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Fully explicit numerical scheme for linearized wave propagation in nearly-incompressible soft hyperelastic solids

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Abstract

The numerical approximation of wave propagation problems in nearly or pure incompressible solids faces several challenges such as locking and stability constraints. In this work we propose a stabilized Leapfrog scheme based on the use of Chebyshev polynomials to relax the stability condition, which is strongly limited by the enforcement of incompressibility. The scheme is fully explicit, second order accurate and energy-preserving. For the space discretization we use a mixed formulation with high-order spectral elements and mass-lumping. A strategy is proposed for an efficient and accurate computation of the pressure contribution with a new definition of the discrete Grad-div operator. Finally, we consider linear wave propagation problems in nearly-incompressible hyperelastic solids subject to static preload.

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51 1 Introduction

52 The accurate and efficient simulation of wave propagation in nearly incompressible
53 solids represents a significant challenge in numerical modeling. Classically, the finite
54 element approximation of solids relies on the displacement-based method, in which it is
55 assumed that the displacement of a body is completely specified by the nodal point
56 displacement [Sussman1987]. However, the enforcement of incompressibility highly
57 affects the accuracy of the solution, such that a finer space discretization will be needed,
58 compared to the compressible case. This is due to the fact that volumetric strains are
59 computed with a lower accuracy with respect to the displacements and are close to zero
60 in the incompressible limit. Any error related to the computation of the volumetric
61 strain will result in a large error in the computation of the stress (pressure) due to the
62 multiplication by a large bulk modulus, and then a wrong prediction of the displacement
63 obtained from the principle of virtual work [Bathe2006]. With a higher spatial
64 resolution, we may circumvent errors in the computation of the displacement field, but
65 we will still have spurious or incorrect pressure. In addition, it is known [Sussman1987,
66 Oden1970, Taylor1968, Caforio2018] that for nearly-incompressible solids, this
67 method suffers from ill-conditioning of the stiffness matrix. In general, we may refer to
68 formulation whose predictive capability is dependent from the bulk modulus as locking
69 formulations [Bathe2006, LeTallec1994]. Nevertheless, efforts have been done to
70 overcome these limitations, since most of large-scale structural analysis programs are
71 based on displacement-based method [ELGUEDJ2008]. We cite in particular the
72 B-scheme for finite-deformation problems [Hughes1977equivalence] and F-scheme
73 for hyperelastic problems, which rely on the reduction of volumetric constraints through
74 projection.

75 Mixed methods form another class of methods for finite element approximations,
76 that have proven to be effective to avoid locking behavior in the framework of elasticity
77 with nearly or pure incompressible materials. These methods are based on the
78 introduction of the pressure as an additional unknown. The problem becomes now
79 a saddle-point problem where the key aspect for the finite-element approximation
80 is the introduction of an auxiliary finite element space for the computation of the
81 pressure field. These methods are proven to be stable under the well-known LBB
82 inf-sup condition derived by Brezzi and Babuska, together with the ellipticity condition

83 [Brezzi1974, Babuvska1973, Boffi2017], for which the choice of the finite element
 84 spaces is fundamental to guarantee stability and convergence of the method. When the
 85 stability is not given for a certain choice of the finite element spaces, it is possible to
 86 apply stabilization techniques such as cross-grid triangulation or with the introduction
 87 of a stabilization in order to relax the incompressibility constraint. We refer the reader
 88 to [Quarteroni2008] for a review of stabilization methods [brezzi1984stabilization]
 89 such as Galerkin-Least Square [HughesFranca], Streamline Upwind Petrov Galerkin
 90 [hughes1986new] and bubble functions [baiocchi1993virtual].

91 In addition, the simulation of dynamic problems must account for an appropriate
 92 discretization strategy in time. For elastic wave propagation problem is generally
 93 prohibitive to use implicit schemes, as they require the inversion of the stiffness matrix
 94 at each time-step, which is too large to be stored even when considering factorization
 95 methods. Explicit is well suited given the phenomena that we are observing. In
 96 particular, the leapfrog (LF) scheme based on centered finite difference of second
 97 order time differential equation benefits of several properties: second-order accuracy,
 98 energy conservation for linear problems and time reversibility [hairer2006geometric].
 99 However, the stability of the scheme is given under the classical CFL condition
 100 for the maximal time-step, which is highly constraining when considering nearly
 101 incompressibility. As the bulk approaches the incompressible limit, the maximal
 102 time-step goes to zero, with resulting endless computations.

103 A possible alternative is to use fractional step projection methods, in which the
 104 idea is to split the original problem in two simpler ones, such that at each time step we
 105 compute implicitly a first half-step considering one sub-problem, and then we correct
 106 the obtained solution with the second half of the problem that we compute explicitly
 107 [quarteroni2009modellistica]. We cite in particular [Caforio2018], which has been
 108 a starting point for this work in order to treat linear elastic wave propagation problems
 109 in nearly or pure incompressible solids. The conservative scheme proposed by Caforio
 110 et al. [Caforio2018] is based on the relaxation of the volumetric component of the
 111 displacement solution, related to the incompressible constraint, by solving at each
 112 iteration in time a Poisson problem. Then, the explicit leapfrog time-discretization
 113 method is used to solve the second part of the problem to account for the "shear" part
 114 of the problem. Solving the Poisson problem must then involve the use of fast solvers,
 115 like FFT solvers.

In this work, we focus on the development of a fully explicit scheme, such that
 only the mass matrix will be inverted together with mass-lumping approximation
 [Cohen2002]. Recently, [Carle2020] have proposed a new class of methods, named
 Leapfrog-Chebyshev (LFC) scheme for the relaxation of the CFL condition. They
 applied Chebyshev polynomials to the linear component of the problem, while the
 non-linear part is computed with the LF scheme. These schemes have proven to be
 effective together with local-time stepping methods (LFC-LTS) [Grote2021], which
 are developed to deal with stability constraint related to local mesh refinement. Here,
 we would like to apply the same idea, but for the incompressibility constraint. We
 propose, indeed, a novel explicit formulation to treat linear wave propagation in
 nearly-incompressible problems, such as follows

$$\underline{u}_h^{n+1} - 2\underline{u}_h^n + \underline{u}_h^{n-1} + \Delta t^2 M_h^{-1} A_h^s \underline{u}_h^n + \mathcal{R}_{m+1}^\nu (\Delta t^2 \lambda M_h^{-1} \bar{A}_h^p) \underline{u}_h^n = \Delta t^2 M_h^{-1} \underline{f}^n$$

where M_h is the lumped mass operator, A_h^s corresponds to the shear-stiffness opera-
 tor (independent of volumetric parameters), \bar{A}_h^p is a discretized Grad-div operator
 multiplied by the bulk modulus λ taken positive and very large, and \mathcal{R}_{m+1}^ν is the

damped Chebyshev polynomial of order $m + 1$, to be later presented. As explained in [Carle2020], it should be interpreted as a multi-rate scheme, such that we evaluate the shear part once, while the volumetric part is iterated as many times as the order $m + 1$ of the Chebyshev polynomial. The scheme is stable under the following condition

$$\Delta t^2 \leq \frac{1}{\rho(M_h^{-1}A_h^s)} \quad \text{and} \quad m + 1 \geq \sqrt{\lambda} \frac{\sqrt{\rho(\Delta t^2 M_h^{-1} \bar{A}_h^p)} e^{\nu/4}}{2}$$

116 where we denote $\rho(\cdot)$ as the spectral radius of the operator. We underline that the new
 117 time-step is now only dependent on the shear operator, while the incompressibility
 118 constraint determines the minimum order of the Chebyshev polynomial to guarantee
 119 stability.

120 This method has been developed in the perspective of reproducing elastographic
 121 measurements of biological tissues, for the analysis of the material properties and
 122 the detection of pathologies. These materials are precisely characterized by a high
 123 water content that makes them almost incompressible, and they exhibit hyperelastic
 124 (nonlinear) behavior. We must also consider that in a physiological framework, these
 125 tissues are generally subjected to certain loading conditions, which then generate a
 126 state of pre-stress and pre-deformation of the tissue that affects wave propagation
 127 during elastographic measurements. In the following work we have derived a novel
 128 formulation for linear wave propagation in nearly incompressible pre-loaded solids, for
 129 which we separate the Grad-div operator (volumetric term) from the stiffness operator.
 130 The isolation of the term related to the enforcement of incompressibility allows us to
 131 tackle the critical aspects related to accuracy, stability and efficiency that we have
 132 mentioned before with the presented Chebyshev Leapfrog scheme.

133 2 Modelling elastic wave propagation in nearly incompress- 134 ible non-linear solid

135 In this section we derive the continuous formulation of wave propagation problem in
 136 the framework of hyperelastic nearly-incompressible solids. In particular, we consider
 137 a small amplitude elastic wave propagating in the solid subjected to a quasi-static
 138 load with large deformations. We follow the procedure of linearization as described in
 139 [Destrade2007, Shams2011, Cafor2019These, Dalmora2023], with the primary
 140 aim of presenting the problem and the related notation. The second objective is to
 141 derive a weak formulation for linear wave propagation problems in nearly-incompressible
 142 preloaded solids, for which the volumetric term is separated from the stiffness operator
 143 and is written as a Grad-div operator.

144 2.1 Kinematics and the non-linear elastodynamic problem

We define $\Omega(t)$ as the domain occupied by a deformable solid at time t . We assume that its boundary $\Gamma(t)$ is sufficiently smooth, such that the outward normal $\underline{n}(t)$ is always defined. For the formulation of the non-linear problem we use a Lagrangian description, which consists in representing the position of points in $\Omega(t)$ with respect to a reference configuration $\hat{\Omega}$. All the material points in $\hat{\Omega}$ are described by the coordinate system $\underline{\xi}$ and we can retrieve the *current* (at time t) position of points in $\Omega(t)$ with the bijective deformation map $\underline{\phi}(\underline{\xi}, t)$:

$$\underline{\phi} : \underline{\xi} \rightarrow \underline{x}(\underline{\xi}, t) = \underline{\phi}(\underline{\xi}, t), \quad \underline{\phi}(\hat{\Omega}) = \Omega(t)$$

From the definition of the bijective map, we define the *displacement field*

$$\underline{u}(\underline{\xi}, t) = \underline{\phi}(\underline{\xi}, t) - \underline{\xi} = \underline{x}(\underline{\xi}, t) - \underline{\xi}$$

and the deformation gradient with its determinant

$$\underline{F}(\underline{u}(\underline{\xi}, t)) = \underline{\nabla}_{\underline{\xi}} \underline{\phi} = \underline{\mathbb{1}} + \underline{\nabla}_{\underline{\xi}} \underline{u} \quad \text{and} \quad J(\underline{u}) = \det \underline{F}(\underline{u})$$

145 The volume element $d\Omega$ in the current configuration is then given by the quantity
 146 $J(\underline{u})d\hat{\Omega}$, while the current area element ds is equal to $J|\underline{F}^{-T} \cdot \underline{n}|d\hat{s}$. We suggest
 147 the reader to look into [Ciarlet1988] for details. Before introducing the relevant
 148 mechanical quantities to describe the deformation of the solid, we underline that for
 149 second order tensors we use the dot-product and double dot-product defined as follows

$$\underline{A} \cdot \underline{B} = A_{ik} B_{kj} \underline{e}_i \otimes \underline{e}_j, \quad \underline{A} : \underline{B} = \text{tr}(A^T \cdot B) = A_{ij} B_{ij}$$

We then introduce the Cauchy-Green deformation tensor $\underline{C}(\underline{u}) = \underline{F}(\underline{u})^T \cdot \underline{F}(\underline{u})$,
 the Green-Lagrange strain tensor $\underline{\underline{e}}$ and its linearized expression $\underline{\underline{\varepsilon}}$

$$\underline{\underline{e}}(\underline{u}) = \frac{1}{2}(\underline{C} - \underline{\mathbb{1}}) = \frac{1}{2}(\underline{\nabla}_{\underline{\xi}} \underline{u} + \underline{\nabla}_{\underline{\xi}} \underline{u}^T + \underline{\nabla}_{\underline{\xi}} \underline{u}^T \cdot \underline{\nabla}_{\underline{\xi}} \underline{u}), \quad \underline{\underline{\varepsilon}}(\underline{u}) = \frac{1}{2}(\underline{\nabla}_{\underline{\xi}} \underline{u} + \underline{\nabla}_{\underline{\xi}} \underline{u}^T)$$

150 Fundamental law of dynamics

151 Before presenting the functional spaces in which we will seek the displacement solutions,
 152 we define the boundary $\Gamma(t) = \Gamma_D \cup \Gamma_N(t)$, where $\Gamma_N(t)$ is the part of the boundary on
 153 which we have a free surface condition, while on the rest of the boundary we consider
 154 Dirichlet's condition, meaning that the displacement vanishes. Then, the functional
 155 spaces of the displacement field are defined as follows

$$\mathcal{V}(\hat{\Omega}) := \{\hat{v} \in H^1(\hat{\Omega})^d \mid \hat{v} = 0 \text{ on } \Gamma_D\}, \quad \mathcal{V}(\Omega) := \{v \in H^1(\Omega)^d \mid v = 0 \text{ on } \Gamma_D\}$$

156 with dimension $d = 2$ or $d = 3$, and

$$\mathcal{W}(\hat{\Omega}) \subset \mathcal{V}(\hat{\Omega}), \quad \mathcal{W}(\Omega) \subset \mathcal{V}(\Omega)$$

157 with \mathcal{W} as the space of admissible displacements for the non-linear problem, which
 158 ensures the existence of the solution. We consider that the solid is subjected to a force
 159 \underline{f} per unit mass. Given the observation time T , we write the standard weak formulation
 160 associated to the wave propagation problem defined in the current configuration Ω :

161 For any $t \in [0, T]$, find $\underline{u}(t) \in \mathcal{W}(\Omega)$ such that

$$\int_{\Omega} \varrho(\partial_{tt} \underline{u} \circ \underline{\phi}^{-1} - \underline{f}) \cdot \underline{v} \, d\Omega + \int_{\Omega} \underline{\underline{\sigma}} : \underline{\underline{\nabla}}_{\underline{x}} \underline{v} \, d\Omega = 0 \quad \forall \underline{v} \in \mathcal{V}(\Omega), \quad (1)$$

162 where ϱ is the density in the current configuration and $\underline{\underline{\sigma}}$ is the Cauchy stress tensor
 163 defined in $\Omega(t)$. In classical continuum mechanics, this problem is named *principle of*
 164 *virtual work* and it is formally equivalent to the *equilibrium equation* with boundary
 165 condition of traction in the current configuration $\Omega(t)$. The focus of this work is the
 166 description of the wave propagation phenomena in materials with non-linear behavior,
 167 namely hyperelastic materials, so that a constitutive law must be introduced to relate
 168 internal stress and deformation of the solid due to external loads. Furthermore, we
 169 remark that the current configuration in which the problem is written is unknown, so
 170 that it is classical to express the problem in a total Lagrangian form with respect to the

171 reference configuration. We may neglect some details as it is a classical development,
 172 for which we refer again the reader to [Ciarlet1988].

