\mathbf{a}} \times \mathbf{n}_F)_F$  is single-valued across every interface  $F \in \mathcal{F}_{\mathbf{a}} \cap \mathcal{F}_h^i$ , we infer that

$$\begin{aligned} T_1 &= - \sum_{F \in \mathcal{F}_{\mathbf{a}} \cap \mathcal{F}_h^i} (\phi - I_{\text{mKM}}^{k,\mathbf{a}}(\phi), \psi_{\mathbf{a}} \llbracket \nabla_{\mathcal{T}_h} u_{\mathcal{T}_h} \rrbracket_F \times \mathbf{n}_F)_F - \sum_{F \in \mathcal{F}_{\mathbf{a}} \cap \mathcal{F}_h^D} (\phi - I_{\text{mKM}}^{k,\mathbf{a}}(\phi), \psi_{\mathbf{a}} \nabla(u_{\mathcal{T}_h} - g_D) \times \mathbf{n}_{\Omega})_F \\ &\quad - \sum_{F \in \mathcal{F}_{\mathbf{a}} \cap \mathcal{F}_h^i} (\phi - I_{\text{mKM}}^{k,\mathbf{a}}(\phi), (\nabla \psi_{\mathbf{a}} \times \mathbf{n}_F) \llbracket u_{\mathcal{T}_h} \rrbracket_F)_F - \sum_{F \in \mathcal{F}_{\mathbf{a}} \cap \mathcal{F}_h^D} (\phi - I_{\text{mKM}}^{k,\mathbf{a}}(\phi), (\nabla \psi_{\mathbf{a}} \times \mathbf{n}_{\Omega})(u_{\mathcal{T}_h} - g_D))_F. \end{aligned}$$

Let  $T_{11}$  and  $T_{12}$  denote, respectively, the terms on the first and second lines on the above right-hand side. Using the Cauchy–Schwarz inequality and  $\|\psi_{\mathbf{a}}\|_{L^\infty(F)} = 1$ , we infer that

$$\begin{aligned} |T_{11}| &\leq \left\{ \sum_{F \in \mathcal{F}_{\mathbf{a}} \cap \mathcal{F}_h^i} \left( \frac{h_F}{k+1} \right) \|\llbracket \nabla u_{\mathcal{T}_h} \rrbracket_F \times \mathbf{n}_F\|_F^2 + \sum_{F \in \mathcal{F}_{\mathbf{a}} \cap \mathcal{F}_h^D} \left( \frac{h_F}{k+1} \right) \|\nabla(u_{\mathcal{T}_h} - g_D) \times \mathbf{n}_{\Omega}\|_F^2 \right\}^{\frac{1}{2}} \\ &\quad \times \left\{ \sum_{F \in \mathcal{F}_{\mathbf{a}} \cap (\mathcal{F}_h^i \cup \mathcal{F}_h^D)} \left( \frac{k+1}{h_F} \right) \|\phi - I_{\text{mKM}}^{k,\mathbf{a}}(\phi)\|_F^2 \right\}^{\frac{1}{2}}. \end{aligned}$$

We now invoke the approximation result (7) on  $I_{\text{mKM}}^{k,\mathbf{a}}$ , which here takes the following form: For all  $\phi \in \mathbf{H}_{0,\text{N}}^1(\omega_{\mathbf{a}})$  and all  $K \in \mathcal{T}_{\mathbf{a}}$ ,

$$\left( \frac{k}{h_K} \right)^2 \|\phi - I_{\text{mKM}}^{k,\mathbf{a}}(\phi)\|_K^2 + \left( \frac{k}{h_K} \right) \|\phi - I_{\text{mKM}}^{k,\mathbf{a}}(\phi)\|_{\partial K}^2 + \|\nabla I_{\text{mKM}}^{k,\mathbf{a}}(\phi)\|_K^2 \leq C_{\text{mKM}}^{\mathbf{a}} \|\nabla \phi\|_{\omega_{\mathbf{a}}}^2, \quad (61)$$

where the constant  $C_{\text{mKM}}^{\mathbf{a}}$  is uniformly bounded in terms of the mesh shape-regularity for all  $\mathbf{a} \in \mathcal{V}_h$ . Notice, in particular, that  $\text{es}(K) = \omega_{\mathbf{a}}$  since  $I_{\text{mKM}}^{k,\mathbf{a}}$  is locally constructed in the vertex star  $\omega_{\mathbf{a}}$ . For every  $F \in \mathcal{F}_{\mathbf{a}} \cap (\mathcal{F}_h^i \cup \mathcal{F}_h^D)$ , we can pick a mesh cell  $K \in \mathcal{T}_{\mathbf{a}}$  of which  $F$  is a face and obtain

$$\left( \frac{k+1}{h_F} \right)^{\frac{1}{2}} \|\phi - I_{\text{mKM}}^{k,\mathbf{a}}(\phi)\|_F \leq C \|\nabla \phi\|_{\omega_{\mathbf{a}}}.$$

