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# Finite element method. Detailed proofs to be formalized in Coq

François Clément, Vincent Martin

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## Finite element method. Detailed proofs to be formalized in Coq

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**Abstract:** To obtain the highest confidence on the correction of numerical simulation programs for the resolution of Partial Differential Equations (PDEs), one has to formalize the mathematical notions and results that allow to establish the soundness of the approach. The finite element method is one of the popular tools for the numerical resolution of a wide range of PDEs. The purpose of this document is to provide the formal proof community with very detailed pen-and-paper proofs for the construction of the Lagrange finite elements of any degree on simplices in positive dimension.

**Key-words:** partial differential equation, finite element method, Lagrange finite elements, simplex, detailed mathematical proof, formal proof in real analysis

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## Méthode des éléments finis. Preuves détaillées en vue d'une formalisation en Coq

**Résumé :** Pour obtenir la plus grande confiance dans l'exactitude des résultats de programmes de simulation numérique pour la résolution d'Équations aux Dérivées Partielles (EDPs), il faut formaliser les notions mathématiques sur lesquelles ils sont basés. La méthode des éléments finis est l'un des outils les plus utilisés pour la résolution de larges gammes d'EDPs. L'objectif de ce document est de fournir à la communauté des chercheurs en preuve formelle des preuves papiers très détaillées pour la construction des éléments finis de Lagrange de tout degré sur des simplexes en dimension quelconque.

**Mots-clés :** équation aux dérivées partielles, méthode des éléments finis, éléments finis de Lagrange, simplexe, preuve mathématique détaillée, preuve formelle en analyse réelle

# Foreword

This document is intended to evolve over time. Last version is release 1.0 (i.e. version 1). It is available at <https://hal.inria.fr/hal-04713897/>.

**Version 1** (release 1.0, 2024/09/30) is the first release.

It covers:

- the general definition of finite element;
- some results about simplicial geometry;
- the construction of simplicial Lagrange finite elements on a segment, including results about  $\mathbb{P}_k^1$  Lagrange polynomials;
- the construction of simplicial Lagrange finite elements in dimension  $d \geq 1$ , including:
  - the construction of multi-indices of given maximum length;
  - results about multivariate polynomials, such as the linear independence of monomials, and the Euclidean division by a monomial;
  - results about  $\mathbb{P}_1^d$  Lagrange polynomials, such as their view as barycentric coordinates;
  - results about affine geometric mappings such as the transformation of  $l$ -faces of dimension  $l \leq d$ ;
  - results about Lagrange nodes and Lagrange linear forms of  $\mathbb{P}_k^d$ ;
  - the proofs of unisolvence of  $\mathbb{P}_k^d$ , and of face unisolvence.



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# Part I

## Overview

# Chapter 1

## Introduction

### 1.1 Formal proof

A formal proof is conducted in a logical framework that provides dedicated computer programs to mechanically check the validity of the proof, the so-called formal proof assistants. Such formal proofs may concern known mathematical theorems, but also properties of some piece of other computer programs, e.g. see [25], and [5, Glossary p. 343]. This field of computer science is extremely popular as it allows to certify with no doubt the behavior of critical programs.

Interactive theorem provers are now known to be able to tackle real analysis. For instance, in the field of Ordinary Differential Equations (ODEs) with Isabelle/HOL [27, 26, 28], and Coq [29], or in the field of Partial Differential Equations (PDEs), again with Isabelle/HOL [1], and Coq [4, 5]. In the latter example, the salient aspect is that the round-off error due to the use of IEEE-754 floating-point arithmetic can also be fully taken into account. But the price to pay is that all the details of the proofs has to be dealt with, and thus the availability of very detailed pen-and-paper proofs is a major asset.

### 1.2 Objective

Our current long-term purpose is to formally prove programs implementing the Finite Element Method (FEM). The FEM is widely used to solve a broad class of PDEs, mainly because of its sound mathematical foundation, see Section 1.3.1. A description of the typical various components that are necessary to solve a partial differential equation with the FEM is shown in Figure 1.1.

Consider a physical stationary problem over a domain  $\Omega \subset \mathbb{R}^d$ , where  $d$  is often 1, 2 or 3. First, a *continuous* variational formulation is set on some functional spaces, typically Hilbert or Banach spaces (complete normed vector spaces, equipped or not with a scalar product) over the domain, denoted by  $V$ . In all cases,  $V$  is infinite-dimensional and thus cannot be represented on a computer. In favorable cases, a theorem, such as the Lax–Milgram theorem (e.g. see [14]), provides existence and uniqueness of the continuous solution. Here, “continuous” means that the solution is a general function defined over the *whole*  $\Omega$ , in contrast to a “discrete” solution that is entirely defined by a *finite* number of data.

Second, a *discrete* variational formulation is built. To do so, a mesh is constructed on  $\Omega$ , i.e. a finite set of polygons that covers  $\Omega$  (with additional properties). These polygons are called geometric elements. A *finite element*, i.e. a space of polynomials up to a certain degree and the type of primary values that are computed (such as values at certain points, means, fluxes), is chosen in these geometric elements. This defines the finite dimensional approximation space  $V_h$  that (inexactly) represents the continuous Hilbert or Banach space  $V$ . The discrete variational formulation is also chosen. In some cases, it is simply the restriction of the continuous variational formula to  $V_h$ . In most cases, additional terms are introduced to improve some numerical features (such as robustness and convergence), this is often called the numerical scheme. A theorem should

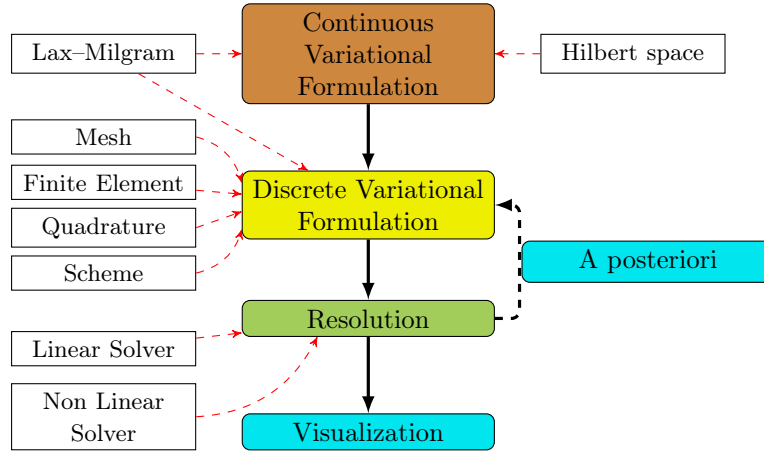


Figure 1.1: Brief recall of the various components of a typical (stationary) FEM formulation. The black thick arrows depict the usual construction steps when solving a problem via the FEM. The black thick dashed arrow shows the possible feedback loop that *a posteriori* analysis allows, to change the mesh, the finite element (i.e. the order of polynomial approximation, the quadrature formula or the numerical scheme). The red dashed arrows depict when a theorem or a numerical tool takes place.

provide existence, uniqueness of the discrete solution, and the convergence in some sense of this discrete solution towards the continuous solution. This discrete formulation also involves generally an approximation to compute the integrals, which is called the quadrature.

Finally, the discrete system (linear or nonlinear) is built and solved numerically. The result is a vector of floating points that can be stored and used to visualize the solution.

In addition, *a posteriori* computations can be devised to improve the results, with a change of mesh for instance. When the problem is unsteady, the discrete part of the procedure is partly repeated: one tries to re-use as much as possible the computed data to avoid numerous expensive computations.

The Lax–Milgram theorem, one of the key ingredients to establish the FEM in some cases, was already addressed, see [14] for a detailed pen-and-paper proof, and [6] for a formal proof in Coq. A part of the Lebesgue integration theory was also addressed, see [15] for detailed pen-and-paper proofs, and [7, 8] for a formalization in Coq.

The present document is a further contribution to our ultimate goal. It focuses only on the definition of the finite element itself as a triple, see Chapter 6, and on the construction of Lagrange finite elements of degree  $k$  on  $d$ -simplices, see Chapter 9. These Lagrange finite elements are defined over simplices (segments, triangles or tetrahedra when  $d = 1, 2, 3$ ), with polynomials having a total degree no greater than  $k$  (denoted  $\mathbb{P}_k^d$  in the sequel), and with nodal values that are evenly distributed over the whole simplex, including its vertices.

### 1.3 The finite elements

In this document,  $d \in \mathbb{N}^*$  denotes the space dimension (typically  $d \in \{1, 2, 3\}$ ), and  $k \in \mathbb{N}$  denotes the *degree of approximation*, i.e. some polynomial degree.

Let us first define two polynomial spaces that are useful in the sequel.  $\mathbb{P}_k^d$ , as already said, is the vector space of polynomials of  $d$  variables and of total degree at most  $k$ ,

$$\mathbb{P}_k^d \stackrel{\text{def.}}{=} \left\{ \left( \mathbf{x} \mapsto \sum_{\alpha \in \mathbb{N}^d, \sum_{i=1}^d \alpha_i \leq k} a_{\alpha} x_1^{\alpha_1} \dots x_d^{\alpha_d} \right) : \mathbb{R}^d \rightarrow \mathbb{R} \mid a_{\alpha} \in \mathbb{R} \right\}.$$

$\mathbb{Q}_k^d$  is the vector space of polynomials of  $d$  variables and of degree at most  $k$  for each variable,

$$\mathbb{Q}_k^d \stackrel{\text{def.}}{=} \left\{ \left( \mathbf{x} \mapsto \sum_{\beta \in \mathbb{N}^d, \forall i \in [1..d], \beta_i \leq k} a_{\beta} x_1^{\beta_1} \dots x_d^{\beta_d} \right) : \mathbb{R}^d \rightarrow \mathbb{R} \mid a_{\beta} \in \mathbb{R} \right\}.$$

$\mathbb{P}_k^d$  is the polynomial approximation space for many simplicial elements, thus it will be used in the whole document, whereas  $\mathbb{Q}_k^d$ , used mainly for hexahedra, is necessary for this introductory part only. For all  $k \in \mathbb{N}$ , we have obviously  $\mathbb{P}_k^1 = \mathbb{Q}_k^1$ , and, for all  $d \geq 2$ ,  $\mathbb{P}_k^d \subsetneq \mathbb{Q}_k^d$ . Note that  $\mathbb{P}_1^d$  is the space of affine functions to  $\mathbb{R}$ .

### 1.3.1 Some references

The mathematical literature dedicated to the FEM is very rich, FEM is a very active field of research. Thus, it is not intended in the present document to be comprehensive, nor to review the various approaches to this method. We limit ourselves to some books among many other references: Ciarlet [13], Ern [18], Ern and Guermond [19, 20, 21, 22], Babuška and Strouboulis [2], Quarteroni and Valli [31], Brenner and Scott [10], and Zienkiewicz, Taylor and Zhu [34]. The first part of the Handbook of Numerical Analysis is dedicated to the finite element methods [12, 33, 32, 3, 23].

The interested reader will find many references in all these books.

### 1.3.2 The finite element as a triple

To define a finite element, in addition to  $d$  and  $k$ , some other notations are necessary:  $q \in \mathbb{N}^*$  denotes the size of the physical unknowns ( $q$  is generally 1 or  $d$ ), and  $n_{\text{sh}} \in \mathbb{N}^*$  represents the number of so-called local *shape functions*, i.e. a number of linear forms on  $P$  (see below). A “correct” finite element needs to satisfy the unisolvence property, that relates  $\Sigma$  (see below) and  $P$  (and thus  $n_{\text{sh}}$  and  $k$ ).

Following [13] and [20], a finite element is mathematically defined as a triple  $(K, P, \Sigma)$ , where:

- $K$  is a geometrical “element”, i.e. a closed bounded set with nonempty interior. Typically,  $K$  is a nondegenerate polygonal set (a polyhedron), or the image of such a set by a regular function.
- $P$  is a nonzero, finite-dimensional vector space of functions from  $\mathbb{R}^d$  to  $\mathbb{R}^q$ . It is meant to be a space of polynomial functions, or the image of such a space via regular functions.
- $\Sigma$  is a collection of linear forms on  $P$ ,  $\Sigma \stackrel{\text{def.}}{=} (\sigma_i)_{i \in [1..n_{\text{sh}}]}$  where each  $\sigma_i$  is a linear form on  $P$ .

The application  $\phi_{\Sigma} : P \rightarrow \mathbb{R}^{n_{\text{sh}}}$  defined by  $\phi_{\Sigma}(p) \stackrel{\text{def.}}{=} (\sigma_i(p))_{i \in [1..n_{\text{sh}}]}$  is an isomorphism.

The bijectivity of  $\phi_{\Sigma}$  is called *unisolvence*. The  $\sigma_i$  are the degrees of freedom, and  $n_{\text{sh}}$  is the number of degrees of freedom. It can be easily seen that the unisolvence requires that the dimension of  $P$  be equal to  $n_{\text{sh}}$ .

A finite element is generally defined over what is called a *reference* geometrical element, a regular polyhedron  $\hat{K}$ : the reference FE is denoted  $(\hat{K}, \hat{P}, \hat{\Sigma})$ . Typically,  $\hat{K}$  is the unit rectangle simplex, see below Section 1.3.3.1 or Definition 1437. It can also be the unit  $d$ -cube. This reference geometrical element  $\hat{K}$  is transported to a so-called *current* element of the mesh denoted by  $K$ , by a geometric mapping that we call  $\varphi_{\text{geo}}^K : \mathbb{R}^d \rightarrow \mathbb{R}^d$ , see a simple 2D example in Figure 9.4. The finite element  $(K, P, \Sigma)$  on  $K$  is then somewhat the image of  $(\hat{K}, \hat{P}, \hat{\Sigma})$  by  $\varphi_{\text{geo}}^K$ , see [20, Sec. 9.1].

The 1D Lagrange finite element is constructed in this way in Chapter 8. Note that in Chapter 9, the general case  $d \geq 1$  is treated the other way around: we deal directly with a general nondegenerate  $d$ -simplex, and then introduce the reference Lagrange finite element as a particular case. The reason for this is explained at the end of Section 3.1.

According to the hypotheses on  $\varphi_{\text{geo}}^K$ ,  $K$  can be a polyhedron with planar faces, or not, and can be convex, or not. Not all geometrical elements are possible, though. In this document, we assume that the mesh is *affine*, which means that  $\varphi_{\text{geo}}^K$  is supposed to be invertible and affine. The non-affine meshes are more tricky to handle, see Section 1.3.3.1 below. Note that the geometric mapping is affine iff each of its components is in  $\mathbb{P}_1^d$ . With a slight abuse of notation, the geometrical element is commonly labelled  $\mathbb{P}_1^d$ ,  $\mathbb{Q}_1^d$  or  $\mathbb{P}_2^d \dots$  when the components of  $\varphi_{\text{geo}}^K$  are in the corresponding space.

In this document, the FE triple is specified in Definition 1424, using the hypothesis of an affine mesh, thus  $K$  is assumed to be a polyhedron.

Note that one can also define a finite element as a quadruple, adding to the previous triple a linear interpolation operator that maps continuously functions in a larger space containing  $P$  (to be defined) to  $P$  itself, see [20, Sec. 5.3.]. The construction of the interpolation operator is not covered in the present version of the document.

### 1.3.3 Lagrange finite elements

There exist various families of finite elements, that are more or less adapted to the considered problem. For each family, there are usually a version for the simplices, and another for the quadrangles (when  $d = 2$ ), or the hexahedra (when  $d = 3$ ). Some finite elements can also be constructed on other geometries such as hexagons ( $d = 2$ ) or prisms ( $d = 3$ ).

First, let us discuss briefly the geometry of the FE. Then, we introduce the most common families of FE, before presenting the nodal finite elements, and in particular Lagrange FE.

#### 1.3.3.1 Typology of geometries: simplex vs quadrangular

When the geometrical element is a simplex, or when it is a  $d$ -cuboid (defined as  $\prod_{i=1}^d [x_i, y_i]$ , for some  $(x_i)_{i \in [1..d]}$  and  $(y_i)_{i \in [1..d]}$  in  $\mathbb{R}$ , with  $x_i < y_i$ ), the geometrical mapping  $\varphi_{\text{geo}}^K$  is affine, thus covered by the assumption in this document. For cuboids, note that the FE can be defined by a tensorization of 1D finite elements, see [20, Sec. 6.4].

However, dealing with non-affine geometric element may be necessary. For instance, to better approximate the geometry of the domain (think of airplane wings for instance), there exist *curved* simplicial finite elements: for instance,  $\mathbb{P}_2^d$ -simplices are simplices where the geometrical mapping is in  $(\mathbb{P}_2^d)^d$ . Thus, when  $d = 2$ , each edge of  $K$  is a part of a parabola, see [20, Chap. 13]. Another important non-affine mesh case in practice, is the  $\mathbb{Q}_1^d$ -quadrangles and hexahedra: in this case,  $\varphi_{\text{geo}}^K$  is in  $(\mathbb{Q}_1^d)^d$  (thus not affine *a priori*). In 3D in particular, this may give complicated geometries, as general nondegenerate hexahedra are not always convex, and their faces are not necessarily planar.

This leads to complex numerical issues: meshing the domain  $\Omega$  can be a very hard task (it is *not* simple, even with simplices). One needs to avoid “wrong” geometrical elements (for instance self-intersecting), the approximation results are also more involved, the integration needs more care, and so on.

To conclude on geometrical element, we say once again that we limit ourselves to affine, invertible geometrical mapping, and more specifically to geometrical elements that are nondegenerate standard simplices.

In this case, the reference simplex  $\hat{K}$  is the unit rectangle simplex, see Definition 1437. It is the convex envelop of the reference vertices  $[\hat{\mathbf{v}}]^d \stackrel{\text{def.}}{=} (\hat{\mathbf{v}}_0, \dots, \hat{\mathbf{v}}_n)$ , where  $\hat{\mathbf{v}}_0$  is the canonical origin  $\mathbf{0} \stackrel{\text{def.}}{=} (0, \dots, 0)^T \in \mathbb{R}^d$ , and for all  $i \in [1..d]$ ,  $\hat{\mathbf{v}}_i$  corresponds to the  $i$ -th canonical basis vector  $\mathbf{e}_i \stackrel{\text{def.}}{=} (0, \dots, 0, 1, 0, \dots, 0)^T \in \mathbb{R}^d$  (the 1 is in position  $i$ ).

Given  $d+1$  points  $[\mathbf{v}]^d \stackrel{\text{def.}}{=} (\mathbf{v}_0, \dots, \mathbf{v}_n)$  in  $\mathbb{R}^d$ , the simplex  $K^{[\mathbf{v}]^d}$  is the convex envelop of  $[\mathbf{v}]^d$ , see Definition 1440. It is proven nondegenerate iff  $[\mathbf{v}]^d$  is affinely independent. The geometrical mapping that maps  $\hat{K}$  to  $K$  is defined as, see Definition 1547,  $\varphi_{\text{geo}}^K(\hat{\mathbf{x}}) \stackrel{\text{def.}}{=} \sum_{i=0}^d \hat{\mathcal{L}}_i^{1,d}(\hat{\mathbf{x}}) \mathbf{v}_i$  where

$(\hat{\mathcal{L}}_i^{1,d})_{i \in [0..d]}$  is the Lagrange basis of  $\mathbb{P}_1^d$ , such that  $\hat{\mathcal{L}}_0^{1,d}(\hat{\mathbf{x}}) \stackrel{\text{def.}}{=} 1 - \sum_{j=0}^d \hat{x}_j$ , and for all  $i \in [1..d]$ ,  $\hat{\mathcal{L}}_i^{1,d}(\hat{\mathbf{x}}) \stackrel{\text{def.}}{=} \hat{x}_i$ . It is easy to see that  $\varphi_{\text{geo}}^K$  is affine.

### 1.3.3.2 Some finite elements

Various approximation spaces  $P$  and linear forms  $\Sigma$  exist in the literature. Just to mention some of the most popular (original references can be found in [20]):

- for the *nodal* FE, the linear forms are the evaluation at some points  $\mathbf{a}_i$ , for  $i \in [1..n_{\text{sh}}]$ , that are called the FE *nodes*:  $\sigma_i(p) = p(\mathbf{a}_i)$ . The Lagrange FE are the most common nodal FE, see [20, Sec. 6.4 and 7.4], and below.
- the Hermite FE are nodal FE, but the linear forms are the evaluation at the nodes of both the function and its derivatives (assuming enough regularity for the function), see [11].
- for the *modal* FE in 1D, the linear forms are the integrals of the product of the function and Legendre polynomials. It can be tensorized for cuboid elements, see [20, Sec. 6.3.2 and 6.4.2].
- canonical hybrid FE mix evaluations at nodal values and integrals of the function over edges, faces, and volumes (in 3D) of the geometrical element, see [20, Sec. 7.6].
- the RT, BDM, BDFM... are *flux* FE, see [20, Chap. 14]. They aim at approximating  $\mathbf{H}(\text{div})$  functions (vector functions in  $(L^2)^d$  whose divergence is in  $L^2$ ). Some of the linear forms are integrals of the normal of the function across the  $(d-1)$ -faces of the simplex.
- the Nedelec family  $\mathbb{N}$  of FE is meant for the approximation of  $\mathbf{H}(\text{curl})$  functions (functions in  $(L^2)^d$  whose curl is in  $(L^2)^d$ ). Some of the linear forms are integrals of the tangential of the function along the edges of the simplex, see [20, Chap. 15].

### 1.3.3.3 Some nodal finite elements, Lagrange finite elements

For the nodal finite elements, there exist various choices of nodes. The Lagrange FE are based on Lagrange nodes: these nodes are evenly distributed in the element, and when  $k \geq 1$ , the vertices are always some of the nodes, see examples for the reference simplices in Figure 9.1. The precise definition using barycentric coordinates is given in Definition 1588. For cuboids, the Lagrange nodes are merely a tensorization of the segment nodes.

For simplices, the approximation space is  $\mathbb{P}_k^d$ , and the Lagrange FE is denoted  ${}^{\text{Lag}}\mathbb{P}_k^d$ . In this document, we clearly separate the notations for the approximation space and for the finite element triple. In cuboids, the approximation space is  $\mathbb{Q}_k^d$ , and the Lagrange FE is denoted  ${}^{\text{Lag}}\mathbb{Q}_k^d$ . In cuboids, it is possible to remove some internal nodes and to reduce the space dimension of the approximation space, which reduces the computational effort, while preserving the accuracy. These FE are called serendipity finite elements, and the approximation space is denoted  $\mathbb{S}_k^d(\subsetneq \mathbb{Q}_k^d)$ , see [20, Sec. 6.4.3].

The polynomial interpolation problem consists in finding a polynomial  $p$  to approximate a given function  $f$ , with the constraint that  $f(\mathbf{x}_i) = p(\mathbf{x}_i)$  for a finite family of chosen points  $(\mathbf{x}_i)_{i \in I}$ . It is well known that choosing Lagrange nodes as interpolation points gives rise to an interpolating polynomial that can have very large oscillations. These oscillations tend to increase when the polynomial degree  $k$  increases. This may lead to poor approximation results and to numerical difficulty for integration. This is why some other nodes may be favored. For instance, in 1D, one can take Gauss–Lobatto nodes, that are the two vertices of the segment and the roots of some Chebyshev polynomials, see [20, Sec. 6.3.5]. This improves drastically the high oscillation effects. The Gauss–Lobatto nodes can be easily tensorized for cuboid elements.

In the present document, we focus on Lagrange simplicial FE, because  ${}^{\text{Lag}}\mathbb{P}_1^d$  and  ${}^{\text{Lag}}\mathbb{P}_2^d$  are the most popular FE, because treating  ${}^{\text{Lag}}\mathbb{P}_k^d$  for all  $k$  and  $d$  is of great interest by itself, and because  $\mathbb{P}_k^d$  is an approximation space that is used for many FE. Thus Lagrange simplicial FE is a natural first step in our development.



## 1.4 Our main sources

Our main source for the proofs in the present text is [20]. The part on affine geometry is inspired by [24].

Note that, because the support for differential calculus is still limited in `Coq`, we choose to favor proof paths using algebraic arguments over those using analysis results. As a consequence, some of our proofs are distinct from those of [20]. In particular, we do not rely on differential calculus to prove the lemmas on geometric mappings that relate various configurations of simplices or faces. As we assume that the meshes are affine (i.e. in our case, made up of simplices that are nondegenerate and have planar hyperfaces), and as we do not need to integrate to define the Lagrange finite element, the results on affine maps and affine spaces are sufficient. Nonetheless, some results of topology (and of analysis) are obviously required to prove that nondegenerate simplices have nonempty interior.

To obtain the dimension of  $\mathbb{P}_k^d$ , defined as the linear span of monomials, we prove the linear independence of these monomials by calculating their partial derivatives and taking the value at  $\mathbf{0}$ . A possible alternative, that is not followed in the `Coq` formalization, is to use the Euclidean division to prove an isomorphism between  $\mathbb{P}_k^d$  and  $\mathbb{P}_{k-1}^d \times \mathbb{P}_k^{d-1}$ , and conclude on the dimension by induction (see Remark 1530).

## 1.5 Contents

The present version of this document covers all material up to the construction of the simplicial Lagrange finite elements in dimension  $d \geq 1$ .

After the general definition of the notion of finite element, and some results about simplicial geometry, it starts with the construction of the simplicial Lagrange finite elements on a segment including results about  $\mathbb{P}_k^1$  Lagrange polynomials; this step is mostly for pedagogical purpose. Then, for any dimension  $d \geq 1$ , it covers the construction of multi-indices of given maximum length. This allows to write the definition of multivariate polynomial spaces as the linear span of monomials, and some results such as the linear independence of monomials using partial derivation and providing the dimension of the considered space of polynomial, the product of polynomials, the composition of polynomials and affine mappings, and the Euclidean division by a monomial. Then, there are some results about  $\mathbb{P}_1^d$  Lagrange polynomials, with in particular their view as barycentric coordinates. This provides results about affine geometric mappings, such as the transformation of  $l$ -faces of dimension  $l \leq d$ . This allows to pass from the reference simplex in dimension  $l$  to a current element in dimension  $d$ . The case  $l = d$  is the most important one and is treated first. Next, some results about Lagrange nodes and Lagrange linear forms of  $\mathbb{P}_k^d$  are given, before the proof of unisolvence of  $\mathbb{P}_k^d$ . This allows to conclude with the construction of the simplicial Lagrange finite elements in dimension  $d \geq 1$ , and the statement and proof of face unisolvence. The formalization in `Coq` of most of these aspects is presented in [30, 9].

It is planned to add more results in a forthcoming version.

## 1.6 Teaching

This document is not primarily meant for teaching usage. The objective was to be as comprehensive as possible in the proofs. This led to very detailed demonstrations, and to a compact style of writing that is not common, and may seem daunting to the uninformed reader.

However, the authors tried to give some insights on the FEM theory and on the proofs and theorems in the introductory chapters. They also strove to give some indications in the proofs when they felt it necessary. They believe that this document could be useful for interested teachers, and dedicated students, in complement to the usual manuals.

## 1.7 Disclaimer

Note that the manuscript itself is not formally proved (and will never be). Indeed,  $\text{\LaTeX}$  compilers are not formal proof tools!

Moreover, formalization is not just straightforward translation of mathematical texts and formulas. Some design choices have to be made and proof paths may differ, mainly to favor usability of Coq theorems and ease formal developments. Thus, there may exist differences between the mathematical setting presented here, and the formal setting developed in Coq [30, 9].

Hence, despite the care taken in its writing, this document might still be prone to errors or holes in the demonstrations. There could also exist simpler paths in the proofs. Please, feel free to inform the authors of any such issue, and to share any comments or suggestions...

## 1.8 Organization

Part I of this document is organized as follows. After the present introductory Chapter 1, the notations are collected in Chapter 2. The chosen proof paths of the main results are then sketched in Chapter 3.

Part II (Chapters 4 to 9) is the core of this document. In this part, the definitions are presented, and the lemmas and theorems are stated with their detailed proofs. Its organization is briefly described at the end of the second introductory Chapter 4.

Chapter 10 concludes and gives some perspectives.

Finally, an appendix gathers the list of statements in Chapter A, and explicit dependencies (both ways) in Chapters B and C. The appendix is not intended for printing!



## Chapter 2

# Notations

In this chapter (as in most of the document), we use the following conventions:

- $d$  and  $l$  denote a dimension, a (usually nonzero) natural number;
- $k$  denotes the order of a finite element, a natural number, e.g. the maximum degree of polynomials in an approximation space;
- the hat diacritical mark “ $\hat{\phantom{x}}$ ” denotes a reference quantity: a reference set, e.g. the reference simplex  $\hat{K}^d$ , or a reference element in a reference set, e.g. a point  $\hat{\mathbf{x}}$  in  $\hat{K}^d$ , see below;
- **boldface** denotes finite families, typically of size  $d$ , and more generally vectors in a vector space, or points in an affine subspace:
  - **boldface** small Greek letters  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  denote multi-indices in  $\mathbb{N}^d$ ;
  - **boldface** small letters, such as  $\mathbf{x}$  and  $\hat{\mathbf{x}}$ , denote points or vectors in  $\mathbb{R}^d$ ;
  - the components of a finite family  $\mathbf{x}$  are usually denoted  $x_i$ , and by default the indices of components start from 1;
- the tilde (resp. check) diacritical mark “ $\tilde{\phantom{x}}$ ” (resp. “ $\check{\phantom{x}}$ ”) denotes a family of size  $d - 1$  that corresponds to the  $d - 1$  first (resp. last) components of the regular family of size  $d$ , or a function of such family. For instance, for  $\boldsymbol{\alpha} \in \mathbb{N}^d$ , the multi-indices  $\tilde{\boldsymbol{\alpha}}$  (resp.  $\check{\boldsymbol{\alpha}}$ ) satisfy  $\boldsymbol{\alpha} = (\alpha_1, \tilde{\boldsymbol{\alpha}}) = (\tilde{\boldsymbol{\alpha}}, \alpha_d)$ , see below.

The following notations and conventions are used throughout this document.

- Logic:
  - Using a compound (tuple of elements of  $X$ , or subset of  $X$ ) in an expression at a location where only a single element makes sense, is a shorthand for the same expression expanded for all elements of the compound; for instance, “ $\forall x, x' \in X$ ” means “ $\forall x \in X, \forall x' \in X$ ”, “ $\forall (x_i)_{i \in I} \in X$ ” means “ $\forall i \in I, x_i \in X$ ”, and “ $x, x' \mathcal{R} x''$ ” means “ $x \mathcal{R} x'$ ” and “ $x' \mathcal{R} x''$ ”;
  - “iff” is a shorthand for “if and only if”.
- Naive set theory:
  - $\text{card}(E)$  denotes the cardinal of some set  $E$ , especially when it is finite, see Lemma 1426;
  - the cartesian products  $E^{d+1}$ ,  $E^d \times E$  and  $E \times E^d$  are assimilated, see Lemma 1533;
  - the set of functions from  $E$  to  $F$  is either denoted  $F^E$ , or through the type annotation “ $E \rightarrow F$ ”. Both compact expressions “let  $f \in F^E$ ” and “let  $f : E \rightarrow F$ ” mean “let  $f$  be a function from  $E$  to  $F$ ”;
  - $\text{rg}(f)$  denotes the range of function  $f : E \rightarrow F$ , i.e. the image  $f(E)$  of its domain, see Lemma 1408.
- Numbers:
  - *positive/negative*, and *increasing/decreasing* are meant in their *strict* sense, i.e. without the possibility of equality. Otherwise, we use *nonnegative/nonpositive*, and *nondecreasing/nonincreasing*;
  - $[n..p]$  denotes the integer interval  $[n, p] \cap \mathbb{N}$  of values from integer  $n$  up to integer  $p$  (both included);

- $\binom{n}{p}$  denotes the binomial coefficient, i.e. the number of subsets of  $p$  elements of a set of  $n$  elements, see Definition 1362;
  - $\delta$  denotes the Kronecker delta function, i.e.  $\delta_{ii} \stackrel{\text{def.}}{=} 1$  for all  $i \in \mathbb{N}$ , and  $\delta_{ij} \stackrel{\text{def.}}{=} 0$  when  $i \neq j$ , see Definition 1365;
  - $\theta_i^d : [0..d] \rightarrow [0..d+1]$  is the “jump” enumeration function that skips  $i$ , see Lemma 1368;
  - $\mathbf{0}$  denotes the family of zeros  $(0, \dots, 0)$ , and  $\mathbf{1}$  denotes the family of ones  $(1, \dots, 1)$ , see Definition 1365;
  - $\mathbf{e}_i$  denotes the  $i$ -th canonical “basis” family with only a 1 in  $i$ -th position, see Definition 1365;
  - $|\alpha|$  denotes the length of a multi-index, i.e. the sum of its components, see Definition 1480;
  - $\alpha!$  denotes the factorial of a multi-index, i.e. the product of the factorials of its components, see Definition 1482;
  - $\delta_{\alpha,\beta}$  denotes the Kronecker delta of two multi-indices, i.e. the product of the Kronecker deltas of their components, see Definition 1484;
  - $\mathcal{A}_k^d$  denotes the set of multi-indices in dimension  $d$  of sum at most  $k$ , see Definitions 1445 (for  $d \stackrel{\text{def.}}{=} 1$ ) and 1486;
  - $\mathcal{C}_k^d$  denotes the set of multi-indices in dimension  $d$  of sum equal to  $k$ , see Definition 1486;
  - $\mathcal{A}_{k,i}^d$  denotes the subset of  $\mathcal{A}_k^d$  of multi-indices with zero  $i$ -th component, see Definition 1486;
  - $\tilde{\alpha}$  denotes the first  $d-1$  components of a multi-index  $\alpha$  in  $\mathbb{N}^d$ , see Remark 1490;
  - $\check{\alpha}$  denotes the last  $d-1$  components of a multi-index  $\alpha$  in  $\mathbb{N}^d$ , see Remark 1490;
  - $\check{\mathcal{S}}_{k,i}^d$  (resp.  $\tilde{\mathcal{S}}_{k,i}^d$ ) denotes the  $i$ -th “vertical” (resp. “horizontal”) slice of  $\mathcal{C}_k^d$ , see Definition 1491;
  - $f_{k,0}^d$  (resp.  $\tilde{f}_{k,0}^d$ ) denotes the function  $\mathcal{A}_k^{d-1} \rightarrow \mathcal{C}_k^d$  on multi-indices that puts  $k$  minus the length at the beginning (resp. at the end), see Lemma 1500;
  - $f_{k,i}^d : \mathcal{A}_k^{d-1} \rightarrow \mathcal{A}_{k,i}^d$  denotes the function on multi-indices that inserts 0 in  $i$ -th position, see Lemma 1501.
- General topology:
    - $\overset{\circ}{K}$  denotes the interior of some subset  $K$  of some topological space, see Definition 1424.
  - Calculus:
    - $\partial^\beta$  denotes the partial derivative of order  $\beta$ , a multi-index, see Lemma 1517;
    - $Df$  denotes the differential of the function  $f$  of appropriate type, see Lemma 1545.
  - Linear and affine algebra:
    - $\text{span}(u_1, \dots, u_n)$  denotes the linear span of the family of vectors  $(u_i)_{i \in [1..n]}$ , with the convention that  $\text{span}(\emptyset) \stackrel{\text{def.}}{=} \{\mathbf{0}\}$ , see Definitions 83 and 1417, and Lemma 1387;
    - $\ker(f)$  denotes the kernel of morphism  $f$  (usually a linear map from a vector space to another), see Definition 101, and Lemma 1369;
    - $\dim(E)$  denotes the finite dimension of a vector space  $E$ , i.e. the common cardinality of its bases, see Lemma 1374;
    - $\mathcal{M}_{n,p}(\mathbb{R})$ , or simply  $\mathcal{M}_{n,p}$ , denotes the space of matrices with  $n$  lines and  $m$  columns, see Lemma 1399;
    - for a matrix  $A \in \mathcal{M}_{n,p}(\mathbb{R})$ , where  $n, p > 0$ , the line  $i \in [1..n]$  of  $A$  is denoted by  $\underline{A}_i$ , and its column  $j \in [1..p]$  is denoted by  $A_j$ , see Lemma 1556;
    - $\mathcal{L}(E, F)$  denotes the vector space of linear maps from vector space  $E$  to vector space  $F$ , see Definition 65, and Lemma 1374;
    - $\mathcal{E}' \stackrel{\text{def.}}{=} \mathbf{x}_0 + E'$  denotes the affine subspace of direction  $E'$ , a vector subspace of  $E$ , and origin  $\mathbf{x}_0 \in E$ , see Definition 1379;
    - $\text{Aff}(E, F)$  denotes the vector space of affine maps from vector space  $E$  to vector space  $F$ , see Lemma 1396.
  - Geometry:

- $[[\mathbf{v}]]^{n,d} \stackrel{\text{def.}}{=} (\mathbf{v}_0, \dots, \mathbf{v}_n)$  denotes a family of  $n + 1$  points in  $\mathbb{R}^d$ , usually indexed by  $[0..n]$ ,  $[[\mathbf{v}]]^d$  is a shorthand for  $[[\mathbf{v}]]^{d,d}$ , see Definition 1365; it is the notation used to represent the vertices of the geometry of finite elements, see Remark 1433;
  - $[[\mathbf{v}_{\pi^l}]]^{l,d}$  is the sub-family  $(\mathbf{v}_{\pi^l(i)})_{i \in [0..l]}$  where  $\pi^l : [0..l] \rightarrow [0..n]$ , see Lemma 1414;
  - $[[\hat{\mathbf{v}}]]^d \stackrel{\text{def.}}{=} (\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_d)$  denotes the family of  $d + 1$  reference points in  $\mathbb{R}^d$ , see Definition 1434;
  - $\mathbf{g}^{[[\mathbf{v}]]^d}$  denotes the isobarycenter of the family of points  $[[\mathbf{v}]]^d$ ; see Definition 1401;
  - $\hat{\mathbf{g}}^d$  denotes the isobarycenter of the family of reference points  $[[\hat{\mathbf{v}}]]^d$ ; see Lemma 1435;
  - $K^{[[\mathbf{v}]]^d}$  denotes the simplex of vertices  $[[\mathbf{v}]]^d$  in  $\mathbb{R}^d$ , see Definition 1440;
  - $\hat{K}^d$  denotes the reference simplex in  $\mathbb{R}^d$ , its vertices are the reference points  $[[\hat{\mathbf{v}}]]^d$ , see Definition 1437;
  - $\{\{a\}\}_{\mathcal{A}_k^1}$  denotes  $k + 1$  nodes in  $\mathbb{R}$ , see Definitions 1432 and 1448;
  - $\{\{\hat{a}\}\}_{\mathcal{A}_k^1}$  denotes the reference Lagrange nodes of  $\mathbb{P}_k^1$  that are equally distributed over the reference simplex  $[0, 1]$  in  $\mathbb{R}$ , see Definition 1453;
  - $\{\{\mathbf{a}\}\}_{\mathcal{B}_d}$  denotes a family of  $\text{card}(\mathcal{B}_d)$  points in  $\mathbb{R}^d$ , see Definition 1432, it is the notation used to represent the nodes of the geometry of nodal finite elements, see Remark 1433;
  - $\{\{\mathbf{a}\}\}_{\mathcal{A}_k^d}$  denotes the Lagrange nodes of  $\mathbb{P}_k^d$  that are equally distributed over the simplex  $K^{[[\mathbf{v}]]^d}$  in  $\mathbb{R}^d$ , see Definition 1588;
  - $\{\{\hat{\mathbf{a}}\}\}_{\mathcal{A}_k^d}$  denotes the reference Lagrange nodes of  $\mathbb{P}_k^d$  that are equally distributed over the reference simplex  $\hat{K}^d$  in  $\mathbb{R}^d$ , see Lemma 1599;
  - $[[\mathbf{v}]]^d$  denotes the sub-vertices of the Lagrange nodes of  $\mathbb{P}_k^d$  with respect to  $\mathbf{v}_0$ , see Definition 1594;
  - $\{\{\mathbf{a}\}\}_{\mathcal{A}_{k-1}^d}$  denotes the Lagrange sub-nodes of  $\mathbb{P}_{k-1}^d$  associated with the sub-vertices  $[[\mathbf{v}]]^d$  with respect to  $\mathbf{v}_0$ , see Lemma 1598;
  - $\varphi_{\text{geo}}^{[[v]]^1}$  denotes the geometric mapping associated with  $[[v]]^1$ , see Definition 1461;
  - $\varphi_{\text{geo}}^{[[\mathbf{v}]]^d}$  denotes the geometric mapping associated with  $[[\mathbf{v}]]^d$ , see Definition 1547;
  - $A_{\text{geo}}^{[[\mathbf{v}]]^d}$  denotes the square matrix  $(\mathbf{v}_1 - \mathbf{v}_0 \quad \mathbf{v}_2 - \mathbf{v}_0 \quad \dots \quad \mathbf{v}_d - \mathbf{v}_0)$ , see Lemma 1550;
  - $\lambda_i^{[[\mathbf{v}]]^d}(\mathbf{x})$  denotes the  $i$ -th barycentric coordinate of  $\mathbf{x}$  with respect to  $[[\mathbf{v}]]^d$ , see Lemma 1559;
  - $\mathcal{H}_i^{[[\mathbf{v}]]^d}$  denotes the  $i$ -th face hyperplane with respect to  $K^{[[\mathbf{v}]]^d}$ , opposite the vertex  $\mathbf{v}_i$ , see Definition 1562;
  - $\hat{\mathcal{H}}_i^d \stackrel{\text{def.}}{=} \mathcal{H}_i^{[[\hat{\mathbf{v}}]]^d}$  denotes the  $i$ -th reference face hyperplane with respect to  $\hat{K}^d$ , opposite the reference vertex  $\hat{\mathbf{v}}_i$ , see Definition 1562;
  - $H_i^{[[\mathbf{v}]]^d}$  denotes the  $i$ -th hyperface of simplex  $K^{[[\mathbf{v}]]^d}$ , opposite the vertex  $\mathbf{v}_i$ , see Definition 1566;
  - $\mathcal{F}_{\pi^l}^{[[\mathbf{v}]]^d}$  denotes the  $l$ -face affine space having the vertices  $[[\mathbf{v}_{\pi^l}]]^{l,d}$ , see Definition 1568;
  - $F_{\pi^l}^{[[\mathbf{v}]]^d}$  denotes the  $l$ -face having the vertices  $[[\mathbf{v}_{\pi^l}]]^{l,d}$ , see Definition 1572;
  - $\phi_{\pi^l}^{[[\mathbf{v}]]^d}$  is the geometric  $l$ -face mapping associated with the  $l$ -face  $F_{\pi^l}^{[[\mathbf{v}]]^d}$ , see Definition 1578.
- Polynomials:
    - $X_k \stackrel{\text{def.}}{=} (x \in \mathbb{R} \mapsto x^k)$  denotes the monomial of a single variable of degree  $k$  in  $\mathbb{R}$  (it is simply denoted 1 when  $k \stackrel{\text{def.}}{=} 0$ ), see Definition 1417;
    - $\mathbf{X}^\alpha \stackrel{\text{def.}}{=} (\mathbf{x} \in \mathbb{R}^d \mapsto \prod_{i=1}^d x_i^{\alpha_i})$  denotes the monomial of  $d$  variables of degree  $\sum_{i=1}^d \alpha_i$ , 1 is a shortcut for  $\mathbf{X}^{\mathbf{0}}$  (the constant function of value 1), and  $X_i$  is a shortcut for  $\mathbf{X}^{\mathbf{e}_i}$ , see Definition 1503;
    - $\mathcal{P}(\mathbb{R}^d)$  denotes the algebra of polynomials of  $d$  variables, see Lemma 1418;
    - $\text{deg}(p)$  denotes the degree of polynomial  $p$ , i.e. the highest degree of its monomials with

- nonzero coefficients, where the degree of a monomial is the sum of the degrees for each variable, see Definition 1510;
- $\mathbb{P}_k^d$  denotes the vector space of polynomials of  $d$  variables and of degree at most  $k$ , see Definition 1505;
  - $\xi^d$  denotes the isomorphism between  $\mathbb{P}_0^d$  and  $\mathbb{P}_0^{d-1}$  that keeps the value of (constant) polynomials, see Lemma 1526;
  - $\zeta_k^d$  denotes the isomorphism between  $\mathbb{P}_k^d$  and  $\mathbb{P}_k^{d-1} \times \mathbb{P}_{k-1}^d$  that realizes the Euclidean division by  $X_d$ , see Lemma 1531;
  - $\mathcal{L}_i^{\{\mathbf{a}\}_{\mathcal{A}_k^1}}$  for  $i \in [0..k]$  denotes the family of Lagrange polynomials in  $\mathbb{P}_k^1$  associated with nodes  $\{\mathbf{a}\}_{\mathcal{A}_k^1} \in \mathbb{R}$ , see Definition 1448;
  - $\hat{\mathcal{L}}_i^{k,1} \stackrel{\text{def.}}{=} \mathcal{L}_i^{\{\hat{\mathbf{a}}\}_{\mathcal{A}_k^1}}$  for  $i \in [0..k]$  denotes the family of reference Lagrange polynomials in  $\mathbb{P}_k^1$  associated with the reference Lagrange nodes  $\{\hat{\mathbf{a}}\}_{\mathcal{A}_k^1} \in \mathbb{R}$ , see Lemma 1455;
  - $\hat{\mathcal{L}}_i^{1,d}$  for  $i \in [0..d]$  denotes the family of reference Lagrange polynomials of  $\mathbb{P}_1^d$ , see Definition 1541;
  - $\mathcal{L}_i^{[\mathbf{v}]^d} \stackrel{\text{def.}}{=} \hat{\mathcal{L}}_i^{1,d} \circ \left( \varphi_{\text{geo}}^{[\mathbf{v}]^d} \right)^{-1}$  for  $i \in [0..d]$  denotes the family of Lagrange polynomials of  $\mathbb{P}_1^d$  associated with  $[\mathbf{v}]^d$ , see Lemma 1552.
- Finite element:
    - $(K, P, \Sigma)$  denotes a finite element where  $K$  represents the geometry,  $P$  the approximation space, and  $\Sigma$  the degrees of freedom, see Definition 1424;
    - $\phi_\Sigma$  denotes the collection of degrees of freedom applications, see Definition 1424;
    - $\Sigma^{\{\mathbf{a}\}_{\mathcal{A}_k^d}} = (\sigma_\alpha)_{\alpha \in \mathcal{A}_k^d}$  denote the Lagrange linear forms associated with the Lagrange nodes of  $\mathbb{P}_k^d$ , see Definition 1608;
    - $\hat{\Sigma}_k^d = (\hat{\sigma}_\alpha)_{\alpha \in \mathcal{A}_k^d}$  denote the reference Lagrange linear forms associated with the reference Lagrange nodes of  $\mathbb{P}_k^d$ , see Definition 1612;
    - $\text{Lag } \mathbb{P}_k^d \stackrel{\text{def.}}{=} \left( K^{[\mathbf{v}]^d}, \mathbb{P}_k^d, \Sigma^{\{\mathbf{a}\}_{\mathcal{A}_k^d}} \right)$  denotes the Lagrange finite element of degree  $k$  associated with vertices  $[\mathbf{v}]^d$ , see Theorems 1629, and 1478 for  $d \stackrel{\text{def.}}{=} 1$ ;
    - $\text{Lag } \hat{\mathbb{P}}_k^d \stackrel{\text{def.}}{=} \left( \hat{K}^d, \mathbb{P}_k^d, \hat{\Sigma}_k^d \right)$  denotes the reference Lagrange finite element of degree  $k$  in dimension  $d$ , see Theorems 1631, and 1460 for  $d \stackrel{\text{def.}}{=} 1$ .

Note that the vector space  $\mathbb{P}_k^d$  of polynomials of total degree at most  $k$  does not have the same notation as the Lagrange finite element  $\text{Lag } \mathbb{P}_k^d$ . We have tried to visually separate the two concepts, although specialists often use the same notation.

## Chapter 3

# Statements and sketches of proofs

This chapter gathers the sketches of the proofs of the main results that are detailed in Part II. Namely: the theorem of unisolvence of  ${}^{\text{Lag}}\mathbb{P}_k^d$ , the lemma of Euclidean division by monomial in  $\mathbb{P}_k^d$ , and a more technical discussion about ordering the multi-indices.

### 3.1 Sketch of the proof of unisolvence of ${}^{\text{Lag}}\mathbb{P}_k^d$

**Theorem of unisolvence of  ${}^{\text{Lag}}\mathbb{P}_k^d$ .**

Let  $d \geq 1$ . Let  $k \geq 1$ . Let  $[[\mathbf{v}]]^d$  be  $d + 1$  affinely independent points in  $\mathbb{R}^d$ .

Then,  $(K[[\mathbf{v}]]^d, \mathbb{P}_k^d, \Sigma\{\mathbf{a}\}_{\mathcal{A}_k^d})$  satisfies the unisolvence property.

See Theorem 1626, and the proof of Lemma 1625. The definition of unisolvence is explained in Section 1.3.2.

The proof of the theorem of unisolvence of  ${}^{\text{Lag}}\mathbb{P}_k^d$  uses a double induction scheme, and the Euclidean division of polynomials (see Section 3.2). The proof of the theorem of unisolvence of  ${}^{\text{Lag}}\mathbb{P}_k^d$  goes as follows:

- prove that  $\phi_{\Sigma\{\mathbf{a}\}_{\mathcal{A}_k^d}}$  is injective by a double induction on  $d, k \geq 1$  (see Lemma 1625).

This amounts to prove that if  $[[\mathbf{v}]]^d$  are  $d + 1$  affinely independent points in  $\mathbb{R}^d$ , and if the polynomial  $p \in \mathbb{P}_k^d$  is zero on all Lagrange nodes  $\{\{\mathbf{a}\}_{\mathcal{A}_k^d}\}$  (i.e. for all  $\alpha \in \mathcal{A}_k^d$ ,  $p(\mathbf{a}_\alpha) = 0$ ), then  $p$  is zero. We recall that the Lagrange nodes  $\{\{\mathbf{a}\}_{\mathcal{A}_k^d}\}$  are evenly distributed over the simplex whose vertices are  $[[\mathbf{v}]]^d$ .

- for  $d = 1$ , the components of any polynomial in  $\mathbb{P}_k^1$  on the Lagrange basis are the values at the Lagrange nodes (see Lemma 1450), thus the cancellation of the latter implies the cancellation of the polynomial itself (see Lemma 1476);
- for  $k = 1$ , the components of any polynomial in  $\mathbb{P}_1^d$  on the Lagrange basis are the values at the vertices (which are the Lagrange nodes in this case, see Lemmas 1593 and 1555), thus the cancellation of the latter implies the cancellation of the polynomial itself (see Lemma 1619);
- for  $d, k \geq 2$ , assume that the injectivity result holds for  $(d - 1, k)$  and  $(d, k - 1)$ , and let  $p \in \mathbb{P}_k^d$  vanishing on  $\{\{\mathbf{a}\}_{\mathcal{A}_k^d}\}$ , the Lagrange nodes of  $\mathbb{P}_k^d$  defined for the vertices  $[[\mathbf{v}]]^d$ , then:

- \* **step 1:** factorization using the injectivity result for  $(d - 1, k)$ . This is done by using the fact that  $p$  is zero on the nodes of a hyperface of the current  $d$ -simplex, and passing this information on the reference  $(d - 1)$ -simplex via an affine bijective mapping between  $\mathbb{R}^{d-1}$  and this hyperface:



- show that  $p_0 \stackrel{\text{def.}}{=} p \circ \phi_{\theta_0^{d-1}}^{\llbracket \mathbf{v} \rrbracket^d}$  vanishes on  $\{\{\hat{\mathbf{a}}\}_{\mathcal{A}_k^{d-1}}\}$ , the reference Lagrange nodes of  $\mathbb{P}_k^{d-1}$ , where  $\phi_{\theta_0^{d-1}}^{\llbracket \mathbf{v} \rrbracket^d}$  is the bijective geometric hyperface mapping (see Lemma 1584) from  $\mathbb{R}^{d-1}$  onto  $\mathcal{H}_0^{\llbracket \mathbf{v} \rrbracket^d}$ , the face hyperplane opposite the vertex  $\mathbf{v}_0$ ;
- then, apply the injectivity result for  $(d-1, k)$  to obtain the cancellation of  $p_0$  on  $\mathbb{R}^{d-1}$ , and thus the cancellation of  $p$  on the whole hyperplane  $\mathcal{H}_0^{\llbracket \mathbf{v} \rrbracket^d}$  (and not only the nodes on this hyperplane);
- finally, use the Euclidean division on  $\mathbb{P}_k^d$  to show the existence of  $q \in \mathbb{P}_{k-1}^d$  such that  $p = \lambda_0^{\llbracket \mathbf{v} \rrbracket^d} \times q$ , where  $\lambda_0^{\llbracket \mathbf{v} \rrbracket^d}$  is the barycentric coordinate associated with the vertex  $\mathbf{v}_0$ , (the hyperplane  $\mathcal{H}_0^{\llbracket \mathbf{v} \rrbracket^d}$  is the place where  $\lambda_0^{\llbracket \mathbf{v} \rrbracket^d}$  vanishes, see Lemmas 1563 and 1623);
- \* **step 2:** cancellation using the injectivity result for  $(d, k-1)$ . The principle here is to observe that  $q \in \mathbb{P}_{k-1}^d$  is zero on all the Lagrange nodes except the ones on  $\mathcal{H}_0^{\llbracket \mathbf{v} \rrbracket^d}$ , and that these cancelling nodes are precisely the Lagrange nodes of  $\mathbb{P}_{k-1}^d$  that are defined from sub-vertices (see Figure 9.8):
  - show that  $q$  vanishes on the first Lagrange nodes of  $\mathbb{P}_k^d$ , indexed by  $\mathcal{A}_{k-1}^d$ ;
  - then, show that those nodes are equal to  $\{\{\underline{\mathbf{a}}\}_{\mathcal{A}_{k-1}^d}\}$ , the Lagrange nodes of  $\mathbb{P}_{k-1}^d$  associated with the sub-vertices with respect to  $\mathbf{v}_0$  (see Lemma 1598);
  - finally, apply the injectivity result for  $(d, k-1)$  to obtain cancellation of  $q$ , then of  $p$ , and thus the injectivity result for  $(d, k)$ ;
- use equivalence between injectivity and bijectivity of linear maps when the dimensions of input and output spaces coincide.

The theorem of unisolvence of  ${}^{\text{Lag}}\mathbb{P}_k^d$  is used in the present document to build the  ${}^{\text{Lag}}\mathbb{P}_d^k$  Lagrange finite element of degree  $k$  associated with simplicial vertices  $\llbracket \mathbf{v} \rrbracket^d$  (see Theorem 1629).

Note that in this proof, one needs in step 2 to take sub-nodes in order to use the induction hypothesis for  $(d, k-1)$ . This requires to pass from the initial simplex to a different simplex. This new simplex is the first one, minus a slice that is comprised between the hyperplane  $\mathcal{H}_0^{\llbracket \mathbf{v} \rrbracket^d}$  opposite  $\mathbf{v}_0$ , and the hyperplane parallel to  $\mathcal{H}_0^{\llbracket \mathbf{v} \rrbracket^d}$  that passes through the nodes closest to the hyperplane, see Figure 9.8. From this remark, note that if the initial simplex is the reference one, the new simplex cannot be the reference simplex. Thus, this proof requires to treat general nondegenerate simplices and not only the *reference* simplex.

This is why the unisolvence is proved directly in all generality for any nondegenerate simplices. The unisolvence for the reference simplex is then simply a particular case.

### 3.2 Sketch of the proof of Euclidean division by monomial in $\mathbb{P}_k^d$

**Lemma of Euclidean division by monomial in  $\mathbb{P}_k^d$ .** *Let  $d \geq 2$ . Let  $k \geq 1$ . Let  $p \in \mathbb{P}_k^d$ . Then, there exist unique  $\tilde{p}_0 \in \mathbb{P}_k^{d-1}$  and  $p_1 \in \mathbb{P}_{k-1}^d$ , such that  $p = \tilde{p}_0 + X_d p_1$ , which also writes for all  $(x_1, \dots, x_d) \in \mathbb{R}^d$ ,  $p(x_1, \dots, x_d) = \tilde{p}_0(x_1, \dots, x_{d-1}) + x_d p_1(x_1, \dots, x_d)$ .*

See Lemma 1529. The proof of the lemma of Euclidean division by monomial in  $\mathbb{P}_k^d$  uses the “ $\sim$ ” notation for functions for which the last variable is dropped (see  $p$  and  $\tilde{p}_0$  above, and Remark 1490). The proof of the lemma of Euclidean division by monomial in  $\mathbb{P}_k^d$  goes as follows:

- **existence:** by induction on  $k \geq 1$ ,
  - for  $k = 1$ , the result is straightforward as  $\mathbb{P}_1^d \stackrel{\text{def.}}{=} \text{span}(1, X_1, X_2, \dots, X_d)$  (the constant on  $X_d$  is  $p_1$ );
  - for  $k \geq 1$ , assume that the result holds for  $k$ , and let  $p = \sum_{\alpha \in \mathcal{A}_{k+1}^d} a_\alpha \mathbf{X}^\alpha \in \mathbb{P}_{k+1}^d$ , then:
    - \* from the partition of the multi-index sets  $\mathcal{A}_{k+1}^d = \mathcal{A}_k^d \uplus \mathcal{C}_{k+1}^d$  (see Definition 1486 and Lemma 1496), we have  $p = q + r$  with  $q = \sum_{\alpha \in \mathcal{A}_k^d} a_\alpha \mathbf{X}^\alpha \in \mathbb{P}_k^d$  and  $r = \sum_{\alpha \in \mathcal{C}_{k+1}^d} a_\alpha \mathbf{X}^\alpha \in \mathbb{P}_{k+1}^d$ ;
    - \* then, apply the result for  $k$  to  $q$ , and obtain existence of  $\tilde{q}_0 \in \mathbb{P}_k^{d-1} \subset \mathbb{P}_{k+1}^{d-1}$  and  $q_1 \in \mathbb{P}_{k-1}^d \subset \mathbb{P}_k^d$  such that  $q = \tilde{q}_0 + X_d q_1$ ;
    - \* for all  $i \in [0..k+1]$ , for all  $\tilde{\alpha}_i \in \mathcal{C}_{k+1-i}^{d-1}$ , let  $b_{\tilde{\alpha}_i} \stackrel{\text{def.}}{=} a_{(\tilde{\alpha}_i, i)}$ , let

$$\tilde{r}_0 \stackrel{\text{def.}}{=} \sum_{\tilde{\alpha}_0 \in \mathcal{C}_{k+1}^{d-1}} b_{\tilde{\alpha}_0} \tilde{\mathbf{X}}^{\tilde{\alpha}_0} \quad \text{and} \quad r_1 \stackrel{\text{def.}}{=} \sum_{i=1}^{k+1} \sum_{\tilde{\alpha}_i \in \mathcal{C}_{k+1-i}^{d-1}} b_{\tilde{\alpha}_i} \tilde{\mathbf{X}}^{\tilde{\alpha}_i} X_d^{i-1},$$

then show that  $\tilde{r}_0 \in \mathbb{P}_{k+1}^{d-1}$ ,  $r_1 \in \mathbb{P}_k^d$  and  $r = \tilde{r}_0 + r_1$ ;

- \* finally show that  $\tilde{p}_0 \stackrel{\text{def.}}{=} \tilde{q}_0 + \tilde{r}_0 \in \mathbb{P}_{k+1}^{d-1}$ ,  $p_1 \stackrel{\text{def.}}{=} q_1 + r_1 \in \mathbb{P}_k^d$  and  $p = \tilde{p}_0 + X_d p_1$ ;

- **uniqueness:** assume that  $\tilde{q}_0 + X_d q_1 = \tilde{r}_0 + X_d r_1$  with  $\tilde{q}_0, \tilde{r}_0 \in \mathbb{P}_k^{d-1}$  and  $q_1, r_1 \in \mathbb{P}_{k-1}^d$ ,
  - applying the equality to  $\mathbf{x} = (\tilde{\mathbf{x}}, 0) \in \mathbb{R}^d$  provides  $\tilde{q}_0 = \tilde{r}_0$ ;
  - let  $\sum_{\alpha \in \mathcal{A}_{k-1}^d} a_\alpha \mathbf{X}^\alpha$  be the decomposition on the monomial basis of  $q_1 - r_1 \in \mathbb{P}_{k-1}^d$ , then the linear independence of the monomial family in  $\mathbb{P}_k^d$  and the cancellation  $X_d(q_1 - r_1) = 0$  provide  $(a_\alpha)_{\alpha \in \mathcal{A}_{k-1}^d} = \mathbf{0}$ , thus  $q_1 = r_1$ .

The lemma of Euclidean division by monomial in  $\mathbb{P}_k^d$  is used in the present document to build an isomorphism between  $\mathbb{P}_k^d$  and  $\mathbb{P}_k^{d-1} \times \mathbb{P}_{k-1}^d$  in Lemma 1531, to express a multivariate polynomial as a polynomial of  $x_d$  in Lemma 1534, to establish the degree of the product of two polynomials in Lemma 1537, and the factorization of a polynomial vanishing in the last reference face hyperplane  $\tilde{H}_d^d$  in Lemma 1621 (used for Lemma 1623, see Section 3.1).

Note that, instead of building  $\mathbb{P}_k^d$  as the linear span of monomials as done here (see Definition 1505), it is possible to build it by incrementing the dimension of the polynomial by setting the components of 1D polynomial as polynomials of other variables. The Lemma 1534 somewhat establishes the link between these two views.

### 3.3 Multi-index ordering

The second part of the present document does not focus on the ordering of multi-indices, but it may be interesting to say a few words on this topic.

#### 3.3.1 Multi-indices and the finite element method

For Lagrange finite elements in  $d$ -simplices, we deal with multi-indices whose sum is at most  $k$ .

**Definition of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$ .** *Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . The set of multi-indices of length at most  $k$  (resp. of length  $k$ ) is denoted  $\mathcal{A}_k^d$  (resp.  $\mathcal{C}_k^d$ ), and is defined by*

$$\mathcal{A}_k^d \stackrel{\text{def.}}{=} \{\boldsymbol{\alpha} \in \mathbb{N}^d \mid |\boldsymbol{\alpha}| \leq k\} \quad (\text{resp. } \mathcal{C}_k^d \stackrel{\text{def.}}{=} \{\boldsymbol{\alpha} \in \mathbb{N}^d \mid |\boldsymbol{\alpha}| = k\}),$$

where  $\forall \boldsymbol{\alpha} \in \mathbb{N}^d$ ,  $|\boldsymbol{\alpha}| \stackrel{\text{def.}}{=} \sum_{i=1}^d \alpha_i$ .

Let  $i \in [1..d]$ .  $\mathcal{A}_{k,i}^d \stackrel{\text{def.}}{=} \{\boldsymbol{\alpha} \in \mathcal{A}_k^d \mid \alpha_i = 0\}$  is the subset of  $\mathcal{A}_k^d$  of multi-indices with zero  $i$ -th component.

(See also Definition 1486.) These multi-indices are used for two purposes in this document. First, they provide the multi-exponent of the multivariate monomial  $\mathbf{X}^\boldsymbol{\alpha} \stackrel{\text{def.}}{=} \prod_{i=1}^d X_i^{\alpha_i}$  (whose total degree is  $|\boldsymbol{\alpha}|$ , see Definition 1503). Thus, a polynomial of total degree at most  $k$  is the sum of such monomials: for  $(a_\boldsymbol{\alpha})_{\boldsymbol{\alpha} \in \mathcal{A}_k^d} \in \mathbb{R}$ ,  $p \stackrel{\text{def.}}{=} \sum_{\boldsymbol{\alpha} \in \mathcal{A}_k^d} a_\boldsymbol{\alpha} \mathbf{X}^\boldsymbol{\alpha}$  (see Definition 1505 and the following statements).

Second, they index the Lagrange nodes in a (nondegenerate)  $d$ -simplex. Let  $K^{\llbracket \mathbf{v} \rrbracket^d}$  be a simplex defined by its  $d+1$  (affinely independent) vertices in  $\mathbb{R}^d$ , denoted by  $\llbracket \mathbf{v} \rrbracket^d \stackrel{\text{def.}}{=} (\mathbf{v}_0, \dots, \mathbf{v}_d)$ . Then, the Lagrange nodes  $\{\{\mathbf{a}\}_{\mathcal{A}_k^d}\}$  are defined by

**Definition of Lagrange nodes.** *Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . Let  $\llbracket \mathbf{v} \rrbracket^d$  be  $d+1$  points in  $\mathbb{R}^d$ . The Lagrange nodes of  $\mathbb{P}_k^d$  are denoted  $\{\{\mathbf{a}\}_{\mathcal{A}_k^d}\} = (\mathbf{a}_\boldsymbol{\alpha})_{\boldsymbol{\alpha} \in \mathcal{A}_k^d}$ , and are defined by*

$$(k=0) \quad \mathbf{a}_\mathbf{0} \stackrel{\text{def.}}{=} \mathbf{g}^{\llbracket \mathbf{v} \rrbracket^d} \left( = \frac{1}{d+1} \sum_{i=0}^d \mathbf{v}_i \right),$$

$$(k \geq 1) \quad \forall \boldsymbol{\alpha} \in \mathcal{A}_k^d, \quad \mathbf{a}_\boldsymbol{\alpha} \stackrel{\text{def.}}{=} \mathbf{v}_0 + \sum_{i=1}^d \frac{\alpha_i}{k} (\mathbf{v}_i - \mathbf{v}_0).$$

(See also Definition 1588.) The case of constant polynomials ( $k=0$ ) is special, as the node is set at the isobarycenter of  $\llbracket \mathbf{v} \rrbracket^d$ . We focus here on the general case when  $k \geq 1$ . In this case, the Lagrange nodes can be written equivalently

$$\mathbf{a}_\boldsymbol{\alpha} = \frac{k-|\boldsymbol{\alpha}|}{k} \mathbf{v}_0 + \sum_{i=1}^d \frac{\alpha_i}{k} \mathbf{v}_i, \quad \text{with } \frac{k-|\boldsymbol{\alpha}|}{k} + \sum_{i=1}^d \frac{\alpha_i}{k} = 1,$$

which exhibits the barycentric coordinates  $(\frac{k-|\boldsymbol{\alpha}|}{k}, \frac{\alpha_1}{k}, \dots, \frac{\alpha_d}{k})$  of  $\mathbf{a}_\boldsymbol{\alpha}$  with respect to  $\llbracket \mathbf{v} \rrbracket^d$  (see Section 9.4). For instance, in the reference simplex  $\hat{K}^d$ , whose reference vertices are  $\hat{\mathbf{v}}_0 \stackrel{\text{def.}}{=} \mathbf{0}$  and for  $i \in [1..d]$ ,  $\hat{\mathbf{v}}_i \stackrel{\text{def.}}{=} \mathbf{e}_i$ , this expression shows that each reference node  $\hat{\mathbf{a}}_\boldsymbol{\alpha}$  has coordinates  $(\hat{\mathbf{a}}_\boldsymbol{\alpha})_i = \frac{\alpha_i}{k}$ . It is thus natural to represent a Lagrange node in a simplex by its multi-index in  $\mathcal{A}_k^d$ , and to link it to the corresponding monomial, see Figure 3.1. These correspondences are used throughout the document.

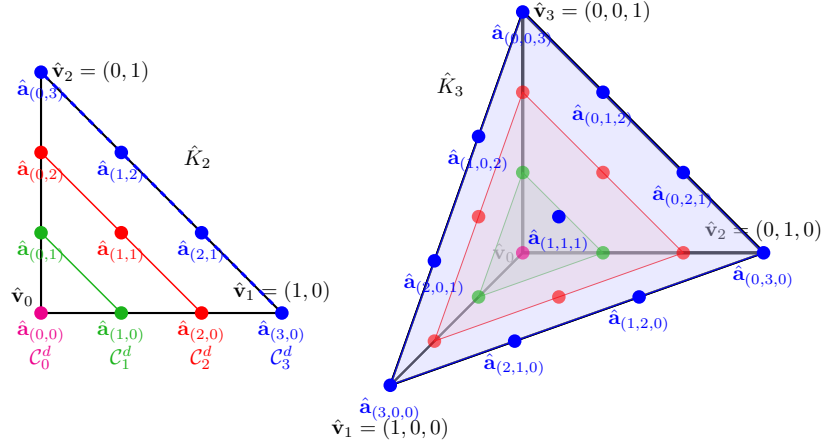


Figure 3.1: Lagrange nodes  $\{\hat{\mathbf{a}}\}_{\mathcal{A}_k^d}$  of the reference simplex  $\hat{K}_d$  when  $d \in \{2, 3\}$  and  $k = 3$  (see Section 3.3.1).

Each node is depicted as a colored ball, and corresponds to a unique element of  $\mathcal{A}_3^d$ . The colors correspond to degrees  $l \leq 3$  of polynomials, or equivalently to lengths of multi-indices (i.e. in  $\mathcal{C}_l^d$  for  $l \leq 3$ ). In magenta, the node  $\hat{\mathbf{a}}_0$  corresponds to constant polynomials (with degree 0) in  $\mathbb{P}_0^d$ , and to the multi-index  $\mathbf{0}$  in the singleton  $\mathcal{C}_0^d$ . In green, the nodes correspond to non-constant affine polynomials (with degree 1), and to the multi-indices  $\mathbf{e}_1, \dots, \mathbf{e}_d$  in  $\mathcal{C}_1^d$ . In red, the nodes correspond to non-affine quadratic polynomials (with degree 2), and to multi-indices in  $\mathcal{C}_2^d$ . In blue, the nodes correspond to non-quadratic cubic polynomials (with degree 3), and to multi-indices in  $\mathcal{C}_3^d$ . We observe in this picture that  $\mathcal{A}_3^d = \mathcal{C}_0^d \uplus \mathcal{C}_1^d \uplus \mathcal{C}_2^d \uplus \mathcal{C}_3^d$ .

### 3.3.2 Monomial order

Once these notations are set, we can present the various possibilities to order multi-indices. Such ordering should be a *monomial order*, i.e. a total order that is compatible with the monoid structure of the monomials: for all  $\alpha, \beta, \gamma \in \mathcal{A}_k^d$ ,  $\mathbf{X}^\alpha < \mathbf{X}^\beta$  implies  $\mathbf{X}^\alpha \mathbf{X}^\gamma < \mathbf{X}^\beta \mathbf{X}^\gamma$ , which means that  $\alpha < \beta$  implies  $\alpha + \gamma < \beta + \gamma$ . It is also generally required that for all  $\alpha \neq \mathbf{0}$ ,  $\mathbf{X}^{\mathbf{0}} = 1 < \mathbf{X}^\alpha$ .

Among various possibilities, e.g. see [17, Chap. 2] and [16], we present commonly used monomial orders, and variants including one that reveals more convenient in our context (and that we call “grsymlex”, see Section 3.3.6).

In the sequel,  $\alpha$  and  $\beta$  denote any multi-indices in  $\mathcal{A}_k^d$ .

Note that we use the following notations when  $d \geq 2$ : the check notation  $\check{\alpha}$  denotes the *last*  $d-1$  components of  $\alpha$ , and the tilde notation  $\tilde{\alpha}$  denotes the *first* ones. Thus, when  $\alpha \stackrel{\text{def.}}{=} (\alpha_1, \dots, \alpha_d)$ , we have  $\alpha = (\alpha_1, \check{\alpha}) = (\tilde{\alpha}, \alpha_d)$ , see also Remark 1490.

### 3.3.3 Lexicographic order

The *lexicographic order*, or simply “*lex*” order, can be recursively defined as

$$(3.1) \quad \alpha <^{\text{lex}} \beta \stackrel{\text{def.}}{\iff} \begin{cases} \alpha_1 < \beta_1, \text{ or} \\ \alpha_1 = \beta_1 \wedge d \geq 2 \wedge \check{\alpha} <^{\text{lex}} \check{\beta}. \end{cases}$$

We have  $\alpha <^{\text{lex}} \beta$  iff  $\alpha_i < \beta_i$  for the *first* index  $i$  for which  $\alpha_i$  and  $\beta_i$  differ.

Starting from the right, the *colexicographic order*, or simply “*colex*” order, can be recursively defined as

$$(3.2) \quad \alpha <^{\text{colex}} \beta \stackrel{\text{def.}}{\iff} \begin{cases} \alpha_d < \beta_d, \text{ or} \\ \alpha_d = \beta_d \wedge d \geq 2 \wedge \tilde{\alpha} <^{\text{colex}} \tilde{\beta}. \end{cases}$$

We have  $\alpha <^{\text{colex}} \beta$  iff  $\alpha_i < \beta_i$  for the *last* index  $i$  for which  $\alpha_i$  and  $\beta_i$  differ. The *colex* order is also called *inverse lexicographic order*, or simply “*invlex*” order, e.g. see [17, p. 61].