173 As a first step towards the Lagrangian form, we remark that for any quantity $\underline{w}(\xi)$
 174 defined in $\widehat{\Omega}$, we have that

$$\int_{\Omega} \underline{w} \circ \phi^{-1}(\underline{x}) \, d\Omega = \int_{\widehat{\Omega}} J \underline{w}(\underline{\xi}) \, d\widehat{\Omega} \quad (2)$$

175 and the gradient of a field in the reference configuration can be rewritten in the current
 176 configuration using

$$(\underline{\nabla}_{\underline{\xi}} \underline{w}) \circ \phi^{-1} = \underline{\nabla}_{\underline{x}} (\underline{w} \circ \phi^{-1}) \cdot (\underline{F} \circ \phi^{-1}). \quad (3)$$

177 We rewrite first the inertial term in $\widehat{\Omega}$

$$\int_{\Omega} \varrho (\partial_{tt} \underline{u} \circ \phi^{-1} - \underline{f}) \cdot \underline{v} \, d\Omega = \int_{\widehat{\Omega}} \varrho \circ \phi(\underline{\xi}) (\partial_{tt} \underline{u} - \underline{f} \circ \phi(\underline{\xi})) \cdot \widehat{v} J \, d\widehat{\Omega} = \int_{\widehat{\Omega}} \widehat{\varrho} (\partial_{tt} \underline{u} - \widehat{f}) \cdot \widehat{v} \, d\widehat{\Omega}. \quad (4)$$

178 where we note the mass density in the reference configuration $\widehat{\varrho} : \widehat{\Omega} \rightarrow \mathbb{R}$, and is given
 179 by $\widehat{\varrho}(\underline{\xi}) = J \varrho \circ \phi$. We also define the body force in the reference configuration as
 180 $\widehat{f} := \underline{f} \circ \phi(\underline{\xi})$.

181 For the stress tensor, we introduce the Second Piola Kirchhoff tensor $\underline{\underline{\Sigma}}$, which
 182 corresponds to the symmetric stress tensor defined in the reference configuration,
 183 related to the Cauchy stress tensor by the following relation:
 184

$$\underline{\underline{\sigma}} = J^{-1} \underline{F} \cdot \underline{\underline{\Sigma}} \cdot \underline{F}^T. \quad (5)$$

We have

$$\int_{\Omega} \underline{\underline{\sigma}} : \underline{\nabla}_{\underline{x}} \underline{v} \, d\Omega = \int_{\widehat{\Omega}} J^{-1} \underline{F} \cdot \underline{\underline{\Sigma}} \cdot \underline{F}^T : \underline{\nabla}_{\underline{\xi}} \widehat{v} \cdot \underline{F}^{-1} J \, d\widehat{\Omega} = \int_{\widehat{\Omega}} \underline{\underline{\Sigma}} : \underline{F}^T \cdot \underline{\nabla}_{\underline{\xi}} \widehat{v} \, d\widehat{\Omega}.$$

185 We then introduce the Gateaux derivative of the Green-Lagrange strain tensor \underline{e}
 186 at \underline{u} in the direction \widehat{v} given by the following equation

$$D\underline{e}(\underline{u}) \cdot \widehat{v} = \frac{1}{2} \left(\underline{\nabla}_{\underline{\xi}} \widehat{v}^T \cdot \underline{F}(\underline{u}) + \underline{F}(\underline{u})^T \cdot \underline{\nabla}_{\underline{\xi}} \widehat{v} \right) \quad (6)$$

for which we recall the definition of the Gateaux derivative

$$DA(\underline{u}) \cdot \underline{w} = \lim_{h \rightarrow 0} \frac{A(\underline{u} + h\underline{w}) - A(\underline{u})}{h}.$$

187 As $\underline{\underline{\Sigma}}$ is a symmetric tensor, we can consider the contraction with the symmetrized
 188 version of $\underline{F} \cdot \underline{\nabla}_{\underline{\xi}}$, that corresponds to

$$\int_{\widehat{\Omega}} \underline{\underline{\Sigma}} : \underline{F}^T \cdot \underline{\nabla}_{\underline{\xi}} \widehat{v} \, d\widehat{\Omega} = \int_{\widehat{\Omega}} \underline{\underline{\Sigma}} : \frac{1}{2} \left(\underline{F}^T \cdot \underline{\nabla}_{\underline{\xi}} \widehat{v} + \underline{\nabla}_{\underline{\xi}} \widehat{v}^T \cdot \underline{F} \right) \, d\widehat{\Omega} = \int_{\widehat{\Omega}} \underline{\underline{\Sigma}} : D\underline{e}(\underline{u}) \cdot \widehat{v} \, d\widehat{\Omega}. \quad (7)$$

189 Combining (4) and (7) we obtain the *weak formulation in the reference configuration*,
 190 that reads:

191 For any $t \in [0; T]$, find $\underline{u} \in \mathcal{W}$ such that

$$\int_{\widehat{\Omega}} \widehat{\varrho} (\partial_{tt} \underline{u} - \widehat{f}) \cdot \widehat{v} \, d\widehat{\Omega} + \int_{\widehat{\Omega}} \underline{\underline{\Sigma}} : D\underline{e}(\underline{u}) \cdot \widehat{v} \, d\widehat{\Omega} = \int_{\widehat{\Gamma}_N} \widehat{t} \cdot \widehat{v} \, d\widehat{s} \quad \forall \widehat{v} \in \mathcal{V}(\widehat{\Omega}) \quad (8)$$

In the framework of hyperelastic materials, the state of stress within the solid depends on the *hyperelastic potential* $\mathcal{W}^e = \mathcal{W}^e(\underline{\underline{C}})$ and, in the case of *isotropic* material, we have

$$\mathcal{W}^e = \mathcal{W}^e(I_1, I_2, I_3) \quad s.t. \quad \underline{\underline{\Sigma}} = \frac{\partial \mathcal{W}^e}{\partial \underline{\underline{e}}}(I_1, I_2, I_3) = 2 \sum_{i=1}^3 \frac{\partial \mathcal{W}^e}{\partial I_i} \frac{\partial I_i}{\partial \underline{\underline{C}}}$$

where I_1, I_2, I_3 are the eigenvalues of $\underline{\underline{C}}(\underline{u})$ known as *invariants*

$$I_1 = \text{tr} \underline{\underline{C}}, \quad I_2 = \text{tr}(\text{cof} \underline{\underline{C}}) = \frac{1}{2}((\text{tr} \underline{\underline{C}})^2 - \text{tr}(\underline{\underline{C}}^2)), \quad I_3 = \det \underline{\underline{C}} = J^2.$$

192 The application of this work concerns hyperelastic nearly-incompressible materials,
193 like soft tissues characterized by high water content. A classical decomposition of the
194 Second Piola Kirchhoff tensor $\underline{\underline{\Sigma}}$ is given by the definition of the hydrostatic pressure
195 p and the deviatoric component $\underline{\underline{\Sigma}}^d$

$$p = -\frac{1}{3J} \underline{\underline{\Sigma}} : \underline{\underline{C}} \quad \text{and} \quad \underline{\underline{\Sigma}}^d = \underline{\underline{\Sigma}} + pJ\underline{\underline{C}}^{-1} \quad s.t. \quad \underline{\underline{\Sigma}} = \underline{\underline{\Sigma}}^d - pJ\underline{\underline{C}}^{-1}. \quad (9)$$

196 The stress tensor is thereby decomposed in its deviatoric and volumetric term, and in
197 particular we have $\underline{\underline{\Sigma}}^d : \underline{\underline{C}} = 0$ (volumetric deformation do not generate any work from
198 the deviatoric component).

199 The nearly-incompressibility is taken into account with an additional strain energy
200 density that constraints volume changes. This term can be interpreted as a dilatation
201 strain energy function that depends only on I_3 , for which a small variation in volume
202 is penalized by a large bulk modulus. However, to achieve a separation in the energy
203 density function between the component related to distortion and dilatation deformation
204 [Penn1970, Chen1996], we cannot use I_1 and I_2 as they are not fully isochoric. We
205 introduce the *reduced invariants*

$$J_1 = I_1 I_3^{-1/3}, \quad J_2 = I_2 I_3^{-2/3}, \quad J_3 = I_3^{1/2} = J \quad (10)$$

and we do the classical splitting

$$\mathcal{W}^e(I_1, I_2, I_3) = \mathcal{W}^d(J_1, J_2) + \mathcal{W}^v(J),$$

where $\mathcal{W}^d(J_1, J_2)$ and $\mathcal{W}^v(J_3)$ correspond respectively to the elastic potential referred to deviatoric and volumetric deformations. We suggest the reader to look in [Chang1991] for further details. From definitions (9), we derive that

$$p(J) = -\frac{d\mathcal{W}^v(J)}{dJ} \quad \text{and} \quad \underline{\underline{\Sigma}}^d = 2 \sum_{i=1}^2 \frac{\partial \mathcal{W}^d}{\partial J_i} \frac{\partial J_i}{\partial \underline{\underline{C}}}.$$

206 Consequently, we can specify an equivalent formulation of (8), with the additional
207 term related to volumetric deformation. It reads:

208 For any $t \in [0; T]$ find $\underline{u}(t) \in \mathcal{W}$ such that

$$\begin{aligned} \int_{\hat{\Omega}} \hat{p} \partial_{tt} \underline{u} \cdot \hat{v} \, d\hat{\Omega} + \int_{\hat{\Omega}} \left(\underline{\underline{\Sigma}}^d(\underline{\underline{e}}(\underline{u})) - pJ\underline{\underline{C}}^{-1} \right) : D\underline{\underline{e}}(\underline{u}) \cdot \hat{v} \, d\hat{\Omega} \\ = \int_{\hat{\Omega}} \hat{p} \hat{f} \cdot \hat{v} \, d\hat{\Omega} + \int_{\hat{\Gamma}_N} \hat{t} \cdot \hat{v} \, d\hat{s} \end{aligned} \quad \forall \hat{v} \in \mathcal{V}(\hat{\Omega}). \quad (11)$$

209 We recall that in (11) the terms p, J and $\underline{\underline{C}}$ are function of the unknown solution \underline{u} ,
210 while \hat{p}, \hat{f} and \hat{t} are given parameters.

2.2 Elastic wave propagation in quasi-static loaded solid

Let us now consider that the load applied to the solid is defined as follows

$$\widehat{f}(\underline{\xi}, t) = \widehat{f}_0(\underline{\xi}) + \delta \widetilde{f}(\underline{\xi}, t) \quad (12)$$

The solid undergoes large deformation due to the application of the quasi-static load $\widehat{f}_0(\underline{\xi})$, and it is subjected to a transient small-amplitude perturbation $\delta \widetilde{f}(\underline{\xi}, t)$ with δ as a small parameter, that generates the wave propagation. Solutions can be written as

$$\underline{u}(\underline{\xi}, t) = \underline{u}_0(\underline{\xi}) + \delta \widetilde{u}(\underline{\xi}, t) + \mathcal{O}(\delta^2), \quad (13)$$

where \underline{u}_0 is the governing solution of the quasi-static non-linear mechanics and $\delta \widetilde{u}$ is the small-amplitude wave propagating in the loaded configuration.

2.2.1 Wave propagation problem in the reference configuration

With the external load defined as in (12) we derive the governing equation for the small perturbation \widetilde{u} , i.e. the wave propagation, by linearizing (8) around the static loaded configuration given by the displacement \underline{u}_0 .

We start the process of linearization with the component related to the deviatoric component of the internal work

$$\int_{\widehat{\Omega}} \underline{\underline{\Sigma}}^d(\underline{e}(\underline{u})) : \mathbb{D}\underline{e}(\underline{u}) \cdot \widehat{v} \, d\widehat{\Omega} \quad (14)$$

First, we linearize the derivative of the Green-Lagrange strain tensor

$$\mathbb{D}\underline{e}(\underline{u}) \cdot \widehat{v} = \mathbb{D}\underline{e}(\underline{u}_0) \cdot \widehat{v} + \delta \mathbb{D}^2 \underline{e}(\widetilde{u}, \widehat{v}) + \mathcal{O}(\delta^2) \quad (15)$$

with

$$\mathbb{D}^2 \underline{e}(\widetilde{u}, \widehat{v}) := \frac{1}{2} \left(\underline{\nabla}_{\underline{\xi}} \widetilde{u}^T \underline{\nabla}_{\underline{\xi}} \widehat{v} + \underline{\nabla}_{\underline{\xi}} \widehat{v}^T \underline{\nabla}_{\underline{\xi}} \widetilde{u} \right)$$

For the linearization of the deviatoric component of the stress tensor $\underline{\underline{\Sigma}}^d$, let us first introduce some useful notation that we will use for the rest of the paper

$$\mathbb{D}\underline{e}_0(\underline{w}) := \mathbb{D}\underline{e}_0(\underline{w}) \circ \phi(\underline{\xi}) := \mathbb{D}\underline{e}(\underline{u}_0) \cdot \underline{w} = \frac{1}{2} \left(\underline{\nabla}_{\underline{\xi}} \underline{w}^T \cdot \underline{F}_0 + \underline{F}_0^T \cdot \underline{\nabla}_{\underline{\xi}} \underline{w} \right) \quad (16)$$

$$\mathbf{C}_0^d : \underline{s} := \mathbb{D}\underline{\underline{\Sigma}}^d(\underline{e}(\underline{u}_0)) : \underline{s} \quad (\text{Gateaux derivative}) \quad (17)$$

where $\underline{F}_0 = \underline{F}(\underline{u}_0)$ and the first order derivative of \underline{e} evaluated in \underline{u}_0 in the direction \underline{w} is now denoted with $\mathbb{D}\underline{e}_0(\underline{w})$. The 4th-order tensor \mathbf{C}_0^d is characterized by minor and major symmetries, and it accounts for the deviatoric behavior of the solid at the quasi-static configuration defined by \underline{u}_0 . The linearization of the deviatoric stress reads

$$\underline{\underline{\Sigma}}^d(\underline{e}(\underline{u})) = \underline{\underline{\Sigma}}_0^d + \delta \widetilde{\underline{\underline{\Sigma}}}^d + \mathcal{O}(\delta^2) \quad \text{with} \quad \begin{cases} \underline{\underline{\Sigma}}_0^d := \underline{\underline{\Sigma}}^d(\underline{e}(\underline{u}_0)) \\ \widetilde{\underline{\underline{\Sigma}}}^d := \mathbf{C}_0^d : \mathbb{D}\underline{e}_0(\widetilde{u}) \end{cases} \quad (18)$$

Finally, due to the symmetry of $\underline{\underline{\Sigma}}^d$ and \mathbf{C}_0^d , we write

$$\begin{aligned} & \int_{\widehat{\Omega}} \underline{\underline{\Sigma}}^d(\underline{e}(\underline{u})) : \mathbb{D}\underline{e}(\underline{u}) \cdot \widehat{v} \, d\widehat{\Omega} = \\ & = \int_{\widehat{\Omega}} \underline{\underline{\Sigma}}_0^d : \mathbb{D}\underline{e}_0(\widehat{v}) \, d\widehat{\Omega} + \delta \left[\int_{\widehat{\Omega}} \mathbb{D}\underline{e}_0(\widetilde{u}) : \mathbf{C}_0^d : \mathbb{D}\underline{e}_0(\widehat{v}) \, d\widehat{\Omega} + \int_{\widehat{\Omega}} \underline{\underline{\Sigma}}_0^d : \underline{\nabla}_{\underline{\xi}} \widetilde{u}^T \underline{\nabla}_{\underline{\xi}} \widehat{v} \, d\widehat{\Omega} \right] + \mathcal{O}(\delta^2). \end{aligned}$$

Let us now present the linearization of the volumetric stress term

$$\int_{\hat{\Omega}} p J \underline{C}^{-1} : \underline{D} \underline{e}(\underline{u}) \cdot \hat{\underline{v}} \, d\hat{\Omega}.$$

We linearize all terms dependent on \underline{u} as follows

$$\begin{aligned} \underline{C}^{-1} &= \underline{C}_0^{-1} + \delta \tilde{\underline{G}} & \text{with} & \begin{cases} \underline{C}_0^{-1} = \underline{C}(\underline{u}_0)^{-1} \\ \tilde{\underline{G}} = -2 \underline{C}_0^{-1} : \underline{D} \underline{e}_0(\tilde{\underline{u}}) \cdot \underline{C}_0^{-1} \end{cases} \\ J &= J_0 + \delta \tilde{J} & \text{with} & \begin{cases} J_0 = [\det(\underline{C}_0)]^{1/2} \\ \tilde{J} = J_0 \underline{C}_0^{-1} : \underline{D} \underline{e}_0(\tilde{\underline{u}}) \end{cases} \\ p &= p_0 + \delta \tilde{p} & \text{with} & \begin{cases} p_0 = p(J_0) = -\frac{d\mathcal{W}^v}{dJ}(J_0) \\ \tilde{p} = \frac{dp}{dJ}(J_0) \tilde{J} = -\frac{d^2\mathcal{W}^v}{dJ^2}(J_0) J_0 \underline{C}_0^{-1} : \underline{D} \underline{e}_0(\tilde{\underline{u}}). \end{cases} \end{aligned}$$