Putting the above two bounds together gives

$$|T_{11}| \leq C \left\{ \sum_{F \in \mathcal{F}_{\mathbf{a}} \cap \mathcal{F}_h^i} \left( \frac{h_F}{k+1} \right) \|\llbracket \nabla u_{\mathcal{T}_h} \rrbracket_F \times \mathbf{n}_F\|_F^2 + \sum_{F \in \mathcal{F}_{\mathbf{a}} \cap \mathcal{F}_h^D} \left( \frac{h_F}{k+1} \right) \|\nabla(u_{\mathcal{T}_h} - g_D) \times \mathbf{n}_{\Omega}\|_F^2 \right\}^{\frac{1}{2}} \|\nabla \phi\|_{\omega_{\mathbf{a}}}.$$

Owing to (60), we infer that

$$\|\nabla \phi\|_{\omega_{\mathbf{a}}} \leq C_{\omega_{\mathbf{a}}} \|\mathbf{curl} \phi\|_{\omega_{\mathbf{a}}} \leq C_{\omega_{\mathbf{a}}} (A_{\mathbf{a}}^{\sharp})^{\frac{1}{2}} \|A^{-\frac{1}{2}} \mathbf{curl} \phi\|_{\omega_{\mathbf{a}}} \leq C_{\omega_{\mathbf{a}}} (A_{\mathbf{a}}^{\sharp})^{\frac{1}{2}} \|A^{\frac{1}{2}} \nabla_{\mathcal{T}_h} e_d\|_{\omega_{\mathbf{a}}}. \quad (62)$$

Combining the above bounds and recalling the definition (37) of  $\chi_{\mathbf{a}}(A)$  and the definition (38) of  $A_F^b$ , we obtain

$$\begin{aligned} |T_{11}| &\leq C \chi_{\mathbf{a}}(A)^{\frac{1}{2}} \left\{ \sum_{F \in \mathcal{F}_{\mathbf{a}} \cap \mathcal{F}_h^i} A_F^b \left( \frac{h_F}{k+1} \right) \|\llbracket \nabla u_{\mathcal{T}_h} \rrbracket_F \times \mathbf{n}_F\|_F^2 + \sum_{F \in \mathcal{F}_{\mathbf{a}} \cap \mathcal{F}_h^D} A_F^b \left( \frac{h_F}{k+1} \right) \|\nabla(u_{\mathcal{T}_h} - g_D) \times \mathbf{n}_{\Omega}\|_F^2 \right\}^{\frac{1}{2}} \\ &\quad \times \|A^{\frac{1}{2}} \nabla_{\mathcal{T}_h} e_d\|_{\omega_{\mathbf{a}}}. \end{aligned}$$

Let us now bound  $T_{12}$ . Letting  $\mathbb{I}$  be the identity operator, we write  $T_{12} = T_{12a} + T_{12b}$  with

$$\begin{aligned} T_{12a} &= - \sum_{F \in \mathcal{F}_a \cap \mathcal{F}_h^i} (\phi - I_{\text{mKM}}^{k,\mathbf{a}}(\phi), (\nabla \psi_{\mathbf{a}} \times \mathbf{n}_F)(\mathbb{I} - \Pi_F^k)[[u_{\mathcal{T}_h}]]_F)_F \\ &\quad - \sum_{F \in \mathcal{F}_a \cap \mathcal{F}_h^D} (\phi - I_{\text{mKM}}^{k,\mathbf{a}}(\phi), (\nabla \psi_{\mathbf{a}} \times \mathbf{n}_\Omega)(\mathbb{I} - \Pi_F^k)(u_{\mathcal{T}_h} - g_D))_F \\ T_{12b} &= - \sum_{F \in \mathcal{F}_a \cap \mathcal{F}_h^i} (\phi - I_{\text{mKM}}^{k,\mathbf{a}}(\phi), (\nabla \psi_{\mathbf{a}} \times \mathbf{n}_F)\Pi_F^k[[u_{\mathcal{T}_h}]]_F)_F \\ &\quad - \sum_{F \in \mathcal{F}_a \cap \mathcal{F}_h^D} (\phi - I_{\text{mKM}}^{k,\mathbf{a}}(\phi), (\nabla \psi_{\mathbf{a}} \times \mathbf{n}_\Omega)(\Pi_F^k(u_{\mathcal{T}_h} - g_D)))_F. \end{aligned}$$