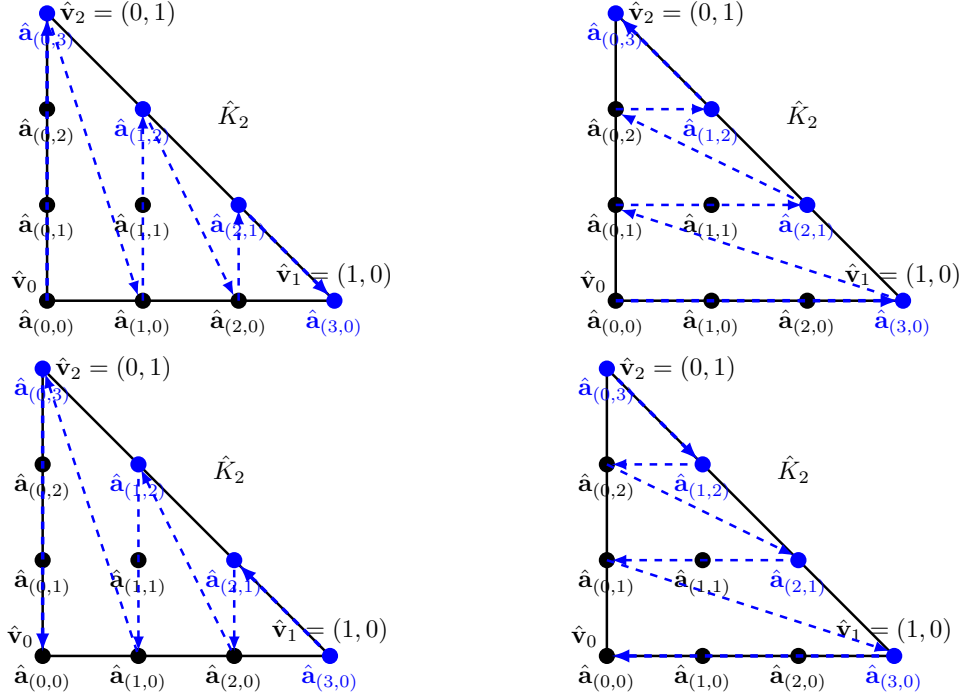


Figure 3.2: Lex (top left), colex (top right), symlex (bottom left), and revlex (bottom right) orderings of  $\mathcal{A}_k^d$  when  $d = 2$  and  $k = 3$  (see Section 3.3.3).

The increase in the order is represented by dashed arrows. For  $\mathcal{A}_3^2$ , we have  $(0,0) <^{\text{lex}} (0,1) <^{\text{lex}} (0,2) <^{\text{lex}} (0,3) <^{\text{lex}} (1,0) <^{\text{lex}} (1,1) <^{\text{lex}} (1,2) <^{\text{lex}} (2,0) <^{\text{lex}} (2,1) <^{\text{lex}} (3,0)$ , and  $(0,0) <^{\text{colex}} (1,0) <^{\text{colex}} (2,0) <^{\text{colex}} (3,0) <^{\text{colex}} (0,1) <^{\text{colex}} (1,1) <^{\text{colex}} (2,1) <^{\text{colex}} (0,2) <^{\text{colex}} (1,2) <^{\text{colex}} (0,3)$ .

The symlex order is the symmetrical of the lex order, and the revlex order is the symmetrical of the colex order. For instance, when the length is 3 (hypotenuse of the triangles, blue nodes), we have  $(0,3) <^{\text{lex}} (1,2) <^{\text{lex}} (2,1) <^{\text{lex}} (3,0)$ , and also  $(0,3) <^{\text{revlex}} (1,2) <^{\text{revlex}} (2,1) <^{\text{revlex}} (3,0)$ .

We may introduce the *symmetrical lexicographic order*, or simply “*symlex*” order, as

$$(3.3) \quad \alpha <^{\text{symlex}} \beta \stackrel{\text{def.}}{\iff} \beta <^{\text{lex}} \alpha.$$

It is the symmetrical of the lex order, we have  $\alpha <^{\text{symlex}} \beta$  iff  $\beta_i < \alpha_i$  for the *first* index  $i$  for which  $\alpha_i$  and  $\beta_i$  differ.

We may also define the *reverse lexicographic order*, or simply “*revlex*” order, as

$$(3.4) \quad \alpha <^{\text{revlex}} \beta \stackrel{\text{def.}}{\iff} \beta <^{\text{colex}} \alpha.$$

It is the symmetrical of the colex order, we have  $\alpha <^{\text{revlex}} \beta$  iff  $\beta_i < \alpha_i$  for the *last* index  $i$  for which  $\alpha_i$  and  $\beta_i$  differ. The *revlex* order is also called *reverse inverse lexicographic order*, or simply “*rinvlex*” order, e.g. see [17, p. 61].

The lex, colex, symlex, and revlex orders are monomial orders. Note also that lex and colex are obviously equivalent when  $d = 1$  (and so are symlex and revlex). Moreover, when  $d = 2$  and the multi-indices have the same length, lex and colex orders are symmetrical. Indeed, assume that  $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$ . Then, we have  $\alpha_1 < \beta_1 \iff \beta_2 < \alpha_2$  and  $\alpha_1 = \beta_1 \iff \alpha_2 = \beta_2$ , i.e.  $(\alpha_1, \alpha_2) <^{\text{lex}} (\beta_1, \beta_2) \iff (\beta_1, \beta_2) <^{\text{colex}} (\alpha_1, \alpha_2)$ . Thus, in that case ( $|\alpha| = |\beta|$  and  $d = 2$ ), lex and revlex orders are equivalent, as well as colex and symlex. See a 2D example in Figure 3.2.

The lex order and its variants are not convenient in practice here, as they do not sort the monomials of a given polynomial according to their total degrees: for instance, for  $d = 2$ , let  $p \stackrel{\text{def.}}{=} X_1^0 X_2^8$  and  $q \stackrel{\text{def.}}{=} X_1^1 X_2^2$ , we have  $p <^{\text{lex}} q$  (as  $0 < 1$ ), but  $\deg(p) = 8 > 3 = \deg(q)$ . Thus, in the sequel, they are only be used to define other monomial orders.

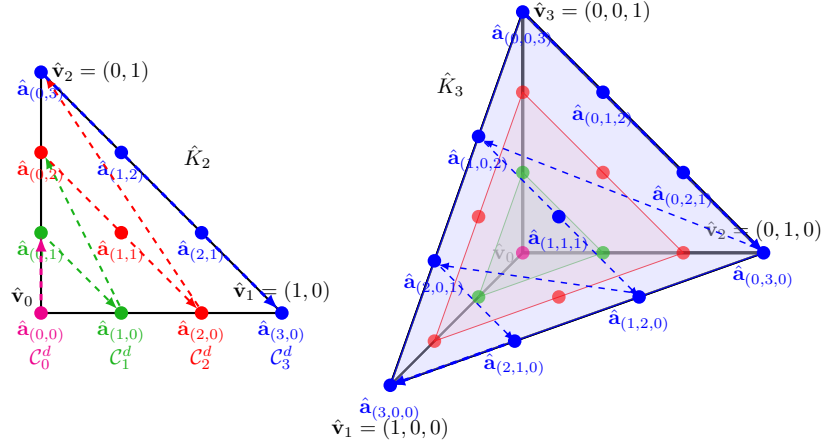


Figure 3.3: Grlex (or deglex) ordering of  $\mathcal{A}_k^d$  when  $d \in \{2, 3\}$  and  $k = 3$  (see Section 3.3.4).

The increase in the order is represented by dashed arrows (only in the case  $l = 3$  when  $d = 3$ , see Figure 3.1).

For  $\mathcal{A}_3^2$  ( $d = 2$ ), we have  $(0, 0) <_{\text{grlex}} (0, 1) <_{\text{grlex}} (1, 0) <_{\text{grlex}} (0, 2) <_{\text{grlex}} (1, 1) <_{\text{grlex}} (2, 0) <_{\text{grlex}} (0, 3) <_{\text{grlex}} (1, 2) <_{\text{grlex}} (2, 1) <_{\text{grlex}} (3, 0)$ .

For  $\mathcal{C}_3^3$  ( $d = 3$ ), we have  $(0, 0, 3) <_{\text{grlex}} (0, 1, 2) <_{\text{grlex}} (0, 2, 1) <_{\text{grlex}} (0, 3, 0) <_{\text{grlex}} (1, 0, 2) <_{\text{grlex}} (1, 1, 1) <_{\text{grlex}} (1, 2, 0) <_{\text{grlex}} (2, 0, 1) <_{\text{grlex}} (2, 1, 0) <_{\text{grlex}} (3, 0, 0)$ .

### 3.3.4 Graded lexicographic order

The *graded lexicographic order*, or simply “*grlex*” order, is defined by

$$(3.5) \quad \alpha <_{\text{grlex}} \beta \iff \begin{cases} |\alpha| < |\beta|, \text{ or} \\ |\alpha| = |\beta| \quad \wedge \quad \alpha <_{\text{lex}} \beta. \end{cases}$$

This amounts to first compare the length of multi-indices, and in case of equality, use the standard lex order (3.1). Thus, when  $|\alpha| = |\beta|$ , we have  $\alpha <_{\text{grlex}} \beta$  iff  $\alpha_i < \beta_i$  for the first index  $i$  for which  $\alpha_i$  and  $\beta_i$  differ.

We have the following equivalence, which may be seen as an alternative recursive definition,

$$(3.6) \quad \alpha <_{\text{grlex}} \beta \iff \begin{cases} |\alpha| < |\beta|, \text{ or} \\ |\alpha| = |\beta| \quad \wedge \quad \alpha_1 < \beta_1, \text{ or} \\ |\alpha| = |\beta| \quad \wedge \quad \alpha_1 = \beta_1 \quad \wedge \quad d \geq 2 \quad \wedge \quad \check{\alpha} <_{\text{grlex}} \check{\beta}, \end{cases}$$

as when  $|\alpha| = |\beta|$  and  $\alpha_1 = \beta_1$ , we have  $|\check{\alpha}| = |\check{\beta}|$ , and lex and grlex are identical. Note that the second case ( $|\alpha| = |\beta|$  and  $\alpha_1 < \beta_1$ ) implies  $d \geq 2$ .

This ordering is a monomial order. It is also called *degree lexicographic order*, or simply “*deglex*” order. See 2D and 3D examples in Figure 3.3.

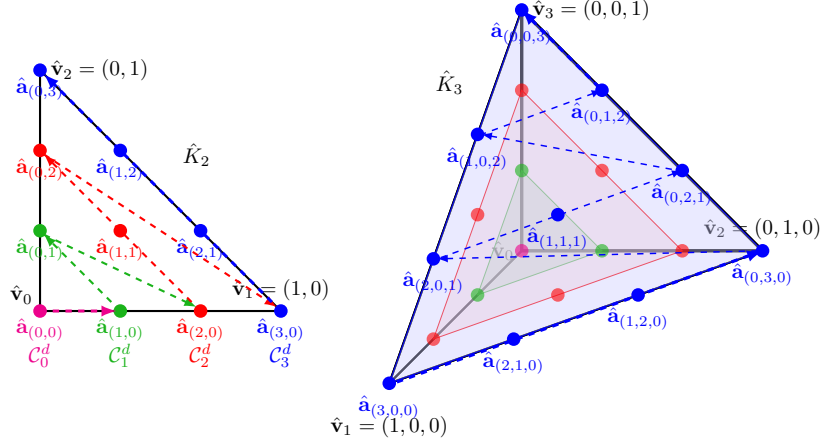


Figure 3.4: Grcolex ordering of  $\mathcal{A}_k^d$  when  $d \in \{2, 3\}$  and  $k = 3$  (see Section 3.3.5).

The increase in the order is represented by dashed arrows (only in the case  $l = 3$  when  $d = 3$ , see Figure 3.1).

For  $\mathcal{A}_3^2$  ( $d = 2$ ), we have  $(0, 0) <_{\text{grcolex}} (1, 0) <_{\text{grcolex}} (0, 1) <_{\text{grcolex}} (2, 0) <_{\text{grcolex}} (1, 1) <_{\text{grcolex}} (0, 2) <_{\text{grcolex}} (3, 0) <_{\text{grcolex}} (2, 1) <_{\text{grcolex}} (1, 2) <_{\text{grcolex}} (0, 3)$ .

For  $\mathcal{C}_3^3$  ( $d = 3$ ), we have  $(3, 0, 0) <_{\text{grcolex}} (2, 1, 0) <_{\text{grcolex}} (1, 2, 0) <_{\text{grcolex}} (0, 3, 0) <_{\text{grcolex}} (2, 0, 1) <_{\text{grcolex}} (1, 1, 1) <_{\text{grcolex}} (0, 2, 1) <_{\text{grcolex}} (1, 0, 2) <_{\text{grcolex}} (0, 1, 2) <_{\text{grcolex}} (0, 0, 3)$ .

### 3.3.5 Graded colexicographic order

We design the *graded colexicographic order*, or simply “*grcolex*” order, defined by

$$(3.7) \quad \alpha <_{\text{grcolex}} \beta \stackrel{\text{def.}}{\iff} \begin{cases} |\alpha| < |\beta|, \text{ or} \\ |\alpha| = |\beta| \quad \wedge \quad \alpha <_{\text{colex}} \beta. \end{cases}$$

This amounts to first compare the length of multi-indices, and in case of equality, use the colex order (3.2). Thus, when  $|\alpha| = |\beta|$ , we have  $\alpha <_{\text{grcolex}} \beta$  iff  $\alpha_i < \beta_i$  for the last index  $i$  where  $\alpha_i$  and  $\beta_i$  differ.

We have the following equivalence, which may be seen as an alternative recursive definition,

$$(3.8) \quad \alpha <_{\text{grcolex}} \beta \iff \begin{cases} |\alpha| < |\beta|, \text{ or} \\ |\alpha| = |\beta| \quad \wedge \quad \alpha_d < \beta_d, \text{ or} \\ |\alpha| = |\beta| \quad \wedge \quad \alpha_d = \beta_d \quad \wedge \quad d \geq 2 \quad \wedge \quad \tilde{\alpha} <_{\text{grcolex}} \tilde{\beta}. \end{cases}$$

as when  $|\alpha| = |\beta|$  and  $\alpha_d = \beta_d$ , we have  $|\tilde{\alpha}| = |\tilde{\beta}|$ , and colex and grcolex are identical. Note that the second case ( $|\alpha| = |\beta|$  and  $\alpha_d < \beta_d$ ) implies  $d \geq 2$ .

This ordering is a monomial order. Compare on the same 2D and 3D examples the grcolex order in Figure 3.4 and the grlex order in Figure 3.3.

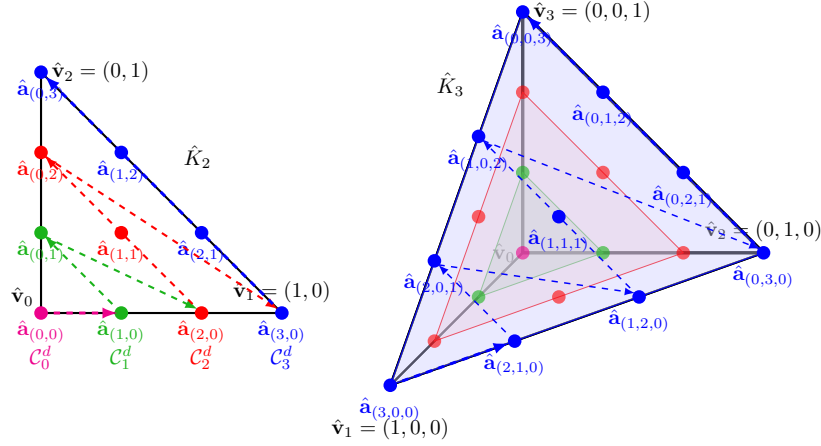


Figure 3.5: Grsymlex ordering of  $\mathcal{A}_k^d$  when  $d \in \{2, 3\}$  and  $k = 3$  (see Section 3.3.6).

The increase in the order is represented by dashed arrows (only in the case  $l = 3$  when  $d = 3$ , see Figure 3.1).

For  $\mathcal{A}_3^2$  ( $d = 2$ ), we have (as in the case of grcolex)  $(0, 0) <_{\text{grsymlex}} (1, 0) <_{\text{grsymlex}} (0, 1) <_{\text{grsymlex}} (2, 0) <_{\text{grsymlex}} (1, 1) <_{\text{grsymlex}} (0, 2) <_{\text{grsymlex}} (3, 0) <_{\text{grsymlex}} (2, 1) <_{\text{grsymlex}} (1, 2) <_{\text{grsymlex}} (0, 3)$ .

When  $d \geq 3$ , grcolex and grsymlex differ. For instance, for  $\mathcal{C}_3^3$  (compare with Figure 3.4), we have  $(3, 0, 0) <_{\text{grsymlex}} (2, 1, 0) <_{\text{grsymlex}} (2, 0, 1) <_{\text{grsymlex}} (1, 2, 0) <_{\text{grsymlex}} (1, 1, 1) <_{\text{grsymlex}} (1, 0, 2) <_{\text{grsymlex}} (0, 3, 0) <_{\text{grsymlex}} (0, 2, 1) <_{\text{grsymlex}} (0, 1, 2) <_{\text{grsymlex}} (0, 0, 3)$ .

### 3.3.6 Graded symmetric lexicographic order

We also design the *graded symmetric lexicographic order*, or simply “*grsymlex*” order, defined by

$$(3.9) \quad \alpha <_{\text{grsymlex}} \beta \stackrel{\text{def.}}{\iff} \begin{cases} |\alpha| < |\beta|, \text{ or} \\ |\alpha| = |\beta| \wedge \alpha <_{\text{symlex}} \beta. \end{cases}$$

This amounts to first compare the length of multi-indices, and in case of equality, use the symlex order (3.3). Thus, when  $|\alpha| = |\beta|$ , we have  $\alpha <_{\text{grsymlex}} \beta$  iff  $\beta_i < \alpha_i$  for the first index  $i$  where  $\alpha_i$  and  $\beta_i$  differ.

We have the following equivalence, which may be seen as an alternative recursive definition,

$$(3.10) \quad \alpha <_{\text{grsymlex}} \beta \iff \begin{cases} |\alpha| < |\beta|, \text{ or} \\ |\alpha| = |\beta| \wedge \beta_1 < \alpha_1, \text{ or} \\ |\alpha| = |\beta| \wedge \beta_1 = \alpha_1 \wedge d \geq 2 \wedge \check{\alpha} <_{\text{grsymlex}} \check{\beta}. \end{cases}$$

as when  $|\alpha| = |\beta|$  and  $\beta_1 = \alpha_1$ , we have  $|\check{\alpha}| = |\check{\beta}|$ , and symlex and grsymlex are identical. Note that the second case ( $|\alpha| = |\beta|$  and  $\beta_1 < \alpha_1$ ) implies  $d \geq 2$ .

Moreover, the recursive definition may be simplified with the following equivalence,

$$(3.11) \quad \alpha <_{\text{grsymlex}} \beta \iff \begin{cases} |\alpha| < |\beta|, \text{ or} \\ |\alpha| = |\beta| \wedge d \geq 2 \wedge \check{\alpha} <_{\text{grsymlex}} \check{\beta}. \end{cases}$$

as when  $|\alpha| = |\beta|$  and  $\beta_1 < \alpha_1$ , we have  $d \geq 2$  and  $|\check{\alpha}| < |\check{\beta}|$ , i.e.  $\check{\alpha} <_{\text{grsymlex}} \check{\beta}$ . Note that this simplification is made possible by the symmetric aspect of symlex. Note also that it is not possible for the grlex and grcolex orders: indeed, for instance for grlex,  $|\alpha| = |\beta| \wedge \alpha_1 < \beta_1$  implies  $|\check{\beta}| < |\check{\alpha}|$ , i.e.  $\check{\beta} <_{\text{grlex}} \check{\alpha}$ , but not  $\check{\alpha} <_{\text{grlex}} \check{\beta}$ .

This ordering is a monomial order. Note that the grcolex and grsymlex orders are identical when  $d = 1$  or  $2$ , but differ as soon as  $d \geq 3$ . Compare on the same 2D and 3D examples the grsymlex order in Figure 3.5, and the grcolex order in Figure 3.4.



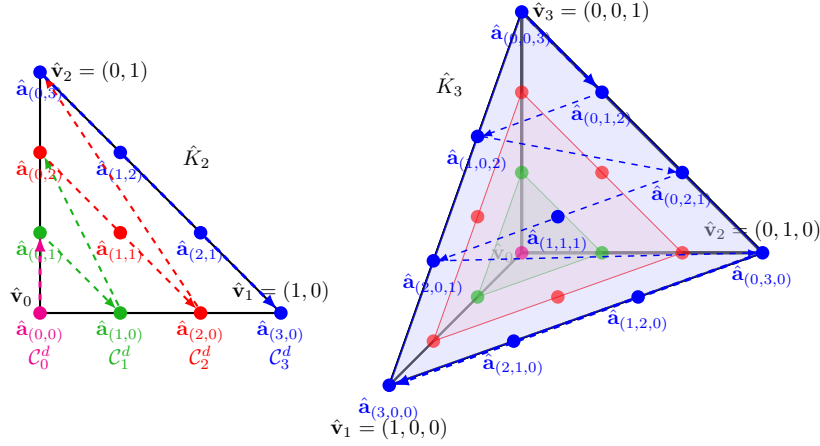


Figure 3.6: Grevlex ordering of  $\mathcal{A}_k^d$  when  $d \in \{2, 3\}$  and  $k = 3$  (see Section 3.3.7).

The increase in the order is represented by dashed arrows (only in the case  $l = 3$  when  $d = 3$ , see Figure 3.1).

For  $\mathcal{A}_3^2$  ( $d = 2$ ), we have (as in the case of grlex)  $(0,0) <_{\text{grevlex}} (0,1) <_{\text{grevlex}} (1,0) <_{\text{grevlex}} (0,2) <_{\text{grevlex}} (1,1) <_{\text{grevlex}} (2,0) <_{\text{grevlex}} (0,3) <_{\text{grlex}} (1,2) <_{\text{grevlex}} (2,1) <_{\text{grevlex}} (3,0)$ .

When  $d \geq 3$ , grlex and grevlex differ. For instance, for  $\mathcal{C}_3^3$  (compare with Figure 3.3), we have  $(0,0,3) <_{\text{grevlex}} (0,1,2) <_{\text{grevlex}} (1,0,2) <_{\text{grevlex}} (0,2,1) <_{\text{grevlex}} (1,1,1) <_{\text{grevlex}} (2,0,1) <_{\text{grevlex}} (0,3,0) <_{\text{grevlex}} (1,2,0) <_{\text{grevlex}} (2,1,0) <_{\text{grevlex}} (3,0,0)$ .

### 3.3.7 Graded reverse lexicographic order

The *graded reverse lexicographic order*, or simply “*grevlex*” order, is defined by

$$(3.12) \quad \alpha <_{\text{grevlex}} \beta \stackrel{\text{def.}}{\iff} \begin{cases} |\alpha| < |\beta|, \text{ or} \\ |\alpha| = |\beta| \quad \wedge \quad \alpha <_{\text{revlex}} \beta. \end{cases}$$

This amounts to first compare the length of multi-indices, and in case of equality, use the revlex order (3.4). Thus, when  $|\alpha| = |\beta|$ , we have  $\alpha <_{\text{grevlex}} \beta$  iff  $\beta_i < \alpha_i$  for the last index  $i$  where  $\alpha_i$  and  $\beta_i$  differ.

We have the following equivalence, which may be seen as an alternative recursive definition,

$$(3.13) \quad \alpha <_{\text{grevlex}} \beta \iff \begin{cases} |\alpha| < |\beta|, \text{ or} \\ |\alpha| = |\beta| \quad \wedge \quad \beta_d < \alpha_d, \text{ or} \\ |\alpha| = |\beta| \quad \wedge \quad \beta_d = \alpha_d \quad \wedge \quad d \geq 2 \quad \wedge \quad \tilde{\alpha} <_{\text{grevlex}} \tilde{\beta}, \end{cases}$$

as when  $|\alpha| = |\beta|$  and  $\beta_d = \alpha_d$ , we have  $|\tilde{\alpha}| = |\tilde{\beta}|$ , and revlex and grevlex are identical. Note that the second case ( $|\alpha| = |\beta|$  and  $\beta_d < \alpha_d$ ) implies  $d \geq 2$ .

Moreover, the recursive definition may be simplified with the following equivalence,

$$(3.14) \quad \alpha <_{\text{grevlex}} \beta \iff \begin{cases} |\alpha| < |\beta|, \text{ or} \\ |\alpha| = |\beta| \quad \wedge \quad d \geq 2 \quad \wedge \quad \tilde{\alpha} <_{\text{grevlex}} \tilde{\beta}, \end{cases}$$

as when  $|\alpha| = |\beta|$  and  $\beta_d < \alpha_d$ , we have  $d \geq 2$  and  $|\tilde{\alpha}| < |\tilde{\beta}|$ , i.e.  $\tilde{\alpha} <_{\text{grevlex}} \tilde{\beta}$ .

This ordering is a monomial order. Note that the grlex and grevlex orders are identical when  $d = 1$  or  $2$ , but differ as soon as  $d \geq 3$ . Compare on the same 2D and 3D examples the grevlex order in Figure 3.6, and the grlex order in Figure 3.3.

### 3.3.8 Discussion

Several orders on  $\mathbb{N}^d$  may be used to number multi-indices and the grevlex order of Section 3.3.7 is known to be well-suited for the division of multivariate polynomials using Gröbner bases, e.g. see [16, 17]. However, in the context of the FEM, other desired properties enter the picture. In this framework, a practical order should be consistent with

- (i) an increase of the degree (from  $k$  to  $k + 1$ ): multi-indices of length at most  $k$  should be numbered before those of length  $k + 1$ , i.e. for all  $\alpha \in \mathcal{C}_k^d \subset \mathcal{A}_k^d$  and for all  $\beta \in \mathcal{C}_{k+1}^d$ , we should have  $\alpha < \beta$ ;
- (ii) an increase of the dimension (from  $d-1$  to  $d$ ): “natural” bijections between  $d-1$ -multi-indices of length at most  $k$  and  $d$ -multi-indices of length  $k$  should be increasing,

- $f_{k,0}^d \stackrel{\text{def.}}{=} (\check{\alpha} \in \mathcal{A}_k^{d-1} \mapsto (k - |\check{\alpha}|, \check{\alpha}) \in \mathcal{C}_k^d)$ , or  
 $\tilde{f}_{k,0}^d \stackrel{\text{def.}}{=} (\tilde{\alpha} \in \mathcal{A}_k^{d-1} \mapsto (\tilde{\alpha}, k - |\tilde{\alpha}|) \in \mathcal{C}_k^d)$  from Lemma 1500, and
- for any  $i \in [1..d]$ ,  $f_{k,i}^d \stackrel{\text{def.}}{=} (\alpha' \in \mathcal{A}_k^{d-1} \mapsto (\alpha'_1, \dots, \alpha'_{i-1}, 0, \alpha'_i, \dots, \alpha'_{d-1}) \in \mathcal{A}_{k,i}^d)$  from Lemma 1501,

i.e.  $\mathcal{A}_k^{d-1}$ ,  $\mathcal{C}_k^d (= f_{k,0}^d(\mathcal{A}_k^{d-1}) = \tilde{f}_{k,0}^d(\mathcal{A}_k^{d-1}))$ , and  $\mathcal{A}_{k,i}^d (= f_{k,i}^d(\mathcal{A}_k^{d-1}))$  share the same numbering;

- (iii) the natural numbering of the multi-indices corresponding to the reference vertices  $(\hat{v}_i)_{i \in [0..d]}$  of the reference simplex, i.e.  $\mathbf{0} < k\mathbf{e}_1 < k\mathbf{e}_2 < \dots < k\mathbf{e}_d$ .

Indeed, condition (i) allows to easily sort monomials with respect to their total degree. Condition (ii) allows to easily relate the face nodes and the volume nodes during the computations. And condition (iii) allows to have *positive* simplices (i.e. positively oriented) when computing the integrals.

Obviously, condition (i) disqualifies the **lex** order and its variants of Section 3.3.3, but all “graded” orders are designed to comply with it.

Obviously, we also have  $\mathbf{0} < k\mathbf{e}_1$  (from condition (iii)) for all “graded” orders.

Conditions (ii) and (iii) are not met for the **grlex** order of Section 3.3.4, which is thus disqualified for our purpose. To see this, here are some counter-examples, cf. Figure 3.3. For (ii), with  $d = k = 3$ , we have  $(1, 0) <^{\text{grlex}} (0, 2)$ , while  $f_{3,0}^3(0, 2) = (1, 0, 2) <^{\text{grlex}} (2, 1, 0) = f_{3,0}^3(1, 0)$ , and  $\tilde{f}_{3,0}^3(0, 2) = (0, 2, 1) <^{\text{grlex}} (1, 0, 2) = \tilde{f}_{3,0}^3(1, 0)$ . For (iii), with  $d = 2$  and  $k = 3$ , the vertex  $\hat{v}_1 = \hat{\mathbf{a}}_{(3,0)}$  is numbered after the vertex  $\hat{v}_2 = \hat{\mathbf{a}}_{(0,3)}$ .

The **grcolex** order of Section 3.3.5 complies with (iii). Indeed, if  $i < j$ , then  $|k\mathbf{e}_i| = |k\mathbf{e}_j| = k$ ,  $j$  is the last index where the components of  $k\mathbf{e}_i$  and  $k\mathbf{e}_j$  differ, and the  $j$ -th component of  $k\mathbf{e}_i$  is 0, whereas  $(k\mathbf{e}_j)_j = k > 0$ , thus  $k\mathbf{e}_i <^{\text{colex}} k\mathbf{e}_j$ , i.e.  $k\mathbf{e}_i <^{\text{grcolex}} k\mathbf{e}_j$  (see Figure 3.4 in the cases  $d \in \{2, 3\}$  and  $k = 3$ ).

However, condition (ii) is not met for the **grcolex** order, which is thus also disqualified for our purpose. Indeed, from Figure 3.4 with  $d = k = 3$ , we have  $(0, 1) <^{\text{grcolex}} (2, 0)$ , while

$$\begin{aligned} f_{3,0}^3(2, 0) &= (1, 2, 0) <^{\text{grcolex}} (2, 0, 1) = f_{3,0}^3(0, 1), \quad \text{and} \\ \tilde{f}_{3,0}^3(2, 0) &= (2, 0, 1) <^{\text{grcolex}} (0, 1, 2) = \tilde{f}_{3,0}^3(0, 1). \end{aligned}$$

The **grevlex** order of Section 3.3.7 satisfies condition (ii) with  $\tilde{f}_{k,0}^d$  and  $f_{k,i}^d$  (for all  $i \in [1..d]$ ). Indeed, we have the following.

- Let  $\tilde{\alpha}, \tilde{\beta} \in \mathcal{A}_k^{d-1}$  such that  $\tilde{\alpha} <^{\text{grevlex}} \tilde{\beta}$ . Let  $\gamma \stackrel{\text{def.}}{=} \tilde{f}_{k,0}^d(\tilde{\alpha})$  and  $\delta \stackrel{\text{def.}}{=} \tilde{f}_{k,0}^d(\tilde{\beta})$ . Both  $\gamma$  and  $\delta$  have length  $k$ , and by removing their last component, we obtain  $\tilde{\gamma} = \tilde{\alpha}$  and  $\tilde{\delta} = \tilde{\beta}$ . Hence, (3.14) yields  $\tilde{f}_{k,0}^d(\tilde{\alpha}) <^{\text{grevlex}} \tilde{f}_{k,0}^d(\tilde{\beta})$ . Thus,  $\tilde{f}_{k,0}^d$  is increasing with respect to **grevlex**.
- Let  $i \in [1..d]$ , and  $\alpha', \beta' \in \mathcal{A}_k^{d-1}$  such that  $\alpha' <^{\text{grevlex}} \beta'$ . We have  $|f_{k,i}^d(\alpha')| = |\alpha'|$  and  $|f_{k,i}^d(\beta')| = |\beta'|$ . Thus, from (3.12), we have two cases:
  - if  $|\alpha'| < |\beta'|$ , then we have  $|f_{k,i}^d(\alpha')| < |f_{k,i}^d(\beta')|$ , and thus  $f_{k,i}^d(\alpha') <^{\text{grevlex}} f_{k,i}^d(\beta')$ .
  - if  $|\alpha'| = |\beta'|$ , then  $\alpha' <^{\text{revlex}} \beta'$ , and  $|f_{k,i}^d(\alpha')| = |f_{k,i}^d(\beta')|$ . The insertion of a “0” in position  $i$  in  $\alpha'$  and  $\beta'$  does not alter the revlex order, and thus we also have  $f_{k,i}^d(\alpha') <^{\text{revlex}} f_{k,i}^d(\beta')$ , i.e.  $f_{k,i}^d(\alpha') <^{\text{grevlex}} f_{k,i}^d(\beta')$ .

Thus, in both cases,  $f_{k,i}^d$  is increasing with respect to **grevlex**.

However, **grevlex** does not comply with condition (iii), as the vertex  $\hat{\mathbf{v}}_1 = \hat{\mathbf{a}}_{(3,0)}$  is numbered after the vertex  $\hat{\mathbf{v}}_2 = \hat{\mathbf{a}}_{(0,3)}$ , see Figure 3.6 (left). For this reason, we also avoided **grevlex**.

Finally, the **grsymlex** order of Section 3.3.6 satisfies the three conditions. Indeed, for condition (ii), using  $f_{k,0}^d$  instead of  $\tilde{f}_{k,0}^d$  as for **grevlex**, we have the following.

- Let  $\check{\alpha}, \check{\beta} \in \mathcal{A}_k^{d-1}$  such that  $\check{\alpha} <^{\text{grsymlex}} \check{\beta}$ . Let  $\gamma \stackrel{\text{def.}}{=} f_{k,0}^d(\check{\alpha})$  and  $\delta \stackrel{\text{def.}}{=} f_{k,0}^d(\check{\beta})$ . Both  $\gamma$  and  $\delta$  have length  $k$ , and by removing their first component, we obtain  $\check{\gamma} = \check{\alpha}$  and  $\check{\delta} = \check{\beta}$ . Hence, (3.11) yields  $f_{k,0}^d(\check{\alpha}) <^{\text{grsymlex}} f_{k,0}^d(\check{\beta})$ . Thus,  $f_{k,0}^d$  is increasing with respect to **grsymlex**.
- Let  $i \in [1..d]$ , and  $\alpha', \beta' \in \mathcal{A}_k^{d-1}$  such that  $\alpha' <^{\text{grsymlex}} \beta'$ . We have  $|f_{k,i}^d(\alpha')| = |\alpha'|$  and  $|f_{k,i}^d(\beta')| = |\beta'|$ . Thus, from (3.9), we have two cases:
  - if  $|\alpha'| < |\beta'|$ , then we have  $|f_{k,i}^d(\alpha')| < |f_{k,i}^d(\beta')|$ , and thus  $f_{k,i}^d(\alpha') <^{\text{grsymlex}} f_{k,i}^d(\beta')$ .
  - if  $|\alpha'| = |\beta'|$ , then  $\alpha' <^{\text{symlex}} \beta'$ , and  $|f_{k,i}^d(\alpha')| = |f_{k,i}^d(\beta')|$ . The insertion of a “0” in position  $i$  in  $\alpha'$  and  $\beta'$  does not alter the symlex order, and thus we also have  $f_{k,i}^d(\alpha') <^{\text{symlex}} f_{k,i}^d(\beta')$ , i.e.  $f_{k,i}^d(\alpha') <^{\text{grsymlex}} f_{k,i}^d(\beta')$ .

Thus, in both cases,  $f_{k,i}^d$  is increasing with respect to **grsymlex**.

The **grsymlex** order also complies with (iii). Indeed, if  $i < j$ , then we have  $|k\mathbf{e}_i| = |k\mathbf{e}_j| = k$ ,  $i$  is the first index where the components of  $k\mathbf{e}_i$  and  $k\mathbf{e}_j$  differ, and the  $i$ -th component of  $k\mathbf{e}_j$  is 0, whereas  $(k\mathbf{e}_i)_i = k > 0$ , thus  $k\mathbf{e}_i <^{\text{symlex}} k\mathbf{e}_j$ , i.e.  $k\mathbf{e}_i <^{\text{grsymlex}} k\mathbf{e}_j$  (see Figure 3.5 in the cases  $d \in \{2, 3\}$  and  $k = 3$ ).

This is why the **grsymlex** order was chosen for the Coq implementation.

Note that the numbering of  $\mathcal{A}_k^d$  is not used in the present version of this document, but is essential in the Coq formalization of the present work in [30, 9].

# Part II

## Detailed proofs



## Chapter 4

# Introduction

Statements are displayed inside colored boxes. Their nature can be identified at a glance by using the following color code:

light gray is for remarks, light green for definitions, light blue for lemmas, and light red for theorems.

Definitions and results have a number and a name. Inside the bodies of proofs, pertinent statements are referenced using both their number and name. When appropriate, some hints are given about the application of the result, either to specify arguments, or to provide justification or consequences; they are underlined. Some useful definitions and results were already stated in [14] or in [15], which were respectively devoted to the detailed proofs of the Lax–Milgram theorem and for Lebesgue integration. Those are numbered up to 1356, and the statements in the present document are numbered starting from 1357.

Furthermore, as in [14, 15], the most basic results are supposed to be known and are not detailed further; they are displayed in **bold dark red**. These include:

- Naive set theory: definition and results about injective and bijective functions.
- Numbers:
  - ordered abelian monoid properties of  $\mathbb{N}$ ;
  - ordered valued field properties of  $\mathbb{R}$ .
- Linear algebra:
  - basic definitions and results about vector spaces and subspaces, such as freedom, generator, linear span, basis, dimension, and linear operations over functions;
  - basic definitions and results about linear maps, such as distributivity of composition over addition, range/rank and kernel/nullity, the rank–nullity theorem, the characterization of surjectivity with the rank and injectivity with the nullity;
  - some results in finite dimension, such as the rules for matrix–vector product, the incomplete basis theorem, and the dimension of the dual space.
- Topology in metric spaces: definitions of interior, open ball, and continuity.
- Calculus: basic definitions and results in  $\mathbb{R}^n$ , such as the usual norms, the differentiability of affine maps ( $\mathcal{C}^\infty$ ), and the rules of differentiation.
- Real analysis: basic definitions and results in  $\mathbb{R}$ , such as the rules of derivation.
- Polynomials: basic definitions about univariate polynomials, such as the degree.

This part is organized as follows. Chapter 5 contains some results from various fields of mathematics (arithmetics, linear and affine algebra, geometry, and univariate polynomials), that

are needed in the proofs for the finite element method. We recall that the material stated in [14, 15] may also be used. Then, Chapter 6 is devoted to the general definition of finite element, and Chapter 7 addresses simplicial geometry. Lagrange finite elements on segments are presented in Chapter 8, and finally Chapter 9 is dedicated to the general case of dimension  $d \geq 1$ .

# Chapter 5

## Complements

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### 5.1 Complements on natural numbers

*Remark 1357.* In Figures 5.1 and 5.2, we provide illustrations for the double inductions used in this document.

In Figure 5.1, a double induction “by diagonal” is presented, see Lemma 1358. The proof of unisolvence of  $\mathbb{P}_k^d$  relies on it, see Lemma 1625. In Figure 5.2, the strong double induction is shown, see Lemma 1360. It is used in Lemma 1537.

**Lemma 1358 (double induction by diagonal).** *Let  $P$  be a predicate on  $\mathbb{N}^2$ . Then, we have*

$$(5.1) \quad \begin{aligned} & (\forall m \in \mathbb{N}, P(m, 0)) \quad \wedge \quad (\forall n \in \mathbb{N}, P(0, n)) \quad \wedge \\ & (\forall m, n \in \mathbb{N}, P(m, n+1) \quad \wedge \quad P(m+1, n) \quad \implies \quad P(m+1, n+1)) \\ & \implies \quad \forall m, n \in \mathbb{N}, P(m, n). \end{aligned}$$

*Proof.* Let  $H_0$ ,  $H_1$ , and  $H_2$  be the three hypotheses in the implication (5.1).

For all  $m \in \mathbb{N}$ , let  $Q(m) \stackrel{\text{def.}}{=} [\forall n \in \mathbb{N}, P(m, n)]$ .

**Induction:  $Q(0)$ .** Trivial (hypothesis  $H_1$ ).

**Induction:  $Q(m) \implies Q(m+1)$ .** Let  $m \in \mathbb{N}$ , assume that  $Q(m)$  holds.

**Induction:  $P(m+1, 0)$ .** The property  $P(m+1, 0)$  holds by hypothesis  $H_0$ .

**Induction:  $P(m+1, n) \implies P(m+1, n+1)$ .** Let  $n \in \mathbb{N}$ , assume that  $P(m+1, n)$  holds.

Then, from  $Q(m)$ , we also have  $P(m, n+1)$ , and thus, from  $H_2$ , we have  $P(m+1, n+1)$ .

This concludes the induction on  $n$ , and we have for all  $n \in \mathbb{N}$ ,  $P(m+1, n)$ , i.e.,  $Q(m+1)$ .

This concludes the induction on  $m$ , and we have for all  $m \in \mathbb{N}$ ,  $Q(m)$ .

Therefore, we have  $\forall m, n \in \mathbb{N}, P(m, n)$ . □



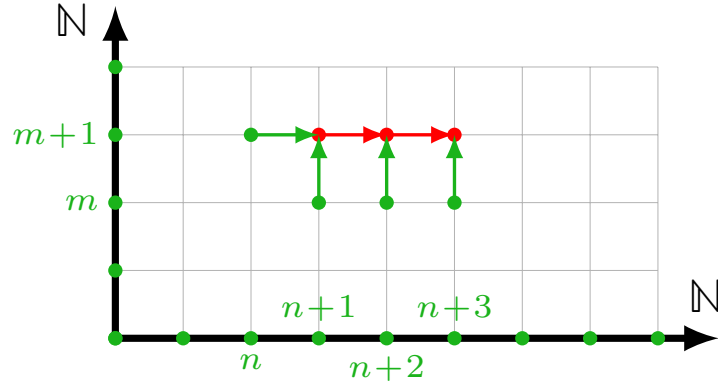


Figure 5.1: Double induction scheme “by diagonal” (see Lemma 1358).

Let a predicate  $P(m, n)$ . Assume the initializations  $P(0, n)$  and  $P(m, 0)$  for all  $m$  and  $n$  (symbolized by green dots along the two axes). Then, if  $P(m, n+1)$  and  $P(m+1, n)$  implies  $P(m+1, n+1)$  (represented by green dots and arrows), then  $P(m, n)$  holds everywhere. It is proven by induction on  $m$ , then on  $n$ . Thus, fixing  $m$ , one proves successively  $P(m+1, k)$ , for all  $k$  (red arrows and dots).

*Remark 1359.* The strong induction principle stipulates that a predicate  $P$  on  $\mathbb{N}$  holds everywhere if it satisfies the property

$$\forall n \in \mathbb{N}, \quad (\forall n_1 < n, \quad P(n_1)) \implies P(n).$$

Note that there is no need to assume the base case  $P(0)$  since it follows from the previous hypothesis. Indeed, for all  $n_1 \in \mathbb{N}$ ,  $n_1 < 0$ , which is false, implies anything, such as  $P(n_1)$ .

In the next lemma, the condition  $m_1 \leq m \wedge n_1 \leq n \wedge (m_1, n_1) \neq (m, n)$  can be reformulated as  $(m_1 < m \wedge n_1 \leq n) \vee (m_1 = m \wedge n_1 < n)$ . As in the strong induction for a single integer, the initialization steps  $P(m, 0)$  and  $P(0, n)$  are not needed for the double strong induction.

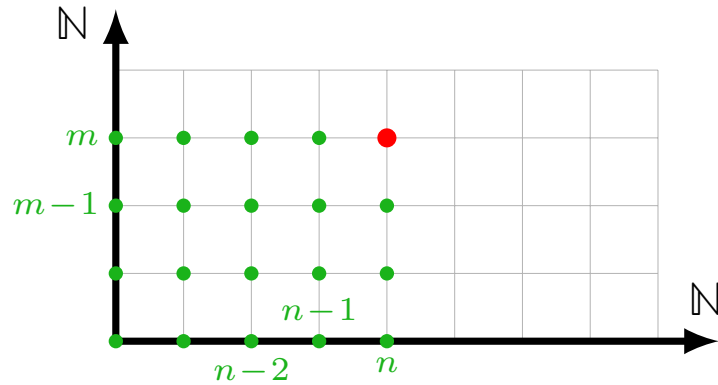


Figure 5.2: Strong double induction scheme (see Lemma 1360).

The induction hypothesis is as follows: if the proposition holds on all the green nodes, then it holds on the red one.

**Lemma 1360 (strong double induction).** *Let  $P$  be a predicate on  $\mathbb{N}^2$ . Then, we have*

$$(5.2) \quad \begin{aligned} & [\forall m, n \in \mathbb{N}, \quad (\forall m_1 \leq m, \forall n_1 \leq n, (m_1, n_1) \neq (m, n) \implies P(m_1, n_1)) \implies P(m, n)] \\ & \implies \forall m, n \in \mathbb{N}, \quad P(m, n). \end{aligned}$$

*Proof.* For all  $m \in \mathbb{N}$ , let  $Q(m) \stackrel{\text{def.}}{=} [\forall n \in \mathbb{N}, P(m, n)]$ .

**Strong induction on  $m$ :**  $(\forall m_1 < m, Q(m_1)) \Rightarrow Q(m)$ .

Let  $m \in \mathbb{N}$ , assume that  $H_1 : \forall m_1 < m, Q(m_1)$  holds. Let us show that  $Q(m)$  holds.

**Strong induction on  $n$ :**  $(\forall n_1 < n, P(m, n_1)) \Rightarrow P(m, n)$ .

Let  $n \in \mathbb{N}$ , assume that  $H_2 : \forall n_1 < n, P(m, n_1)$  holds. Let us show that  $P(m, n)$  holds.

Let  $m_1 \leq m$  and  $n_1 \leq n$ , assume that  $(m_1, n_1) \neq (m, n)$ . **Case  $m_1 < m$ .** Then,  $P(m_1, n_1)$  holds by  $H_1$ . **Case  $m_1 = m$ .** Then,  $n_1 < n$ , and  $P(m_1, n_1) = P(m, n_1)$  holds by  $H_2$ . Thus, in both cases,  $P(m_1, n_1)$  holds, and from the hypothesis in (5.2), we have  $P(m, n)$ .

This concludes the strong induction on  $n$ , and we have for all  $n \in \mathbb{N}$ ,  $P(m, n)$ , i.e.,  $Q(m)$ .

This concludes the strong induction on  $m$ , and we have for all  $m \in \mathbb{N}$ ,  $Q(m)$ .

Therefore, we have  $\forall m, n \in \mathbb{N}, P(m, n)$ . □

*Remark 1361.*

Note that, for  $n, p \in \mathbb{N}$ , we use the convention  $n - p \stackrel{\text{def.}}{=} 0$  when  $n < p$  (to have a result in  $\mathbb{N}$ ).

**Definition 1362 (binomial coefficient).**

Let  $n, p \in \mathbb{N}$ . Then, the *binomial coefficient* is defined as

$$(5.3) \quad \binom{n}{p} \stackrel{\text{def.}}{=} \frac{n!}{p!(n-p)!} \text{ when } p \leq n, \quad \text{and} \quad \binom{n}{p} \stackrel{\text{def.}}{=} 0 \text{ otherwise.}$$

*Remark 1363.* Note that the formula with factorials on the left of (5.3) can be extended to the irregular case when  $n < p$  (and yields the value 0) provided the use of the Euclidean division. Indeed, in this case  $\binom{n}{p} = \frac{n!}{p!0!}$ , whose Euclidean division is 0 because  $n! < p!$ .

In the Pascal triangle formula (5.7), the regular case  $1 \leq p \leq n - 1$  (with  $n \geq 1$ ) is extended with the convention of (5.3) and the remark that  $\binom{0-1}{p} = \binom{0}{p}$ , for  $p \in \mathbb{N}$ .

**Lemma 1364 (properties of the binomial coefficient).**

Let  $n, p \in \mathbb{N}$ . Then, we have

$$(5.4) \quad \binom{n}{0} = \binom{n}{n} = 1,$$

$$(5.5) \quad n \geq 1 \quad \Rightarrow \quad \binom{n}{1} = \binom{n}{n-1} = n,$$

$$(5.6) \quad p \leq n \quad \Rightarrow \quad \binom{n}{n-p} = \binom{n}{p},$$

$$(5.7) \quad (n \neq 0 \vee p \neq 1) \quad \wedge \quad p \neq 0 \quad \Rightarrow \quad \binom{n}{p} = \binom{n-1}{p-1} + \binom{n-1}{p},$$

$$(5.8) \quad p \geq 1 \quad \Rightarrow \quad \sum_{j=0}^n \binom{j+p-1}{p-1} = \binom{n+p}{p},$$

$$(5.9) \quad \binom{n}{p} \in \mathbb{N}.$$

*Proof.*

Properties (5.4), (5.5) and (5.6) are direct consequences of Definition 1362 (*binomial coefficient*).

(5.7). **Case  $p < n$ .** Then,  $n \geq 1$  and  $1 \leq p \leq n - 1$ , and all binomials are regular. Thus, from Definition 1362 (*binomial coefficient*), we have

$$\binom{n-1}{p-1} + \binom{n-1}{p} = \frac{(n-1)!}{(p-1)!(n-p)!} + \frac{(n-1)!}{p!(n-p-1)!} = \frac{(n-1)!(p+(n-p))}{p!(n-p)!} = \binom{n}{p}.$$

**Case  $p = n$ .** Then,  $n \geq 1$ ,  $\binom{n-1}{n} = 0$ , and from (5.4), the equality holds.

**Case  $n < p$ .** Then,  $p \geq 2$ , all three binomials are equal to 0, and the equality holds.

(5.8). Let  $p \geq 1$ . For all  $n \in \mathbb{N}$ , let  $P(n) \stackrel{\text{def.}}{=} \left[ \sum_{j=0}^n \binom{j+p-1}{p-1} = \binom{n+p}{p} \right]$ .

**Induction:  $P(0)$ .** Direct consequence of (5.4).

**Induction:  $P(n) \Rightarrow P(n+1)$ .** Let  $n \in \mathbb{N}$ , assume that  $P(n)$  holds. Then, from **abelian monoid properties of  $\mathbb{N}$** , and (5.7) (with  $(n+p+1, p) \neq (0, 1)$  and  $p \neq 0$ ), we have

$$\sum_{j=0}^{n+1} \binom{j+p-1}{p-1} = \sum_{j=0}^n \binom{j+p-1}{p-1} + \binom{n+p}{p-1} = \binom{n+p}{p} + \binom{n+p}{p-1} = \binom{n+1+p}{p}.$$

This concludes the induction on  $n$ , and we have for all  $n \in \mathbb{N}$ ,  $P(n)$ .

(5.9). Let  $P(n) \stackrel{\text{def.}}{=} \left[ \forall p \in \mathbb{N}, \binom{n}{p} \in \mathbb{N} \right]$ . **Strong induction on  $n$ :  $(\forall n_1 < n, P(n_1)) \Rightarrow P(n)$ .**

**Case  $n = 0$  and  $p = 1$ .** Then, from Definition 1362 (*binomial coefficient*), we have  $\binom{0}{p} = 0 \in \mathbb{N}$ .

**Case  $p = 0$ .** Then, from (5.4), we have  $\binom{n}{p} = 1 \in \mathbb{N}$ .

**Case  $(n \neq 0 \vee p \neq 1)$  and  $p \neq 0$ .** Let  $n_1 < n$ , assume that  $\binom{n_1}{p} \in \mathbb{N}$  for all  $p \in \mathbb{N}$ .

Then, from (5.7), and **monoid properties of  $\mathbb{N}$** , we have  $\binom{n}{p} = \binom{n-1}{p-1} + \binom{n-1}{p} \in \mathbb{N}$ .

This concludes the strong induction on  $n$ , and we have for all  $n \in \mathbb{N}$ ,  $P(n)$ .  $\square$

**Definition 1365 (canonic families).**

Let  $d \geq 1$ .

The notation  $\mathbf{0}$ , resp.  $\mathbf{1}$ , represents the constant family of value 0, resp. 1, either in  $\mathbb{N}^d$  or in  $\mathbb{R}^d$ . For all  $i \in [1..d]$ ,  $\mathbf{e}_i$  denotes the element of the canonical ‘‘basis’’ family in  $\mathbb{N}^d$  or in  $\mathbb{R}^d$ , such that  $(\mathbf{e}_i)_j = \delta_{ij}$  for all  $j \in [1..d]$ .

Let  $n \in \mathbb{N}$ . The notation  $[[\mathbf{v}]]^{n,d} \stackrel{\text{def.}}{=} (\mathbf{v}_0, \dots, \mathbf{v}_n)$  represents a family of  $n+1$  points in  $\mathbb{R}^d$ . When  $n \stackrel{\text{def.}}{=} d$ , it is simply written  $[[\mathbf{v}]]^d$ .

**Lemma 1366 (circular permutation).**

Let  $d \in \mathbb{N}$ , and  $i \in [0..d]$ . Let  $c_i^d$  be the function defined by

$$(5.10) \quad \forall j \in [0..d], \quad c_i^d(j) \stackrel{\text{def.}}{=} \begin{cases} j+i+1 & \text{when } j < d-i, \\ j-(d-i) & \text{otherwise.} \end{cases}$$

Then,  $c_i^d$  is bijective from  $[0..d]$  onto  $[0..d]$ , and we have  $c_i^d(d) = i$  and  $d = (c_i^d)^{-1}(i)$ .

In particular, we have  $c_d^d = \text{Id}_{[0..d]}$ .

*Proof.* From **monoid properties of  $\mathbb{N}$** , the fact that **monotony on disjoint parts implies injectivity**, and the fact that **injectivity and cardinal equality imply bijectivity**, the permutation  $c_i^d$  is increasing from  $[0..d-i-1]$  to  $[i+1..d]$  and from  $[d-i..d]$  to  $[0..i]$ , both  $[0..d-i-1] \cap [d-i..d]$  and  $[i+1..d] \cap [0..i]$  are empty, and  $[0..d] = [0..d-i-1] \cup [d-i..d]$  equals  $[i+1..d] \cup [0..i]$ , thus it is bijective from  $[0..d]$  onto itself.

If  $j = d$ , then  $j \geq d-i$ , and we have  $c_i^d(d) = d - (d-i) = i$ .

If  $i = d$ , the set  $\{j < d-d\}$  is empty, and we have  $c_d^d(j) = j - (d-d) = j$  for all  $j \in [0..d]$ .  $\square$

**Lemma 1367 (transposition).** Let  $d \in \mathbb{N}$ , and  $i \in [0..d]$ . Let  $\tau_i^d$  be the function defined by

$$(5.11) \quad \forall j \in [0..d], \quad \tau_i^d(j) \stackrel{\text{def.}}{=} \begin{cases} d & \text{when } j \stackrel{\text{def.}}{=} i, \\ i & \text{when } j \stackrel{\text{def.}}{=} d, \\ j & \text{otherwise.} \end{cases}$$

Then,  $\tau_i^d$  is involutive and bijective from  $[0..d]$  onto  $[0..d]$ .

In particular, we have  $\tau_d^d = \text{Id}_{[0..d]}$ .

*Proof.* From **monoid properties of  $\mathbb{N}$** , we have  $\tau_i^d([0..d]) \subset [0..d]$ , and  $\tau_i^d \circ \tau_i^d = \text{Id}_{[0..d]}$ , thus it is involutive and bijective from  $[0..d]$  onto itself.

If  $i = d$ , we have  $\tau_d^d(d) = d$ , and, for all  $j \in [0..d-1]$ ,  $\tau_d^d(j) = j$ , hence the result.  $\square$

**Lemma 1368 (jump enumeration).**

Let  $d \in \mathbb{N}$ , and  $i \in [0..d+1]$ . Let  $\theta_i^d$  be the function defined by

$$(5.12) \quad \forall j \in [0..d], \quad \theta_i^d(j) \stackrel{\text{def.}}{=} \begin{cases} j & \text{when } j < i, \\ j+1 & \text{otherwise.} \end{cases}$$

Then,  $\theta_i^d$  is injective from  $[0..d]$  to  $[0..d+1]$ , and we have  $\theta_i^d([0..d]) = [0..d+1] \setminus \{i\}$ .

In particular, we have  $\theta_0^d = \text{Id}_{[0..d]} + 1$  and  $\theta_{d+1}^d = \text{Id}_{[0..d]}$ .

*Proof. Injectivity.* Let  $j, k \in [0..d]$ . Assume that  $r \stackrel{\text{def.}}{=} \theta_i^d(j) = \theta_i^d(k)$ .

**Case  $i \leq j, k$ .** Then  $r = j + 1 = k + 1$ , and thus, from **monoid properties of  $\mathbb{N}$** ,  $j = k$ .

**Case  $j < i \leq k$ .** Then  $r = j = k + 1$ , which contradicts  $j < k$ .

**Case  $k < i \leq j$ .** Then  $r = j + 1 = k$ , which contradicts  $k < j$ .

**Case  $j, k < i$ .** Then  $r = j = k$ .

Thus, from **the definition of injectivity**, we have  $j = k$  in all cases, and  $\theta_i^d$  is injective.

**Image.** Direct consequence of **monotony of image**, and **monoid properties of  $\mathbb{N}$** , with  $[0..d] = [0..i-1] \uplus [i..d]$ ,  $\theta_i^d([0..i-1]) = [0..i-1]$ ,  $\theta_i^d([i..d]) = [i+1..d+1]$ , and  $[0..i-1] \cup [i+1..d+1]$  equals  $[0..d+1] \setminus \{i\}$ .

**Cases  $i = 0$  and  $i = d$ .** Direct consequence of **monoid properties of  $\mathbb{N}$** .  $\square$

## 5.2 Complements on monoids

**Lemma 1369 (image of ker is included in ker).** Let  $E, F$  be sets, and  $G$  be a monoid. Let  $f : E \rightarrow F$  and  $g : F \rightarrow G$  be functions. Then, we have

$$(5.13) \quad f(\ker(g \circ f)) \subset \ker(g).$$

*Proof.* Let  $x \in \ker(g \circ f)$ . Then, from Definition 101 (**kernel**, extended to functions taking values in a monoid), we have  $g(f(x)) = 0$ , and thus,  $f(x) \in \ker(g)$ .  $\square$

**Lemma 1370 (image of ker is ker).** Let  $E, F$  be sets, and  $G$  be a monoid. Let  $f : E \rightarrow F$  and  $g : F \rightarrow G$  be functions. Assume that  $f$  is surjective. Then, we have

$$(5.14) \quad f(\ker(g \circ f)) = \ker(g).$$

*Proof.* From Lemma 1369 (**image of ker is included in ker**), we have  $f(\ker(g \circ f)) \subset \ker(g)$ .

Reciprocally, let  $y \in \ker(g)$ . Then, from Definition 101 (**kernel**,  $g(y) = 0$ ), and **the definition of surjectivity (let  $x$  such that  $y = f(x)$ )**, we have  $g(f(x)) = g(y) = 0$ , and  $x \in \ker(g \circ f)$ . Thus,  $y = f(x)$  belongs to  $f(\ker(g \circ f))$ .  $\square$

## 5.3 Complements on linear algebra

*Remark 1371.*

In this section, vector spaces are supposed to be defined over some scalar field  $\mathbb{K}$ , such as  $\mathbb{R}$  or  $\mathbb{C}$ .

**Lemma 1372 (vector subspace is invariant by translation).** Let  $E$  be a vector space. Let  $E'$  be a vector subspace of  $E$ . Let  $\mathbf{u}'_0 \in E'$ . Then, we have  $E' + \mathbf{u}'_0 \stackrel{\text{def.}}{=} \{\mathbf{u}' + \mathbf{u}'_0 \mid \mathbf{u}' \in E'\} = E'$ .

*Proof.* Direct consequence of Lemma 81 (**closed under vector operations is subspace**).  $\square$

**Lemma 1373 (range of linear map is vector subspace).**

Let  $E$  and  $F$  be vector spaces. Let  $f \in \mathcal{L}(E, F)$ . Then, the range  $f(E)$  is a vector subspace of  $F$ .

*Proof.* Direct consequence of Definition 64 (*linear map*), **the definition of the range of a function**, and Lemma 81 (*closed under vector operations is subspace*).  $\square$

**Lemma 1374 (*injectivity or surjectivity and dimension implies bijectivity*).**

Let  $E$  and  $F$  be vector spaces. Let  $f \in \mathcal{L}(E, F)$  be a linear map from  $E$  to  $F$ . Then, we have

$$(5.15) \quad (\infty > \dim E \geq \dim F \quad \wedge \quad f \text{ injective}) \implies f \text{ bijective},$$

$$(5.16) \quad (\dim E \leq \dim F < \infty \quad \wedge \quad f \text{ surjective}) \implies f \text{ bijective}.$$

*Proof.* Direct consequences of **the rank–nullity theorem**, **the characterization of surjectivity with the rank**, Lemma 103 (*injective linear map has zero kernel*), and  $\dim\{0\} = 0$ .  $\square$

**Lemma 1375 (*inverse of isomorphism is linear map*).** Let  $E$  and  $F$  be vector spaces. Let

$f \in \mathcal{L}(E, F)$  be a linear map from  $E$  to  $F$ . Assume that  $f$  is an isomorphism from  $E$  onto  $F$ .

Then,  $f^{-1}$  is a linear map from  $F$  to  $E$ .

*Proof.* From Definition 100 (*isomorphism*,  $f$  is bijective), there exists a mapping  $f^{-1}$  from  $F$  to  $E$ . Let  $\lambda, \lambda' \in \mathbb{K}$  be scalars. Let  $v, v' \in F$  be vectors. Let  $u, u' \in E$  be the preimages of  $v$  and  $v'$  ( $f(u) = v$  and  $f(u') = v'$ ). Then, from Definition 100 (*isomorphism*,  $f$  is linear), and Lemma 95 (*linear map preserves linear combinations*), we have

$$f(\lambda u + \lambda' u') = \lambda f(u) + \lambda' f(u') = \lambda v + \lambda' v'.$$

Thus,  $\lambda u + \lambda' u'$  is the preimage of  $\lambda v + \lambda' v'$ , i.e.

$$f^{-1}(\lambda v + \lambda' v') = \lambda u + \lambda' u' = \lambda f^{-1}(v) + \lambda' f^{-1}(v').$$

Therefore, from Lemma 95 (*linear map preserves linear combinations*),  $f^{-1}$  is a linear map.  $\square$

**Lemma 1376 (*free family of dim elements is basis*).**

Let  $E$  be a vector space of dimension  $n > 0$ . Then, any free family of  $n$  elements is a basis of  $E$ .

*Proof.* Direct consequence of **the incomplete basis theorem** in the case where the free family is completed with an empty family.  $\square$

## 5.4 Complements on affine algebra

*Remark 1377.* In this section, vector spaces are supposed to be defined over some scalar field  $\mathbb{K}$ , but some results are only stated for  $\mathbb{K} \stackrel{\text{def.}}{=} \mathbb{R}$ . Even though most of the latter can be extended to any field of characteristic not 2, such as  $\mathbb{C}$ , this is not considered here.

*Remark 1378.* Note that the notion of affine space associated with a vector space can be defined in an abstract way, either through a function that builds a vector from any two points, or equivalently, through a *translation* function that builds a point from any point and any vector. Barycenters correspond to linear combinations with coefficients of sum 1. Then, affine subspaces are those closed under barycenter, and affine functions (from an affine space to another) are those that preserve barycenters. Affine functions between two given affine spaces can be proved to form an affine space.

Actually, any vector space can be equipped with an affine structure, and vice versa. Moreover, in the present work, in most cases, we encounter affine structures associated to the vector spaces  $\mathbb{R}^n$ , and the abstract vision is not mandatory. Thus, we choose to only define affine subspaces of a vector space, and affine maps between two affine subspaces. Elements are called *vectors* to emphasize their linear properties, or *points* to emphasize their affine properties.

### 5.4.1 Affine subspaces

**Definition 1379 (affine subspace).**

Let  $E$  be a vector space. A subset  $\mathcal{E}'$  of  $E$  is said to be an *affine subspace* of  $E$  iff there exists a vector subspace  $E'$  of  $E$  and  $\mathbf{x}_0 \in E$  such that  $\mathcal{E}' = \mathbf{x}_0 + E' \stackrel{\text{def.}}{=} \{\mathbf{x}_0 + \mathbf{u}' \mid \mathbf{u}' \in E'\}$ .

Then,  $E'$  is called the *direction* of the affine subspace  $\mathcal{E}'$ , and  $\mathbf{x}_0$  its *origin*. Elements of affine subspaces are called *points*.

**Lemma 1380 (origin is in affine subspace).**

Let  $E$  be a vector space. Let  $E'$  be a vector subspace of  $E$ , and  $\mathbf{x}_0 \in E$ . Then, we have  $\mathbf{x}_0 \in \mathcal{E}'$ .

*Proof.* Direct consequence of Definition 1379 (affine subspace), and Lemma 81 (closed under vector operations is subspace,  $0 \in E'$ ).  $\square$

**Lemma 1381 (equivalent definition of affine subspace).**

Let  $E$  be a vector space. Let  $\mathcal{E}' \subset E$ . Let  $\mathbf{x}'_0 \in \mathcal{E}'$ .

Then,  $\mathcal{E}'$  is an affine subspace of  $E$  iff  $\mathcal{E}' - \mathbf{x}'_0 \stackrel{\text{def.}}{=} \{\mathbf{x}' - \mathbf{x}'_0 \mid \mathbf{x}' \in \mathcal{E}'\}$  is a vector subspace of  $E$ .

Moreover, let  $E'$  be a vector subspace of  $E$ , and let  $\mathbf{x}_0 \in E$ , then we have

$$(5.17) \quad \forall \mathbf{x} \in E, \quad \mathbf{x} \in \mathbf{x}_0 + E' \iff \mathbf{x} - \mathbf{x}_0 \in E'.$$

*Proof. First equivalence.* Direct consequence of Definition 1379 (affine subspace, thus  $\mathcal{E}'$  equals  $\mathbf{x}_0 + E'$  with  $E' \subset E$  subspace and  $\mathbf{x}_0 \in E$ ,  $\mathbf{x}'_0 = \mathbf{x}_0 + \mathbf{u}'_0$  and  $\mathbf{x}' = \mathbf{x}_0 + \mathbf{u}'$  for all  $\mathbf{x}' \in \mathcal{E}'$  with  $\mathbf{u}'_0, \mathbf{u}' \in E'$ ), Lemma 1372 (vector subspace is invariant by translation, with  $-\mathbf{u}'_0$ ), and Definition 61 (vector space, additive group properties, thus  $\mathbf{x}' - \mathbf{x}'_0 = \mathbf{u}' - \mathbf{u}'_0$ , and  $\mathcal{E}' - \mathbf{x}'_0 = E' - \mathbf{u}'_0 = E'$ ).

(5.17). Direct consequence of Definition 1379 (affine subspace), Lemma 81 (closed under vector operations is subspace).  $\square$

**Lemma 1382 (affine subspace is invariant by change of origin).**

Let  $E$  be a vector space. Let  $E'$  be a vector subspace of  $E$  and  $\mathbf{x}_0 \in E$ . Let  $\mathbf{y}' \in \mathbf{x}_0 + E'$ .

Then, we have  $\mathbf{x}_0 + E' = \mathbf{y}' + E'$ .

*Proof.* Direct consequence of Definition 1379 (affine subspace, thus  $\mathbf{y}' = \mathbf{x}_0 + \mathbf{u}'_0$  with  $\mathbf{u}'_0 \in E'$ ), Lemma 81 (closed under vector operations is subspace), Lemma 1372 (vector subspace is invariant by translation, with  $-\mathbf{u}'_0$  thus  $E' - \mathbf{u}'_0 = E'$ ), and Lemma 1381 (equivalent definition of affine subspace, thus  $\mathbf{x} \in \mathbf{y}' + E'$  iff  $(\mathbf{x} - \mathbf{x}_0) - \mathbf{u}'_0 \in E' = E' - \mathbf{u}'_0$  iff  $\mathbf{x} - \mathbf{x}_0 \in E'$  iff  $\mathbf{x} \in \mathbf{x}_0 + E'$ ).  $\square$

**Lemma 1383 (vector subspace is affine subspace).**

Let  $E$  be a vector space. Let  $E'$  be a vector subspace of  $E$ .

Then,  $E'$  is an affine subspace of  $E$ , of direction itself and of origin any of its points.

*Proof.* Direct consequence of Lemma 1372 (vector subspace is invariant by translation), and Lemma 1381 (equivalent definition of affine subspace).  $\square$

*Remark 1384.* For instance, note that  $E$  is also an affine subspace of itself, e.g. with origin  $\mathbf{0}$ .

**Lemma 1385 (affine subspace plus vector subspace is affine subspace).**

Let  $E$  be a vector space. Let  $E'$  be a vector subspace of  $E$ , let  $\mathbf{x}_0 \in E$ , and let  $\mathcal{E}' \stackrel{\text{def.}}{=} \mathbf{x}_0 + E'$ .

Let  $F'$  be a vector subspace of  $E'$ . Then, we have  $\mathcal{E}' + F' \stackrel{\text{def.}}{=} \{\mathbf{x}' + \mathbf{v}' \mid \mathbf{x}' \in \mathcal{E}' \wedge \mathbf{v}' \in F'\} = \mathcal{E}'$ .

*Proof.*  $\mathcal{E}' + F' \subset \mathcal{E}'$ . Direct consequence of Definition 61 (vector space, additive abelian group properties, thus  $\mathbf{x}' + \mathbf{v}' = \mathbf{x}_0 + ((\mathbf{x}' - \mathbf{x}_0) + \mathbf{v}')$ ), Lemma 1381 (equivalent definition of affine subspace, (5.17), thus  $\mathbf{x}' - \mathbf{x}_0 \in E'$ ), and Lemma 81 (closed under vector operations is subspace, thus  $(\mathbf{x}' - \mathbf{x}_0) + \mathbf{v}' \in E'$ ).

$\mathcal{E}' + F' \supset \mathcal{E}'$ . Direct consequence of Definition 1379 (affine subspace), and Lemma 81 (closed under vector operations is subspace, thus  $0 \in F'$ ).  $\square$

**Lemma 1386 (closed under barycenter is affine subspace).** Let  $E$  be a real vector space. Let  $\mathcal{E}' \subset E$  be nonempty. Then,  $\mathcal{E}'$  is an affine subspace of  $E$  iff it is closed under barycenter,

$$(5.18) \quad \forall n \in \mathbb{N}, \forall (\mathbf{v}_i)_{i \in [0..n]} \in \mathcal{E}', \forall (\mu_i)_{i \in [0..n]} \in \mathbb{R}, \sum_{i=0}^n \mu_i = 1 \implies \sum_{i=0}^n \mu_i \mathbf{v}_i \in \mathcal{E}'.$$

*Proof. From left to right.* Direct consequence of Definition 1379 (affine subspace, thus  $\mathcal{E}'$  is of the form  $\mathbf{x}_0 + E'$ ), Lemma 1381 (equivalent definition of affine subspace, (5.17)), Definition 61 (vector space, thus  $\sum_{i=0}^n \mu_i \mathbf{v}_i - \mathbf{x}_0 = \sum_{i=0}^n \mu_i (\mathbf{v}_i - \mathbf{x}_0)$ ), and Lemma 82 (closed under linear combination is subspace, with  $\mathbf{v}_i - \mathbf{x}_0 \in E'$ ).

**From right to left.** Let  $\mathbf{x}'_0 \in \mathcal{E}'$ . Assume that  $\mathcal{E}'$  is closed under barycenter.

1.  $\mathbf{0} \in \mathcal{E}' - \mathbf{x}'_0$ . Trivial.

2.  $\mathcal{E}' - \mathbf{x}'_0$  is closed under scalar multiplication.