227 From these results and the previous linearization of the deformation tensor, the
228 volumetric term reads

$$\begin{aligned} \int_{\hat{\Omega}} p J \underline{C}^{-1} : \underline{D} \underline{e}(\underline{u}) \cdot \hat{\underline{v}} \, d\hat{\Omega} &= \int_{\hat{\Omega}} p_0 J_0 \underline{C}_0^{-1} : \underline{D} \underline{e}_0(\hat{\underline{v}}) \, d\hat{\Omega} + \\ &+ \delta \left[\int_{\hat{\Omega}} (\tilde{p} J_0 + p_0 \tilde{J}) \underline{C}_0^{-1} : \underline{D} \underline{e}_0(\hat{\underline{v}}) \, d\hat{\Omega} + \int_{\hat{\Omega}} p_0 J_0 \tilde{\underline{G}} : \underline{D} \underline{e}_0(\hat{\underline{v}}) \, d\hat{\Omega} \right. \\ &\left. + \int_{\hat{\Omega}} p_0 J_0 \underline{C}_0^{-1} : \underline{\nabla}_{\xi} \tilde{\underline{u}}^T \underline{\nabla}_{\xi} \hat{\underline{v}} \, d\hat{\Omega} \right] + \mathcal{O}(\delta^2). \end{aligned} \quad (20)$$

The linearization of the inertial term takes into account the static condition of the non-linear solution, hence we have:

$$\int_{\hat{\Omega}} \hat{\rho} \partial_{tt} \underline{u} \cdot \hat{\underline{v}} \, d\hat{\Omega} = \delta \int_{\hat{\Omega}} \hat{\rho} \partial_{tt} \tilde{\underline{u}} \cdot \hat{\underline{v}} \, d\hat{\Omega}.$$

229 Let us now separate the non-linear static part from the linear part of the problem,
230 in order to distinguish the different physics of the problem for which we will need
231 different methods of approximation. We also assume no surface traction for the sake
232 of simplicity ($\underline{t} = 0$).

Quasi-static problem *The non-linear problem in the reference configuration reads:*

Find $\underline{u}_0 \in \mathcal{W}(\hat{\Omega})$ such that

$$\begin{cases} \int_{\hat{\Omega}} \left(\underline{\Sigma}_0^d - p_0 J_0 \underline{C}_0^{-1} \right) : \underline{D} \underline{e}_0(\hat{\underline{v}}) \, d\hat{\Omega} = \int_{\hat{\Omega}} \hat{\rho} \hat{\underline{f}}_0 \cdot \hat{\underline{v}} \, d\hat{\Omega} \quad \forall \hat{\underline{v}} \in \mathcal{V}(\hat{\Omega}) \\ p_0 = -\frac{d\mathcal{W}^v}{dJ}(J_0) \end{cases} \quad (21)$$

233

234 We recall that the Second Piola Kirchhoff evaluated at \underline{u}_0 , that will be referred to
235 as the prestress is

$$\underline{\Sigma}_0 := \underline{\Sigma}(\underline{u}_0) = \underline{\Sigma}_0^d - p_0 J_0 \underline{C}_0^{-1}. \quad (22)$$

236 Once the non-linear solution is given, we obtain the wave propagation in the quasi-static
 237 loaded solid by solving the linearized problem, following

Wave equation in reference configuration For any $t \in [0, T]$, find $\tilde{\underline{u}}(t) \in \mathcal{V}(\hat{\Omega})$ such that

$$\left\{ \begin{array}{l} \int_{\hat{\Omega}} \hat{\rho} \partial_{tt} \tilde{\underline{u}} \cdot \hat{\underline{v}} \, d\hat{\Omega} + \int_{\hat{\Omega}} \mathbf{D}_{\underline{e}_0}(\tilde{\underline{u}}) : \mathbf{C}_0^d : \mathbf{D}_{\underline{e}_0}(\hat{\underline{v}}) \, d\hat{\Omega} \\ \quad + 2 \int_{\hat{\Omega}} p_0 J_0 \underline{C}_0^{-1} \cdot \mathbf{D}_{\underline{e}_0}(\tilde{\underline{u}}) : \underline{C}_0^{-1} \cdot \mathbf{D}_{\underline{e}_0}(\hat{\underline{v}}) \, d\hat{\Omega} \\ \quad + \int_{\hat{\Omega}} \lambda^* \underline{C}_0^{-1} : \mathbf{D}_{\underline{e}_0}(\tilde{\underline{u}}) \underline{C}_0^{-1} : \mathbf{D}_{\underline{e}_0}(\hat{\underline{v}}) \, d\hat{\Omega} \\ \quad + \int_{\hat{\Omega}} \underline{\Sigma}_0 : \underline{\nabla}_{\underline{\xi}} \tilde{\underline{u}}^T \underline{\nabla}_{\underline{\xi}} \hat{\underline{v}} \, d\hat{\Omega} = \int_{\hat{\Omega}} \hat{\underline{f}} \cdot \hat{\underline{v}} \, d\hat{\Omega} \end{array} \right. \quad \forall \hat{\underline{v}} \in \mathcal{V}(\hat{\Omega}) \quad (23)$$

where

$$\lambda^* := -J_0^2 \frac{dp}{dJ}(J_0) - J_0 p_0 = J_0 \left(J_0 \frac{d^2 \mathcal{W}^v}{dJ^2}(J_0) + \frac{d\mathcal{W}^v}{dJ}(J_0) \right)$$

with initial condition

$$\tilde{\underline{u}}(0) = 0, \quad \partial_t \tilde{\underline{u}}(0) = 0$$

238

239 We remark that the parameter λ^* will have a high value as the bulk modulus of
 240 the potential \mathcal{W}^v is taken very large to enforce the incompressibility.

241 2.2.2 Wave propagation problem in the apparent configuration

In an application framework, the wave propagation is measured in the loaded configuration described by \underline{u}_0 . It is useful, indeed, to reformulate the linearized problem in the quasi-static loaded configuration denoted with Ω_0 , whose set of points \underline{x}_0 can be retrieved with the bijective deformation map $\phi_0(\underline{\xi})$:

$$\phi_0 : \underline{\xi} \rightarrow \underline{x}_0(\underline{\xi}) = \phi_0(\underline{\xi})$$

$$\phi_0(\underline{\xi}) = \underline{u}_0(\underline{\xi}) + \underline{\xi}$$

242

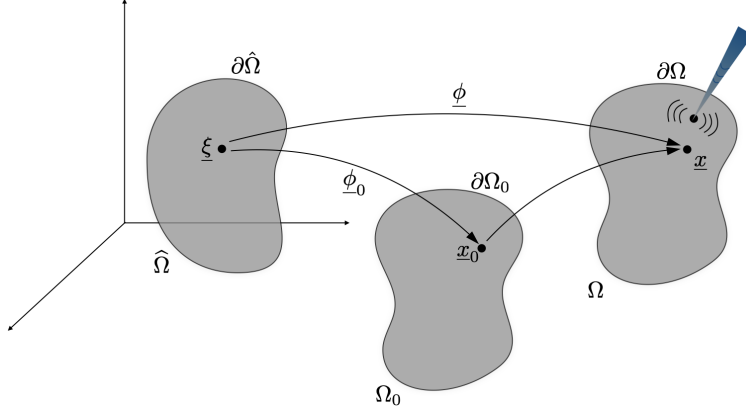


Figure 1. Reference $\hat{\Omega}$, apparent Ω_0 and deformed Ω configurations related by the respective deformation maps

243 We also introduce the functional space $\mathcal{V}(\Omega_0) := \{v \in H^1(\Omega_0)^d\}$ where we seek the
 244 solution of the linear solution. Given a function \underline{w} defined in Ω_0 , we can define the
 245 linear strain tensor in the apparent configuration as

$$\underline{\underline{\varepsilon}}_{\underline{x}_0}(\underline{w}) = \frac{1}{2}(\underline{\nabla}_{\underline{x}_0} \underline{w} + \underline{\nabla}_{\underline{x}_0}^T \underline{w}^T). \quad (24)$$

We now want to refer the mechanical quantities of the wave equation (23) with respect to Ω_0 . We combine (3) and (16) to write the first order derivative of Green-Lagrange strain tensor evaluated in \underline{u}_0 w.r.t. the apparent configuration Ω_0

$$(\underline{D}_{\underline{e}_0}(\underline{w} \circ \underline{\phi}_0) \circ \underline{\phi}_0^{-1})(\underline{x}_0) = \frac{1}{2} \left(\underline{F}_0^T \cdot \underline{\nabla}_{\underline{x}_0} \underline{w}^T \cdot \underline{F}_0 + \underline{F}_0^T \cdot \underline{\nabla}_{\underline{x}_0} \underline{w} \cdot \underline{F}_0 \right) = \underline{F}_0^T \cdot \underline{\underline{\varepsilon}}_{\underline{x}_0}(\underline{w}) \cdot \underline{F}_0$$

From this relation one can derive thw two equation below that are used to treat the terms related to the linearization of the volumetric work, in the apparent configuration we have

$$\begin{aligned} (\underline{C}_0^{-1} : \underline{D}_{\underline{e}_0}(\underline{w} \circ \underline{\phi}_0) \circ \underline{\phi}_0^{-1}) &= \underline{\underline{1}} : \underline{\underline{\varepsilon}}_{\underline{x}_0}(\underline{w}) = \text{div}_{\underline{x}_0} \underline{w}, \\ (\underline{C}_0^{-1} \cdot \underline{D}_{\underline{e}_0}(\tilde{\underline{u}}) : \underline{C}_0^{-1} \cdot \underline{D}_{\underline{e}_0}(\underline{w} \circ \underline{\phi}_0) \circ \underline{\phi}_0^{-1}) &= \underline{\underline{\varepsilon}}_{\underline{x}_0}(\tilde{\underline{u}} \circ \underline{\phi}_0^{-1}) : \underline{\underline{\varepsilon}}_{\underline{x}_0}(\underline{w}). \end{aligned}$$

With the relation between the Cauchy tensor and the second Piola Kirckhoff (5), we bring in the apparent configuration the term depending on the pre-stress $\underline{\underline{\Sigma}}_0$

$$(\underline{\underline{\Sigma}}_0 : \underline{\nabla}_{\underline{\xi}} \tilde{\underline{u}}^T \underline{\nabla}_{\underline{\xi}}(\underline{w} \circ \underline{\phi}_0)) \circ \underline{\phi}_0^{-1} = J_0 \underline{\nabla}_{\underline{x}_0}(\tilde{\underline{u}} \circ \underline{\phi}_0^{-1}) \cdot \underline{\underline{\sigma}}_0 : \underline{\nabla}_{\underline{x}_0} \underline{w}$$

$$\text{with } \underline{\underline{\sigma}}_0 := \underline{\underline{\sigma}}(\underline{u}_0) \circ \underline{\phi}_0^{-1} = (J_0^{-1} \underline{F}_0 \cdot \underline{\underline{\Sigma}}_0 \cdot \underline{F}_0^T) \circ \underline{\phi}_0^{-1}$$

246 For the sake of lightening the reading, we will now drop the notation " $\cdot \circ \underline{\phi}_0^{-1}$ ".

247

248 We now use (2) considering the deformation map $\underline{\phi}_0$ between $\hat{\Omega}$ and Ω_0 , in order
 249 to obtain the formulation of the elastic wave propagation problem in the apparent
 250 configuration, which reads:

Wave equation in apparent configuration Ω_0 : For any $t \in [0, T]$, find $\tilde{\underline{u}}(t) \in \mathcal{V}(\Omega_0)$ such that

$$\left\{ \begin{array}{l} \int_{\Omega_0} \varrho_0 \partial_{tt} \tilde{\underline{u}} \cdot \underline{v} \, d\Omega_0 + \int_{\Omega_0} \underline{\underline{\varepsilon}}_{\underline{x}_0}(\tilde{\underline{u}}) : \tilde{\underline{\mathbf{C}}}_0 : \underline{\underline{\varepsilon}}_{\underline{x}_0}(\underline{v}) \, d\Omega_0 \\ + \int_{\Omega_0} \underline{\underline{\nabla}}_{\underline{x}_0} \tilde{\underline{u}} \underline{\underline{\sigma}}_0 : \underline{\underline{\nabla}}_{\underline{x}_0} \underline{v} \, d\Omega_0 \\ + \int_{\Omega_0} \lambda_0^* \operatorname{div}_{\underline{x}_0}(\tilde{\underline{u}}) \operatorname{div}_{\underline{x}_0}(\underline{v}) \, d\Omega_0 = \int_{\Omega_0} \varrho_0 \tilde{\underline{f}} \cdot \underline{v} \, d\Omega_0 \end{array} \right. \quad \forall \underline{v} \in \mathcal{V}(\Omega_0) \quad (25)$$

where

$$\begin{aligned} \tilde{\underline{\mathbf{C}}}_0 : \underline{\underline{s}} &:= J_0^{-1} \underline{\underline{F}}_0 \left(\underline{\underline{\mathbf{C}}}_0^d : \underline{\underline{F}}_0^T \underline{\underline{s}} \underline{\underline{F}}_0 \right) \underline{\underline{F}}_0^T + 2 p_0 \underline{\underline{s}} \\ \lambda_0^* &:= J_0^{-1} \lambda^* = J_0 \frac{d^2 \mathcal{W}^v}{dJ^2}(J_0) + \frac{d\mathcal{W}^v}{dJ}(J_0) \\ \varrho_0 &:= J_0^{-1} \hat{\varrho} \end{aligned} \quad (26)$$

with initial condition

$$\tilde{\underline{u}}(0) = 0 \quad \partial_t \tilde{\underline{u}}(0) = 0$$

251

252 We can then distinguish three stiffness terms that affect the propagation of the
253 wave:

- the material stiffness given by the linearized material behavior around a loaded configuration

$$\int_{\Omega_0} \underline{\underline{\varepsilon}}_{\underline{x}_0}(\tilde{\underline{u}}) : \tilde{\underline{\mathbf{C}}}_0 : \underline{\underline{\varepsilon}}_{\underline{x}_0}(\underline{v}) \, d\Omega_0$$

- the geometric stiffness depending on the prestress resulting from the non-linear static problem

$$\int_{\Omega_0} \underline{\underline{\nabla}}_{\underline{x}_0} \tilde{\underline{u}} \underline{\underline{\sigma}}(\underline{u}_0) : \underline{\underline{\nabla}}_{\underline{x}_0} \underline{v} \, d\Omega_0$$

- the volumetric stiffness that corresponds to the Grad-div operator

$$\int_{\Omega_0} \lambda^* \operatorname{div}_{\underline{x}_0}(\tilde{\underline{u}}) \operatorname{div}_{\underline{x}_0}(\underline{v}) \, d\Omega_0$$

254

255

where λ^* only depends on the chosen energy density function to penalize volumetric deformations.

256

257

258

259

As we mentioned at the beginning, the formulation of the problem in the apparent configuration Ω_0 is useful to isolate the term related to the nearly-incompressibility and develop a numerical strategy capable of overcoming the limitations related to this constraint.