Using the Cauchy–Schwarz inequality and  $\|\nabla \psi_{\mathbf{a}} \times \mathbf{n}_F\|_{L^\infty(F)} \leq Ch_F^{-1}$  gives

$$\begin{aligned} |T_{12a}| &\leq C \left\{ \sum_{F \in \mathcal{F}_a \cap \mathcal{F}_h^i} \left( \frac{h_F}{k+1} \right) h_F^{-2} \|(\mathbb{I} - \Pi_F^k)[[u_{\mathcal{T}_h}]]_F\|_F^2 + \sum_{F \in \mathcal{F}_a \cap \mathcal{F}_h^D} \left( \frac{h_F}{k+1} \right) h_F^{-2} \|(\mathbb{I} - \Pi_F^k)(u_{\mathcal{T}_h} - g_D)\|_F^2 \right\}^{\frac{1}{2}} \\ &\quad \times \left\{ \sum_{F \in \mathcal{F}_a \cap (\mathcal{F}_h^i \cup \mathcal{F}_h^D)} \left( \frac{k+1}{h_F} \right) \|\phi - I_{\text{mKM}}^{k,\mathbf{a}}(\phi)\|_F^2 \right\}^{\frac{1}{2}}. \end{aligned}$$

Invoking the Poincaré inequalities  $\|(\mathbb{I} - \Pi_F^k)[[u_{\mathcal{T}_h}]]_F\|_F \leq Ch_F \|[\nabla u_{\mathcal{T}_h}]_F \times \mathbf{n}_F\|_F$  for all  $F \in \mathcal{F}_a \cap \mathcal{F}_h^i$  and  $\|(\mathbb{I} - \Pi_F^k)(u_{\mathcal{T}_h} - g_D)\|_F \leq Ch_F \|\nabla(u_{\mathcal{T}_h} - g_D) \times \mathbf{n}_\Omega\|_F$  for all  $F \in \mathcal{F}_a \cap \mathcal{F}_h^D$ , we infer that

$$\begin{aligned} |T_{12a}| &\leq C \left\{ \sum_{F \in \mathcal{F}_a \cap \mathcal{F}_h^i} \left( \frac{h_F}{k+1} \right) \|[\nabla u_{\mathcal{T}_h}]_F \times \mathbf{n}_F\|_F^2 + \sum_{F \in \mathcal{F}_a \cap \mathcal{F}_h^D} \left( \frac{h_F}{k+1} \right) \|\nabla(u_{\mathcal{T}_h} - g_D) \times \mathbf{n}_\Omega\|_F^2 \right\}^{\frac{1}{2}} \\ &\quad \times \left\{ \sum_{F \in \mathcal{F}_a \cap (\mathcal{F}_h^i \cup \mathcal{F}_h^D)} \left( \frac{k+1}{h_F} \right) \|\phi - I_{\text{mKM}}^{k,\mathbf{a}}(\phi)\|_F^2 \right\}^{\frac{1}{2}}. \end{aligned}$$

Invoking the same arguments as above to bound the term involving  $\phi$ , we conclude that

$$\begin{aligned} |T_{12a}| &\leq C \chi_{\mathbf{a}}(A)^{\frac{1}{2}} \left\{ \sum_{F \in \mathcal{F}_a \cap \mathcal{F}_h^i} A_F^b \left( \frac{h_F}{k+1} \right) \|[\nabla u_{\mathcal{T}_h}]_F \times \mathbf{n}_F\|_F^2 + \sum_{F \in \mathcal{F}_a \cap \mathcal{F}_h^D} A_F^b \left( \frac{h_F}{k+1} \right) \|\nabla(u_{\mathcal{T}_h} - g_D) \times \mathbf{n}_\Omega\|_F^2 \right\}^{\frac{1}{2}} \\ &\quad \times \|A^{\frac{1}{2}} \nabla_{\mathcal{T}_h} e_d\|_{\omega_{\mathbf{a}}}. \end{aligned}$$

Turning our attention to  $T_{12b}$ , we observe that  $u_F$  is single-valued at every interface  $F \in \mathcal{F}_a \cap \mathcal{F}_h^i$ , and that  $u_F = \Pi_F^k(g_D|_F)$  on every Dirichlet boundary face  $F \in \mathcal{F}_a \cap \mathcal{F}_h^D$ . Therefore, we have

$$\begin{aligned} T_{12b} &= - \sum_{F \in \mathcal{F}_a \cap \mathcal{F}_h^i} (\phi - I_{\text{mKM}}^{k,\mathbf{a}}(\phi), (\nabla \psi_{\mathbf{a}} \times \mathbf{n}_F)\Pi_F^k[[u_{\mathcal{T}_h} - u_F]]_F)_F \\ &\quad - \sum_{F \in \mathcal{F}_a \cap \mathcal{F}_h^D} (\phi - I_{\text{mKM}}^{k,\mathbf{a}}(\phi), (\nabla \psi_{\mathbf{a}} \times \mathbf{n}_\Omega)(\Pi_F^k(u_{\mathcal{T}_h} - u_F)))_F. \end{aligned}$$