Let  $a \in \mathbb{R}$  and  $\mathbf{u} \in \mathcal{E}' - \mathbf{x}'_0$ , i.e.  $\mathbf{x}'_0 + \mathbf{u} \in \mathcal{E}'$ . Then, from Definition 61 (vector space), and hypothesis (with  $(1-a) + a = 1$ ), we have  $\mathbf{x}'_0 + a\mathbf{u} = (1-a)\mathbf{x}'_0 + a(\mathbf{x}'_0 + \mathbf{u}) \in \mathcal{E}'$ . Thus,  $a\mathbf{u} \in \mathcal{E}' - \mathbf{x}'_0$ .

3.  $\mathcal{E}' - \mathbf{x}'_0$  is closed under addition.

Let  $\mathbf{u}, \mathbf{v} \in \mathcal{E}' - \mathbf{x}'_0$ , i.e.  $\mathbf{x}'_0 + \mathbf{u}, \mathbf{x}'_0 + \mathbf{v} \in \mathcal{E}'$ . Then, from Definition 61 (vector space), and hypothesis (with  $\frac{1}{2} + \frac{1}{2} = 1$ ), we have  $\mathbf{x}'_0 + \frac{1}{2}(\mathbf{u} + \mathbf{v}) = \frac{1}{2}(\mathbf{x}'_0 + \mathbf{u}) + \frac{1}{2}(\mathbf{x}'_0 + \mathbf{v}) \in \mathcal{E}'$ . Thus,  $\frac{1}{2}(\mathbf{u} + \mathbf{v}) \in \mathcal{E}' - \mathbf{x}'_0$ , and from 2 (with  $a \stackrel{\text{def.}}{=} 2$ ), we have  $\mathbf{u} + \mathbf{v} = 2(\frac{1}{2}(\mathbf{u} + \mathbf{v})) \in \mathcal{E}' - \mathbf{x}'_0$ .

Finally, from Lemma 81 (closed under vector operations is subspace), and Lemma 1381 (equivalent definition of affine subspace),  $\mathcal{E}' - \mathbf{x}'_0$  is a vector subspace, and  $\mathcal{E}'$  is an affine subspace.  $\square$

**Lemma 1387 (barycenter closure is affine subspace).**

Let  $d \geq 1$ . Let  $n \in \mathbb{N}$ . Let  $[[\mathbf{v}]]^{n,d}$  be  $n+1$  points in  $\mathbb{R}^d$ . Let  $\mathcal{F} \stackrel{\text{def.}}{=} \mathbf{v}_0 + \text{span}(\mathbf{v}_j - \mathbf{v}_0)_{j \in [1..n]}$ . Then,  $\mathcal{F}$  is an affine subspace of  $\mathbb{R}^d$ , and we have

$$(5.19) \quad \forall i \in [0..n], \quad \mathcal{F} = \mathbf{v}_i + \text{span}(\mathbf{v}_j - \mathbf{v}_i)_{j \in [0..n] \setminus \{i\}},$$

$$(5.20) \quad \mathcal{F} = \left\{ \sum_{j=0}^n \mu_j \mathbf{v}_j \in \mathbb{R}^d \mid (\mu_j)_{j \in [0..n]} \in \mathbb{R} \quad \wedge \quad \sum_{j=0}^n \mu_j = 1 \right\},$$

with the convention that  $\text{span}(\emptyset) \stackrel{\text{def.}}{=} \{\mathbf{0}\}$ , thus  $\mathcal{F} = \{\mathbf{v}_0\}$  when  $n \stackrel{\text{def.}}{=} 0$ .

*Proof. Case  $n = 0$ .* Direct consequence of Definition 1379 (affine subspace, with  $E' = \{\mathbf{0}\}$ ), and Lemma 80 (trivial subspaces).

**Case  $n \geq 1$ .** From Definition 1379 (affine subspace),  $\mathcal{F}$  is an affine subspace.

Let  $i \in [0..n]$ . Let  $\mathcal{F}_i \stackrel{\text{def.}}{=} \mathbf{v}_i + \text{span}(\mathbf{v}_j - \mathbf{v}_i)_{j \in [0..n] \setminus \{i\}}$  and

$$\mathcal{H} \stackrel{\text{def.}}{=} \left\{ \sum_{j=0}^n \mu_j \mathbf{v}_j \in \mathbb{R}^d \mid \forall j \in [0..n], \mu_j \in \mathbb{R} \quad \wedge \quad \sum_{j=0}^n \mu_j = 1 \right\}.$$

Then, from Definition 85 (*finite dimensional subspace*), and **ring properties of  $\mathbb{R}$** , we have

$$\begin{aligned}
\mathbf{x} \in \mathcal{F}_i &\iff \exists (\lambda_j)_{j \in [0..n] \setminus \{i\}} \in \mathbb{R}^n, \quad \mathbf{x} = \mathbf{v}_i + \sum_{j=0, j \neq i}^n \lambda_j (\mathbf{v}_j - \mathbf{v}_i) \\
&\iff \exists (\lambda_j)_{j \in [0..n] \setminus \{i\}} \in \mathbb{R}^n, \quad \mathbf{x} = \left(1 - \sum_{j=0, j \neq i}^n \lambda_j\right) \mathbf{v}_i + \sum_{j=0, j \neq i}^n \lambda_j \mathbf{v}_j \\
&\iff \exists (\mu_j)_{j \in [0..n]} \in \mathbb{R}^{n+1}, \quad \mathbf{x} = \sum_{j=0}^n \mu_j \mathbf{v}_j \in \mathbb{R}^d \quad \wedge \quad \sum_{j=0}^n \mu_j = 1 \\
&\iff \mathbf{x} \in \mathcal{H},
\end{aligned}$$

where  $\mu_j \stackrel{\text{def.}}{=} \lambda_j$  for all  $j \in [0..n] \setminus \{i\}$ , and  $\mu_i \stackrel{\text{def.}}{=} (1 - \sum_{j=0, j \neq i}^n \lambda_j)$ , so that  $\sum_{j=0}^n \mu_j = 1$  (from left to right), and  $\lambda_j \stackrel{\text{def.}}{=} \mu_j$  for all  $j \in [0..n] \setminus \{i\}$  (from right to left). Therefore, we have the two equalities.  $\square$

## 5.4.2 Affine maps and submaps

**Definition 1388 (affine map).** Let  $E$  and  $F$  be vector spaces. Let  $E'$  be a vector subspace of  $E$  and  $\mathbf{x}_0 \in E$ . Let  $\mathcal{E}' \stackrel{\text{def.}}{=} \mathbf{x}_0 + E'$ . A function  $f : \mathcal{E}' \rightarrow F$  is said to be an *affine map* iff there exists  $\phi \in \mathcal{L}(E', F)$  and  $\mathbf{c} \in F$  such that

$$(5.21) \quad \forall \mathbf{u}' \in E', \quad f(\mathbf{x}_0 + \mathbf{u}') = \mathbf{c} + \phi(\mathbf{u}'),$$

also denoted  $f = \mathbf{c} + \phi$  with origin  $\mathbf{x}_0$  (and we may omit the origin when there is no ambiguity).

*Remark 1389.* For instance, note that a function  $f : E \rightarrow F$  (i.e. with  $\mathcal{E}' \stackrel{\text{def.}}{=} E' \stackrel{\text{def.}}{=} E$ , and origin  $\mathbf{0}$ ) is an affine mapping iff there exists  $\phi \in \mathcal{L}(E, F)$  and  $\mathbf{c} \in F$  such that  $f = \mathbf{c} + \phi$ , i.e. for all vector  $\mathbf{u} \in E$ ,  $f(\mathbf{0} + \mathbf{u}) = \mathbf{c} + \phi(\mathbf{u})$ , or equivalently, for all point  $\mathbf{x} \in E$ ,  $f(\mathbf{x}) = \mathbf{c} + \phi(\mathbf{x} - \mathbf{0})$ .

**Lemma 1390 (equivalent definition of affine map).** Let  $E$  and  $F$  be vector spaces. Let  $E'$  be a vector subspace of  $E$  and  $\mathbf{x}_0 \in E$ . Let  $\mathcal{E}' \stackrel{\text{def.}}{=} \mathbf{x}_0 + E'$ . Let  $f : \mathcal{E}' \rightarrow F$ . Then,  $f$  is an affine map iff there exists  $\phi \in \mathcal{L}(E', F)$  and  $\mathbf{c} \in F$  such that

$$(5.22) \quad \forall \mathbf{x}' \in \mathcal{E}', \quad f(\mathbf{x}') = \mathbf{c} + \phi(\mathbf{x}' - \mathbf{x}_0).$$

*Proof.* Direct consequence of Definition 1388 (*affine map*), Definition 1379 (*affine subspace*), and Lemma 1381 (*equivalent definition of affine subspace*, (5.17), thus  $\mathbf{x}' - \mathbf{x}_0 \in E'$ ).  $\square$

**Lemma 1391 (change of origin in affine map).** Let  $E$  and  $F$  be vector spaces. Let  $E'$  be a vector subspace of  $E$  and  $\mathbf{x}_0 \in E$ . Let  $\mathcal{E}' \stackrel{\text{def.}}{=} \mathbf{x}_0 + E'$ . Let  $\phi \in \mathcal{L}(E', F)$  and  $\mathbf{c} \in F$ . Let  $f \stackrel{\text{def.}}{=} \mathbf{c} + \phi$  be an affine map (with origin  $\mathbf{x}_0$ ). Let  $\mathbf{y}'_0 \in \mathcal{E}'$ . Then, we have

$$(5.23) \quad \forall \mathbf{u}' \in E', \quad f(\mathbf{y}'_0 + \mathbf{u}') = (\mathbf{c} + \phi(\mathbf{y}'_0 - \mathbf{x}_0)) + \phi(\mathbf{u}').$$

Thus, setting  $\mathbf{c}' \stackrel{\text{def.}}{=} \mathbf{c} + \phi(\mathbf{y}'_0 - \mathbf{x}_0)$ , we have  $f = \mathbf{c}' + \phi$  with origin  $\mathbf{y}'_0$ .

*Proof.* Direct consequence of Definition 1388 (*affine map*), Definition 1379 (*affine subspace*), Lemma 1381 (*equivalent definition of affine subspace*, (5.17), thus  $\mathbf{y}'_0 - \mathbf{x}_0 \in E'$ ), Lemma 81 (*closed under vector operations is subspace*, thus  $\mathbf{y}'_0 - \mathbf{x}_0 + \mathbf{u}' \in E'$ ), Lemma 95 (*linear map preserves linear combinations*), and Definition 61 (*vector space*, additive abelian group properties).  $\square$



*Remark 1392.* One can thus change the origin of an affine mapping defined over  $E$ . Indeed, let an affine mapping  $f : E \rightarrow F$  (i.e. with  $\mathcal{E}' \stackrel{\text{def.}}{=} E' \stackrel{\text{def.}}{=} E$ , and origin  $\mathbf{0}$ ) such that  $f = \mathbf{c} + \phi$ . Thus, we have for all point  $\mathbf{x}_0 \in E$  and for all vector  $\mathbf{u} \in E$ ,  $f(\mathbf{x}_0 + \mathbf{u}) = (\mathbf{c} + \phi(\mathbf{x}_0 - \mathbf{0})) + \phi(\mathbf{u})$ .

**Lemma 1393 (range of affine map is affine subspace).**

Let  $E$  and  $F$  be vector spaces. Let  $E'$  be a vector subspace of  $E$  and  $\mathbf{x}_0 \in E$ . Let  $\mathcal{E}' \stackrel{\text{def.}}{=} \mathbf{x}_0 + E'$ . Let  $\phi \in \mathcal{L}(E', F)$  and  $\mathbf{c} \in F$ . Let  $f \stackrel{\text{def.}}{=} \mathbf{c} + \phi : \mathcal{E}' \rightarrow F$  be an affine map. Then,  $f(\mathcal{E}') = \mathbf{c} + \phi(E')$  is an affine subspace of  $F$ .

*Proof.* Direct consequence of Definition 1388 (affine map), Lemma 1373 (range of linear map is vector subspace), and Definition 1379 (affine subspace).  $\square$

*Remark 1394.* Thus, affine maps can be considered as functions from an affine subspace  $\mathcal{E}'$  of  $E$  to another affine subspace  $\mathcal{F}'$  of  $F$ .

**Definition 1395 (set of affine maps).**

Let  $E$  and  $F$  be vector spaces. Let  $E'$  (resp.  $F'$ ) be a vector subspace of  $E$  (resp.  $F$ ). Let  $\mathcal{E}'$  (resp.  $\mathcal{F}'$ ) be an affine subspace of  $E$  (resp.  $F$ ) of direction  $E'$  (resp.  $F'$ ). The set of affine maps from  $\mathcal{E}'$  to  $\mathcal{F}'$  is  $\text{Aff}(\mathcal{E}', \mathcal{F}') \stackrel{\text{def.}}{=} \{f : \mathcal{E}' \rightarrow \mathcal{F}' \mid f \text{ is affine map}\}$ .

**Lemma 1396 (space of affine maps).**

Let  $E$  and  $F$  be vector spaces. Then,  $\text{Aff}(E, F)$  is a vector subspace of  $F^E$ .

*Proof.* Direct consequence of Definition 1395 (set of affine maps, with  $\mathcal{E}' \stackrel{\text{def.}}{=} E' \stackrel{\text{def.}}{=} E$  and  $\mathcal{F}' \stackrel{\text{def.}}{=} F' \stackrel{\text{def.}}{=} F$ ), Lemma 93 (space of functions to space,  $\text{Aff}(E, F) \subset F^E$ ), Lemma 82 (closed under linear combination is subspace,  $\lambda f + \lambda' f' = (\lambda \mathbf{c}_0 + \lambda' \mathbf{c}'_0) + (\lambda \phi + \lambda' \phi')$ ), and Lemma 96 (space of linear maps).  $\square$

**Lemma 1397 (output restriction of affine map).**

Let  $E$  and  $F$  be vector spaces.

Let  $E'$  (resp.  $F'$ ) be a vector subspace of  $E$  (resp.  $F$ ). Let  $\mathbf{x}_0 \in E$  and  $\mathbf{y}_0 \in F$ .

Let  $\mathcal{E}' \stackrel{\text{def.}}{=} \mathbf{x}_0 + E'$  and  $\mathcal{F}' \stackrel{\text{def.}}{=} \mathbf{y}_0 + F'$ . Let  $\phi \in \mathcal{L}(E', F)$  and  $\mathbf{c} \in F$ . Let  $f \stackrel{\text{def.}}{=} \mathbf{c} + \phi \in \text{Aff}(\mathcal{E}', F)$ . Assume that  $\phi(E') \subset F'$  and  $\mathbf{c} \in \mathcal{F}'$ . Then, we have  $f(\mathcal{E}') \subset \mathcal{F}'$  and  $f \in \text{Aff}(\mathcal{E}', \mathcal{F}')$ .

*Proof.* Direct consequence of Lemma 1393 (range of affine map is affine subspace), Lemma 1382 (affine subspace is invariant by change of origin, thus  $f(\mathcal{E}') = \mathbf{c} + \phi(E') \subset \mathbf{c} + F' = \mathcal{F}'$ ), and Definition 1395 (set of affine maps).  $\square$

**Lemma 1398 (affine submap).**

Let  $E$  and  $F$  be vector spaces.

Let  $\phi \in \mathcal{L}(E, F)$  and  $\mathbf{c} \in F$ . Let  $f \stackrel{\text{def.}}{=} \mathbf{c} + \phi \in \text{Aff}(E, F)$  with origin  $\mathbf{0}$ .

Let  $E'$  (resp.  $F'$ ) be a vector subspace of  $E$  (resp.  $F$ ). Let  $\mathbf{x}_0 \in E$  and  $\mathbf{y}_0 \in F$ . Let  $\mathcal{E}' \stackrel{\text{def.}}{=} \mathbf{x}_0 + E'$  and  $\mathcal{F}' \stackrel{\text{def.}}{=} \mathbf{y}_0 + F'$ . Let  $\phi' \stackrel{\text{def.}}{=} \phi|_{E'}$  and  $f' \stackrel{\text{def.}}{=} f|_{\mathcal{E}'}$ . Let  $\mathbf{c}' \stackrel{\text{def.}}{=} \mathbf{c} + \phi(\mathbf{x}_0 - \mathbf{0}) \in F$ .

Then, we have  $f' = \mathbf{c}' + \phi'$  with origin  $\mathbf{x}_0$ , and  $f' \in \text{Aff}(\mathcal{E}', F)$ .

Moreover, if  $\phi'(E') \subset F'$  and  $\mathbf{c}' \in \mathcal{F}'$ , then we have  $f' \in \text{Aff}(\mathcal{E}', \mathcal{F}')$ , and  $f'$  is called the affine submap of  $f$  on  $\mathcal{E}'$ .

*Proof.*  $f' = \mathbf{c}' + \phi' \in \text{Aff}(\mathcal{E}', F)$ . Let  $\mathbf{u}' \in E'$ . Then, from Definition 1379 (affine subspace, thus  $\mathbf{x}_0 + \mathbf{u}' \in \mathcal{E}'$ ), Definition 1388 (affine map, with  $\mathcal{E}' \stackrel{\text{def.}}{=} E' \stackrel{\text{def.}}{=} E$  and origin  $\mathbf{0}$ ), and Lemma 1391 (change of origin in affine map, for  $f$ , with  $\mathbf{x}_0 \stackrel{\text{def.}}{=} \mathbf{0}$  and  $\mathbf{y}'_0 \stackrel{\text{def.}}{=} \mathbf{x}_0$ ), we have

$$f'(\mathbf{x}_0 + \mathbf{u}') = f(\mathbf{x}_0 + \mathbf{u}') = (\mathbf{c} + \phi(\mathbf{x}_0 - \mathbf{0})) + \phi(\mathbf{u}') = \mathbf{c}' + \phi'(\mathbf{u}').$$

Then, from **restriction to subspace preserves linearity (thus  $\phi'$  belongs to  $\mathcal{L}(E', F)$ )**, Definition 1388 (**affine map**, with  $f', \phi', \mathbf{c}', \mathcal{E}' \stackrel{\text{def.}}{=} \mathbf{x}_0 + E'$ ), and Definition 1395 (**set of affine maps**, with  $\mathcal{F}' \stackrel{\text{def.}}{=} F' \stackrel{\text{def.}}{=} F$ ), we have  $f' = \mathbf{c}' + \phi'$  (with origin  $\mathbf{x}_0$ ) and  $f' \in \text{Aff}(\mathcal{E}', F)$ .

$f' \in \text{Aff}(\mathcal{E}', \mathcal{F}')$ .

Direct consequence of Lemma 1397 (**output restriction of affine map**, with  $f', \phi'$  and  $\mathbf{c}'$ ).  $\square$

**Lemma 1399 (equivalent definition of affine map (finite dimension)).**

Let  $p, q \geq 1$ . Let  $f : \mathbb{R}^p \rightarrow \mathbb{R}^q$ . Then, we have

$$(5.24) \quad f \in \text{Aff}(\mathbb{R}^p, \mathbb{R}^q) \iff \exists A \in \mathcal{M}_{q,p}, \exists \mathbf{c} \in \mathbb{R}^q, \forall \mathbf{x} \in \mathbb{R}^p, f(\mathbf{x}) = \mathbf{c} + A\mathbf{x}.$$

*Proof.* Direct consequence of Definition 1388 (**affine map**, with  $\mathcal{E}' \stackrel{\text{def.}}{=} E' \stackrel{\text{def.}}{=} \mathbb{R}^p, F \stackrel{\text{def.}}{=} \mathbb{R}^q, \mathbf{x}_0 \stackrel{\text{def.}}{=} \mathbf{0}$ , and  $\phi \stackrel{\text{def.}}{=} (\mathbf{u} \mapsto A\mathbf{u})$ ), and **the definition of matrix–vector product**.  $\square$

**Lemma 1400 (affine map preserves barycenter).** Let  $E$  and  $F$  be real vector spaces.

Let  $f \in \text{Aff}(E, F)$ . Let  $n \in \mathbb{N}$ . Let  $[\mathbf{v}]^n$  be  $n+1$  points in  $E$ . Let  $(\mu_i)_{i \in [0..n]} \in \mathbb{R}$ . Then, we have

$$(5.25) \quad \sum_{i=0}^n \mu_j = 1 \implies f\left(\sum_{i=0}^n \mu_j \mathbf{v}_j\right) = \sum_{i=0}^n \mu_j f(\mathbf{v}_j).$$

*Proof.* Assume that  $\sum_{i=0}^n \mu_j = 1$ . Let  $\phi \in \mathcal{L}(E, F)$  and  $\mathbf{c} \in F$ . Then, from Definition 1388 (**affine map**), Lemma 95 (**linear map preserves linear combinations**), and Definition 61 (**vector space**,  $(F, +)$  is an abelian group, distributivity), we have

$$\begin{aligned} f\left(\sum_{i=0}^n \mu_j \mathbf{v}_j\right) &= \mathbf{c} + \phi\left(\sum_{i=0}^n \mu_j \mathbf{v}_j\right) = \left(\sum_{i=0}^n \mu_j\right) \mathbf{c} + \sum_{i=0}^n \mu_j \phi(\mathbf{v}_j) \\ &= \sum_{i=0}^n \mu_j (\mathbf{c} + \phi(\mathbf{v}_j)) = \sum_{i=0}^n \mu_j f(\mathbf{v}_j). \end{aligned}$$

$\square$

**Definition 1401 (isobarycenter).**

Let  $E$  be a real vector space. Let  $n \in \mathbb{N}$ . Let  $[\mathbf{v}]^n$  be  $n+1$  points in  $E$ . The **isobarycenter** of  $[\mathbf{v}]^n$  is denoted  $\mathbf{g}^{[\mathbf{v}]^n}$ , and is defined by

$$(5.26) \quad \mathbf{g}^{[\mathbf{v}]^n} \stackrel{\text{def.}}{=} \frac{1}{n+1} \sum_{i=0}^n \mathbf{v}_i.$$

**Lemma 1402 (affine map preserves isobarycenter).** Let  $E$  and  $F$  be real vector spaces.

Let  $f \in \text{Aff}(E, F)$ . Let  $n \in \mathbb{N}$ . Let  $[\mathbf{v}]^n$  be  $n+1$  points in  $E$ . Let  $[f(\mathbf{v})]^n \stackrel{\text{def.}}{=} (f(\mathbf{v}_i))_{i \in [0..n]} \in F$ . Then, we have  $f(\mathbf{g}^{[\mathbf{v}]^n}) = \mathbf{g}^{[f(\mathbf{v})]^n}$ .

*Proof.* Direct consequence of Definition 1401 (**isobarycenter**), Definition 61 (**vector space**, properties of linear operations), **field properties of  $\mathbb{R}$  (with  $n+1 > 0$ , thus  $\sum_{i=0}^n \frac{1}{n+1} = 1$ )**, and

Lemma 1400 (**affine map preserves barycenter**, with  $\mu_i \stackrel{\text{def.}}{=} \frac{1}{n+1}$ ).  $\square$

**Lemma 1403 (affine maps are closed by composition).**

Let  $E, F$ , and  $G$  be vector spaces. Let  $\phi \in \mathcal{L}(E, F)$ ,  $\mathbf{c} \in F$ ,  $\psi \in \mathcal{L}(F, G)$  and  $\mathbf{d} \in G$ .

Let  $f \stackrel{\text{def.}}{=} \mathbf{c} + \phi \in \text{Aff}(E, F)$  and  $g \stackrel{\text{def.}}{=} \mathbf{d} + \psi \in \text{Aff}(F, G)$ , both with origin  $\mathbf{0}$ .

Then, we have

$$(5.27) \quad g \circ f = \mathbf{d} + \psi(\mathbf{c} - \mathbf{0}) + \psi \circ \phi \text{ with origin } \mathbf{0} \quad \wedge \quad g \circ f \in \text{Aff}(E, G).$$

*Proof.* Direct consequence of **the definition of composition of functions**, Definition 1388 (*affine map*, with  $f$  and  $\mathbf{x}_0 \stackrel{\text{def.}}{=} \mathbf{0}$ ), Lemma 1390 (*equivalent definition of affine map*, with  $g$ ,  $\mathbf{x}_0 \stackrel{\text{def.}}{=} \mathbf{0}$  and  $\mathbf{x}' \stackrel{\text{def.}}{=} (\mathbf{c} - \mathbf{0}) + \phi(\mathbf{u})$  for any  $\mathbf{u} \in E$ ), Definition 61 (*vector space*, properties of linear operations), Definition 64 (*linear map*), Lemma 99 (*composition of linear maps is bilinear*, thus  $\psi \circ \phi \in \mathcal{L}(E, G)$ ), and Definition 1395 (*set of affine maps*).  $\square$

**Lemma 1404 (continuous affine map is continuous linear map).**

Let  $(E, \|\cdot\|_E)$  and  $(F, \|\cdot\|_F)$  be normed vector spaces. Let  $\phi \in \mathcal{L}(E, F)$ ,  $\mathbf{x}_0 \in E$  and  $\mathbf{c} \in F$ .

Let  $f \stackrel{\text{def.}}{=} \mathbf{c} + \phi \in \text{Aff}(E, F)$  with origin  $\mathbf{x}_0$ . Then,  $f$  is continuous iff  $\phi \in \mathcal{L}_c(E, F)$ .

Moreover, in this case,  $f$  is  $\|\phi\|_{F,E}$ -Lipchitz continuous.

*Proof.* Direct consequence of Lemma 1390 (*equivalent definition of affine map*), Definition 61 (*vector space*, properties of linear operations, thus  $f(\mathbf{x}') - f(\mathbf{x}) = \phi(\mathbf{x}' - \mathbf{x}_0) - \phi(\mathbf{x} - \mathbf{x}_0)$ , and  $\mathbf{x} - \mathbf{x}_0 = \mathbf{x} - \mathbf{0} - (\mathbf{x}_0 - \mathbf{0})$ ), Definition 64 (*linear map*, thus  $f(\mathbf{x}') - f(\mathbf{x}) = \phi(\mathbf{x}' - \mathbf{0}) - \phi(\mathbf{x} - \mathbf{0})$ ), Definition 44 (*pointwise continuity*), Definition 43 (*continuity in a point*), Theorem 145 (*continuous linear map*), and Theorem 150 (*normed vector space of continuous linear maps*).  $\square$

**Lemma 1405 (injective affine submap is zero linear kernel).**

Let  $E$  and  $F$  be vector spaces. Let  $\phi \in \mathcal{L}(E, F)$  and  $\mathbf{c} \in F$ . Let  $f \stackrel{\text{def.}}{=} \mathbf{c} + \phi \in \text{Aff}(E, F)$  with origin  $\mathbf{0}$ . Let  $E'$  (resp.  $F'$ ) be a vector subspace of  $E$  (resp.  $F$ ). Let  $\mathbf{x}_0 \in E$  and  $\mathbf{y}_0 \in F$ .

Let  $\mathcal{E}' \stackrel{\text{def.}}{=} \mathbf{x}_0 + E'$  and  $\mathcal{F}' \stackrel{\text{def.}}{=} \mathbf{y}_0 + F'$ . Let  $\phi' \stackrel{\text{def.}}{=} \phi|_{E'}$  and  $f' \stackrel{\text{def.}}{=} f|_{\mathcal{E}'}$ . Let  $\mathbf{c}' \stackrel{\text{def.}}{=} \mathbf{c} + \phi(\mathbf{x}_0 - \mathbf{0}) \in F$ . Let  $f' = \mathbf{c}' + \phi'$  with origin  $\mathbf{x}_0$  be the affine submap of  $f$  on  $\mathcal{E}'$ .

Then,  $f'$  is injective iff  $\ker(\phi') = \{0\}$ .

*Proof.* Direct consequence of Lemma 1398 (*affine submap*), Definition 61 (*vector space*, properties of linear operations, thus  $f'(\mathbf{x}_0 + \mathbf{v}') - f'(\mathbf{x}_0 + \mathbf{u}') = \phi'(\mathbf{v}') - \phi'(\mathbf{u}')$ ), Definition 64 (*linear map*, thus  $\phi'(\mathbf{v}') - \phi'(\mathbf{u}') = \phi'(\mathbf{v}' - \mathbf{u}')$ ), **the definition of injectivity**, and Lemma 103 (*injective linear map has zero kernel*).  $\square$

**Lemma 1406 (injective affine map is zero linear kernel).** Let  $E$  and  $F$  be vector spaces.

Let  $\phi \in \mathcal{L}(E, F)$  and  $\mathbf{c} \in F$ . Let  $f \stackrel{\text{def.}}{=} \mathbf{c} + \phi \in \text{Aff}(E, F)$  with origin  $\mathbf{0}$ .

Then,  $f$  is injective iff  $\ker(\phi) = \{0\}$ .

*Proof.* Direct consequence of Lemma 1405 (*injective affine submap is zero linear kernel*, with  $\mathcal{E}' \stackrel{\text{def.}}{=} E' \stackrel{\text{def.}}{=} E$ ,  $\mathcal{F}' \stackrel{\text{def.}}{=} F' \stackrel{\text{def.}}{=} F$  and  $\mathbf{x}_0 = \mathbf{y}_0 \stackrel{\text{def.}}{=} \mathbf{0}$ , thus  $\phi' = \phi$ ,  $f' = f$  and  $\mathbf{c}' = \mathbf{c}$ ).  $\square$

**Lemma 1407 (surjective affine submap is full linear range).**

Let  $E$  and  $F$  be vector spaces. Let  $\phi \in \mathcal{L}(E, F)$  and  $\mathbf{c} \in F$ . Let  $f \stackrel{\text{def.}}{=} \mathbf{c} + \phi \in \text{Aff}(E, F)$  with origin  $\mathbf{0}$ . Let  $E'$  (resp.  $F'$ ) be a vector subspace of  $E$  (resp.  $F$ ). Let  $\mathbf{x}_0 \in E$  and  $\mathbf{y}_0 \in F$ .

Let  $\mathcal{E}' \stackrel{\text{def.}}{=} \mathbf{x}_0 + E'$  and  $\mathcal{F}' \stackrel{\text{def.}}{=} \mathbf{y}_0 + F'$ . Let  $\phi' \stackrel{\text{def.}}{=} \phi|_{E'}$  and  $f' \stackrel{\text{def.}}{=} f|_{\mathcal{E}'}$ . Let  $\mathbf{c}' \stackrel{\text{def.}}{=} \mathbf{c} + \phi(\mathbf{x}_0 - \mathbf{0}) \in F$ . Assume that  $\phi'(E') \subset F'$  and  $\mathbf{c}' \in F'$ .

Let  $f' = \mathbf{c}' + \phi'$  with origin  $\mathbf{x}_0$  be the affine submap of  $f$  on  $\mathcal{E}'$ .

Then,  $f'$  is surjective iff  $\phi'(E') = F'$ .

*Proof. From left to right.* Assume that  $f'$  is surjective onto  $\mathcal{F}'$ .

Let  $\mathbf{v}' \in F'$ . Then, from Definition 1379 (*affine subspace*, with  $\mathcal{F}'$ ), and Lemma 1382 (*affine subspace is invariant by change of origin*, with  $\mathbf{y}' \stackrel{\text{def.}}{=} \mathbf{c}' \in \mathbf{y}_0 + F'$ ), we have  $\mathbf{c}' + \mathbf{v}' \in \mathbf{c}' + F' = \mathcal{F}'$ . Thus, from **the definition of surjectivity**, Lemma 1390 (*equivalent definition of affine map*), Lemma 82 (*closed under linear combination is subspace*), and Lemma 1381 (*equivalent definition of affine subspace*, (5.17)), there exists  $\mathbf{x}' \in \mathcal{E}'$  such that  $f'(\mathbf{x}') = \mathbf{c}' + \mathbf{v}'$ , and  $\phi'(\mathbf{x}' - \mathbf{x}_0) = f'(\mathbf{x}') - \mathbf{c}' = \mathbf{v}'$  with  $\mathbf{x}' - \mathbf{x}_0 \in E'$ .

Therefore, we have  $F' \subset \phi'(E')$ , and thus  $\phi'(E') = F'$ .

**From right to left.** Assume that  $\phi'(E') = F'$ . Let  $\mathbf{y}' \in \mathcal{F}'$ . Then, from Definition 61 (*vector space, properties of linear operations*), Lemma 1381 (*equivalent definition of affine subspace, (5.17)* with  $\mathbf{y}'$  and  $\mathbf{c}'$ ), Lemma 82 (*closed under linear combination is subspace*), and Definition 1388 (*affine map*), and Definition 1379 (*affine subspace, with  $\mathcal{E}'$* ), we have  $\mathbf{y}' - \mathbf{c}' = (\mathbf{y}' - \mathbf{y}_0) - (\mathbf{c}' - \mathbf{y}_0) \in F'$ , thus there exists  $\mathbf{u}' \in E'$  such that  $\phi'(\mathbf{u}') = \mathbf{y}' - \mathbf{c}'$ , and  $f'(\mathbf{x}_0 + \mathbf{u}') = \mathbf{c}' + \phi'(\mathbf{u}') = \mathbf{y}'$  with  $\mathbf{x}_0 + \mathbf{u}' \in \mathcal{E}'$ .

Therefore, from **the definition of surjectivity**,  $f'$  is surjective.  $\square$

**Lemma 1408 (surjective affine map is full linear range).** Let  $E$  and  $F$  be vector spaces. Let  $\phi \in \mathcal{L}(E, F)$  and  $\mathbf{c} \in F$ . Let  $f \stackrel{\text{def.}}{=} \mathbf{c} + \phi \in \text{Aff}(E, F)$  with origin  $\mathbf{0}$ . Then,  $f$  is surjective iff  $\phi(E) = F$ .

*Proof.* Direct consequence of Lemma 1407 (*surjective affine submap is full linear range, with  $\mathcal{E}' \stackrel{\text{def.}}{=} E' \stackrel{\text{def.}}{=} E$ ,  $\mathcal{F}' \stackrel{\text{def.}}{=} F' \stackrel{\text{def.}}{=} F$  and  $\mathbf{x}_0 = \mathbf{y}_0 \stackrel{\text{def.}}{=} \mathbf{0}$ , thus  $\phi' = \phi$ ,  $f' = f$ , and  $\mathbf{c}' = \mathbf{c} \in F$ , and  $\phi(E) \subset F$ ).  $\square$*

**Lemma 1409 (inverse of affine submap is affine submap).**

Let  $E$  and  $F$  be vector spaces. Let  $\phi \in \mathcal{L}(E, F)$  and  $\mathbf{c} \in F$ . Let  $f \stackrel{\text{def.}}{=} \mathbf{c} + \phi \in \text{Aff}(E, F)$  with origin  $\mathbf{0}$ . Let  $E'$  (resp.  $F'$ ) be a vector subspace of  $E$  (resp.  $F$ ). Let  $\mathbf{x}_0 \in E$  and  $\mathbf{y}_0 \in F$ .

Let  $\mathcal{E}' \stackrel{\text{def.}}{=} \mathbf{x}_0 + E'$  and  $\mathcal{F}' \stackrel{\text{def.}}{=} \mathbf{y}_0 + F'$ . Let  $\phi' \stackrel{\text{def.}}{=} \phi|_{E'}$  and  $f' \stackrel{\text{def.}}{=} f|_{\mathcal{E}'}$ . Let  $\mathbf{c}' \stackrel{\text{def.}}{=} \mathbf{c} + \phi(\mathbf{x}_0 - \mathbf{0}) \in F$ . Assume that  $\phi'(E') \subset F'$  and  $\mathbf{c}' \in \mathcal{F}'$ .

Let  $f' = \mathbf{c}' + \phi'$  with origin  $\mathbf{x}_0$  be the affine submap of  $f$  on  $\mathcal{E}'$ .

Then,  $f'$  is bijective in  $\text{Aff}(\mathcal{E}', \mathcal{F}')$  iff  $\phi'$  is an isomorphism in  $\mathcal{L}(E', F')$ .

Moreover, in this case, we have  $(f')^{-1} \in \text{Aff}(\mathcal{F}', \mathcal{E}')$ , and with  $\mathbf{x}'_0 \stackrel{\text{def.}}{=} \mathbf{x}_0 + (\phi')^{-1}(\mathbf{y}_0 - \mathbf{c}') \in \mathcal{E}'$ ,

$$(5.28) \quad \forall \mathbf{y}' \in \mathcal{F}', \quad (f')^{-1}(\mathbf{y}') = \mathbf{x}_0 + (\phi')^{-1}(\mathbf{y}' - \mathbf{c}') = \mathbf{x}'_0 + (\phi')^{-1}(\mathbf{y}' - \mathbf{y}_0),$$

$$(5.29) \quad \forall \mathbf{v}' \in F', \quad (\phi')^{-1}(\mathbf{v}') = (f')^{-1}(\mathbf{c}' + \mathbf{v}') - \mathbf{x}_0.$$

*Proof. Equivalence.* Direct consequence of **the definition of bijectivity**, Lemma 1405 (*injective affine submap is zero linear kernel*), Lemma 1407 (*surjective affine submap is full linear range*), Lemma 103 (*injective linear map has zero kernel*), **the characterization of surjectivity with full range**, and Definition 100 (*isomorphism*).

(5.28) and  $(f')^{-1} \in \text{Aff}(\mathcal{F}', \mathcal{E}')$ . Let  $\mathbf{y}' \in \mathcal{F}'$ . Let  $\mathbf{x}' \stackrel{\text{def.}}{=} (f')^{-1}(\mathbf{y}') \in \mathcal{E}'$ . Then, from **the definition of the inverse**, Lemma 1390 (*equivalent definition of affine map*), Lemma 1381 (*equivalent definition of affine subspace, thus  $\mathbf{c}' - \mathbf{y}_0, \mathbf{y}' - \mathbf{y}_0 \in F'$* ), Lemma 81 (*closed under vector operations is subspace, thus  $\mathbf{y}_0 - \mathbf{c}' = -(\mathbf{c}' - \mathbf{y}_0) \in F'$  and  $\mathbf{y}' - \mathbf{c}' = (\mathbf{y}_0 - \mathbf{c}') + (\mathbf{y}' - \mathbf{y}_0) \in F'$* ), Definition 1379 (*affine subspace, with  $(\phi')^{-1}(\mathbf{y}_0 - \mathbf{c}') \in E'$ , thus  $\mathbf{x}'_0 \in \mathcal{E}'$* ), Definition 61 (*vector space, additive abelian group properties for  $E'$  and  $F'$* ), Lemma 1375 (*inverse of isomorphism is linear map*), and Definition 1388 (*affine map*), we have successively  $\mathbf{y}' = f'(\mathbf{x}') = \mathbf{c}' + \phi'(\mathbf{x}' - \mathbf{x}_0)$ ,  $\phi'(\mathbf{x}' - \mathbf{x}_0) = \mathbf{y}' - \mathbf{c}' = (\mathbf{y}_0 - \mathbf{c}') + (\mathbf{y}' - \mathbf{y}_0)$ , and thus

$$(f')^{-1}(\mathbf{y}') = \mathbf{x}' = \mathbf{x}_0 + (\phi')^{-1}(\mathbf{y}' - \mathbf{c}') = \mathbf{x}'_0 + (\phi')^{-1}(\mathbf{y}' - \mathbf{y}_0) \quad \text{and} \quad (f')^{-1} \in \text{Aff}(\mathcal{F}', \mathcal{E}').$$

(5.29). Direct consequence of (5.28), and Lemma 82 (*closed under linear combination is subspace, for both  $E$  and  $F$* ).  $\square$

**Lemma 1410 (inverse of affine map is affine map).** Let  $E$  and  $F$  be vector spaces.

Let  $\phi \in \mathcal{L}(E, F)$  and  $\mathbf{c} \in F$ . Let  $f \stackrel{\text{def.}}{=} \mathbf{c} + \phi \in \text{Aff}(E, F)$  with origin  $\mathbf{0}$ .

Then,  $f$  is bijective iff  $\phi$  is an isomorphism.

Moreover, in this case, we have  $f^{-1} \in \text{Aff}(F, E)$ , and with  $\mathbf{x}'_0 \stackrel{\text{def.}}{=} \mathbf{0} + \phi^{-1}(\mathbf{0} - \mathbf{c}) \in E$ ,

$$(5.30) \quad \forall \mathbf{y} \in F, \quad f^{-1}(\mathbf{y}) = \mathbf{0} + \phi^{-1}(\mathbf{y} - \mathbf{c}) = \mathbf{x}'_0 + \phi^{-1}(\mathbf{y} - \mathbf{0}),$$

$$(5.31) \quad \forall \mathbf{v} \in F, \quad \phi^{-1}(\mathbf{v}) = f^{-1}(\mathbf{c} + \mathbf{v}) - \mathbf{0}.$$

*Proof.* Direct consequence of Lemma 1409 (*inverse of affine submap is affine submap*, with  $\mathcal{E}' \stackrel{\text{def.}}{=} E' \stackrel{\text{def.}}{=} E$ ,  $\mathcal{F}' \stackrel{\text{def.}}{=} F' \stackrel{\text{def.}}{=} F$  and  $\mathbf{x}_0 = \mathbf{y}_0 \stackrel{\text{def.}}{=} \mathbf{0}$ , thus  $\phi' = \phi$ ,  $f' = f$ , and  $\mathbf{c}' = \mathbf{c} \in F$ , and  $\phi(E) \subset F$ ).  $\square$

## 5.5 Complements on real affine geometry

**Definition 1411** (*affinely independent family*).

Let  $d \geq 1$ . Let  $n \in \mathbb{N}$ . Let  $[[\mathbf{v}]]^{n,d}$  be  $n + 1$  points in  $\mathbb{R}^d$ .

Then,  $[[\mathbf{v}]]^{n,d}$  is said *affinely independent* iff  $(\mathbf{v}_1 - \mathbf{v}_0, \dots, \mathbf{v}_n - \mathbf{v}_0)$  is free in the vector space  $\mathbb{R}^d$ .

**Lemma 1412** (*equivalent definition of affinely independent family*).

Let  $d \geq 1$ . Let  $n \in \mathbb{N}$ . Let  $[[\mathbf{v}]]^{n,d}$  be  $n + 1$  points in  $\mathbb{R}^d$ . Let  $j \in [0..n]$ .

Then,  $[[\mathbf{v}]]^{n,d}$  is affinely independent iff  $(\mathbf{v}_i - \mathbf{v}_j)_{i \in [0..n] \setminus \{j\}}$  is free in the vector space  $\mathbb{R}^d$ .

*Proof.* **Case  $n = 0$ .** Direct consequence of Definition 1411 (*affinely independent family*), and **the definition of freedom (the empty family in any vector space is free)**.

**Case  $j = 0$ .** Direct consequence of Definition 1411 (*affinely independent family*).

**Case  $n \geq 1$  and  $j \neq 0$ . (1).** Let  $k, l \in [0..n]$ , such that  $k \neq l$ . Let  $(\mu_i)_{i \in [0..n] \setminus \{k\}} \in \mathbb{R}^n$ . Then, from Definition 61 (*vector space, properties of linear operations*), we have

$$\begin{aligned} \sum_{i=0, i \neq k}^n \mu_i (\mathbf{v}_i - \mathbf{v}_k) &= \sum_{\substack{i=0 \\ i \neq k, i \neq l}}^n \mu_i \mathbf{v}_i - \left( \sum_{i=0, i \neq k}^n \mu_i \right) \mathbf{v}_k + \mu_l \mathbf{v}_l = \sum_{i=0, i \neq l}^n \tilde{\mu}_i \mathbf{v}_i + \mu_l \mathbf{v}_l \\ &= \sum_{i=0, i \neq l}^n \tilde{\mu}_i (\mathbf{v}_i - \mathbf{v}_l) + \left( \mu_l + \sum_{i=0, i \neq l}^n \tilde{\mu}_i \right) \mathbf{v}_l \\ &= \sum_{i=0, i \neq l}^n \tilde{\mu}_i (\mathbf{v}_i - \mathbf{v}_l) + \left( \tilde{\mu}_k + \sum_{i=0, i \neq k}^n \mu_i \right) \mathbf{v}_l = \sum_{i=0, i \neq l}^n \tilde{\mu}_i (\mathbf{v}_i - \mathbf{v}_l), \end{aligned}$$

where we have set  $\tilde{\mu}_k \stackrel{\text{def.}}{=} -\sum_{i=0, i \neq k}^n \mu_i$ , and  $\tilde{\mu}_i \stackrel{\text{def.}}{=} \mu_i$  for  $i \in [0..n] \setminus \{k, l\}$ .

**(2)  $(\mathbf{v}_i - \mathbf{v}_l)_{i \in [0..n] \setminus \{l\}}$  free  $\Rightarrow (\mathbf{v}_i - \mathbf{v}_k)_{i \in [0..n] \setminus \{k\}}$  free.**

Let  $k, l \in [0..n]$ , such that  $k \neq l$ . Assume that  $(\mathbf{v}_i - \mathbf{v}_l)_{i \in [0..n] \setminus \{l\}}$  is free.

Let  $(\mu_i)_{i \in [0..n] \setminus \{k\}} \in \mathbb{R}^n$ , such that  $\sum_{i=0, i \neq k}^n \mu_i (\mathbf{v}_i - \mathbf{v}_k) = \mathbf{0}$ . Then, from (1), and **the definition of freedom**, we have  $\sum_{i=0, i \neq l}^n \tilde{\mu}_i (\mathbf{v}_i - \mathbf{v}_l) = \mathbf{0}$ , and

$$\forall i \in [0..n] \setminus \{k, l\}, \tilde{\mu}_i = \mu_i = 0, \quad \text{and} \quad \tilde{\mu}_k = -\sum_{i=0, i \neq k}^n \mu_i = -\mu_l = 0.$$

Thus, for all  $i \in [0..n] \setminus \{k\}$ ,  $\mu_i = 0$ , and the family  $(\mathbf{v}_i - \mathbf{v}_k)_{i \in [0..n] \setminus \{k\}}$  is free.

**(3).** The equivalence is a direct consequence of (2), first with  $k \stackrel{\text{def.}}{=} j$  and  $l \stackrel{\text{def.}}{=} 0$  (from left to right), then with  $k \stackrel{\text{def.}}{=} 0$  and  $l \stackrel{\text{def.}}{=} j$  (from right to left).  $\square$

**Lemma 1413** (*affinely independent family of 2 elements*).

Let  $d \geq 1$ .

Let  $[[\mathbf{v}]]^{1,d} \stackrel{\text{def.}}{=} (\mathbf{v}_0, \mathbf{v}_1)$  be 2 points in  $\mathbb{R}^d$ . Then,  $[[\mathbf{v}]]^{1,d}$  is affinely independent iff  $\mathbf{v}_0 \neq \mathbf{v}_1$ .

*Proof.* Direct consequence of Definition 1411 (*affinely independent family*), and **a free vector is nonzero**.  $\square$

**Lemma 1414 (affinely independent family is closed by sub-family).**

Let  $d \geq 1$ . Let  $n \in \mathbb{N}$ . Let  $[\mathbf{v}]^{n,d}$  be  $n+1$  affinely independent points in  $\mathbb{R}^d$ . Let  $l \in [0..n]$ , and let  $\pi^l$  be an injective map from  $[0..l]$  into  $[0..n]$ . Let  $j \in [0..l]$ . Then,  $(\mathbf{v}_{\pi^l(i)} - \mathbf{v}_{\pi^l(j)})_{i \in [0..l] \setminus \{j\}}$  is free in the vector space  $\mathbb{R}^d$ .

In other terms, the family  $[\mathbf{v}_{\pi^l}]^{l,d} \stackrel{\text{def.}}{=} (\mathbf{v}_{\pi^l(i)})_{i \in [0..l]}$  is affinely independent in  $\mathbb{R}^d$ .

*Proof.* Let  $(\mu_i)_{i \in [0..l] \setminus \{j\}}$  in  $\mathbb{R}^l$  such that  $\sum_{i \in [0..l] \setminus \{j\}} \mu_i (\mathbf{v}_{\pi^l(i)} - \mathbf{v}_{\pi^l(j)}) = 0$ .

Let  $I \stackrel{\text{def.}}{=} \pi^l([0..l]) \subset [0..n]$ , and  $J \stackrel{\text{def.}}{=} [0..n] \setminus I$ . Thus, from **the definition of bijectivity**,  $\pi^l$  is bijective from  $[0..l]$  to  $I$ , and  $[0..n] = I \uplus J$  is a partition. Let  $j' \stackrel{\text{def.}}{=} \pi^l(j) \in I$ . Thus, we have

$$0 = \sum_{i' \in I \setminus \{j'\}} \mu_{(\pi^l)^{-1}(i')} (\mathbf{v}_{i'} - \mathbf{v}_{j'}) + \sum_{i' \in J} 0 (\mathbf{v}_{i'} - \mathbf{v}_{j'}) = \sum_{i' \in [0..n] \setminus \{j'\}} \tilde{\mu}_{i'} (\mathbf{v}_{i'} - \mathbf{v}_{j'}),$$

where we have set  $\forall i' \in I \setminus \{j'\}$ ,  $\tilde{\mu}_{i'} \stackrel{\text{def.}}{=} \mu_{(\pi^l)^{-1}(i')}$ , and  $\forall i' \in J$ ,  $\tilde{\mu}_{i'} \stackrel{\text{def.}}{=} 0$ . Thus, from Lemma 1412 (equivalent definition of affinely independent family, with  $j \stackrel{\text{def.}}{=} j'$ ), and **the definition of freedom**, we deduce that for all  $i' \in I \setminus \{j'\}$ ,  $\mu_{(\pi^l)^{-1}(i')} = 0$ , and thus for all  $i \in [0..l] \setminus \{j\}$ ,  $\mu_i = 0$ . Thus, from **the definition of freedom**, and Definition 1411 (affinely independent family),  $(\mathbf{v}_{\pi^l(i)} - \mathbf{v}_{\pi^l(j)})_{i \in [0..l] \setminus \{j\}}$  is free, and  $[\mathbf{v}_{\pi^l}]^{l,d}$  is affinely independent.  $\square$

## 5.6 Complements on univariate polynomials

*Remark 1415.* See also Sections 8.2 and 9.2.

**Definition 1416 (monomial of a single variable).** Let  $\alpha \in \mathbb{N}$ . The monomial of degree  $\alpha$  of a single variable is denoted  $X^\alpha$ , and is defined by  $X^\alpha \stackrel{\text{def.}}{=} (x \in \mathbb{R} \mapsto x^\alpha \in \mathbb{R})$ .

**Definition 1417 (polynomial space  $\mathbb{P}_k^1$ ).** Let  $k \in \mathbb{N}$ . The space of polynomials of degree at most  $k$  of a single variable is denoted  $\mathbb{P}_k^1$ , and is defined by

$$(5.32) \quad \mathbb{P}_k^1 \stackrel{\text{def.}}{=} \text{span}(1, X, X^2, \dots, X^k) = \left\{ \left( x \mapsto \sum_{\alpha=0}^k a_\alpha x^\alpha \right) : \mathbb{R} \rightarrow \mathbb{R} \mid (a_\alpha)_{\alpha \in [0..k]} \in \mathbb{R} \right\}.$$

**Lemma 1418 ( $\mathbb{P}_k^1$  is space of degree at most  $k$ ).** Let  $\mathcal{P}(\mathbb{R})$  be the infinite-dimensional space of polynomials of a single variable on  $\mathbb{R}$ . Let  $k \in \mathbb{N}$ . Then, we have

$$(5.33) \quad \mathbb{P}_k^1 = \{p \in \mathcal{P}(\mathbb{R}) \mid \deg p \leq k\}.$$

For instance, we have

$$(5.34) \quad \mathbb{P}_0^1 = \text{span}(1) = \{(x \mapsto a_0) \mid a_0 \in \mathbb{R}\},$$

$$(5.35) \quad \mathbb{P}_1^1 = \text{span}(1, X) = \{(x \mapsto a_0 + a_1 x) \mid a_0, a_1 \in \mathbb{R}\} = \mathcal{A}ff(\mathbb{R}, \mathbb{R}).$$

*Proof.* Direct consequence of **the definition of the degree a univariate polynomial**, Definition 1417 (polynomial space  $\mathbb{P}_k^1$ ), Definition 1416 (monomial of a single variable), Lemma 1399 (equivalent definition of affine map (finite dimension), with  $p = q = 1$ ), and Lemma 1396 (space of affine maps).  $\square$

**Lemma 1419 (monomials are free in  $\mathbb{P}_k^1$ ).** Let  $k \in \mathbb{N}$ . Then,  $(1, X, \dots, X^k)$  is free in  $\mathbb{P}_k^1$ .

*Proof.* Let  $(a_\alpha)_{\alpha \in [0..k]}$  such that for all  $x \in \mathbb{R}$ ,  $\sum_{\alpha=0}^k a_\alpha x^\alpha = 0$ . Let  $l \in [0..k]$ . Then, from **the linearity of derivation**, **the positivity of factorial**, **the zero-product property in  $\mathbb{R}$** , and **the definition of freedom**, the  $l$ -th derivative taken in  $x = 0$  provides  $l! a_l = 0$ , thus  $a_l = 0$ , and the family is free.  $\square$

**Lemma 1420 (dimension of  $\mathbb{P}_k^1$ ).**

Let  $k \in \mathbb{N}$ . Then,  $\mathbb{P}_k^1$  is a vector space of dimension  $\dim \mathbb{P}_k^1 = k + 1$ .

*Proof.* Direct consequence of Definition 1417 (polynomial space  $\mathbb{P}_k^1$ ), Lemma 1419 (monomials are free in  $\mathbb{P}_k^1$ ), Lemma 1376 (free family of dim elements is basis), and **the definition of the dimension of a vector space**.  $\square$

*Remark 1421.* Note in the next lemma that when  $p$  or  $q$  is 0, then by convention, its degree is  $-\infty$  (which is absorbing for addition in  $\{-\infty\} \uplus \mathbb{N}$ ), and the formula  $\deg(pq) = -\infty = \deg(p) + \deg(q)$  is still valid.

**Lemma 1422 (product of two univariate polynomials).**

Let  $k, l \in \mathbb{N}$ . Let  $p \in \mathbb{P}_k^1$ , and  $q \in \mathbb{P}_l^1$ . Then, we have  $pq \in \mathbb{P}_{k+l}^1$ .

Moreover, if  $\deg(p) = k$  and  $\deg(q) = l$ , then  $\deg(pq) = k + l$ .

*Proof.* From **the definition of univariate polynomial**, let  $(a_i)_{i \in [0..k]} \in \mathbb{R}^{k+1}$  and  $(b_j)_{j \in [0..l]}$  in  $\mathbb{R}^{l+1}$  such that we have  $p = \sum_{i=0}^k a_i X^i$  and  $q = \sum_{j=0}^l b_j X^j$ . Thus, from **commutative ring properties of  $\mathbb{R}$** , and **the fact that  $a_j b_{n-j}$  is a priori nonzero iff  $0 \leq j \leq k$  and  $0 \leq n - j \leq l$  (i.e.  $n - l \leq j \leq n$ )**, we obtain

$$(5.36) \quad pq = \sum_{n=0}^{k+l} \sum_{j=\max\{0, n-l\}}^{\min\{n, k\}} a_j b_{n-j} X^n.$$

Thus,  $pq \in \mathbb{P}_{k+l}^1$ .

**Degree formula.**

Direct consequence of **the definition of univariate polynomial**, **the zero-product property in  $\mathbb{R}$** , and formula (5.36) (coefficients of highest degree,  $a_k$ ,  $b_l$  and  $a_k b_l$  are all nonzero).  $\square$

## Chapter 6

# Finite element

*Remark 1423.* The next definition is from from Ciarlet [13], it is taken from [20, pp. 50–52].

**Definition 1424 (finite element triple).** Let  $d \geq 1$ . Let  $n_{\text{sh}} \geq 1$ . Let  $q \in \{1, d\}$ . Let  $K \subset \mathbb{R}^d$ . Let  $P \subset \mathcal{F}(K, \mathbb{R}^q)$ . Let  $\Sigma \stackrel{\text{def.}}{=} (\sigma_i)_{i \in [1..n_{\text{sh}}]} \in \mathcal{L}(P, \mathbb{R})$ . The triple  $(K, P, \Sigma)$  is called *finite element* iff

1.  $K$  is a nontrivial polyhedron,  $\overset{\circ}{K} \neq \emptyset$ .  
The polyhedron  $K$  represents the *geometry* of the finite element.
2.  $P$  is a nontrivial finite-dimensional vector space of functions,  $P \neq \{0\}$ .  
The vector space  $P$  is called *approximation space* of the finite element.
3.  $\Sigma$  is such that  $\phi_\Sigma : P \rightarrow \mathbb{R}^{n_{\text{sh}}}$  defined by  $\phi_\Sigma(p) \stackrel{\text{def.}}{=} (\sigma_i(p))_{i \in [1..n_{\text{sh}}]}$  is an isomorphism.  
The linear forms  $\sigma_i$  are called *degrees of freedom* of the finite element,  $n_{\text{sh}}$  is the *number of degrees of freedom*, and the bijectivity of  $\phi_\Sigma$  is called *unisolvence*.

*Remark 1425.*

The functions of  $P$  are typically polynomial functions, possibly composed with some smooth diffeomorphism. For any finite collection of linear forms  $\Sigma$ , the mapping  $\phi_\Sigma$  is obviously linear.

**Lemma 1426 (injectivity implies unisolvence).** Let  $d \geq 1$ . Let  $n_{\text{sh}} \geq 1$ . Let  $q \in \{1, d\}$ . Let  $K \subset \mathbb{R}^d$  be a nontrivial polyhedron. Let  $P \subset \mathcal{F}(K, \mathbb{R}^q)$  be a nontrivial finite-dimensional vector space. Let  $\Sigma \stackrel{\text{def.}}{=} (\sigma_i)_{i \in [1..n_{\text{sh}}]} \in \mathcal{L}(P, \mathbb{R})$  be linear forms on  $P$ . If  $\dim P \geq n_{\text{sh}} = \text{card}(\Sigma)$  and  $\phi_\Sigma$  is injective, then  $(K, P, \Sigma)$  is unisolvent

*Proof.* Direct consequence of Definition 1424 (*finite element triple*), and Lemma 1374 (*injectivity or surjectivity and dimension implies bijectivity*).  $\square$

**Lemma 1427 (dimension of approximation space).**

Let  $(K, P, \Sigma)$  be a finite element. Then, we have  $\dim P = n_{\text{sh}} = \text{card}(\Sigma)$ .

*Proof.* Direct consequence of **the rank–nullity theorem**.  $\square$

**Lemma 1428 (degrees of freedom are basis).**

Let  $(K, P, \Sigma)$  be a finite element. Then,  $\Sigma$  is a basis of the dual vector space  $\mathcal{L}(P, \mathbb{R})$ .

*Proof.*

From Lemma 1376 (*free family of dim elements is basis*), Lemma 1427 (*dimension of approximation*



*space*), and **the fact that  $\dim(\mathcal{L}(P, \mathbb{R})) = \dim P$** , it suffices to show that  $\Sigma$  is free.

Let  $a \stackrel{\text{def.}}{=} (a_i)_{i \in [1..n_{\text{sh}}]} \in \mathbb{R}$  such that  $\sum_{i \in [1..n_{\text{sh}}]} a_i \sigma_i = 0$ . Then, with  $p \stackrel{\text{def.}}{=} \phi_{\Sigma}^{-1}(a)$ , we have

$$\sum_{i \in [1..n_{\text{sh}}]} a_i \sigma_i(p) = \sum_{i \in [1..n_{\text{sh}}]} a_i^2 = 0,$$

and thus for all  $i \in [1..n_{\text{sh}}]$ ,  $a_i = 0$ . Therefore,  $\Sigma$  is free, and it is a basis of  $\mathcal{L}(P, \mathbb{R})$ .  $\square$

*Remark 1429.* In the next definition, the *predual basis* of a basis  $\mathcal{B}' = (b'_i)_{i \in [1..n]}$  of  $E' \stackrel{\text{def.}}{=} \mathcal{L}(E, \mathbb{K})$ , the dual of some vector space  $E$  of dimension  $n$ , is a basis  $\mathcal{B} = (b_i)_{i \in [1..n]}$  of  $E$  such that its dual basis is  $\mathcal{B}'$ . This means that, for all  $i, j \in [1..n]$ , we have  $b'_i(b_j) = \delta_{ij}$ .

**Definition 1430 (*shape function*).**

Let  $(K, P, \Sigma)$  be a finite element. The elements of the predual basis of  $\Sigma$  are called *shape functions*.

## Chapter 7

# Simplicial geometry

*Remark 1431.* This is an important example, see [20, Chap 7, pp. 75–86].

In this section,  $d \in \mathbb{N}^*$  denotes the dimension and  $k \in \mathbb{N}$  the maximal degree of the polynomial approximation. The case  $k = 0$  corresponds to constant functions and is often excluded. In the same way, the case  $d = 0$  (points) is a limit case that is not considered (at least in a first step).

**Definition 1432 (family of points).**

Let  $d \geq 1$ . Let  $\mathcal{B}_d \subset \mathbb{N}^d$  be a finite set. For all  $\beta \in \mathcal{B}_d$ , let  $\mathbf{a}_\beta \in \mathbb{R}^d$ . Then, the whole family is denoted  $\{\{\mathbf{a}\}_{\mathcal{B}_d} \stackrel{\text{def.}}{=} (\mathbf{a}_\beta)_{\beta \in \mathcal{B}_d}$ .

*Remark 1433.* The notation  $\{\{\mathbf{a}\}_{\mathcal{B}_d}$  is used to collect the *nodes* (i.e. points where the nodal degrees of freedom are computed through the linear forms  $\sigma_\beta$ , with  $\beta \in \mathcal{B}_d$ ). They are numbered with multi-indices in  $\mathbb{N}^d$ .

In contrast, the notation  $[[\mathbf{v}]]^d$  defined in Definition 1365 are typically used to collect the *vertices* of the geometry, with the canonical numbering of  $[0..n]$ .

**Definition 1434 (family of reference points).**

Let  $d \geq 1$ . The family of *reference points* (in  $\mathbb{R}^d$ ) is denoted  $[[\hat{\mathbf{v}}]]^d$ , and is defined by  $\hat{\mathbf{v}}_0 \stackrel{\text{def.}}{=} \mathbf{0}$  and for all  $i \in [1..d]$ ,  $\hat{\mathbf{v}}_i \stackrel{\text{def.}}{=} \mathbf{e}_i$ .

**Lemma 1435 (reference isobarycenter).**

Let  $d \geq 1$ .

The reference isobarycenter (in  $\mathbb{R}^d$ ) is denoted by  $\hat{\mathbf{g}}^d$  and is defined as the isobarycenter of  $[[\hat{\mathbf{v}}]]^d$ . Thus, we have  $\hat{\mathbf{g}}^d = \frac{1}{d+1} \sum_{i=0}^d \hat{\mathbf{v}}_i = \frac{1}{d+1} \sum_{i=1}^d \mathbf{e}_i$ , and for all  $i \in [1..d]$ ,  $(\hat{\mathbf{g}}^d)_i = \frac{1}{d+1}$ .

*Proof.* Direct consequence of Definition 1401 (*isobarycenter*), and Definition 1434 (*family of reference points*).  $\square$

**Lemma 1436 (family of reference points is affinely independent).**

Let  $d \geq 1$ . Then, the family of reference points  $[[\hat{\mathbf{v}}]]^d$  is affinely independent.

*Proof.* Direct consequence of Definition 1434 (*family of reference points*) Definition 1365 (*canonical families*), Definition 1411 (*affinely independent family*), **the canonical family is basis**, and **basis is free**.  $\square$

**Definition 1437 (reference simplex).**

Let  $d \geq 1$ . The *reference simplex* is denoted  $\hat{K}^d$ , and is defined as the unit rectangular simplex in  $\mathbb{R}^d$ ,

$$(7.1) \quad \hat{K}^d \stackrel{\text{def.}}{=} \left\{ \hat{\mathbf{x}} \in \mathbb{R}^d \mid \forall i \in [1..d], \hat{x}_i \geq 0 \quad \wedge \quad \sum_{i=1}^d \hat{x}_i \leq 1 \right\}.$$

Its vertices are the reference points  $[[\hat{\mathbf{v}}]]^d$ .

**Lemma 1438 (coordinates in reference simplex are smaller than 1).**

Let  $d \geq 1$ . Let  $\hat{\mathbf{x}} \in \hat{K}^d$ . Then, we have for all  $i \in [1..d]$ ,  $\hat{x}_i \leq 1$ .

*Proof.*

By contradiction, assume that there exists  $i$  such that  $\hat{x}_i > 1$ , then from Definition 1437 (*reference simplex*), and **monotony of addition in  $\mathbb{R}_+$** , we have  $\sum_{j=1}^d \hat{x}_j \geq \hat{x}_i > 1$ , which is impossible.  $\square$

**Lemma 1439 (nontrivial reference simplex).** Let  $d \geq 1$ . Then,  $\hat{K}^d$  has nonempty interior.

*Proof.* Let  $\mathcal{B} \stackrel{\text{def.}}{=} \mathcal{B}_\infty^o((\frac{1}{2d}, \dots, \frac{1}{2d}), \frac{1}{2d})$ . Let  $\hat{\mathbf{x}} \in \mathcal{B}$ . Then, from **the definition of the  $\infty$ -norm, valued field properties of  $\mathbb{R}$** , and Definition 1437 (*reference simplex*), we have  $\forall i \in [1..d]$ ,  $|\hat{x}_i - \frac{1}{2d}| < \frac{1}{2d}$ , i.e.  $0 < \hat{x}_i < \frac{1}{d}$ , thus  $\hat{x}_i > 0$  and  $\sum_{i=0}^d \hat{x}_i < 1$ , and  $\hat{\mathbf{x}} \in \hat{K}^d$ . Therefore, from **open ball is open in metric space, open ball with positive radius is nonempty (with  $\frac{1}{2d} > 0$ )**, and **the definition of the interior**,  $\mathcal{B} \neq \emptyset$ , is open, and included in of  $\hat{K}^d$ , and thus  $\hat{K}^d$  has a nonempty interior.  $\square$

**Definition 1440 (simplex).**

Let  $d \geq 1$ . Let  $[[\mathbf{v}]]^d$  be  $d + 1$  points in  $\mathbb{R}^d$ .

The *simplex of vertices*  $[[\mathbf{v}]]^d$  is denoted  $K^{[[\mathbf{v}]]^d}$ , and is defined as the convex envelop of  $[[\mathbf{v}]]^d$ ,

$$(7.2) \quad K^{[[\mathbf{v}]]^d} \stackrel{\text{def.}}{=} \left\{ \mathbf{x} = \sum_{i=0}^d \mu_i \mathbf{v}_i \in \mathbb{R}^d \mid \forall i \in [0..d], \mu_i \geq 0 \quad \wedge \quad \sum_{i=0}^d \mu_i = 1 \right\}.$$

*Remark 1441.* The proof that a simplex  $K^{[[\mathbf{v}]]^d}$  with affinely independent vertices has nonempty interior is given later, once the geometrical transformation that maps the reference simplex  $\hat{K}^d$  to  $K^{[[\mathbf{v}]]^d}$  is defined, see Lemma 1557.

**Lemma 1442 (coordinates in simplex are smaller than 1).** Let  $d \geq 1$ . Let  $[[\mathbf{v}]]^d$  be  $d + 1$  points in  $\mathbb{R}^d$ . Let  $\mathbf{x} = \sum_{i=0}^d \mu_i \mathbf{v}_i \in K^{[[\mathbf{v}]]^d}$ . Then, we have for all  $i \in [0..d]$ ,  $\mu_i \leq 1$ .

*Proof.* By contradiction, assume that there exists  $i$  such that  $\mu_i > 1$ , then from Definition 1440 (*simplex*), and **monotony of addition in  $\mathbb{R}_+$** , we have  $\sum_{i=0}^d \mu_i \geq \mu_i > 1$ , which is impossible.  $\square$

**Lemma 1443 (simplex of reference vertices is reference simplex).**

Let  $d \geq 1$ . Then, we have  $K^{[[\hat{\mathbf{v}}]]^d} = \hat{K}^d$ .

*Proof.* Let  $\hat{\mathbf{x}} \in \mathbb{R}^d$ . Then, since **the canon family is basis**, we have  $\hat{\mathbf{x}} = \sum_{i=1}^d \hat{x}_i \mathbf{e}_i$ .

Let  $\mu_j \stackrel{\text{def.}}{=} \hat{x}_j$  for all  $j \in [1..d]$ , and  $\mu_0 \stackrel{\text{def.}}{=} 1 - \sum_{j=1}^d \hat{x}_j$ . Then, from Definition 1434 (*family of reference points*), we also have  $\hat{\mathbf{x}} = \sum_{i=0}^d \mu_i \hat{\mathbf{v}}_i$  with  $\sum_{i=0}^d \mu_i = 1$ . Thus, from **ordered field properties of  $\mathbb{R}$** , we obviously have the equivalence

$$\forall i \in [0..d], \mu_i \geq 0 \iff \sum_{j=1}^d \hat{x}_j \leq 1, \quad \text{and} \quad \forall i \in [1..d], \hat{x}_i \geq 0.$$

And finally, from Definition 1437 (*reference simplex*), and Definition 1440 (*simplex*), we have  $\hat{\mathbf{x}} \in K^{[[\hat{\mathbf{v}}]]^d}$  iff  $\hat{\mathbf{x}} \in \hat{K}^d$ , hence the equality.  $\square$

## Chapter 8

# $Lag \mathbb{P}_k^1$ Lagrange finite element on a segment

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*Remark 1444.* In this chapter, we set  $d \stackrel{\text{def.}}{=} 1$ . General results for  $d \geq 1$  are provided in Chapter 9.

### 8.1 Multi-indices in dimension $d = 1$

**Definition 1445** (*multi-indices  $\mathcal{A}_k^1$* ).

Let  $k \in \mathbb{N}$ . The set of multi-indices in dimension 1 is denoted  $\mathcal{A}_k^1$ , and is defined by

$$(8.1) \quad \mathcal{A}_k^1 \stackrel{\text{def.}}{=} \{\alpha \in \mathbb{N} \mid \alpha \leq k\} = [0..k].$$

**Lemma 1446** (*cardinal of  $\mathcal{A}_k^1$* ).

Let  $k \in \mathbb{N}$ . The number of elements of  $\mathcal{A}_k^1$  is

$$(8.2) \quad \text{card}(\mathcal{A}_k^1) = \binom{k+1}{1} = k+1.$$

*Proof.* Direct consequence of Definition 1445 (*multi-indices  $\mathcal{A}_k^1$* ), and Lemma 1364 (*properties of the binomial coefficient, (5.5)*).  $\square$

### 8.2 $\mathbb{P}_k^1$ Lagrange polynomials

*Remark 1447.* In the sequel, for a set of indices  $I$ , and for a family of elements  $(a_i)_{i \in I}$ , the expression “there is no double in  $(a_i)_{i \in I}$ ” means that the indexation is injective, i.e. for all  $i, j \in I$ ,  $i \neq j$  implies  $a_i \neq a_j$ .

**Definition 1448 (Lagrange polynomials of  $\mathbb{P}_k^1$ ).**

Let  $k \in \mathbb{N}$ . Let  $\{\{a\}_{\mathcal{A}_k^1}\}$  be  $k + 1$  points in  $\mathbb{R}$ . Assume that there is no double in  $\{\{a\}_{\mathcal{A}_k^1}\}$ . The Lagrange polynomials associated with  $\{\{a\}_{\mathcal{A}_k^1}\}$  are denoted  $\left(\mathcal{L}_i^{\{\{a\}_{\mathcal{A}_k^1}\}}\right)_{i \in [0..k]}$ , and are defined by

$$(8.3) \quad \forall x \in \mathbb{R}, \quad \mathcal{L}_i^{\{\{a\}_{\mathcal{A}_k^1}\}}(x) \stackrel{\text{def.}}{=} \prod_{j=0, j \neq i}^k \frac{x - a_j}{a_i - a_j},$$

with the convention that a product indexed by the empty set equals 1 (i.e. when  $k \stackrel{\text{def.}}{=} 0$ ).

**Lemma 1449 (Lagrange polynomials is basis of  $\mathbb{P}_k^1$ ).**

Let  $k \in \mathbb{N}$ . Let  $\{\{a\}_{\mathcal{A}_k^1}\}$  be  $k + 1$  points in  $\mathbb{R}$ . Assume that there is no double in  $\{\{a\}_{\mathcal{A}_k^1}\}$ . Then, the Lagrange polynomials associated with  $\{\{a\}_{\mathcal{A}_k^1}\}$  form a basis of  $\mathbb{P}_k^1$ , that satisfies

$$(8.4) \quad \forall i \in [0..k], \quad \deg \mathcal{L}_i^{\{\{a\}_{\mathcal{A}_k^1}\}} = k,$$

$$(8.5) \quad \forall i, j \in [0..k], \quad \mathcal{L}_i^{\{\{a\}_{\mathcal{A}_k^1}\}}(a_j) = \delta_{ij},$$

$$(8.6) \quad \sum_{i=0}^k \mathcal{L}_i^{\{\{a\}_{\mathcal{A}_k^1}\}} = 1.$$

*Proof.* **Degree (8.4), in  $\mathbb{P}_k^1$ , and identity (8.5).** Direct consequence of Definition 1448 (Lagrange polynomials of  $\mathbb{P}_k^1$ ), Lemma 1422 (product of two univariate polynomials), Lemma 1418 ( $\mathbb{P}_k^1$  is space of degree at most  $k$ ), and **field properties of  $\mathbb{R}$** .