260

2.2.3 An example of strain energy function for nearly incompressible material
Let us now consider an example of penalization term for nearly-incompressible solids, as the one proposed in [Ciarlet1982lois]

$$\mathcal{W}^v = k(J - 1) - k \log(J)$$

where $k \in \mathbb{R}^+$ is the bulk modulus taken very large with respect to other material parameters. The stress related to the volumetric deformation will be defined

$$\underline{\underline{\Sigma}}^v = -pJ\underline{\underline{C}}^{-1} \quad \text{with} \quad p = k \frac{1-J}{J}$$

and its linearization results in

$$\underline{\underline{\Sigma}}^v = -pJ\underline{\underline{C}}^{-1} = k[(J_0\underline{\underline{C}}_0^{-1} - \underline{\underline{C}}_0^{-1}) + \delta((J_0 - 1)\tilde{\underline{\underline{G}}} + \tilde{J}\underline{\underline{C}}_0^{-1})]$$

We derive from (21) and (23) that

$$p_0 = k \frac{1 - J_0}{J_0} \quad \text{and} \quad \lambda^* = k ,$$

such that the shear term will be characterized by the following material stiffness

$$\tilde{\underline{\underline{C}}}_0 : \underline{\underline{\varepsilon}}_{\underline{\underline{x}}_0}(\tilde{\underline{\underline{u}}}) := J_0^{-1} \underline{\underline{F}}_0 \left(\underline{\underline{C}}_0^d : \underline{\underline{F}}_0^T \underline{\underline{\varepsilon}}_{\underline{\underline{x}}_0}(\tilde{\underline{\underline{u}}}) \underline{\underline{F}}_0 \right) \underline{\underline{F}}_0^T + 2k \frac{1 - J_0}{J_0} \underline{\underline{\varepsilon}}_{\underline{\underline{x}}_0}(\tilde{\underline{\underline{u}}})$$

and the term related to the pressure propagation corresponds to

$$\int_{\Omega_0} k \operatorname{div}_{\underline{\underline{x}}_0}(\tilde{\underline{\underline{u}}}) \operatorname{div}_{\underline{\underline{x}}_0}(\underline{\underline{v}}) d\Omega_0 .$$

261 We remark that the internal pressure p_0 could be a source of instability, since it is
 262 negative for non-linear problems characterized by volumetric increase. Furthermore, it
 263 has been seen that wave propagation problems obtained by linearization may suffer
 264 from loss of coercivity, depending on the chosen elastic potential and the static solution
 265 $\underline{\underline{u}}_0$. To avoid a possible buckling issue, a check of the smallest eigenvalues of the
 266 stiffness operator should be performed. Further discussions concerning these aspects
 267 can be found in [Dalmora2023].

268 For the purpose of our problem, we will strongly penalize any volumetric deformation
 269 due to nearly-incompressibility, hence we can assume the total tangent stiffness term
 270 to be coercive.

271 3 A fully explicit numerical scheme

272 The scope of this section is to propose a numerical scheme for the computation of the
 273 elastic wave propagation problem in a nearly-incompressible linearized solid. We will
 274 present the scheme considering the solid at, rest in order to simplify the presentation
 275 and highlight the main aspects of the scheme. We remark that this assumption doesn't
 276 affect the applicability of the scheme, since the operator related to the enforcement
 277 of incompressibility (the grad-div operator) is not affected by the preload, beside the
 278 parameter λ^* which – although it may vary in space in general – is homogeneous
 279 for the specific case presented Section 2.2.3. In more complicated situations, since
 280 in practice the parameter λ^* is large it acts as a penalization parameter and can be
 281 approximated by a constant value. Therefore, in what follows we assume that

$$\lambda^* = \lambda \quad \text{with} \quad \nabla \lambda = 0.$$

282

283

284 Given the linearity of the problem it is equivalent to write equation with lagrangian
 285 or eulerian formalism, such that we will write the wave propagation problem in the
 286 configuration Ω . Once we have introduced the linear continuous operators associated
 287 to our problem, we will introduce a strategy for space discretization focused on the
 288 accuracy and the efficiency of the numerical approximation. Finally, a fully explicit
 289 method for the time-discretization is given. The method is based on a classical LF
 290 scheme together with the use of Chebyshev polynomials (as proposed in [Carle2020]),
 291 for the relaxation of the CFL condition in the novel application of nearly-incompressible
 292 solids.

293 3.1 Linearization at rest condition

294 For the presentation of the numerical scheme, we consider the case of a hyperelastic
 295 solid, linearized around at rest condition, such that the displacement field (13) reduces
 296 to $\underline{u}(\underline{x}, t) = \tilde{\underline{u}}(\underline{x}, t)$.

297 We rewrite the formulation (25) as a set of equations in \mathcal{V}' , the dual space of the
 298 Hilbert space

$$\mathcal{V} = \mathcal{V}(\Omega) = \{\underline{v} \in H^1(\Omega)^d \mid \underline{v} = 0 \text{ on } \Gamma_D\}.$$

Note that the scalar product on \mathcal{V} is the standard $H^1(\Omega)$ scalar product. Let us define
 the bilinear form $m : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$ and related linear continuous operator $M : \mathcal{V} \rightarrow \mathcal{V}'$ as

$$m(\underline{u}, \underline{v}) := \langle M\underline{u}, \underline{v} \rangle_{\mathcal{V}', \mathcal{V}} := \int_{\Omega} \underline{\rho} \underline{u} \cdot \underline{v} \, d\Omega.$$

299 defined in $L^2(\Omega)$ space. Notice that this bilinear form is symmetric and coercive in
 300 $L^2(\Omega)^3$.

The bilinear form $a : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$ is rewritten as the sum of two bilinear forms,
 one contributing to the shear-waves propagation $a_s : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$ and another one
 contributing to pressure wave propagation $a_p : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$. To these bilinear forms we
 associate linear continuous operators A^s and $A^p : \mathcal{V} \rightarrow \mathcal{V}'$ defined by

$$a_s(\underline{u}, \underline{v}) := \langle A^s \underline{u}, \underline{v} \rangle_{\mathcal{V}', \mathcal{V}} := \int_{\Omega} \underline{\underline{\varepsilon}}(\underline{u}) : \mathbf{C}^d : \underline{\underline{\varepsilon}}(\underline{v}) \, d\Omega,$$

$$a_p(\underline{u}, \underline{v}) := \langle A^p \underline{u}, \underline{v} \rangle_{\mathcal{V}', \mathcal{V}} := \int_{\Omega} \operatorname{div}(\underline{u}) \operatorname{div}(\underline{v}) \, d\Omega.$$

The bilinear form $a(\cdot, \cdot) = a_s(\cdot, \cdot) + a_p(\cdot, \cdot)$ is continuous and weakly coercive i.e.
 $\exists c, C \in \mathbb{R}^+$ such that

$$c \|\underline{v}\|_{\mathcal{V}}^2 \leq \|\underline{v}\|_{L^2(\Omega)}^2 + |a(\underline{v}, \underline{v})| \quad \text{and} \quad |a(\underline{u}, \underline{v})| \leq C \|\underline{u}\|_{\mathcal{V}} \|\underline{v}\|_{\mathcal{V}}$$

301 while the symmetry is easily verified by looking at definition of $a_s(\cdot, \cdot)$ and $a_p(\cdot, \cdot)$.
 302 Under these assumptions, the wave propagation problem of (25) linearized around at
 303 rest condition reads:

304 For any $t \in [0, T]$, find $\underline{u}(t) \in \mathcal{V}$ such that

$$M \partial_{tt} \underline{u} + A^s \underline{u} + \lambda A^p \underline{u} = \underline{f} \quad \text{in } \mathcal{V}' \quad \text{and} \quad \underline{u}(0) = \partial_t \underline{u}(0) = 0 \quad \text{in } \Omega. \quad (\text{NI})$$

305 Existence results of a unique solution for this problem are given for instance in
 306 [joly2007numerical].

Let us now present an equivalent formulation with the introduction of the pressure
 $p := \lambda \operatorname{div} \underline{u}$ as a second variable of the problem. We denote with \mathcal{L} the function space
 of the scalar pressure field

$$\mathcal{L} := \{q \in L^2(\Omega)\},$$

and we define the bilinear form $d : \mathcal{L} \times \mathcal{V} \rightarrow \mathbb{R}$, with the related divergence operator $D : \mathcal{V} \rightarrow \mathcal{L}$ and its adjoint gradient operator $D^* : \mathcal{L} \rightarrow \mathcal{V}'$, such that $\forall(\underline{u}, q) \in \mathcal{V} \times \mathcal{L}$

$$d(q, \underline{u}) := \langle D^* q, \underline{u} \rangle_{\mathcal{V}', \mathcal{V}} := (q, D \underline{u})_{\mathcal{L}} := \int_{\Omega} q \operatorname{div}(\underline{u}) \, d\Omega.$$

307 Note that the scalar product on \mathcal{L} is the standard $L^2(\Omega)$ scalar product. Then, the
308 *mixed formulation* reads: For any $t \in [0, T]$, find $(\underline{u}(t), p(t)) \in \mathcal{V} \times \mathcal{L}$ such that

$$\begin{cases} M \partial_{tt} \underline{u} + A^s \underline{u} + D^* p = \underline{f} & \text{in } \mathcal{V}' \\ D \underline{u} = \lambda^{-1} M^p p & \text{in } \mathcal{L} \end{cases} \quad \text{and} \quad \underline{u}(0) = \partial_t \underline{u}(0) = 0 \quad \text{in } \Omega \quad (\text{NIM})$$

309 with M^p is actually the identity map from $\mathcal{L} \rightarrow \mathcal{L}$. As we are dealing with a nearly-
310 incompressible solid, it is interesting to notice that this latter formulation corresponds
311 to a sort of penalization method for the resolution of the pure incompressible problem,
312 where the pressure p would be a Lagrange multiplier [Caforio2018].

313 Comparing (NI) and (NIM) one can observe that

$$A^p = D^* (M^p)^{-1} D. \quad (27)$$

314 This observation has a discrete counterpart as shown in the Section below.

3.2 Space discretization

3.2.1 Semi-discret formulation

317 The classical method to solve finite-element approximation problem is known as
318 displacement-based, which relies on the assumption that nodal displacements completely
319 define the displacement of the body and are indeed the unknowns of the problem
320 [Bathe2006, Sussman1987]. This nodal displacements belongs to finite-dimensional
321 space $\mathcal{V}_h \subset \mathcal{V}$ that is a space of piecewise smooth and globally continuous functions.
322 The space \mathcal{V}_h is equipped with the scalar product

$$(\underline{u}_h, \underline{v}_h)_{\mathcal{H}} := \int_{\Omega} \underline{u}_h \cdot \underline{v}_h \, d\Omega.$$

In order to derive the semi-discretized problem, we define the linear operators
 $M_h : \mathcal{V}_h \rightarrow \mathcal{V}_h$, $A_h^s : \mathcal{V}_h \rightarrow \mathcal{V}_h$ and $A_h^p : \mathcal{V}_h \rightarrow \mathcal{V}_h$ such that $\forall(\underline{u}_h, \underline{v}_h) \in \mathcal{V}_h \times \mathcal{V}_h$

$$(M_h \underline{u}_h, \underline{v}_h)_{\mathcal{H}} = \oint_{\Omega} \varrho \underline{u}_h \cdot \underline{v}_h \, d\Omega \quad (A_h^s \underline{u}_h, \underline{v}_h)_{\mathcal{H}} = \int_{\Omega} \mathbf{C}^d \underline{\underline{\varepsilon}}(\underline{u}_h) : \underline{\underline{\varepsilon}}(\underline{v}_h) \, d\Omega$$

$$(A_h^p \underline{u}_h, \underline{v}_h)_{\mathcal{H}} = \int_{\Omega} \operatorname{div} \underline{u}_h \operatorname{div} \underline{v}_h \, d\Omega$$

323 where the notation \oint stands for the use of a quadrature formula for the computation
324 of the integrals. The adequate quadrature formula to use depends on the choice of the
325 finite element space \mathcal{V}_h and is discussed later. The semi-discretized problem of (NI)
326 reads: For any $t \in [0, T]$, find $\underline{u}_h(t) \in \mathcal{V}_h$ such that

$$M_h \partial_{tt} \underline{u}_h + A_h^s \underline{u}_h + \lambda A_h^p \underline{u}_h = \underline{f}_h \quad \text{in } \mathcal{V}_h \quad (\text{NI}_d)$$

327 with null initial conditions and f_h an approximation of f in \mathcal{V}_h . However, it is well-
328 known in finite element methods that the approximation of nearly-incompressible
329 problems with a displacement based method suffers from several limitations, such as

330 need of highly-refined mesh, numerical locking (especially with low-order elements), ill-
 331 conditioning of the stiffness matrix and spurious pressure [LeTallec1994, Bathe2006,
 332 Sussman1987].

333 The classical technique to overcome these limitations is the *mixed method*, which
 334 relies on the interpolation of the pressure field in a separate finite element space,
 335 denoted here as $\mathcal{L}_h \subset \mathcal{L}$ equipped with the scalar product $(\cdot, \cdot)_{\mathcal{V}}$. Let us then consider
 336 the space discretization of the equivalent continuous problem defined with the mixed
 337 formulation (NIM). We introduce the discrete divergence operator $D_h : \mathcal{V}_h \rightarrow \mathcal{L}_h$ and
 338 its adjoint gradient operator $D_h^* : \mathcal{L}_h \rightarrow \mathcal{V}_h$ such that $\forall (\underline{u}_h, p_h) \in \mathcal{V}_h \times \mathcal{L}_h$

$$(D_h \underline{u}_h, p_h)_{\mathcal{L}} = (\underline{u}_h, D_h^* p_h)_{\mathcal{H}} \simeq \int_{\Omega} p_h \operatorname{div} \underline{u}_h \, d\Omega. \quad (28)$$

339 Note that we are very vague at this point on the actual definition of the operator D_h .
 340 A precise description is given later in the manuscript. The spatial discretization of
 341 problem (NIM) reads:

342 For any time $t \in [0, T]$, find $(\underline{u}_h(t), p_h(t)) \in \mathcal{V}_h \times \mathcal{L}_h$ such that

$$\begin{cases} M_h \partial_{tt} \underline{u}_h + A_h^s \underline{u}_h + D_h^* p_h = \underline{f}_h & \text{in } \mathcal{V}_h \\ D_h \underline{u}_h = \lambda^{-1} M_h^P p_h & \text{in } \mathcal{L}_h \end{cases} \quad (29)$$

343 with null initial data and where M_h^P is the linear operator $\mathcal{L}_h \rightarrow \mathcal{L}_h$ given by

$$\forall (p_h, q_h) \in \mathcal{L}_h \times \mathcal{L}_h, \quad (M_h^P p_h, q_h)_{\mathcal{L}} = \oint_{\Omega} p_h q_h \, d\Omega,$$

where – once again – a quadrature formula, adapted to the space \mathcal{L}_h is used to compute
 the integral. By means of *Schur complement* method, we first compute the pressure
 field

$$p_h = \lambda (M_h^P)^{-1} D_h \underline{u}_h$$

344 and we obtain the *semi-discretized* scheme

$$M_h \partial_{tt} \underline{u}_h + A_h^s \underline{u}_h + \lambda \bar{A}_h^P \underline{u}_h = \underline{f}_h \quad \text{with} \quad \bar{A}_h^P := D_h^* (M_h^P)^{-1} D_h. \quad (\text{NIMd})$$

345 First observe that \bar{A}_h^P is define using a discrete counterpart of the formula (27) for
 346 defining A^P . However – in general – the operator \bar{A}_h^P differs from A_h^P unless some
 347 specific conditions are met. For instance, the two operators coincide when integral are
 348 computed exactly and $\operatorname{div} \mathcal{V}_h \subset \mathcal{L}_h$.

349 3.2.2 Spectral finite elements

350 Let us now assume our spatial domain Ω to be a geometry, such that we construct a
 351 quasi-uniform regular partition \mathcal{T}_h of Ω , made of quadrangles ($d = 2$) or hexahedra
 352 ($d = 3$). Each deformed element $K \in \mathcal{T}_h$ is obtained with an invertible transformation
 353 $\underline{\phi}_K$ applied to the reference element $\hat{K} = [0, 1]^d$.