Using the Cauchy–Schwarz inequality and  $\|\nabla \psi_{\mathbf{a}} \times \mathbf{n}_F\|_{L^\infty(F)} \leq Ch_F^{-1}$  gives

$$|T_{12b}| \leq C \left\{ \sum_{K \in \mathcal{T}_a} \sum_{F \in \mathcal{F}_{\partial K}} \left( \frac{h_F}{k+1} \right) h_F^{-2} \|\Pi_{\partial K}^k(u_K - u_F)\|_F^2 \right\}^{\frac{1}{2}} \left\{ \sum_{F \in \mathcal{F}_a \cap (\mathcal{F}_h^i \cup \mathcal{F}_h^D)} \left( \frac{k+1}{h_F} \right) \|\phi - I_{\text{mKM}}^{k,\mathbf{a}}(\phi)\|_F^2 \right\}^{\frac{1}{2}},$$

where, in the first factor, we re-organized the sum over the mesh faces in  $\mathcal{F}_a$  as a sum over the mesh cells in  $\mathcal{T}_a$ . Invoking the same arguments as above to bound the factor involving  $\phi$  and recalling the definition (12) of the stabilization bilinear form  $S_{\partial K}$ , we obtain

$$\begin{aligned} |T_{12b}| &\leq C \chi_{\mathbf{a}}(A)^{\frac{1}{2}} (k+1)^{-\frac{3}{2}} \left\{ \sum_{K \in \mathcal{T}_a} A_K S_{\partial K}(\hat{u}_K, \hat{u}_K) \right\}^{\frac{1}{2}} \|A^{\frac{1}{2}} \nabla_{\mathcal{T}_h} e_d\|_{\omega_{\mathbf{a}}} \\ &\leq C \chi_{\mathbf{a}}(A)^{\frac{1}{2}} \left\{ \sum_{K \in \mathcal{T}_a} A_K S_{\partial K}(\hat{u}_K, \hat{u}_K) \right\}^{\frac{1}{2}} \|A^{\frac{1}{2}} \nabla_{\mathcal{T}_h} e_d\|_{\omega_{\mathbf{a}}}. \end{aligned}$$



Notice that we dropped the (favorable) factor  $(k+1)^{-\frac{3}{2}}$  since the bound on  $T_2$  derived below does not involve this factor. Putting everything together, we conclude that

$$|T_1| \leq C\chi_{\mathbf{a}}(A)^{\frac{1}{2}} \left\{ \sum_{F \in \mathcal{F}_{\mathbf{a}} \cap \mathcal{F}_h^i} A_F^b \left( \frac{h_F}{k+1} \right) \|\llbracket \nabla u_{\mathcal{T}_h} \rrbracket_F \times \mathbf{n}_F\|_F^2 + \sum_{F \in \mathcal{F}_{\mathbf{a}} \cap \mathcal{F}_h^D} A_F^b \left( \frac{h_F}{k+1} \right) \|\nabla(u_{\mathcal{T}_h} - g_D) \times \mathbf{n}_\Omega\|_F^2 \right. \\ \left. + \sum_{K \in \mathcal{T}_{\mathbf{a}}} A_K S_{\partial K}(\hat{u}_K, \hat{u}_K) \right\}^{\frac{1}{2}} \times \|A^{\frac{1}{2}} \nabla_{\mathcal{T}_h} e_d\|_{\omega_{\mathbf{a}}}.$$

(2) Bound on  $T_2$ . Since  $I_{\text{mKM}}^{k,\mathbf{a}}(\phi) \in \mathbf{H}_{0,\text{N}}^1(\omega_{\mathbf{a}})$ ,  $\llbracket \mathbf{curl} I_{\text{mKM}}^{k,\mathbf{a}}(\phi) \cdot \mathbf{n} \rrbracket_F = 0$  across every interface  $F \in \mathcal{F}_{\mathbf{a}} \cap \mathcal{F}_h^i$ ,  $\mathbf{curl} I_{\text{mKM}}^{k,\mathbf{a}}(\phi) \cdot \mathbf{n}_F = 0$  on every boundary face  $F \in \partial\omega_{\mathbf{a}} \cap \Gamma_{\text{N}}$ , and  $\nabla \cdot \mathbf{curl} I_{\text{mKM}}^{k,\mathbf{a}}(\phi) = 0$  on  $\omega_{\mathbf{a}}$ , integrating by parts the gradient operator gives

$$(\nabla u_c^{\mathbf{a}}, \mathbf{curl} I_{\text{mKM}}^{k,\mathbf{a}}(\phi))_{\omega_{\mathbf{a}}} = (\psi_{\mathbf{a}} g_D, \mathbf{curl} I_{\text{mKM}}^{k,\mathbf{a}}(\phi) \cdot \mathbf{n}_\Omega)_{\partial\omega_{\mathbf{a}} \cap \Gamma_D} = \sum_{K \in \mathcal{T}_{\mathbf{a}}} (\psi_{\mathbf{a}} g_D, \mathbf{curl} I_{\text{mKM}}^{k,\mathbf{a}}(\phi) \cdot \mathbf{n}_\Omega)_{\partial K^D}.$$