**Basis.** Let  $(\lambda_i)_{i \in [0..k]}$  in  $\mathbb{R}$ , such that  $\sum_{i=0}^k \lambda_i \mathcal{L}_i^{\{\{a\}_{\mathcal{A}_k^1}\}} = 0$ . This implies that for all  $x \in \mathbb{R}$ ,  $\sum_{i=0}^k \lambda_i \mathcal{L}_i^{\{\{a\}_{\mathcal{A}_k^1}\}}(x) = 0$ . Then, from (8.5), and **ring properties of  $\mathbb{R}$** , taking  $x \stackrel{\text{def.}}{=} a_j$  provides  $\sum_{i=0}^k \lambda_i \delta_{ij} = \lambda_j = 0$ . Thus, from **the definition of freedom**, Lemma 1420 (dimension of  $\mathbb{P}_k^1$ ), and Lemma 1376 (free family of dim elements is basis), the Lagrange polynomials form a basis of  $\mathbb{P}_k^1$ .

**Identity (8.6).** From Definition 1417 (polynomial space  $\mathbb{P}_k^1$ ,  $\underline{1}$  is in  $\mathbb{P}_k^1$ ), the previous point (basis), and **the definition of basis (basis is generator)**, there exists  $(\alpha_i)_{i \in [0..k]}$  in  $\mathbb{R}$  such that  $1 = \sum_{i=0}^k \alpha_i \mathcal{L}_i^{\{\{a\}_{\mathcal{A}_k^1}\}}$ , and thus for all  $x \in \mathbb{R}$ ,  $1 = \sum_{i=0}^k \alpha_i \mathcal{L}_i^{\{\{a\}_{\mathcal{A}_k^1}\}}(x)$ . Let  $j \in [0..k]$ . Then, from (8.5), taking  $x \stackrel{\text{def.}}{=} a_j$  provides  $1 = \sum_{i=0}^k \alpha_i \delta_{ij} = \alpha_j$ , and we obtain (8.6).  $\square$

**Lemma 1450 (decomposition of  $\mathbb{P}_k^1$  polynomial in Lagrange basis).**

Let  $k \in \mathbb{N}$ .

Let  $\{\{a\}_{\mathcal{A}_k^1}\}$  be  $k + 1$  points in  $\mathbb{R}$ . Assume that there is no double in  $\{\{a\}_{\mathcal{A}_k^1}\}$ . Then, we have

$$(8.7) \quad \forall p \in \mathbb{P}_k^1, \quad p = \sum_{i=0}^k p(a_i) \mathcal{L}_i^{\{\{a\}_{\mathcal{A}_k^1}\}}.$$

*Proof.* Let  $p \in \mathbb{P}_k^1$ . Then, from Lemma 1449 (Lagrange polynomials is basis of  $\mathbb{P}_k^1$ , basis), there exists a unique  $(\alpha_i)_{i \in [0..k]}$  in  $\mathbb{R}$  such that for all  $x \in \mathbb{R}$ ,  $p(x) = \sum_{i=0}^k \alpha_i \mathcal{L}_i^{\{\{a\}_{\mathcal{A}_k^1}\}}(x)$ .

Let  $j \in [0..k]$ . Then, from Lemma 1449 (Lagrange polynomials is basis of  $\mathbb{P}_k^1$ , (8.5)), taking  $x \stackrel{\text{def.}}{=} a_j$  provides  $p(a_j) = \sum_{i=0}^k \alpha_i \delta_{ij} = \alpha_j$ . Thus, the result.  $\square$

*Remark 1451.* In the following, instead of directly defining the finite element on a current cell, we choose to define the finite element on the reference cell first, then obtain the one on any current cell by the affine geometric transformation.

### 8.3 $\overset{\text{Lag}}{\hat{\mathbb{P}}}_k^1$ Lagrange finite element on the reference segment

**Lemma 1452 (reference simplex is nontrivial in  $\mathbb{R}$ ).** *The reference simplex  $\hat{K}^1$  is the segment  $[0, 1]$  of  $\mathbb{R}$ , whose vertices are  $\hat{v}_0 = 0$  and  $\hat{v}_1 = 1$  in  $\mathbb{R}$ . Its interior  $\overset{\circ}{\hat{K}}^1$  is nonempty.*

*Proof.* Direct consequence of Definition 1437 (reference simplex), and Lemma 1439 (nontrivial reference simplex).  $\square$

**Definition 1453 (reference Lagrange nodes of  $\mathbb{P}_k^1$ ).**

Let  $k \in \mathbb{N}$ . The reference Lagrange nodes of  $\mathbb{P}_k^1$  are denoted  $\{\{\hat{a}\}\}_{\mathcal{A}_k^1}$ , and are defined by

$$(8.8) \quad (k = 0) \quad \hat{a}_0 \stackrel{\text{def.}}{=} \hat{g}^1 = \frac{1}{2},$$

$$(8.9) \quad (k \geq 1) \quad \forall i \in [0..k], \quad \hat{a}_i \stackrel{\text{def.}}{=} i \hat{h} \quad \text{where } \hat{h} \stackrel{\text{def.}}{=} \frac{1}{k}.$$

**Lemma 1454 (reference Lagrange nodes are distinct).**

*Let  $k \in \mathbb{N}$ . Let  $i, j \in \mathbb{N}$ . Then, the  $(k+1)$  reference Lagrange nodes  $\{\{\hat{a}\}\}_{\mathcal{A}_k^1}$  are distinct, and  $0 \leq i < j \leq k$  implies  $\hat{a}_i < \hat{a}_j$ .*

*Proof.* Direct consequence of Definition 1453 (reference Lagrange nodes of  $\mathbb{P}_k^1$ ), and **ordered field properties of  $\mathbb{R}$** .  $\square$

**Lemma 1455 (reference Lagrange basis of  $\mathbb{P}_k^1$ ).**

*Let  $k \in \mathbb{N}$ . Then, the Lagrange polynomials associated with the reference Lagrange nodes form a basis of  $\mathbb{P}_k^1$  that satisfies (8.4), (8.5) and (8.6).*

*They are called reference Lagrange polynomials in  $\mathbb{P}_k^1$ , and for all  $i \in [0..k]$ , we define the shorthand notation  $\hat{\mathcal{L}}_i^{k,1} \stackrel{\text{def.}}{=} \mathcal{L}_i^{\{\hat{a}\}_{\mathcal{A}_k^1}}$ .*

*Proof.* Direct consequence of Lemma 1454 (reference Lagrange nodes are distinct), and Lemma 1449 (Lagrange polynomials is basis of  $\mathbb{P}_k^1$ ).  $\square$

**Definition 1456 (reference Lagrange linear forms for  $\mathbb{P}_k^1$ ).**

Let  $k \in \mathbb{N}$ . The reference Lagrange linear forms associated with the reference Lagrange nodes of  $\mathbb{P}_k^1$  are denoted  $\hat{\Sigma}_k^1 = (\hat{\sigma}_i)_{i \in [0..k]}$ , and are defined by

$$(8.10) \quad \forall i \in [0..k], \quad \forall \hat{f} : \mathbb{R} \rightarrow \mathbb{R}, \quad \hat{\sigma}_i(\hat{f}) \stackrel{\text{def.}}{=} \hat{f}(\hat{a}_i).$$

**Lemma 1457 (reference Lagrange linear forms for  $\mathbb{P}_k^1$  are linear).**

*Let  $k \in \mathbb{N}$ . Let  $i \in [0..k]$ . Then, the reference Lagrange linear form  $\hat{\sigma}_i$  is linear.*

*Proof.* Direct consequence of Definition 1456 (reference Lagrange linear forms for  $\mathbb{P}_k^1$ ), and **the definition of linear operations over functions**.  $\square$

**Lemma 1458 (reference Lagrange linear forms for  $\mathbb{P}_k^1$  are injective).**

*Let  $k \in \mathbb{N}$ . Then,  $\phi_{\hat{\Sigma}_k^1}$  is injective.*

*Proof.* Direct consequence of Definition 1424 (finite element triple), Lemma 1454 (reference Lagrange nodes are distinct), Definition 1456 (reference Lagrange linear forms for  $\mathbb{P}_k^1$ ), thus for all  $\hat{p} \in \mathbb{P}_k^1$ ,  $\hat{\sigma}_i(\hat{p})$  is zero implies  $\hat{p}(\hat{a}_i) = 0$ , Lemma 1450 (decomposition of  $\mathbb{P}_k^1$  polynomial in Lagrange basis, thus  $\hat{p} = 0$ ), Definition 101 (kernel), and Lemma 103 (injective linear map has zero kernel).  $\square$

**Lemma 1459 (unisolvence of  $\mathbb{P}_k^1$  (reference)).**

Let  $k \in \mathbb{N}$ . Then,  $(\hat{K}^1, \mathbb{P}_k^1, \hat{\Sigma}_k^1)$  satisfies the unisolvence property.

*Proof.* Direct consequence of Lemma 1420 (dimension of  $\mathbb{P}_k^1$ ), Lemma 1454 (reference Lagrange nodes are distinct), **the definition of order (reflexivity, thus  $\dim \mathbb{P}_k^1 = k + 1$  is greater than or equal to  $k + 1 = \text{card}(\hat{\Sigma}_k^1)$ )**, Lemma 1458 (reference Lagrange linear forms for  $\mathbb{P}_k^1$  are injective), and Lemma 1426 (injectivity implies unisolvence).  $\square$

**Theorem 1460 ( $\overset{\text{Lag}}{\hat{\mathbb{P}}}_k^1$  reference Lagrange finite element).**

Let  $k \in \mathbb{N}$ . Then,  $\overset{\text{Lag}}{\hat{\mathbb{P}}}_k^1 \stackrel{\text{def.}}{=} (\hat{K}^1, \mathbb{P}_k^1, \hat{\Sigma}_k^1)$  is a finite element.

It is called the reference Lagrange finite element of degree  $k$  in dimension 1.

*Proof.* Direct consequence of Lemma 1452 (reference simplex is nontrivial in  $\mathbb{R}$ ), Lemma 1420 (dimension of  $\mathbb{P}_k^1$ ), Lemma 1459 (unisolvence of  $\mathbb{P}_k^1$  (reference)), and Definition 1424 (finite element triple, with  $q \stackrel{\text{def.}}{=} 1$ ).  $\square$

## 8.4 $\overset{\text{Lag}}{\mathbb{P}}_k^1$ Lagrange finite element on a current segment

**Definition 1461 (geometric mapping in dimension 1).** Let  $[[v]]^1$  be two points in  $\mathbb{R}$ .

The geometric mapping associated with  $[[v]]^1$  is denoted  $\varphi_{\text{geo}}^{[[v]]^1}$ , and is defined by

$$(8.11) \quad \forall \hat{x} \in \mathbb{R}, \quad \varphi_{\text{geo}}^{[[v]]^1}(\hat{x}) \stackrel{\text{def.}}{=} (v_1 - v_0)\hat{x} + v_0.$$

**Lemma 1462 (properties of geometric mapping in dimension 1).**

Let  $[[v]]^1$  be two points in  $\mathbb{R}$ . Then,  $\varphi_{\text{geo}}^{[[v]]^1} \in \mathbb{P}_1^1$ , i.e. is affine, and we have

$$(8.12) \quad \forall \hat{x} \in \mathbb{R}, \quad \varphi_{\text{geo}}^{[[v]]^1}(\hat{x}) = (1 - \hat{x})v_0 + \hat{x}v_1 = \hat{\mathcal{L}}_0^{1,1}(\hat{x})v_0 + \hat{\mathcal{L}}_1^{1,1}(\hat{x})v_1,$$

$$(8.13) \quad \forall \hat{x} \in \mathbb{R}, \quad \left(\varphi_{\text{geo}}^{[[v]]^1}\right)'(\hat{x}) = v_1 - v_0.$$

Moreover, if  $v_0 \neq v_1$ , then  $\varphi_{\text{geo}}^{[[v]]^1}$  is bijective,  $\left(\varphi_{\text{geo}}^{[[v]]^1}\right)^{-1}$  is affine, and we have

$$(8.14) \quad \forall x \in \mathbb{R}, \quad \left(\varphi_{\text{geo}}^{[[v]]^1}\right)^{-1}(x) = \frac{x - v_0}{v_1 - v_0}.$$

*Proof.* Direct consequence of Definition 1461 (geometric mapping in dimension 1), Lemma 1399 (equivalent definition of affine map (finite dimension)), Lemma 1418 ( $\mathbb{P}_k^1$  is space of degree at most  $k$ ), Definition 1453 (reference Lagrange nodes of  $\mathbb{P}_k^1$ , with  $k \stackrel{\text{def.}}{=} 1$ , thus  $\{\{\hat{a}\}_{\mathcal{A}_k^1} \stackrel{\text{def.}}{=} [[\hat{v}]]^1\}$ ), Lemma 1455 (reference Lagrange basis of  $\mathbb{P}_k^1$ , with  $k \stackrel{\text{def.}}{=} 1$ ), **the rules of derivation in  $\mathbb{R}$** , and Lemma 1410 (inverse of affine map is affine map, with  $\phi^{-1}(y) \stackrel{\text{def.}}{=} \frac{y - v_0}{v_1 - v_0}$ ).  $\square$

**Lemma 1463 (current simplex is nontrivial in  $\mathbb{R}$ ).** Let  $[[v]]^1$  be two points in  $\mathbb{R}$ .

Then, the current simplex  $K^{[[v]]^1}$  is the segment  $[\min\{v_0, v_1\}, \max\{v_0, v_1\}]$  of  $\mathbb{R}$ .

Moreover, we have

$$(8.15) \quad v_0 \neq v_1 \iff K^{\overset{\circ}{[[v]]^1}} \neq \emptyset.$$

*Proof.*  $K^{\llbracket v \rrbracket^1}$ . Direct consequence of Definition 1440 (*simplex*), and **the definition of a segment**.

**Equivalence.** Let  $w_0 \stackrel{\text{def.}}{=} \min\{v_0, v_1\}$  and  $w_1 \stackrel{\text{def.}}{=} \max\{v_0, v_1\}$ . Then, from **ordered field properties of  $\mathbb{R}$** , and **the interior of a segment**, we have

$$v_0 \neq v_1 \iff w_0 < w_1 \iff (w_0, w_1) \neq \emptyset \iff K^{\circ\llbracket v \rrbracket^1} \neq \emptyset.$$

□

**Lemma 1464 (current simplex is image of reference in  $\mathbb{R}$ ).**

Let  $\llbracket v \rrbracket^1$  be two points in  $\mathbb{R}$ . Then, we have  $K^{\llbracket v \rrbracket^1} = \varphi_{\text{geo}}^{\llbracket v \rrbracket^1}(\hat{K}^1)$ .

*Proof.* Direct consequence of Definition 1461 (*geometric mapping in dimension 1*), Lemma 1452 (*reference simplex is nontrivial in  $\mathbb{R}$* ), and Lemma 1463 (*current simplex is nontrivial in  $\mathbb{R}$* ). □

**Definition 1465 (Lagrange nodes of  $\mathbb{P}_k^1$ ).** Let  $k \in \mathbb{N}$ . Let  $\llbracket v \rrbracket^1$  be two points in  $\mathbb{R}$ . The Lagrange nodes of  $\mathbb{P}_k^1$  are denoted  $\{\{a\}_{\mathcal{A}_k^1}\}$ , and are defined by

$$(8.16) \quad (k = 0) \quad a_0 \stackrel{\text{def.}}{=} g^{\llbracket v \rrbracket^1} = \frac{v_0 + v_1}{2},$$

$$(8.17) \quad (k \geq 1) \quad \forall i \in [0..k], \quad a_i \stackrel{\text{def.}}{=} v_0 + i h \quad \text{where } h \stackrel{\text{def.}}{=} \frac{v_1 - v_0}{k}.$$

*Remark 1466.* The isobarycenter  $g^{\llbracket v \rrbracket^1}$  is defined in Definition 1401.

**Lemma 1467 (Lagrange nodes are distinct).**

Let  $k \in \mathbb{N}$ . Let  $i, j \in \mathbb{N}$ . Let  $\llbracket v \rrbracket^1$  be two distinct points in  $\mathbb{R}$ .

Then, the  $(k + 1)$  Lagrange nodes  $\{\{a\}_{\mathcal{A}_k^1}\}$  are distinct, and  $0 \leq i < j \leq k$  implies  $a_i \neq a_j$ .

Moreover, if  $v_0 < v_1$ , then  $h > 0$ , and  $0 \leq i < j \leq k$  implies  $a_i < a_j$ , and if  $v_0 > v_1$ , then  $h < 0$ , and  $0 \leq i < j \leq k$  implies  $a_i > a_j$ .

*Proof.* Direct consequence of Definition 1465 (*Lagrange nodes of  $\mathbb{P}_k^1$* ), and **ordered field properties of  $\mathbb{R}$** . □

**Lemma 1468 (Lagrange nodes of  $\mathbb{P}_k^1$  are images of reference).**

Let  $k \in \mathbb{N}$ . Let  $\llbracket v \rrbracket^1$  be two points in  $\mathbb{R}$ . Then, we have  $\forall i \in [0..k], a_i = \varphi_{\text{geo}}^{\llbracket v \rrbracket^1}(\hat{a}_i)$ .

*Proof.* Direct consequence of Definition 1461 (*geometric mapping in dimension 1*), Definition 1453 (*reference Lagrange nodes of  $\mathbb{P}_k^1$* ), and Definition 1465 (*Lagrange nodes of  $\mathbb{P}_k^1$* ). □

**Lemma 1469 (Lagrange basis of  $\mathbb{P}_k^1$ ).** Let  $k \in \mathbb{N}$ . Let  $\llbracket v \rrbracket^1$  be two distinct points in  $\mathbb{R}$ . Then, the Lagrange polynomials associated with the Lagrange nodes form a basis of  $\mathbb{P}_k^1$  that satisfies (8.4), (8.5) and (8.6). They are called current Lagrange polynomials in  $\mathbb{P}_k^1$ .

*Proof.* Direct consequence of Lemma 1467 (*Lagrange nodes are distinct*), and Lemma 1449 (*Lagrange polynomials is basis of  $\mathbb{P}_k^1$* ). □

**Lemma 1470 (Lagrange polynomials of  $\mathbb{P}_k^1$  are images of reference).**

Let  $k \in \mathbb{N}$ . Let  $\llbracket v \rrbracket^1$  be two distinct points in  $\mathbb{R}$ . Then, we have for all  $i \in [0..k]$ ,

$$(8.18) \quad \mathcal{L}_i^{\{a\}_{\mathcal{A}_k^1}} = \hat{\mathcal{L}}_i^{k,1} \circ \left( \varphi_{\text{geo}}^{\llbracket v \rrbracket^1} \right)^{-1} \quad \text{and} \quad \hat{\mathcal{L}}_i^{k,1} = \mathcal{L}_i^{\{a\}_{\mathcal{A}_k^1}} \circ \varphi_{\text{geo}}^{\llbracket v \rrbracket^1}.$$



*Proof.* Direct consequence of Definition 1448 (*Lagrange polynomials of  $\mathbb{P}_k^1$* ), Lemma 1455 (*reference Lagrange basis of  $\mathbb{P}_k^1$* ), Lemma 1469 (*Lagrange basis of  $\mathbb{P}_k^1$* ), Lemma 1468 (*Lagrange nodes of  $\mathbb{P}_k^1$  are images of reference*), Definition 1461 (*geometric mapping in dimension 1*), and **field properties of  $\mathbb{R}$  (providing the following identities)**

$$\frac{x - a_j}{a_i - a_j} = \frac{(v_1 - v_0)\hat{x} + v_0 - (v_1 - v_0)\hat{a}_j - v_0}{(v_1 - v_0)\hat{a}_i + v_0 - (v_1 - v_0)\hat{a}_j - v_0} = \frac{\hat{x} - \hat{a}_j}{\hat{a}_i - \hat{a}_j},$$

$$\forall f : \mathbb{R} \rightarrow \mathbb{R}, \quad 1 = 1 \circ f).$$

□

**Lemma 1471 (*geometric mapping of  $\mathbb{P}_k^1$  is  $\mathbb{P}_k^1$* ).** Let  $k \in \mathbb{N}$ . Let  $\llbracket v \rrbracket^1$  be two points in  $\mathbb{R}$ . Then, for all  $p \in \mathbb{P}_k^1$ ,  $p \circ \varphi_{\text{geo}}^{\llbracket v \rrbracket^1} \in \mathbb{P}_k^1$ . Moreover, if  $v_0 \neq v_1$ , then,  $p \circ \left(\varphi_{\text{geo}}^{\llbracket v \rrbracket^1}\right)^{-1} \in \mathbb{P}_k^1$ .

*Proof.* Let  $p \in \mathbb{P}_k^1$ .

**Case  $v_0 = v_1$ .** Direct consequence of Definition 1461 (*geometric mapping in dimension 1, thus,  $\varphi_{\text{geo}}^{\llbracket v \rrbracket^1} = v_0$  is constant*), and Definition 1417 (*polynomial space  $\mathbb{P}_k^1$* , thus, we have  $p \circ \varphi_{\text{geo}}^{\llbracket v \rrbracket^1} = p(v_0)$  lives in  $\mathbb{P}_0^1 \subset \mathbb{P}_k^1$ ).

**Case  $v_0 \neq v_1$ .** Then, from Lemma 1469 (*Lagrange basis of  $\mathbb{P}_k^1$ , basis*), Lemma 1470 (*Lagrange polynomials of  $\mathbb{P}_k^1$  are images of reference*), and Lemma 1455 (*reference Lagrange basis of  $\mathbb{P}_k^1$ , basis*), there exists  $(\alpha_i)_{i \in [0..k]}$  in  $\mathbb{R}$  such that  $p = \sum_{i=0}^k \alpha_i \mathcal{L}_i^{\{\mathbf{a}\}_{\mathcal{A}_k^1}}$ , and we have

$$p \circ \varphi_{\text{geo}}^{\llbracket v \rrbracket^1} = \sum_{i=0}^k \alpha_i \mathcal{L}_i^{\{\mathbf{a}\}_{\mathcal{A}_k^1}} \circ \varphi_{\text{geo}}^{\llbracket v \rrbracket^1} = \sum_{i=0}^k \alpha_i \hat{\mathcal{L}}_i^{k,1} \in \mathbb{P}_k^1.$$

Similarly, from Lemma 1455 (*reference Lagrange basis of  $\mathbb{P}_k^1$ , basis*), Lemma 1470 (*Lagrange polynomials of  $\mathbb{P}_k^1$  are images of reference*), and Lemma 1469 (*Lagrange basis of  $\mathbb{P}_k^1$ , basis*), there exists  $(\alpha_i)_{i \in [0..k]}$  in  $\mathbb{R}$  such that  $p = \sum_{i=0}^k \alpha_i \hat{\mathcal{L}}_i^{k,1}$ , and we have

$$p \circ \left(\varphi_{\text{geo}}^{\llbracket v \rrbracket^1}\right)^{-1} = \sum_{i=0}^k \alpha_i \hat{\mathcal{L}}_i^{k,1} \circ \left(\varphi_{\text{geo}}^{\llbracket v \rrbracket^1}\right)^{-1} = \sum_{i=0}^k \alpha_i \mathcal{L}_i^{\{\mathbf{a}\}_{\mathcal{A}_k^1}} \in \mathbb{P}_k^1.$$

□

**Definition 1472 (*Lagrange linear forms for  $\mathbb{P}_k^1$* ).**

Let  $k \in \mathbb{N}$ . Let  $\llbracket v \rrbracket^1$  be two points in  $\mathbb{R}$ . The *Lagrange linear forms* associated with the Lagrange nodes of  $\mathbb{P}_k^1$  are denoted  $\Sigma^{\{\mathbf{a}\}_{\mathcal{A}_k^1}} = (\sigma_i)_{i \in [0..k]}$ , and are defined by

$$(8.19) \quad \forall i \in [0..k], \forall f : \mathbb{R} \rightarrow \mathbb{R}, \quad \sigma_i(f) \stackrel{\text{def.}}{=} f(a_i).$$

**Lemma 1473 ( $\mathbb{P}_k^1$  Lagrange linear forms are linear).**

Let  $k \in \mathbb{N}$ .

Let  $\llbracket v \rrbracket^1$  be two points in  $\mathbb{R}$ . Let  $i \in [0..k]$ . Then, the Lagrange linear form  $\sigma_i$  is linear.

*Proof.* Direct consequence of Definition 1472 (*Lagrange linear forms for  $\mathbb{P}_k^1$* ), and **the definition of linear operations over functions.** □

**Lemma 1474 ( $\mathbb{P}_k^1$  Lagrange linear forms are images of reference).** Let  $k \in \mathbb{N}$ . Let  $\llbracket v \rrbracket^1$  be two points in  $\mathbb{R}$ . Then, we have  $\forall i \in [0..k], \forall f : \mathbb{R} \rightarrow \mathbb{R}, \sigma_i(f) = \hat{\sigma}_i(\hat{f})$ , where  $\hat{f} = f \circ \varphi_{\text{geo}}^{\llbracket v \rrbracket^1}$ .

*Proof.* Direct consequence of Definition 1472 (*Lagrange linear forms for  $\mathbb{P}_k^1$* ), Lemma 1468 (*Lagrange nodes of  $\mathbb{P}_k^1$  are images of reference*), and Definition 1456 (*reference Lagrange linear forms for  $\mathbb{P}_k^1$* ). □

*Remark 1475.* Note that Lemmas 1464, 1471, and 1474 imply that the current  $\mathbb{P}_k^1$  FE  $(K^{\llbracket v \rrbracket^1}, \mathbb{P}_k^1, \Sigma^{\{\mathbf{a}\}_{\mathcal{A}_k^1}})$  can be considered as the image of the reference  $\mathbb{P}_k^1$  FE  $(\hat{K}^1, \mathbb{P}_k^1, \hat{\Sigma}_k^1)$  by  $\varphi_{\text{geo}}^{\llbracket v \rrbracket^1}$ .

**Lemma 1476 (Lagrange linear forms for  $\mathbb{P}_k^1$  are injective).**

Let  $k \in \mathbb{N}$ . Let  $\llbracket v \rrbracket^1$  be two distinct points in  $\mathbb{R}$ . Then,  $\phi_{\Sigma^{\{\mathbf{a}\}_{\mathcal{A}_k^1}}}$  is injective.

*Proof.* Direct consequence of Definition 1424 (finite element triple), Lemma 1467 (Lagrange nodes are distinct), Definition 1472 (Lagrange linear forms for  $\mathbb{P}_k^1$ , thus for all  $p \in \mathbb{P}_k^1$ ,  $\sigma_i(p) = 0$  implies  $p(a_i) = 0$ ), Lemma 1450 (decomposition of  $\mathbb{P}_k^1$  polynomial in Lagrange basis, thus  $p = 0$ ), Definition 101 (kernel), and Lemma 103 (injective linear map has zero kernel).  $\square$

**Lemma 1477 (unisolvence of  $\mathbb{P}_k^1$ ).**

Let  $k \in \mathbb{N}$ . Let  $\llbracket v \rrbracket^1$  be two distinct points in  $\mathbb{R}$ . Then,  $(K^{\llbracket v \rrbracket^1}, \mathbb{P}_k^1, \Sigma^{\{\mathbf{a}\}_{\mathcal{A}_k^1}})$  satisfies the unisolvence property.

*Proof.* Direct consequence of Lemma 1420 (dimension of  $\mathbb{P}_k^1$ ), Lemma 1467 (Lagrange nodes are distinct), the definition of order (reflexivity, thus  $\dim \mathbb{P}_k^1 = k + 1$  is greater than or equal to  $k + 1 = \text{card}(\Sigma^{\{\mathbf{a}\}_{\mathcal{A}_k^1}})$ ), Lemma 1476 (Lagrange linear forms for  $\mathbb{P}_k^1$  are injective), and Lemma 1426 (injectivity implies unisolvence).  $\square$

**Theorem 1478 ( ${}^{\text{Lag}}\mathbb{P}_k^1$  Lagrange finite element).** Let  $k \in \mathbb{N}$ . Let  $\llbracket v \rrbracket^1$  be two distinct points in  $\mathbb{R}$ . Then,  ${}^{\text{Lag}}\mathbb{P}_k^1 \stackrel{\text{def.}}{=} (K^{\llbracket v \rrbracket^1}, \mathbb{P}_k^1, \Sigma^{\{\mathbf{a}\}_{\mathcal{A}_k^1}})$  is a finite element.

It is called the Lagrange finite element of degree  $k$  in dimension 1 associated with vertices  $\llbracket v \rrbracket^1$ .

*Proof.* Direct consequence of Lemma 1463 (current simplex is nontrivial in  $\mathbb{R}$ ), Lemma 1420 (dimension of  $\mathbb{P}_k^1$ ), Lemma 1477 (unisolvence of  $\mathbb{P}_k^1$ ), and Definition 1424 (finite element triple, with  $q \stackrel{\text{def.}}{=} 1$ ).  $\square$



## Chapter 9

# $Lag \mathbb{P}_k^d$ Lagrange finite element in dimension $d \geq 1$

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*Remark 1479.* In this chapter, we consider a simplex in  $\mathbb{R}^d$  with  $d \geq 1$ . The case  $d = 1$  corresponds to a segment, and is treated in detail in Chapter 8.

In Figure 9.1, we present the Lagrange nodes on simplices for several dimensions and degrees.

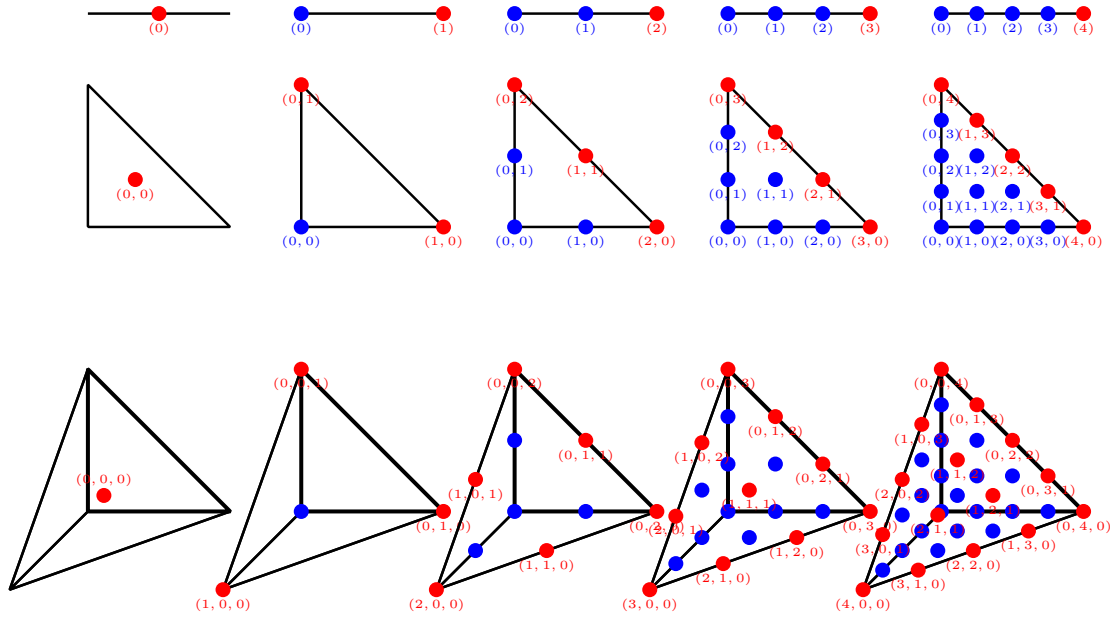


Figure 9.1: Lagrange nodes of the reference simplex for  $d = 1, 2, 3$  (from top to bottom), and  $k = 0, 1, 2, 3, 4$  (from left to right). The nodes corresponding to the highest degree are depicted in red, and the others in blue. The multi-indices are indicated in all cases when  $d = 1, 2$ , and only for the highest degree when  $d = 3$ .

## 9.1 Multi-indices

### 9.1.1 Some useful notations for multi-indices

**Definition 1480 (length of multi-indices).** Let  $d \geq 1$ . Let  $\alpha \in \mathbb{N}^d$  be a multi-index. The length of  $\alpha$  is denoted  $|\alpha|$ , and is defined by  $|\alpha| \stackrel{\text{def.}}{=} \sum_{i=1}^d \alpha_i \in \mathbb{N}$ .

**Lemma 1481 (length of multi-indices is additive).** Let  $d \geq 1$ . Then, the length of multi-indices is additive, for all  $\alpha, \beta \in \mathbb{N}^d$ ,  $|\alpha + \beta| = |\alpha| + |\beta|$ .

*Proof.* Direct consequence of Definition 1480 (length of multi-indices), and **abelian monoid properties of  $\mathbb{N}$** . □

**Definition 1482 (factorial of multi-indices).** Let  $d \geq 1$ . Let  $\alpha \in \mathbb{N}^d$  be a multi-index. The factorial of  $\alpha$  is denoted  $\alpha!$ , and is defined by  $\alpha! \stackrel{\text{def.}}{=} \prod_{i=1}^d (\alpha_i!) \in \mathbb{N}^*$ .

**Lemma 1483 (factorial of multi-index is positive).** Let  $d \geq 1$ . Let  $\alpha \in \mathbb{N}^d$ . Then,  $\alpha! > 0$ .

*Proof.* Direct consequence of Definition 1482 (factorial of multi-indices), **the positivity of factorial**, and **closedness of multiplication in  $\mathbb{N}^*$** . □

**Definition 1484 (Kronecker delta of multi-indices).** Let  $d \geq 1$ . Let  $\alpha, \beta \in \mathbb{N}^d$ . The Kronecker delta of  $\alpha$  and  $\beta$  is denoted  $\delta_{\alpha, \beta}$ , and is defined by  $\delta_{\alpha, \beta} \stackrel{\text{def.}}{=} \prod_{i=1}^d \delta_{\alpha_i \beta_i}$ .

**Lemma 1485 (value of Kronecker delta of multi-indices).** Let  $d \geq 1$ . Let  $\alpha, \beta \in \mathbb{N}^d$ . Then, we have  $\delta_{\alpha, \beta}$  equals 1 when  $\alpha = \beta$ , and 0 otherwise.

*Proof.* Direct consequence of Definition 1484 (Kronecker delta of multi-indices), the definition of the (scalar) Kronecker delta, and the zero-product property in  $\mathbb{N}$ .  $\square$

### 9.1.2 Sets $\mathcal{A}_k^d$ and $\mathcal{C}_k^d$ of multi-indices

**Definition 1486 (sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$ ).** Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . The set of multi-indices of length at most  $k$  (resp. of length  $k$ ) is denoted  $\mathcal{A}_k^d$  (resp.  $\mathcal{C}_k^d$ ), and is defined by

$$(9.1) \quad \mathcal{A}_k^d \stackrel{\text{def.}}{=} \{\alpha \in \mathbb{N}^d \mid |\alpha| \leq k\},$$

$$(9.2) \quad \mathcal{C}_k^d \stackrel{\text{def.}}{=} \{\alpha \in \mathbb{N}^d \mid |\alpha| = k\}.$$

Let  $i \in [1..d]$ . The subset of  $\mathcal{A}_k^d$  of multi-indices with zero  $i$ -th component is defined by

$$(9.3) \quad \mathcal{A}_{k,i}^d \stackrel{\text{def.}}{=} \{\alpha \in \mathcal{A}_k^d \mid \alpha_i = 0\}.$$

**Lemma 1487 (multi-indices  $\mathcal{A}_k^d$  for  $d = 1$  is  $\mathcal{A}_k^1$ ).**

Let  $k \in \mathbb{N}$ . Then,  $\mathcal{A}_k^d$  for  $d \stackrel{\text{def.}}{=} 1$  and  $\mathcal{A}_k^1$  from Definition 1445 coincide.

*Proof.* Direct consequence of Definition 1486 (sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$ , (9.1) with  $d \stackrel{\text{def.}}{=} 1$ ), Definition 1480 (length of multi-indices, with  $d \stackrel{\text{def.}}{=} 1$ , thus  $|\alpha| = \alpha_1$ ), and Definition 1445 (multi-indices  $\mathcal{A}_k^1$ ).  $\square$

**Lemma 1488 (indices are smaller than maximal length).**

Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . Let  $\alpha \in \mathcal{A}_k^d \cup \mathcal{C}_k^d$ . Let  $i \in [1..d]$ . Then, we have  $\alpha_i \leq k$ .

*Proof.*

Direct consequence of Definition 1480 (length of multi-indices, thus  $\alpha_i \leq |\alpha|$ ), Definition 1486 (sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$ , thus  $|\alpha| \leq k$ ), and the definition of order (transitivity).  $\square$

**Lemma 1489 (first  $\mathcal{C}_k^d$ ).**

Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . Then, we have

$$(9.4) \quad (d = 1) \quad \mathcal{C}_k^1 = \{k\}, \quad \text{and} \quad \text{card}(\mathcal{C}_k^1) = 1,$$

$$(9.5) \quad (d = 2) \quad \mathcal{C}_k^2 = \{(k-i, i) \in \mathbb{N}^2 \mid i \in [0..k]\}, \quad \text{and} \quad \text{card}(\mathcal{C}_k^2) = k + 1,$$

$$(9.6) \quad (k = 0) \quad \mathcal{C}_0^d = \{\mathbf{0} \in \mathbb{N}^d\}, \quad \text{and} \quad \text{card}(\mathcal{C}_0^d) = 1,$$

$$(9.7) \quad (k = 1) \quad \mathcal{C}_1^d = \{\mathbf{e}_i \in \mathbb{N}^d \mid i \in [1..d]\}, \quad \text{and} \quad \text{card}(\mathcal{C}_1^d) = d.$$

*Proof.* Direct consequence of Definition 1365 (canonic families), Definition 1486 (sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$ ), and Lemma 1488 (indices are smaller than maximal length).  $\square$

### 9.1.3 Slices of $\mathcal{C}_k^d$ and its cardinal

*Remark 1490.* In the sequel, when  $d \geq 2$ , the check notation  $\check{\alpha}$  denotes the last  $d-1$  components of the multi-index  $\alpha \in \mathbb{N}^d$ , and the tilde notation  $\tilde{\alpha}$  denotes the first ones.

Thus, for all  $\alpha \stackrel{\text{def.}}{=} (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ , we have  $\alpha = (\alpha_1, \check{\alpha}) = (\tilde{\alpha}, \alpha_d)$ .

The notation naturally extends to vectors in  $\mathbb{R}^d$ , and other objects such as subsets and functions may also inherit the check or tilde decoration, e.g. see Definition 1491 and Lemma 1494.

**Definition 1491 (slices  $\check{S}_{k,i}^d$  and  $\tilde{S}_{k,i}^d$ ).** Let  $d \geq 2$ . Let  $k \in \mathbb{N}$ . Let  $i \in [0..k]$ . The  $i$ -th “vertical” (resp. “horizontal”) slice of  $\mathcal{C}_k^d$  is denoted  $\check{S}_{k,i}^d$  (resp.  $\tilde{S}_{k,i}^d$ ), and is defined by

$$(9.8) \quad \check{S}_{k,i}^d \stackrel{\text{def.}}{=} \{(i, \check{\alpha}) \in \mathbb{N}^d \mid \check{\alpha} \in \mathcal{C}_{k-i}^{d-1}\}, \quad \text{and} \quad \tilde{S}_{k,i}^d \stackrel{\text{def.}}{=} \{(\tilde{\alpha}, i) \in \mathbb{N}^d \mid \tilde{\alpha} \in \mathcal{C}_{k-i}^{d-1}\}.$$

*Remark 1492.* Geometrically, the “vertical” slice  $\check{S}_{k,i}^d$  is the intersection of  $\mathcal{C}_k^d$  (subset in  $\mathbb{N}^d$  of the hyperplane of equation  $\sum_{i=1}^d \alpha_i = k$ ) with the (vertical) first canonical hyperplane of equation  $\alpha_1 = i$ , see Figure 9.2. The order discussed in Section 3.3.8 is consistent with the decomposition of  $\mathcal{C}_k^d$  in slices as  $\mathcal{C}_k^d = \bigsqcup_{i=k}^0 \check{S}_{k,i}^d$  (decreasing from  $k$  to 0), see Lemma 1493.

In the same way, the “horizontal” slice  $\tilde{S}_{k,i}^d$  is the intersection of  $\mathcal{C}_k^d$  with the (horizontal) last canonical hyperplane of equation  $\alpha_d = i$ .

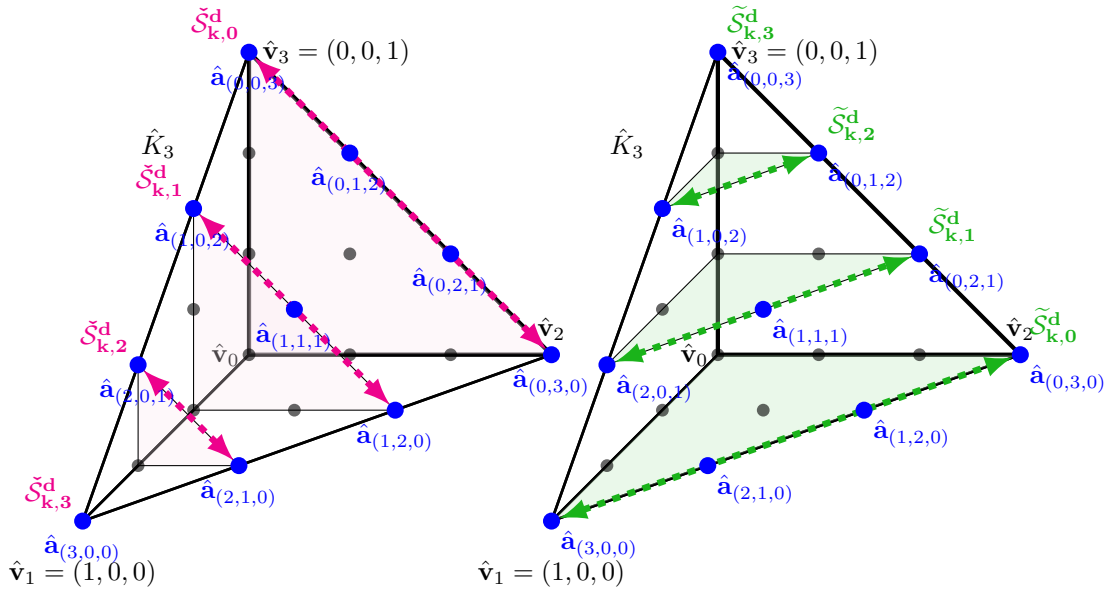


Figure 9.2: Vertical slices  $\check{S}_{k,i}^d$  (left) and horizontal slices  $\tilde{S}_{k,i}^d$  (right), in the case  $d = k = 3$  (see Definition 1491). The reference Lagrange node  $\hat{a}_{(\alpha_1, \alpha_2, \alpha_3)}$  in blue corresponds to the element  $(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{C}_k^d$ . For instance, the set  $\check{S}_{k,1}^d = \{(1, 2, 0), (1, 1, 1), (1, 0, 2)\}$  is depicted by the nodes linked by a dashed arrow.

**Lemma 1493 (slices of  $\mathcal{C}_k^d$ ).**

Let  $d \geq 2$ . Let  $k \in \mathbb{N}$ . Let  $i \in [0..k]$ . Then,  $\tilde{S}_{k,i}^d$  and  $\check{S}_{k,i}^d$  are subsets of  $\mathcal{C}_k^d$ , and we have

$$(9.9) \quad \mathcal{C}_k^d = \bigsqcup_{i=0}^k \check{S}_{k,i}^d = \bigsqcup_{i=0}^k \tilde{S}_{k,i}^d.$$

*Proof. Case of  $\check{S}_{k,i}^d$ .* Let  $\alpha \in \check{S}_{k,i}^d$ . Then, from Definition 1491 (slices  $\check{S}_{k,i}^d$  and  $\tilde{S}_{k,i}^d$ ), and Definition 1486 (sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$ ), there exists  $\check{\alpha} \in \mathcal{C}_{k-i}^{d-1}$  such that  $\alpha = (i, \check{\alpha})$ , and we have  $|\alpha| = i + |\check{\alpha}| = i + k - i = k$ , i.e.  $\alpha \in \mathcal{C}_k^d$ , and  $\check{S}_{k,i}^d \subset \mathcal{C}_k^d$ . Thus, we have  $\bigcup_{i=0}^k \check{S}_{k,i}^d \subset \mathcal{C}_k^d$ .

Let  $i, j \in [0..k]$ . Let  $\alpha \in \check{S}_{k,i}^d \cap \check{S}_{k,j}^d$ . Then, from Definition 1491 (slices  $\check{S}_{k,i}^d$  and  $\tilde{S}_{k,i}^d$ ), there exist  $\check{\alpha}^i \in \mathcal{C}_{k-i}^{d-1}$  and  $\check{\alpha}^j \in \mathcal{C}_{k-j}^{d-1}$  such that  $\alpha = (i, \check{\alpha}^i) = (j, \check{\alpha}^j)$ . Thus, we have  $i = j$  and  $\check{\alpha}^i = \check{\alpha}^j$ , and by contrapositive, we have  $\check{S}_{k,i}^d \cap \check{S}_{k,j}^d = \emptyset$  when  $i \neq j$ .

Let  $\alpha \in \mathcal{C}_k^d$ . Let  $\tilde{\alpha} \stackrel{\text{def.}}{=} (\alpha_2, \dots, \alpha_d) \in \mathbb{N}^{d-1}$ . Then, from Lemma 1488 (*indices are smaller than maximal length*), and Definition 1486 (*sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$* ), we have  $0 \leq \alpha_1 \leq k$ ,  $\tilde{\alpha} \in \mathcal{C}_{k-\alpha_1}^{d-1}$ , and  $\alpha = (\alpha_1, \tilde{\alpha}) \in \mathcal{S}_{k, \alpha_1}^d$ . Thus, we have  $\mathcal{C}_k^d \subset \bigsqcup_{i=0}^k \mathcal{S}_{k,i}^d$ .

Therefore, we have  $\mathcal{C}_k^d = \bigsqcup_{i=0}^k \mathcal{S}_{k,i}^d$ .

**Case of  $\tilde{\mathcal{S}}_{k,i}^d$ .** The proof is very similar. Let  $\alpha \in \tilde{\mathcal{S}}_{k,i}^d$ . Then, from Definition 1491 (*slices  $\tilde{\mathcal{S}}_{k,i}^d$  and  $\tilde{\mathcal{S}}_{k,i}^d$* ), and Definition 1486 (*sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$* ), there exists  $\tilde{\alpha} \in \mathcal{C}_{k-i}^{d-1}$  such that  $\alpha = (\tilde{\alpha}, i)$ , and we have  $|\alpha| = |\tilde{\alpha}| + i = k - i + i = k$ , i.e.  $\alpha \in \mathcal{C}_k^d$ , and  $\tilde{\mathcal{S}}_{k,i}^d \subset \mathcal{C}_k^d$ . Thus, we have  $\bigcup_{i=0}^k \tilde{\mathcal{S}}_{k,i}^d \subset \mathcal{C}_k^d$ .

Let  $i, j \in [0..k]$ . Let  $\alpha \in \tilde{\mathcal{S}}_{k,i}^d \cap \tilde{\mathcal{S}}_{k,j}^d$ . Then, from Definition 1491 (*slices  $\tilde{\mathcal{S}}_{k,i}^d$  and  $\tilde{\mathcal{S}}_{k,i}^d$* ), there exist  $\tilde{\alpha}^i \in \mathcal{C}_{k-i}^{d-1}$  and  $\tilde{\alpha}^j \in \mathcal{C}_{k-j}^{d-1}$  such that  $\alpha = (\tilde{\alpha}^i, i) = (\tilde{\alpha}^j, j)$ . Thus, we have  $\tilde{\alpha}^i = \tilde{\alpha}^j$  and  $i = j$ , and by contrapositive, we have  $\tilde{\mathcal{S}}_{k,i}^d \cap \tilde{\mathcal{S}}_{k,j}^d = \emptyset$  when  $i \neq j$ .

Let  $\alpha \in \mathcal{C}_k^d$ . Let  $\tilde{\alpha} \stackrel{\text{def.}}{=} (\alpha_1, \dots, \alpha_{d-1}) \in \mathbb{N}^{d-1}$ . Then, from Lemma 1488 (*indices are smaller than maximal length*), and Definition 1486 (*sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$* ), we have  $0 \leq \alpha_d \leq k$ ,  $\tilde{\alpha} \in \mathcal{C}_{k-\alpha_d}^{d-1}$ , and  $\alpha = (\tilde{\alpha}, \alpha_d) \in \tilde{\mathcal{S}}_{k, \alpha_d}^d$ . Thus, we have  $\mathcal{C}_k^d \subset \bigsqcup_{i=0}^k \tilde{\mathcal{S}}_{k,i}^d$ .

Therefore, we have  $\mathcal{C}_k^d = \bigsqcup_{i=0}^k \tilde{\mathcal{S}}_{k,i}^d$ .  $\square$

**Lemma 1494 (cardinal of slices of  $\mathcal{C}_k^d$ ).** *Let  $d \geq 2$ . Let  $k \in \mathbb{N}$ . Let  $i \in [0..k]$ . Let  $\tilde{\varphi}_{k,d}^i \stackrel{\text{def.}}{=} (\alpha = (i, \tilde{\alpha}) \in \tilde{\mathcal{S}}_{k,i}^d \mapsto \tilde{\alpha} \in \mathcal{C}_{k-i}^{d-1})$ , and  $\tilde{\varphi}_{k,d}^i \stackrel{\text{def.}}{=} (\alpha = (\tilde{\alpha}, i) \in \tilde{\mathcal{S}}_{k,i}^d \mapsto \tilde{\alpha} \in \mathcal{C}_{k-i}^{d-1})$ . Then,  $\tilde{\varphi}_{k,d}^i$  and  $\tilde{\varphi}_{k,d}^i$  are bijections, and we have  $\text{card}(\tilde{\mathcal{S}}_{k,i}^d) = \text{card}(\tilde{\mathcal{S}}_{k,i}^d) = \text{card}(\mathcal{C}_{k-i}^{d-1})$ .*

*Proof.* From Definition 1491 (*slices  $\tilde{\mathcal{S}}_{k,i}^d$  and  $\tilde{\mathcal{S}}_{k,i}^d$* ), the applications  $\tilde{\varphi}_{k,d}^i$  and  $\tilde{\varphi}_{k,d}^i$  are well defined, and obviously surjective. Moreover, let  $\alpha, \beta \in \tilde{\mathcal{S}}_{k,i}^d$ , such that  $\tilde{\alpha} = \tilde{\varphi}_{k,d}^i(\alpha) = \tilde{\varphi}_{k,d}^i(\beta) = \tilde{\beta}$ . Thus, we have  $(i, \tilde{\alpha}) = (i, \tilde{\beta})$ , and  $\tilde{\varphi}_{k,d}^i$  is injective. In the same manner, let  $\alpha, \beta \in \tilde{\mathcal{S}}_{k,i}^d$ , such that  $\tilde{\alpha} = \tilde{\varphi}_{k,d}^i(\alpha)$  equals  $\tilde{\varphi}_{k,d}^i(\beta) = \tilde{\beta}$ . Thus, we have  $(\tilde{\alpha}, i) = (\tilde{\beta}, i)$ , and  $\tilde{\varphi}_{k,d}^i$  is injective.

Therefore,  $\tilde{\varphi}_{k,d}^i$  and  $\tilde{\varphi}_{k,d}^i$  are bijective, and we have the result.  $\square$

**Lemma 1495 (cardinal of  $\mathcal{C}_k^d$ ).** *Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . Then, the number of elements of  $\mathcal{C}_k^d$  is*

$$(9.10) \quad \text{card}(\mathcal{C}_k^d) = \binom{k+d-1}{d-1} = \binom{k+d-1}{k} = \frac{(k+d-1)!}{k!(d-1)!}.$$

*Proof.* From Lemma 1364 (*properties of the binomial coefficient, (5.6)*), and Definition 1362 (*binomial coefficient*), we have the last two equalities.

**First equality.** For all  $d \in \mathbb{N}^*$ , let  $P(d) \stackrel{\text{def.}}{=} [\text{card}(\mathcal{C}_k^d) = \binom{k+d-1}{d-1}]$ .

**Induction:  $P(1)$ .** Direct consequence of Lemma 1489 (*first  $\mathcal{C}_k^d$ , (9.4)*), and Lemma 1364 (*properties of the binomial coefficient, (5.4)*).

**Induction:  $P(d) \Rightarrow P(d+1)$ .**

Assume that  $P(d)$  holds. Then, from Lemma 1493 (*slices of  $\mathcal{C}_k^d$* ), Lemma 1494 (*cardinal of slices of  $\mathcal{C}_k^d$* ), and Lemma 1364 (*properties of the binomial coefficient, (5.8)*), we have

$$\begin{aligned} \text{card}(\mathcal{C}_k^{d+1}) &= \sum_{i=0}^k \text{card}(\tilde{\mathcal{S}}_{k,i}^{d+1}) = \sum_{i=0}^k \text{card}(\mathcal{C}_{k-i}^d) \\ &= \sum_{i=0}^k \binom{k-i+d-1}{d-1} = \sum_{j=0}^k \binom{j+d-1}{d-1} = \binom{k+d}{d}. \end{aligned}$$

This concludes the induction on  $d$ , and we have for all  $d \geq 1$ ,  $P(d)$ .  $\square$



### 9.1.4 Layers of $\mathcal{A}_k^d$ and its cardinal

**Lemma 1496** ( *$\mathcal{C}_k^d$  are layers of  $\mathcal{A}_k^d$* ). Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . Then, we have

$$(9.11) \quad \mathcal{A}_0^d = \mathcal{C}_0^d \quad \text{and} \quad \mathcal{A}_{k+1}^d = \mathcal{A}_k^d \uplus \mathcal{C}_{k+1}^d,$$

thus,  $\mathcal{A}_k^d = \bigsqcup_{l=0}^k \mathcal{C}_l^d$  and the sequence  $(\mathcal{A}_k^d)_{k \in \mathbb{N}}$  is increasing.

*Proof.* Direct consequence of Definition 1486 (*sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$* ), and **the definition of (disjoint) union**.  $\square$

**Lemma 1497** (*first multi-indices  $\mathcal{A}_k^d$* ). Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . Then, we have

$$(9.12) \quad (d = 1) \quad \mathcal{A}_k^1 = [0..k], \quad \text{and} \quad \text{card}(\mathcal{A}_k^1) = k + 1,$$

$$(9.13) \quad (k = 0) \quad \mathcal{A}_0^d = \{\mathbf{0} \in \mathbb{N}^d\}, \quad \text{and} \quad \text{card}(\mathcal{A}_0^d) = 1,$$

$$(9.14) \quad (k = 1) \quad \mathcal{A}_1^d = \{\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_d\}, \quad \text{and} \quad \text{card}(\mathcal{A}_1^d) = d + 1.$$

*Proof.*

Direct consequence of Definition 1496 ( *$\mathcal{C}_k^d$  are layers of  $\mathcal{A}_k^d$* ), and Definition 1489 (*first  $\mathcal{C}_k^d$* ).  $\square$

**Lemma 1498** (*cardinal of  $\mathcal{A}_k^d$* ). Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . Then, the number of elements of  $\mathcal{A}_k^d$  is

$$(9.15) \quad \text{card}(\mathcal{A}_k^d) = \binom{k+d}{d} = \binom{k+d}{k} = \frac{(k+d)!}{k! d!}.$$

By extension, for  $d = 0$ , we also set  $\text{card}(\mathcal{A}_k^0) \stackrel{\text{def.}}{=} \binom{k}{0} = \binom{k}{k} = 1$ .

*Proof.* Direct consequence of Lemma 1496 ( *$\mathcal{C}_k^d$  are layers of  $\mathcal{A}_k^d$* ), thus,  $\text{card}(\mathcal{A}_k^d) = \sum_{l=0}^k \text{card}(\mathcal{C}_l^d)$ , Lemma 1495 (*cardinal of  $\mathcal{C}_k^d$* ), Lemma 1364 (*properties of the binomial coefficient, (5.8)*), and Lemma 1364 (*properties of the binomial coefficient, (5.6), (5.4)*), thus last two equalities hold, and with  $d = 0$  too).  $\square$

### 9.1.5 Other cardinals

*Remark 1499.* In Figure 9.3, we plot examples of the mappings  $f_{k,0}^d$  and  $f_{k,1}^d$ , defined over multi-index sets in Lemmas 1500 and 1501.

These functions are used for the transfer of nodes from a  $(d-1)$ -simplex to the hyperface of a  $d$ -simplex, see Lemmas 1605 and 1607. The transfer is made via the geometric applications  $\phi_{\theta_i}^{\llbracket \mathbf{v} \rrbracket^d}$  (for  $i \in [0..d]$ ), that map an hyperface to a simplex, see Lemma 1584, and Figure 9.6.

**Lemma 1500** (*cardinal of  $\mathcal{C}_k^d$  and  $\mathcal{A}_k^{d-1}$* ). Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ .

Let  $f_{k,0}^d \stackrel{\text{def.}}{=} (\tilde{\alpha} \in \mathcal{A}_k^{d-1} \mapsto (k - |\tilde{\alpha}|, \tilde{\alpha}) \in \mathcal{C}_k^d)$ , and  $\tilde{f}_{k,0}^d \stackrel{\text{def.}}{=} (\tilde{\alpha} \in \mathcal{A}_k^{d-1} \mapsto (\tilde{\alpha}, k - |\tilde{\alpha}|) \in \mathcal{C}_k^d)$ .

Then,  $f_{k,0}^d$  and  $\tilde{f}_{k,0}^d$  are bijections, and we have  $\text{card}(\mathcal{C}_k^d) = \text{card}(\mathcal{A}_k^{d-1})$ .

*Proof. Identity on the cardinals.*

Direct consequence of Lemma 1495 (*cardinal of  $\mathcal{C}_k^d$* ), and Lemma 1498 (*cardinal of  $\mathcal{A}_k^d$* ).

**Well-defined applications.** Let  $\tilde{\alpha}, \tilde{\alpha} \in \mathcal{A}_k^{d-1}$ . Then, from Definition 1486 (*sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$* ), and Definition 1480 (*length of multi-indices*), we have  $k - |\tilde{\alpha}|, k - |\tilde{\alpha}| \geq 0$ , and  $|f_{k,0}^d(\tilde{\alpha})| = (k - |\tilde{\alpha}|) + |\tilde{\alpha}| = k = |\tilde{\alpha}| + (k - |\tilde{\alpha}|) = |\tilde{f}_{k,0}^d(\tilde{\alpha})|$ . Thus,  $f_{k,0}^d$  and  $\tilde{f}_{k,0}^d$  are well defined.

**Bijections.** Direct consequence of the definition of  $f_{k,0}^d$  and  $\tilde{f}_{k,0}^d$  (obviously injective), and **the fact that injectivity and cardinal equality imply bijectivity**.  $\square$

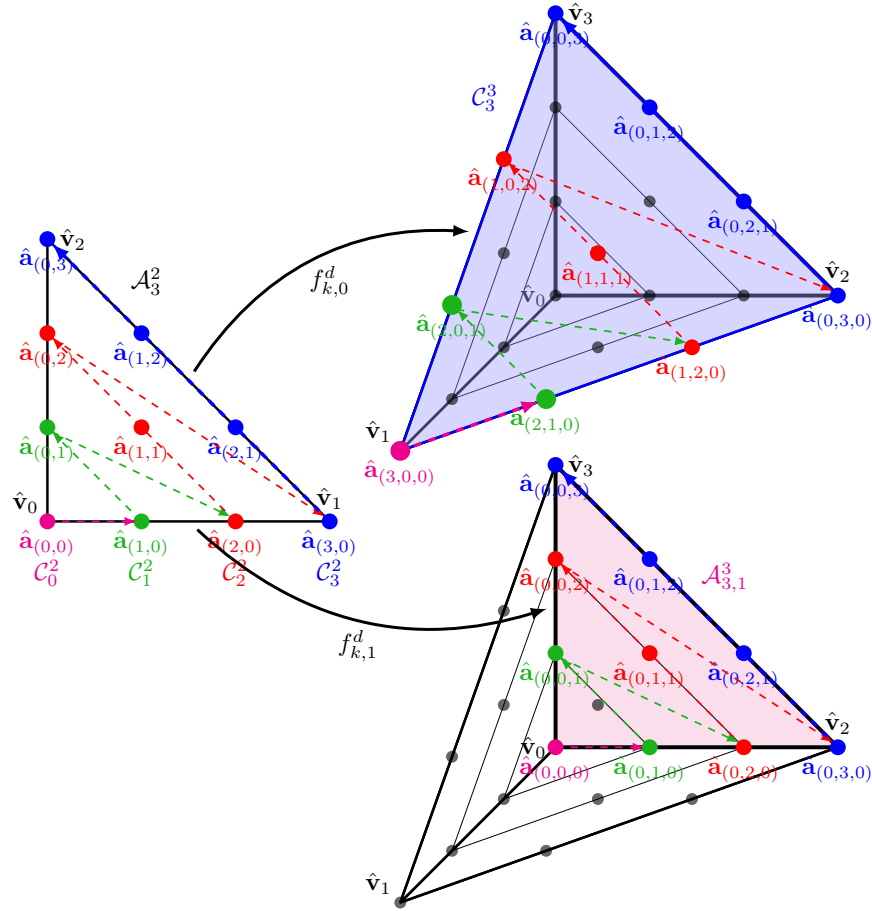


Figure 9.3: Mappings  $f_{k,0}^d$  and  $f_{k,1}^d$  in the case  $d = k = 3$  (see Lemmas 1500 and 1501).

The multi-indices in  $\mathcal{A}_3^2$  are mapped to multi-indices of  $\mathcal{C}_3^3$  (for  $f_{k,0}^d$ ) or to multi-indices of  $\mathcal{A}_{3,1}^3$  (for  $f_{k,1}^d$ ). This is illustrated geometrically with the representation of the nodes in the triangle or the tetrahedra.

For  $f_{k,0}^d$ , the reference triangle nodes are mapped onto the nodes of the blue face of the tetrahedron. This face, opposite vertex  $\hat{v}_0$ , contains the nodes having indices in  $\mathcal{C}_k^d$ . The coloring of the nodes is intended to help see the mapping: for all  $(i, j) \in \mathcal{A}_2^3$ , we have  $f_{k,0}^d(i, j) = (3 - (i + j), i, j)$ .

For  $f_{k,1}^d$ , the reference triangle nodes are mapped onto the nodes of the magenta face of the tetrahedron. This face, opposite vertex  $\hat{v}_1$ , contains the nodes having indices in  $\mathcal{A}_{k,1}^d$ . For all  $(i, j) \in \mathcal{A}_2^3$ , we have  $f_{k,1}^d(i, j) = (0, i, j)$ .

**Lemma 1501 (cardinal of  $\mathcal{A}_{k,i}^d$  and  $\mathcal{A}_k^{d-1}$ ).** Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . Let  $i \in [1..d]$ .  
 Let  $f_{k,i}^d \stackrel{\text{def.}}{=} (\alpha' \in \mathcal{A}_k^{d-1} \mapsto (\alpha'_1, \dots, \alpha'_{i-1}, 0, \alpha'_i, \dots, \alpha'_{d-1}) \in \mathcal{A}_{k,i}^d)$ .  
 Then,  $f_{k,i}^d$  is a bijection that preserves length, and we have  $\text{card}(\mathcal{A}_{k,i}^d) = \text{card}(\mathcal{A}_k^{d-1})$ .

*Proof.* (1) **Well-defined application and preservation of length.**

Let  $\alpha' \in \mathcal{A}_k^{d-1}$ . Then, from Definition 1480 (*length of multi-indices*), and Definition 1486 (*sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$* ), we have  $|f_{k,i}^d(\alpha')| = |\alpha'| \leq k$ , and  $f_{k,i}^d(\alpha') \in \mathcal{A}_{k,i}^d$ .

(2) **Injectivity.** Direct consequence of the definition of  $f_{k,i}^d$ .

(3) **Surjectivity.**

Let  $\alpha \in \mathcal{A}_{k,i}^d$ . Let  $\alpha' \stackrel{\text{def.}}{=} (\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_d) \in \mathbb{N}^{d-1}$ . Then, from (1), and Definition 1486 (*sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$* ), we have  $f_{k,i}^d(\alpha') = \alpha$  and  $|\alpha'| = |\alpha| \leq k$ . Thus,  $\alpha' \in \mathcal{A}_k^{d-1}$ .

Therefore, from (2), (3), **the definition of bijectivity**, and **the definition of cardinal**,  $f_{k,i}^d$  is bijective, and we have the equality of cardinals.  $\square$

## 9.2 Multivariate polynomials

*Remark 1502.* See also univariate polynomials in Section 5.6.

In the following statements,  $\mathbb{P}_k^d$  is defined as the space of polynomials of total degree at most  $k$ . Note that the notion of degree is fully defined *afterwards*.

### 9.2.1 Monomials and polynomials of $\mathbb{P}_k^d$

**Definition 1503 (monomial in  $d$  variables).** Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . Let  $\alpha \in \mathcal{C}_k^d$ . The monomial of degree  $k$  in  $d$  variables of multi-exponent  $\alpha$  is denoted  $\mathbf{X}^\alpha$ , and is defined by

$$(9.16) \quad \mathbf{X}^\alpha \stackrel{\text{def.}}{=} \left( \mathbf{x} \in \mathbb{R}^d \mapsto \prod_{i=1}^d x_i^{\alpha_i} \in \mathbb{R} \right).$$

Moreover, 1 is a shortcut for  $\mathbf{X}^{\mathbf{0}}$  (the constant function of value 1, which may be omitted in a multiplicative context), and for all  $i \in [1..d]$ ,  $X_i$  is a shortcut for  $\mathbf{X}^{e_i} = (\mathbf{x} \mapsto x_i)$ .

**Lemma 1504 (monomial in  $d$  variables for  $d = 1$  is monomial of a single variable).**

Let  $k \in \mathbb{N}$ . Then,  $\mathbf{X}^\alpha$  for  $\alpha \in \mathcal{C}_k^d$  and  $d \stackrel{\text{def.}}{=} 1$ , and  $X^k$  from Definition 1416 coincide.

*Proof.*

Direct consequence of Definition 1503 (monomial in  $d$  variables, (9.16) with  $d \stackrel{\text{def.}}{=} 1$ ), Lemma 1489 (first  $\mathcal{C}_k^d$ , with  $d \stackrel{\text{def.}}{=} 1$ ), and Definition 1416 (monomial of a single variable, with  $\alpha \stackrel{\text{def.}}{=} k$ ).  $\square$

**Definition 1505 (polynomial space  $\mathbb{P}_k^d$ ).** Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . The space of polynomials of degree at most  $k$  of  $d$  variables is denoted  $\mathbb{P}_k^d$ , and is defined by

$$(9.17) \quad \mathbb{P}_k^d \stackrel{\text{def.}}{=} \text{span}(\mathbf{X}^\alpha)_{\alpha \in \mathcal{A}_k^d} = \left\{ \left( \mathbf{x} \mapsto \sum_{\alpha \in \mathcal{A}_k^d} a_\alpha x_1^{\alpha_1} \dots x_d^{\alpha_d} \right) : \mathbb{R}^d \rightarrow \mathbb{R} \mid (a_\alpha)_{\alpha \in \mathcal{A}_k^d} \in \mathbb{R} \right\}.$$

**Lemma 1506 (polynomial space  $\mathbb{P}_k^d$  for  $d = 1$  is  $\mathbb{P}_k^1$ ).**

Let  $k \in \mathbb{N}$ . Then,  $\mathbb{P}_k^d$  for  $d \stackrel{\text{def.}}{=} 1$ , and  $\mathbb{P}_k^1$  from Definition 1417 coincide.

*Proof.* Direct consequence of Definition 1505 (polynomial space  $\mathbb{P}_k^d$ , (9.17) with  $d \stackrel{\text{def.}}{=} 1$ ), Lemma 1497 (first multi-indices  $\mathcal{A}_k^d$ , with  $d \stackrel{\text{def.}}{=} 1$ ), and Definition 1417 (polynomial space  $\mathbb{P}_k^1$ ).  $\square$

**Lemma 1507 ( $\mathbb{P}_k^d$  is vector space).** Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . Then,  $\mathbb{P}_k^d$  is a vector space.

*Proof.* Direct consequence of Definition 1505 (polynomial space  $\mathbb{P}_k^d$ ), and **the definition of the linear span**.  $\square$

**Lemma 1508 ( $\mathbb{P}_k^d$  is nondecreasing sequence in  $k$ ).**

Let  $d \geq 1$ . Then, the sequence  $(\mathbb{P}_k^d)_{k \in \mathbb{N}}$  is nondecreasing for the inclusion.

*Proof.* Direct consequence of Definition 1505 (polynomial space  $\mathbb{P}_k^d$ ), Lemma 1496 ( $\mathcal{C}_k^d$  are layers of  $\mathcal{A}_k^d$ , increasing sequence), and **monotonicity of the linear span**.  $\square$

**Lemma 1509 (constant and affine spaces  $\mathbb{P}_0^d$  and  $\mathbb{P}_1^d$ ).**

Let  $d \geq 1$ . The spaces of polynomials of  $d$  variables of degree at most 0 and 1 are respectively the spaces of constant and affine maps,

$$(9.18) \quad \mathbb{P}_0^d = \text{span}(1) = \{(\mathbf{x} \mapsto a_0) \mid a_0 \in \mathbb{R}\},$$

$$(9.19) \quad \begin{aligned} \mathbb{P}_1^d &= \text{span}(1, X_1, X_2, \dots, X_d) \\ &= \{(\mathbf{x} \mapsto a_0 + a_{\mathbf{e}_1}x_1 + \dots + a_{\mathbf{e}_d}x_d) \mid a_0, a_{\mathbf{e}_1}, \dots, a_{\mathbf{e}_d} \in \mathbb{R}\} \\ &= \mathcal{A}ff(\mathbb{R}^d, \mathbb{R}). \end{aligned}$$

Thus,  $p \in \mathbb{P}_1^d$  iff there exists  $a_0 \in \mathbb{R}$  and  $A \in \mathcal{M}_{1,d}(\mathbb{R})$  such that, for all  $\mathbf{x} \in \mathbb{R}^d$ ,  $p(\mathbf{x}) = a_0 + A\mathbf{x}$ .

*Proof.* Direct consequence of Definition 1505 (polynomial space  $\mathbb{P}_k^d$ ), Definition 1365 (canonic families), Lemma 1497 (first multi-indices  $\mathcal{A}_k^d$ ), Lemma 1399 (equivalent definition of affine map (finite dimension)).  $\square$

**Definition 1510 (degree of polynomial).**

Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . Let  $(a_\alpha)_{\alpha \in \mathcal{A}_k^d} \in \mathbb{R}$ .

Let  $p \stackrel{\text{def.}}{=} \sum_{\alpha \in \mathcal{A}_k^d} a_\alpha \mathbf{X}^\alpha \in \mathbb{P}_k^d$ . The degree of  $p$  is denoted  $\deg p$ , and is defined by

$$(9.20) \quad \deg p \stackrel{\text{def.}}{=} \max(|\alpha|)_{a_\alpha \neq 0},$$

with the convention that the maximum of an empty family is  $-\infty$ , i.e. when  $p \stackrel{\text{def.}}{=} 0$ .

**Lemma 1511 (values of degree of polynomial).**

Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . Let  $p \in \mathbb{P}_k^d$ . Then, we have  $\deg p \in \{-\infty\} \cup [0..k]$ .

*Proof.* Direct consequence of Definition 1510 (degree of polynomial).  $\square$

**Lemma 1512 (monomials of  $\mathcal{C}_k^d$  have degree  $k$ ).**

Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . Let  $\alpha \in \mathcal{C}_k^d$ . Then, we have  $\deg \mathbf{X}^\alpha = |\alpha| = k$ .