354
 355 This setting allows the use of high-order spectral finite element, for which we can
 356 apply the mass-lumping approach. The method [Cohen2002] consists in obtaining a
 357 diagonal mass matrix – which is trivial to invert – by choosing a quadrature rule which
 358 has quadrature points at the locations of the interpolation points of a Lagrangian
 359 basis of \mathcal{V}_h . The lagrangian basis is – as usual for finite elements – constructed locally
 360 on the reference element. On the reference element the basis is constructed using

361 Gauss-Lobatto points.

362

363 Furthermore, we recall that mixed variational problems are stable only for certain
364 choices of finite element spaces that satisfy the *inf-sup* condition from the so-called
365 Ladyzenskaya-Babuska-Brezzi theorem [Babuvska1973, Brezzi1974, LeTallec1994].
366 Unstable mixed methods are characterized by the appearance of nonphysical oscilla-
367 tions of the pressure field. The demonstration of the *inf-sup* condition is quite complex
368 and out of the scope of this work. Hence, we rely on the analysis held in [Pena2009],
369 in which a numerical analysis of convergences properties is presented for various choice
370 of finite element spaces for \mathcal{V}_h and \mathcal{L}_h .

371

372 We consider the $\mathcal{Q}_n - \mathcal{Q}_{n-2}^{disc}$ method, proposed by Maday et al. [BernardiMaday1999].
373 More precisely

$$\mathcal{V}_h = \{\varphi \in C^0(\Omega)^d \mid \varphi|_K \circ \underline{\phi}_K \in \mathcal{Q}_n(\hat{K})^d\}, \quad (30)$$

374

$$\mathcal{L}_h = \{\psi \in L^2(\Omega) \mid \psi|_K \circ \underline{\phi}_K \in \mathcal{Q}_{n-2}(\hat{K})\}. \quad (31)$$

375 This choice benefits of higher efficiency for the computation of the pressure field, given
376 the lower order for the pressure field and the discontinuity of elements, for which
377 the pressure contribution is computed locally. We remark that in 3D problems, the
378 lower order of the shape functions for the pressure field results in the loss of one order
379 of accuracy, that is comparable with the loss of one order already happening on
380 non-cartesian mesh due to mass-lumping [Durufle2009].

381

382 Note that, with the discussed choices, the operator M_h and M_h^P are computed using
383 respectively a quadrature formula on the set of $(n+1)^d$ and $(n-1)^d$ Gauss-Lobatto
384 points on the reference element.

385 **Remark.** *A first alternative would be the choice $\mathcal{Q}_n - \mathcal{Q}_{n-1}$ more precisely \mathcal{L}_h is*
386 *chosen as*

$$\mathcal{L}_h = \{\psi \in C^0(\Omega) \mid \psi|_K \circ \underline{\phi}_K \in \mathcal{Q}_{n-1}(\hat{K})\}. \quad (32)$$

387 *This choice is analyzed in [brezzi1991stability], which are an extension of the widely*
388 *used Taylor-Hood elements. They are characterized by optimal convergence properties*
389 *but it more costly when computing local contributions. A second alternative is the*
390 *choice $\mathcal{Q}_n - \mathcal{P}_{n-1}^{disc}$, for which*

$$\mathcal{L}_h = \{\psi \in C^0(\Omega) \mid \psi|_K \circ \underline{\phi}_K \in \mathcal{P}_{n-1}(\hat{K})\}. \quad (33)$$

391 *The choice has been discarded since it does not allow important optimization procedure*
392 *related to the use of finite elements constructed on $\hat{K} = [0, 1]^d$ by cartesian product*
393 *(such as the spectral finite elements).*

394 3.2.3 Definition of divergence operator

395 We now detail the definition of the discrete operator D_h and its adjoint D_h^* , developed
396 to increase the efficiency for the computation of the volumetric contribution. When

397 considering the computation of the pressure term at the discrete level, we can write

$$\begin{aligned}
\int_{\Omega} q_h \operatorname{div} \underline{w}_h \, d\Omega &= \sum_{K \in \mathcal{T}_h} \int_K q_h \operatorname{div} \underline{w}_h \, dK \\
&= \sum_{K \in \mathcal{T}_h} \int_{\widehat{K}} \widehat{q}_h \operatorname{div}(J_K \underline{\nabla} \phi_K^{-1} \widehat{w}_h) \, d\widehat{K} \\
&= \sum_{K \in \mathcal{T}_h} \int_{\widehat{K}} \widehat{q}_h \operatorname{cof}(\underline{\nabla} \phi_K) : \underline{\nabla} \widehat{w}_h \, d\widehat{K}
\end{aligned} \tag{34}$$

398 where

$$J_K = \det \underline{\nabla} \phi_K \quad \text{and} \quad \operatorname{cof}(\underline{\nabla} \phi_K) = J_K \underline{\nabla} \phi_K^{-T}.$$

399 Then, we have the displacement \widehat{w}_h and pressure \widehat{q}_h w.r.t the reference element \widehat{K} .
400 The computation performed on the element \widehat{K} is a classical procedure in finite-element
401 analysis as it allows simplifying and reducing computations. The last line of (34) can
402 be rewritten with the application of the *Green's formula*, and we obtain

$$\int_{\Omega} q_h \operatorname{div} \underline{w}_h \, d\Omega = - \sum_{K \in \mathcal{T}_h} \int_{\widehat{K}} \operatorname{cof}(\underline{\nabla} \phi_K) \cdot \underline{\nabla} \widehat{q}_h \cdot \widehat{w}_h \, d\widehat{\Omega}_K + \sum_{K \in \mathcal{T}_h} \int_{\partial \widehat{K}} \operatorname{cof}(\underline{\nabla} \phi_K) \widehat{q}_h \widehat{w}_h \cdot \widehat{n} \, d\widehat{s} \tag{35}$$

403 On this reformulation we base the definition of discrete operators D_h and D_h^* , given
404 by the application of a quadrature formula (Gauss Lobatto), to compute the integrals,
405 so that we have

$$\begin{aligned}
(D_h \underline{w}_h, q_h)_{\mathcal{L}} &:= (\underline{w}_h, D_h^* q_h)_{\mathcal{H}} \\
&:= - \sum_{K \in \mathcal{T}_h} \int_{\widehat{K}} \operatorname{cof}(\underline{\nabla} \phi_K) \cdot \underline{\nabla} \widehat{q}_h \cdot \widehat{w}_h \, d\widehat{\Omega}_K + \sum_{K \in \mathcal{T}_h} \int_{\partial \widehat{K}} \operatorname{cof}(\underline{\nabla} \phi_K) \widehat{q}_h \widehat{w}_h \cdot \widehat{n} \, d\widehat{s}
\end{aligned} \tag{36}$$

406 For the computation of the volumetric term we use a quadrature based on $(n+1)^d$
407 Gauss-Lobatto points. Moreover, we use a quadrature based on $(n+1)^{d-1}$ Gauss-
408 Lobatto points on each facet of the reference element (4 when $d=2$, 6 when $d=3$).
409

410 An alternative would be to introduce \widetilde{D}_h and \widetilde{D}_h^* by

$$(\widetilde{D}_h \underline{w}_h, q_h)_{\mathcal{L}} := (\underline{w}_h, \widetilde{D}_h^* q_h)_{\mathcal{H}} := \sum_{K \in \mathcal{T}_h} \int_{\widehat{K}} \widehat{q}_h \operatorname{cof}(\underline{\nabla} \phi_K) : \underline{\nabla} \widehat{w}_h \, d\widehat{K} \tag{37}$$

411 using a quadrature based on $(n+1)^d$ Gauss-Lobatto points.
412

413 We now compare the computational cost associated to the application of \widetilde{D}_h or
414 D_h . The application of the operator \widetilde{D}_h implies the computation of 9 derivatives for
415 the gradient of the displacement field, plus the interpolation of the pressure field into
416 \mathcal{Q}_n . The use of the proposed operator D_h , instead, requires the computation of 3
417 derivatives of the \mathcal{Q}_{n-2} pressure field, 3 interpolations for the resulting gradient, and
418 the computation of the boundary term that has a lower complexity. Without going
419 into details our computations show that asymptotically, using D_h over \widetilde{D}_h offers a
420 reduction of 40% in terms of computational cost. Nevertheless, higher efficiency is
421 expected for some specific choice of n as we explain in the following section.

422 3.2.4 The $\mathcal{Q}_4 - \mathcal{Q}_2^{disc}$ element

423 In practice to further increase the efficiency of the numerical scheme, we use for the
 424 order of \mathcal{V}_h and \mathcal{L}_h the couple spaces such that the DoFs of the pressure element
 425 correspond to a subset of the DoFs of the displacement element. By taking $n = 4$, we
 426 have the highest element order with the mentioned feature, reducing the number of
 427 operations for the interpolation from \mathcal{L}_h to \mathcal{V}_h .

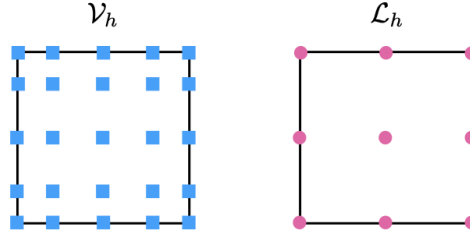


Figure 2. Representation of spectral elements choice with related DoFs distribution. From the left, $\mathcal{Q}_4(\hat{K})$ element of displacement space \mathcal{V}_h and $\mathcal{Q}_2^{disc}(\hat{K})$ element for pressure space \mathcal{L}_h .

428 The computation is performed within a home-made software and the interpolation of
 429 the pressure field is specifically optimized in order to further speed up the computations.
 430 In our implementaton, no matrix is assembled since all the computations are done
 431 locally, which allow to strongly reduce the CPU storage and cost. We measure
 432 numerically a speed-up of 70% for the computation of the pressure contribution
 433 compared to the use of (37).

434 **3.3 Time discretization**

435 3.3.1 The leap-frog method

436 We now develop a fully-discretized scheme associated to the mixed formulation (NIMd).
 437 Given the time interval $[0, T]$, with $T > 0$, we define the partition $t^n = n\Delta t$, with
 438 $n \in \{0, 1, \dots, N\}$ and the time-step defined as $\Delta t = T/N$, we can introduce a time-
 439 discretization based on Leap-Frog method

$$M_h \frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2} + (A_h^s + \lambda \bar{A}_h^p) u_h^n = \underline{f}_h^n \quad \text{in } \mathcal{V}_h \quad (\text{LF})$$

440 The LF method is characterized by second order accuracy and energy conservation.
 441 It is fully explicit, which makes it useful to deal with large scale problem such as
 442 elastodynamic problems. However, the scheme is stable under the well-known CFL
 443 condition [joly2007numerical]

$$0 \leq \Delta t^2 A_h^s + \Delta t^2 \lambda \bar{A}_h^p \leq 4M_h \quad \iff \quad \Delta t^2 \leq \frac{4}{\rho(M_h^{-1} A_h^s + \lambda M_h^{-1} \bar{A}_h^p)}, \quad (38)$$

444 where here $\rho(\cdot)$ now denotes the spectral radius of an operator. The enforcement
 445 of nearly-incompressibility corresponds to a large value of the bulk modulus λ . As
 446 a result, the maximal time-step to guarantee the stability of the scheme is strongly
 447 reduced, the number of iterations N to reach the end of the simulation at time T will
 448 proportionally increase.

449 For a physical interpretation, we could consider that with a large bulk modulus,
 450 the speed of pressure waves is much larger than shear waves (respectively $1500ms^{-1}$

451 and $10ms^{-1}$ in biological tissues) and our explicit scheme should have a time step
 452 adapted to the the fastest wave propagation phenomena. Then, the pressure wave
 453 strongly limits the time-step and affects the efficiency of the computations.

454 3.3.2 Stabilised leapfrog method

455 As mentionned, the pressure operator $\lambda\bar{A}_h^p$ limits the time-step Δt , but at the same
 456 time it has a low computational cost, thanks to the proposed operators D_h and D_h^* . On
 457 the other side, the shear operator A^s has a lower spectral radius, around two order of
 458 magnitudes smaller compared to $\lambda\bar{A}_h^p$ in soft tissues, but it has a higher computational
 459 cost as it involves the computation of the gradient of displacements and accounts for
 460 all the complexities of the material (typically heterogeneities and anisotropy). In this
 461 framework, we would like to separate the computation of shear and pressure wave
 462 operator. The shear waves is computed at a "global" larger time step, whereas pressure
 463 wave operator is computed as if "local" time-step were defined. This general idea is the
 464 basis of the Chebyshev Leapfrog method [Carle2020], such that at each time-step the
 465 shear part is iterated only once and the volumetric operator is iterated $m + 1$ times.

466 In order to use the strategy suggested above, the pressure operator is approximated
 467 with a polynomial $\mathcal{P}_m(x)$ of order m , given the condition $\mathcal{P}_m(0) = 1$ to be consistent
 468 with the leap-frog scheme,

$$M_h \frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2} + A^s u_h^n + \mathcal{P}_m(\Delta t^2 \lambda \bar{A}_h^p M_h^{-1}) \lambda \bar{A}_h^p u_h^n = \underline{f}_h^n. \quad (39)$$

469 We rewrite the stability condition (38) for our modified scheme

$$0 \leq \Delta t^2 A_h^s + \Delta t^2 \mathcal{P}_m(\Delta t^2 \lambda \bar{A}_h^p M_h^{-1}) \lambda \bar{A}_h^p \leq 4M_h \quad (40)$$

470 from which we derive the following theorem for a polynomial \mathcal{P}_m yet unspecified

Theorem 3.1. *A sufficient condition for the stability of the scheme (39) is*

$$\Delta t^2 \leq \frac{1}{\rho(M_h^{-1} A_h^s)} \quad (41)$$

and

$$\mathcal{P}_m(x)x \in [0, 3] \quad \forall x \in [0, \rho(\Delta t^2 \lambda M_h^{-1} \bar{A}_h^p)] \quad (42)$$

471

472 *Proof.* From the stability condition of the modified scheme (40), we derive a sufficient
 473 condition

$$\begin{cases} \Delta t^2 A_h^s \leq M_h \\ 0 \leq \Delta t^2 \mathcal{P}_m(\Delta t^2 \lambda \bar{A}_h^p M_h^{-1}) \lambda \bar{A}_h^p \leq 3M_h. \end{cases} \quad (43)$$

The first condition (41) is then trivially obtained. Given that the pressure operators is self-adjoint and positive, we can then apply the Spectral Mapping theorem [Harte1972] to reduce the analysis to a spectral analysis. More precisely, for condition (42), we see that the second equation of (43) can be written in terms of the spectrum σ

$$\sigma\left(\Delta t^2 M_h^{-1} \mathcal{P}_m(\Delta t^2 \lambda \bar{A}_h^p M_h^{-1}) \lambda \bar{A}_h^p\right) \in [0, 3]$$

which is equivalent to

$$\sigma\left(\mathcal{P}_m(\Delta t^2 \lambda M_h^{-1} \bar{A}_h^p) \Delta t^2 \lambda M_h^{-1} \bar{A}_h^p\right) \in [0, 3]$$

474 Hence, the spectrum lies in $[0, 3]$ if the polynomial \mathcal{P}_m is such that

$$\mathcal{P}_m(x)x \in [0, 3] \quad \forall x \in [0, \rho(\Delta t^2 \lambda M_h^{-1} \bar{A}_h^p)] \quad (44)$$

475 concluding the proof. \square

476 Notice that the implementation of the polynomial \mathcal{P}_m should be obtained with
477 an algorithm, for example like the Horner's rule or recurrence rules, so that the
478 computation of the volumetric contribution of the displacement would correspond to
479 the application of the argument of the polynomial for $m + 1$ times.