Similarly, we have

$$(\nabla_{\mathcal{T}_h}(\psi_{\mathbf{a}} u_{\mathcal{T}_h}), \mathbf{curl} I_{\text{mKM}}^{k,\mathbf{a}}(\phi))_{\omega_{\mathbf{a}}} = \sum_{K \in \mathcal{T}_{\mathbf{a}}} (\psi_{\mathbf{a}} u_K, \mathbf{curl} I_{\text{mKM}}^{k,\mathbf{a}}(\phi) \cdot \mathbf{n}_K)_{\partial K}.$$

Recalling that  $T_2 = (\nabla_{\mathcal{T}_h} e_d^{\mathbf{a}}, \mathbf{curl} I_{\text{mKM}}^{k,\mathbf{a}}(\phi))_{\omega_{\mathbf{a}}}$  and  $e_d^{\mathbf{a}} = u_c^{\mathbf{a}} - \psi_{\mathbf{a}} u_{\mathcal{T}_h}$ , we infer that

$$T_2 = \sum_{K \in \mathcal{T}_{\mathbf{a}}} \left\{ (g_D - u_K, \psi_{\mathbf{a}} \mathbf{curl} I_{\text{mKM}}^{k,\mathbf{a}}(\phi) \cdot \mathbf{n}_\Omega)_{\partial K^D} - (u_K, \psi_{\mathbf{a}} \mathbf{curl} I_{\text{mKM}}^{k,\mathbf{a}}(\phi) \cdot \mathbf{n}_K)_{\partial K^i} \right\} \\ = \sum_{K \in \mathcal{T}_{\mathbf{a}}} \left\{ (\Pi_{\partial K}^k(g_D - u_K), \psi_{\mathbf{a}} \mathbf{curl} I_{\text{mKM}}^{k,\mathbf{a}}(\phi) \cdot \mathbf{n}_\Omega)_{\partial K^D} - (\Pi_{\partial K}^k(u_K), \psi_{\mathbf{a}} \mathbf{curl} I_{\text{mKM}}^{k,\mathbf{a}}(\phi) \cdot \mathbf{n}_K)_{\partial K^i} \right\}.$$

Notice that we used here that  $\psi_{\mathbf{a}} \mathbf{curl} I_{\text{mKM}}^{k,\mathbf{a}}(\phi) \cdot \mathbf{n}_K$  is in  $\mathbb{P}^k(\mathcal{F}_{\partial K})$  to introduce the  $L^2$ -orthogonal projection  $\Pi_{\partial K}^k$  in both terms on the right-hand side. Since  $u_{\partial K}$  is single-valued on every interface  $F \in \mathcal{F}_{\partial K^i}$  and  $u_{\partial K}|_F = \Pi_F^k(g_D|_F)$  on every boundary face  $F \in \mathcal{F}_{\partial K^D}$ , and since  $\|\psi_{\mathbf{a}}\|_{L^\infty(F)} = 1$ , we obtain

$$T_2 = \sum_{K \in \mathcal{T}_{\mathbf{a}}} -(\Pi_{\partial K}^k(u_K - u_{\partial K}), \psi_{\mathbf{a}} \mathbf{curl} I_{\text{mKM}}^{k,\mathbf{a}}(\phi) \cdot \mathbf{n}_K)_{\partial K^i \cup \partial K^D}.$$

Invoking the Cauchy–Schwarz inequality, the discrete trace inverse inequality (3), and the definition of the stabilization bilinear form  $S_{\partial K}$ , we infer that