*Proof.* Direct consequence of Definition 1510 (degree of polynomial), Definition 1503 (monomial in  $d$  variables), Definition 1486 (sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$ ).  $\square$

**Lemma 1513 ( $\mathbb{P}_k^d$  is space of degree at most  $k$ ).**

Let  $d \geq 1$ .

Let  $\mathcal{P}(\mathbb{R}^d)$  be the infinite-dimensional space of polynomials of  $d$  variables on  $\mathbb{R}$ . Let  $k \in \mathbb{N}$ .

Then,  $\mathbb{P}_k^d$  is the space of polynomials of degree at most  $k$ ,

$$(9.21) \quad \mathbb{P}_k^d = \{p \in \mathcal{P}(\mathbb{R}^d) \mid \deg p \leq k\}.$$

*Proof.* Direct consequence of Definition 1505 (polynomial space  $\mathbb{P}_k^d$ ), Definition 1510 (degree of polynomial), and Lemma 1496 ( $\mathcal{C}_k^d$  are layers of  $\mathcal{A}_k^d$ ).  $\square$

## 9.2.2 Product of polynomials

**Lemma 1514 (product of monomials).**

Let  $d \geq 1$ . Let  $k, l \in \mathbb{N}$ . Let  $\alpha \in \mathcal{C}_k^d$  and  $\beta \in \mathcal{C}_l^d$ . Then, we have

$$(9.22) \quad \mathbf{X}^\alpha \mathbf{X}^\beta = \mathbf{X}^{\alpha+\beta}, \quad \text{and} \quad \deg(\mathbf{X}^\alpha \mathbf{X}^\beta) = \deg \mathbf{X}^\alpha + \deg \mathbf{X}^\beta = |\alpha| + |\beta| = k + l.$$

*Proof.* Direct consequence of Definition 1503 (monomial in  $d$  variables), **commutative ring properties of  $\mathbb{R}$**  (thus  $\prod_{i=1}^d x_i^{\alpha_i} \times \prod_{i=1}^d x_i^{\beta_i} = \prod_{i=1}^d x_i^{\alpha_i + \beta_i}$ ), Lemma 1512 (monomials of  $\mathcal{C}_k^d$  have degree  $k$ ), and Lemma 1481 (length of multi-indices is additive).  $\square$

**Lemma 1515 (product of monomial and polynomial).**

Let  $d \geq 1$ . Let  $k, l \in \mathbb{N}$ . Let  $\alpha \in \mathcal{A}_k^d$ . Let  $q \in \mathbb{P}_l^d$ . Then, we have  $\mathbf{X}^\alpha q \in \mathbb{P}_{k+l}^d$ .

*Proof.* From Definition 1505 (polynomial space  $\mathbb{P}_k^d$ ), commutative ring properties of  $\mathbb{R}$ , Lemma 1514 (product of monomials), and Definition 1486 (sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$ , thus  $|\alpha| \leq k$  and  $|\beta| \leq l$ ), there exists  $(b_\beta)_{\beta \in \mathcal{A}_l^d} \in \mathbb{R}$  such that

$$\mathbf{X}^\alpha q = \mathbf{X}^\alpha \left( \sum_{\beta \in \mathcal{A}_l^d} b_\beta \mathbf{X}^\beta \right) = \sum_{\beta \in \mathcal{A}_l^d} b_\beta \mathbf{X}^{\alpha+\beta}, \quad \text{and} \quad |\alpha + \beta| \leq k + l.$$

Thus, from Definition 1505 (polynomial space  $\mathbb{P}_k^d$ ), we have  $\mathbf{X}^\alpha q \in \mathbb{P}_{k+l}^d$ .  $\square$

**Lemma 1516 (product of two polynomials).**

Let  $d \geq 1$ . Let  $k, l \in \mathbb{N}$ . Let  $p \in \mathbb{P}_k^d$  and  $q \in \mathbb{P}_l^d$ . Then, we have  $pq \in \mathbb{P}_{k+l}^d$ .

*Proof.* From Definition 1505 (polynomial space  $\mathbb{P}_k^d$ ), and ring properties of  $\mathbb{R}$ , there exists coefficients  $(a_\alpha)_{\alpha \in \mathcal{A}_k^d} \in \mathbb{R}$  such that  $pq = \sum_{\alpha \in \mathcal{A}_k^d} a_\alpha (\mathbf{X}^\alpha q)$ . Thus, from Lemma 1515 (product of monomial and polynomial), and Lemma 1507 ( $\mathbb{P}_k^d$  is vector space), we have  $pq \in \mathbb{P}_{k+l}^d$ .  $\square$

**9.2.3 Linear independence of monomials****Lemma 1517 (partial derivative of monomials).**

Let  $d \geq 1$ . Let  $k, l \in \mathbb{N}$ .

Let  $\alpha \in \mathcal{A}_k^d$  and  $\beta \in \mathcal{A}_l^d$ . Then, the partial derivative of order  $\beta$  of the monomial  $\mathbf{X}^\alpha$  is

$$(9.23) \quad \partial^\beta \mathbf{X}^\alpha = \begin{cases} \prod_{i=1}^d \binom{\beta_i-1}{\alpha_i-j} X_i^{\alpha_i-\beta_i} & \text{when } \forall i \in [1..d], \beta_i \leq \alpha_i, \\ 0 & \text{otherwise.} \end{cases}$$

Thus,  $\partial^\beta \mathbf{X}^\alpha \in \mathbb{P}_k^d$ .

Moreover, if for all  $i \in [1..d]$ ,  $\beta_i \leq \alpha_i$ , then we have  $\deg(\partial^\beta \mathbf{X}^\alpha) = |\alpha| - |\beta| (= k - l)$ .

*Proof.* Direct consequence of Definition 1503 (monomial in  $d$  variables), Lemma 1512 (monomials of  $\mathcal{C}_k^d$  have degree  $k$ ), and the differentiation rules for univariate polynomials,

$$\forall i \in [1..d], \quad \frac{\partial^{\beta_i}(x_i^{\alpha_i})}{\partial x_i^{\beta_i}} = \begin{cases} x_i^{\alpha_i} & \text{when } \beta_i = 0, \\ \binom{\beta_i-1}{\alpha_i-j} X_i^{\alpha_i-\beta_i} & \text{when } 0 < \beta_i \leq \alpha_i, \\ 0 & \text{when } \alpha_i < \beta_i. \end{cases}$$

$\square$

**Remark 1518.**

Note that in the first clause of (9.23), we use the convention that a product indexed by the empty set equals 1, the identity element for multiplication, i.e. when some  $\beta_i$  is zero.

Note also that the second clause may be omitted, provided the use of the convention that subtraction is closed in  $\mathbb{N}$  (i.e.  $n - p = 0$  when  $n < p$ ). Indeed, when there exists some  $i \in [1..d]$  such that  $\alpha_i < \beta_i$ , then  $x_i^{\alpha_i-\beta_i} = 1$  and  $\prod_{j=0}^{\beta_i-1} (\alpha_i - j) = 0$ .

**Lemma 1519 (partial derivative is linear).** *Let  $d \geq 1$ . Let  $k, l \in \mathbb{N}$ . Let  $\beta \in \mathcal{A}_l^d$ . Then,  $\partial^\beta \in \mathcal{L}(\mathbb{P}_k^d, \mathbb{P}_k^d)$ , i.e.  $\partial^\beta$  is linear from  $\mathbb{P}_k^d$  to  $\mathbb{P}_k^d$ , and for all  $(a_\alpha)_{\alpha \in \mathcal{A}_k^d} \in \mathbb{R}$ , we have*

$$(9.24) \quad \partial^\beta \left( \sum_{\alpha \in \mathcal{A}_k^d} a_\alpha \mathbf{X}^\alpha \right) = \sum_{\alpha \in \mathcal{A}_k^d} a_\alpha \partial^\beta \mathbf{X}^\alpha = \sum_{\substack{\alpha \in \mathcal{A}_k^d \\ \forall i \in [1..d], \beta_i \leq \alpha_i}} a_\alpha \prod_{i=1}^d \binom{\beta_i - 1}{\alpha_i - j} X_i^{\alpha_i - \beta_i}.$$

*Proof.* Direct consequence of **the linearity of partial derivative**, Definition 1505 (*polynomial space  $\mathbb{P}_k^d$* ), and Lemma 1517 (*partial derivative of monomials*).  $\square$

**Lemma 1520 (partial derivative of 0).**

*Let  $d \geq 1$ . Let  $l \in \mathbb{N}$ . Let  $\beta \in \mathcal{A}_l^d$ . Then, we have  $\partial^\beta 0 = 0$ .*

*Proof.* Direct consequence of Lemma 1519 (*partial derivative is linear*).  $\square$

**Lemma 1521 (derivating more than degree is 0).**

*Let  $d \geq 1$ . Let  $k, l \in \mathbb{N}$ . Let  $\alpha \in \mathcal{C}_k^d$  and  $\beta \in \mathcal{C}_l^d$ . Assume that  $k < l$ . Then, we have  $\partial^\beta \mathbf{X}^\alpha = 0$ .*

*Proof.* **Case  $\forall i \in [1..d], \beta_i \leq \alpha_i$ .** Then, from Definition 1486 (*sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$* ), and **the monotonicity of addition in  $\mathbb{N}$** , we have  $l = |\beta| \leq |\alpha| = k$ , which is impossible.

**Case  $\exists i \in [1..d], \alpha_i < \beta_i$ .** Direct consequence of Lemma 1517 (*partial derivative of monomials*).  $\square$

**Lemma 1522 (partial derivative of monomials at 0).**

*Let  $d \geq 1$ . Let  $k, l \in \mathbb{N}$ . Let  $\alpha \in \mathcal{A}_k^d$  and  $\beta \in \mathcal{A}_l^d$ . Then, we have  $\partial^\beta \mathbf{X}^\alpha(\mathbf{0}) = \alpha! \delta_{\alpha, \beta}$ .*

*Proof.* From Definition 1482 (*factorial of multi-indices*), and Definition 1484 (*Kronecker delta of multi-indices*), we have  $\alpha! \delta_{\alpha, \beta} = \prod_{i=1}^d \alpha_i!$  when  $\alpha = \beta$ , and 0 otherwise.

**Case  $\alpha = \beta$ .** Direct consequence of Lemma 1517 (*partial derivative of monomials*, since for all  $i \in [1..d], x_i^{\alpha_i - \beta_i} = 1$  and  $\prod_{j=0}^{\beta_i - 1} (\alpha_i - j) = \alpha_i!$ ).

**Case  $\exists i \in [1..d], \alpha_i \neq \beta_i$ .**

**Case  $\alpha_i < \beta_i$ .** Direct consequence of Lemma 1517 (*partial derivative of monomials*,  $\partial^\beta \mathbf{X}^\alpha = 0$ ).

**Case  $\beta_i < \alpha_i$ .** Direct consequence of Lemma 1517 (*partial derivative of monomials*, since  $x_i^{\alpha_i - \beta_i} \stackrel{\text{def.}}{=} 0$ ).  $\square$

**Lemma 1523 (monomials are free in  $\mathbb{P}_k^d$ ).**

*Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . Then,  $(\mathbf{X}^\alpha)_{\alpha \in \mathcal{A}_k^d}$  is free in  $\mathbb{P}_k^d$ .*

*Proof.* Let  $(\lambda_\alpha)_{\alpha \in \mathcal{A}_k^d} \in \mathbb{R}$  such that  $\sum_{\alpha \in \mathcal{A}_k^d} \lambda_\alpha \mathbf{X}^\alpha = 0$ . Let  $\beta \in \mathcal{A}_k^d$ .

Then, from Lemma 1519 (*partial derivative is linear*), Lemma 1520 (*partial derivative of 0*), and Lemma 1522 (*partial derivative of monomials at 0, with  $\mathbf{x} \stackrel{\text{def.}}{=} \mathbf{0}$* ), we have

$$0 = \sum_{\alpha \in \mathcal{A}_k^d} \lambda_\alpha \partial^\beta \mathbf{X}^\alpha(\mathbf{0}) = \sum_{\alpha \in \mathcal{A}_k^d} \lambda_\alpha \beta! \delta_{\alpha, \beta} = \lambda_\beta \beta!.$$

Thus, from Lemma 1483 (*factorial of multi-index is positive*), **the zero-product property in  $\mathbb{R}$** , Definition 1505 (*polynomial space  $\mathbb{P}_k^d$* , thus  $\mathbf{X}^\alpha \in \mathbb{P}_k^d$ ), and **the definition of freedom**, we have  $\lambda_\beta = 0$ , and  $(\mathbf{X}^\alpha)_{\alpha \in \mathcal{A}_k^d}$  is free in  $\mathbb{P}_k^d$ .  $\square$

**Lemma 1524 (monomials are a basis of  $\mathbb{P}_k^d$ ).**

*Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . Then,  $(\mathbf{X}^\alpha)_{\alpha \in \mathcal{A}_k^d}$  is a basis of  $\mathbb{P}_k^d$ .*

*Proof.* Direct consequence of Definition 1505 (*polynomial space  $\mathbb{P}_k^d$* , thus monomials are generators), Lemma 1523 (*monomials are free in  $\mathbb{P}_k^d$* ), and **the definition of basis**.  $\square$

**Lemma 1525** (*dimension of  $\mathbb{P}_k^d$* ).

Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . Then,  $\mathbb{P}_k^d$  is a vector space of dimension

$$(9.25) \quad \dim \mathbb{P}_k^d = \binom{k+d}{d} = \binom{k+d}{k} = \frac{(k+d)!}{k! d!} \quad (= \text{card}(\mathcal{A}_k^d)).$$

For instance, we have

$$(9.26) \quad \begin{cases} \dim \mathbb{P}_0^d = 1, \\ \dim \mathbb{P}_1^d = d + 1, \\ \dim \mathbb{P}_2^d = \frac{1}{2}(d+1)(d+2), \end{cases} \quad \text{and} \quad \begin{cases} \dim \mathbb{P}_k^1 = (k+1), \\ \dim \mathbb{P}_k^2 = \frac{1}{2}(k+1)(k+2), \\ \dim \mathbb{P}_k^3 = \frac{1}{6}(k+1)(k+2)(k+3). \end{cases}$$

*Proof.* Direct consequence of Lemma 1524 (*monomials are a basis of  $\mathbb{P}_k^d$* ), **the definition of the dimension**, and Lemma 1498 (*cardinal of  $\mathcal{A}_k^d$* ).  $\square$

## 9.2.4 Decomposition of polynomials, isomorphisms of $\mathbb{P}_k^d$

**Lemma 1526** (*isomorphism between  $\mathbb{P}_0^d$  and  $\mathbb{P}_0^{d-1}$* ).

Let  $d \geq 2$ . Let  $\xi^d \stackrel{\text{def.}}{=} ((\mathbf{x} \mapsto a_0) \in \mathbb{P}_0^d \mapsto (\tilde{\mathbf{x}} \mapsto a_0) \in \mathbb{P}_0^{d-1})$ .

Then, for all  $p \in \mathbb{P}_0^d$ , for all  $\mathbf{x} \in \mathbb{R}^d$ , we have  $\xi^d(p)(\tilde{\mathbf{x}}) = p(\mathbf{x})$ , and  $\xi^d$  is an isomorphism.

*Proof.* Direct consequence of Lemma 1509 (*constant and affine spaces  $\mathbb{P}_0^d$  and  $\mathbb{P}_1^d$* , thus  $\xi^d$  is well-defined, and the identity holds), Lemma 1525 (*dimension of  $\mathbb{P}_k^d$ ,  $\dim \mathbb{P}_0^d = \dim \mathbb{P}_0^{d-1} = 1$* ), **the definition of order (reflexivity)**, Lemma 1374 (*injectivity or surjectivity and dimension implies bijectivity*,  $\xi^d$  is obviously linear and injective), and Definition 100 (*isomorphism*).  $\square$

*Remark 1527.*

Note in the next lemma the use of the tilde notation of Remark 1490, such that  $\mathbf{x} = (\tilde{\mathbf{x}}, x_d)$ .

Note also that the decomposition extracts  $x_d$ , and thus uses the horizontal slices  $\tilde{\mathcal{S}}_{k,i}^d$ . A similar result, exhibiting  $x_1$  and using vertical slices  $\check{\mathcal{S}}_{k,i}^d$ , could be written,  $p(\mathbf{x}) = \check{p}_0(\tilde{\mathbf{x}}) + x_1 p_1(\mathbf{x})$ , with  $\mathbf{x} = (x_1, \tilde{\mathbf{x}})$  and obvious modifications in the notations.

*Remark 1528.* See the sketch of the next proof in Section 3.2.

**Lemma 1529** (*decomposition of  $\mathbb{P}_k^d$* ).

Let  $d \geq 2$ . Let  $k \geq 1$ . Let  $p \in \mathbb{P}_k^d$ .

Then, there exist unique  $\tilde{p}_0 \in \mathbb{P}_{k-1}^{d-1}$  and  $p_1 \in \mathbb{P}_{k-1}^d$  such that

$$(9.27) \quad \forall \mathbf{x} \in \mathbb{R}^d, \quad p(\mathbf{x}) = \tilde{p}_0(\tilde{\mathbf{x}}) + x_d p_1(\mathbf{x}) \quad (\text{i.e. } p = \tilde{p}_0 + X_d p_1),$$

which also writes  $p(x_1, \dots, x_d) = \tilde{p}_0(x_1, \dots, x_{d-1}) + x_d p_1(x_1, \dots, x_d)$ .

*Proof. Existence.* For all  $k \geq 1$ , let  $P(k) \stackrel{\text{def.}}{=} [\forall p \in \mathbb{P}_k^d, \exists \tilde{p}_0 \in \mathbb{P}_{k-1}^{d-1}, \exists p_1 \in \mathbb{P}_{k-1}^d, p = \tilde{p}_0 + \mathbf{X}_d p_1]$ .

**Induction:  $P(1)$ .** Let  $p \in \mathbb{P}_1^d$ . Then, from Lemma 1509 (*constant and affine spaces  $\mathbb{P}_0^d$  and  $\mathbb{P}_1^d$* ), and **commutative ring properties of  $\mathbb{R}$**  there exist

$$a_0, (a_{\mathbf{e}_i})_{i \in [1..d]} \in \mathbb{R}, \quad \tilde{p}_0 \stackrel{\text{def.}}{=} a_0 + \sum_{i=1}^{d-1} a_{\mathbf{e}_i} X_i \in \mathbb{P}_1^{d-1}, \quad \text{and} \quad p_1 \stackrel{\text{def.}}{=} a_{\mathbf{e}_d} \in \mathbb{P}_0^d,$$

such that  $p = a_0 + \sum_{i=1}^d a_{\mathbf{e}_i} X_i = \tilde{p}_0 + X_d p_1$ .

**Induction:  $P(k) \Rightarrow P(k+1)$ .** Assume that  $P(k)$  holds. Let  $p \in \mathbb{P}_{k+1}^d$ .

Then, from Definition 1505 (*polynomial space  $\mathbb{P}_k^d$* ), and Lemma 1496 ( *$\mathcal{C}_k^d$  are layers of  $\mathcal{A}_k^d$* ), there exist  $(a_\alpha)_{\alpha \in \mathcal{A}_{k+1}^d} \in \mathbb{R}$ ,  $q \stackrel{\text{def.}}{=} \sum_{\alpha \in \mathcal{A}_k^d} a_\alpha \mathbf{X}^\alpha \in \mathbb{P}_k^d$ , and  $r \stackrel{\text{def.}}{=} \sum_{\alpha \in \mathcal{C}_{k+1}^d} a_\alpha \mathbf{X}^\alpha \in \mathbb{P}_{k+1}^d$  such that  $p = \sum_{\alpha \in \mathcal{A}_{k+1}^d} a_\alpha \mathbf{X}^\alpha = q + r$ . By induction hypothesis, and from Lemma 1508 ( *$\mathbb{P}_k^d$  is nondecreasing sequence in  $k$* ), there exist  $\tilde{q}_0 \in \mathbb{P}_k^{d-1} \subset \mathbb{P}_{k+1}^{d-1}$  and  $q_1 \in \mathbb{P}_{k-1}^d \subset \mathbb{P}_k^d$ , such that  $q = \tilde{q}_0 + X_d q_1$ . For all  $i \in [0..k+1]$ , for all  $\tilde{\alpha}_i \in \mathcal{C}_{k+1-i}^{d-1}$ , let  $b_{\tilde{\alpha}_i} \stackrel{\text{def.}}{=} a_{(\tilde{\alpha}_i, i)} = a_{(\tilde{\varphi}_{k+1, d}^i)^{-1}(\tilde{\alpha}_i)}$ . Then, from Lemma 1493 (*slices of  $\mathcal{C}_k^d$* ), Lemma 1494 (*cardinal of slices of  $\mathcal{C}_k^d$* ), and Definition 1491 (*slices  $\tilde{\mathcal{S}}_{k,i}^d$  and  $\tilde{\mathcal{S}}_{k,i}^d$* ), we have

$$\begin{aligned} r &= \sum_{\alpha \in \tilde{\mathcal{S}}_{k+1,0}^d} a_\alpha \mathbf{X}^\alpha + \sum_{i=1}^{k+1} \sum_{\alpha \in \tilde{\mathcal{S}}_{k+1,i}^d} a_\alpha \mathbf{X}^\alpha \\ &= \underbrace{\sum_{\tilde{\alpha}_0 \in \mathcal{C}_{k+1}^{d-1}} b_{\tilde{\alpha}_0} \tilde{\mathbf{X}}^{\tilde{\alpha}_0}}_{\stackrel{\text{def.}}{=} \tilde{r}_0} + X_d \underbrace{\sum_{i=1}^{k+1} \sum_{\tilde{\alpha}_i \in \mathcal{C}_{k+1-i}^{d-1}} b_{\tilde{\alpha}_i} \tilde{\mathbf{X}}^{\tilde{\alpha}_i} X_d^{i-1}}_{\stackrel{\text{def.}}{=} r_1}. \end{aligned}$$

Let  $i \in [1..k+1]$ . Let  $\tilde{\alpha}_i \in \mathcal{C}_{k+1-i}^{d-1}$ . Then, from Lemma 1512 (*monomials of  $\mathcal{C}_k^d$  have degree  $k$* ), we have  $\deg(\tilde{\mathbf{X}}^{\tilde{\alpha}_i} X_d^{i-1}) = (k+1-i) + (i-1) = k$ . Thus, from Lemma 1496 ( *$\mathcal{C}_k^d$  are layers of  $\mathcal{A}_k^d$* ), and Definition 1505 (*polynomial space  $\mathbb{P}_k^d$* ), we have  $\tilde{r}_0 \in \mathbb{P}_{k+1}^{d-1}$  and  $r_1 \in \mathbb{P}_k^d$ .

Let  $\tilde{p}_0 \stackrel{\text{def.}}{=} \tilde{q}_0 + \tilde{r}_0$  and  $p_1 \stackrel{\text{def.}}{=} q_1 + r_1$ . Then, from **ring properties of  $\mathbb{R}$** , and Lemma 1507 ( *$\mathbb{P}_k^d$  is vector space*), we have  $\tilde{p}_0 \in \mathbb{P}_{k+1}^{d-1}$ ,  $p_1 \in \mathbb{P}_k^d$ , and  $p = \tilde{p}_0 + X_d p_1$ , i.e.  $P(k+1)$  holds. This concludes the induction on  $k$ , and we have existence for all  $k \geq 1$ .

**Uniqueness.** Let  $p \in \mathbb{P}_k^d$ ,  $\tilde{q}_0, \tilde{r}_0 \in \mathbb{P}_k^{d-1}$ , and  $q_1, r_1 \in \mathbb{P}_{k-1}^d$ , such that,

$$\forall \mathbf{x} \in \mathbb{R}^d, \quad p(\mathbf{x}) = \tilde{q}_0(\tilde{\mathbf{x}}) + x_d q_1(\mathbf{x}) = \tilde{r}_0(\tilde{\mathbf{x}}) + x_d r_1(\mathbf{x}).$$

Let  $\tilde{\mathbf{x}} \in \mathbb{R}^{d-1}$ . Let  $\mathbf{x} \stackrel{\text{def.}}{=} (\tilde{\mathbf{x}}, 0)$ . Then, we have  $\tilde{q}_0(\tilde{\mathbf{x}}) = \tilde{r}_0(\tilde{\mathbf{x}})$ , i.e.  $\tilde{q}_0 = \tilde{r}_0$ , thus  $X_d(q_1 - r_1) = 0$ . Thus, from Lemma 1507 ( *$\mathbb{P}_k^d$  is vector space*), Definition 1505 (*polynomial space  $\mathbb{P}_k^d$* ), Lemma 1523 (*monomials are free in  $\mathbb{P}_k^d$* ), and **the definition of freedom**, we have  $q_1 - r_1 \in \mathbb{P}_{k-1}^d$  and there exists  $(a_\alpha)_{\alpha \in \mathcal{A}_{k-1}^d} \in \mathbb{R}$  such that  $q_1 - r_1 = \sum_{\alpha \in \mathcal{A}_{k-1}^d} a_\alpha \mathbf{X}^\alpha$ , and thus

$$X_d(q_1 - r_1) = \sum_{\alpha \in \mathcal{A}_{k-1}^d} a_\alpha \mathbf{X}^{\alpha + \mathbf{e}_d} = 0 \text{ (in } \mathbb{P}_k^d\text{),}$$

which implies  $a_\alpha = 0$  for all  $\alpha \in \mathcal{A}_{k-1}^d$ , i.e.  $q_1 = r_1$ .

Therefore, we have uniqueness.  $\square$

*Remark 1530.* Lemmas 1529 and 1531 allow to write  $\dim(\mathbb{P}_k^d) = \dim(\mathbb{P}_k^{d-1}) + \dim(\mathbb{P}_{k-1}^d)$ . Actually, this could be a way to prove (by induction) that  $\dim(\mathbb{P}_k^d) = \binom{k+d}{d}$ . But, we preferred to prove directly that  $\text{card}(\mathcal{C}_k^d) = \binom{k+d-1}{d-1}$ , by working on slices of  $\mathcal{C}_k^d$ .

**Lemma 1531 (isomorphism between  $\mathbb{P}_k^d$  and  $\mathbb{P}_k^{d-1} \times \mathbb{P}_{k-1}^d$ ).**

Let  $d \geq 2$ . Let  $k \geq 1$ . Let  $\zeta_k^d \stackrel{\text{def.}}{=} (p \in \mathbb{P}_k^d \mapsto (\tilde{p}_0, p_1) \in \mathbb{P}_k^{d-1} \times \mathbb{P}_{k-1}^d)$  with  $p = \tilde{p}_0 + X_d p_1$ . Then,  $\zeta_k^d$  is an isomorphism.

*Proof.* From Lemma 1529 (*decomposition of  $\mathbb{P}_k^d$* ),  $\zeta_k^d$  is well-defined for all  $p \in \mathbb{P}_k^d$ .

**Linearity.** Let  $p, q \in \mathbb{P}_k^d$ , and  $\lambda, \mu \in \mathbb{R}$ . Then, from Lemma 1529 (*decomposition of  $\mathbb{P}_k^d$* ), **commutative ring properties of  $\mathbb{R}$** , Lemma 1507 ( *$\mathbb{P}_k^d$  is vector space*), and Lemma 82 (*closed under*



(*linear combination is subspace*), there exist unique  $\tilde{p}_0, \tilde{q}_0 \in \mathbb{P}_k^{d-1}$  and  $p_1, q_1 \in \mathbb{P}_{k-1}^d$  such that  $p = \tilde{p}_0 + X_d p_1$  and  $q = \tilde{q}_0 + X_d q_1$ , thus  $\lambda p + \mu q = \lambda \tilde{p}_0 + \mu \tilde{q}_0 + X_d(\lambda p_1 + \mu q_1)$  with  $\lambda \tilde{p}_0 + \mu \tilde{q}_0 \in \mathbb{P}_k^{d-1}$  and  $\lambda p_1 + \mu q_1 \in \mathbb{P}_{k-1}^d$ . Hence, from Lemma 1529 (*decomposition of  $\mathbb{P}_k^d$ , for  $\lambda p + \mu q$ , uniqueness*), Definition 89 (*product vector operations*), Lemma 90 (*product is space*), and Lemma 95 (*linear map preserves linear combinations*), we have

$$\zeta_k^d(\lambda p + \mu q) = (\lambda \tilde{p}_0 + \mu \tilde{q}_0, \lambda p_1 + \mu q_1) = \lambda(\tilde{p}_0, p_1) + \mu(\tilde{q}_0, q_1) = \lambda \zeta_k^d(p) + \mu \zeta_k^d(q),$$

and  $\zeta_k^d$  is a linear map from  $\mathbb{P}_k^d$  to the product vector space  $\mathbb{P}_k^{d-1} \times \mathbb{P}_{k-1}^d$ .

**Injectivity.** Let  $p \in \mathbb{P}_k^d$ . Assume that  $\zeta_k^d(p) = 0$ . Then, from Lemma 1529 (*decomposition of  $\mathbb{P}_k^d$* ), Definition 101 (*kernel*), and Lemma 103 (*injective linear map has zero kernel*), there exist unique  $\tilde{p}_0 \in \mathbb{P}_k^{d-1}$  and  $p_1 \in \mathbb{P}_{k-1}^d$  such that  $p = \tilde{p}_0 + X_d p_1$ , thus  $\zeta_k^d(p) = (\tilde{p}_0, p_1) = 0$ ,  $p = 0 + X_d 0 = 0$ , and  $\zeta_k^d$  is injective.

**Dimension.** From Lemma 1525 (*dimension of  $\mathbb{P}_k^d$ , three times*), Lemma 1364 (*properties of the binomial coefficient, (5.7)*), and **the rule of dimension for product of spaces**, we have

$$\dim(\mathbb{P}_k^d) = \binom{k+d}{d} = \binom{k+d-1}{d-1} + \binom{k-1+d}{d} = \dim(\mathbb{P}_k^{d-1}) + \dim(\mathbb{P}_{k-1}^d) = \dim(\mathbb{P}_k^{d-1} \times \mathbb{P}_{k-1}^d).$$

**Isomorphism.** Direct consequence of **the definition of order (reflexivity)**, Lemma 1374 (*injectivity or surjectivity and dimension implies bijectivity*), and Definition 100 (*isomorphism*).  $\square$

*Remark 1532.* Note in the next lemma that, by an abuse of notation, for all  $p \in \mathbb{P}_k^d$ , the function that sends  $(\mathbf{x}, x_{d+1})$  to  $p(\mathbf{x})$  is sometimes still denoted  $p$ .

**Lemma 1533** ( *$\mathbb{P}_k^d$  is nondecreasing sequence in  $d$* ). Let  $k \in \mathbb{N}$ .  
Then, the sequence  $(\mathbb{P}_k^d)_{d \geq 1}$  is nondecreasing for the inclusion, in the sense that, for all  $d \geq 1$ , for all  $p \in \mathbb{P}_k^d$ , the function  $(\mathbf{x}, x_{d+1}) \in \mathbb{R}^d \times \mathbb{R} \mapsto p(\mathbf{x}) \in \mathbb{R}$  belongs to  $\mathbb{P}_k^{d+1}$ .

*Proof.* Let  $d \geq 1$ . Let  $p \in \mathbb{P}_k^d$ .

**Case  $k = 0$ .** Direct consequence of Lemma 1509 (*constant and affine spaces  $\mathbb{P}_0^d$  and  $\mathbb{P}_1^d$* ).

**Case  $k \geq 1$ .** Let  $\tilde{p}_0 \stackrel{\text{def.}}{=} p \in \mathbb{P}_k^d$  and  $p_1 \stackrel{\text{def.}}{=} 0$ . Let  $\mathbf{x} \in \mathbb{R}^d$  and  $x_{d+1} \in \mathbb{R}$ . Then, from Lemma 1507 ( *$\mathbb{P}_k^d$  is vector space, thus  $p_1 = 0 \in \mathbb{P}_{k-1}^{d+1}$* ), Lemma 1531 (*isomorphism between  $\mathbb{P}_k^d$  and  $\mathbb{P}_k^{d-1} \times \mathbb{P}_{k-1}^d$* ), and Definition 1505 (*polynomial space  $\mathbb{P}_k^d$* ), we have  $\hat{p} = (\zeta_k^{d+1})^{-1}(\tilde{p}_0, p_1) \in \mathbb{P}_k^{d+1}$ , and  $\hat{p}(\mathbf{x}, x_{d+1}) = p(\mathbf{x}) + x_{d+1}0 = p(\mathbf{x})$ .  $\square$

**Lemma 1534** (*expression of  $\mathbb{P}_k^d$  as polynomial of  $x_d$* ). Let  $d \geq 2$ . Let  $k \in \mathbb{N}$ . Let  $p \in \mathbb{P}_k^d$ . Then, for all  $i \in [0..k]$ , there exists a unique  $\tilde{r}_i \in \mathbb{P}_{k-i}^{d-1}$ , such that

$$(9.28) \quad \forall \mathbf{x} \in \mathbb{R}^d, \quad p(\mathbf{x}) = \sum_{i=0}^k \tilde{r}_i(\tilde{\mathbf{x}}) x_d^i \quad \left( \text{i.e. } p = \sum_{i=0}^k \tilde{r}_i X_d^i \right).$$

Thus, for all  $\tilde{\mathbf{x}} \in \mathbb{R}^{d-1}$ , the function  $(x_d \mapsto p(\tilde{\mathbf{x}}, x_d))$  belongs to  $\mathbb{P}_k^1$ .

*Proof. Existence.* For all  $k \in \mathbb{N}$ , let  $P(k) \stackrel{\text{def.}}{=} [\forall p \in \mathbb{P}_k^d, (\forall i \in [0..k], \exists \tilde{r}_i \in \mathbb{P}_{k-i}^{d-1}), p = \sum_{i=0}^k \tilde{r}_i X_d^i]$ .

**Induction:  $P(0)$ .** Let  $p \in \mathbb{P}_0^d$ . Then, from Lemma 1526 (*isomorphism between  $\mathbb{P}_0^d$  and  $\mathbb{P}_0^{d-1}$* ),  $\tilde{r}_0 \stackrel{\text{def.}}{=} \zeta^d(p) \in \mathbb{P}_0^{d-1}$  is such that, for all  $\tilde{\mathbf{x}} \in \mathbb{R}^{d-1}$ , for all  $x_d \in \mathbb{R}$ ,  $p(\tilde{\mathbf{x}}, x_d) = \tilde{r}_0(\tilde{\mathbf{x}})$ , i.e.  $P(0)$  holds.

**Induction:  $P(1)$ .** Let  $p \in \mathbb{P}_1^d$ . Then, from Lemma 1529 (*decomposition of  $\mathbb{P}_k^d$* ), there exists  $\tilde{r}_0 \in \mathbb{P}_1^{d-1}$  and  $q_1 \in \mathbb{P}_0^d$ , such that  $p = \tilde{r}_0 + X_d q_1$ . Thus, from Lemma 1526 (*isomorphism between*

$\mathbb{P}_0^d$  and  $\mathbb{P}_0^{d-1}$ ),  $\tilde{r}_1 \stackrel{\text{def.}}{=} \xi^d(q_1) \in \mathbb{P}_0^{d-1}$  is such that, for all  $\tilde{\mathbf{x}} \in \mathbb{R}^{d-1}$ , for all  $x_d \in \mathbb{R}$ ,  $q_1(\tilde{\mathbf{x}}, x_d) = \tilde{r}_1(\tilde{\mathbf{x}})$ , hence  $p = \tilde{r}_0 + X_d \tilde{r}_1$  with  $\tilde{r}_0 \in \mathbb{P}_1^{d-1}$  and  $\tilde{r}_1 \in \mathbb{P}_0^{d-1}$ , i.e.  $P(1)$  holds.

**Induction:  $P(k) \Rightarrow P(k+1)$ .** Let  $k \geq 1$ . Assume that  $P(k)$  holds. Let  $p \in \mathbb{P}_{k+1}^d$ . Then, from Lemma 1529 (*decomposition of  $\mathbb{P}_k^d$* ), there exist  $\tilde{r}_0 \in \mathbb{P}_{k+1}^{d-1}$  and  $q_1 \in \mathbb{P}_k^d$ , such that  $p = \tilde{r}_0 + X_d q_1$ . By induction hypothesis, for all  $i \in [0..k]$ , there exist  $\tilde{s}_i \in \mathbb{P}_{k-i}^{d-1}$ , such that  $q_1 = \sum_{i=0}^k \tilde{s}_i X_d^i$ . For all  $i \in [1..k+1]$ , let  $\tilde{r}_i \stackrel{\text{def.}}{=} \tilde{s}_{i-1} \in \mathbb{P}_{k+1-i}^{d-1}$ . Then, from **commutative ring properties of  $\mathbb{R}$** , we have  $p = \sum_{i=0}^{k+1} \tilde{r}_i X_d^i$ , i.e.  $P(k+1)$  holds.

This concludes the induction on  $k$ , and we have existence for all  $k \in \mathbb{N}$ .

**Uniqueness.** Let  $p \in \mathbb{P}_k^d$ , and for all  $i \in [0..k]$ , let  $\tilde{r}_i, \tilde{s}_i \in \mathbb{P}_{k-i}^{d-1}$  such that

$$p = \sum_{i=0}^k \tilde{r}_i X_d^i = \sum_{i=0}^k \tilde{s}_i X_d^i.$$

Let  $\tilde{\mathbf{x}} \in \mathbb{R}^{d-1}$ . Then, from **ring properties of  $\mathbb{R}$** , and Lemma 1419 (*monomials are free in  $\mathbb{P}_k^1$* ), we have  $\sum_{i=0}^k (\tilde{r}_i(\tilde{\mathbf{x}}) - \tilde{s}_i(\tilde{\mathbf{x}})) X_d^i = 0$ , and thus, for all  $i \in [0..k]$ ,  $\tilde{r}_i(\tilde{\mathbf{x}}) - \tilde{s}_i(\tilde{\mathbf{x}}) = 0$ , i.e.  $\tilde{r}_i = \tilde{s}_i$ .

Therefore, we have uniqueness.  $\square$

*Remark 1535.* Lemma 1534 expresses a  $d$ -multivariate polynomial as an univariate polynomial with coefficients that are  $(d-1)$ -multivariate polynomials. The proof uses Horner's rule,

$$p = \tilde{r}_0 + X_d(\tilde{r}_1 + X_d(\tilde{r}_2 + X_d(\dots + X_d(\tilde{r}_{k-1} + X_d \tilde{r}_k) \dots))).$$

## 9.2.5 Product of polynomials (alternate)

*Remark 1536.* The next lemma provides an alternate proof for Lemma 1516. This second proof avoids the double sum on multi-indices that is not explicitly formulated in the first proof. It relies on the decomposition of polynomials of Lemma 1529 and on a strong double induction.

**Lemma 1537 (product of two polynomials (alternate proof)).**

Let  $d \geq 1$ . Let  $k, l \in \mathbb{N}$ . Let  $p \in \mathbb{P}_k^d$  and  $q \in \mathbb{P}_l^d$ . Then, we have  $pq \in \mathbb{P}_{k+l}^d$ .

*Proof.* For all  $d \geq 1$ , for all  $n \in \mathbb{N}$ , let  $P$  be the property defined by

$$P(d, n) \stackrel{\text{def.}}{=} [\forall k, l \in \mathbb{N}, \forall p \in \mathbb{P}_k^d, \forall q \in \mathbb{P}_l^d, \quad k+l \leq n \implies pq \in \mathbb{P}_n^d].$$

Then, from Lemma 1508 ( $\mathbb{P}_k^d$  is nondecreasing sequence in  $k$ , thus  $k+l \leq n \implies \mathbb{P}_{k+l}^d \subset \mathbb{P}_n^d$ ), the result to prove is equivalent to establish  $P$ . Let  $d \geq 1$  and  $n \in \mathbb{N}$ .

**Strong double induction.**

Assume that for all  $d_1, n_1 \in \mathbb{N}$ ,  $1 \leq d_1 \leq d$ ,  $n_1 \leq n$ , and  $(d_1, n_1) \neq (d, n)$  implies  $P(d_1, n_1)$ . Let us show  $P(d, n)$ . Let  $k, l \in \mathbb{N}$ . Let  $p \in \mathbb{P}_k^d$  and  $q \in \mathbb{P}_l^d$ . Assume that  $k+l \leq n$ . Let us show  $pq \in \mathbb{P}_n^d$ .

**Case  $d = 1$ .** Direct consequence of Lemma 1422 (*product of two univariate polynomials*).

**Case  $k = 0$  or  $l = 0$ .** Direct consequence of Lemma 1509 (*constant and affine spaces  $\mathbb{P}_0^d$  and  $\mathbb{P}_1^d$ , thus  $p = a_0 \mathbf{X}^0$  (resp.  $q = b_0 \mathbf{X}^0$ ) is constant*), **commutative ring properties of  $\mathbb{R}$  (thus  $pq = a_0 q$  (resp.  $pq = b_0 p$ ))**, Lemma 1507 ( $\mathbb{P}_k^d$  is vector space), Lemma 81 (*closed under vector operations is subspace*, closed under scalar multiplication, thus  $pq$  is in  $\mathbb{P}_{k+l}^d$ ), and Lemma 1508 ( $\mathbb{P}_k^d$  is nondecreasing sequence in  $k$ , thus  $\mathbb{P}_{k+l}^d \subset \mathbb{P}_n^d$ ).

**Case  $k+l < n$ .** Then, by hypothesis,  $P(d, k+l)$  holds, i.e.  $pq \in \mathbb{P}_n^d$ .

**Case  $d \geq 2$ ,  $k \neq 0$ ,  $l \neq 0$  and  $k+l = n \geq 1$ .**

Then, from Lemma 1529 (*decomposition of  $\mathbb{P}_k^d$* ), and **commutative ring properties of  $\mathbb{R}$** , there exist  $\tilde{p}_0 \in \mathbb{P}_k^{d-1}$ ,  $p_1 \in \mathbb{P}_{k-1}^d$ ,  $\tilde{q}_0 \in \mathbb{P}_l^{d-1}$  and  $q_1 \in \mathbb{P}_{l-1}^d$ , such that

$$p = \tilde{p}_0 + X_d p_1, \quad q = \tilde{q}_0 + X_d q_1, \quad \text{thus} \quad pq = \tilde{p}_0 \tilde{q}_0 + X_d (\tilde{p}_0 q_1 + p_1 \tilde{q}_0) + X_d^2 p_1 q_1.$$

Then, by hypothesis (with  $d_1 \stackrel{\text{def.}}{=} d - 1 < d$  and  $n_1 \stackrel{\text{def.}}{=} n$ ), and from Lemma 1533 ( $\mathbb{P}_k^d$  is nondecreasing sequence in  $d$ ),  $P(d - 1, n)$  holds, i.e.  $\tilde{p}_0 \tilde{q}_0 \in \mathbb{P}_n^{d-1}$ , and thus

$$\left( \mathbf{x} \stackrel{\text{def.}}{=} (\tilde{\mathbf{x}}, x_d) \in \mathbb{R}^d \mapsto (\tilde{p}_0 \tilde{q}_0)(\tilde{\mathbf{x}}) \right) \in \mathbb{P}_n^d.$$

Next, from Lemma 1533 ( $\mathbb{P}_k^d$  is nondecreasing sequence in  $d$ ), and by hypothesis (with  $d_1 \stackrel{\text{def.}}{=} d$  and  $n_1 \stackrel{\text{def.}}{=} n - 1 < n$ ), we have  $(\mathbf{x} \mapsto \tilde{p}_0(\tilde{\mathbf{x}})) \in \mathbb{P}_k^d$ ,  $(\mathbf{x} \mapsto \tilde{q}_0(\tilde{\mathbf{x}})) \in \mathbb{P}_1^d$ , and  $P(d, n - 1)$  holds, i.e.  $(\mathbf{x} \mapsto \tilde{p}_0(\tilde{\mathbf{x}})q_1(\mathbf{x}))$  and  $(\mathbf{x} \mapsto p_1(\mathbf{x})\tilde{q}_0(\tilde{\mathbf{x}}))$  belong to  $\mathbb{P}_{n-1}^d$ . Hence, from Lemma 1507 ( $\mathbb{P}_k^d$  is vector space), Lemma 81 (closed under vector operations is subspace), and Lemma 1515 (product of monomial and polynomial, with  $\boldsymbol{\alpha} \stackrel{\text{def.}}{=} \mathbf{e}_d$ ), we have  $\tilde{p}_0 q_1 + p_1 \tilde{q}_0 \in \mathbb{P}_{n-1}^d$ , and  $X_d(\tilde{p}_0 q_1 + p_1 \tilde{q}_0) \in \mathbb{P}_n^d$ . Moreover, by hypothesis (with  $d_1 \stackrel{\text{def.}}{=} d$  and  $n_1 \stackrel{\text{def.}}{=} n - 2 < n$ ), and from Lemma 1515 (product of monomial and polynomial, with  $\boldsymbol{\alpha} \stackrel{\text{def.}}{=} 2\mathbf{e}_d$ ),  $P(d, n - 2)$  holds, i.e.  $p_1 q_1 \in \mathbb{P}_{n-2}^d$ , and  $X_d^2 p_1 q_1 \in \mathbb{P}_n^d$ . Finally, from Lemma 1507 ( $\mathbb{P}_k^d$  is vector space), and Lemma 81 (closed under vector operations is subspace), we have  $pq \in \mathbb{P}_n^d$ .

Therefore, we always have  $pq \in \mathbb{P}_n^d$ , and from Lemma 1360 (strong double induction), the property holds for all  $d \geq 1$  and for all  $n \in \mathbb{N}$ .  $\square$

**Lemma 1538 (product of polynomials).** Let  $d, n \geq 1$ . Let  $(k_i)_{i \in [1..n]} \in \mathbb{N}$ .  
Let  $k \stackrel{\text{def.}}{=} \sum_{i=1}^n k_i$ . For all  $i \in [1..n]$ , let  $p_i \in \mathbb{P}_{k_i}^d$ . Then, we have  $\prod_{i=1}^n p_i \in \mathbb{P}_k^d$ .

*Proof.* Direct consequence (by induction on  $n$ ) of Lemma 1516 (product of two polynomials), or Lemma 1537 (product of two polynomials (alternate proof)).  $\square$

## 9.2.6 Composition of polynomials and affine mappings

**Lemma 1539 (affine mapping of monomials is  $\mathbb{P}_k^l$ ).**  
Let  $l, d \geq 1$ . Let  $k \in \mathbb{N}$ . Let  $f \in \text{Aff}(\mathbb{R}^l, \mathbb{R}^d)$ . Let  $\boldsymbol{\alpha} \in \mathcal{C}_k^d$ . Then, we have  $\mathbf{X}^\alpha \circ f \in \mathbb{P}_k^l$ .

*Proof.* Let  $q \stackrel{\text{def.}}{=} \mathbf{X}^\alpha \circ f$ .

For all  $i \in [1..d]$ , let  $f_i$  be the  $i$ -th component of  $f$ . Let  $\mathbf{y} \in \mathbb{R}^l$ . Then, from Lemma 1399 (equivalent definition of affine map (finite dimension)), Lemma 1509 (constant and affine spaces  $\mathbb{P}_0^d$  and  $\mathbb{P}_1^d$ ), and Definition 1503 (monomial in  $d$  variables),  $f_i$  belongs to  $\mathbb{P}_1^l$ , and we have

$$q(\mathbf{y}) = \prod_{i=1}^d [f_i(\mathbf{y})]^{\alpha_i}, \quad \text{and} \quad [f_i(\mathbf{y})]^{\alpha_i} = \prod_{j=1}^{\alpha_i} f_i(\mathbf{y}).$$

Thus, from Lemma 1538 (product of polynomials, with  $n \stackrel{\text{def.}}{=} \alpha_i$  and  $p_j \stackrel{\text{def.}}{=} f_i \in \mathbb{P}_1^l$  for all  $j \in [1..\alpha_i]$ ), then with  $n \stackrel{\text{def.}}{=} d$  and  $p_i \stackrel{\text{def.}}{=} f_i^{\alpha_i} \in \mathbb{P}_{\alpha_i}^l$  for all  $i \in [1..d]$ , thus  $q \in \mathbb{P}_{\sum_{i=1}^d \alpha_i}^l$ , Definition 1486 (sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$ , thus  $|\boldsymbol{\alpha}| = k$ ), and Definition 1480 (length of multi-indices), we have  $q \in \mathbb{P}_k^l$ .  $\square$

**Lemma 1540 (affine mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^l$ ).**  
Let  $l, d \geq 1$ . Let  $k \in \mathbb{N}$ . Let  $f \in \text{Aff}(\mathbb{R}^l, \mathbb{R}^d)$ . Let  $p \in \mathbb{P}_k^d$ . Then, we have  $p \circ f \in \mathbb{P}_k^l$ .

*Proof.* From Definition 1505 (polynomial space  $\mathbb{P}_k^d$ ), **distributivity of composition over addition**, Lemma 1496 ( $\mathcal{C}_k^d$  are layers of  $\mathcal{A}_k^d$ ), Lemma 1539 (affine mapping of monomials is  $\mathbb{P}_k^l$ , thus  $\mathbf{X}^\alpha \circ f \in \mathbb{P}_k^l$  for all  $i \in [0..k]$  and for all  $\boldsymbol{\alpha} \in \mathcal{C}_i^d$ ), and Lemma 1508 ( $\mathbb{P}_k^d$  is nondecreasing sequence in  $k$ , thus  $\mathbb{P}_i^d \subset \mathbb{P}_k^d$  for all  $i \in [0..k]$ ), and Lemma 1507 ( $\mathbb{P}_k^d$  is vector space), we have

$$q \stackrel{\text{def.}}{=} p \circ f = \left( \sum_{\boldsymbol{\alpha} \in \mathcal{A}_k^d} a_\alpha \mathbf{X}^\alpha \right) \circ f = \sum_{\boldsymbol{\alpha} \in \mathcal{A}_k^d} a_\alpha (\mathbf{X}^\alpha \circ f) = \sum_{i=0}^k \sum_{\boldsymbol{\alpha} \in \mathcal{C}_i^d} a_\alpha (\mathbf{X}^\alpha \circ f),$$

and  $q \in \mathbb{P}_k^l$ .  $\square$

## 9.3 $\mathbb{P}_1^d$ affine polynomials and affine geometric mapping

### 9.3.1 Reference affine Lagrange polynomials

**Definition 1541** (*reference Lagrange polynomials of  $\mathbb{P}_1^d$* ). Let  $d \geq 1$ . Let  $i \in [0..d]$ . Let  $\hat{\mathbf{x}} \in \mathbb{R}^d$ . The  $i$ -th reference Lagrange polynomials of  $\mathbb{P}_1^d$  is denoted  $\hat{\mathcal{L}}_i^{1,d}$ , and is defined by

$$(9.29) \quad (i = 0) \quad \hat{\mathcal{L}}_0^{1,d}(\hat{\mathbf{x}}) \stackrel{\text{def.}}{=} 1 - \sum_{j=1}^d \hat{x}_j,$$

$$(9.30) \quad (i \in [1..d]) \quad \hat{\mathcal{L}}_i^{1,d}(\hat{\mathbf{x}}) \stackrel{\text{def.}}{=} \hat{x}_i.$$

**Lemma 1542** (*reference Lagrange polynomials of  $\mathbb{P}_1^d$  for  $d = 1$  are reference Lagrange polynomials of  $\mathbb{P}_k^1$  for  $k = 1$* ).

The reference Lagrange polynomials of  $\mathbb{P}_1^d$  for  $d \stackrel{\text{def.}}{=} 1$  and the reference Lagrange polynomials of  $\mathbb{P}_k^1$  for  $k \stackrel{\text{def.}}{=} 1$  from Lemma 1455 (*reference Lagrange basis of  $\mathbb{P}_k^1$* ) coincide.

*Proof.* Direct consequence of Definition 1541 (*reference Lagrange polynomials of  $\mathbb{P}_1^d$ , with  $d \stackrel{\text{def.}}{=} 1$* ), Definition 1448 (*Lagrange polynomials of  $\mathbb{P}_k^1$ , with  $k \stackrel{\text{def.}}{=} 1$* ), Definition 1453 (*reference Lagrange nodes of  $\mathbb{P}_k^1$ , (8.9) with  $k \stackrel{\text{def.}}{=} 1$* ), Lemma 1455 (*reference Lagrange basis of  $\mathbb{P}_k^1$ , with  $k \stackrel{\text{def.}}{=} 1$* ), and **field properties of  $\mathbb{R}$  (with  $1 - 0 \neq 0$  and  $0 - 1 \neq 0$  for  $k \stackrel{\text{def.}}{=} 1$ )**.  $\square$

**Lemma 1543** (*reference Lagrange polynomials is basis of  $\mathbb{P}_1^d$* ). Let  $d \geq 1$ . Then, the reference Lagrange polynomials  $(\hat{\mathcal{L}}_i^{1,d})_{i \in [0..d]}$  form a basis of  $\mathbb{P}_1^d$ , that satisfies

$$(9.31) \quad \forall i \in [0..d], \quad \deg \hat{\mathcal{L}}_i^{1,d} = 1,$$

$$(9.32) \quad \forall i, j \in [0..d], \quad \hat{\mathcal{L}}_i^{1,d}(\hat{\mathbf{v}}_j) = \delta_{ij},$$

$$(9.33) \quad \sum_{i=0}^d \hat{\mathcal{L}}_i^{1,d} = 1.$$

*Proof.* Degree (9.31), in  $\mathbb{P}_1^d$ , and identities (9.32) and (9.33).

Direct consequence of Definition 1541 (*reference Lagrange polynomials of  $\mathbb{P}_1^d$* ), Lemma 1509 (*constant and affine spaces  $\mathbb{P}_0^d$  and  $\mathbb{P}_1^d$* ), Definition 1510 (*degree of polynomial*), **additive group properties of  $\mathbb{R}$** , and Definition 1434 (*family of reference points*).

**Basis.** Let  $(\lambda_i)_{i \in [0..d]} \in \mathbb{R}$  such that, for all  $\hat{\mathbf{x}} \in \mathbb{R}^d$ ,  $\sum_{i=0}^d \lambda_i \hat{\mathcal{L}}_i^{1,d}(\hat{\mathbf{x}}) = 0$ . Then, from (9.32), and **ring properties of  $\mathbb{R}$** , taking  $\hat{\mathbf{x}} \stackrel{\text{def.}}{=}} \hat{\mathbf{v}}_j$  provides  $\sum_{i=0}^d \lambda_i \delta_{ij} = \lambda_j = 0$ . Thus, from **the definition of freedom**, Lemma 1525 (*dimension of  $\mathbb{P}_k^d$ , with  $k \stackrel{\text{def.}}{=} 1$* ), and Lemma 1376 (*free family of elements is basis*), the reference Lagrange polynomials form a basis of  $\mathbb{P}_1^d$ .  $\square$

*Remark 1544.* In the next lemma, the linear maps from  $\mathbb{R}^n$  to  $\mathbb{R}^p$  are assimilated to their matrix relative to the canonical bases.

**Lemma 1545** (*differential of reference Lagrange polynomials*). Let  $d \geq 1$ . Let  $\hat{\mathbf{x}} \in \mathbb{R}^d$ . Let  $i \in [0..d]$ . Then, we have  $\hat{\mathcal{L}}_i^{1,d} \in C^\infty(\mathbb{R}^d, \mathbb{R})$  and

$$(9.34) \quad (i = 0) \quad D\hat{\mathcal{L}}_0^{1,d}(\hat{\mathbf{x}}) = -\mathbf{1} \in \mathcal{M}_{1,d}(\mathbb{R}),$$

$$(9.35) \quad (i \in [1..d]) \quad D\hat{\mathcal{L}}_i^{1,d}(\hat{\mathbf{x}}) = \mathbf{e}_i \in \mathcal{M}_{1,d}(\mathbb{R}).$$

*Proof.* Direct consequence of Lemma 1543 (*reference Lagrange polynomials is basis of  $\mathbb{P}_1^d$* ), Lemma 1509 (*constant and affine spaces  $\mathbb{P}_0^d$  and  $\mathbb{P}_1^d$ , thus reference Lagrange polynomials are affine maps*), **affine maps in  $\mathbb{R}^d$  are  $\mathcal{C}^\infty$** , and **the rules of derivation (thus affine maps have constant differential and the differential is the linear part)**.  $\square$

### 9.3.2 Affine geometric mapping from $\mathbb{R}^d$ to $\mathbb{R}^d$

*Remark 1546.* In Figure 9.4, we plot an example of  $\varphi_{\text{geo}}^{\llbracket \mathbf{v} \rrbracket^d}$ , the affine geometric mapping from the reference simplex in  $\mathbb{R}^d$  to the current simplex in  $\mathbb{R}^d$ . This mapping transforms the  $i$ -th hyperface (i.e. opposite the vertex  $i \in [0..d]$ ) of the reference simplex to the  $i$ -th hyperface of the current simplex, see Figure 9.5.

Other affine geometric mappings are also introduced. First, in Definition 1578, the affine geometric mapping  $\phi_{\pi^l}^{\llbracket \mathbf{v} \rrbracket^d} : \mathbb{R}^l \rightarrow \mathbb{R}^d$  (for  $l \in [1..d]$ ) is a generalization of  $\varphi_{\text{geo}}^{\llbracket \mathbf{v} \rrbracket^d}$ , that allows to pass from the reference  $l$ -simplex to the current  $d$ -simplex. The injective mapping  $\pi^l : [0..l] \rightarrow [0..d]$  is used to select the chosen  $l$ -face of the  $d$ -simplex.

Next, one can define the geometric mapping  $\phi_{\theta_i^{d-1}}^{\llbracket \mathbf{v} \rrbracket^d} : \mathbb{R}^{d-1} \rightarrow \mathbb{R}^d$  (for  $i \in [0..d]$ ), see Lemma 1584. It transfers the reference  $(d-1)$ -simplex to the  $i$ -th hyperface of the current  $d$ -simplex, see Figure 9.6.

Finally, the geometric mapping  $\phi_{\pi^d}^{\llbracket \mathbf{v} \rrbracket^d} : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is defined in Lemma 1586. It transforms the reference  $d$ -simplex to a “permutation” of the current  $d$ -simplex (i.e. with a permutation of the vertices), see Figure 9.7 in the case of a circular permutation (defined in Lemma 1366). It is used in the proof of the factorization of polynomial Lemma 1623.

**Definition 1547 (geometric mapping).** Let  $d \geq 1$ . Let  $\llbracket \mathbf{v} \rrbracket^d$  be  $d+1$  points in  $\mathbb{R}^d$ . The *geometric mapping associated with  $\llbracket \mathbf{v} \rrbracket^d$*  is denoted  $\varphi_{\text{geo}}^{\llbracket \mathbf{v} \rrbracket^d}$ , and is defined by

$$(9.36) \quad \forall \hat{\mathbf{x}} \in \mathbb{R}^d, \quad \varphi_{\text{geo}}^{\llbracket \mathbf{v} \rrbracket^d}(\hat{\mathbf{x}}) = \sum_{i=0}^d \hat{\mathcal{L}}_i^{1,d}(\hat{\mathbf{x}}) \mathbf{v}_i \in \mathbb{R}^d.$$

**Lemma 1548 (geometric mapping for  $d=1$  is geometric mapping in dimension 1).**

Let  $\llbracket \mathbf{v} \rrbracket^1 = \llbracket v \rrbracket^1$  be two points in  $\mathbb{R}^1$ .

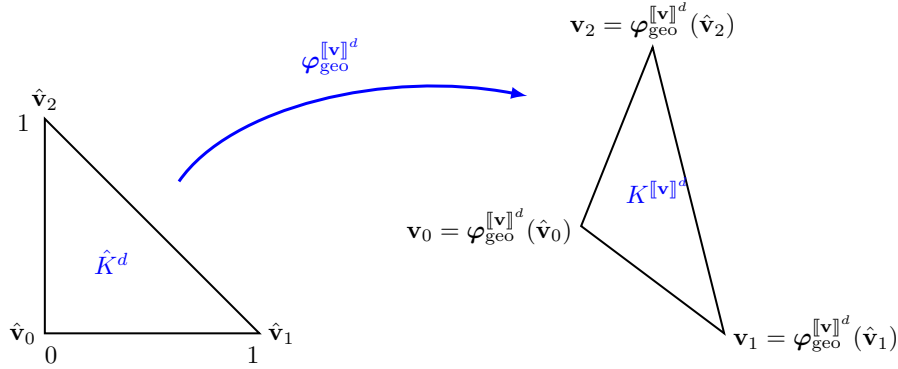
Then,  $\varphi_{\text{geo}}^{\llbracket \mathbf{v} \rrbracket^d}$  for  $d \stackrel{\text{def.}}{=} 1$  and  $\varphi_{\text{geo}}^{\llbracket v \rrbracket^1}$  from Definition 1461 coincide.

*Proof.* Direct consequence of Definition 1547 (*geometric mapping, with  $d \stackrel{\text{def.}}{=} 1$* ), Definition 1541 (*reference Lagrange polynomials of  $\mathbb{P}_1^d$ , with  $d \stackrel{\text{def.}}{=} 1$* ), Definition 1461 (*geometric mapping in dimension 1*), and **commutative ring properties of  $\mathbb{R}$** .  $\square$

**Lemma 1549 (reference geometric mapping is identity).**

Let  $d \geq 1$ . Then, the geometric mapping for reference vertices  $\llbracket \hat{\mathbf{v}} \rrbracket^d$  is the identity,  $\varphi_{\text{geo}}^{\llbracket \hat{\mathbf{v}} \rrbracket^d} = \text{Id}_{\mathbb{R}^d}$ .

*Proof.* Direct consequence of Definition 1547 (*geometric mapping*), Definition 1541 (*reference Lagrange polynomials of  $\mathbb{P}_1^d$* ), and Definition 1434 (*family of reference points*).  $\square$

Figure 9.4: Geometric mapping  $\varphi_{\text{geo}}^{[\mathbf{v}]^d}$  in the case  $d = 2$ , see Definition 1547.

**Lemma 1550 (properties of geometric mapping).** Let  $d \geq 1$ .

Let  $[\mathbf{v}]^d$  be  $d + 1$  points in  $\mathbb{R}^d$ . Let  $A_{\text{geo}}^{[\mathbf{v}]^d} \stackrel{\text{def.}}{=} (\mathbf{v}_1 - \mathbf{v}_0 \quad \mathbf{v}_2 - \mathbf{v}_0 \quad \dots \quad \mathbf{v}_d - \mathbf{v}_0)$  in  $\mathcal{M}_{d,d}$ .

Then,  $\varphi_{\text{geo}}^{[\mathbf{v}]^d}$  belongs to  $\text{Aff}(\mathbb{R}^d, \mathbb{R}^d)$ , and we have

(9.37)  $\forall j \in [0..d], \quad \forall \hat{\mathbf{x}} \in \mathbb{R}^d, \quad \varphi_{\text{geo}}^{[\mathbf{v}]^d}(\hat{\mathbf{x}}) = \mathbf{v}_j + \sum_{i=0, i \neq j}^d \hat{\mathcal{L}}_i^{1,d}(\hat{\mathbf{x}})(\mathbf{v}_i - \mathbf{v}_j),$

(9.38)  $\forall \hat{\mathbf{x}} \in \mathbb{R}^d, \quad \varphi_{\text{geo}}^{[\mathbf{v}]^d}(\hat{\mathbf{x}}) = \mathbf{v}_0 + \sum_{i=1}^d \hat{x}_i(\mathbf{v}_i - \mathbf{v}_0) = \mathbf{v}_0 + A_{\text{geo}}^{[\mathbf{v}]^d} \hat{\mathbf{x}},$

(9.39)  $\forall i \in [0..d], \quad \varphi_{\text{geo}}^{[\mathbf{v}]^d}(\hat{\mathbf{v}}_i) = \mathbf{v}_i,$

(9.40)  $\varphi_{\text{geo}}^{[\mathbf{v}]^d}(\hat{K}^d) = K^{[\mathbf{v}]^d}.$

Moreover, if  $[\mathbf{v}]^d$  is affinely independent, then  $\varphi_{\text{geo}}^{[\mathbf{v}]^d}$  is bijective,  $(\varphi_{\text{geo}}^{[\mathbf{v}]^d})^{-1}$  is affine, and

(9.41)  $\forall \mathbf{x} \in \mathbb{R}^d, \quad (\varphi_{\text{geo}}^{[\mathbf{v}]^d})^{-1}(\mathbf{x}) = (A_{\text{geo}}^{[\mathbf{v}]^d})^{-1}(\mathbf{x} - \mathbf{v}_0),$

(9.42)  $\forall i \in [0..d], \quad (\varphi_{\text{geo}}^{[\mathbf{v}]^d})^{-1}(\mathbf{v}_i) = \hat{\mathbf{v}}_i.$

*Proof.* **Identities (9.37) and (9.38), affine.** Let  $j \in [0..d]$ . Then, from Definition 1547 (*geometric mapping*), Lemma 1543 (*reference Lagrange polynomials is basis of  $\mathbb{P}_1^d$* , (9.33), extracting  $j$ ), and **ring properties of  $\mathbb{R}$** , identity (9.37) holds. Thus, from (9.37) (with  $j \stackrel{\text{def.}}{=} 0$ ), Definition 1541 (*reference Lagrange polynomials of  $\mathbb{P}_1^d$* , (9.30)), **the columnwise rule for matrix-vector product, commutative ring properties of  $\mathbb{R}$** , and Definition 1388 (*affine map*, with  $\phi \stackrel{\text{def.}}{=} A_{\text{geo}}^{[\mathbf{v}]^d}$  and  $\mathbf{y}_0 \stackrel{\text{def.}}{=} \mathbf{v}_0$ ), identities (9.38) hold, and  $\varphi_{\text{geo}}^{[\mathbf{v}]^d}$  is affine.

**Identity (9.39).** Direct consequence of Definition 1547 (*geometric mapping*), and Lemma 1543 (*reference Lagrange polynomials is basis of  $\mathbb{P}_1^d$* , (9.32)).

**Identity (9.40).**

$\varphi_{\text{geo}}^{[\mathbf{v}]^d}(\hat{K}^d) \subset K^{[\mathbf{v}]^d}$ . Let  $\hat{\mathbf{x}} \in \hat{K}^d$ . Then, from Definition 1541 (*reference Lagrange polynomials of  $\mathbb{P}_1^d$* ), Definition 1437 (*reference simplex*), **ordered field properties of  $\mathbb{R}$** , Lemma 1543 (*reference Lagrange polynomials is basis of  $\mathbb{P}_1^d$* , (9.33)), Definition 1547 (*geometric mapping*), and Definition 1440 (*simplex*), we have, for all  $i \in [1..d]$ ,  $\hat{\mathcal{L}}_i^{1,d}(\hat{\mathbf{x}}) = \hat{x}_i \geq 0$ , and  $\hat{\mathcal{L}}_0^{1,d}(\hat{\mathbf{x}}) = 1 - \sum_{i=1}^d \hat{\mathcal{L}}_i^{1,d} \geq 0$ ,

thus  $\varphi_{\text{geo}}^{\llbracket \mathbf{v} \rrbracket^d}(\hat{\mathbf{x}})$  is a convex combination of the  $\mathbf{v}_i$ 's, and it belongs to  $K^{\llbracket \mathbf{v} \rrbracket^d}$ .

$K^{\llbracket \mathbf{v} \rrbracket^d} \subset \varphi_{\text{geo}}^{\llbracket \mathbf{v} \rrbracket^d}(\hat{K}^d)$ . Let  $\mathbf{x} \in K^{\llbracket \mathbf{v} \rrbracket^d}$ . Then, from Definition 1440 (*simplex*), there exists  $(\mu_i)_{i \in [0..d]} \in \mathbb{R}$  such that, for all  $i \in [0..d]$ ,  $\mu_i \geq 0$ ,  $\sum_{i=0}^d \mu_i = 1$  and  $\mathbf{x} = \sum_{i=0}^d \mu_i \mathbf{v}_i$ .

Let  $\hat{\mathbf{x}} \stackrel{\text{def.}}{=} (\mu_i)_{i \in [1..d]} \in \mathbb{R}^d$ . Then, from Definition 1541 (*reference Lagrange polynomials of  $\mathbb{P}_1^d$* ), **ordered field properties of  $\mathbb{R}$** , Definition 1437 (*reference simplex*), and Definition 1547 (*geometric mapping*), we have, for all  $i \in [1..d]$ ,  $\hat{\mathcal{L}}_i^{1,d}(\hat{\mathbf{x}}) = \mu_i \geq 0$  and  $\hat{\mathcal{L}}_0^{1,d}(\hat{\mathbf{x}}) = 1 - \sum_{j=1}^d \mu_j = \mu_0 \geq 0$ , thus  $\hat{\mathbf{x}} \in \hat{K}^d$  and  $\varphi_{\text{geo}}^{\llbracket \mathbf{v} \rrbracket^d}(\hat{\mathbf{x}}) = \mathbf{x}$ , i.e.  $\mathbf{x} \in \varphi_{\text{geo}}^{\llbracket \mathbf{v} \rrbracket^d}(\hat{K}^d)$ .

Therefore, we have equality.

**Bijection,  $(\varphi_{\text{geo}}^{\llbracket \mathbf{v} \rrbracket^d})^{-1}$  affine, and identity (9.41).** Direct consequence of (9.38), Lemma 1406 (*injective affine map is zero linear kernel*), Definition 101 (*kernel*), Definition 1411 (*affinely independent family*), Lemma 1374 (*injectivity or surjectivity and dimension implies bijectivity*), and Lemma 1410 (*inverse of affine map is affine map*).

**Identity (9.42).** Direct consequence of (9.39).  $\square$

**Lemma 1551 (*differential of geometric mapping*).**

Let  $d \geq 1$ . Let  $\llbracket \mathbf{v} \rrbracket^d$  be  $d+1$  points in  $\mathbb{R}^d$ . Then,  $\varphi_{\text{geo}}^{\llbracket \mathbf{v} \rrbracket^d}$  belongs to  $\mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$ , and we have

$$(9.43) \quad \forall \hat{\mathbf{x}} \in \mathbb{R}^d, \quad D\varphi_{\text{geo}}^{\llbracket \mathbf{v} \rrbracket^d}(\hat{\mathbf{x}}) = A_{\text{geo}}^{\llbracket \mathbf{v} \rrbracket^d}.$$

Moreover, if  $\llbracket \mathbf{v} \rrbracket^d$  is affinely independent,  $(\varphi_{\text{geo}}^{\llbracket \mathbf{v} \rrbracket^d})^{-1}$  also belongs to  $\mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$ , and we have

$$(9.44) \quad \forall \mathbf{x} \in \mathbb{R}^d, \quad D\left(\varphi_{\text{geo}}^{\llbracket \mathbf{v} \rrbracket^d}\right)^{-1}(\mathbf{x}) = \left(A_{\text{geo}}^{\llbracket \mathbf{v} \rrbracket^d}\right)^{-1}.$$

*Proof.* Direct consequence of Lemma 1550 (*properties of geometric mapping*), **affine maps in  $\mathbb{R}^d$  are  $\mathcal{C}^\infty$** , and **the rules of derivation (thus affine maps have constant differential and the differential is the linear part)**.  $\square$

### 9.3.3 Current affine Lagrange polynomials

**Lemma 1552 (*Lagrange polynomials of  $\mathbb{P}_1^d$* ).**

Let  $d \geq 1$ . Let  $\llbracket \mathbf{v} \rrbracket^d$  be  $d+1$  affinely independent points in  $\mathbb{R}^d$ . Let  $i \in [0..d]$ .

Then,  $\mathcal{L}_i^{\llbracket \mathbf{v} \rrbracket^d} \stackrel{\text{def.}}{=} \hat{\mathcal{L}}_i^{1,d} \circ (\varphi_{\text{geo}}^{\llbracket \mathbf{v} \rrbracket^d})^{-1}$  is well-defined, and we have  $\hat{\mathcal{L}}_i^{1,d} = \mathcal{L}_i^{\llbracket \mathbf{v} \rrbracket^d} \circ \varphi_{\text{geo}}^{\llbracket \mathbf{v} \rrbracket^d}$ .

$\mathcal{L}_i^{\llbracket \mathbf{v} \rrbracket^d}$  is called the  $i$ -th Lagrange polynomial of  $\mathbb{P}_1^d$  associated with points  $\llbracket \mathbf{v} \rrbracket^d$ .

*Proof. Existence.* Direct consequence of Lemma 1550 (*properties of geometric mapping*).

**Property.** Direct consequence of **associativity of composition of functions**, and **the definition of the inverse**.  $\square$

**Lemma 1553 (*Lagrange polynomials of reference vertices are reference Lagrange polynomials of  $\mathbb{P}_1^d$* ).** Let  $d \geq 1$ . Let  $i \in [0..d]$ . Then, we have  $\mathcal{L}_i^{\llbracket \hat{\mathbf{v}} \rrbracket^d} = \hat{\mathcal{L}}_i^{1,d}$ .