480 3.3.3 Leapfrog Chebyshev Methods

481 First note that – following (41) – the time step Δt is chosen independently of the
482 choice of \mathcal{P}_m . We set

$$\Delta t = (1 - \epsilon) \frac{1}{\sqrt{\rho(M_h^{-1} A_h^s)}} \quad \text{with } \epsilon \ll 1. \quad (45)$$

483 For the choice of the polynomial, we use the so-called *stabilized* or *damped Chebyshev*
484 *Polynomials* \mathcal{P}_m^ν , proposed by [Carle2020], and obtained from scaling and shifting of
485 Chebyshev polynomials of first kind $\mathcal{T}_m(x)$:

$$\mathcal{P}_m^\nu(x) = \frac{2}{x} \left[1 - \frac{\mathcal{T}_{m+1}(\delta_m^\nu - \frac{x}{\omega_m^\nu})}{\mathcal{T}_{m+1}(\delta_m^\nu)} \right] \quad (46)$$

with parameters δ_m^ν and ω_m^ν

$$\delta_m^\nu = 1 + \frac{\nu}{(m+1)^2}, \quad \omega_m^\nu = 2 \frac{\mathcal{T}'_{m+1}(\delta_m^\nu)}{\mathcal{T}_m(\delta_m^\nu)},$$

486 and first order derivative $\mathcal{T}'_{m+1}(x) = (m+1)\mathcal{U}_m(x)$ where \mathcal{U}_m is the Chebyshev of
487 second kind. We see that consistency with respect to the original scheme is guaranteed,
488 since

$$\mathcal{P}_m^\nu(x)x = 2 \left[1 - \frac{\mathcal{T}_{m+1}(\delta_m^\nu - \frac{x}{\omega_m^\nu})}{\mathcal{T}_{m+1}(\delta_m^\nu)} \right] = x + \mathcal{O}(x^2) \quad (47)$$

489 and the Leap-Frog-Chebyshev scheme reads

$$M_h \frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2} + A_h^s u_h^n + \mathcal{P}_m^\nu(\Delta t^2 \lambda \bar{A}_h^p M_h^{-1}) \lambda \bar{A}_h^p u_h^n = \underline{f}_h^n \quad (\text{LFC})$$

490 Chebyshev polynomials are sequences of polynomial functions that can be computed
491 with recurrence relations and benefit of optimal properties for the uniform distribution
492 of its roots within the interval $[-1, 1]$. The formulation (46) allows to change the
493 interval in which Chebyshev polynomials oscillates, with the following properties
494 derived in [Carle2020, Grote2021].

Property 3.1. For $\nu \in [0, 1]$:

$$\sup_{x \in [0, \alpha_m]} |x \mathcal{P}_m^\nu(x)| < 4 - \frac{2\nu}{1 + \nu} \quad \text{with} \quad \alpha_m = \left(2 + \frac{\nu}{(m+1)^2}\right) \omega_m^\nu \quad (48)$$

and $\forall \beta \leq \alpha_m$

$$\inf_{x \in [\beta, \alpha_m]} |x \mathcal{P}_m^\nu(x)| > \min\left(\frac{\beta}{1 + \nu}, \frac{2\nu}{1 + \nu}\right) \quad (49)$$

495

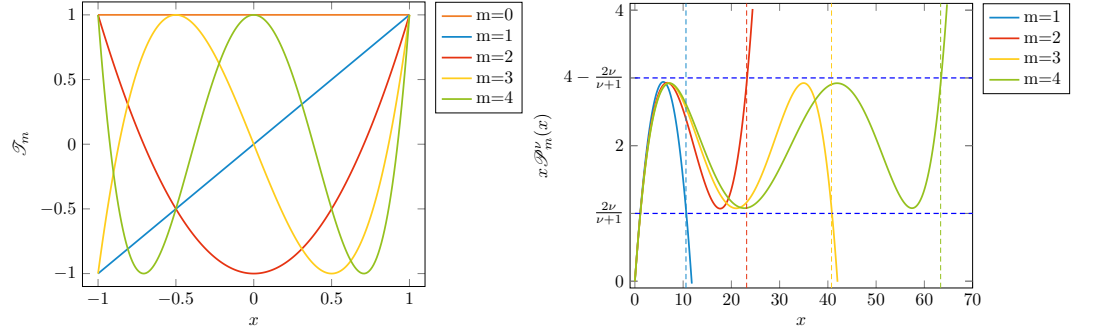


Figure 3. On the left: Chebyshev Polynomials of first kind with order m from 0 to 4. On the right: Stabilized Chebyshev polynomials $x \mathcal{P}_m^\nu$ with order m from 1 to 4. Dashed vertical lines correspond to α_m which guarantees that for $x \in [0, \alpha_m]$ we have $x \mathcal{P}_m^\nu < 4 - \frac{2\nu}{1+\nu}$. In the figure, $\nu = 1$.

496 This choice of polynomials allows to have the largest stability region (in x) in which
 497 the Chebyshev $x \mathcal{P}_m^\nu(x)$ is comprised between the inferior and the superior limit, as
 498 given by Property 3.1.

499 We now present the constructed solution in order to respect the sufficient condition
 500 of Theorem 3.1. There are two parameters to choose: the parameter ν that defines the
 501 inferior and superior limit of \mathcal{P}_m^ν , and the polynomial order m that depends on the
 502 spectral radius of the pressure operator through the parameter α_m .

503 By taking $\nu = 1$, we define the superior limit of the polynomial within the stability
 504 region $[0, \alpha_m]$.

$$\sup_{x \in [0, \alpha_m]} |x \mathcal{P}_m^1(x)| < 3 \quad (50)$$

505 which partially corresponds to the second condition (42) of the stability condition of
 506 theorem 3.1.

507 Still, we remark that the stabilised Chebyshev polynomial is characterized by a
 508 superior limit only within the stability region $x \in [0, \alpha_m]$, for which we have to ensure
 509 that

$$\sigma(\Delta t^2 \lambda M_h^{-1} \bar{A}_h^p) \in [0, \alpha_m] \quad (51)$$

510 in order to guarantee condition (42). Since the positivity of the spectrum is given, we
 511 only need to respect

$$\rho(\Delta t^2 \lambda M_h^{-1} \bar{A}_h^p) \leq \alpha_m \quad (52)$$

From an estimation of α_m , we see that for $\nu > 0.7$ we have

$$4(m+1)^2 e^{-\nu/2} \leq \alpha_m$$

512 By substitution, we derive the minimum order of the damped Chebyshev polynomial

$$m + 1 \geq \sqrt{\lambda} \frac{\sqrt{\rho(\Delta t^2 M_h^{-1} \bar{A}_h^p) e^{1/4}}}{2}. \quad (53)$$

513 To sum up, the stability of the scheme (LFC) is given by the following proposition

Proposition 3.2. *A sufficient condition to guarantee the stability of the Leapfrog Chebyshev scheme (LFC) is given by*

Maximum time step

$$\Delta t \leq (1 - \epsilon) \frac{1}{\sqrt{\rho(M_h^{-1} A_h^s)}} \quad \text{with } \epsilon \ll 1 \quad (54)$$

Minimum order of polynomial $x \mathcal{P}_m^1(x)$

$$m + 1 = \left\lceil \sqrt{\lambda} \frac{\sqrt{\rho(\Delta t^2 M_h^{-1} \bar{A}_h^p) e^{1/4}}}{2} \right\rceil \quad (55)$$

514

515 Note that in Proposition 3.2 the operator $\lceil \cdot \rceil$ outputs the smallest larger integer.
 516 With the introduction of polynomial $\mathcal{R}_{m+1}(x) := x \mathcal{P}_m^1(x)$, the fully-discretized scheme
 517 reads

$$\underline{u}_h^{n+1} - 2\underline{u}_h^n + \underline{u}_h^n + \Delta t^2 M_h^{-1} A_h^s \underline{u}_h^n + \mathcal{R}_{m+1}(\Delta t^2 \lambda M_h^{-1} \bar{A}_h^p) \underline{u}_h^n = \Delta t^2 M_h^{-1} \underline{f}^n \quad (\text{LFC})$$

518

In conclusion, we have developed a stable explicit scheme for linear elastic wave
 519 propagation in incompressible media. The stability of the scheme is based on the
 520 definition of a time step Δt from the shear operator only and on the order of a stabilised
 521 Chebyshev polynomials. One way to interpret this is to say that the *global* time step
 522 Δt is chosen using the shear operator only, while the pressure terms are computed
 523 with a *local* time-step $\Delta t_m = \Delta t / (m + 1)$. As we will see in the Results section, the
 524 efficiency of the scheme w.r.t. the LF method relies on two points: First, the low
 525 computational cost of the pressure operator compared to the shear operator ; Second,
 526 the value of λ , the higher λ is, higher will be the potential gain given by this strategy.

527

528 The implemented algorithm for computing the solution to (LFC) is given Algo-
 529 rithm 1. It is based on the recurrence relations of the Chebyshev polynomials. An
 530 auxiliary variable \underline{z}_l^n is introduced, with $l = 0, \dots, m + 1$, which corresponds to the *local*
 531 contribution of the pressure operator within each time step n .

532 3.4 Efficiency estimation the LFC scheme

533 Let us now present a theoretical estimation of the computational gain that we obtain
 534 with the (LFC) scheme w.r.t. to the classical Leap-Frog scheme (LF).

We denote C_S the computational time for the application of the shear operator, and
 C_P the computational time for the application of the pressure operator. The objective
 is to compare the simulation time of the LF method and of the LFC method, based
 on C_S and C_P and on the time-steps Δt_{LF} and Δt_{LFC} derived from the respective
 stability condition. The quantity that we would like to analyze is the computational

Algorithm 1 Algorithm for the (LFC) scheme

$$\underline{z}_0^n \leftarrow \underline{u}_h^n$$

$$\underline{z}_1^n \leftarrow \delta_m^1 \underline{u}_h^n - \frac{\Delta t^2 \lambda}{\omega_m^1} M_h^{-1} \bar{A}_h^p \underline{u}_h^n$$

for $l = 2 : m + 1$ **do**

$$\underline{z}_l^n \leftarrow 2 \left(\delta_m^1 I - \frac{\lambda \Delta t^2}{\omega_m^1} M_h^{-1} \bar{A}_h^p \right) \underline{z}_{l-1}^n - \underline{z}_{l-2}^n$$

end for

$$\mathcal{R}_{m+1}(\Delta t^2 \lambda M_h^{-1} \bar{A}_h^p) \underline{u}_h^n \leftarrow 2 \underline{u}_h^n - \frac{2 \underline{z}_{m+1}^n}{\mathcal{T}_{m+1}(\delta_m^1)}$$

$$\underline{u}_h^{n+1} \leftarrow \Delta t^2 M_h^{-1} \underline{f}^n - \underline{u}_h^{n-1} + 2 \underline{u}_h^n - \Delta t^2 M_h^{-1} A_h^s \underline{u}_h^n - \mathcal{R}_{m+1}(\Delta t^2 \lambda M_h^{-1} \bar{A}_h^p) \underline{u}_h^n$$

gain G defined as follows

$$G = \frac{N_{LF} C_{LF}}{N_{CH} C_{LFC}}$$

535 where N_{LF} and N_{LFC} are the number of iterations to complete a simulation, while
 536 C_{LF} and C_{LFC} are the computational cost of the time step for each method.

537 We start from the classic LF scheme considering that, with the property that the
 538 shear and pressure operators are positive and self-adjoint, we have

$$\rho(M_h^{-1} A_h^s + \lambda M_h^{-1} \bar{A}_h^p) \leq \rho(M_h^{-1} A_h^s) + \rho(\lambda M_h^{-1} \bar{A}_h^p) \quad (56)$$

539 Then, the CFL condition (38) is respected for the following time-step

$$\Delta t_{LF}^2 = \frac{4}{\rho(M_h^{-1} A_h^s) + \rho(\lambda M_h^{-1} \bar{A}_h^p)} = \rho(M_h^{-1} A_h^s) \frac{4}{1 + \lambda R} \quad R := \frac{\rho(M_h^{-1} \bar{A}_h^p)}{\rho(M_h^{-1} A_h^s)}. \quad (57)$$

540 The choice of time step given by (57) may appear suboptimal because (56) is in general
 541 a strict inequality. However for large λ the difference between, $\rho(M_h^{-1} A_h^s + \lambda M_h^{-1} \bar{A}_h^p)$
 542 and the sum of the spectral radii is small with respect to λ . The computational time
 543 of a single time step in the LF scheme is trivially given by

$$C_{LF} = C_S + C_P. \quad (58)$$

544 For the LFC scheme, we have constructed the stability condition by choosing

$$\Delta t_{LFC}^2 = \frac{1}{\rho(M_h^{-1} A_h^s)}. \quad (59)$$

Notice that with respect to the time-step defined in (45), we forget the term $(1 - \epsilon)$ for the sake of simplicity. For the evaluation of the computational cost of a time-step with the LFC method, we have

$$C_{LFC} = C_S + (m + 1) C_P$$

where we take $m + 1$ as the minimal value to guarantee the stability condition (53), and we combine with (59) in order to obtain

$$C_{LFC} = C_S + \left[\Delta t_{LFC} \frac{\sqrt{\lambda \rho (M_h^{-1} \bar{A}_h^p) e^{1/4}}}{2} \right] C_P = C_S + \left[\frac{e^{1/4}}{2} \sqrt{\lambda R} C_P \right].$$

Now that we have all the elements, we rewrite the computational gain G as

$$G = \frac{\frac{1}{\Delta t_{LF}} C_{LF}}{\frac{1}{\Delta t_{LFC}} C_{LFC}} = \frac{\Delta t_{LFC} C_{LF}}{\Delta t_{LF} C_{LFC}}$$

We notice that we can write the ratio between Δt_{LFC} and Δt_{LF} as a function of the quantity R from relations (57) and (59)

$$\frac{\Delta t_{LFC}}{\Delta t_{LF}} = \frac{\sqrt{1 + \lambda R}}{2}$$

545 Finally, the computational gain G is given by the following relation

$$G = \frac{\sqrt{1 + \lambda R}}{2} \frac{C_S + C_P}{C_S + \left[\frac{e^{1/4}}{2} \sqrt{\lambda R} C_P \right]} \quad (60)$$

Considering the incompressible limit ($\lambda \rightarrow \infty$), we have

$$\lim_{\lambda \rightarrow \infty} G = \frac{\sqrt{\lambda R}}{2} \frac{C_S + C_P}{\frac{e^{1/4}}{2} \sqrt{\lambda R} C_P} = e^{-1/4} \left(\frac{C_S}{C_P} + 1 \right)$$

546 for which we obtain that the LFC scheme is more efficient than the LF scheme (i.e.
547 $G > 1$) if the computational cost of the pressure operator is sufficiently smaller than
548 the shear operator, in particular

$$C_S/C_P > e^{1/4} - 1 \approx 0.2840. \quad (61)$$

549 In practice, this condition is always met, for which we show that at the incompressible
550 limit the proposed (LFC) scheme is more efficient than the (LF).

551 In addition, we point out that the strategy developed for the space discretization
552 Section 3.2 has an impact also on the efficiency estimation that we have propose by G .
553 Indeed, the optimization for the computation of the pressure contribution corresponds
554 to a decrease the computational cost for the pressure operator C_P , i.e. a higher
555 efficiency of the scheme through relation (61), given that C_S is the same.