$$|T_2| \leq (A_{\mathbf{a}}^b)^{-\frac{1}{2}} \left\{ \sum_{K \in \mathcal{T}_{\mathbf{a}}} A_K \frac{(k+1)^2}{h_K} \|\Pi_{\partial K}^k(u_K - u_{\partial K})\|_{\partial K^i \cup \partial K^D}^2 \right\}^{\frac{1}{2}} \left\{ \sum_{K \in \mathcal{T}_{\mathbf{a}}} \frac{h_K}{(k+1)^2} \|\mathbf{curl} I_{\text{mKM}}^{k,\mathbf{a}}(\phi)\|_{\partial K^i \cup \partial K^D}^2 \right\}^{\frac{1}{2}} \\ \leq C(A_{\mathbf{a}}^b)^{-\frac{1}{2}} \left\{ \sum_{K \in \mathcal{T}_{\mathbf{a}}} A_K S_{\partial K}(\hat{u}_K, \hat{u}_K) \right\}^{\frac{1}{2}} \|\mathbf{curl} I_{\text{mKM}}^{k,\mathbf{a}}(\phi)\|_{\omega_{\mathbf{a}}} \\ \leq C'(A_{\mathbf{a}}^b)^{-\frac{1}{2}} \left\{ \sum_{K \in \mathcal{T}_{\mathbf{a}}} A_K S_{\partial K}(\hat{u}_K, \hat{u}_K) \right\}^{\frac{1}{2}} \|\nabla \phi\|_{\omega_{\mathbf{a}}},$$

where the last bound follows from  $\|\mathbf{curl} I_{\text{mKM}}^{k,\mathbf{a}}(\phi)\|_{\omega_{\mathbf{a}}} \leq (d-1)\|\nabla I_{\text{mKM}}^{k,\mathbf{a}}(\phi)\|_{\omega_{\mathbf{a}}}$  and the  $H^1$ -stability of  $I_{\text{mKM}}^{k,\mathbf{a}}$  (see (61)). Proceeding as above to bound  $\|\nabla \phi\|_{\omega_{\mathbf{a}}}$ , we conclude that

$$|T_2| \leq C\chi_{\mathbf{a}}(A)^{\frac{1}{2}} \left\{ \sum_{K \in \mathcal{T}_{\mathbf{a}}} A_K S_{\partial K}(\hat{u}_K, \hat{u}_K) \right\}^{\frac{1}{2}} \times \|A^{\frac{1}{2}} \nabla_{\mathcal{T}_h} e_d\|_{\omega_{\mathbf{a}}}.$$

(3) Putting the bounds on  $T_1$  and  $T_2$  together and invoking the triangle inequality to introduce the oscillation term on  $g_D$  proves the assertion.

**Remark 7.1** ( $k = 0$ ). *The assumption  $k \geq 1$  is needed to ensure that  $\psi_{\mathbf{a}}(\mathbf{n}_F \cdot \mathbf{curl} I_{\text{mKM}}^k(\phi))|_F \in \mathbb{P}^k(F)$ . In the case  $k = 0$ , the nonconformity error can be bounded by using a (piecewise affine) nodal-averaging operator.*

## 7.4 Proof of Theorem 5.9

We consider the following bubble functions from [41]: For all  $K \in \mathcal{T}_h$  and all  $F \in \mathcal{F}_h$ , we set

$$b_K(\mathbf{x}) := h_K^{-1} \text{dist}(\mathbf{x}, \partial K), \quad \forall \mathbf{x} \in K, \quad b_F(\mathbf{x}) := h_F^{-1} \text{dist}(\mathbf{x}, \partial F), \quad \forall \mathbf{x} \in F. \quad (63)$$

We recall the following  $hp$ -inverse inequalities from [41, Theorem 2.5]: For all  $v \in \mathbb{P}^{k+1}(K)$ , all  $K \in \mathcal{T}_h$ , and all  $F \in \mathcal{F}_{\partial K}$ , we have

$$\|v\|_K \leq C(k+1) \|b_K^{\frac{1}{2}} v\|_K, \quad (64a)$$

$$\|\nabla(vb_K)\|_K \leq C\left(\frac{k+1}{h_K}\right) \|b_K^{\frac{1}{2}} v\|_K, \quad (64b)$$

$$\|v\|_F \leq C(k+1) \|b_F^{\frac{1}{2}} v\|_F, \quad (64c)$$

We also recall the following  $hp$ -polynomial extension result from [41, Lemma 2.6] (with  $\varepsilon := (k+1)^{-2}$  and  $\alpha := 1$ ): For all  $F \in \mathcal{F}_{\partial K}$  and all  $K \in \mathcal{T}_h$ , there exists an extension operator  $E_{K,F} : H_0^1(F) \rightarrow H^1(K)$  such that, for all  $v \in \mathbb{P}^k(F)$ ,

$$E_{K,F}(b_F v)|_F = (b_F v), \quad E_{K,F}(b_F v)|_{\partial K \setminus F} = 0, \quad \|\nabla E_{K,F}(b_F v)\|_K \leq C\left(\frac{(k+1)^2}{h_K}\right)^{\frac{1}{2}} \|b_F^{\frac{1}{2}} v\|_F. \quad (65)$$