*Proof.* Direct consequence of Lemma 1552 (*Lagrange polynomials of  $\mathbb{P}_1^d$* ), and Lemma 1549 (*reference geometric mapping is identity*).  $\square$

**Lemma 1554 (Lagrange polynomials is basis of  $\mathbb{P}_1^d$ ).**

Let  $d \geq 1$ . Let  $[\mathbf{v}]^d$  be  $d + 1$  affinely independent points in  $\mathbb{R}^d$ .

Then, the Lagrange polynomials  $(\mathcal{L}_i^{[\mathbf{v}]^d})_{i \in [0..d]}$  form a basis of  $\mathbb{P}_1^d$ , that satisfies

$$(9.45) \quad \forall i, j \in [0..d], \quad \mathcal{L}_i^{[\mathbf{v}]^d}(\mathbf{v}_j) = \delta_{ij},$$

$$(9.46) \quad \sum_{i=0}^d \mathcal{L}_i^{[\mathbf{v}]^d} = 1.$$

*Proof.* In  $\mathbb{P}_1^d$ . Direct consequence of Lemma 1552 (Lagrange polynomials of  $\mathbb{P}_1^d$ ), Lemma 1509 (constant and affine spaces  $\mathbb{P}_0^d$  and  $\mathbb{P}_1^d$ , thus  $\mathbb{P}_1^d = \mathcal{Aff}(\mathbb{R}^d, \mathbb{R})$ ), Lemma 1543 (reference Lagrange polynomials is basis of  $\mathbb{P}_1^d$ , thus  $\hat{\mathcal{L}}_i^{1,d}$  is affine), Lemma 1550 (properties of geometric mapping, thus  $(\varphi_{\text{geo}}^{[\mathbf{v}]^d})^{-1}$  is affine), and Lemma 1403 (affine maps are closed by composition).

**Identities (9.45) and (9.46).**

Direct consequence of Lemma 1552 (Lagrange polynomials of  $\mathbb{P}_1^d$ ), Lemma 1550 (properties of geometric mapping, (9.42)), Lemma 1543 (reference Lagrange polynomials is basis of  $\mathbb{P}_1^d$ , (9.32) and (9.33)), and **distributivity of composition over addition**.

**Basis.** Let  $(\lambda_i)_{i \in [0..d]} \in \mathbb{R}$ , such that for all  $\mathbf{x} \in \mathbb{R}^d$ ,  $\sum_{i=0}^d \lambda_i \mathcal{L}_i^{[\mathbf{v}]^d}(\mathbf{x}) = 0$ . Then, from (9.45), and **ring properties of  $\mathbb{R}$** , taking  $\mathbf{x} \stackrel{\text{def.}}{=} \mathbf{v}_j$  provides  $\sum_{i=0}^d \lambda_i \delta_{ij} = \lambda_j = 0$ . Thus, from **the definition of freedom**, Lemma 1525 (dimension of  $\mathbb{P}_k^d$ , with  $k \stackrel{\text{def.}}{=} 1$ ), and Lemma 1376 (free family of dim elements is basis), the Lagrange polynomials form a basis of  $\mathbb{P}_1^d$ .  $\square$

**Lemma 1555 (decomposition of  $\mathbb{P}_1^d$  polynomial in Lagrange basis).**

Let  $d \geq 1$ . Let  $[\mathbf{v}]^d$  be  $d + 1$  affinely independent points in  $\mathbb{R}^d$ . Let  $p \in \mathbb{P}_1^d$ . Then, we have

$$(9.47) \quad p = \sum_{i=0}^d p(\mathbf{v}_i) \mathcal{L}_i^{[\mathbf{v}]^d}.$$

*Proof.* Direct consequence of Lemma 1554 (Lagrange polynomials is basis of  $\mathbb{P}_1^d$ , basis, then (9.45)), and **ring properties of  $\mathbb{R}$** .  $\square$

**Lemma 1556 (differential of Lagrange polynomials  $\mathbb{P}_1^d$ ).**

Let  $d \geq 1$ . Let  $[\mathbf{v}]^d$  be  $d + 1$  affinely independent points in  $\mathbb{R}^d$ . Let  $\mathbf{x} \in \mathbb{R}^d$ . Let  $i \in [0..d]$ .

Then, we have  $\mathcal{L}_i^{[\mathbf{v}]^d} \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})$  and

$$(9.48) \quad (i = 0) \quad D\mathcal{L}_0^{[\mathbf{v}]^d}(\mathbf{x}) = - \sum_{j=1}^d \frac{(A_{\text{geo}}^{[\mathbf{v}]^d})^{-1}}{j} \in \mathcal{M}_{1,d}(\mathbb{R}),$$

$$(9.49) \quad (i \in [1..d]) \quad D\mathcal{L}_i^{[\mathbf{v}]^d}(\mathbf{x}) = \frac{(A_{\text{geo}}^{[\mathbf{v}]^d})^{-1}}{i} \in \mathcal{M}_{1,d}(\mathbb{R}),$$

where, we recall that for any matrix  $B$ ,  $\underline{B}_i$  denotes its  $i$ -th line.

*Proof.* Direct consequence of Lemma 1554 (Lagrange polynomials is basis of  $\mathbb{P}_1^d$ ), Lemma 1509 (constant and affine spaces  $\mathbb{P}_0^d$  and  $\mathbb{P}_1^d$ , thus Lagrange polynomials are affine maps), **affine maps in  $\mathbb{R}^d$  are  $\mathcal{C}^\infty$** , Lemma 1545 (differential of reference Lagrange polynomials), Lemma 1551 (differential of geometric mapping), and **the rules derivation for composition**.  $\square$



### 9.3.4 Nontrivial current simplex

**Lemma 1557 (nontrivial simplex).** *Let  $d \geq 1$ . Let  $[\mathbf{v}]^d$  be  $d + 1$  affinely independent points in  $\mathbb{R}^d$ . Then,  $K^{[\mathbf{v}]^d}$  has a nonempty interior. It is then said to be nontrivial.*

*Proof.* From Lemma 1439 (nontrivial reference simplex), and the definition of the interior, let  $\hat{U}$  be a nonempty open set included in  $\hat{K}^d$ . Then, from Lemma 1551 (differential of geometric mapping),  $\mathcal{C}^\infty$  is continuous (thus  $\varphi_{\text{geo}}^{[\mathbf{v}]^d}$  is an homeomorphism), and homeomorphisms are open maps,  $U \stackrel{\text{def.}}{=} \varphi_{\text{geo}}^{[\mathbf{v}]^d}(\hat{U})$  is open. Therefore, from image of  $\emptyset$  is  $\emptyset$  (contrapositive), image is nondecreasing, Lemma 1550 (properties of geometric mapping, (9.40)), and the definition of the interior, we have  $\emptyset \neq U \subset K^{[\mathbf{v}]^d}$ , and  $K^{[\mathbf{v}]^d}$  has a nonempty interior.  $\square$

## 9.4 Barycentric coordinates

*Remark 1558.* In the next lemma, the uniqueness of the barycentric coordinates actually makes them functions of the point  $\mathbf{x}$ . In particular, in (9.51), they do not depend on the choice for  $j$ .

**Lemma 1559 (barycentric coordinate).**

*Let  $d \geq 1$ . Let  $[\mathbf{v}]^d$  be  $d + 1$  affinely independent points in  $\mathbb{R}^d$ . Let  $\mathbf{x} \in \mathbb{R}^d$ .*

*Then, there exists unique  $(\lambda_i^{[\mathbf{v}]^d})_{i \in [0..d]} \in \mathbb{R}$  satisfying the next two equivalent decompositions*

$$(9.50) \quad \mathbf{x} = \sum_{i=0}^d \lambda_i^{[\mathbf{v}]^d} \mathbf{v}_i \quad \text{with} \quad \sum_{i=0}^d \lambda_i^{[\mathbf{v}]^d} = 1,$$

$$(9.51) \quad \forall j \in [0..d], \quad \mathbf{x} = \mathbf{v}_j + \sum_{i=0, i \neq j}^d \lambda_i^{[\mathbf{v}]^d} (\mathbf{v}_i - \mathbf{v}_j) \quad \text{with} \quad \lambda_j^{[\mathbf{v}]^d} = 1 - \sum_{i=0, i \neq j}^d \lambda_i^{[\mathbf{v}]^d}.$$

*The  $\lambda_i^{[\mathbf{v}]^d}$ 's are called barycentric coordinates of  $\mathbf{x}$  with respect to  $[\mathbf{v}]^d$ , they are functions of  $\mathbf{x}$ .*

*Proof. Existence.* Let  $j \in [0..d]$ . Then, from Lemma 81 (closed under vector operations is subspace, thus  $\mathbf{x} - \mathbf{v}_j$  belongs to the vector space  $\mathbb{R}^d$ ), Lemma 1412 (equivalent definition of affinely independent family), Lemma 1376 (free family of dim elements is basis), and the definition of basis, there exist unique real numbers  $(\mu_i^j)_{i \in [0..d] \setminus \{j\}}$  such that  $\mathbf{x} - \mathbf{v}_j = \sum_{i=0, i \neq j}^d \mu_i^j (\mathbf{v}_i - \mathbf{v}_j)$ , i.e. the decomposition (9.51) exists, and is unique (but still depends on  $j$ ).

Let  $\mu_j^j \stackrel{\text{def.}}{=} 1 - \sum_{i=0, i \neq j}^d \mu_i^j$ . Then, from commutative ring properties of  $\mathbb{R}$ , we have  $\mathbf{x} = \sum_{i=0}^d \mu_i^j \mathbf{v}_i$  and  $\sum_{i=0}^d \mu_i^j = 1$ , i.e. the decomposition (9.50) exists.

**Uniqueness.** Let  $(\lambda_i)_{i \in [0..d]}, (\mu_i)_{i \in [0..d]} \in \mathbb{R}$ , such that

$$\mathbf{x} = \sum_{i=0}^d \lambda_i \mathbf{v}_i = \sum_{i=0}^d \mu_i \mathbf{v}_i \quad \text{and} \quad \sum_{i=0}^d \lambda_i = \sum_{i=0}^d \mu_i = 1.$$

For all  $i \in [0..d]$ , let  $\xi_i \stackrel{\text{def.}}{=} \lambda_i - \mu_i$ . Then, from ring properties of  $\mathbb{R}$ , we have successively  $\sum_{i=0}^d \xi_i \mathbf{v}_i = 0$ ,  $\sum_{i=0}^d \xi_i = 0$ ,  $\xi_0 = -\sum_{i=1}^d \xi_i$ , and  $\sum_{i=1}^d \xi_i (\mathbf{v}_i - \mathbf{v}_0) = 0$ . Thus, from Definition 1411 (affinely independent family), the definition of freedom, and additive group properties of  $\mathbb{R}$ , we have, for all  $i \in [1..d]$ ,  $\xi_i = 0$ , and  $\xi_0 = 0$ . Therefore, the decomposition (9.50) is unique.

**Equivalence and independence on  $j$ .**

Let  $j \in [0..d]$ . Then, from (9.50), and ring properties of  $\mathbb{R}$ , we have

$$\mathbf{x} = \left( 1 - \sum_{i=0, i \neq j}^d \lambda_i^{[\mathbf{v}]^d} \right) \mathbf{v}_j + \sum_{i=0, i \neq j}^d \lambda_i^{[\mathbf{v}]^d} \mathbf{v}_i,$$

and thus  $\mathbf{x} - \mathbf{v}_j = \sum_{i=0, i \neq j}^d \lambda_i^{[\mathbf{v}]^d} (\mathbf{v}_i - \mathbf{v}_j)$ , i.e. equivalence of the two decompositions. Moreover, from the uniqueness of the  $\mu_i^j$ 's defined above, we have, for all  $i \in [0..d] \setminus \{j\}$ ,  $\mu_i^j = \lambda_i^{[\mathbf{v}]^d}$ , i.e. the decomposition (9.51) is independent on  $j$ .  $\square$

**Lemma 1560 (Lagrange polynomials of  $\mathbb{P}_1^d$  are barycentric coordinate).**

Let  $d \geq 1$ . Let  $[\mathbf{v}]^d$  be  $d + 1$  affinely independent points in  $\mathbb{R}^d$ .

Then, barycentric coordinates and Lagrange polynomials coincide, for all  $i \in [0..d]$ ,  $\lambda_i^{[\mathbf{v}]^d} = \mathcal{L}_i^{[\mathbf{v}]^d}$ . Thus, we have, for all  $\mathbf{x} \in \mathbb{R}^d$ ,  $\mathbf{x} = \sum_{i=0}^d \mathcal{L}_i^{[\mathbf{v}]^d}(\mathbf{x}) \mathbf{v}_i$  with  $\sum_{i=0}^d \mathcal{L}_i^{[\mathbf{v}]^d}(\mathbf{x}) = 1$ .

Moreover, let  $\boldsymbol{\lambda}^{[\mathbf{v}]^d} \stackrel{\text{def.}}{=} (\lambda_i^{[\mathbf{v}]^d})_{i \in [1..d]}$ . Then, we have  $\boldsymbol{\lambda}^{[\mathbf{v}]^d} = (\varphi_{\text{geo}}^{[\mathbf{v}]^d})^{-1} \in \text{Aff}(\mathbb{R}^d, \mathbb{R}^d)$ , i.e.

$$(9.52) \quad \forall \mathbf{x} \in \mathbb{R}^d, \quad \mathbf{x} = \varphi_{\text{geo}}^{[\mathbf{v}]^d}(\boldsymbol{\lambda}^{[\mathbf{v}]^d}(\mathbf{x})), \quad \text{or equivalently} \quad \forall \hat{\mathbf{x}} \in \mathbb{R}^d, \quad \hat{\mathbf{x}} = \boldsymbol{\lambda}^{[\mathbf{v}]^d}(\varphi_{\text{geo}}^{[\mathbf{v}]^d}(\hat{\mathbf{x}})).$$

*Proof.* (i) Equality  $\boldsymbol{\lambda}^{[\mathbf{v}]^d} = (\varphi_{\text{geo}}^{[\mathbf{v}]^d})^{-1}$ . Let  $\mathbf{x} \in \mathbb{R}^d$ . Then, from Lemma 1559 (barycentric coordinate, (9.51) with  $j \stackrel{\text{def.}}{=} 0$ ), additive group properties of  $\mathbb{R}$ , the columnwise rule for matrix-vector product, the rules of matrix-vector product (inverse matrix), and Lemma 1550 (properties of geometric mapping), we have

$$\begin{aligned} \mathbf{x} - \mathbf{v}_0 &= \sum_{i=1}^d \lambda_i^{[\mathbf{v}]^d}(\mathbf{x})(\mathbf{v}_i - \mathbf{v}_0) = A_{\text{geo}}^{[\mathbf{v}]^d} \boldsymbol{\lambda}^{[\mathbf{v}]^d}(\mathbf{x}), \\ \boldsymbol{\lambda}^{[\mathbf{v}]^d}(\mathbf{x}) &= \left(A_{\text{geo}}^{[\mathbf{v}]^d}\right)^{-1}(\mathbf{x} - \mathbf{v}_0) = \left(\varphi_{\text{geo}}^{[\mathbf{v}]^d}\right)^{-1}(\mathbf{x}), \end{aligned}$$

and thus  $\boldsymbol{\lambda}^{[\mathbf{v}]^d} = \left(\varphi_{\text{geo}}^{[\mathbf{v}]^d}\right)^{-1} \in \text{Aff}(\mathbb{R}^d, \mathbb{R}^d)$ .

(ii) Identities (9.52). Direct consequence of (i), and the definition of the inverse.

(iii) Identity  $\lambda_i^{[\mathbf{v}]^d} = \mathcal{L}_i^{[\mathbf{v}]^d}$ .

Let  $\mathbf{x} \in \mathbb{R}^d$ . Then, from Lemma 1559 (barycentric coordinate, (9.50), then uniqueness), (ii), Definition 1547 (geometric mapping), (i), and Lemma 1552 (Lagrange polynomials of  $\mathbb{P}_1^d$ ), we have

$$\begin{aligned} \sum_{i=0}^d \lambda_i^{[\mathbf{v}]^d}(\mathbf{x}) \mathbf{v}_i &= \mathbf{x} = \varphi_{\text{geo}}^{[\mathbf{v}]^d}(\boldsymbol{\lambda}^{[\mathbf{v}]^d}(\mathbf{x})) \\ &= \sum_{i=0}^d \hat{\mathcal{L}}_i^{1,d}(\boldsymbol{\lambda}^{[\mathbf{v}]^d}(\mathbf{x})) \mathbf{v}_i = \sum_{i=0}^d \hat{\mathcal{L}}_i^{1,d} \left( \left(\varphi_{\text{geo}}^{[\mathbf{v}]^d}\right)^{-1}(\mathbf{x}) \right) \mathbf{v}_i = \sum_{i=0}^d \mathcal{L}_i^{[\mathbf{v}]^d}(\mathbf{x}) \mathbf{v}_i, \end{aligned}$$

and thus, for all  $i \in [0..d]$ ,  $\lambda_i^{[\mathbf{v}]^d} = \mathcal{L}_i^{[\mathbf{v}]^d}$ .  $\square$

**Lemma 1561 (decomposition of  $\mathbb{P}_1^d$  polynomial with barycentric coordinates).**

Let  $d \geq 1$ . Let  $[\mathbf{v}]^d$  be  $d + 1$  affinely independent points in  $\mathbb{R}^d$ . Let  $p \in \mathbb{P}_1^d$ . Then, we have

$$(9.53) \quad p = \sum_{i=0}^d p(\mathbf{v}_i) \lambda_i^{[\mathbf{v}]^d}.$$

*Proof.* Direct consequence of Lemma 1555 (decomposition of  $\mathbb{P}_1^d$  polynomial in Lagrange basis), and Lemma 1560 (Lagrange polynomials of  $\mathbb{P}_1^d$  are barycentric coordinate).  $\square$

## 9.5 Hyperplanes and $l$ -faces

### 9.5.1 Hyperplanes and $d - 1$ -faces

**Definition 1562 (face hyperplane).**

Let  $d \geq 1$ . Let  $[\mathbf{v}]^d$  be  $d + 1$  points in  $\mathbb{R}^d$ . Let  $i \in [0..d]$ . The  $i$ -th face hyperplane opposite vertex  $\mathbf{v}_i$  is denoted  $\mathcal{H}_i^{[\mathbf{v}]^d}$ , and is defined as the affine subspace of  $\mathbb{R}^d$

$$(9.54) \quad (i = 0) \quad \mathcal{H}_0^{[\mathbf{v}]^d} \stackrel{\text{def.}}{=} \mathbf{v}_1 + \text{span}(\mathbf{v}_j - \mathbf{v}_1)_{j \in [2..d]},$$

$$(9.55) \quad (i \in [1..d]) \quad \mathcal{H}_i^{[\mathbf{v}]^d} \stackrel{\text{def.}}{=} \mathbf{v}_0 + \text{span}(\mathbf{v}_j - \mathbf{v}_0)_{j \in [1..d] \setminus \{i\}}.$$

For the reference vertices  $[\hat{\mathbf{v}}]^d$ , the  $i$ -th reference face hyperplane is denoted  $\hat{\mathcal{H}}_i^d \stackrel{\text{def.}}{=} \mathcal{H}_i^{[\hat{\mathbf{v}}]^d}$ .

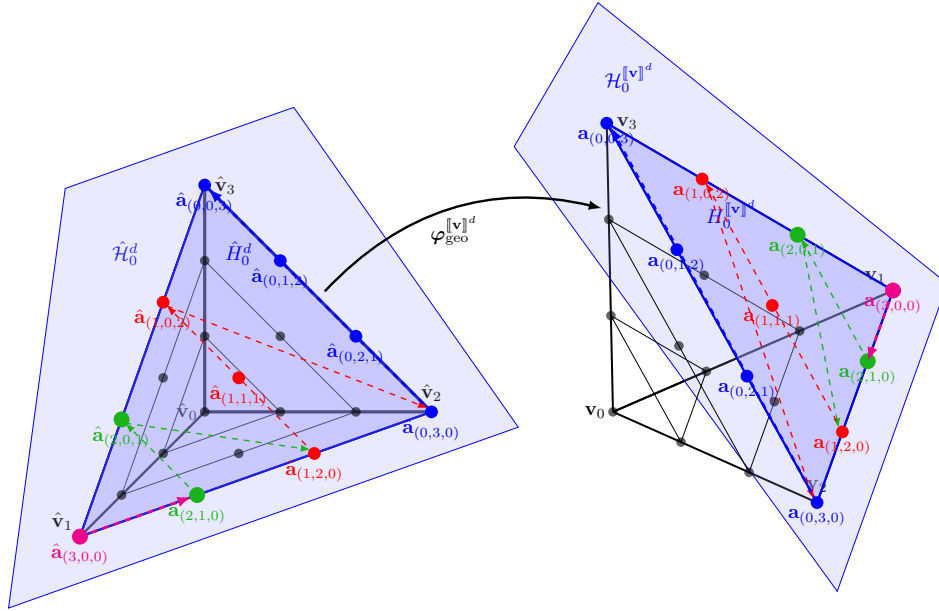


Figure 9.5: Geometric transformation  $\varphi_{\text{geo}}^{[\mathbf{v}]^d}$  in the case  $d = k = 3$ .

The reference simplex  $\hat{K}^d$  is mapped to the current simplex  $K^{[\mathbf{v}]^d}$ , and reference nodes  $\hat{\mathbf{a}}_\alpha$  onto current nodes  $\mathbf{a}_\alpha$ , see Lemmas 1550 and 1604. For instance, we show that the reference hyperplane  $\hat{\mathcal{H}}_0^d$  containing the face  $\hat{H}_0^d$  is mapped onto  $\mathcal{H}_0^{[\mathbf{v}]^d}$  containing the face  $H_0^{[\mathbf{v}]^d}$ . The nodes in these two faces are colored in order to help see the correspondence: for all  $\alpha \in C_3^3$ , we have  $\mathbf{a}_\alpha = \varphi_{\text{geo}}^{[\mathbf{v}]^d}(\hat{\mathbf{a}}_\alpha)$ .

**Lemma 1563 (equivalent definition of face hyperplane).** Let  $d \geq 1$ . Let  $[\mathbf{v}]^d$  be  $d + 1$  points in  $\mathbb{R}^d$ . Let  $i \in [0..d]$ . Then, the  $i$ -th face hyperplane is characterized by

$$(9.56) \quad \mathcal{H}_i^{[\mathbf{v}]^d} = \left\{ \sum_{j=0, j \neq i}^d \mu_j \mathbf{v}_j \in \mathbb{R}^d \mid \forall j \in [0..d] \setminus \{i\}, \mu_j \in \mathbb{R} \wedge \sum_{j=0, j \neq i}^d \mu_j = 1 \right\}.$$

Moreover, if  $[\mathbf{v}]^d$  is affinely independent, then we have

$$(9.57) \quad \mathcal{H}_i^{[\mathbf{v}]^d} = \ker(\lambda_i^{[\mathbf{v}]^d}),$$

where  $\lambda_i^{[\mathbf{v}]^d}$  is the  $i$ -th barycentric coordinate application.

*Proof. Identity (9.56).* Direct consequence of Lemma 1387 (*barycenter closure is affine subspace*, with  $n \stackrel{\text{def.}}{=} d - 1 \geq 0$ , and  $\mathbf{v}_j$  for  $j \in [0..d] \setminus \{i\}$ ).

**Identity (9.57). Case  $i = 0$ .** Then, from Definition 1562 (*face hyperplane*), **ring properties of  $\mathbb{R}$** , Lemma 1559 (*barycentric coordinate*, (9.51) with  $j \stackrel{\text{def.}}{=} 1$ ), we have

$$\begin{aligned} \mathbf{x} \in \mathcal{H}_0^{\llbracket \mathbf{v} \rrbracket^d} &\iff \exists (\tilde{\mu}_j)_{j \in [2..d]} \in \mathbb{R}, \quad \mathbf{x} = \mathbf{v}_1 + 0(\mathbf{v}_0 - \mathbf{v}_1) + \sum_{j=2}^d \tilde{\mu}_j (\mathbf{v}_j - \mathbf{v}_1) \\ &\iff \lambda_0^{\llbracket \mathbf{v} \rrbracket^d}(\mathbf{x}) = 0. \end{aligned}$$

**Case  $i \in [1..d]$ .** Then, from Definition 1562 (*face hyperplane*), **ring properties of  $\mathbb{R}$** , Lemma 1559 (*barycentric coordinate*, (9.51) with  $j \stackrel{\text{def.}}{=} 0$ ), we have

$$\begin{aligned} \mathbf{x} \in \mathcal{H}_i^{\llbracket \mathbf{v} \rrbracket^d} &\iff \exists (\tilde{\mu}_j)_{j \in [1..d] \setminus \{i\}} \in \mathbb{R}, \quad \mathbf{x} = \mathbf{v}_0 + 0(\mathbf{v}_i - \mathbf{v}_0) + \sum_{j=1, j \neq i}^d \tilde{\mu}_j (\mathbf{v}_j - \mathbf{v}_0) \\ &\iff \lambda_i^{\llbracket \mathbf{v} \rrbracket^d}(\mathbf{x}) = 0. \end{aligned}$$

Therefore, from Definition 101 (*kernel*), the identity always holds.  $\square$

**Lemma 1564 (reference face hyperplane).** *Let  $d \geq 1$ . Let  $i \in [0..d]$ . Then, we have*

$$(9.58) \quad (i = 0) \quad \hat{\mathcal{H}}_0^d = \left\{ \hat{\mathbf{x}} \in \mathbb{R}^d \mid \sum_{j=1}^d \hat{x}_j = 1 \right\} = \ker \left( \hat{\mathcal{L}}_0^{1,d} \right),$$

$$(9.59) \quad (i \in [1..d]) \quad \hat{\mathcal{H}}_i^d = \{ \hat{\mathbf{x}} \in \mathbb{R}^d \mid \hat{x}_i = 0 \} = \ker \left( \hat{\mathcal{L}}_i^{1,d} \right).$$

*Proof.* Direct consequence of Lemma 1436 (*family of reference points is affinely independent*), Lemma 1563 (*equivalent definition of face hyperplane*, (9.57)), Lemma 1560 (*Lagrange polynomials of  $\mathbb{P}_1^d$  are barycentric coordinate*), Lemma 1553 (*Lagrange polynomials of reference vertices are reference Lagrange polynomials of  $\mathbb{P}_1^d$* ), Definition 1541 (*reference Lagrange polynomials of  $\mathbb{P}_1^d$* ), and Definition 101 (*kernel*).  $\square$

**Lemma 1565 (face hyperplane is image of reference face hyperplane).**

*Let  $d \geq 1$ . Let  $\llbracket \mathbf{v} \rrbracket^d$  be  $d + 1$  affinely independent points in  $\mathbb{R}^d$ . Let  $i \in [0..d]$ . Then, we have*

$$(9.60) \quad \mathcal{H}_i^{\llbracket \mathbf{v} \rrbracket^d} = \varphi_{\text{geo}}^{\llbracket \mathbf{v} \rrbracket^d} \left( \hat{\mathcal{H}}_i^d \right).$$

*Proof. Case  $i = 0$ .* Then, from Definition 1562 (*face hyperplane*), **ring properties of  $\mathbb{R}$** , Definition 1541 (*reference Lagrange polynomials of  $\mathbb{P}_1^d$* , thus  $\hat{\mathcal{L}}_j^{1,d}(\hat{\boldsymbol{\mu}}) = \mu_j$  for all  $j \in [2..d]$ ,  $\hat{\mathcal{L}}_1^{1,d}(\hat{\boldsymbol{\mu}})$  equals  $(1 - \sum_{j=2}^d \mu_j)$  and  $\hat{\mathcal{L}}_0^{1,d}(\hat{\boldsymbol{\mu}}) = 0$ ), Definition 1547 (*geometric mapping*), and Definition 1434

(family of reference points), we have

$$\begin{aligned}
\mathbf{x} \in \mathcal{H}_0^{\llbracket \mathbf{v} \rrbracket^d} &\iff \exists (\mu_j)_{j \in [2..d]} \in \mathbb{R}, \quad \mathbf{x} = \mathbf{v}_1 + \sum_{j=2}^d \mu_j (\mathbf{v}_j - \mathbf{v}_1) \\
&\iff \exists (\mu_j)_{j \in [2..d]} \in \mathbb{R}, \quad \mathbf{x} = 0\mathbf{v}_0 + \left(1 - \sum_{j=2}^d \mu_j\right) \mathbf{v}_1 + \sum_{j=2}^d \mu_j \mathbf{v}_j \\
&\iff \exists (\mu_j)_{j \in [2..d]} \in \mathbb{R}, \quad \mu_1 \stackrel{\text{def.}}{=} 1 - \sum_{j=2}^d \mu_j \wedge \mathbf{x} = \left(1 - \sum_{j=1}^d \mu_j\right) \mathbf{v}_0 + \sum_{j=1}^d \mu_j \mathbf{v}_j \\
&\iff \exists (\mu_j)_{j \in [2..d]} \in \mathbb{R}, \quad \hat{\boldsymbol{\mu}} \stackrel{\text{def.}}{=} \left(1 - \sum_{j=2}^d \mu_j\right) \mathbf{e}_1 + \sum_{j=2}^d \mu_j \mathbf{e}_j \in \mathbb{R}^d \wedge \mathbf{x} = \varphi_{\text{geo}}^{\llbracket \mathbf{v} \rrbracket^d}(\hat{\boldsymbol{\mu}}) \\
&\iff \exists (\mu_j)_{j \in [2..d]} \in \mathbb{R}, \quad \hat{\boldsymbol{\mu}} \stackrel{\text{def.}}{=} \hat{\mathbf{v}}_1 + \sum_{j=2}^d \mu_j (\hat{\mathbf{v}}_j - \hat{\mathbf{v}}_1) \in \hat{\mathcal{H}}_0^d \wedge \mathbf{x} = \varphi_{\text{geo}}^{\llbracket \mathbf{v} \rrbracket^d}(\hat{\boldsymbol{\mu}}).
\end{aligned}$$

Therefore, we have the equality.

**Case  $i \in [1..d]$ .** Then, from Definition 1562 (*face hyperplane*), **ring properties of  $\mathbb{R}$** , Definition 1541 (*reference Lagrange polynomials of  $\mathbb{P}_1^d$* ), thus  $\hat{\mathcal{L}}_j^{1,d}(\hat{\boldsymbol{\mu}}) = \mu_j$  for all  $j \in [1..d] \setminus \{i\}$ ,  $\hat{\mathcal{L}}_0^{1,d}(\hat{\boldsymbol{\mu}}) = (1 - \sum_{l=1, l \neq i}^d \mu_l)$  and  $\hat{\mathcal{L}}_i^{1,d}(\hat{\boldsymbol{\mu}}) = 0$ , Definition 1547 (*geometric mapping*), and Definition 1434 (*family of reference points*), we have

$$\begin{aligned}
\mathbf{x} \in \mathcal{H}_i^{\llbracket \mathbf{v} \rrbracket^d} &\iff \exists (\mu_j)_{j \in [1..d] \setminus \{i\}} \in \mathbb{R}, \quad \mathbf{x} = \mathbf{v}_0 + \sum_{j=1, j \neq i}^d \mu_j (\mathbf{v}_j - \mathbf{v}_0) \\
&\iff \exists (\mu_j)_{j \in [1..d] \setminus \{i\}} \in \mathbb{R}, \quad \mathbf{x} = \left(1 - \sum_{j=1, j \neq i}^d \mu_j\right) \mathbf{v}_0 + 0\mathbf{v}_i + \sum_{j=1, j \neq i}^d \mu_j \mathbf{v}_j \\
&\iff \exists (\mu_j)_{j \in [1..d] \setminus \{i\}} \in \mathbb{R}, \quad \mu_i \stackrel{\text{def.}}{=} 0 \wedge \mathbf{x} = \left(1 - \sum_{j=1}^d \mu_j\right) \mathbf{v}_0 + \sum_{j=1}^d \mu_j \mathbf{v}_j \\
&\iff \exists (\mu_j)_{j \in [1..d] \setminus \{i\}} \in \mathbb{R}, \quad \hat{\boldsymbol{\mu}} \stackrel{\text{def.}}{=} 0\mathbf{e}_i + \sum_{j=1, j \neq i}^d \mu_j \mathbf{e}_j \in \mathbb{R}^d \wedge \mathbf{x} = \varphi_{\text{geo}}^{\llbracket \mathbf{v} \rrbracket^d}(\hat{\boldsymbol{\mu}}) \\
&\iff \exists (\mu_j)_{j \in [1..d] \setminus \{i\}} \in \mathbb{R}, \quad \hat{\boldsymbol{\mu}} \stackrel{\text{def.}}{=} \hat{\mathbf{v}}_0 + \sum_{j=1, j \neq i}^d \mu_j (\hat{\mathbf{v}}_j - \hat{\mathbf{v}}_0) \in \hat{\mathcal{H}}_i^d \wedge \mathbf{x} = \varphi_{\text{geo}}^{\llbracket \mathbf{v} \rrbracket^d}(\hat{\boldsymbol{\mu}}).
\end{aligned}$$

Therefore, we have the equality. □

**Definition 1566 (*hyperface*).**

Let  $d \geq 1$ . Let  $\llbracket \mathbf{v} \rrbracket^d$  be  $d + 1$  points in  $\mathbb{R}^d$ . Let  $i \in [0..d]$ . The  $i$ -th *hyperface* opposite vertex  $\mathbf{v}_i$ , having the vertices  $(\mathbf{v}_j)_{j \in [0..d] \setminus \{i\}}$ , is denoted  $H_i^{\llbracket \mathbf{v} \rrbracket^d}$ , and is defined by

$$(9.61) \quad H_i^{\llbracket \mathbf{v} \rrbracket^d} \stackrel{\text{def.}}{=} \left\{ \sum_{j=0, j \neq i}^d \mu_j \mathbf{v}_j \in \mathbb{R}^d \mid \forall j \in [0..d] \setminus \{i\}, \quad \mu_j \geq 0 \wedge \sum_{j=0, j \neq i}^d \mu_j = 1 \right\}.$$

**Lemma 1567 (*hyperface is included in face hyperplane*).**

Let  $d \geq 1$ . Let  $\llbracket \mathbf{v} \rrbracket^d$  be  $d + 1$  points in  $\mathbb{R}^d$ . Let  $i \in [0..d]$ . Then, we have  $H_i^{\llbracket \mathbf{v} \rrbracket^d} \subset \mathcal{H}_i^{\llbracket \mathbf{v} \rrbracket^d}$ .

*Proof.* Direct consequence of Definition 1566 (*hyperface*), and Lemma 1563 (*equivalent definition of face hyperplane*). □

### 9.5.2 $l$ -face affine spaces and $l$ -faces

**Definition 1568 ( $l$ -face affine space).** Let  $d \geq 1$ . Let  $[\mathbf{v}]^d$  be  $d+1$  points in  $\mathbb{R}^d$ . Let  $l \in [0..d]$ . Let  $\pi^l$  be an injective map from  $[0..l]$  to  $[0..d]$ . The  $l$ -face affine space associated with  $[\mathbf{v}]^d$  and having the vertices  $[\mathbf{v}_{\pi^l}]^{l,d} \stackrel{\text{def.}}{=} (\mathbf{v}_{\pi^l(j)})_{j \in [0..l]}$  is denoted  $\mathcal{F}_{\pi^l}^{[\mathbf{v}]^d}$ , and is defined by

$$(9.62) \quad \mathcal{F}_{\pi^l}^{[\mathbf{v}]^d} \stackrel{\text{def.}}{=} \left\{ \sum_{j=0}^l \mu_j \mathbf{v}_{\pi^l(j)} \in \mathbb{R}^d \mid \forall j \in [0..l], \mu_j \in \mathbb{R} \wedge \sum_{j=0}^l \mu_j = 1 \right\}.$$

**Lemma 1569 (equivalent definition of  $l$ -face affine space).** Let  $d \geq 1$ . Let  $[\mathbf{v}]^d$  be  $d+1$  points in  $\mathbb{R}^d$ . Let  $l \in [0..d]$ . Let  $\pi^l$  be an injective map from  $[0..l]$  to  $[0..d]$ . Then,  $\mathcal{F}_{\pi^l}^{[\mathbf{v}]^d}$  is an affine space, that can be equivalently defined for any  $i \in [0..l]$  by

$$(9.63) \quad \mathcal{F}_{\pi^l}^{[\mathbf{v}]^d} = \mathbf{v}_{\pi^l(i)} + \text{span}(\mathbf{v}_{\pi^l(j)} - \mathbf{v}_{\pi^l(i)})_{j \in [0..l] \setminus \{i\}}.$$

*Proof.* Direct consequence of Definition 1568 ( $l$ -face affine space), and Lemma 1387 (barycenter closure is affine subspace, with  $n \stackrel{\text{def.}}{=} l$  and  $\mathbf{v}_j \stackrel{\text{def.}}{=} \mathbf{v}_{\pi^l(j)}$  for  $j \in [0..l]$ ).  $\square$

**Lemma 1570 ( $d$ -face affine space is full space).** Let  $d \geq 1$ . Let  $[\mathbf{v}]^d$  be  $d+1$  affinely independent points in  $\mathbb{R}^d$ . Let  $\pi^d$  be an injective map from  $[0..d]$  to  $[0..d]$ . Then,  $\mathcal{F}_{\pi^d}^{[\mathbf{v}]^d} = \mathbb{R}^d$ .

*Proof.* Direct consequence of Lemma 1569 (equivalent definition of  $l$ -face affine space, with  $i \stackrel{\text{def.}}{=} 0$ ), Lemma 1414 (affinely independent family is closed by sub-family, with  $j \stackrel{\text{def.}}{=} 0$ ), Definition 1411 (affinely independent family, with  $[\mathbf{v}_{\pi^d}]^d$ ), Lemma 1376 (free family of dim elements is basis), and  $\mathbb{R}^d$  is affine space with any origin.  $\square$

**Lemma 1571 (0-face affine space is vertex).** Let  $d \geq 1$ . Let  $[\mathbf{v}]^d$  be  $d+1$  points in  $\mathbb{R}^d$ . Let  $\pi^0$  be a map from  $\{0\}$  to  $[0..d]$ . Then,  $\mathcal{F}_{\pi^0}^{[\mathbf{v}]^d}$  is the affine space  $\{\mathbf{v}_{\pi^0(0)}\} = \mathbf{v}_{\pi^0(0)} + \{\mathbf{0}\}$ .

*Proof.* Direct consequence of the injectivity of a function defined on singleton, and Lemma 1569 (equivalent definition of  $l$ -face affine space, with  $l \stackrel{\text{def.}}{=} 0$ ).  $\square$

**Definition 1572 ( $l$ -face).** Let  $d \geq 1$ . Let  $[\mathbf{v}]^d$  be  $d+1$  points in  $\mathbb{R}^d$ . Let  $l \in [0..d]$ . Let  $\pi^l$  an injective map from  $[0..l]$  to  $[0..d]$ . The  $l$ -face associated with  $[\mathbf{v}]^d$  and having the vertices  $[\mathbf{v}_{\pi^l}]^{l,d}$  is denoted  $F_{\pi^l}^{[\mathbf{v}]^d}$ , and is defined by

$$(9.64) \quad F_{\pi^l}^{[\mathbf{v}]^d} \stackrel{\text{def.}}{=} \left\{ \sum_{j=0}^l \mu_j \mathbf{v}_{\pi^l(j)} \in \mathbb{R}^d \mid \forall j \in [0..l], \mu_j \geq 0 \wedge \sum_{j=0}^l \mu_j = 1 \right\}.$$

**Lemma 1573 ( $l$ -face is included in  $l$ -face affine space).** Let  $d \geq 1$ . Let  $[\mathbf{v}]^d$  be  $d+1$  points in  $\mathbb{R}^d$ . Let  $l \in [0..d]$ . Let  $\pi^l$  an injective map from  $[0..l]$  to  $[0..d]$ . Then, we have  $F_{\pi^l}^{[\mathbf{v}]^d} \subset \mathcal{F}_{\pi^l}^{[\mathbf{v}]^d}$ .

*Proof.* Direct consequence of Definition 1572 ( $l$ -face), and Definition 1568 ( $l$ -face affine space).  $\square$

**Lemma 1574 ( $d$ -face is simplex).** Let  $d \geq 1$ . Let  $[\mathbf{v}]^d$  be  $d+1$  points in  $\mathbb{R}^d$ . Let  $\pi^d$  be an injective map from  $[0..d]$  to  $[0..d]$ . Then,  $\pi^d$  is a bijective permutation of  $[0..d]$ , and we have  $F_{\pi^d}^{[\mathbf{v}]^d} = K^{[\mathbf{v}]^d}$ .

*Proof.* **Bijectivity.**

Direct consequence of **the fact that injectivity and cardinal equality imply bijectivity.**

**Equality.** From Definition 1572 (*l-face*), and Definition 1440 (*simplex*), we have

$$\begin{aligned} F_{\pi^d}^{\llbracket \mathbf{v} \rrbracket^d} &= \left\{ \sum_{j=0}^d \mu_j \mathbf{v}_{\pi^d(j)} \in \mathbb{R}^d \mid \forall j \in [0..d], \mu_j \geq 0 \wedge \sum_{j=0}^d \mu_j = 1 \right\} \\ &= \left\{ \sum_{i=0}^d \lambda_i \mathbf{v}_i \in \mathbb{R}^d \mid \forall i \in [0..d], \lambda_i \geq 0 \wedge \sum_{i=0}^d \lambda_i = 1 \right\} = K^{\llbracket \mathbf{v} \rrbracket^d}, \end{aligned}$$

with  $i \stackrel{\text{def.}}{=} \pi^d(j) \Leftrightarrow j = (\pi^d)^{-1}(i)$  ( $\pi^d$  bijective in  $[0..d]$ ) and  $\lambda_i \stackrel{\text{def.}}{=} \mu_{(\pi^d)^{-1}(i)}$ .  $\square$

*Remark 1575.* In the next lemma, the injection  $\theta_i^{d-1}$  from  $[0..d-1]$  to  $[0..d]$  is the “jump” enumeration function defined in Lemma 1368. The  $(d-1)$ -face is used to build the geometric hyperface mapping in Lemma 1584, see also Figure 9.6.

**Lemma 1576 (*(d-1)-face is hyperface*).**

Let  $d \geq 1$ , and  $i \in [0..d]$ . Let  $\llbracket \mathbf{v} \rrbracket^d$  be  $d+1$  points in  $\mathbb{R}^d$ .

Then, we have  $\llbracket \mathbf{v}_{\theta_i^{d-1}} \rrbracket^{d-1,d} = (\mathbf{v}_j)_{j \in [0..d] \setminus \{i\}}$ .

Moreover, if  $\llbracket \mathbf{v} \rrbracket^d$  is affinely independent, then we have  $\mathcal{H}_i^{\llbracket \mathbf{v} \rrbracket^d} = \mathcal{F}_{\theta_i^{d-1}}^{\llbracket \mathbf{v} \rrbracket^d}$  and  $H_i^{\llbracket \mathbf{v} \rrbracket^d} = F_{\theta_i^{d-1}}^{\llbracket \mathbf{v} \rrbracket^d}$ .

*Proof.*

Direct consequence of Lemma 1368 (*jump enumeration*, with  $n \stackrel{\text{def.}}{=} d-1$ , injectivity and image of  $\theta_i^{d-1}$ ), Definition 1568 (*l-face affine space*, with  $l \stackrel{\text{def.}}{=} d-1$  and  $\pi^l \stackrel{\text{def.}}{=} \theta_i^{d-1}$ ), Lemma 1563 (*equivalent definition of face hyperplane*), Definition 1566 (*hyperface*), and Definition 1572 (*l-face*).  $\square$

### 9.5.3 Geometric *l*-face mappings

*Remark 1577.* In this section, we assume that  $l \geq 1$  to avoid treating polynomials over 0-dimensional affine spaces.

**Definition 1578 (*geometric l-face mapping*).** Let  $d \geq 1$ . Let  $\llbracket \mathbf{v} \rrbracket^d$  be  $d+1$  points in  $\mathbb{R}^d$ . Let  $l \in [1..d]$ . Let  $\pi^l$  be an injective map from  $[0..l]$  to  $[0..d]$ . The *geometric l-face mapping* associated with  $\llbracket \mathbf{v} \rrbracket^d$  and the *l-face*  $F_{\pi^l}^{\llbracket \mathbf{v} \rrbracket^d}$  is denoted  $\phi_{\pi^l}^{\llbracket \mathbf{v} \rrbracket^d}$ , and is defined by

$$(9.65) \quad \forall \hat{\mathbf{x}} \in \mathbb{R}^l, \quad \phi_{\pi^l}^{\llbracket \mathbf{v} \rrbracket^d}(\hat{\mathbf{x}}) \stackrel{\text{def.}}{=} \sum_{j=0}^l \hat{\mathcal{L}}_j^{1,l}(\hat{\mathbf{x}}) \mathbf{v}_{\pi^l(j)} \in \mathbb{R}^d.$$

**Lemma 1579 (*geometric d-face mapping is geometric mapping*).**

Let  $d \geq 1$ . Let  $\llbracket \mathbf{v} \rrbracket^d$  be  $d+1$  points in  $\mathbb{R}^d$ . Then, we have  $\phi_{\text{Id}_{\mathbb{R}^d}}^{\llbracket \mathbf{v} \rrbracket^d} = \varphi_{\text{geo}}^{\llbracket \mathbf{v} \rrbracket^d}$ .

*Proof.* Direct consequence of **injectivity of the identity map**, Definition 1578 (*geometric l-face mapping*), and Definition 1547 (*geometric mapping*).  $\square$

*Remark 1580.* The next lemma (on  $\phi_{\pi^l}^{\llbracket \mathbf{v} \rrbracket^d}$ ) is an extension of Lemma 1550 (on  $\varphi_{\text{geo}}^{\llbracket \mathbf{v} \rrbracket^d}$ ).

**Lemma 1581 (properties of geometric  $l$ -face mapping).** Let  $d \geq 1$ . Let  $[\mathbf{v}]^d$  be  $d+1$  points in  $\mathbb{R}^d$ . Let  $l \in [1..d]$ . Let  $\pi^l$  an injective map from  $[0..l]$  to  $[0..d]$ .

Let  $B_{\pi^l}^{[\mathbf{v}]^d} \stackrel{\text{def.}}{=} \begin{pmatrix} \mathbf{v}_{\pi^l(1)} - \mathbf{v}_{\pi^l(0)} & \mathbf{v}_{\pi^l(2)} - \mathbf{v}_{\pi^l(0)} & \dots & \mathbf{v}_{\pi^l(l)} - \mathbf{v}_{\pi^l(0)} \end{pmatrix}$  in  $\mathcal{M}_{d,l}$ .

Then,  $\phi_{\pi^l}^{[\mathbf{v}]^d}$  belongs to  $\text{Aff}(\mathbb{R}^l, \mathbb{R}^d)$ , and we have

$$(9.66) \quad \forall j \in [0..l], \quad \forall \hat{\mathbf{x}} \in \mathbb{R}^l, \quad \phi_{\pi^l}^{[\mathbf{v}]^d}(\hat{\mathbf{x}}) = \mathbf{v}_{\pi^l(j)} + \sum_{k=0, k \neq j}^l \hat{\mathcal{L}}_k^{1,l}(\hat{\mathbf{x}})(\mathbf{v}_{\pi^l(k)} - \mathbf{v}_{\pi^l(j)}),$$

$$(9.67) \quad \forall \hat{\mathbf{x}} \in \mathbb{R}^l, \quad \phi_{\pi^l}^{[\mathbf{v}]^d}(\hat{\mathbf{x}}) = \mathbf{v}_{\pi^l(0)} + \sum_{j=1}^l \hat{x}_j (\mathbf{v}_{\pi^l(j)} - \mathbf{v}_{\pi^l(0)}) = \mathbf{v}_{\pi^l(0)} + B_{\pi^l}^{[\mathbf{v}]^d} \hat{\mathbf{x}},$$

$$(9.68) \quad \forall j \in [0..l], \quad \phi_{\pi^l}^{[\mathbf{v}]^d}(\hat{\mathbf{v}}_j^l) = \mathbf{v}_{\pi^l(j)} \quad (\text{where } \hat{\mathbf{v}}_j^l \text{ is a reference vertex in } \mathbb{R}^{l-1}),$$

$$(9.69) \quad \phi_{\pi^l}^{[\mathbf{v}]^d}(\mathbb{R}^l) = \mathcal{F}_{\pi^l}^{[\mathbf{v}]^d},$$

$$(9.70) \quad \phi_{\pi^l}^{[\mathbf{v}]^d}(\hat{K}^l) = F_{\pi^l}^{[\mathbf{v}]^d}.$$

Moreover, if  $[\mathbf{v}]^d$  is affinely independent, then  $\phi_{\pi^l}^{[\mathbf{v}]^d}$  is bijective from  $\mathbb{R}^l$  onto  $\mathcal{F}_{\pi^l}^{[\mathbf{v}]^d}$ , its inverse  $(\phi_{\pi^l}^{[\mathbf{v}]^d})^{-1}$  from  $\mathcal{F}_{\pi^l}^{[\mathbf{v}]^d}$  onto  $\mathbb{R}^l$  is affine, and we have

$$(9.71) \quad \forall j \in [0..l], \quad (\phi_{\pi^l}^{[\mathbf{v}]^d})^{-1}(\mathbf{v}_{\pi^l(j)}) = \hat{\mathbf{v}}_j^l.$$

*Proof.* Identities (9.66) and (9.67),  $\phi_{\pi^l}^{[\mathbf{v}]^d}$  affine. Let  $j \in [0..l]$ . Then, from Definition 1578 (geometric  $l$ -face mapping), Lemma 1543 (reference Lagrange polynomials is basis of  $\mathbb{P}_1^d$ , (9.33) with  $d \stackrel{\text{def.}}{=} l$ , extracting  $j$ ), and **ring properties of  $\mathbb{R}$** , identity (9.66) holds. Thus, from (9.66) (with  $j \stackrel{\text{def.}}{=} 0$ ), Definition 1541 (reference Lagrange polynomials of  $\mathbb{P}_1^d$ , (9.30)), **the columnwise rule for matrix-vector product, commutative ring properties of  $\mathbb{R}$** , and Definition 1388 (affine map, with  $\phi \stackrel{\text{def.}}{=} B_{\pi^l}^{[\mathbf{v}]^d}$  and  $\mathbf{y}_0 \stackrel{\text{def.}}{=} \mathbf{v}_{\pi^l(0)}$ ), identities (9.67) hold, and  $\phi_{\pi^l}^{[\mathbf{v}]^d}$  is affine.

**Identity (9.68).** Direct consequence of Definition 1578 (geometric  $l$ -face mapping), and Lemma 1543 (reference Lagrange polynomials is basis of  $\mathbb{P}_1^d$ , (9.32)).

**Identity (9.69).**  $\phi_{\pi^l}^{[\mathbf{v}]^d}(\mathbb{R}^l) \subset \mathcal{F}_{\pi^l}^{[\mathbf{v}]^d}$ .

Direct consequence of Definition 1578 (geometric  $l$ -face mapping), Lemma 1543 (reference Lagrange polynomials is basis of  $\mathbb{P}_1^d$ , (9.33)), and Definition 1568 ( $l$ -face affine space).

$\mathcal{F}_{\pi^l}^{[\mathbf{v}]^d} \subset \phi_{\pi^l}^{[\mathbf{v}]^d}(\mathbb{R}^l)$ . From Definition 1568 ( $l$ -face affine space), let  $\mathbf{x} = \sum_{j=0}^l \mu_j \mathbf{v}_{\pi^l(j)} \in \mathcal{F}_{\pi^l}^{[\mathbf{v}]^d}$  with  $\sum_{j=0}^l \mu_j = 1$ . Let  $\hat{\mathbf{x}} \stackrel{\text{def.}}{=} (\mu_j)_{j \in [1..l]} \in \mathbb{R}^l$ . Then, from Definition 1541 (reference Lagrange polynomials of  $\mathbb{P}_1^d$ ), and Definition 1578 (geometric  $l$ -face mapping), we have for all  $j \in [1..l]$ ,  $\hat{\mathcal{L}}_j^{1,l}(\hat{\mathbf{x}}) = \mu_j$  and  $\hat{\mathcal{L}}_0^{1,l}(\hat{\mathbf{x}}) = 1 - \sum_{j=1}^l \mu_j = \mu_0$ , thus  $\mathbf{x} = \phi_{\pi^l}^{[\mathbf{v}]^d}(\hat{\mathbf{x}}) \in \phi_{\pi^l}^{[\mathbf{v}]^d}(\mathbb{R}^l)$ . Therefore, we have the equality.

**Identity (9.70).**  $\phi_{\pi^l}^{[\mathbf{v}]^d}(\hat{K}^l) \subset F_{\pi^l}^{[\mathbf{v}]^d}$ . Direct consequence of Definition 1578 (geometric  $l$ -face mapping), Definition 1437 (reference simplex), Definition 1541 (reference Lagrange polynomials of  $\mathbb{P}_1^d$ , thus for all  $\hat{\mathbf{x}} \in \hat{K}^l$ , for all  $i \in [0..l]$ ,  $\hat{\mathcal{L}}_i^{1,l}(\hat{\mathbf{x}}) \geq 0$ ), and Lemma 1543 (reference Lagrange polynomials is basis of  $\mathbb{P}_1^d$ , (9.33)), and Definition 1572 ( $l$ -face).

$F_{\pi^l}^{[\mathbf{v}]^d} \subset \phi_{\pi^l}^{[\mathbf{v}]^d}(\hat{K}^l)$ . From Definition 1572 ( $l$ -face), let  $\mathbf{x} = \sum_{j=0}^l \mu_j \mathbf{v}_{\pi^l(j)} \in F_{\pi^l}^{[\mathbf{v}]^d}$  with for all  $j \in [0..l]$ ,  $\mu_j \geq 0$  and  $\sum_{j=0}^l \mu_j = 1$ . Let  $\hat{\mathbf{x}} \stackrel{\text{def.}}{=} (\mu_j)_{j \in [1..l]} \in \mathbb{R}^l$ . Then, from Definition 1541 (reference Lagrange polynomials of  $\mathbb{P}_1^d$ ), Definition 1437 (reference simplex), and Definition 1578



(*geometric l-face mapping*), we have

$$\forall j \in [1..l], \hat{x}_j = \hat{\mathcal{L}}_j^{1,l}(\hat{\mathbf{x}}) = \mu_j \geq 0 \quad \text{and} \quad 1 - \sum_{j=1}^l \hat{x}_j = \hat{\mathcal{L}}_0^{1,l}(\hat{\mathbf{x}}) = 1 - \sum_{j=1}^l \mu_j = \mu_0 \geq 0,$$

thus  $\hat{\mathbf{x}} \in \hat{K}^l$  and  $\mathbf{x} = \phi_{\pi^l}^{\llbracket \mathbf{v} \rrbracket^d}(\hat{\mathbf{x}}) \in \phi_{\pi^l}^{\llbracket \mathbf{v} \rrbracket^d}(\hat{K}^l)$ .

Therefore, we have the equality.

**Bijection.** Direct consequence of (9.67), Lemma 1406 (*injective affine map is zero linear kernel*), Definition 101 (*kernel*), Lemma 1414 (*affinely independent family is closed by sub-family, with  $j \stackrel{\text{def.}}{=} 0$ , thus  $(\mathbf{v}_{\pi^l(i)} - \mathbf{v}_{\pi^l(0)})_{i \in [1..l]}$  is free*), (9.69), and **the definition of bijectivity**.

$(\phi_{\pi^l}^{\llbracket \mathbf{v} \rrbracket^d})^{-1}$  **affine.** Direct consequence of (9.67), Lemma 1409 (*inverse of affine submap is affine submap*, with  $E = E' = \mathcal{E}' \stackrel{\text{def.}}{=} \mathbb{R}^l$  ( $\mathbf{x}_0 \stackrel{\text{def.}}{=} \mathbf{0}$ ),  $F \stackrel{\text{def.}}{=} \mathbb{R}^d$ ,  $\mathcal{F}' \stackrel{\text{def.}}{=} \mathcal{F}_{\pi^l}^{\llbracket \mathbf{v} \rrbracket^d}$ , and  $\mathbf{y}'_0 \stackrel{\text{def.}}{=} \mathbf{v}_{\pi^l(0)} \in \mathcal{F}_{\pi^l}^{\llbracket \mathbf{v} \rrbracket^d}$ ), and Lemma 1569 (*equivalent definition of l-face affine space*).

**Equation (9.71).** Direct consequence of Equation (9.68).  $\square$

**Lemma 1582 (*geometric l-face mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^l$* ).**

Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . Let  $\llbracket \mathbf{v} \rrbracket^d$  be  $d+1$  points in  $\mathbb{R}^d$ . Let  $l \in [1..d]$ . Let  $\pi^l$  an injective map from  $[0..l]$  to  $[0..d]$ . Let  $p \in \mathbb{P}_k^d$ . Then, we have  $p \circ \phi_{\pi^l}^{\llbracket \mathbf{v} \rrbracket^d} \in \mathbb{P}_k^l$ .

*Proof.* Direct consequence of Lemma 1581 (*properties of geometric l-face mapping, thus  $\phi_{\pi^l}^{\llbracket \mathbf{v} \rrbracket^d}$  belongs to  $\text{Aff}(\mathbb{R}^l, \mathbb{R}^d)$* ), and Lemma 1540 (*affine mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^l$* ).  $\square$

**Lemma 1583 (*geometric mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^d$* ).**

Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . Let  $\llbracket \mathbf{v} \rrbracket^d$  be  $d+1$  points in  $\mathbb{R}^d$ . Let  $p \in \mathbb{P}_k^d$ . Then, we have  $p \circ \varphi_{\text{geo}}^{\llbracket \mathbf{v} \rrbracket^d} \in \mathbb{P}_k^d$ .  
Moreover, if  $\llbracket \mathbf{v} \rrbracket^d$  is affinely independent, then  $p \circ (\varphi_{\text{geo}}^{\llbracket \mathbf{v} \rrbracket^d})^{-1}$  also belongs to  $\mathbb{P}_k^d$ .

*Proof.* Direct consequence of Lemma 1579 (*geometric d-face mapping is geometric mapping*), Lemma 1582 (*geometric l-face mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^l$ , with  $l \stackrel{\text{def.}}{=} d$  and  $\pi^l \stackrel{\text{def.}}{=} \text{Id}_{\mathbb{R}^d}$ ), Lemma 1550 (*properties of geometric mapping, thus  $(\varphi_{\text{geo}}^{\llbracket \mathbf{v} \rrbracket^d})^{-1}$  is affine*), and Lemma 1540 (*affine mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^l$ , with  $l \stackrel{\text{def.}}{=} d$* ).  $\square$*

### 9.5.4 Geometric hyperface mapping

**Lemma 1584 (*geometric hyperface mapping*).** Let  $d \geq 1$ . Let  $\llbracket \mathbf{v} \rrbracket^d$  be  $d+1$  points in  $\mathbb{R}^d$ . Let  $i \in [0..d]$ . Let  $\hat{\mathbf{x}} \in \mathbb{R}^{d-1}$ . Then, we have  $\phi_{\theta_i^{d-1}}^{\llbracket \mathbf{v} \rrbracket^d} \in \text{Aff}(\mathbb{R}^{d-1}, \mathbb{R}^d)$  and

$$(9.72) \quad (i = 0) \quad \phi_{\theta_0^{d-1}}^{\llbracket \mathbf{v} \rrbracket^d}(\hat{\mathbf{x}}) = \mathbf{v}_1 + \sum_{j=1}^{d-1} \hat{x}_j (\mathbf{v}_{j+1} - \mathbf{v}_1),$$

$$(9.73) \quad (i \in [1..d-1]) \quad \phi_{\theta_i^{d-1}}^{\llbracket \mathbf{v} \rrbracket^d}(\hat{\mathbf{x}}) = \mathbf{v}_0 + \sum_{j=1}^{i-1} \hat{x}_j (\mathbf{v}_j - \mathbf{v}_0) + \sum_{j=i}^{d-1} \hat{x}_j (\mathbf{v}_{j+1} - \mathbf{v}_0),$$

$$(9.74) \quad \forall j \in [0..d] \quad \phi_{\theta_i^{d-1}}^{\llbracket \mathbf{v} \rrbracket^d}(\hat{\mathbf{v}}_j^{d-1}) = \begin{cases} \mathbf{v}_j & \text{if } j < i, \\ \mathbf{v}_{j+1} & \text{if } j \geq i. \end{cases}$$

Moreover, if  $\llbracket \mathbf{v} \rrbracket^d$  is affinely independent, then  $\phi_{\theta_i^{d-1}}^{\llbracket \mathbf{v} \rrbracket^d}$  is bijective from  $\mathbb{R}^{d-1}$  onto  $\mathcal{H}_i^{\llbracket \mathbf{v} \rrbracket^d}$ , and its inverse  $(\phi_{\theta_i^{d-1}}^{\llbracket \mathbf{v} \rrbracket^d})^{-1}$  from  $\mathcal{H}_i^{\llbracket \mathbf{v} \rrbracket^d}$  onto  $\mathbb{R}^{d-1}$  is affine.

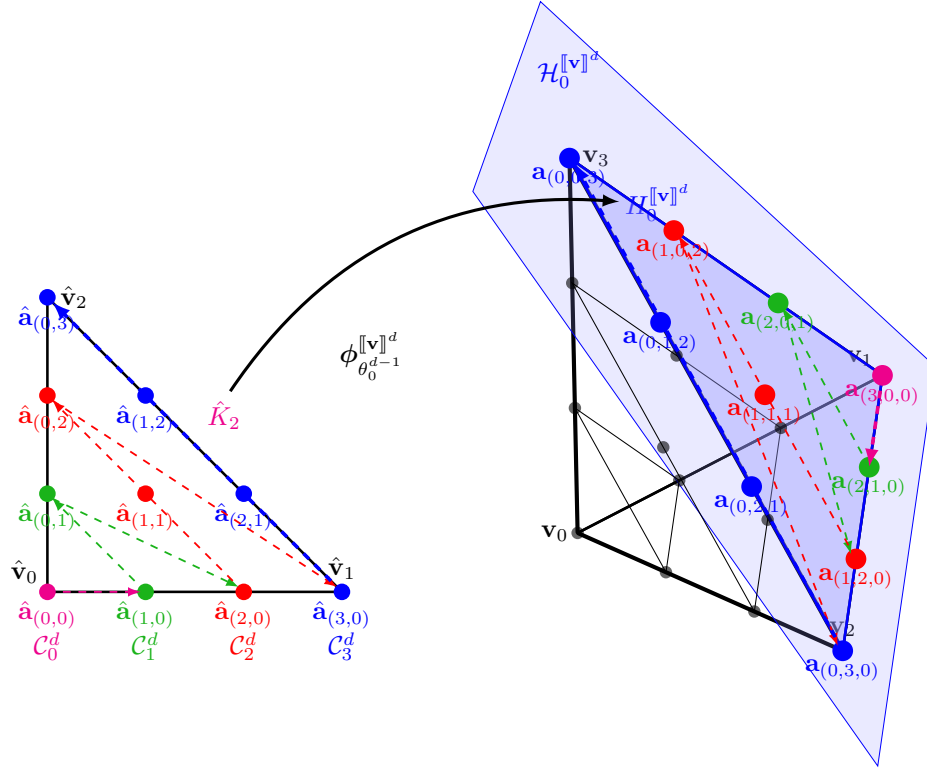


Figure 9.6: Geometric hyperface mapping in the case  $d = k = 3$  (see Lemma 1584).

The reference triangle  $\hat{K}_2$  is mapped onto  $H_0^{[[\mathbf{v}]]^d}$ , the 0-th face opposite vertex  $\mathbf{v}_0$  that is depicted in blue. The correspondence between the reference nodes in the triangle ( $d = 2$ ) and the face nodes of the tetrahedron ( $d = 3$ ) is illustrated by the colors: we have  $\phi_{\theta_0^{d-1}}^{[[\mathbf{v}]]^d}(\hat{\mathbf{a}}_{i,j}) = \mathbf{a}_{(3-(i+j),i,j)}$ , for all  $(i,j) \in \mathcal{A}_2^3$  (see Lemma 1607).

*Proof.* Direct consequence of Lemma 1576 ( *$(d-1)$ -face is hyperface*), and Lemma 1581 (*properties of geometric  $l$ -face mapping*).  $\square$

**Lemma 1585** (*hyperface geometric mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^{d-1}$* ). *Let  $d \geq 2$ . Let  $k \in \mathbb{N}$ . Let  $[[\mathbf{v}]]^d$  be  $d+1$  points in  $\mathbb{R}^d$ . Let  $i \in [0..d]$ . Let  $p \in \mathbb{P}_k^d$ . Then, we have  $p \circ \phi_{\theta_i^{d-1}}^{[[\mathbf{v}]]^d} \in \mathbb{P}_k^{d-1}$ .*

*Proof.* Direct consequence of Lemma 1576 ( *$(d-1)$ -face is hyperface*), and Lemma 1582 (*geometric  $l$ -face mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^l$ , with  $l \stackrel{\text{def.}}{=} d-1$  and  $\pi^l \stackrel{\text{def.}}{=} \theta_i^{d-1}$* ).  $\square$

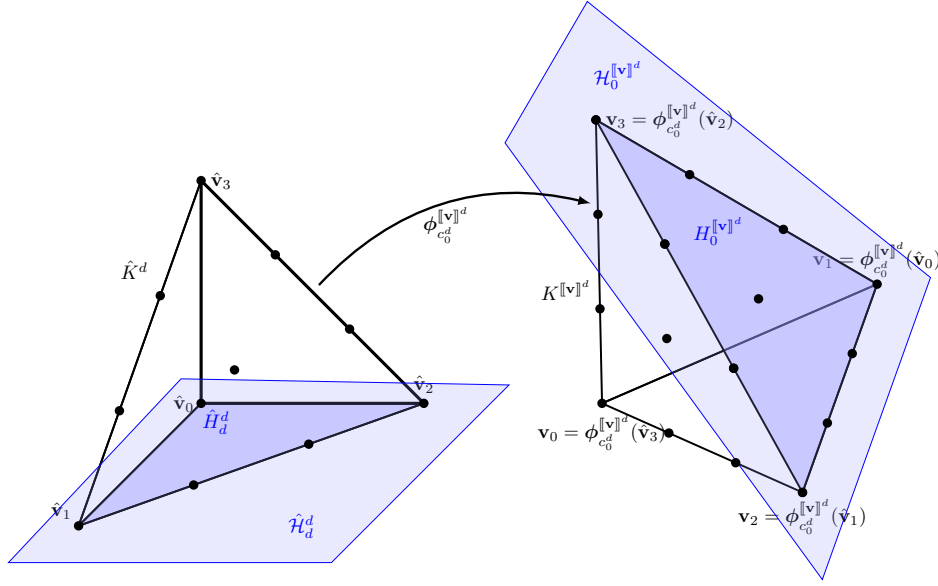


Figure 9.7: Geometric transformation  $\phi_{c_0^d}^{[v]^d}$  in the case  $d = k = 3$ , with the circular permutation  $c_0^d$  that maps  $(0, 1, 2, 3)$  to  $(1, 2, 3, 0)$ , see Lemma 1366.

The reference simplex  $\hat{K}^d$  is mapped onto the current simplex  $K^{[v]^d}$ , see Lemma 1586. The reference vertices are mapped onto the current vertices with a change of indices: for all  $j \in [0..d]$ , we have  $\phi_{c_0^d}^{[v]^d}(\hat{v}_j) = \mathbf{v}_{c_0^d(j)}$ , see Lemma 1581. Note that as  $c_0^d(d) = 0$ , the reference hyperplane  $\hat{H}_d^d$  (containing the face  $\hat{H}_d^d$  opposite vertex  $\hat{v}_d$ ) is mapped onto  $\mathcal{H}_0^{[v]^d}$  (containing the face  $H_0^{[v]^d}$  opposite vertex  $\mathbf{v}_0$ ). In this figure, only the visible nodes are depicted.

### 9.5.5 Geometric mapping with permutation

**Lemma 1586 (geometric mapping with permutation).** *Let  $d \geq 1$ . Let  $[\mathbf{v}]^d$  be  $d + 1$  affinely independent points in  $\mathbb{R}^d$ . Let  $\pi^d$  be an injective mapping from  $[0..d]$  to  $[0..d]$ . Let  $\hat{\mathbf{x}} \in \mathbb{R}^d$ . Then,  $\pi^d$  is a bijective permutation of  $[0..d]$ ,  $\phi_{\pi^d}^{[v]^d} \in \text{Aff}(\mathbb{R}^d, \mathbb{R}^d)$  is bijective, and we have*

$$(9.75) \quad \phi_{\pi^d}^{[v]^d}(\hat{\mathbf{x}}) = \sum_{i=0}^d \hat{\mathcal{L}}_{(\pi^d)^{-1}(i)}^{1,d}(\hat{\mathbf{x}}) \mathbf{v}_i$$

$$(9.76) \quad \forall i \in [0..d], \quad \phi_{\pi^d}^{[v]^d}(\hat{\mathbf{x}}) = \mathbf{v}_i + \sum_{k=0, k \neq i}^d \hat{\mathcal{L}}_{(\pi^d)^{-1}(k)}^{1,d}(\hat{\mathbf{x}})(\mathbf{v}_k - \mathbf{v}_i),$$

$$(9.77) \quad \forall i \in [0..d], \quad \lambda_i^{[v]^d} = \mathcal{L}_i^{[v]^d} = \hat{\mathcal{L}}_{(\pi^d)^{-1}(i)}^{1,d} \circ \left( \phi_{\pi^d}^{[v]^d} \right)^{-1},$$

$$(9.78) \quad \forall j \in [0..d], \quad \phi_{\pi^d}^{[v]^d}(\hat{\mathcal{H}}_j^d) = \mathcal{H}_{\pi^d(j)}^{[v]^d},$$

$$(9.79) \quad \phi_{\pi^d}^{[v]^d}(\hat{K}^d) = K^{[v]^d}.$$

*Proof.* **Bijections,  $\phi_{\pi^d}^{[v]^d}$  affine, identities (9.75), (9.76).** Direct consequence of Lemma 1574 (*d-face is simplex*), Lemma 1581 (*properties of geometric l-face mapping, with  $l \stackrel{\text{def}}{=} d$* ), Lemma 1570 (*d-face affine space is full space*), and Definition 1578 (*geometric l-face mapping*).

**Identities (9.77).** Let  $\mathbf{x} \in \mathbb{R}^d$ . Let  $\hat{\mathbf{y}} \stackrel{\text{def}}{=} \left( \phi_{\pi^d}^{[v]^d} \right)^{-1}(\mathbf{x}) \in \mathbb{R}^d$ , i.e. such that  $\phi_{\pi^d}^{[v]^d}(\hat{\mathbf{y}}) = \mathbf{x}$ . From Lemma 1559 (*barycentric coordinate, decomposition, then uniqueness*), (9.75), Lemma 1543 (*reference Lagrange polynomials is basis of  $\mathbb{F}_1^d$ , (9.33)*), and Lemma 1560 (*Lagrange polynomials*

of  $\mathbb{P}_1^d$  are barycentric coordinate), we have

$$\mathbf{x} = \sum_{i=0}^d \lambda_i^{[\mathbf{v}]^d}(\mathbf{x}) \mathbf{v}_i = \sum_{i=0}^d \hat{\mathcal{L}}_{(\pi^d)^{-1}(i)}^{1,d}(\hat{\mathbf{y}}) \mathbf{v}_i = \sum_{i=0}^d \hat{\mathcal{L}}_{(\pi^d)^{-1}(i)}^{1,d} \circ \left( \phi_{\pi^d}^{[\mathbf{v}]^d} \right)^{-1}(\mathbf{x}) \mathbf{v}_i$$

and  $\sum_{i=0}^d \hat{\mathcal{L}}_{(\pi^d)^{-1}(i)}^{1,d} = 1$ . Thus, for all  $i \in [0..d]$ ,  $\hat{\mathcal{L}}_{(\pi^d)^{-1}(i)}^{1,d} \circ \left( \phi_{\pi^d}^{[\mathbf{v}]^d} \right)^{-1} = \lambda_i^{[\mathbf{v}]^d} = \mathcal{L}_i^{[\mathbf{v}]^d}$ .