556 4 Numerical Results

This section is devoted to the presentation of numerical results in order to validate the properties of the proposed scheme (LFC). Before presenting details of the simulations, we consider the non-dimensionalization of the wave propagation problem. As in [Caforio2018], we introduce the characteristic length L of the domain Ω , the characteristic observation time T and the shear parameter μ . The non-dimensionalized displacement and pressure field reads respectively:

$$\underline{u}_a(t_a, \underline{x}_a) := L^{-1} \underline{u}(T t_a, L \underline{x}_a) \quad \text{and} \quad p_c(t_a, \underline{x}_a) := \mu^{-1} p(T t_a, L \underline{x}_a)$$

with $t_a \in [0, 1]$. We then derive the non-dimensionalized quantities

$$\varrho_a = \frac{L^2 \varrho}{T^2 \mu}, \quad \mathbf{C}_a^d = \mu^{-1} \mathbf{C}^d, \quad \lambda_a = \frac{\lambda}{\mu}, \quad \underline{f}_a = \frac{L}{\mu} \underline{f}$$

557 and with these, we rewrite our wave propagation problem in a non-dimensional form.
 558 For the rest of the paper, we drop the notation $(\cdot)_a$ that represents non-dimensionalized
 559 quantities.

560 In the following results, the source term used to generate the wave propagation
 561 has always the same form. It is the product of a spatial and a temporal term. The
 562 spatial excitation is a C^∞ function with a compact support, localized in the center of
 563 our solid \underline{x}_C . For the temporal push, we have a Ricker wavelet with a small offset t_0 .
 564 The source term reads:

$$f(\underline{x}, t) = \begin{cases} A_m \cdot \exp \left[\frac{1}{2\sigma_s} \left(\frac{r_0^2}{d^2 - r_0^2} + 1 \right) \right] \cdot \left[\left(\frac{t-t_0}{\sigma_\tau} \right)^2 - \frac{1}{\sigma_\tau} \right] \exp \left[-\frac{(t-t_0)^2}{\sigma_\tau} \right] & \text{if } d^2 < r_0^2 \\ 0 & \text{otherwise} \end{cases} \quad (62)$$

565 with $d^2 = \sum_{i=1}^{dim} (x_i - x_{C,i})^2$ with d the distance from the center, r_0 the maximum
 566 width of the source, A_m the maximum amplitude, σ_τ and σ_s the covariance.

567 4.1 Non-locking mixed formulation

568 We compare simulation results of the nearly-incompressible wave propagation problem
 569 considering the two semi-discretized schemes (NId) and (NIMd) respectively obtained
 570 with the displacement-based and mixed method. At this point we do not use the
 571 strategy developed with Chebyshev polynomials and we use the classic LF scheme for
 572 the time discretization.

573 Let us now provide numerical and physical parameters of simulations. We consider
 574 a 2D circular domain with non-dimensionalized density $\varrho = 1$. We consider isotropic
 575 homogeneous behavior, characterized by the constitutive relation (non-dimensionalized)

$$\underline{\underline{\sigma}} = 2 \underline{\underline{\varepsilon}}(\underline{u}) + \lambda \text{tr}(\underline{\underline{\varepsilon}}(\underline{u})). \quad (63)$$

576 We choose the non-dimensionalized second Lamé parameters as $\lambda = 2000$. The function
 577 described in (62) is used with $\underline{x}_C = (0, 0)$. A vector field is generate by assuming that
 578 the source is polarized in the y-direction: $\underline{f} = f \underline{e}_y$. The other chosen parameters of
 579 the source are: $A_m = 10^6$, $r_0 = 0.05$, $\sigma_s = 0.5$, $\sigma_\tau = 2.5 \cdot 10^{-4}$.

580 For the space discretization, we compare the two schemes by using \mathcal{Q}_4 elements for
 581 the displacement-based method (NId), while for the mixed method we discretize with
 582 $\mathcal{Q}_4 - \mathcal{Q}_2^{disc}$ elements, as presented in (30) and (31). In both cases the mesh is highly
 583 refined, with 150k DoFs.

In figure 4 we have the magnitude of the displacement field obtained from the
 mixed method, in which we can appreciate only the propagation of the shear wave. In
 an isotropic homogeneous material the velocity of propagation of shear wave (v_s) and
 of pressure wave (v_p) are related to the material parameters such that

$$v_s^2 = \frac{\mu}{\varrho}, \quad v_p^2 = \frac{\lambda + 2\mu}{\varrho}$$

584 With the present choice of material parameters, we have that $v_p \approx 45 v_s$. Then, when
 585 looking at the resulting displacement field, we only see a propagating shear wave and

586 not the pressure one (in figure 4), as the pressure is too fast for the present timescale.
 587 We show the displacement results only of the mixed method (NIMd), as no qualitative
 588 difference is present with respect to the solution of the displacement-based method.

Then, we extract the volumetric component from the solution (which corresponds to $\lambda \nabla \text{div } \underline{u}$) a posteriori, by computing the quantity

$$-\frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2} - M_h^{-1} A_h^s u_h^n$$

589 for both displacement-based and mixed method. The limitations of the displacement-
 590 based method previously mentioned is visible. In figure 5 we observe nonphysical
 591 oscillations in the pressure field that propagates with the shear wave, which is a
 592 numerical effect due to enforcement of incompressibility. This phenomenon disappears
 with the implemented mixed method, as we can see from figure 6.

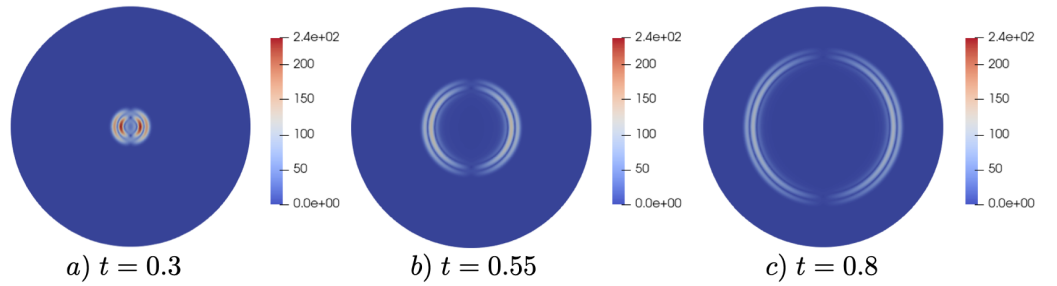


Figure 4. Elastic wave propagation in homogeneous isotropic solid. Magnitude of displacement field obtained with the mixed scheme. The 2D circular mesh is discretized with 150k Dofs.

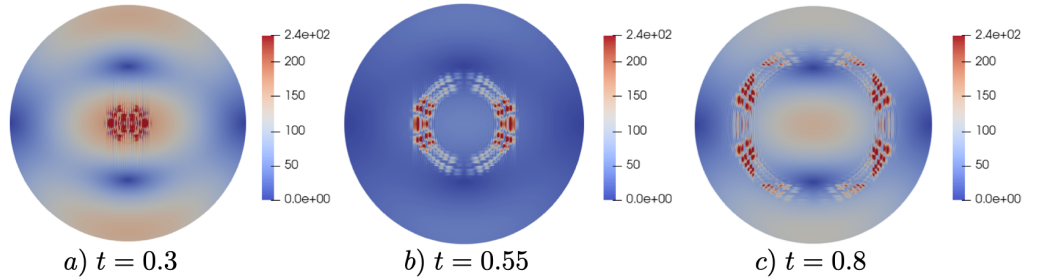


Figure 5. Volumetric component of the elastic wave propagation in homogeneous isotropic solid, with incompressible parameter $\lambda = 2000$. The 2D circular mesh is discretized with 150k Dofs. Solution obtained with displacement based method (NId), characterized by the presence of nonphysical oscillations coupled with the shear wave

593

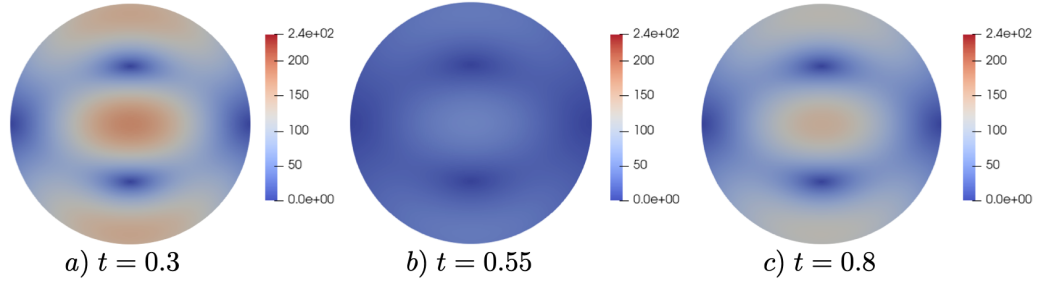


Figure 6. Volumetric component of the elastic wave propagation in homogeneous isotropic solid, with incompressible parameter $\lambda = 2000$. The 2D circular mesh is discretized with 150k Dofs. Solution obtained with displacement based method (NIMd). In this case, no spurious pressure wave is observed.

594 One important remark is that in this simulation we are far from the nearly-
 595 incompressible case. In biological tissues, the pressure wave has a velocity that is
 596 at least 2 orders of magnitude higher than the shear wave velocity. Yet, in our case
 597 we can already see the importance of approximating this type of problems with an
 598 appropriate mixed formulation. Nevertheless, these are well-known results when dealing
 599 with nearly-incompressible problems [Bathe2006, brezzi2014basic]. The primary
 600 objective of the space discretization strategy was to compute efficiently the pressure
 601 contribution, ensuring the stability of the method and the accuracy of the computation
 602 for both displacements and pressure.

603 4.2 Leapfrog Chebyshev Methods

604 4.2.1 Efficiency

605 We perform simulations of elastic wave propagation in isotropic homogeneous solids
 606 within different domains: square, circle and cube. For the material, we take $\rho = 1$, $\lambda =$
 607 1000 and the same parameters of the source that we have used in the previous section.
 608 In Table 1 we present the associated numerical results. We measure the simulation
 609 time η_{LF} and η_{LFC} with the LF and LFC scheme. The average computational cost
 610 of C_P and C_S is measured over 1000 iterations, and the ratio C_S/C_P can be then
 611 compared to the measured computational gain $G_{eff} = \eta_{LF}/\eta_{LFC}$. We underline that
 612 the effective computational gain G_{eff} is related to simulations with $\lambda = 1000$, while
 613 the estimated G corresponds to the limit case of incompressibility $\lambda \rightarrow \infty$. Yet, we see
 614 a significant gain in the order of 3 for 2D solid and in the order of 2 for 3D solid. We
 615 remark that a 3D simulation with 1,5M of DoFs of elastic wave propagating within a
 616 nearly-incompressible isotropic homogeneous cube takes around 6 minutes.

617 In addition, the present case of study is given by an isotropic material linearized at
 618 rest condition, which is a very simplified case compared to possible realistic application.
 619 If we consider the biomedical application, tissues often undergoes large load and present
 620 anisotropic behavior. If we were supposed to model this type of problems, then the
 621 shear operator will be much more complex and the ratio C_S/C_P would be higher and
 622 so the computational gain G with respect to the classical LF method.

623 4.2.2 Space-time Convergence analysis

624 As a final validation of the proposed numerical scheme, we present a convergence
 625 analysis over space and time, comparing different boundary conditions. We consider a
 626 2D square with $\Omega = [0, 1]^2$, discretized with N elements for each direction, from $N = 8$

Shape	η_{LF} [s]	η_{LFC} [s]	G_{eff}	C_S/C_P	$NDoF$
Square	12.24	5.23	2.34	1.99	51,842
Circle	41.82	14.76	2.83	2.77	154,242
Cube	799.0	377.18	2.12	1.76	1,594,323

Table 1. Numerical results of time efficiency of LF-Chebyshev Scheme w.r.t. to Leap-Frog scheme, considering different domains. η_{LF} and η_{LFC} are the measured simulation time for the computation of the elastic wave propagation in an isotropic homogeneous solid. G_{eff} is the effective computational gain measured.

627 to 256. The parameters of the source are the following: $A_m = 10^3$, $r_0 = 0.05$, $\sigma_s = 0.5$,
628 $\sigma_\tau = 0.5 \cdot 10^{-4}$.

629 We compute the $L^\infty(L^2(\Omega))$ norm and the $L^\infty(H^1(\Omega))$ norm of the displacement
630 error taking the LF scheme as a reference, and we plot it with respect to the space-
631 step h . In these simulations we take $\lambda = 1000$. For each image, we consider the
632 convergence of simulations with final times of simulation $T = 0.6$ and $T = 1.0$. In
633 shorter simulations ($T = 0.6$) the shear wave almost reaches the boundary, but is not
634 reflected, while with the longer simulations $T = 1.0$, we have reflection and possible
635 effects of boundary conditions. Figure 7 represents the results obtained with a free
636 boundary ($\underline{t} = 0$). Results show that second order convergence is maintained for both
637 $L^2(\Omega)$ and $H^1(\Omega)$ norms (in space) for the shortest simulations ($T = 0.6$), (i.e. no
638 reflection). With longer simulations ($T = 1.0$), we see instead a strong deterioration
639 of convergence rate, which is not the same for the two norms: the order is two times
640 smaller for $L^2(\Omega)$ norm and four times smaller for $H^1(\Omega)$.

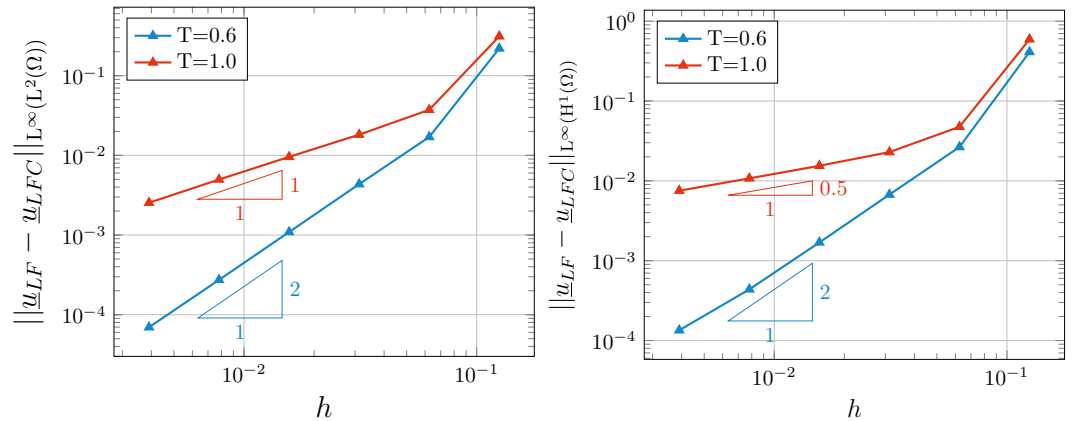


Figure 7. Convergence of scheme LFC with respect to the space step h – with $\underline{t} = 0$ on Γ (*free boundary condition*). On the left: we plot the $L^\infty(L^2(\Omega))$ -displacement error for a final simulation time $T = 0.6$ (before the shear wave touches the boundary) and $T = 1.0$ (after reflection on the boundary). On the right we have an equivalent plot but with $L^\infty(H^1(\Omega))$ -displacement error.

641 Then, we choose Dirichlet boundary condition $\underline{u} = 0$ on Γ , and we present the
642 related convergence rate in figure 8. The simulations are obtained with the same
643 material and source parameters as before. The convergence rate are now not affected
644 by the boundary condition, as we see that no deterioration appears between simulations
645 with different length in time. In addition, we see that the error is homogeneous between
646 the two space norms $L^2(\Omega)$ and $H^1(\Omega)$, as expected. We point out that with Dirichlet

647 boundary condition, the proposed (LFC) scheme maintains the second order accuracy
 648 of the classical (LF) scheme.