(1) Proof of (51a). Set  $v_K := \Pi_K^{k+1}(f) + A_K \Delta u_K \in \mathbb{P}^{k+1}(K)$ ,  $w_K := b_K v_K$ , observe that  $w_K$  vanishes on  $\partial K$ , and let  $w$  be the zero-extension  $w_K$  to  $\Omega$ . Since  $(f, w_K)_K - (A_K \nabla u, \nabla w_K)_K = (f, w)_K - (A_K \nabla u, \nabla w)_K = 0$ , an integration by parts for  $\Delta u_K$  gives

$$\begin{aligned} \|b_K^{\frac{1}{2}} v_K\|_K^2 &= \|b_K^{\frac{1}{2}} (\Pi_K^{k+1}(f) + A_K \Delta u_K)\|_K^2 \\ &= (\Pi_K^{k+1}(f) + A_K \Delta u_K, w_K)_K \\ &= ((\Pi_K^{k+1}(f) - f), w_K)_K + (A_K \nabla e, \nabla w_K)_K \\ &\leq \|\Pi_K^{k+1}(f) - f\|_K \|b_K v_K\|_K + A_K^{\frac{1}{2}} \|A^{\frac{1}{2}} \nabla e\|_K \|\nabla(b_K v_K)\|_K \\ &\leq \left( \|\Pi_K^{k+1}(f) - f\|_K + C A_K^{\frac{1}{2}} \left(\frac{k+1}{h_K}\right) \|A^{\frac{1}{2}} \nabla e\|_K \right) \|b_K^{\frac{1}{2}} v_K\|_K, \end{aligned}$$

where the last bound follows from  $b_K \leq 1$  and (64b). Hence,

$$\|b_K^{\frac{1}{2}} v_K\|_K \leq \|\Pi_K^{k+1}(f) - f\|_K + C A_K^{\frac{1}{2}} \left(\frac{k+1}{h_K}\right) \|A^{\frac{1}{2}} \nabla e\|_K = A_K^{\frac{1}{2}} \left(\frac{k+1}{h_K}\right) (O_K(f) + C \|A^{\frac{1}{2}} \nabla e\|_K),$$

where we used the definition (36a) of  $O_K(f)$ . Invoking (64a), we infer that

$$\eta_{K,\text{res}} = A_K^{-\frac{1}{2}} \left(\frac{h_K}{k+1}\right) \|v_K\|_K \leq C(k+1) A_K^{-\frac{1}{2}} \left(\frac{h_K}{k+1}\right) \|b_K^{\frac{1}{2}} v_K\|_K.$$

Combining the above two bounds proves (51a).

(2) Proof of (51b). For every interface  $F \in \mathcal{F}_{\partial K^i}$ , using the local conservation property (20) and the triangle inequality gives

$$\| \llbracket A \nabla R_{\mathcal{T}_h}^{k+1}(\hat{u}_h) \rrbracket_{F \cdot \mathbf{n}_F} \|_F^2 \leq \sum_{\mathcal{K} \in \{K, K'\}} 2 \left( \frac{A_{\mathcal{K}}(k+1)^2}{h_{\mathcal{K}}} \right)^2 \|\Pi_{\partial \mathcal{K}}^k(u_{\mathcal{K}} - u_{\partial \mathcal{K}})\|_F^2,$$

where  $K'$  denotes the mesh cell sharing  $F$  with  $K$ . Using the mesh shape-regularity, we infer that

$$\begin{aligned} A_K^{-1} \left(\frac{h_K}{k+1}\right) \| \llbracket A \nabla R_{\mathcal{T}_h}^{k+1}(\hat{u}_h) \rrbracket_{F \cdot \mathbf{n}_F} \|_F^2 &\leq \sum_{\mathcal{K} \in \{K, K'\}} 2 \left( \frac{A_{\mathcal{K}}}{A_K} \right) \left( \frac{(k+1)h_K}{h_{\mathcal{K}}} \right) \left( \frac{A_{\mathcal{K}}(k+1)^2}{h_{\mathcal{K}}} \right) \|\Pi_{\partial \mathcal{K}}^k(u_{\mathcal{K}} - u_{\partial \mathcal{K}})\|_F^2 \\ &\leq C \chi'_K(A) (k+1) \sum_{\mathcal{K} \in \{K, K'\}} A_{\mathcal{K}} S_{\partial \mathcal{K}}(\hat{u}_{\mathcal{K}}, \hat{u}_{\mathcal{K}}). \end{aligned}$$

Moreover, for every boundary face  $F \in \mathcal{F}_{\partial K^N}$ , we have

$$\begin{aligned} A_K^{-1} \left( \frac{h_K}{k+1} \right) \|A \nabla R_K^{k+1}(\hat{u}_K) \cdot \mathbf{n}_F - \Pi_{\partial K}^k(g_N)\|_F^2 &= (k+1) A_K \frac{(k+1)^2}{h_K} \|\Pi_{\partial K}^k(u_K - u_{\partial K})\|_F^2 \\ &\leq (k+1) A_K S_{\partial K}(\hat{u}_K, \hat{u}_K). \end{aligned}$$

Summing over all the faces  $F \in \mathcal{F}_{\partial K^i} \cup \mathcal{F}_{\partial K^N}$  completes the proof of (51b).