**Identity (9.78).** Let  $j \in [0..d]$ . Let  $i \stackrel{\text{def.}}{=} \pi^d(j)$ , i.e. such that  $(\pi^d)^{-1}(i) = j$ . Then, from Lemma 1564 (reference face hyperplane), (9.77) (thus  $\hat{\mathcal{L}}_j^{1,d} = \lambda_i^{[\mathbf{v}]^d} \circ \phi_{\pi^d}^{[\mathbf{v}]^d}$ ), **the definition of bijectivity**, Lemma 1370 (image of ker is ker, with  $f \stackrel{\text{def.}}{=} \phi_{\pi^d}^{[\mathbf{v}]^d}$  surjective, and  $g \stackrel{\text{def.}}{=} \lambda_i^{[\mathbf{v}]^d}$ ), and Lemma 1563 (equivalent definition of face hyperplane), we have

$$\phi_{\pi^d}^{[\mathbf{v}]^d}(\hat{\mathcal{H}}_j^d) = \phi_{\pi^d}^{[\mathbf{v}]^d}(\ker(\hat{\mathcal{L}}_j^{1,d})) = \ker(\lambda_i^{[\mathbf{v}]^d}) = \mathcal{H}_i^{[\mathbf{v}]^d}.$$

**Identity (9.79).** Direct consequence of Lemma 1581 (properties of geometric  $l$ -face mapping, (9.70) with  $l \stackrel{\text{def.}}{=} d$ ), and Lemma 1574 ( $d$ -face is simplex).  $\square$

## 9.6 Lagrange nodes for $\text{Lag}\mathbb{P}_k^d$

*Remark 1587.* We recall the distinct notations for vertices  $[[\mathbf{v}]]^d$ , and for nodes  $\{\{\mathbf{a}\}\}_{\mathcal{A}_k^d}$  (see Remark 1433), as well as Definition 1401 for isobarycenters.

### 9.6.1 Lagrange nodes for the current element

**Definition 1588 (Lagrange nodes of  $\mathbb{P}_k^d$ ).** Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . Let  $[[\mathbf{v}]]^d$  be  $d+1$  points in  $\mathbb{R}^d$ . The Lagrange nodes of  $\mathbb{P}_k^d$  are denoted  $\{\{\mathbf{a}\}\}_{\mathcal{A}_k^d} = (\mathbf{a}_\alpha)_{\alpha \in \mathcal{A}_k^d}$ , and are defined by

$$(9.80) \quad (k=0) \quad \mathbf{a}_0 \stackrel{\text{def.}}{=} \mathbf{g}^{[\mathbf{v}]^d} \left( = \frac{1}{d+1} \sum_{i=0}^d \mathbf{v}_i \right),$$

$$(9.81) \quad (k \geq 1) \quad \forall \alpha \in \mathcal{A}_k^d, \quad \mathbf{a}_\alpha \stackrel{\text{def.}}{=} \mathbf{v}_0 + \sum_{i=1}^d \frac{\alpha_i}{k} (\mathbf{v}_i - \mathbf{v}_0).$$

**Lemma 1589 (Lagrange nodes of  $\mathbb{P}_k^d$  for  $d=1$  are Lagrange nodes of  $\mathbb{P}_k^1$ ).**

Let  $k \in \mathbb{N}$ . Let  $[[\mathbf{v}]]^1 = [[v]]^1$  be two points in  $\mathbb{R}^1$ .

Then,  $\{\{\mathbf{a}\}\}_{\mathcal{A}_k^d}$  for  $d \stackrel{\text{def.}}{=} 1$ , and  $\{\{a\}\}_{\mathcal{A}_k^1}$  from Definition 1465 coincide.

*Proof.* Direct consequence of Definition 1588 (Lagrange nodes of  $\mathbb{P}_k^d$ , with  $d \stackrel{\text{def.}}{=} 1$ ), Lemma 1497 (first multi-indices  $\mathcal{A}_k^d$ , with  $d \stackrel{\text{def.}}{=} 1$ ), and Definition 1465 (Lagrange nodes of  $\mathbb{P}_k^1$ ).  $\square$

**Lemma 1590 (number of Lagrange nodes of  $\mathbb{P}_k^d$ ).**

Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . Let  $[[\mathbf{v}]]^d$  be  $d+1$  affinely independent points in  $\mathbb{R}^d$ . Let  $\alpha, \beta \in \mathcal{A}_k^d$ .

Then,  $\alpha \neq \beta$  implies  $\mathbf{a}_\alpha \neq \mathbf{a}_\beta$ . Thus, there are  $\binom{k+d}{d}$  distinct Lagrange nodes in  $\{\{\mathbf{a}\}\}_{\mathcal{A}_k^d}$ .

*Proof.* Direct consequence of Definition 1588 (Lagrange nodes of  $\mathbb{P}_k^d$ ), Lemma 1559 (barycentric coordinate, uniqueness in (9.51) with  $j \stackrel{\text{def.}}{=} 0$ ), and Lemma 1498 (cardinal of  $\mathcal{A}_k^d$ ).  $\square$

**Lemma 1591 (barycentric coordinates of Lagrange nodes of  $\mathbb{P}_k^d$ ).** Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . Let  $[\mathbf{v}]^d$  be  $d+1$  affinely independent points in  $\mathbb{R}^d$ . Let  $\boldsymbol{\alpha} \in \mathcal{A}_k^d$ . Let  $i \in [0..d]$ . Then, the barycentric coordinates of the Lagrange node  $\mathbf{a}_\alpha$  are

$$(9.82) \quad (k=0) \quad \lambda_i^{[\mathbf{v}]^d}(\mathbf{a}_0) = \frac{1}{d+1} \in [0, 1],$$

$$(9.83) \quad (k \geq 1) \quad \begin{cases} (i=0) & \lambda_0^{[\mathbf{v}]^d}(\mathbf{a}_\alpha) = 1 - \frac{|\boldsymbol{\alpha}|}{k} \in [0, 1], \\ (i \in [1..d]) & \lambda_i^{[\mathbf{v}]^d}(\mathbf{a}_\alpha) = \frac{\alpha_i}{k} \in [0, 1]. \end{cases}$$

Moreover, if  $k \geq 1$ , then the Lagrange node is equivalently defined by

$$(9.84) \quad \mathbf{a}_\alpha = \left(1 - \frac{|\boldsymbol{\alpha}|}{k}\right) \mathbf{v}_0 + \sum_{i=1}^d \frac{\alpha_i}{k} \mathbf{v}_i \quad \text{with} \quad \left(1 - \frac{|\boldsymbol{\alpha}|}{k}\right) + \sum_{i=1}^d \frac{\alpha_i}{k} = 1.$$

*Proof.* Direct consequence of Definition 1588 (Lagrange nodes of  $\mathbb{P}_k^d$ ), Definition 1401 (isobarycenter, for  $k=0$ ), Lemma 1559 (barycentric coordinate, (9.50) for  $k=0$ , (9.50) and (9.51) with  $j \stackrel{\text{def.}}{=} 0$  for  $k \geq 1$ , and uniqueness), Definition 1486 (sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$ , for  $k \geq 1$ ), Definition 1480 (length of multi-indices, for  $k \geq 1$ ), Lemma 1488 (indices are smaller than maximal length, for  $k \geq 1$ ), and **ordered field properties of  $\mathbb{R}$  (with  $d+1 > 0$  for  $k=0$ , and  $k > 0$  for  $k \geq 1$ )**.  $\square$

**Lemma 1592 (vertices are Lagrange nodes of  $\mathbb{P}_k^d$ ).**

Let  $d \geq 1$ . Let  $k \geq 1$ . Let  $[\mathbf{v}]^d$  be  $d+1$  points in  $\mathbb{R}^d$ . Then, the vertices are nodes of  $\{\{\mathbf{a}\}\}_{\mathcal{A}_k^d}$ ,

$$(9.85) \quad \mathbf{v}_0 = \mathbf{a}_0 \quad \text{and} \quad \forall i \in [1..d], \quad \mathbf{v}_i = \mathbf{a}_{k\mathbf{e}_i}.$$

*Proof.* **Case  $i=0$ .** Direct consequence of Definition 1588 (Lagrange nodes of  $\mathbb{P}_k^d$ , with  $k > 0$ ).

**Case  $i \in [1..d]$ .** From Definition 1365 (canonic families), Definition 1480 (length of multi-indices), Definition 1486 (sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$ ), and Definition 1588 (Lagrange nodes of  $\mathbb{P}_k^d$ , with  $k > 0$ ), we have  $|k\mathbf{e}_i| = k \sum_{j=1}^d \delta_{ij} = k \leq k$ , thus  $k\mathbf{e}_i \in \mathcal{A}_k^d$ , and

$$\mathbf{a}_{k\mathbf{e}_i} = \mathbf{v}_0 + \sum_{j=1}^d \frac{k\delta_{ij}}{k} (\mathbf{v}_j - \mathbf{v}_0) = \mathbf{v}_0 + (\mathbf{v}_i - \mathbf{v}_0) = \mathbf{v}_i.$$

Therefore, the identities hold.  $\square$

**Lemma 1593 (Lagrange nodes of  $\mathbb{P}_1^d$  are vertices).**

Let  $d \geq 1$ . Let  $[\mathbf{v}]^d$  be  $d+1$  affinely independent points in  $\mathbb{R}^d$ . Then, the nodes of  $\{\{\mathbf{a}\}\}_{\mathcal{A}_1^d}$  are the vertices,

$$(9.86) \quad \mathbf{a}_0 = \mathbf{v}_0 \quad \text{and} \quad \forall i \in [1..d], \quad \mathbf{a}_{\mathbf{e}_i} = \mathbf{v}_i.$$

*Proof.* Direct consequence of Lemma 1497 (first multi-indices  $\mathcal{A}_k^d$ , with  $k=1$ ), and Lemma 1592 (vertices are Lagrange nodes of  $\mathbb{P}_k^d$ , with  $k=1$ ).  $\square$

## 9.6.2 Sub-vertices and sub-nodes

**Definition 1594 (sub-vertices of Lagrange nodes of  $\mathbb{P}_k^d$ ).**

Let  $d \geq 1$ . Let  $k \geq 1$ . Let  $[\mathbf{v}]^d$  be  $d+1$  points in  $\mathbb{R}^d$ . Let  $i \in [0..d]$ . The sub-vertices of the Lagrange nodes of  $\mathbb{P}_k^d$  with respect to  $\mathbf{v}_0$  are denoted  $[\mathbf{v}]^d = (\mathbf{v}_i)_{i \in [0..d]}$ , and are defined by

$$(9.87) \quad (i=0) \quad \mathbf{v}_0 \stackrel{\text{def.}}{=} \mathbf{v}_0,$$

$$(9.88) \quad (i \in [1..d]) \quad \mathbf{v}_i \stackrel{\text{def.}}{=} \mathbf{a}_{(k-1)\mathbf{e}_i}.$$

**Lemma 1595 (equivalent definition of sub-vertices of Lagrange nodes of  $\mathbb{P}_k^d$ ).**

Let  $d \geq 1$ . Let  $k \geq 1$ . Let  $[\mathbf{v}]^d$  be  $d+1$  points in  $\mathbb{R}^d$ . Let  $i \in [0..d]$ . Then, we have

$$(9.89) \quad \mathbf{v}_i = \frac{1}{k} \mathbf{v}_0 + \frac{k-1}{k} \mathbf{v}_i.$$

*Proof.* Direct consequence of Definition 1594 (sub-vertices of Lagrange nodes of  $\mathbb{P}_k^d$ ), **ordered field properties of  $\mathbb{R}$  (with  $k > 0$ )**, thus  $\frac{1}{k} + \frac{k-1}{k} = 1$  (case  $i = 0$ ),  $1 - \frac{k-1}{k} = \frac{1}{k}$ , and  $0 \leq k-1 < k$ , Definition 1365 (canonic families), Definition 1480 (length of multi-indices, thus  $|(k-1)\mathbf{e}_i| = k-1$ ), Definition 1486 (sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$ , thus  $(k-1)\mathbf{e}_i \in \mathcal{A}_k^d$ ), and Lemma 1591 (barycentric coordinates of Lagrange nodes of  $\mathbb{P}_k^d$ , (9.84)).  $\square$

*Remark 1596.* An illustration of the sub-vertices (see Definition 1594) and of  $\mathbb{P}_{k-1}^d$  sub-nodes (see Lemma 1598) in dimension  $d = 3$  and for  $k = 3$  can be seen in Figure 9.8.

The sub-vertices  $[\mathbf{v}]^d$  are  $\mathbf{v}_0$  and the nodes of  $\mathbb{P}_k^d$  that are the closest to the vertices along the axes. The  $\mathbb{P}_{k-1}^d$  sub-nodes are the nodes of  $\mathbb{P}_k^d$ , except the nodes indexed by  $\alpha \in \mathcal{C}_k^d$  (that lie in the face  $H_0^{[\mathbf{v}]^d}$ ). These sub-vertices and sub-nodes are used in the step 2 of the proof of the unisolvence Lemma 1625.

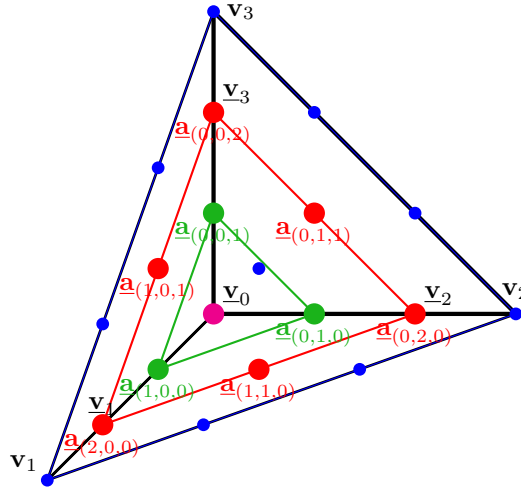


Figure 9.8: Sub-vertices  $[\mathbf{v}]^d$  with respect to  $\mathbf{v}_0$  and Lagrange  $\mathbb{P}_{k-1}^d$  sub-nodes  $\{\{\mathbf{a}\}_{\mathcal{A}_{k-1}^d}\}$ , in the case  $d = k = 3$ .

The Lagrange  $\mathbb{P}_k^d$  nodes  $\{\{\mathbf{a}\}_{\mathcal{A}_k^d}\}$  are set in the tetrahedron defined by the vertices  $[\mathbf{v}]^3 = (\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ .

The sub-vertices are  $[\mathbf{v}]^3 \stackrel{\text{def.}}{=} (\mathbf{v}_0, \mathbf{a}_{2\mathbf{e}_1}, \mathbf{a}_{2\mathbf{e}_2}, \mathbf{a}_{2\mathbf{e}_3})$ , see Definition 1594 and Remark 1596.

The  $\mathbb{P}_{k-1}^d$  sub-nodes  $\{\{\mathbf{a}\}_{\mathcal{A}_{k-1}^d}\}$  are defined with respect to the tetrahedron whose vertices are  $[\mathbf{v}]^3$ . These are the nodes  $\{\{\mathbf{a}\}_{\mathcal{A}_3^d}\}$ , except the (small blue) nodes corresponding to  $k = 3$  (with indices in  $\mathcal{C}_3^d$ ), see Lemma 1598.

Thus, geometrically, passing from  $\mathbb{P}_k^d$  to  $\mathbb{P}_{k-1}^d$  amounts to remove the face  $H_0^{[\mathbf{v}]^d}$  from the tetrahedron defined by  $[\mathbf{v}]^d$ .

**Lemma 1597 (sub-vertices are affinely independent).**

Let  $d \geq 1$ . Let  $k \geq 2$ . Let  $[\mathbf{v}]^d$  be  $d+1$  affinely independent points in  $\mathbb{R}^d$ .

Then, the sub-vertices  $[\mathbf{v}]^d$  are affinely independent.

*Proof.* Let  $(\mu_i)_{i \in [1..d]}$  such that  $\sum_{i=1}^d \mu_i (\mathbf{v}_i - \mathbf{v}_0) = 0$ . Then, from Lemma 1595 (equivalent definition of sub-vertices of Lagrange nodes of  $\mathbb{P}_k^d$ ), and **field properties of  $\mathbb{R}$  (with  $k \neq 0$ )**,

$$\sum_{i=1}^d \mu_i (\mathbf{v}_i - \mathbf{v}_0) = \sum_{i=1}^d \mu_i \left( \frac{k-1}{k} \mathbf{v}_i + \left( \frac{1}{k} - 1 \right) \mathbf{v}_0 \right) = \sum_{i=1}^d \mu_i \frac{k-1}{k} (\mathbf{v}_i - \mathbf{v}_0) = 0.$$

Thus, from Definition 1411 (*affinely independent family*), **the zero-product property in  $\mathbb{R}$**  (with  $\frac{k-1}{k} \neq 0$ ), and **the definition of freedom**, we have  $\mu_i = 0$  for all  $i \in [1..d]$ , thus  $(\mathbf{v}_i - \mathbf{v}_0)_{i \in [1..d]}$  is a free family, and  $[\mathbf{v}]^d$  is affinely independent.  $\square$

**Lemma 1598** ( $\mathbb{P}_{k-1}^d$  *sub-nodes of sub-vertices are some nodes of  $\mathbb{P}_{k-1}^d$* ).

Let  $d \geq 1$ . Let  $k \geq 2$ . Let  $[\mathbf{v}]^d$  be  $d+1$  affinely independent points in  $\mathbb{R}^d$ . The Lagrange nodes of  $\mathbb{P}_{k-1}^d$  with respect to the sub-vertices  $[\mathbf{v}]^d$  are called  $\mathbb{P}_{k-1}^d$  sub-nodes and denoted as  $\{\{\mathbf{a}\}\}_{\mathcal{A}_{k-1}^d}$ . Then, these sub-nodes are some of the nodes  $\{\{\mathbf{a}\}\}_{\mathcal{A}_k^d}$  of  $\mathbb{P}_k^d$  (defined with respect to  $[\mathbf{v}]^d$ ),

$$(9.90) \quad \forall \alpha \in \mathcal{A}_{k-1}^d, \quad \mathbf{a}_\alpha = \mathbf{a}_\alpha.$$

*Proof.* Let  $\alpha \in \mathcal{A}_{k-1}^d$ . Then, from Lemma 1597 (*sub-vertices are affinely independent*), Lemma 1591 (*barycentric coordinates of Lagrange nodes of  $\mathbb{P}_k^d$* , (9.84), first for  $\{\{\mathbf{a}\}\}_{\mathcal{A}_{k-1}^d}$  with  $k-1 \geq 1$  and  $[\mathbf{v}]^d$ , then for  $\{\{\mathbf{a}\}\}_{\mathcal{A}_k^d}$  with  $k \geq 1$  and  $[\mathbf{v}]^d$ ), Lemma 1595 (*equivalent definition of sub-vertices of Lagrange nodes of  $\mathbb{P}_k^d$* ), **field properties of  $\mathbb{R}$**  (with  $k \geq 2$ , thus  $k-1 \neq 0$ ), Definition 1480 (*length of multi-indices*), and Lemma 1496 ( *$\mathcal{C}_k^d$  are layers of  $\mathcal{A}_k^d$* , thus  $\alpha \in \mathcal{A}_k^d$ ), we have

$$\begin{aligned} \mathbf{a}_\alpha &= \left(1 - \frac{|\alpha|}{k-1}\right) \mathbf{v}_0 + \sum_{i=1}^d \frac{\alpha_i}{k-1} \mathbf{v}_i = \left(1 - \frac{|\alpha|}{k-1}\right) \mathbf{v}_0 + \sum_{i=1}^d \frac{\alpha_i}{k-1} \left(\frac{1}{k} \mathbf{v}_0 + \frac{k-1}{k} \mathbf{v}_i\right) \\ &= \left(1 - \frac{|\alpha|}{k-1} + \frac{|\alpha|}{(k-1)k}\right) \mathbf{v}_0 + \sum_{i=1}^d \frac{\alpha_i}{k} \mathbf{v}_i = \left(1 - \frac{|\alpha|}{k}\right) \mathbf{v}_0 + \sum_{i=1}^d \frac{\alpha_i}{k} \mathbf{v}_i = \mathbf{a}_\alpha. \end{aligned}$$

$\square$

### 9.6.3 Lagrange nodes for the reference element

**Lemma 1599** (*reference Lagrange nodes of  $\mathbb{P}_k^d$* ).

Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ .

The reference Lagrange nodes of  $\mathbb{P}_k^d$  are denoted  $\{\{\hat{\mathbf{a}}\}\}_{\mathcal{A}_k^d}$ , and are defined as the Lagrange nodes of  $\mathbb{P}_k^d$  for the reference vertices. Let  $\alpha \in \mathcal{A}_k^d$ . Then, we have

$$(9.91) \quad (k=0) \quad \hat{\mathbf{a}}_0 = \hat{\mathbf{g}}^d,$$

$$(9.92) \quad (k \geq 1) \quad \begin{aligned} \hat{\mathbf{a}}_\alpha &= \hat{\mathbf{v}}_0 + \sum_{i=1}^d \frac{\alpha_i}{k} (\hat{\mathbf{v}}_i - \hat{\mathbf{v}}_0) = \sum_{i=1}^d \frac{\alpha_i}{k} \mathbf{e}_i \\ &= \left(1 - \frac{|\alpha|}{k}\right) \hat{\mathbf{v}}_0 + \sum_{i=1}^d \frac{\alpha_i}{k} \hat{\mathbf{v}}_i \quad \text{with} \quad \left(1 - \frac{|\alpha|}{k}\right) + \sum_{i=1}^d \frac{\alpha_i}{k} = 1. \end{aligned}$$

*Proof.* Direct consequence of Definition 1588 (*Lagrange nodes of  $\mathbb{P}_k^d$* ), Lemma 1435 (*reference isobarycenter*, for  $k=0$ ), Lemma 1591 (*barycentric coordinates of Lagrange nodes of  $\mathbb{P}_k^d$* , (9.84) for  $k \geq 1$ ), and Definition 1434 (*family of reference points*, for  $k \geq 1$ ).  $\square$

**Lemma 1600** (*reference Lagrange nodes of  $\mathbb{P}_k^d$  for  $d=1$  are reference Lagrange nodes of  $\mathbb{P}_k^1$* ). Let  $k \in \mathbb{N}$ . Then,  $\{\{\hat{\mathbf{a}}\}\}_{\mathcal{A}_k^d}$  for  $d \stackrel{\text{def.}}{=} 1$  and  $\{\{\hat{\mathbf{a}}\}\}_{\mathcal{A}_k^1}$  from Definition 1453 coincide.

*Proof.* Direct consequence of Lemma 1599 (*reference Lagrange nodes of  $\mathbb{P}_k^d$* , with  $d \stackrel{\text{def.}}{=} 1$ ), Definition 1365 (*canonic families*,  $\mathbf{e}_i$ ), Lemma 1497 (*first multi-indices  $\mathcal{A}_k^d$* , with  $d \stackrel{\text{def.}}{=} 1$ ), and Definition 1453 (*reference Lagrange nodes of  $\mathbb{P}_k^1$* ).  $\square$

**Lemma 1601 (equivalent definition of reference Lagrange nodes of  $\mathbb{P}_k^d$ ).**

Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . Let  $\alpha \in \mathcal{A}_k^d$ . Let  $i \in [1..d]$ . Then, we have

$$(9.93) \quad (k = 0) \quad (\hat{\mathbf{a}}_0)_i = \frac{1}{d+1},$$

$$(9.94) \quad (k \geq 1) \quad (\hat{\mathbf{a}}_\alpha)_i = \frac{\alpha_i}{k}.$$

*Proof.* Direct consequence of Lemma 1599 (reference Lagrange nodes of  $\mathbb{P}_k^d$ ), Lemma 1435 (reference isobarycenter, for  $k = 0$ ), and Definition 1365 (canonic families, for  $k \geq 1$ ).  $\square$

**Lemma 1602 (number of reference Lagrange nodes of  $\mathbb{P}_k^d$ ).**

Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ .

Then, there are  $\binom{k+d}{d}$  distinct reference Lagrange nodes  $\{\{\hat{\mathbf{a}}\}_{\mathcal{A}_k^d}\}$ .

*Proof.* Direct consequence of Lemma 1599 (reference Lagrange nodes of  $\mathbb{P}_k^d$ ), Lemma 1436 (family of reference points is affinely independent), Lemma 1559 (barycentric coordinate, uniqueness in (9.51) with  $j \stackrel{\text{def.}}{=} 0$  and  $\llbracket \hat{\mathbf{v}} \rrbracket^d$ ), and Lemma 1498 (cardinal of  $\mathcal{A}_k^d$ ).  $\square$

*Remark 1603.* In the next lemma, the proof could also have been done using Lemma 1400.

**Lemma 1604 (Lagrange nodes of  $\mathbb{P}_k^d$  are image of reference).**

Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ .

Let  $\llbracket \mathbf{v} \rrbracket^d$  be  $d+1$  points in  $\mathbb{R}^d$ . Let  $\alpha \in \mathcal{A}_k^d$ . Then, we have  $\mathbf{a}_\alpha = \varphi_{\text{geo}}^{\llbracket \mathbf{v} \rrbracket^d}(\hat{\mathbf{a}}_\alpha)$ .

*Proof.* Direct consequence of Lemma 1550 (properties of geometric mapping, (9.38)), Lemma 1601 (equivalent definition of reference Lagrange nodes of  $\mathbb{P}_k^d$ ), **field properties of  $\mathbb{R}$  (with  $d+1 \neq 0$ ) for  $k = 0$** , and Definition 1588 (Lagrange nodes of  $\mathbb{P}_k^d$ ).  $\square$

## 9.6.4 Lagrange nodes and face hyperplanes

**Lemma 1605 (face hyperplanes of Lagrange nodes of  $\mathbb{P}_k^d$ ).**

Let  $d \geq 1$ . Let  $k \geq 1$ .

Let  $\llbracket \mathbf{v} \rrbracket^d$  be  $d+1$  affinely independent points in  $\mathbb{R}^d$ . Let  $\alpha \in \mathcal{A}_k^d$ . Let  $i \in [0..d]$ . Then, we have

$$(9.95) \quad (i = 0) \quad \mathbf{a}_\alpha \in \mathcal{H}_0^{\llbracket \mathbf{v} \rrbracket^d} \iff |\alpha| = k \iff \alpha \in \mathcal{C}_k^d,$$

$$(9.96) \quad (i \in [1..d]) \quad \mathbf{a}_\alpha \in \mathcal{H}_i^{\llbracket \mathbf{v} \rrbracket^d} \iff \alpha_i = 0 \iff \alpha \in \mathcal{A}_{k,i}^d.$$

Thus, we have  $\text{card} \left\{ \mathbf{a}_\alpha \in \mathcal{H}_i^{\llbracket \mathbf{v} \rrbracket^d} \mid \alpha \in \mathcal{A}_k^d \right\} = \binom{k+d-1}{d-1}$ .

*Proof. Equivalences.* Direct consequence of Lemma 1563 (equivalent definition of face hyperplane, (9.57)), Lemma 1591 (barycentric coordinates of Lagrange nodes of  $\mathbb{P}_k^d$ , (9.83)), **field properties of  $\mathbb{R}$  (with  $k > 0$ )**, Definition 1486 (sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$ ), Lemma 1501 (cardinal of  $\mathcal{A}_{k,i}^d$  and  $\mathcal{A}_k^{d-1}$ ), and Lemma 1498 (cardinal of  $\mathcal{A}_k^d$ ).

**Cardinal. Case  $i = 0$ .** Direct consequence of Definition 1486 (sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$ ), and Lemma 1495 (cardinal of  $\mathcal{C}_k^d$ ).

**Case  $i \in [1..d]$ .** Direct consequence of Lemma 1590 (number of Lagrange nodes of  $\mathbb{P}_k^d$ , injectivity, thus  $\text{card} \{ \mathbf{a}_\alpha \in \mathcal{H}_i^{\llbracket \mathbf{v} \rrbracket^d} \mid \alpha \in \mathcal{A}_k^d \} = \text{card} \{ \mathcal{A}_{k,i}^d \}$ ), Lemma 1501 (cardinal of  $\mathcal{A}_{k,i}^d$  and  $\mathcal{A}_k^{d-1}$ ), Lemma 1500 (cardinal of  $\mathcal{C}_k^d$  and  $\mathcal{A}_k^{d-1}$ ), and Lemma 1495 (cardinal of  $\mathcal{C}_k^d$ ).  $\square$

*Remark 1606.* In the next lemma, we recall that  $\theta_i^{d-1}$  is defined in Lemma 1368,  $f_{k,i}^d$  in Lemmas 1500 ( $i = 0$ ) and 1501 ( $i \in [1..d]$ ), and  $\phi_{\pi^i}^{\llbracket \mathbf{v} \rrbracket^d}$  in Definition 1578. See also Figure 9.6.



**Lemma 1607 (image of nodes by geometric hyperface mapping).** Let  $d \geq 2$ . Let  $k \geq 1$ . Let  $[[\mathbf{v}]]^d$  be  $d + 1$  affinely independent points in  $\mathbb{R}^d$ . Let  $\{\{\hat{\mathbf{a}}^{d-1}\}_{\mathcal{A}_k^{d-1}}\}$  be the reference Lagrange nodes of  $\mathbb{P}_k^{d-1}$  and  $\{\{\mathbf{a}\}_{\mathcal{A}_k^d}\}$  be the Lagrange nodes of  $\mathbb{P}_k^d$  for  $[[\mathbf{v}]]^d$ . Let  $i \in [0..d]$ . Let  $\alpha' \in \mathcal{A}_k^{d-1}$ . Then, we have

$$(9.97) \quad \phi_{\theta_i^{d-1}}^{[[\mathbf{v}]]^d}(\hat{\mathbf{a}}_{\alpha'}^{d-1}) = \mathbf{a}_{f_{k,i}^d(\alpha')}.$$

*Proof.* From Lemma 1599 (reference Lagrange nodes of  $\mathbb{P}_k^d$ , (9.92) with  $d - 1 \geq 1$ ), we have

$$\hat{\mathbf{a}}_{\alpha'}^{d-1} = \left(1 - \frac{|\alpha'|}{k}\right) \hat{\mathbf{v}}_0^{d-1} + \sum_{j=1}^{d-1} \frac{\alpha'_j}{k} \hat{\mathbf{v}}_j^{d-1} \quad \text{with} \quad \left(1 - \frac{|\alpha'|}{k}\right) + \sum_{j=1}^{d-1} \frac{\alpha'_j}{k} = 1.$$

**Case  $i = 0$ .** Then, from Lemma 1500 (cardinal of  $\mathcal{C}_k^d$  and  $\mathcal{A}_k^{d-1}$ ), Definition 1486 (sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$ ), Lemma 1591 (barycentric coordinates of Lagrange nodes of  $\mathbb{P}_k^d$ , (9.84)), and **field properties of  $\mathbb{R}$  (with  $k \neq 0$ )**, we have  $f_{k,0}^d(\alpha') = (k - |\alpha'|, \alpha') \in \mathcal{C}_k^d \subset \mathcal{A}_k^d$ ,  $|f_{k,0}^d(\alpha')| = k$ ,

$$\mathbf{a}_{f_{k,0}^d(\alpha')} = \left(1 - \frac{|\alpha'|}{k}\right) \mathbf{v}_1 + \sum_{j=1}^{d-1} \frac{\alpha'_j}{k} \mathbf{v}_{j+1}.$$

**Case  $i \in [1..d]$ .** Then, from Lemma 1501 (cardinal of  $\mathcal{A}_{k,i}^d$  and  $\mathcal{A}_k^{d-1}$ ), Definition 1486 (sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$ ), and Lemma 1591 (barycentric coordinates of Lagrange nodes of  $\mathbb{P}_k^d$ , (9.84)), we have  $f_{k,i}^d(\alpha') = (\alpha'_1, \dots, \alpha'_{i-1}, 0, \alpha'_i, \dots, \alpha'_{d-1}) \in \mathcal{A}_{k,i}^d \subset \mathcal{A}_k^d$ ,  $|f_{k,i}^d(\alpha')| = |\alpha'|$ , and

$$\mathbf{a}_{f_{k,i}^d(\alpha')} = \left(1 - \frac{|\alpha'|}{k}\right) \mathbf{v}_0 + \sum_{j=1}^{i-1} \frac{\alpha'_j}{k} \mathbf{v}_j + \sum_{j=i}^{d-1} \frac{\alpha'_j}{k} \mathbf{v}_{j+1}.$$

Therefore, in both cases,  $\hat{\mathbf{a}}_{\alpha'}^{d-1}$  is the barycenter of  $(\hat{\mathbf{v}}_j^{d-1})_{j \in [0..d-1]}$  and  $\mathbf{a}_{f_{k,i}^d(\alpha')}$  is the barycenter of  $(\mathbf{v}_j)_{j \in [0..d] \setminus \{i\}}$  with the same coefficients. Thus, from Lemma 1584 (geometric hyperface mapping, affineness and (9.74), thus  $\phi_{\theta_i^{d-1}}^{[[\mathbf{v}]]^d}(\hat{\mathbf{v}}_j^{d-1}) = \mathbf{v}_j$  for  $j < i$  and  $\phi_{\theta_i^{d-1}}^{[[\mathbf{v}]]^d}(\hat{\mathbf{v}}_j^{d-1}) = \mathbf{v}_{j+1}$  for  $j \geq i$ ), and Lemma 1400 (affine map preserves barycenter), we have the equality.  $\square$

## 9.7 Lagrange linear forms for $\text{Lag} \mathbb{P}_k^d$

**Definition 1608 (Lagrange linear forms for  $\mathbb{P}_k^d$ ).**

Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . Let  $[[\mathbf{v}]]^d$  be  $d + 1$  points in  $\mathbb{R}^d$ . The Lagrange linear forms associated with the Lagrange nodes of  $\mathbb{P}_k^d$  are denoted  $\Sigma^{\{\mathbf{a}\}_{\mathcal{A}_k^d}} = (\sigma_{\alpha})_{\alpha \in \mathcal{A}_k^d}$ , and are defined by

$$(9.98) \quad \forall \alpha \in \mathcal{A}_k^d, \quad \forall f : \mathbb{R}^d \rightarrow \mathbb{R}, \quad \sigma_{\alpha}(f) \stackrel{\text{def.}}{=} f(\mathbf{a}_{\alpha}).$$

**Lemma 1609 (Lagrange linear forms for  $\mathbb{P}_k^d$  for  $d=1$  are Lagrange linear forms for  $\mathbb{P}_k^1$ ).** Let  $k \in \mathbb{N}$ . Let  $[[\mathbf{v}]]^1 = [[v]]^1$  be two points in  $\mathbb{R}^1$ .

Then,  $\Sigma^{\{\mathbf{a}\}_{\mathcal{A}_k^d}}$  for  $d \stackrel{\text{def.}}{=} 1$  and  $\Sigma^{\{\mathbf{a}\}_{\mathcal{A}_k^1}}$  from Definition 1472 coincide.

*Proof.* Direct consequence of Definition 1608 (Lagrange linear forms for  $\mathbb{P}_k^d$ , with  $d \stackrel{\text{def.}}{=} 1$ ), Lemma 1589 (Lagrange nodes of  $\mathbb{P}_k^d$  for  $d = 1$  are Lagrange nodes of  $\mathbb{P}_k^1$ ), Lemma 1497 (first multi-indices  $\mathcal{A}_k^d$ , with  $d \stackrel{\text{def.}}{=} 1$ ), and Definition 1472 (Lagrange linear forms for  $\mathbb{P}_k^1$ ).  $\square$

**Lemma 1610 (Lagrange linear forms of  $\mathbb{P}_k^d$  are linear).**

Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . Let  $[[\mathbf{v}]]^d$  be  $d + 1$  points in  $\mathbb{R}^d$ . Let  $\alpha \in \mathcal{A}_k^d$ . Then,  $\sigma_\alpha$  is linear.

*Proof.* Direct consequence of Definition 1608 (Lagrange linear forms for  $\mathbb{P}_k^d$ ), and **the definition of linear operations over functions**.  $\square$

**Lemma 1611 (cardinal of Lagrange linear forms of  $\mathbb{P}_k^d$ ).**

Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . Let  $[[\mathbf{v}]]^d$  be  $d + 1$  affinely independent points in  $\mathbb{R}^d$ . Then,  $\text{card}(\Sigma_{\{\mathbf{a}\}_{\mathcal{A}_k^d}}) = \binom{k+d}{d}$ .

*Proof.* Direct consequence of Definition 1608 (Lagrange linear forms for  $\mathbb{P}_k^d$ ), and Lemma 1590 (number of Lagrange nodes of  $\mathbb{P}_k^d$ ).  $\square$

**Definition 1612 (reference Lagrange linear forms for  $\mathbb{P}_k^d$ ).**

Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . The reference Lagrange linear forms associated with the reference Lagrange nodes of  $\mathbb{P}_k^d$  are denoted  $\hat{\Sigma}_k^d = (\hat{\sigma}_\alpha)_{\alpha \in \mathcal{A}_k^d}$ , and are defined by

$$(9.99) \quad \forall \alpha \in \mathcal{A}_k^d, \quad \forall \hat{f} : \mathbb{R}^d \rightarrow \mathbb{R}, \quad \hat{\sigma}_\alpha(\hat{f}) \stackrel{\text{def.}}{=} \hat{f}(\hat{\mathbf{a}}_\alpha).$$

**Lemma 1613 (reference Lagrange linear forms for  $\mathbb{P}_k^d$  for  $d = 1$  are reference Lagrange linear forms for  $\mathbb{P}_k^1$ ).** Let  $k \in \mathbb{N}$ . Then,  $\hat{\Sigma}_k^d$  for  $d \stackrel{\text{def.}}{=} 1$  and  $\hat{\Sigma}_k^1$  from Definition 1456 coincide.

*Proof.* Direct consequence of Definition 1612 (reference Lagrange linear forms for  $\mathbb{P}_k^d$ , with  $d \stackrel{\text{def.}}{=} 1$ ), Lemma 1600 (reference Lagrange nodes of  $\mathbb{P}_k^d$  for  $d = 1$  are reference Lagrange nodes of  $\mathbb{P}_k^1$ ), Lemma 1497 (first multi-indices  $\mathcal{A}_k^d$ , with  $d \stackrel{\text{def.}}{=} 1$ ), and Definition 1456 (reference Lagrange linear forms for  $\mathbb{P}_k^1$ ).  $\square$

**Lemma 1614 (Lagrange linear forms of  $\mathbb{P}_k^d$  are images of reference).**

Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . Let  $[[\mathbf{v}]]^d$  be  $d + 1$  points in  $\mathbb{R}^d$ . Let  $\alpha \in \mathcal{A}_k^d$ . Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ .

Then, we have  $\sigma_\alpha(f) = \hat{\sigma}_\alpha(\hat{f})$  where  $\hat{f} = f \circ \varphi_{\text{geo}}^d[[\mathbf{v}]]^d$ .

*Proof.* Direct consequence of Definition 1608 (Lagrange linear forms for  $\mathbb{P}_k^d$ ), Lemma 1604 (Lagrange nodes of  $\mathbb{P}_k^d$  are image of reference), and Definition 1612 (reference Lagrange linear forms for  $\mathbb{P}_k^d$ ).  $\square$

## 9.8 Unisolvence for $\text{Lag } \mathbb{P}_0^d$

**Lemma 1615 (Lagrange linear forms for  $\mathbb{P}_0^d$  are injective).**

Let  $d \geq 1$ . Let  $[[\mathbf{v}]]^d$  be  $d + 1$  points of  $\mathbb{R}^d$ . Then,  $\phi_{\Sigma_{\{\mathbf{a}\}_{\mathcal{A}_0^d}}}$  is injective.

*Proof.* Direct consequence of Definition 1424 (finite element triple), Lemma 1509 (constant and affine spaces  $\mathbb{P}_0^d$  and  $\mathbb{P}_1^d$ ), thus for all  $p \in \mathbb{P}_0^d$ , there exists  $a_0 \in \mathbb{R}$  such that  $p = (\mathbf{x} \mapsto a_0)$ , Definition 1608 (Lagrange linear forms for  $\mathbb{P}_k^d$ , thus  $\sigma_\alpha(p) = p(\mathbf{a}_\alpha) = a_0 = 0$ , and  $p = 0$ ), Definition 101 (kernel), and Lemma 103 (injective linear map has zero kernel).  $\square$

*Remark 1616.* In the next Lemma, the hypothesis of affinely independent points is not necessary. In fact, Lemmas 1611 and 1590 do not require this hypothesis when  $k \stackrel{\text{def.}}{=} 0$ , as there is a single node set at the isobarycenter of  $[[\mathbf{v}]]^d$ , which is always defined.

However, to have a proper FE, it is required to have a nondegenerate simplex, and thus the hypothesis remains necessary at that stage. This is why we did not relax it here (otherwise, the statements in Lemmas 1611 and 1590 would have been more intricate).

**Lemma 1617 (unisolvence of  $\mathbb{P}_0^d$ ).**

Let  $d \geq 1$ . Let  $[[\mathbf{v}]]^d$  be  $d + 1$  affinely independent points of  $\mathbb{R}^d$ .

Then,  $(K[[\mathbf{v}]]^d, \mathbb{P}_0^d, \Sigma^{\{\mathbf{a}\}_{\mathcal{A}_0^d}})$  satisfies the unisolvence property.

*Proof.* Direct consequence of Lemma 1525 (*dimension of  $\mathbb{P}_k^d$* ), Lemma 1611 (*cardinal of Lagrange linear forms of  $\mathbb{P}_k^d$ , with  $k \stackrel{\text{def.}}{=} 0$* ), Lemma 1364 (*properties of the binomial coefficient, (5.4)*), **the definition of order (reflexivity, thus  $\dim \mathbb{P}_0^d = 1 \geq 1 = \text{card}(\Sigma^{\{\mathbf{a}\}_{\mathcal{A}_0^d}})$** ), Lemma 1615 (*Lagrange linear forms for  $\mathbb{P}_0^d$  are injective*), and Lemma 1426 (*injectivity implies unisolvence*).  $\square$

## 9.9 Unisolvence for $\text{Lag} \mathbb{P}_1^d$

**Lemma 1618 (decomposition of  $\mathbb{P}_1^d$  polynomial with  $\sigma_\alpha$ ).**

Let  $d \geq 1$ . Let  $[[\mathbf{v}]]^d$  be  $d + 1$  affinely independent points in  $\mathbb{R}^d$ . Let  $p \in \mathbb{P}_1^d$ . Then, we have

$$(9.100) \quad p = \sigma_{\mathbf{0}}(p) \lambda_0^{[[\mathbf{v}]]^d} + \sum_{i=1}^d \sigma_{\mathbf{e}_i}(p) \lambda_i^{[[\mathbf{v}]]^d} = \sigma_{\mathbf{0}}(p) \mathcal{L}_0^{[[\mathbf{v}]]^d} + \sum_{i=1}^d \sigma_{\mathbf{e}_i}(p) \mathcal{L}_i^{[[\mathbf{v}]]^d}.$$

*Proof.* Direct consequence of Definition 1608 (*Lagrange linear forms for  $\mathbb{P}_k^d$* ), Lemma 1593 (*Lagrange nodes of  $\mathbb{P}_1^d$  are vertices, thus  $\sigma_{\mathbf{0}}(p) = p(\mathbf{v}_0)$ , and  $\sigma_{\mathbf{e}_i}(p) = p(\mathbf{v}_i)$  for all  $i \in [1..d]$* ), Lemma 1561 (*decomposition of  $\mathbb{P}_1^d$  polynomial with barycentric coordinates*), and Lemma 1555 (*decomposition of  $\mathbb{P}_1^d$  polynomial in Lagrange basis*).  $\square$

**Lemma 1619 (Lagrange linear forms for  $\mathbb{P}_1^d$  are injective).**

Let  $d \geq 1$ . Let  $[[\mathbf{v}]]^d$  be  $d + 1$  affinely independent points in  $\mathbb{R}^d$ . Then,  $\phi_{\Sigma^{\{\mathbf{a}\}_{\mathcal{A}_1^d}}}$  is injective.

*Proof.* Direct consequence of Definition 1424 (*finite element triple*), Lemma 1497 (*first multi-indices  $\mathcal{A}_k^d$ , with  $k \stackrel{\text{def.}}{=} 1$* ), Lemma 1618 (*decomposition of  $\mathbb{P}_1^d$  polynomial with  $\sigma_\alpha$ , thus  $\forall \alpha \in \mathcal{A}_1^d$ ,  $\sigma_\alpha(p) = 0$  implies  $p = 0$* ), Definition 101 (*kernel*), and Lemma 103 (*injective linear map has zero kernel*).  $\square$

**Lemma 1620 (unisolvence of  $\mathbb{P}_1^d$ ).** Let  $d \geq 1$ . Let  $[[\mathbf{v}]]^d$  be  $d + 1$  affinely independent points in  $\mathbb{R}^d$ . Then,  $(K[[\mathbf{v}]]^d, \mathbb{P}_1^d, \Sigma^{\{\mathbf{a}\}_{\mathcal{A}_1^d}})$  satisfies the unisolvence property.

*Proof.* Direct consequence of Lemma 1525 (*dimension of  $\mathbb{P}_k^d$* ), Lemma 1611 (*cardinal of Lagrange linear forms of  $\mathbb{P}_k^d$ , with  $k \stackrel{\text{def.}}{=} 1$* ), Lemma 1364 (*properties of the binomial coefficient, (5.5) with  $n \stackrel{\text{def.}}{=} d + 1$* ), **the definition of order (reflexivity, thus  $\dim \mathbb{P}_1^d = d + 1$  is greater than or equal to  $d + 1 = \text{card}(\Sigma^{\{\mathbf{a}\}_{\mathcal{A}_1^d})$** ), Lemma 1619 (*Lagrange linear forms for  $\mathbb{P}_1^d$  are injective*), and Lemma 1426 (*injectivity implies unisolvence*).  $\square$

## 9.10 Unisolvence for $\text{Lag} \mathbb{P}_k^d$

**Lemma 1621 (factorization of zero polynomial on last reference hyperplane).**

Let  $d \geq 1$ . Let  $k \geq 1$ . Let  $\hat{p} \in \mathbb{P}_k^d$ . Then, we have the equivalence

$$(9.101) \quad \hat{p}|_{\hat{\mathcal{H}}_d^d} = 0 \iff \exists \hat{q} \in \mathbb{P}_{k-1}^d, \quad \hat{p} = \hat{\mathcal{L}}_d^{1,d} \hat{q}.$$

*Proof. From right to left.* Direct consequence of Lemma 1564 (*reference face hyperplane*, (9.59) with  $i \stackrel{\text{def.}}{=} d$ ), Definition 101 (*kernel*), and **the zero-product property in  $\mathbb{R}$** .

**From left to right.** From Lemma 1529 (*decomposition of  $\mathbb{P}_k^d$* ), Lemma 1564 (*reference face hyperplane*, (9.59) with  $i \stackrel{\text{def.}}{=} d$ ), and Definition 1541 (*reference Lagrange polynomials of  $\mathbb{P}_1^d$* , (9.30) with  $i \stackrel{\text{def.}}{=} d$ ), there exists  $\tilde{p}_0 \in \mathbb{P}_{k-1}^{d-1}$  and  $\hat{q} \in \mathbb{P}_{k-1}^d$  such that  $\hat{p} = \tilde{p}_0 + X_d \hat{q}$ , thus for all  $\hat{\mathbf{x}} = (\hat{x}_1, \dots, \hat{x}_{d-1}, 0) \in \hat{\mathcal{H}}_d^d$ , we have

$$0 = \hat{p}(\hat{\mathbf{x}}) = \tilde{p}_0(\hat{x}_1, \dots, \hat{x}_{d-1}) + 0 \hat{q}(\hat{x}_1, \dots, \hat{x}_{d-1}, 0) = \tilde{p}_0(\hat{x}_1, \dots, \hat{x}_{d-1}),$$

i.e.  $\tilde{p}_0 = 0$ , and  $\hat{p} = \hat{\mathcal{L}}_d^{1,d} \hat{q}$ . □

*Remark 1622.* In the second part of the proof of the next lemma, we used the circular permutation  $c_i^d$ , see Lemma 1366. The geometric mapping with permutation  $\phi_{c_i^d}^{\llbracket \mathbf{v} \rrbracket^d}$  is illustrated in Figure 9.7 when  $i = 0$  and  $d = 3$ .

Note that one could also use for  $\pi^d$  the transposition  $\tau_i^d$  exchanging  $i$  and  $d$ , see Lemma 1367, which is also injective and satisfies  $\tau_i^d(d) = i$ .

**Lemma 1623 (*factorization of zero polynomial on hyperplane  $\mathbb{P}_k^d$* ).** Let  $d \geq 1$ .

Let  $k \geq 1$ . Let  $\llbracket \mathbf{v} \rrbracket^d$  be  $d + 1$  affinely independent points in  $\mathbb{R}^d$ . Let  $p \in \mathbb{P}_k^d$ . Let  $i \in [0..d]$ . Then, we have the equivalence

$$(9.102) \quad p|_{\mathcal{H}_i^{\llbracket \mathbf{v} \rrbracket^d}} = 0 \iff \exists q \in \mathbb{P}_{k-1}^d, \quad p = \lambda_i^{\llbracket \mathbf{v} \rrbracket^d} q.$$

*Proof. From right to left.* Direct consequence of Lemma 1563 (*equivalent definition of face hyperplane*, (9.57) with  $i \stackrel{\text{def.}}{=} 0$ ), Definition 101 (*kernel*), and **the zero-product property in  $\mathbb{R}$** .

**From left to right.** Assume that  $p|_{\mathcal{H}_i^{\llbracket \mathbf{v} \rrbracket^d}} = 0$ . Let  $\hat{p} \stackrel{\text{def.}}{=} p \circ \phi_{c_i^d}^{\llbracket \mathbf{v} \rrbracket^d}$ .

Then, from Lemma 1366 (*circular permutation*, thus  $c_i^d$  bijective,  $c_i^d(d) = i$  and  $(c_i^d)^{-1}(i) = d$ ), **the definition of bijectivity (implies injectivity)**, Lemma 1586 (*geometric mapping with permutation*, with  $\pi^d \stackrel{\text{def.}}{=} c_i^d$ , (9.78) with  $j \stackrel{\text{def.}}{=} d$ , thus  $\phi_{c_i^d}^{\llbracket \mathbf{v} \rrbracket^d}(\hat{\mathcal{H}}_d^d) = \mathcal{H}_i^{\llbracket \mathbf{v} \rrbracket^d}$  and  $\hat{p}|_{\hat{\mathcal{H}}_d^d} = 0$ ), Lemma 1621 (*factorization of zero polynomial on last reference hyperplane*), Lemma 1586 (*geometric mapping with permutation*, thus  $\phi_{c_i^d}^{\llbracket \mathbf{v} \rrbracket^d}$  is affine and bijective, and (9.77) with  $i \stackrel{\text{def.}}{=} i$ ), **the rules of composition with a bijective function**, Lemma 1410 (*inverse of affine map is affine map*), and Lemma 1540 (*affine mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^l$* , with  $l \stackrel{\text{def.}}{=} d$ ,  $k \stackrel{\text{def.}}{=} k - 1$  and  $f \stackrel{\text{def.}}{=} (\phi_{c_i^d}^{\llbracket \mathbf{v} \rrbracket^d})^{-1}$  affine), there exists  $\hat{q} \in \mathbb{P}_{k-1}^d$  such that  $\hat{p} = \hat{\mathcal{L}}_d^{1,d} \hat{q}$ , and thus we have

$$p = \left( \hat{\mathcal{L}}_{(c_i^d)^{-1}(i)}^{1,d} \circ \left( \phi_{c_i^d}^{\llbracket \mathbf{v} \rrbracket^d} \right)^{-1} \right) \left( \hat{q} \circ \left( \phi_{c_i^d}^{\llbracket \mathbf{v} \rrbracket^d} \right)^{-1} \right) = \lambda_i^{\llbracket \mathbf{v} \rrbracket^d} q,$$

with  $q \stackrel{\text{def.}}{=} \hat{q} \circ \left( \phi_{c_i^d}^{\llbracket \mathbf{v} \rrbracket^d} \right)^{-1} \in \mathbb{P}_{k-1}^d$ . □

*Remark 1624.* See the sketch of the next two proofs in Section 3.1.

The first step dealing with the induction from dimension  $d - 1$  to  $d$  (with constant degree  $k$ ) is illustrated in Figure 9.6. The second step that passes from degree  $k - 1$  to  $k$  (with constant dimension  $d$ ) is illustrated in Figure 9.8.

**Lemma 1625 (*Lagrange linear forms for  $\mathbb{P}_k^d$  are injective*).** Let  $d \geq 1$ . Let  $k \geq 1$ .

Let  $\llbracket \mathbf{v} \rrbracket^d$  be  $d + 1$  affinely independent points in  $\mathbb{R}^d$ . Then,  $\phi_{\Sigma_{\mathbf{a}} \mathcal{A}_k^d}$  is injective.

*Proof.* For all  $d \geq 1$  and  $k \geq 1$ , let  $P$  be the predicate defined by

$$P(d, k) \stackrel{\text{def.}}{=} \left[ \forall [\mathbf{v}]^d \in \mathbb{R}^d, \quad [\mathbf{v}]^d \text{ affinely independent} \implies \phi_{\Sigma \{\mathbf{a}\}_{\mathcal{A}_k^d}} \text{ is injective} \right]$$

where the application  $\phi_{\Sigma \{\mathbf{a}\}_{\mathcal{A}_k^d}}$  is defined in Definition 1424 (*finite element triple*), the Lagrange linear forms  $\Sigma \{\mathbf{a}\}_{\mathcal{A}_k^d}$  are defined in Definition 1608 (*Lagrange linear forms for  $\mathbb{P}_k^d$* ), and the Lagrange nodes  $\{\{\mathbf{a}\}_{\mathcal{A}_k^d}\}$  are defined in Definition 1588 (*Lagrange nodes of  $\mathbb{P}_k^d$* ).

**Double induction:  $P(1, k)$ , for all  $k \geq 1$ .**

Direct consequence of Lemma 1413 (*affinely independent family of 2 elements, thus  $v_0 \neq v_1$* ), Lemma 1476 (*Lagrange linear forms for  $\mathbb{P}_k^1$  are injective*), and Lemma 1609 (*Lagrange linear forms for  $\mathbb{P}_k^d$  for  $d=1$  are Lagrange linear forms for  $\mathbb{P}_k^1$* ).

**Double induction:  $P(d, 1)$ , for all  $d \geq 1$ .**

Direct consequence of Lemma 1619 (*Lagrange linear forms for  $\mathbb{P}_1^d$  are injective*).

**Double induction:  $P(d-1, k) \wedge P(d, k-1) \implies P(d, k)$ , for all  $d, k \geq 2$ .**

Let  $d \geq 2$ . Let  $k \geq 2$ . Assume that  $P(d-1, k)$  and  $P(d, k-1)$  hold.

Let  $[\mathbf{v}]^d$  be  $d+1$  affinely independent points in  $\mathbb{R}^d$ . Let  $p \in \mathbb{P}_k^d$  such that for all  $\alpha \in \mathcal{A}_k^d$ ,  $p(\mathbf{a}_\alpha) = 0$  where the Lagrange nodes  $\{\{\mathbf{a}\}_{\mathcal{A}_k^d}\}$  are defined in Definition 1588 (*Lagrange nodes of  $\mathbb{P}_k^d$* ).

**Step 1: factorization using  $P(d-1, k)$ .** Let  $p_0 \stackrel{\text{def.}}{=} p \circ \phi_{\theta_0^{d-1}}^{[\mathbf{v}]^d}$ .

Let  $\alpha' \in \mathcal{A}_k^{d-1}$ . Then, from Lemma 1585 (*hyperface geometric mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^{d-1}$ , with  $i \stackrel{\text{def.}}{=} 0$ , thus  $p_0 \in \mathbb{P}_k^{d-1}$* ), Lemma 1496 ( *$\mathcal{C}_k^d$  are layers of  $\mathcal{A}_k^d$* ), Lemma 1500 (*cardinal of  $\mathcal{C}_k^d$  and  $\mathcal{A}_k^{d-1}$ , thus  $f_{k,0}^d(\alpha') \in \mathcal{C}_k^d \subset \mathcal{A}_k^d$* ), Lemma 1607 (*image of nodes by geometric hyperface mapping, with  $i \stackrel{\text{def.}}{=} 0$* ), and the assumption on  $p$ , we have  $p_0(\hat{\mathbf{a}}_{\alpha'}^{d-1}) = p(\mathbf{a}_{f_{k,0}^d(\alpha')}) = 0$ .

Then, from Lemma 1436 (*family of reference points is affinely independent, with  $d-1 \geq 1$* ), and the property  $P(d-1, k)$  applied to the reference vertices  $[\hat{\mathbf{v}}^{d-1}]^{d-1}$  (thus using the reference nodes  $\{\{\hat{\mathbf{a}}\}_{\mathcal{A}_k^{d-1}}\}$ ),  $\phi_{\Sigma \{\hat{\mathbf{a}}\}_{\mathcal{A}_k^{d-1}}}$  is injective. Hence, from Lemma 103 (*injective linear map has zero kernel*), Definition 101 (*kernel*), Definition 1424 (*finite element triple*), and Definition 1608 (*Lagrange linear forms for  $\mathbb{P}_k^d$* ), we have  $\phi_{\Sigma \{\hat{\mathbf{a}}\}_{\mathcal{A}_k^{d-1}}}(p_0) = (p_0(\hat{\mathbf{a}}_{\alpha'}^{d-1}))_{\alpha' \in \mathcal{A}_k^{d-1}} = 0$  implies  $p_0 = 0$ .

Finally, from Lemma 1584 (*geometric hyperface mapping, with  $i \stackrel{\text{def.}}{=} 0$ , bijection from  $\mathbb{R}^{d-1}$  onto  $\mathcal{H}_0^{[\mathbf{v}]^d}$* ), **the rules of composition with a bijective function**, and Lemma 1623 (*factorization of zero polynomial on hyperplane  $\mathbb{P}_k^d$ , with  $i \stackrel{\text{def.}}{=} 0$* ), we have

$$\forall \mathbf{x} \in \mathcal{H}_0^{[\mathbf{v}]^d}, \quad p(\mathbf{x}) = p_0 \circ \left( \phi_{\theta_0^{d-1}}^{[\mathbf{v}]^d} \right)^{-1}(\mathbf{x}) = 0,$$

i.e.  $p|_{\mathcal{H}_0^{[\mathbf{v}]^d}} = 0$ , and there exists  $q \in \mathbb{P}_{k-1}^d$  such that  $p = \lambda_0^{[\mathbf{v}]^d} q$ .

**Step 2: cancellation using  $P(d, k-1)$ .** Let  $\alpha \in \mathcal{A}_{k-1}^d$ .

Then, from Lemma 1496 ( *$\mathcal{C}_k^d$  are layers of  $\mathcal{A}_k^d$ , thus  $\alpha \in \mathcal{A}_k^d$  and  $\mathcal{A}_k^d \setminus \mathcal{C}_k^d = \mathcal{A}_{k-1}^d$* ), Lemma 1605 (*face hyperplanes of Lagrange nodes of  $\mathbb{P}_k^d$ , (9.95), contrapositive, thus  $\mathbf{a}_\alpha \notin \mathcal{H}_0^{[\mathbf{v}]^d}$* ), Definition 101 (*kernel*), Lemma 1563 (*equivalent definition of face hyperplane, (9.57) with  $i \stackrel{\text{def.}}{=} 0$ , contrapositive, thus  $\lambda_0^{[\mathbf{v}]^d}(\mathbf{a}_\alpha) \neq 0$* ), the assumption on  $p$ , and **the zero-product property in  $\mathbb{R}$  (applied to  $\lambda_0^{[\mathbf{v}]^d}(\mathbf{a}_\alpha) q(\mathbf{a}_\alpha) = 0$ )**, we have  $q(\mathbf{a}_\alpha) = 0$ . Thus, from Lemma 1598 ( *$\mathbb{P}_{k-1}^d$  sub-nodes of sub-vertices are some nodes of  $\mathbb{P}_{k-1}^d$* ), we have  $q(\mathbf{a}_\alpha) = 0$ .

Finally, from Lemma 1597 (*sub-vertices are affinely independent*), and the property  $P(d, k-1)$  applied to the sub-vertices  $[\mathbf{v}]^d$  (thus using the  $\mathbb{P}_{k-1}^d$  sub-nodes  $\{\{\mathbf{a}\}_{\mathcal{A}_{k-1}^d}\}$ ),  $\phi_{\Sigma \{\mathbf{a}\}_{\mathcal{A}_{k-1}^d}}$  is injective. Hence, from Lemma 103 (*injective linear map has zero kernel*), Definition 101 (*kernel*), and

Definition 1424 (*finite element triple*), we have

$$\phi_{\Sigma, \{\mathbf{a}\}_{\mathcal{A}_{k-1}^d}}(q) = (q(\mathbf{a}_\alpha))_{\alpha \in \mathcal{A}_{k-1}^d} = 0 \implies q = 0.$$

Thus, we also have  $p = 0$ , and from Lemma 103 (*injective linear map has zero kernel*), Definition 101 (*kernel*), and Definition 1424 (*finite element triple*),  $\phi_{\Sigma, \{\mathbf{a}\}_{\mathcal{A}_k^d}}$  is injective, i.e. the property  $P(d, k)$  holds, which concludes the double induction.

Therefore, from Lemma 1358 (*double induction by diagonal*, starting from 1 for both  $d$  and  $k$ ), the property  $P(d, k)$  holds for all  $d, k \geq 1$ .  $\square$

**Theorem 1626 (unisolvence of  $\mathbb{P}_k^d$ ).** *Let  $d \geq 1$ . Let  $k \geq 1$ . Let  $[\mathbf{v}]^d$  be  $d+1$  affinely independent points in  $\mathbb{R}^d$ . Then,  $(K[\mathbf{v}]^d, \mathbb{P}_k^d, \Sigma^{\{\mathbf{a}\}_{\mathcal{A}_k^d}})$  satisfies the unisolvence property.*

*Proof.* Direct consequence of Lemma 1525 (*dimension of  $\mathbb{P}_k^d$* ), Lemma 1611 (*cardinal of Lagrange linear forms of  $\mathbb{P}_k^d$* ), **the definition of order (reflexivity, thus  $\dim \mathbb{P}_k^d \geq \text{card}(\Sigma^{\{\mathbf{a}\}_{\mathcal{A}_k^d}})$** ), Definition 1608 (*Lagrange linear forms for  $\mathbb{P}_k^d$* ), Lemma 1625 (*Lagrange linear forms for  $\mathbb{P}_k^d$  are injective*), and Lemma 1426 (*injectivity implies unisolvence*).  $\square$

*Remark 1627.*

The proof of next lemma (from left to right) is similar to step 1 in the proof of Lemma 1625.

This face unisolvence result allows to ensure the continuity of functions that are piecewise  $\mathbb{P}_k^d$  in each element (assuming that the mesh is conforming). Indeed, let  $f$  be a function that is in  $\mathbb{P}_k^d$  in two adjacent mesh cells  $K$  and  $K'$ , sharing a face  $\mathcal{F} = K \cap K'$ . To have  $f$  continuous, it suffices to enforce that the two polynomials  $f|_K$  and  $f|_{K'}$  take the same values at the nodes of the face  $\mathcal{F}$ .

**Lemma 1628 (face unisolvence of  $\mathbb{P}_k^d$ ).** *Let  $d \geq 2$ . Let  $k \geq 1$ . Let  $\mathcal{A}_{k,0}^d \stackrel{\text{def.}}{=} \mathcal{C}_k^d$ . Let  $[\mathbf{v}]^d$  be  $d+1$  affinely independent points in  $\mathbb{R}^d$ . Let  $p \in \mathbb{P}_k^d$ . Let  $i \in [0..d]$ . Then, we have the face unisolvence property:*

$$(9.103) \quad (\forall \alpha \in \mathcal{A}_{k,i}^d, \quad p(\mathbf{a}_\alpha) = 0) \iff p|_{\mathcal{H}_i^d[\mathbf{v}]^d} = 0.$$

*Proof. From right to left.*

Direct consequence of Lemma 1605 (*face hyperplanes of Lagrange nodes of  $\mathbb{P}_k^d$* , thus  $\mathbf{a}_\alpha \in \mathcal{H}_i^d[\mathbf{v}]^d$ ).

**From left to right.** Assume that for all  $\alpha \in \mathcal{A}_{k,i}^d$ , we have  $p(\mathbf{a}_\alpha) = 0$ . Let  $p_i \stackrel{\text{def.}}{=} p \circ \phi_{\theta_i^{d-1}}^d$ .

Let  $\alpha' \in \mathcal{A}_k^{d-1}$ . Then, from Lemma 1585 (*hyperface geometric mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^{d-1}$* , thus  $p_i \in \mathbb{P}_k^{d-1}$ ), Lemma 1500 (*cardinal of  $\mathcal{C}_k^d$  and  $\mathcal{A}_k^{d-1}$* , for  $i = 0$ , thus  $f_{k,0}^d(\alpha') \in \mathcal{A}_{k,0}^d$ ), Lemma 1501 (*cardinal of  $\mathcal{A}_{k,i}^d$  and  $\mathcal{A}_k^{d-1}$* , for  $i \in [1..d]$ , thus  $f_{k,i}^d(\alpha') \in \mathcal{A}_{k,i}^d$ ), Lemma 1607 (*image of nodes by geometric hyperface mapping*), and the assumption on  $p$ , we have  $p_i(\hat{\mathbf{a}}_{\alpha'}^{d-1}) = p(\mathbf{a}_{f_{k,i}^d(\alpha')}) = 0$ .

Then, from Lemma 1436 (*family of reference points is affinely independent*, with  $d-1 \geq 1$ ), Lemma 1625 (*Lagrange linear forms for  $\mathbb{P}_k^d$  are injective*, with  $d-1 \geq 1$  and  $[\hat{\mathbf{v}}^{d-1}]^{d-1}$ ), Lemma 103 (*injective linear map has zero kernel*), Definition 101 (*kernel*), Definition 1424 (*finite element triple*), and Definition 1608 (*Lagrange linear forms for  $\mathbb{P}_k^d$* ),  $\phi_{\Sigma, \{\mathbf{a}\}_{\mathcal{A}_k^{d-1}}}$  is injective, and thus  $\phi_{\Sigma, \{\mathbf{a}\}_{\mathcal{A}_k^{d-1}}}(p_i) = (p_i(\hat{\mathbf{a}}_{\alpha'}^{d-1}))_{\alpha' \in \mathcal{A}_k^{d-1}} = 0$  implies  $p_i = 0$ .

Finally, from Lemma 1584 (*geometric hyperface mapping*, bijection from  $\mathbb{R}^{d-1}$  onto  $\mathcal{H}_i^d[\mathbf{v}]^d$ ), and **the rules of composition with a bijective function**, we have

$$\forall \mathbf{x} \in \mathcal{H}_i^d[\mathbf{v}]^d, \quad p(\mathbf{x}) = p_i \circ \left( \phi_{\theta_i^{d-1}}^d \right)^{-1}(\mathbf{x}) = 0,$$

i.e.  $p|_{\mathcal{H}_i^d[\mathbf{v}]^d} = 0$ .  $\square$

## 9.11 ${}^{\text{Lag}}\mathbb{P}_k^d$ Lagrange finite element

**Theorem 1629** ( ${}^{\text{Lag}}\mathbb{P}_k^d$  Lagrange finite element). *Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . Let  $[\mathbf{v}]^d$  be  $d+1$  affinely independent points of  $\mathbb{R}^d$ . Then,  ${}^{\text{Lag}}\mathbb{P}_k^d \stackrel{\text{def.}}{=} \left( K^{[\mathbf{v}]^d}, \mathbb{P}_k^d, \Sigma^{\{\mathbf{a}\}_{\mathcal{A}_k^d}} \right)$  is a finite element.*

*It is called the Lagrange finite element of degree  $k$  associated with vertices  $[\mathbf{v}]^d$ .*

*Proof.* Direct consequence of Lemma 1557 (*nontrivial simplex*), Lemma 1525 (*dimension of  $\mathbb{P}_k^d$ , thus  $\dim \mathbb{P}_k^d \geq 1$  and  $\mathbb{P}_k^d \neq \{0\}$* ), Lemma 1617 (*unisolvence of  $\mathbb{P}_0^d$ , for  $k=0$* ), Theorem 1626 (*unisolvence of  $\mathbb{P}_k^d$ , for  $k \geq 1$* ), and Definition 1424 (*finite element triple, with  $q \stackrel{\text{def.}}{=} 1$* ).  $\square$

**Lemma 1630** ( ${}^{\text{Lag}}\mathbb{P}_k^d$  Lagrange finite element for  $d=1$  is Lagrange finite element on a current segment). *Let  $k \in \mathbb{N}$ . Let  $[\mathbf{v}]^1 = [v]^1$  be two points in  $\mathbb{R}^1$ .*

*Then,  ${}^{\text{Lag}}\mathbb{P}_k^d$  for  $d \stackrel{\text{def.}}{=} 1$  and  ${}^{\text{Lag}}\mathbb{P}_k^1$  from Theorem 1478 coincide.*

*Proof.* Direct consequence of Theorem 1629 ( ${}^{\text{Lag}}\mathbb{P}_k^d$  Lagrange finite element, with  $d \stackrel{\text{def.}}{=} 1$ ), Lemma 1609 (*Lagrange linear forms for  $\mathbb{P}_k^d$  for  $d=1$  are Lagrange linear forms for  $\mathbb{P}_k^1$* ), and Theorem 1478 ( ${}^{\text{Lag}}\mathbb{P}_k^1$  Lagrange finite element).  $\square$

**Theorem 1631** ( ${}^{\text{Lag}}\hat{\mathbb{P}}_k^d$  reference Lagrange finite element).

*Let  $d \geq 1$ . Let  $k \in \mathbb{N}$ . Then,  ${}^{\text{Lag}}\hat{\mathbb{P}}_k^d \stackrel{\text{def.}}{=} \left( \hat{K}^d, \mathbb{P}_k^d, \hat{\Sigma}_k^d \right)$  is a finite element.*

*It is called the reference Lagrange finite element of degree  $k$  in dimension  $d$ .*

*Proof.* Direct consequence of Lemma 1436 (*family of reference points is affinely independent*), Lemma 1443 (*simplex of reference vertices is reference simplex*), Lemma 1599 (*reference Lagrange nodes of  $\mathbb{P}_k^d$* ), Definition 1612 (*reference Lagrange linear forms for  $\mathbb{P}_k^d$ , thus  $\hat{\Sigma}_k^d = \Sigma^{\{\mathbf{a}\}_{\mathcal{A}_k^d}}$* ), and Theorem 1629 ( ${}^{\text{Lag}}\mathbb{P}_k^d$  Lagrange finite element).  $\square$

**Lemma 1632** ( ${}^{\text{Lag}}\hat{\mathbb{P}}_k^d$  reference Lagrange finite element for  $d=1$  is Lagrange finite element on the reference segment).

*Let  $k \in \mathbb{N}$ . Then,  ${}^{\text{Lag}}\hat{\mathbb{P}}_k^d$  for  $d \stackrel{\text{def.}}{=} 1$  and  ${}^{\text{Lag}}\hat{\mathbb{P}}_k^1$  from Theorem 1460 coincide.*

*Proof.* Direct consequence of Theorem 1631 ( ${}^{\text{Lag}}\hat{\mathbb{P}}_k^d$  reference Lagrange finite element, with  $d \stackrel{\text{def.}}{=} 1$ ), Lemma 1613 (*reference Lagrange linear forms for  $\mathbb{P}_k^d$  for  $d=1$  are reference Lagrange linear forms for  $\mathbb{P}_k^1$* ), and Theorem 1460 ( ${}^{\text{Lag}}\hat{\mathbb{P}}_k^1$  reference Lagrange finite element).  $\square$

## Chapter 10

# Conclusions, perspectives

We have presented very detailed proofs for the building of the simplicial Lagrange finite element (FE) in any nonzero dimension  $d$ , and of any order  $k$ . This includes the general definition of FE, some results about simplicial geometry, the simplified construction in the monodimensional case on a segment (mainly for didactic purposes, but also to prepare the tensorization in  $d$ -cuboids), the construction of multi-indices of given maximum length, results about multi-variate polynomials,  $\mathbb{P}_1^d$  Lagrange polynomials, affine geometric mappings, Lagrange nodes and Lagrange linear forms of  ${}^{Lag}\mathbb{P}_k^d$ , and finally the proofs of unisolvence of  ${}^{Lag}\mathbb{P}_k^d$ , and of face unisolvence.