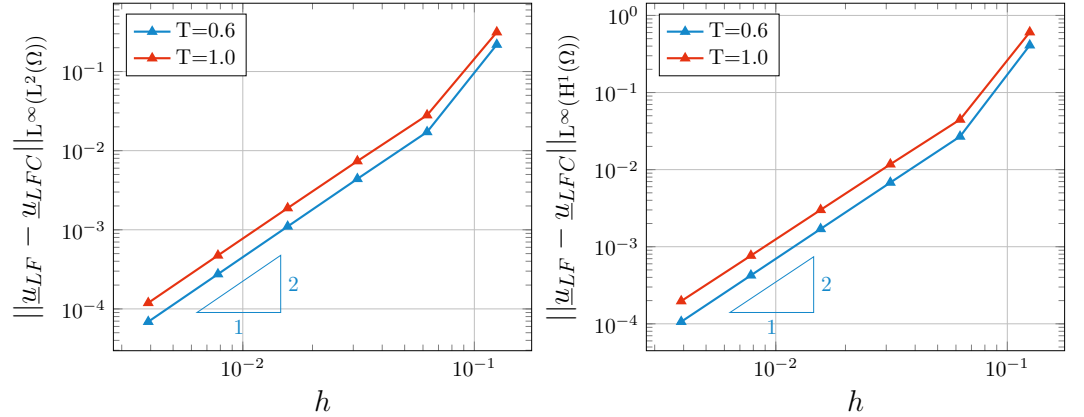


Figure 8. Convergence of scheme LFC with respect to the space step h – with $\underline{u} = 0$ on Γ (*Dirichlet boundary condition*). On the left, we plot the $L^\infty(L^2(\Omega))$ norm of displacement error. On the right, we plot the $L^\infty(H^1(\Omega))$ norm of the displacement error. In both case we consider a final simulation time $t_f = 0.5$ – before the shear wave touches the boundary – and $t_f = 1.0$ – after reflection on the boundary.

649 5 Numerical application: elastic wave propagation in an 650 incompressible hyperelastic stretched cube

651 In the previous section, we presented the numerical scheme for wave propagation
 652 problem in nearly-incompressible isotropic solids linearized around at rest condition.
 653 We now want to present the applicability of the LFC scheme for the approximation
 654 of linear wave propagation problems within solids with large deformations. The
 655 formulation obtained in section 2 allows to give a physical interpretation of the wave
 656 propagation, depending on material and prestress effect.

657 In the following applications, the solution of the non-linear nearly-incompressible
 658 static problem is approximated by the solution of the of the pure incompressible
 659 problem (64) with a non-homogeneous Dirichlet boundary condition yet unspecified.
 660 We solve the following static non-linear problem :

661 Find $(\underline{u}_0, p_0) \in \mathcal{W} \times \mathcal{L}$ such that

$$\begin{cases} \int_{\hat{\Omega}} \underline{\Sigma}_0^d : D\underline{e}_0(\hat{\underline{v}}) \, d\hat{\Omega} - \int_{\hat{\Omega}} p_0 J_0 \underline{C}_0^{-1} : D\underline{e}_0(\hat{\underline{v}}) \, d\hat{\Omega} = 0 & \forall \hat{\underline{v}} \in \mathcal{V}(\hat{\Omega}) \\ \int_{\hat{\Omega}} \hat{\underline{v}}(J_0 - 1) \cdot \underline{q} \, d\hat{\Omega} = 0 & \forall \underline{q} \in \mathcal{L}(\hat{\Omega}) \end{cases} \quad (64)$$

662 where p_0 is now a *Lagrangian multiplier*.

663 Let us consider a hyperelastic isotropic material with the Neo-Hookean elastic
 664 potential

$$\mathcal{W}^d = \mu(J_1 - 3) \quad \text{with} \quad J_1 = I_1 I_3^{-\frac{1}{3}} \quad (65)$$

665 where J_1 is the first reduced invariant presented in (10), and $\mu \in \mathbb{R}^+$ is the material
 666 parameter. As presented in the previous section, we consider a non-dimensionalization
 667 of the problem with respect to a characteristic length L , an observation time T and
 668 the material parameter μ .

The state of stress in the reference configuration is given by the following relation

$$\underline{\underline{\Sigma}}(\underline{u}, p) = \underline{\underline{\Sigma}}_d - pJ\underline{\underline{C}}^{-1} = \frac{\partial \mathcal{W}^d}{\partial \underline{\underline{e}}} - pJ\underline{\underline{C}}^{-1} \quad (67)$$

where the hydrostatic pressure p is now defined as a lagrangian multiplier. We can derive analytically the state of stress in the apparent configuration, taking into account that in the transverse direction there is no constraint. Then, the non-dimensionalized Cauchy tensor is in the form $\underline{\underline{\sigma}}_0 = \sigma_0 \underline{e}_1 \otimes \underline{e}_1$ with

$$p_0 = -\frac{2}{3} \left(\bar{\lambda}_1^2 - \frac{1}{\lambda_1} \right)$$

$$\sigma_0 = \frac{4}{3} \left(\bar{\lambda}_1^2 - \frac{1}{\lambda_1} \right) - p_0 = 2 \left(\bar{\lambda}_1^2 - \frac{1}{\lambda_1} \right)$$

Once we have the solution of the non-linear static problem, we can derive the parameters for the wave propagation problem (66). The geometric stiffness is given by $\underline{\underline{\sigma}}_0$, while the material stiffness is obtained by linearization and it reduces to

$$\zeta = \frac{4}{3} \text{tr}(\underline{\underline{C}}_0) + 2p_0 = \frac{4}{\bar{\lambda}_1} \quad (68)$$

From these preliminary results we can already observe that increasing the imposed stretch $\bar{\lambda}_1$ results in an increase of the geometric stiffness and a reduction of the material stiffness parameter ζ .

The volumetric transient source for the excitation of the wave is defined with the previous formula (62) and polarized in the \underline{e}_3 -direction such that $\underline{f} = f(\underline{x}_0, t) \cdot \underline{e}_3$. The spatial excitation is localized in the center of the stretched cube. We take $r_0 = 0.05$, $\sigma_s = 10$ and $A_m = 5$. For the temporal push we have a Ricker wavelet with a small offset $t_0 = 0.04$ and $\sigma_\tau = 200$.

In figure 10, we see an example of propagation of the elastic wave within the incompressible cube. The mesh is discretized with 30 hexahedra in each direction, with $\mathcal{Q}_4 - \mathcal{Q}_2^{disc}$ elements and 5.3M DoFs. The average time to run these simulations in a normal machine with visual results is of 6 minutes.

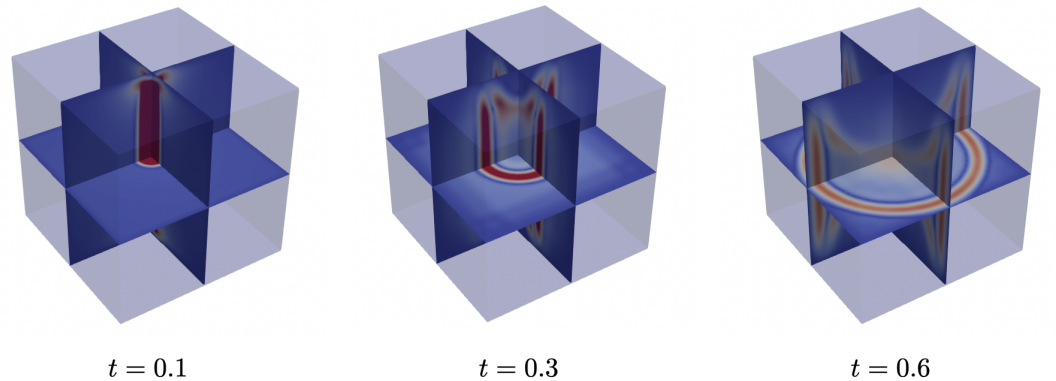


Figure 10. Simulation of 3D elastic wave propagation within an incompressible isotropic cube. In the images we observe the propagation in time (from left to right) of the magnitude of the displacement field. The transient source is focalized in the center of the cube and polarized in the vertical direction. Magnitude displacement in $[0.0, 0.1]$

701 In order to see the effect of loaded configuration over the propagation of the wave,
702 we consider three cases of stretch $\bar{\lambda}_1 = [1.0, 1.1, 1.2]$. In figure 11, we present results of
703 wave propagation in the mid $x_1 - x_2$ plane of the cube. In the first case, where the
704 cube has no pre-stretch, the wave propagates with a circular symmetric shape as the
705 one given by the source. By increasing the static stretch $\bar{\lambda}_1$, we observe a deformation
706 of the wave front, for which the wave is faster in the direction x_1 of the imposed stretch
707 and slower in the perpendicular direction x_2 . As we noticed before, the material
708 stiffness has actually decreased since the parameter ζ , related to the Neo-Hookean
709 density function, is inversely proportional to the pre-stretch λ_1 . As a consequence,
710 a major role for the variation in the wave propagation is played by the geometric
711 stiffness, which depends on the non-linear variation of strains and the pre-stress which
712 has the only non-zero component in the \underline{x} direction.

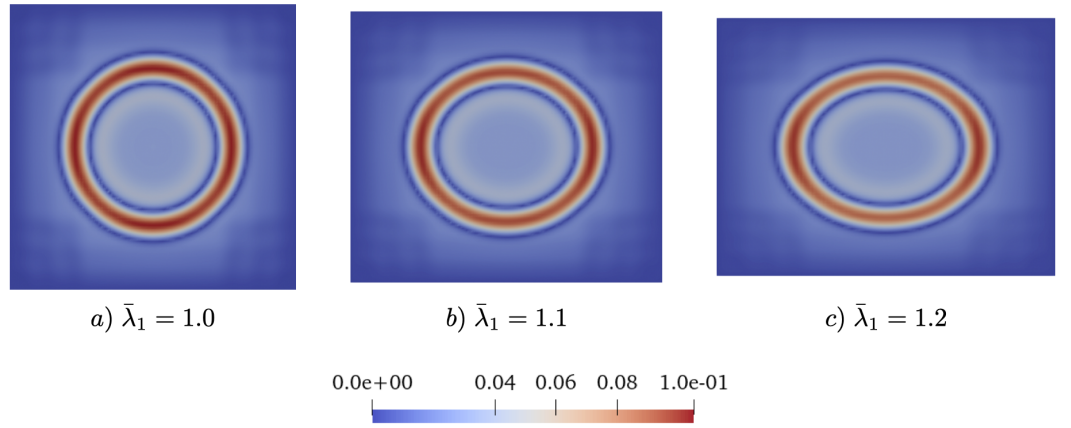


Figure 11. Increasing effect of pre-stretch over elastic wave propagation in nearly-incompressible cube with non-linear isotropic behavior subjected to uniaxial tension. Displacement magnitude in the middle $x_1 - x_2$ plane. From left $\bar{\lambda}_1 = [1.0, 1.1, 1.2]$. Mesh discretized with 5.3M DoFs. Wave observed at $t = 0.3$

713 To have a final validation of these results, we looked at the elastic wave propagation
714 in the cube artificially turning off and on different terms of the problem, in order
715 to compare the different effects. In figure 12(a) we show the reference case, wave
716 propagation in the cube subjected to a prestretch $\bar{\lambda}_1 = 1.2$, which corresponds to
717 the simulation previously discussed (figure 11(c)). Then, we consider (figure 12(b))
718 the wave propagation in a domain that corresponds to the one obtained from the
719 non-linear static problem with $\bar{\lambda}_1 = 1.2$, but in the simulation we do not account for
720 any pre-stretch effect on the material and geometric stiffness. The simulation in figure
721 12(c) show the wave propagation with *only the prestress effect* due to $\bar{\lambda}_1 = 1.2$, while
722 we "turn off" the material stiffness term taking $\zeta = 4$. Finally, in figure 12(d), we have
723 taken $\sigma_0 = 0$ and $\zeta = 4/\bar{\lambda}_1$ with $\bar{\lambda}_1 = 1.2$, which corresponds to a material behavior
724 with the linearized material stiffness term and *no prestress* effect is considered.

725 The resulting displacement fields reflect what we expected by looking at the physical
726 terms of our problem (66). When considering only the linearized material behavior
727 (figure 12(d)), we see that the wave propagates homogeneously slower with respect the
728 unloaded behavior given in (figure 12(a)). As mentioned before, the material stiffness
729 parameter ζ is inversionally proportional to the pre-stretch (68), which has a visible

730 effect on the propagation of the shear wave. Obviously, this is related to the choice of
 731 the non-linear potential. The neo-hookean potential (65) presents a decreasing stiffness
 732 (slope) with the increasing loading condition, which corresponds to a softer response
 733 of the linearized material.

734 Still, the overall results that we have observed in figure 11 is an anisotropic response
 735 of the solid, with a relevant increase of the wave speed in the direction of the pre-
 736 stretch and a decrease in the perpendicular direction. As we can see in figure 12(c), the
 737 geometric stiffness (prestress) is the most relevant term in the propagation of the shear
 738 wave. Indeed, if we compare this result with the "real" solution of the pre-loaded wave
 739 propagation problem 12(a), the wave with only the geometric stiffness (c) actually
 740 propagates faster. Trivially, this result comes from the sum of the geometric effect
 741 with the material effect, where the latter tends to lower the stiffness of the material
 742 and so the speed of the wave front.

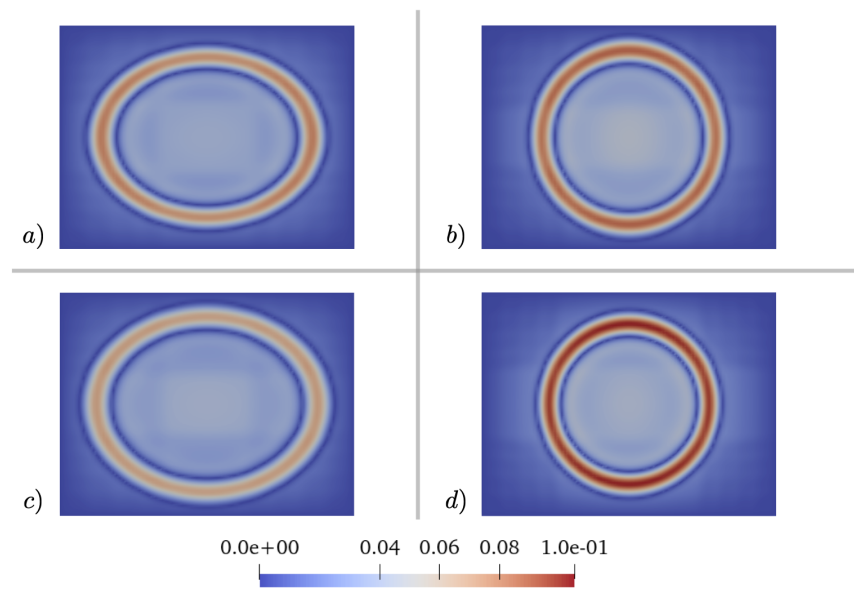


Figure 12. Comparison of physical effects over wave propagation at time $t = 0.3$. (a) Reference case: linear wave propagation without any pre-load. (b) Deformed geometry as for $\bar{\lambda}_1 = 1.2$, but no pre-load effect accounted in the problem. (c) Solution for $\bar{\lambda}_1 = 1.2$ with related material stiffness, while the geometric stiffness term is neglected ($\sigma_0 = 0$). (d) Solution for $\bar{\lambda}_1 = 1.2$ with related geometric stiffness, while the material stiffness term is neglected ($\zeta = 4$).

743 6 Conclusion

744 In conclusion, we have presented a stable, robust and efficient explicit scheme for the
 745 simulation of linear elastic wave propagation in nearly-incompressible solids. Further-
 746 more, the scheme allows the approximation of solids with non-cartesian geometries
 747 subjected to large static deformations.

748 The efficiency of the scheme is based on the use of *damped Chebyshev polynomials*,
 749 applied here for the first time to treat dynamic problems with nearly-incompressible
 750 solids. This method, together with the space discretization strategy, guarantees a
 751 reduction of simulation time of at least a factor 2 with respect to the Leap-Frog
 752 method, second order convergence and preservation of energy. Finally, we have

753 presented the applicability of the scheme for wave propagation in non-linear solids
754 with large deformations.