(3) Proof of (51c). For every interface  $F \in \mathcal{F}_{\partial K^i}$ , we set  $v_F := \llbracket \nabla u_{\mathcal{T}_h} \rrbracket_F \times \mathbf{n}_F$  and  $w_{\mathcal{K},F} := E_{\mathcal{K},F}(b_F v_F)$  for all  $\mathcal{K} \in \{K, K'\}$ , where  $K'$  denotes, as above, the mesh cell sharing  $F$  with  $K$ . Since  $\llbracket \nabla u_{\mathcal{T}_h} \rrbracket_F \times \mathbf{n}_F = \sum_{\mathcal{K} \in \{K, K'\}} \nabla u_{\mathcal{K}} \times \mathbf{n}_{\mathcal{K}}$  and  $w_{\mathcal{K},F}|_F = b_F v_F$  owing to (65), we infer that

$$\|b_F^{\frac{1}{2}} v_F\|_F^2 = \sum_{\mathcal{K} \in \{K, K'\}} (w_{\mathcal{K},F}, \nabla u_{\mathcal{K}} \times \mathbf{n}_{\mathcal{K}})_F = \sum_{\mathcal{K} \in \{K, K'\}} (\mathbf{curl} w_{\mathcal{K},F}, \nabla u_{\mathcal{K}})_{\mathcal{K}} = \sum_{\mathcal{K} \in \{K, K'\}} (\mathbf{curl} w_{\mathcal{K},F}, \nabla(u_{\mathcal{K}} - u))_{\mathcal{K}},$$

where the second equality follows by integration by parts and the third equality additionally uses that  $\llbracket \nabla u \rrbracket_F \times \mathbf{n}_F = \mathbf{0}$ . The Cauchy–Schwarz inequality

$$\|b_F^{\frac{1}{2}} v_F\|_F^2 \leq (A_F^b)^{-\frac{1}{2}} \sum_{\mathcal{K} \in \{K, K'\}} \|A^{\frac{1}{2}} \nabla e\|_{\mathcal{K}} \|\mathbf{curl} w_{\mathcal{K},F}\|_{\mathcal{K}}.$$

Invoking (65) gives

$$\|\mathbf{curl} w_{\mathcal{K},F}\|_{\mathcal{K}} \leq C \|\nabla E_{\mathcal{K},F}(b_F v_F)\|_{\mathcal{K}} \leq C(k+1) h_K^{-\frac{1}{2}} \|b_F^{\frac{1}{2}} v_F\|_F.$$

Hence,

$$\|b_F^{\frac{1}{2}} v_F\|_F \leq C (A_F^b)^{-\frac{1}{2}} (k+1) h_K^{-\frac{1}{2}} \sum_{\mathcal{K} \in \{K, K'\}} \|A^{\frac{1}{2}} \nabla e\|_{\mathcal{K}}.$$

Since  $\|v_F\|_F \leq C(k+1) \|b_F^{\frac{1}{2}} v_F\|_F$  owing to (64c), combining the above bounds gives

$$\|v_F\|_F \leq C (A_F^b)^{-\frac{1}{2}} (k+1)^2 h_K^{-\frac{1}{2}} \sum_{\mathcal{K} \in \{K, K'\}} \|A^{\frac{1}{2}} \nabla e\|_{\mathcal{K}}.$$

This shows that

$$(A_F^b)^{\frac{1}{2}} \left( \frac{h_K}{k+1} \right)^{\frac{1}{2}} \|\llbracket \nabla u_{\mathcal{T}_h} \rrbracket_F \times \mathbf{n}_F\|_F = (A_F^b)^{\frac{1}{2}} \left( \frac{h_K}{k+1} \right)^{\frac{1}{2}} \|v_F\|_F \leq C(k+1)^{\frac{3}{2}} \sum_{\mathcal{K} \in \{K, K'\}} \|A^{\frac{1}{2}} \nabla e\|_{\mathcal{K}}.$$

A similar bound can be established for all  $F \in \mathcal{F}_{\partial K^D}$ . Summing over all  $F \in \mathcal{F}_{\partial K^i} \cup \mathcal{F}_{\partial K^D}$  completes the proof.

## Acknowledgment

The use of the Cleps computing platform at INRIA Paris is gratefully acknowledged.

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