The short-term purpose of this work was to help the formalization in a formal proof assistant such as Coq of the basic concepts of FE, and the building of the simplest example:  ${}^{Lag}\mathbb{P}_k^d$ , that is emblematic and widely used in practice. First milestones towards this will be [30, 9] where special attention will be paid to the formalization of multi-indices.

Our mid-term purpose is now to continue up to the formalization of the quadrangular and hexahedric Lagrange FE, then of other (non-nodal) families of FE such as the face-flux driven Raviart-Thomas FE.

The long-term purpose of these studies is the formal proof of programs implementing the finite element method. As a consequence, after having addressed the formalization of the Lax–Milgram theorem [14, 6], and the formalization of parts of measure theory and Lebesgue integration of nonnegative functions [15, 7, 8], we will also have to write very detailed pen-and-paper proofs for  $L^p$  Lebesgue spaces and  $W^{m,p}$  Sobolev spaces as Banach spaces, including parts of the distribution theory, and the concepts and results of the interpolation and approximation theory to define the Finite Element Method itself.





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# Appendix A

## Lists of statements

This appendix collects the references (name and number) for all statements present in Part II. These are split into definitions, lemmas, and theorems.

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## Appendix B

# The proof cites explicitly...

This appendix gathers the explicit citations of the statements listed in Appendix A that appear in the proof of each result (lemmas and theorems). Statements from [14, 15] are anonymized.

The dual graph is described in Appendix C.

Printing is not advised!

**The proof of Lemma 1358** (*double induction by diagonal*)

has no explicit citation.

**The proof of Lemma 1360** (*strong double induction*)

has no explicit citation.

**The proof of Lemma 1364** (*properties of the binomial coefficient*)

cites explicitly:

Definition 1362 (*binomial coefficient*).

**The proof of Lemma 1366** (*circular permutation*)

has no explicit citation.

**The proof of Lemma 1367** (*transposition*)

has no explicit citation.

**The proof of Lemma 1368** (*jump enumeration*)

has no explicit citation.

**The proof of Lemma 1369** (*image of ker is included in ker*)

cites explicitly:

Statement(s) from [14].

**The proof of Lemma 1370** (*image of ker is ker*)

cites explicitly:

Statement(s) from [14],

Lemma 1369 (*image of ker is included in ker*).

**The proof of Lemma 1372** (*vector subspace is invariant by translation*)

cites explicitly:

Statement(s) from [14].

**The proof of Lemma 1373** (*range of linear map is vector subspace*)

cites explicitly:

Statement(s) from [14].

**The proof of Lemma 1374** (*injectivity or surjectivity and dimension implies bijectivity*)

cites explicitly:  
Statement(s) from [14].

**The proof of Lemma 1375** (*inverse of isomorphism is linear map*)

cites explicitly:  
Statement(s) from [14].

**The proof of Lemma 1376** (*free family of dim elements is basis*)

has no explicit citation.

**The proof of Lemma 1380** (*origin is in affine subspace*)

cites explicitly:  
Statement(s) from [14],  
Definition 1379 (*affine subspace*).

**The proof of Lemma 1381** (*equivalent definition of affine subspace*)

cites explicitly:  
Statement(s) from [14],  
Lemma 1372 (*vector subspace is invariant by translation*),  
Definition 1379 (*affine subspace*).

**The proof of Lemma 1382** (*affine subspace is invariant by change of origin*)

cites explicitly:  
Statement(s) from [14],  
Lemma 1372 (*vector subspace is invariant by translation*),  
Definition 1379 (*affine subspace*),  
Lemma 1381 (*equivalent definition of affine subspace*).

**The proof of Lemma 1383** (*vector subspace is affine subspace*)

cites explicitly:  
Lemma 1372 (*vector subspace is invariant by translation*),  
Lemma 1381 (*equivalent definition of affine subspace*).

**The proof of Lemma 1385** (*affine subspace plus vector subspace is affine subspace*)

cites explicitly:  
Statement(s) from [14],  
Definition 1379 (*affine subspace*),  
Lemma 1381 (*equivalent definition of affine subspace*).

**The proof of Lemma 1386** (*closed under barycenter is affine subspace*)

cites explicitly:  
Statement(s) from [14],  
Definition 1379 (*affine subspace*),  
Lemma 1381 (*equivalent definition of affine subspace*).

**The proof of Lemma 1387** (*barycenter closure is affine subspace*)

cites explicitly:  
Statement(s) from [14],  
Definition 1379 (*affine subspace*).

**The proof of Lemma 1390** (*equivalent definition of affine map*)

cites explicitly:  
Definition 1379 (*affine subspace*),  
Lemma 1381 (*equivalent definition of affine subspace*),  
Definition 1388 (*affine map*).

**The proof of Lemma 1391 (*change of origin in affine map*)**

cites explicitly:

Statement(s) from [14],

Definition 1379 (*affine subspace*),

Lemma 1381 (*equivalent definition of affine subspace*),

Definition 1388 (*affine map*).

**The proof of Lemma 1393 (*range of affine map is affine subspace*)**

cites explicitly:

Lemma 1373 (*range of linear map is vector subspace*),

Definition 1379 (*affine subspace*),

Definition 1388 (*affine map*).

**The proof of Lemma 1396 (*space of affine maps*)**

cites explicitly:

Statement(s) from [14],

Definition 1395 (*set of affine maps*).

**The proof of Lemma 1397 (*output restriction of affine map*)**

cites explicitly:

Lemma 1382 (*affine subspace is invariant by change of origin*),

Lemma 1393 (*range of affine map is affine subspace*),

Definition 1395 (*set of affine maps*).

**The proof of Lemma 1398 (*affine submap*)**

cites explicitly:

Definition 1379 (*affine subspace*),

Definition 1388 (*affine map*),

Lemma 1391 (*change of origin in affine map*),

Definition 1395 (*set of affine maps*),

Lemma 1397 (*output restriction of affine map*).

**The proof of Lemma 1399 (*equivalent definition of affine map (finite dimension)*)**

cites explicitly:

Definition 1388 (*affine map*).

**The proof of Lemma 1400 (*affine map preserves barycenter*)**

cites explicitly:

Statement(s) from [14],

Definition 1388 (*affine map*).

**The proof of Lemma 1402 (*affine map preserves isobarycenter*)**

cites explicitly:

Statement(s) from [14],

Lemma 1400 (*affine map preserves barycenter*),

Definition 1401 (*isobarycenter*).

**The proof of Lemma 1403 (*affine maps are closed by composition*)**

cites explicitly:

Statement(s) from [14],

Definition 1388 (*affine map*),

Lemma 1390 (*equivalent definition of affine map*),

Definition 1395 (*set of affine maps*).

**The proof of Lemma 1404 (*continuous affine map is continuous linear map*)**

cites explicitly:

Statement(s) from [14],

Lemma 1390 (*equivalent definition of affine map*).

- The proof of Lemma 1405** (*injective affine submap is zero linear kernel*)  
 cites explicitly:  
 Statement(s) from [14],  
 Lemma 1398 (*affine submap*).
- The proof of Lemma 1406** (*injective affine map is zero linear kernel*)  
 cites explicitly:  
 Lemma 1405 (*injective affine submap is zero linear kernel*).
- The proof of Lemma 1407** (*surjective affine submap is full linear range*)  
 cites explicitly:  
 Statement(s) from [14],  
 Definition 1379 (*affine subspace*),  
 Lemma 1381 (*equivalent definition of affine subspace*),  
 Lemma 1382 (*affine subspace is invariant by change of origin*),  
 Definition 1388 (*affine map*),  
 Lemma 1390 (*equivalent definition of affine map*).
- The proof of Lemma 1408** (*surjective affine map is full linear range*)  
 cites explicitly:  
 Lemma 1407 (*surjective affine submap is full linear range*).
- The proof of Lemma 1409** (*inverse of affine submap is affine submap*)  
 cites explicitly:  
 Statement(s) from [14],  
 Lemma 1375 (*inverse of isomorphism is linear map*),  
 Definition 1379 (*affine subspace*),  
 Lemma 1381 (*equivalent definition of affine subspace*),  
 Definition 1388 (*affine map*),  
 Lemma 1390 (*equivalent definition of affine map*),  
 Lemma 1405 (*injective affine submap is zero linear kernel*),  
 Lemma 1407 (*surjective affine submap is full linear range*).
- The proof of Lemma 1410** (*inverse of affine map is affine map*)  
 cites explicitly:  
 Lemma 1409 (*inverse of affine submap is affine submap*).
- The proof of Lemma 1412** (*equivalent definition of affinely independent family*)  
 cites explicitly:  
 Statement(s) from [14],  
 Definition 1411 (*affinely independent family*).
- The proof of Lemma 1413** (*affinely independent family of 2 elements*)  
 cites explicitly:  
 Definition 1411 (*affinely independent family*).
- The proof of Lemma 1414** (*affinely independent family is closed by sub-family*)  
 cites explicitly:  
 Definition 1411 (*affinely independent family*),  
 Lemma 1412 (*equivalent definition of affinely independent family*).
- The proof of Lemma 1418** ( $\mathbb{P}_k^1$  *is space of degree at most k*)  
 cites explicitly:  
 Lemma 1396 (*space of affine maps*),  
 Lemma 1399 (*equivalent definition of affine map (finite dimension)*),  
 Definition 1416 (*monomial of a single variable*),  
 Definition 1417 (*polynomial space  $\mathbb{P}_k^1$* ).

- The proof of Lemma 1419** (*monomials are free in  $\mathbb{P}_k^1$* )  
has no explicit citation.
- The proof of Lemma 1420** (*dimension of  $\mathbb{P}_k^1$* )  
cites explicitly:  
Lemma 1376 (*free family of dim elements is basis*),  
Definition 1417 (*polynomial space  $\mathbb{P}_k^1$* ),  
Lemma 1419 (*monomials are free in  $\mathbb{P}_k^1$* ).
- The proof of Lemma 1422** (*product of two univariate polynomials*)  
has no explicit citation.
- The proof of Lemma 1426** (*injectivity implies unisolvence*)  
cites explicitly:  
Lemma 1374 (*injectivity or surjectivity and dimension implies bijectivity*),  
Definition 1424 (*finite element triple*).
- The proof of Lemma 1427** (*dimension of approximation space*)  
has no explicit citation.
- The proof of Lemma 1428** (*degrees of freedom are basis*)  
cites explicitly:  
Lemma 1376 (*free family of dim elements is basis*),  
Lemma 1427 (*dimension of approximation space*).
- The proof of Lemma 1435** (*reference isobarycenter*)  
cites explicitly:  
Definition 1401 (*isobarycenter*),  
Definition 1434 (*family of reference points*).
- The proof of Lemma 1436** (*family of reference points is affinely independent*)  
cites explicitly:  
Definition 1365 (*canonic families*),  
Definition 1411 (*affinely independent family*),  
Definition 1434 (*family of reference points*).
- The proof of Lemma 1438** (*coordinates in reference simplex are smaller than 1*)  
cites explicitly:  
Definition 1437 (*reference simplex*).
- The proof of Lemma 1439** (*nontrivial reference simplex*)  
cites explicitly:  
Definition 1437 (*reference simplex*).
- The proof of Lemma 1442** (*coordinates in simplex are smaller than 1*)  
cites explicitly:  
Definition 1440 (*simplex*).
- The proof of Lemma 1443** (*simplex of reference vertices is reference simplex*)  
cites explicitly:  
Definition 1434 (*family of reference points*),  
Definition 1437 (*reference simplex*),  
Definition 1440 (*simplex*).
- The proof of Lemma 1446** (*cardinal of  $\mathcal{A}_k^1$* )  
cites explicitly:  
Lemma 1364 (*properties of the binomial coefficient*),  
Definition 1445 (*multi-indices  $\mathcal{A}_k^1$* ).

**The proof of Lemma 1449 (*Lagrange polynomials is basis of  $\mathbb{P}_k^1$* )**

cites explicitly:

Lemma 1376 (*free family of dim elements is basis*),

Definition 1417 (*polynomial space  $\mathbb{P}_k^1$* ),

Lemma 1418 ( *$\mathbb{P}_k^1$  is space of degree at most  $k$* ),

Lemma 1420 (*dimension of  $\mathbb{P}_k^1$* ),

Lemma 1422 (*product of two univariate polynomials*),

Definition 1448 (*Lagrange polynomials of  $\mathbb{P}_k^1$* ).

**The proof of Lemma 1450 (*decomposition of  $\mathbb{P}_k^1$  polynomial in Lagrange basis*)**

cites explicitly:

Lemma 1449 (*Lagrange polynomials is basis of  $\mathbb{P}_k^1$* ).

**The proof of Lemma 1452 (*reference simplex is nontrivial in  $\mathbb{R}$* )**

cites explicitly:

Definition 1437 (*reference simplex*),

Lemma 1439 (*nontrivial reference simplex*).

**The proof of Lemma 1454 (*reference Lagrange nodes are distinct*)**

cites explicitly:

Definition 1453 (*reference Lagrange nodes of  $\mathbb{P}_k^1$* ).

**The proof of Lemma 1455 (*reference Lagrange basis of  $\mathbb{P}_k^1$* )**

cites explicitly:

Lemma 1449 (*Lagrange polynomials is basis of  $\mathbb{P}_k^1$* ),

Lemma 1454 (*reference Lagrange nodes are distinct*).

**The proof of Lemma 1457 (*reference Lagrange linear forms for  $\mathbb{P}_k^1$  are linear*)**

cites explicitly:

Definition 1456 (*reference Lagrange linear forms for  $\mathbb{P}_k^1$* ).

**The proof of Lemma 1458 (*reference Lagrange linear forms for  $\mathbb{P}_k^1$  are injective*)**

cites explicitly:

Statement(s) from [14],

Definition 1424 (*finite element triple*),

Lemma 1450 (*decomposition of  $\mathbb{P}_k^1$  polynomial in Lagrange basis*),

Lemma 1454 (*reference Lagrange nodes are distinct*),

Definition 1456 (*reference Lagrange linear forms for  $\mathbb{P}_k^1$* ).

**The proof of Lemma 1459 (*unisolvence of  $\mathbb{P}_k^1$  (reference)*)**

cites explicitly:

Lemma 1420 (*dimension of  $\mathbb{P}_k^1$* ),

Lemma 1426 (*injectivity implies unisolvence*),

Lemma 1454 (*reference Lagrange nodes are distinct*),

Lemma 1458 (*reference Lagrange linear forms for  $\mathbb{P}_k^1$  are injective*).

**The proof of Theorem 1460 ( *$\overset{\text{Lag}}{\widehat{\mathbb{P}}}_k^1$  reference Lagrange finite element*)**

cites explicitly:

Lemma 1420 (*dimension of  $\mathbb{P}_k^1$* ),

Definition 1424 (*finite element triple*),

Lemma 1452 (*reference simplex is nontrivial in  $\mathbb{R}$* ),

Lemma 1459 (*unisolvence of  $\mathbb{P}_k^1$  (reference)*).

**The proof of Lemma 1462 (*properties of geometric mapping in dimension 1*)**

cites explicitly:

Lemma 1399 (*equivalent definition of affine map (finite dimension)*),

Lemma 1410 (*inverse of affine map is affine map*),

Lemma 1418 ( $\mathbb{P}_k^1$  is space of degree at most  $k$ ),  
 Definition 1453 (reference Lagrange nodes of  $\mathbb{P}_k^1$ ),  
 Lemma 1455 (reference Lagrange basis of  $\mathbb{P}_k^1$ ),  
 Definition 1461 (geometric mapping in dimension 1).

**The proof of Lemma 1463 (current simplex is nontrivial in  $\mathbb{R}$ )**  
 cites explicitly:  
 Definition 1440 (simplex).

**The proof of Lemma 1464 (current simplex is image of reference in  $\mathbb{R}$ )**  
 cites explicitly:  
 Lemma 1452 (reference simplex is nontrivial in  $\mathbb{R}$ ),  
 Definition 1461 (geometric mapping in dimension 1),  
 Lemma 1463 (current simplex is nontrivial in  $\mathbb{R}$ ).

**The proof of Lemma 1467 (Lagrange nodes are distinct)**  
 cites explicitly:  
 Definition 1465 (Lagrange nodes of  $\mathbb{P}_k^1$ ).

**The proof of Lemma 1468 (Lagrange nodes of  $\mathbb{P}_k^1$  are images of reference)**  
 cites explicitly:  
 Definition 1453 (reference Lagrange nodes of  $\mathbb{P}_k^1$ ),  
 Definition 1461 (geometric mapping in dimension 1),  
 Definition 1465 (Lagrange nodes of  $\mathbb{P}_k^1$ ).

**The proof of Lemma 1469 (Lagrange basis of  $\mathbb{P}_k^1$ )**  
 cites explicitly:  
 Lemma 1449 (Lagrange polynomials is basis of  $\mathbb{P}_k^1$ ),  
 Lemma 1467 (Lagrange nodes are distinct).

**The proof of Lemma 1470 (Lagrange polynomials of  $\mathbb{P}_k^1$  are images of reference)**  
 cites explicitly:  
 Definition 1448 (Lagrange polynomials of  $\mathbb{P}_k^1$ ),  
 Lemma 1455 (reference Lagrange basis of  $\mathbb{P}_k^1$ ),  
 Definition 1461 (geometric mapping in dimension 1),  
 Lemma 1468 (Lagrange nodes of  $\mathbb{P}_k^1$  are images of reference),  
 Lemma 1469 (Lagrange basis of  $\mathbb{P}_k^1$ ).

**The proof of Lemma 1471 (geometric mapping of  $\mathbb{P}_k^1$  is  $\mathbb{P}_k^1$ )**  
 cites explicitly:  
 Definition 1417 (polynomial space  $\mathbb{P}_k^1$ ),  
 Lemma 1455 (reference Lagrange basis of  $\mathbb{P}_k^1$ ),  
 Definition 1461 (geometric mapping in dimension 1),  
 Lemma 1469 (Lagrange basis of  $\mathbb{P}_k^1$ ),  
 Lemma 1470 (Lagrange polynomials of  $\mathbb{P}_k^1$  are images of reference).

**The proof of Lemma 1473 ( $\mathbb{P}_k^1$  Lagrange linear forms are linear)**  
 cites explicitly:  
 Definition 1472 (Lagrange linear forms for  $\mathbb{P}_k^1$ ).

**The proof of Lemma 1474 ( $\mathbb{P}_k^1$  Lagrange linear forms are images of reference)**  
 cites explicitly:  
 Definition 1456 (reference Lagrange linear forms for  $\mathbb{P}_k^1$ ),  
 Lemma 1468 (Lagrange nodes of  $\mathbb{P}_k^1$  are images of reference),  
 Definition 1472 (Lagrange linear forms for  $\mathbb{P}_k^1$ ).



**The proof of Lemma 1476 (*Lagrange linear forms for  $\mathbb{P}_k^1$  are injective*)**

cites explicitly:

Statement(s) from [14],

Definition 1424 (*finite element triple*),

Lemma 1450 (*decomposition of  $\mathbb{P}_k^1$  polynomial in Lagrange basis*),

Lemma 1467 (*Lagrange nodes are distinct*),

Definition 1472 (*Lagrange linear forms for  $\mathbb{P}_k^1$* ).

**The proof of Lemma 1477 (*unisolvence of  $\mathbb{P}_k^1$* )**

cites explicitly:

Lemma 1420 (*dimension of  $\mathbb{P}_k^1$* ),

Lemma 1426 (*injectivity implies unisolvence*),

Lemma 1467 (*Lagrange nodes are distinct*),

Lemma 1476 (*Lagrange linear forms for  $\mathbb{P}_k^1$  are injective*).

**The proof of Theorem 1478 ( *$L_{\text{eq}}\mathbb{P}_k^1$  Lagrange finite element*)**

cites explicitly:

Lemma 1420 (*dimension of  $\mathbb{P}_k^1$* ),

Definition 1424 (*finite element triple*),

Lemma 1463 (*current simplex is nontrivial in  $\mathbb{R}$* ),

Lemma 1477 (*unisolvence of  $\mathbb{P}_k^1$* ).

**The proof of Lemma 1481 (*length of multi-indices is additive*)**

cites explicitly:

Definition 1480 (*length of multi-indices*).

**The proof of Lemma 1483 (*factorial of multi-index is positive*)**

cites explicitly:

Definition 1482 (*factorial of multi-indices*).

**The proof of Lemma 1485 (*value of Kronecker delta of multi-indices*)**

cites explicitly:

Definition 1484 (*Kronecker delta of multi-indices*).

**The proof of Lemma 1487 (*multi-indices  $\mathcal{A}_k^d$  for  $d = 1$  is  $\mathcal{A}_k^1$* )**

cites explicitly:

Definition 1445 (*multi-indices  $\mathcal{A}_k^1$* ),

Definition 1480 (*length of multi-indices*),

Definition 1486 (*sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$* ).

**The proof of Lemma 1488 (*indices are smaller than maximal length*)**

cites explicitly:

Definition 1480 (*length of multi-indices*),

Definition 1486 (*sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$* ).

**The proof of Lemma 1489 (*first  $\mathcal{C}_k^d$* )**

cites explicitly:

Definition 1365 (*canonic families*),

Definition 1486 (*sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$* ),

Lemma 1488 (*indices are smaller than maximal length*).

**The proof of Lemma 1493 (*slices of  $\mathcal{C}_k^d$* )**

cites explicitly:

Definition 1486 (*sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$* ),

Lemma 1488 (*indices are smaller than maximal length*),

Definition 1491 (*slices  $\hat{S}_{k,i}^d$  and  $\tilde{S}_{k,i}^d$* ).

**The proof of Lemma 1494 (cardinal of slices of  $C_k^d$ )**

cites explicitly:

Definition 1491 (slices  $\check{S}_{k,i}^d$  and  $\tilde{S}_{k,i}^d$ ).**The proof of Lemma 1495 (cardinal of  $C_k^d$ )**

cites explicitly:

Definition 1362 (binomial coefficient),

Lemma 1364 (properties of the binomial coefficient),

Lemma 1489 (first  $C_k^d$ ),Lemma 1493 (slices of  $C_k^d$ ),Lemma 1494 (cardinal of slices of  $C_k^d$ ).**The proof of Lemma 1496 ( $C_k^d$  are layers of  $A_k^d$ )**

cites explicitly:

Definition 1486 (sets of multi-indices  $A_k^d$  and  $C_k^d$ ).**The proof of Lemma 1497 (first multi-indices  $A_k^d$ )**

cites explicitly:

Lemma 1489 (first  $C_k^d$ ),Lemma 1496 ( $C_k^d$  are layers of  $A_k^d$ ).**The proof of Lemma 1498 (cardinal of  $A_k^d$ )**

cites explicitly:

Lemma 1364 (properties of the binomial coefficient),

Lemma 1495 (cardinal of  $C_k^d$ ),Lemma 1496 ( $C_k^d$  are layers of  $A_k^d$ ).**The proof of Lemma 1500 (cardinal of  $C_k^d$  and  $A_k^{d-1}$ )**

cites explicitly:

Definition 1480 (length of multi-indices),

Definition 1486 (sets of multi-indices  $A_k^d$  and  $C_k^d$ ),Lemma 1495 (cardinal of  $C_k^d$ ),Lemma 1498 (cardinal of  $A_k^d$ ).**The proof of Lemma 1501 (cardinal of  $A_{k,i}^d$  and  $A_k^{d-1}$ )**

cites explicitly:

Definition 1480 (length of multi-indices),

Definition 1486 (sets of multi-indices  $A_k^d$  and  $C_k^d$ ).**The proof of Lemma 1504 (monomial in  $d$  variables for  $d = 1$  is monomial of a single variable)**

cites explicitly:

Definition 1416 (monomial of a single variable),

Lemma 1489 (first  $C_k^d$ ),Definition 1503 (monomial in  $d$  variables).**The proof of Lemma 1506 (polynomial space  $\mathbb{P}_k^d$  for  $d = 1$  is  $\mathbb{P}_k^1$ )**

cites explicitly:

Definition 1417 (polynomial space  $\mathbb{P}_k^1$ ),Lemma 1497 (first multi-indices  $A_k^d$ ),Definition 1505 (polynomial space  $\mathbb{P}_k^d$ ).**The proof of Lemma 1507 ( $\mathbb{P}_k^d$  is vector space)**

cites explicitly:

Definition 1505 (polynomial space  $\mathbb{P}_k^d$ ).

**The proof of Lemma 1508** ( *$\mathbb{P}_k^d$  is nondecreasing sequence in  $k$* )

cites explicitly:

Lemma 1496 ( *$C_k^d$  are layers of  $\mathcal{A}_k^d$* ),

Definition 1505 (*polynomial space  $\mathbb{P}_k^d$* ).

**The proof of Lemma 1509** (*constant and affine spaces  $\mathbb{P}_0^d$  and  $\mathbb{P}_1^d$* )

cites explicitly:

Definition 1365 (*canonic families*),

Lemma 1399 (*equivalent definition of affine map (finite dimension)*),

Lemma 1497 (*first multi-indices  $\mathcal{A}_k^d$* ),

Definition 1505 (*polynomial space  $\mathbb{P}_k^d$* ).

**The proof of Lemma 1511** (*values of degree of polynomial*)

cites explicitly:

Definition 1510 (*degree of polynomial*).

**The proof of Lemma 1512** (*monomials of  $C_k^d$  have degree  $k$* )

cites explicitly:

Definition 1486 (*sets of multi-indices  $\mathcal{A}_k^d$  and  $C_k^d$* ),

Definition 1503 (*monomial in  $d$  variables*),

Definition 1510 (*degree of polynomial*).

**The proof of Lemma 1513** ( *$\mathbb{P}_k^d$  is space of degree at most  $k$* )

cites explicitly:

Lemma 1496 ( *$C_k^d$  are layers of  $\mathcal{A}_k^d$* ),

Definition 1505 (*polynomial space  $\mathbb{P}_k^d$* ),

Definition 1510 (*degree of polynomial*).

**The proof of Lemma 1514** (*product of monomials*)

cites explicitly:

Lemma 1481 (*length of multi-indices is additive*),

Definition 1503 (*monomial in  $d$  variables*),

Lemma 1512 (*monomials of  $C_k^d$  have degree  $k$* ).

**The proof of Lemma 1515** (*product of monomial and polynomial*)

cites explicitly:

Definition 1486 (*sets of multi-indices  $\mathcal{A}_k^d$  and  $C_k^d$* ),

Definition 1505 (*polynomial space  $\mathbb{P}_k^d$* ),

Lemma 1514 (*product of monomials*).

**The proof of Lemma 1516** (*product of two polynomials*)

cites explicitly:

Definition 1505 (*polynomial space  $\mathbb{P}_k^d$* ),

Lemma 1507 ( *$\mathbb{P}_k^d$  is vector space*),

Lemma 1515 (*product of monomial and polynomial*).

**The proof of Lemma 1517** (*partial derivative of monomials*)

cites explicitly:

Definition 1503 (*monomial in  $d$  variables*),

Lemma 1512 (*monomials of  $C_k^d$  have degree  $k$* ).

**The proof of Lemma 1519** (*partial derivative is linear*)

cites explicitly:

Definition 1505 (*polynomial space  $\mathbb{P}_k^d$* ),

Lemma 1517 (*partial derivative of monomials*).

**The proof of Lemma 1520 (partial derivative of 0)**

cites explicitly:

Lemma 1519 (partial derivative is linear).

**The proof of Lemma 1521 (derivating more than degree is 0)**

cites explicitly:

Definition 1486 (sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$ ),

Lemma 1517 (partial derivative of monomials).

**The proof of Lemma 1522 (partial derivative of monomials at 0)**

cites explicitly:

Definition 1482 (factorial of multi-indices),

Definition 1484 (Kronecker delta of multi-indices),

Lemma 1517 (partial derivative of monomials).

**The proof of Lemma 1523 (monomials are free in  $\mathbb{P}_k^d$ )**

cites explicitly:

Lemma 1483 (factorial of multi-index is positive),

Definition 1505 (polynomial space  $\mathbb{P}_k^d$ ),

Lemma 1519 (partial derivative is linear),

Lemma 1520 (partial derivative of 0),

Lemma 1522 (partial derivative of monomials at 0).

**The proof of Lemma 1524 (monomials are a basis of  $\mathbb{P}_k^d$ )**

cites explicitly:

Definition 1505 (polynomial space  $\mathbb{P}_k^d$ ),Lemma 1523 (monomials are free in  $\mathbb{P}_k^d$ ).**The proof of Lemma 1525 (dimension of  $\mathbb{P}_k^d$ )**

cites explicitly:

Lemma 1498 (cardinal of  $\mathcal{A}_k^d$ ),Lemma 1524 (monomials are a basis of  $\mathbb{P}_k^d$ ).**The proof of Lemma 1526 (isomorphism between  $\mathbb{P}_0^d$  and  $\mathbb{P}_0^{d-1}$ )**

cites explicitly:

Statement(s) from [14],

Lemma 1374 (injectivity or surjectivity and dimension implies bijectivity),

Lemma 1509 (constant and affine spaces  $\mathbb{P}_0^d$  and  $\mathbb{P}_1^d$ ),Lemma 1525 (dimension of  $\mathbb{P}_k^d$ ).**The proof of Lemma 1529 (decomposition of  $\mathbb{P}_k^d$ )**

cites explicitly:

Definition 1491 (slices  $\check{S}_{k,i}^d$  and  $\tilde{S}_{k,i}^d$ ),Lemma 1493 (slices of  $\mathcal{C}_k^d$ ),Lemma 1494 (cardinal of slices of  $\mathcal{C}_k^d$ ),Lemma 1496 ( $\mathcal{C}_k^d$  are layers of  $\mathcal{A}_k^d$ ),Definition 1505 (polynomial space  $\mathbb{P}_k^d$ ),Lemma 1507 ( $\mathbb{P}_k^d$  is vector space),Lemma 1508 ( $\mathbb{P}_k^d$  is nondecreasing sequence in  $k$ ),Lemma 1509 (constant and affine spaces  $\mathbb{P}_0^d$  and  $\mathbb{P}_1^d$ ),Lemma 1512 (monomials of  $\mathcal{C}_k^d$  have degree  $k$ ),Lemma 1523 (monomials are free in  $\mathbb{P}_k^d$ ).**The proof of Lemma 1531 (isomorphism between  $\mathbb{P}_k^d$  and  $\mathbb{P}_k^{d-1} \times \mathbb{P}_{k-1}^d$ )**

cites explicitly:

Statement(s) from [14],

- Lemma 1364 (*properties of the binomial coefficient*),  
 Lemma 1374 (*injectivity or surjectivity and dimension implies bijectivity*),  
 Lemma 1507 ( $\mathbb{P}_k^d$  is vector space),  
 Lemma 1525 (*dimension of  $\mathbb{P}_k^d$* ),  
 Lemma 1529 (*decomposition of  $\mathbb{P}_k^d$* ).

**The proof of Lemma 1533 ( $\mathbb{P}_k^d$  is nondecreasing sequence in  $d$ )**

cites explicitly:

- Definition 1505 (*polynomial space  $\mathbb{P}_k^d$* ),  
 Lemma 1507 ( $\mathbb{P}_k^d$  is vector space),  
 Lemma 1509 (*constant and affine spaces  $\mathbb{P}_0^d$  and  $\mathbb{P}_1^d$* ),  
 Lemma 1531 (*isomorphism between  $\mathbb{P}_k^d$  and  $\mathbb{P}_k^{d-1} \times \mathbb{P}_{k-1}^d$* ).

**The proof of Lemma 1534 (*expression of  $\mathbb{P}_k^d$  as polynomial of  $x_d$* )**

cites explicitly:

- Lemma 1419 (*monomials are free in  $\mathbb{P}_k^1$* ),  
 Lemma 1526 (*isomorphism between  $\mathbb{P}_0^d$  and  $\mathbb{P}_0^{d-1}$* ),  
 Lemma 1529 (*decomposition of  $\mathbb{P}_k^d$* ).

**The proof of Lemma 1537 (*product of two polynomials (alternate proof)*)**

cites explicitly:

- Statement(s) from [14],  
 Lemma 1360 (*strong double induction*),  
 Lemma 1422 (*product of two univariate polynomials*),  
 Lemma 1507 ( $\mathbb{P}_k^d$  is vector space),  
 Lemma 1508 ( $\mathbb{P}_k^d$  is nondecreasing sequence in  $k$ ),  
 Lemma 1509 (*constant and affine spaces  $\mathbb{P}_0^d$  and  $\mathbb{P}_1^d$* ),  
 Lemma 1515 (*product of monomial and polynomial*),  
 Lemma 1529 (*decomposition of  $\mathbb{P}_k^d$* ),  
 Lemma 1533 ( $\mathbb{P}_k^d$  is nondecreasing sequence in  $d$ ).

**The proof of Lemma 1538 (*product of polynomials*)**

cites explicitly:

- Lemma 1516 (*product of two polynomials*),  
 Lemma 1537 (*product of two polynomials (alternate proof)*).

**The proof of Lemma 1539 (*affine mapping of monomials is  $\mathbb{P}_k^l$* )**

cites explicitly:

- Lemma 1399 (*equivalent definition of affine map (finite dimension)*),  
 Definition 1480 (*length of multi-indices*),  
 Definition 1486 (*sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$* ),  
 Definition 1503 (*monomial in  $d$  variables*),  
 Lemma 1509 (*constant and affine spaces  $\mathbb{P}_0^d$  and  $\mathbb{P}_1^d$* ),  
 Lemma 1538 (*product of polynomials*).

**The proof of Lemma 1540 (*affine mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^l$* )**

cites explicitly:

- Lemma 1496 ( $\mathcal{C}_k^d$  are layers of  $\mathcal{A}_k^d$ ),  
 Definition 1505 (*polynomial space  $\mathbb{P}_k^d$* ),  
 Lemma 1507 ( $\mathbb{P}_k^d$  is vector space),  
 Lemma 1508 ( $\mathbb{P}_k^d$  is nondecreasing sequence in  $k$ ),  
 Lemma 1539 (*affine mapping of monomials is  $\mathbb{P}_k^l$* ).

**The proof of Lemma 1542 (*reference Lagrange polynomials of  $\mathbb{P}_1^d$  for  $d = 1$  are reference Lagrange polynomials of  $\mathbb{P}_k^1$  for  $k = 1$* )**

cites explicitly:

Definition 1448 (*Lagrange polynomials of  $\mathbb{P}_k^1$* ),  
 Definition 1453 (*reference Lagrange nodes of  $\mathbb{P}_k^1$* ),  
 Lemma 1455 (*reference Lagrange basis of  $\mathbb{P}_k^1$* ),  
 Definition 1541 (*reference Lagrange polynomials of  $\mathbb{P}_1^d$* ).

**The proof of Lemma 1543 (*reference Lagrange polynomials is basis of  $\mathbb{P}_1^d$* )**

cites explicitly:

Lemma 1376 (*free family of dim elements is basis*),  
 Definition 1434 (*family of reference points*),  
 Lemma 1509 (*constant and affine spaces  $\mathbb{P}_0^d$  and  $\mathbb{P}_1^d$* ),  
 Definition 1510 (*degree of polynomial*),  
 Lemma 1525 (*dimension of  $\mathbb{P}_k^d$* ),  
 Definition 1541 (*reference Lagrange polynomials of  $\mathbb{P}_1^d$* ).

**The proof of Lemma 1545 (*differential of reference Lagrange polynomials*)**

cites explicitly:

Lemma 1509 (*constant and affine spaces  $\mathbb{P}_0^d$  and  $\mathbb{P}_1^d$* ),  
 Lemma 1543 (*reference Lagrange polynomials is basis of  $\mathbb{P}_1^d$* ).

**The proof of Lemma 1548 (*geometric mapping for  $d = 1$  is geometric mapping in dimension 1*)**

cites explicitly:

Definition 1461 (*geometric mapping in dimension 1*),  
 Definition 1541 (*reference Lagrange polynomials of  $\mathbb{P}_1^d$* ),  
 Definition 1547 (*geometric mapping*).

**The proof of Lemma 1549 (*reference geometric mapping is identity*)**

cites explicitly:

Definition 1434 (*family of reference points*),  
 Definition 1541 (*reference Lagrange polynomials of  $\mathbb{P}_1^d$* ),  
 Definition 1547 (*geometric mapping*).

**The proof of Lemma 1550 (*properties of geometric mapping*)**

cites explicitly:

Statement(s) from [14],  
 Lemma 1374 (*injectivity or surjectivity and dimension implies bijectivity*),  
 Definition 1388 (*affine map*),  
 Lemma 1406 (*injective affine map is zero linear kernel*),  
 Lemma 1410 (*inverse of affine map is affine map*),  
 Definition 1411 (*affinely independent family*),  
 Definition 1437 (*reference simplex*),  
 Definition 1440 (*simplex*),  
 Definition 1541 (*reference Lagrange polynomials of  $\mathbb{P}_1^d$* ),  
 Lemma 1543 (*reference Lagrange polynomials is basis of  $\mathbb{P}_1^d$* ),  
 Definition 1547 (*geometric mapping*).

**The proof of Lemma 1551 (*differential of geometric mapping*)**

cites explicitly:

Lemma 1550 (*properties of geometric mapping*).

**The proof of Lemma 1552 (*Lagrange polynomials of  $\mathbb{P}_1^d$* )**

cites explicitly:

Lemma 1550 (*properties of geometric mapping*).

**The proof of Lemma 1553 (*Lagrange polynomials of reference vertices are reference Lagrange polynomials of  $\mathbb{P}_1^d$* )**

cites explicitly:

Lemma 1549 (*reference geometric mapping is identity*),  
 Lemma 1552 (*Lagrange polynomials of  $\mathbb{P}_1^d$* ).

**The proof of Lemma 1554 (*Lagrange polynomials is basis of  $\mathbb{P}_1^d$* )**

cites explicitly:

Lemma 1376 (*free family of dim elements is basis*),  
 Lemma 1403 (*affine maps are closed by composition*),  
 Lemma 1509 (*constant and affine spaces  $\mathbb{P}_0^d$  and  $\mathbb{P}_1^d$* ),  
 Lemma 1525 (*dimension of  $\mathbb{P}_k^d$* ),  
 Lemma 1543 (*reference Lagrange polynomials is basis of  $\mathbb{P}_1^d$* ),  
 Lemma 1550 (*properties of geometric mapping*),  
 Lemma 1552 (*Lagrange polynomials of  $\mathbb{P}_1^d$* ).

**The proof of Lemma 1555 (*decomposition of  $\mathbb{P}_1^d$  polynomial in Lagrange basis*)**

cites explicitly:

Lemma 1554 (*Lagrange polynomials is basis of  $\mathbb{P}_1^d$* ).

**The proof of Lemma 1556 (*differential of Lagrange polynomials  $\mathbb{P}_1^d$* )**

cites explicitly:

Lemma 1509 (*constant and affine spaces  $\mathbb{P}_0^d$  and  $\mathbb{P}_1^d$* ),  
 Lemma 1545 (*differential of reference Lagrange polynomials*),  
 Lemma 1551 (*differential of geometric mapping*),  
 Lemma 1554 (*Lagrange polynomials is basis of  $\mathbb{P}_1^d$* ).

**The proof of Lemma 1557 (*nontrivial simplex*)**

cites explicitly:

Lemma 1439 (*nontrivial reference simplex*),  
 Lemma 1550 (*properties of geometric mapping*),  
 Lemma 1551 (*differential of geometric mapping*).

**The proof of Lemma 1559 (*barycentric coordinate*)**

cites explicitly:

Statement(s) from [14],  
 Lemma 1376 (*free family of dim elements is basis*),  
 Definition 1411 (*affinely independent family*),  
 Lemma 1412 (*equivalent definition of affinely independent family*).

**The proof of Lemma 1560 (*Lagrange polynomials of  $\mathbb{P}_1^d$  are barycentric coordinate*)**

cites explicitly:

Definition 1547 (*geometric mapping*),  
 Lemma 1550 (*properties of geometric mapping*),  
 Lemma 1552 (*Lagrange polynomials of  $\mathbb{P}_1^d$* ),  
 Lemma 1559 (*barycentric coordinate*).

**The proof of Lemma 1561 (*decomposition of  $\mathbb{P}_1^d$  polynomial with barycentric coordinates*)**

cites explicitly:

Lemma 1555 (*decomposition of  $\mathbb{P}_1^d$  polynomial in Lagrange basis*),  
 Lemma 1560 (*Lagrange polynomials of  $\mathbb{P}_1^d$  are barycentric coordinate*).

**The proof of Lemma 1563 (*equivalent definition of face hyperplane*)**

cites explicitly:

Statement(s) from [14],  
 Lemma 1387 (*barycenter closure is affine subspace*),  
 Lemma 1559 (*barycentric coordinate*),  
 Definition 1562 (*face hyperplane*).

**The proof of Lemma 1564 (reference face hyperplane)**

cites explicitly:

Statement(s) from [14],

Lemma 1436 (family of reference points is affinely independent),

Definition 1541 (reference Lagrange polynomials of  $\mathbb{P}_1^d$ ),

Lemma 1553 (Lagrange polynomials of reference vertices are reference Lagrange polynomials of  $\mathbb{P}_1^d$ ),

Lemma 1560 (Lagrange polynomials of  $\mathbb{P}_1^d$  are barycentric coordinate),

Lemma 1563 (equivalent definition of face hyperplane).

**The proof of Lemma 1565 (face hyperplane is image of reference face hyperplane)**

cites explicitly:

Definition 1434 (family of reference points),

Definition 1541 (reference Lagrange polynomials of  $\mathbb{P}_1^d$ ),

Definition 1547 (geometric mapping),

Definition 1562 (face hyperplane).

**The proof of Lemma 1567 (hyperface is included in face hyperplane)**

cites explicitly:

Lemma 1563 (equivalent definition of face hyperplane),

Definition 1566 (hyperface).

**The proof of Lemma 1569 (equivalent definition of  $l$ -face affine space)**

cites explicitly:

Lemma 1387 (barycenter closure is affine subspace),

Definition 1568 ( $l$ -face affine space).

**The proof of Lemma 1570 ( $d$ -face affine space is full space)**

cites explicitly:

Lemma 1376 (free family of dim elements is basis),

Definition 1411 (affinely independent family),

Lemma 1414 (affinely independent family is closed by sub-family),

Lemma 1569 (equivalent definition of  $l$ -face affine space).

**The proof of Lemma 1571 (0-face affine space is vertex)**

cites explicitly:

Lemma 1569 (equivalent definition of  $l$ -face affine space).

**The proof of Lemma 1573 ( $l$ -face is included in  $l$ -face affine space)**

cites explicitly:

Definition 1568 ( $l$ -face affine space),

Definition 1572 ( $l$ -face).

**The proof of Lemma 1574 ( $d$ -face is simplex)**

cites explicitly:

Definition 1440 (simplex),

Definition 1572 ( $l$ -face).

**The proof of Lemma 1576 ( $(d-1)$ -face is hyperface)**

cites explicitly:

Lemma 1368 (jump enumeration),

Lemma 1563 (equivalent definition of face hyperplane),

Definition 1566 (hyperface),

Definition 1568 ( $l$ -face affine space),

Definition 1572 ( $l$ -face).



**The proof of Lemma 1579 (*geometric d-face mapping is geometric mapping*)**

cites explicitly:

Definition 1547 (*geometric mapping*),

Definition 1578 (*geometric l-face mapping*).

**The proof of Lemma 1581 (*properties of geometric l-face mapping*)**

cites explicitly:

Statement(s) from [14],

Definition 1388 (*affine map*),

Lemma 1406 (*injective affine map is zero linear kernel*),

Lemma 1409 (*inverse of affine submap is affine submap*),

Lemma 1414 (*affinely independent family is closed by sub-family*),

Definition 1437 (*reference simplex*),

Definition 1541 (*reference Lagrange polynomials of  $\mathbb{P}_1^d$* ),

Lemma 1543 (*reference Lagrange polynomials is basis of  $\mathbb{P}_1^d$* ),

Definition 1568 (*l-face affine space*),

Lemma 1569 (*equivalent definition of l-face affine space*),

Definition 1572 (*l-face*),

Definition 1578 (*geometric l-face mapping*).

**The proof of Lemma 1582 (*geometric l-face mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^l$* )**

cites explicitly:

Lemma 1540 (*affine mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^l$* ),

Lemma 1581 (*properties of geometric l-face mapping*).

**The proof of Lemma 1583 (*geometric mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^d$* )**

cites explicitly:

Lemma 1540 (*affine mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^l$* ),

Lemma 1550 (*properties of geometric mapping*),

Lemma 1579 (*geometric d-face mapping is geometric mapping*),

Lemma 1582 (*geometric l-face mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^l$* ).

**The proof of Lemma 1584 (*geometric hyperface mapping*)**

cites explicitly:

Lemma 1576 ( *$(d-1)$ -face is hyperface*),

Lemma 1581 (*properties of geometric l-face mapping*).

**The proof of Lemma 1585 (*hyperface geometric mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^{d-1}$* )**

cites explicitly:

Lemma 1576 ( *$(d-1)$ -face is hyperface*),

Lemma 1582 (*geometric l-face mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^l$* ).

**The proof of Lemma 1586 (*geometric mapping with permutation*)**

cites explicitly:

Lemma 1370 (*image of ker is ker*),

Lemma 1543 (*reference Lagrange polynomials is basis of  $\mathbb{P}_1^d$* ),

Lemma 1559 (*barycentric coordinate*),

Lemma 1560 (*Lagrange polynomials of  $\mathbb{P}_1^d$  are barycentric coordinate*),

Lemma 1563 (*equivalent definition of face hyperplane*),

Lemma 1564 (*reference face hyperplane*),

Lemma 1570 (*d-face affine space is full space*),

Lemma 1574 (*d-face is simplex*),

Definition 1578 (*geometric l-face mapping*),

Lemma 1581 (*properties of geometric l-face mapping*).

**The proof of Lemma 1589 (*Lagrange nodes of  $\mathbb{P}_k^d$  for  $d = 1$  are Lagrange nodes of  $\mathbb{P}_k^1$* )**

cites explicitly:

Definition 1465 (*Lagrange nodes of  $\mathbb{P}_k^1$* ),

Lemma 1497 (*first multi-indices  $\mathcal{A}_k^d$* ),

Definition 1588 (*Lagrange nodes of  $\mathbb{P}_k^d$* ).

**The proof of Lemma 1590 (*number of Lagrange nodes of  $\mathbb{P}_k^d$* )**

cites explicitly:

Lemma 1498 (*cardinal of  $\mathcal{A}_k^d$* ),

Lemma 1559 (*barycentric coordinate*),

Definition 1588 (*Lagrange nodes of  $\mathbb{P}_k^d$* ).

**The proof of Lemma 1591 (*barycentric coordinates of Lagrange nodes of  $\mathbb{P}_k^d$* )**

cites explicitly:

Definition 1401 (*isobarycenter*),

Definition 1480 (*length of multi-indices*),

Definition 1486 (*sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$* ),

Lemma 1488 (*indices are smaller than maximal length*),

Lemma 1559 (*barycentric coordinate*),

Definition 1588 (*Lagrange nodes of  $\mathbb{P}_k^d$* ).

**The proof of Lemma 1592 (*vertices are Lagrange nodes of  $\mathbb{P}_k^d$* )**

cites explicitly:

Definition 1365 (*canonic families*),

Definition 1480 (*length of multi-indices*),

Definition 1486 (*sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$* ),

Definition 1588 (*Lagrange nodes of  $\mathbb{P}_k^d$* ).

**The proof of Lemma 1593 (*Lagrange nodes of  $\mathbb{P}_1^d$  are vertices*)**

cites explicitly:

Lemma 1497 (*first multi-indices  $\mathcal{A}_k^d$* ),

Lemma 1592 (*vertices are Lagrange nodes of  $\mathbb{P}_k^d$* ).

**The proof of Lemma 1595 (*equivalent definition of sub-vertices of Lagrange nodes of  $\mathbb{P}_k^d$* )**

cites explicitly:

Definition 1365 (*canonic families*),

Definition 1480 (*length of multi-indices*),

Definition 1486 (*sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$* ),

Lemma 1591 (*barycentric coordinates of Lagrange nodes of  $\mathbb{P}_k^d$* ),

Definition 1594 (*sub-vertices of Lagrange nodes of  $\mathbb{P}_k^d$* ).

**The proof of Lemma 1597 (*sub-vertices are affinely independent*)**

cites explicitly:

Definition 1411 (*affinely independent family*),

Lemma 1595 (*equivalent definition of sub-vertices of Lagrange nodes of  $\mathbb{P}_k^d$* ).

**The proof of Lemma 1598 ( *$\mathbb{P}_{k-1}^d$  sub-nodes of sub-vertices are some nodes of  $\mathbb{P}_{k-1}^d$* )**

cites explicitly:

Definition 1480 (*length of multi-indices*),

Lemma 1496 ( *$\mathcal{C}_k^d$  are layers of  $\mathcal{A}_k^d$* ),

Lemma 1591 (*barycentric coordinates of Lagrange nodes of  $\mathbb{P}_k^d$* ),

Lemma 1595 (*equivalent definition of sub-vertices of Lagrange nodes of  $\mathbb{P}_k^d$* ),

Lemma 1597 (*sub-vertices are affinely independent*).

**The proof of Lemma 1599 (*reference Lagrange nodes of  $\mathbb{P}_k^d$* )**

cites explicitly:

Definition 1434 (*family of reference points*),

Lemma 1435 (*reference isobarycenter*),  
 Definition 1588 (*Lagrange nodes of  $\mathbb{P}_k^d$* ),  
 Lemma 1591 (*barycentric coordinates of Lagrange nodes of  $\mathbb{P}_k^d$* ).

**The proof of Lemma 1600** (*reference Lagrange nodes of  $\mathbb{P}_k^d$  for  $d = 1$  are reference Lagrange nodes of  $\mathbb{P}_k^1$* )

cites explicitly:  
 Definition 1365 (*canonic families*),  
 Definition 1453 (*reference Lagrange nodes of  $\mathbb{P}_k^1$* ),  
 Lemma 1497 (*first multi-indices  $\mathcal{A}_k^d$* ),  
 Lemma 1599 (*reference Lagrange nodes of  $\mathbb{P}_k^d$* ).

**The proof of Lemma 1601** (*equivalent definition of reference Lagrange nodes of  $\mathbb{P}_k^d$* )

cites explicitly:  
 Definition 1365 (*canonic families*),  
 Lemma 1435 (*reference isobarycenter*),  
 Lemma 1599 (*reference Lagrange nodes of  $\mathbb{P}_k^d$* ).

**The proof of Lemma 1602** (*number of reference Lagrange nodes of  $\mathbb{P}_k^d$* )

cites explicitly:  
 Lemma 1436 (*family of reference points is affinely independent*),  
 Lemma 1498 (*cardinal of  $\mathcal{A}_k^d$* ),  
 Lemma 1559 (*barycentric coordinate*),  
 Lemma 1599 (*reference Lagrange nodes of  $\mathbb{P}_k^d$* ).

**The proof of Lemma 1604** (*Lagrange nodes of  $\mathbb{P}_k^d$  are image of reference*)

cites explicitly:  
 Lemma 1550 (*properties of geometric mapping*),  
 Definition 1588 (*Lagrange nodes of  $\mathbb{P}_k^d$* ),  
 Lemma 1601 (*equivalent definition of reference Lagrange nodes of  $\mathbb{P}_k^d$* ).

**The proof of Lemma 1605** (*face hyperplanes of Lagrange nodes of  $\mathbb{P}_k^d$* )

cites explicitly:  
 Definition 1486 (*sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$* ),  
 Lemma 1495 (*cardinal of  $\mathcal{C}_k^d$* ),  
 Lemma 1498 (*cardinal of  $\mathcal{A}_k^d$* ),  
 Lemma 1500 (*cardinal of  $\mathcal{C}_k^d$  and  $\mathcal{A}_k^{d-1}$* ),  
 Lemma 1501 (*cardinal of  $\mathcal{A}_{k,i}^d$  and  $\mathcal{A}_k^{d-1}$* ),  
 Lemma 1563 (*equivalent definition of face hyperplane*),  
 Lemma 1590 (*number of Lagrange nodes of  $\mathbb{P}_k^d$* ),  
 Lemma 1591 (*barycentric coordinates of Lagrange nodes of  $\mathbb{P}_k^d$* ).

**The proof of Lemma 1607** (*image of nodes by geometric hyperface mapping*)

cites explicitly:  
 Lemma 1400 (*affine map preserves barycenter*),  
 Definition 1486 (*sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$* ),  
 Lemma 1500 (*cardinal of  $\mathcal{C}_k^d$  and  $\mathcal{A}_k^{d-1}$* ),  
 Lemma 1501 (*cardinal of  $\mathcal{A}_{k,i}^d$  and  $\mathcal{A}_k^{d-1}$* ),  
 Lemma 1584 (*geometric hyperface mapping*),  
 Lemma 1591 (*barycentric coordinates of Lagrange nodes of  $\mathbb{P}_k^d$* ),  
 Lemma 1599 (*reference Lagrange nodes of  $\mathbb{P}_k^d$* ).

**The proof of Lemma 1609** (*Lagrange linear forms for  $\mathbb{P}_k^d$  for  $d=1$  are Lagrange linear forms for  $\mathbb{P}_k^1$* )

cites explicitly:  
 Definition 1472 (*Lagrange linear forms for  $\mathbb{P}_k^1$* ),

Lemma 1497 (*first multi-indices  $\mathcal{A}_k^d$* ),  
 Lemma 1589 (*Lagrange nodes of  $\mathbb{P}_k^d$  for  $d = 1$  are Lagrange nodes of  $\mathbb{P}_k^1$* ),  
 Definition 1608 (*Lagrange linear forms for  $\mathbb{P}_k^d$* ).

**The proof of Lemma 1610 (*Lagrange linear forms of  $\mathbb{P}_k^d$  are linear*)**  
 cites explicitly:  
 Definition 1608 (*Lagrange linear forms for  $\mathbb{P}_k^d$* ).

**The proof of Lemma 1611 (*cardinal of Lagrange linear forms of  $\mathbb{P}_k^d$* )**  
 cites explicitly:  
 Lemma 1590 (*number of Lagrange nodes of  $\mathbb{P}_k^d$* ),  
 Definition 1608 (*Lagrange linear forms for  $\mathbb{P}_k^d$* ).

**The proof of Lemma 1613 (*reference Lagrange linear forms for  $\mathbb{P}_k^d$  for  $d = 1$  are reference Lagrange linear forms for  $\mathbb{P}_k^1$* )**  
 cites explicitly:  
 Definition 1456 (*reference Lagrange linear forms for  $\mathbb{P}_k^1$* ),  
 Lemma 1497 (*first multi-indices  $\mathcal{A}_k^d$* ),  
 Lemma 1600 (*reference Lagrange nodes of  $\mathbb{P}_k^d$  for  $d = 1$  are reference Lagrange nodes of  $\mathbb{P}_k^1$* ),  
 Definition 1612 (*reference Lagrange linear forms for  $\mathbb{P}_k^d$* ).

**The proof of Lemma 1614 (*Lagrange linear forms of  $\mathbb{P}_k^d$  are images of reference*)**  
 cites explicitly:  
 Lemma 1604 (*Lagrange nodes of  $\mathbb{P}_k^d$  are image of reference*),  
 Definition 1608 (*Lagrange linear forms for  $\mathbb{P}_k^d$* ),  
 Definition 1612 (*reference Lagrange linear forms for  $\mathbb{P}_k^d$* ).

**The proof of Lemma 1615 (*Lagrange linear forms for  $\mathbb{P}_0^d$  are injective*)**  
 cites explicitly:  
 Statement(s) from [14],  
 Definition 1424 (*finite element triple*),  
 Lemma 1509 (*constant and affine spaces  $\mathbb{P}_0^d$  and  $\mathbb{P}_1^d$* ),  
 Definition 1608 (*Lagrange linear forms for  $\mathbb{P}_k^d$* ).

**The proof of Lemma 1617 (*unisolvence of  $\mathbb{P}_0^d$* )**  
 cites explicitly:  
 Lemma 1364 (*properties of the binomial coefficient*),  
 Lemma 1426 (*injectivity implies unisolvence*),  
 Lemma 1525 (*dimension of  $\mathbb{P}_k^d$* ),  
 Lemma 1611 (*cardinal of Lagrange linear forms of  $\mathbb{P}_k^d$* ),  
 Lemma 1615 (*Lagrange linear forms for  $\mathbb{P}_0^d$  are injective*).

**The proof of Lemma 1618 (*decomposition of  $\mathbb{P}_1^d$  polynomial with  $\sigma_\alpha$* )**  
 cites explicitly:  
 Lemma 1555 (*decomposition of  $\mathbb{P}_1^d$  polynomial in Lagrange basis*),  
 Lemma 1561 (*decomposition of  $\mathbb{P}_1^d$  polynomial with barycentric coordinates*),  
 Lemma 1593 (*Lagrange nodes of  $\mathbb{P}_1^d$  are vertices*),  
 Definition 1608 (*Lagrange linear forms for  $\mathbb{P}_k^d$* ).

**The proof of Lemma 1619 (*Lagrange linear forms for  $\mathbb{P}_1^d$  are injective*)**  
 cites explicitly:  
 Statement(s) from [14],  
 Definition 1424 (*finite element triple*),  
 Lemma 1497 (*first multi-indices  $\mathcal{A}_k^d$* ),  
 Lemma 1618 (*decomposition of  $\mathbb{P}_1^d$  polynomial with  $\sigma_\alpha$* ).

**The proof of Lemma 1620 (unisolvence of  $\mathbb{P}_1^d$ )**

cites explicitly:

- Lemma 1364 (*properties of the binomial coefficient*),
- Lemma 1426 (*injectivity implies unisolvence*),
- Lemma 1525 (*dimension of  $\mathbb{P}_k^d$* ),
- Lemma 1611 (*cardinal of Lagrange linear forms of  $\mathbb{P}_k^d$* ),
- Lemma 1619 (*Lagrange linear forms for  $\mathbb{P}_1^d$  are injective*).

**The proof of Lemma 1621 (factorization of zero polynomial on last reference hyperplane)**

cites explicitly:

- Statement(s) from [14],
- Lemma 1529 (*decomposition of  $\mathbb{P}_k^d$* ),
- Definition 1541 (*reference Lagrange polynomials of  $\mathbb{P}_1^d$* ),
- Lemma 1564 (*reference face hyperplane*).

**The proof of Lemma 1623 (factorization of zero polynomial on hyperplane  $\mathbb{P}_k^d$ )**

cites explicitly:

- Statement(s) from [14],
- Lemma 1366 (*circular permutation*),
- Lemma 1410 (*inverse of affine map is affine map*),
- Lemma 1540 (*affine mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^1$* ),
- Lemma 1563 (*equivalent definition of face hyperplane*),
- Lemma 1586 (*geometric mapping with permutation*),
- Lemma 1621 (*factorization of zero polynomial on last reference hyperplane*).

**The proof of Lemma 1625 (Lagrange linear forms for  $\mathbb{P}_k^d$  are injective)**

cites explicitly:

- Statement(s) from [14],
- Lemma 1358 (*double induction by diagonal*),
- Lemma 1413 (*affinely independent family of 2 elements*),
- Definition 1424 (*finite element triple*),
- Lemma 1436 (*family of reference points is affinely independent*),
- Lemma 1476 (*Lagrange linear forms for  $\mathbb{P}_k^1$  are injective*),
- Lemma 1496 ( *$\mathcal{C}_k^d$  are layers of  $\mathcal{A}_k^d$* ),
- Lemma 1500 (*cardinal of  $\mathcal{C}_k^d$  and  $\mathcal{A}_k^{d-1}$* ),
- Lemma 1563 (*equivalent definition of face hyperplane*),
- Lemma 1584 (*geometric hyperface mapping*),
- Lemma 1585 (*hyperface geometric mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^{d-1}$* ),
- Definition 1588 (*Lagrange nodes of  $\mathbb{P}_k^d$* ),
- Lemma 1597 (*sub-vertices are affinely independent*),
- Lemma 1598 ( *$\mathbb{P}_{k-1}^d$  sub-nodes of sub-vertices are some nodes of  $\mathbb{P}_{k-1}^d$* ),
- Lemma 1605 (*face hyperplanes of Lagrange nodes of  $\mathbb{P}_k^d$* ),
- Lemma 1607 (*image of nodes by geometric hyperface mapping*),
- Definition 1608 (*Lagrange linear forms for  $\mathbb{P}_k^d$* ),
- Lemma 1609 (*Lagrange linear forms for  $\mathbb{P}_k^d$  for  $d=1$  are Lagrange linear forms for  $\mathbb{P}_k^1$* ),
- Lemma 1619 (*Lagrange linear forms for  $\mathbb{P}_1^d$  are injective*),
- Lemma 1623 (*factorization of zero polynomial on hyperplane  $\mathbb{P}_k^d$* ).

**The proof of Theorem 1626 (unisolvence of  $\mathbb{P}_k^d$ )**

cites explicitly:

- Lemma 1426 (*injectivity implies unisolvence*),
- Lemma 1525 (*dimension of  $\mathbb{P}_k^d$* ),
- Definition 1608 (*Lagrange linear forms for  $\mathbb{P}_k^d$* ),

Lemma 1611 (*cardinal of Lagrange linear forms of  $\mathbb{P}_k^d$* ),  
 Lemma 1625 (*Lagrange linear forms for  $\mathbb{P}_k^d$  are injective*).

**The proof of Lemma 1628 (*face unisolvence of  $\mathbb{P}_k^d$* )**

cites explicitly:

Statement(s) from [14],

Definition 1424 (*finite element triple*),

Lemma 1436 (*family of reference points is affinely independent*),

Lemma 1500 (*cardinal of  $\mathcal{C}_k^d$  and  $\mathcal{A}_k^{d-1}$* ),

Lemma 1501 (*cardinal of  $\mathcal{A}_{k,i}^d$  and  $\mathcal{A}_k^{d-1}$* ),

Lemma 1584 (*geometric hyperface mapping*),

Lemma 1585 (*hyperface geometric mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^{d-1}$* ),

Lemma 1605 (*face hyperplanes of Lagrange nodes of  $\mathbb{P}_k^d$* ),

Lemma 1607 (*image of nodes by geometric hyperface mapping*),

Definition 1608 (*Lagrange linear forms for  $\mathbb{P}_k^d$* ),

Lemma 1625 (*Lagrange linear forms for  $\mathbb{P}_k^d$  are injective*).

**The proof of Theorem 1629 ( *${}^{Lag}\mathbb{P}_k^d$  Lagrange finite element*)**

cites explicitly:

Definition 1424 (*finite element triple*),

Lemma 1525 (*dimension of  $\mathbb{P}_k^d$* ),

Lemma 1557 (*nontrivial simplex*),

Lemma 1617 (*unisolvence of  $\mathbb{P}_0^d$* ),

Theorem 1626 (*unisolvence of  $\mathbb{P}_k^d$* ).

**The proof of Lemma 1630 ( *${}^{Lag}\mathbb{P}_k^d$  Lagrange finite element for  $d = 1$  is Lagrange finite element on a current segment*)**

cites explicitly:

Theorem 1478 ( *${}^{Lag}\mathbb{P}_k^1$  Lagrange finite element*),

Lemma 1609 (*Lagrange linear forms for  $\mathbb{P}_k^d$  for  $d=1$  are Lagrange linear forms for  $\mathbb{P}_k^1$* ),

Theorem 1629 ( *${}^{Lag}\mathbb{P}_k^d$  Lagrange finite element*).

**The proof of Theorem 1631 ( *${}^{Lag}\hat{\mathbb{P}}_k^d$  reference Lagrange finite element*)**

cites explicitly:

Lemma 1436 (*family of reference points is affinely independent*),

Lemma 1443 (*simplex of reference vertices is reference simplex*),

Lemma 1599 (*reference Lagrange nodes of  $\mathbb{P}_k^d$* ),

Definition 1612 (*reference Lagrange linear forms for  $\mathbb{P}_k^d$* ),

Theorem 1629 ( *${}^{Lag}\mathbb{P}_k^d$  Lagrange finite element*).

**The proof of Lemma 1632 ( *${}^{Lag}\hat{\mathbb{P}}_k^d$  reference Lagrange finite element for  $d = 1$  is Lagrange finite element on the reference segment*)**

cites explicitly:

Theorem 1460 ( *${}^{Lag}\hat{\mathbb{P}}_k^1$  reference Lagrange finite element*),

Lemma 1613 (*reference Lagrange linear forms for  $\mathbb{P}_k^d$  for  $d = 1$  are reference Lagrange linear forms for  $\mathbb{P}_k^1$* ),

Theorem 1631 ( *${}^{Lag}\hat{\mathbb{P}}_k^d$  reference Lagrange finite element*).



## Appendix C

# Is explicitly cited in the proof of...

This appendix gathers the explicit citations that appear in the proof of results (lemmas and theorems) for each statement listed in Appendix A. Statements from [14, 15] are anonymized.

The dual graph is described in Appendix B.

Printing is not advised!

### Statement(s) from [14]

are explicitly cited in the proof of:

- Lemma 1369 (*image of ker is included in ker*),
- Lemma 1370 (*image of ker is ker*),
- Lemma 1372 (*vector subspace is invariant by translation*),
- Lemma 1373 (*range of linear map is vector subspace*),
- Lemma 1374 (*injectivity or surjectivity and dimension implies bijectivity*),
- Lemma 1375 (*inverse of isomorphism is linear map*),
- Lemma 1380 (*origin is in affine subspace*),
- Lemma 1381 (*equivalent definition of affine subspace*),
- Lemma 1382 (*affine subspace is invariant by change of origin*),
- Lemma 1385 (*affine subspace plus vector subspace is affine subspace*),
- Lemma 1386 (*closed under barycenter is affine subspace*),
- Lemma 1387 (*barycenter closure is affine subspace*),
- Lemma 1391 (*change of origin in affine map*),
- Lemma 1396 (*space of affine maps*),
- Lemma 1400 (*affine map preserves barycenter*),
- Lemma 1402 (*affine map preserves isobarycenter*),
- Lemma 1403 (*affine maps are closed by composition*),
- Lemma 1404 (*continuous affine map is continuous linear map*),
- Lemma 1405 (*injective affine submap is zero linear kernel*),
- Lemma 1407 (*surjective affine submap is full linear range*),
- Lemma 1409 (*inverse of affine submap is affine submap*),
- Lemma 1412 (*equivalent definition of affinely independent family*),
- Lemma 1458 (*reference Lagrange linear forms for  $\mathbb{P}_k^1$  are injective*),
- Lemma 1476 (*Lagrange linear forms for  $\mathbb{P}_k^1$  are injective*),
- Lemma 1526 (*isomorphism between  $\mathbb{P}_0^d$  and  $\mathbb{P}_0^{d-1}$* ),
- Lemma 1531 (*isomorphism between  $\mathbb{P}_k^d$  and  $\mathbb{P}_k^{d-1} \times \mathbb{P}_{k-1}^d$* ),
- Lemma 1537 (*product of two polynomials (alternate proof)*),
- Lemma 1550 (*properties of geometric mapping*),
- Lemma 1559 (*barycentric coordinate*),
- Lemma 1563 (*equivalent definition of face hyperplane*),
- Lemma 1564 (*reference face hyperplane*),



- Lemma 1581 (*properties of geometric l-face mapping*),
- Lemma 1615 (*Lagrange linear forms for  $\mathbb{P}_0^d$  are injective*),
- Lemma 1619 (*Lagrange linear forms for  $\mathbb{P}_1^d$  are injective*),
- Lemma 1621 (*factorization of zero polynomial on last reference hyperplane*),
- Lemma 1623 (*factorization of zero polynomial on hyperplane  $\mathbb{P}_k^d$* ),
- Lemma 1625 (*Lagrange linear forms for  $\mathbb{P}_k^d$  are injective*),
- Lemma 1628 (*face unisolvence of  $\mathbb{P}_k^d$* ).

**Statement(s) from [15]**

is not yet used.

**Lemma 1358 (*double induction by diagonal*)**

is explicitly cited in the proof of:

Lemma 1625 (*Lagrange linear forms for  $\mathbb{P}_k^d$  are injective*).

**Lemma 1360 (*strong double induction*)**

is explicitly cited in the proof of:

Lemma 1537 (*product of two polynomials (alternate proof)*).

**Definition 1362 (*binomial coefficient*)**

is explicitly cited in the proof of:

Lemma 1364 (*properties of the binomial coefficient*),

Lemma 1495 (*cardinal of  $\mathcal{C}_k^d$* ).

**Lemma 1364 (*properties of the binomial coefficient*)**

is explicitly cited in the proof of:

Lemma 1446 (*cardinal of  $\mathcal{A}_k^1$* ),

Lemma 1495 (*cardinal of  $\mathcal{C}_k^d$* ),

Lemma 1498 (*cardinal of  $\mathcal{A}_k^d$* ),

Lemma 1531 (*isomorphism between  $\mathbb{P}_k^d$  and  $\mathbb{P}_k^{d-1} \times \mathbb{P}_{k-1}^d$* ),

Lemma 1617 (*unisolvence of  $\mathbb{P}_0^d$* ),

Lemma 1620 (*unisolvence of  $\mathbb{P}_1^d$* ).

**Definition 1365 (*canonic families*)**

is explicitly cited in the proof of:

Lemma 1436 (*family of reference points is affinely independent*),

Lemma 1489 (*first  $\mathcal{C}_k^d$* ),

Lemma 1509 (*constant and affine spaces  $\mathbb{P}_0^d$  and  $\mathbb{P}_1^d$* ),

Lemma 1592 (*vertices are Lagrange nodes of  $\mathbb{P}_k^d$* ),

Lemma 1595 (*equivalent definition of sub-vertices of Lagrange nodes of  $\mathbb{P}_k^d$* ),

Lemma 1600 (*reference Lagrange nodes of  $\mathbb{P}_k^d$  for  $d = 1$  are reference Lagrange nodes of  $\mathbb{P}_k^1$* ),

Lemma 1601 (*equivalent definition of reference Lagrange nodes of  $\mathbb{P}_k^d$* ).

**Lemma 1366 (*circular permutation*)**

is explicitly cited in the proof of:

Lemma 1623 (*factorization of zero polynomial on hyperplane  $\mathbb{P}_k^d$* ).

**Lemma 1367 (*transposition*)**

is not yet used.

**Lemma 1368 (*jump enumeration*)**

is explicitly cited in the proof of:

Lemma 1576 ( *$(d - 1)$ -face is hyperface*).

**Lemma 1369 (*image of ker is included in ker*)**

is explicitly cited in the proof of:

Lemma 1370 (*image of ker is ker*).

**Lemma 1370** (*image of ker is ker*)

is explicitly cited in the proof of:

Lemma 1586 (*geometric mapping with permutation*).

**Lemma 1372** (*vector subspace is invariant by translation*)

is explicitly cited in the proof of:

Lemma 1381 (*equivalent definition of affine subspace*),

Lemma 1382 (*affine subspace is invariant by change of origin*),

Lemma 1383 (*vector subspace is affine subspace*).

**Lemma 1373** (*range of linear map is vector subspace*)

is explicitly cited in the proof of:

Lemma 1393 (*range of affine map is affine subspace*).

**Lemma 1374** (*injectivity or surjectivity and dimension implies bijectivity*)

is explicitly cited in the proof of:

Lemma 1426 (*injectivity implies unisolvence*),

Lemma 1526 (*isomorphism between  $\mathbb{P}_0^d$  and  $\mathbb{P}_0^{d-1}$* ),

Lemma 1531 (*isomorphism between  $\mathbb{P}_k^d$  and  $\mathbb{P}_k^{d-1} \times \mathbb{P}_{k-1}^d$* ),

Lemma 1550 (*properties of geometric mapping*).

**Lemma 1375** (*inverse of isomorphism is linear map*)

is explicitly cited in the proof of:

Lemma 1409 (*inverse of affine submap is affine submap*).

**Lemma 1376** (*free family of dim elements is basis*)

is explicitly cited in the proof of:

Lemma 1420 (*dimension of  $\mathbb{P}_k^1$* ),

Lemma 1428 (*degrees of freedom are basis*),

Lemma 1449 (*Lagrange polynomials is basis of  $\mathbb{P}_k^1$* ),

Lemma 1543 (*reference Lagrange polynomials is basis of  $\mathbb{P}_1^d$* ),

Lemma 1554 (*Lagrange polynomials is basis of  $\mathbb{P}_1^d$* ),

Lemma 1559 (*barycentric coordinate*),

Lemma 1570 (*d-face affine space is full space*).

**Definition 1379** (*affine subspace*)

is explicitly cited in the proof of:

Lemma 1380 (*origin is in affine subspace*),

Lemma 1381 (*equivalent definition of affine subspace*),

Lemma 1382 (*affine subspace is invariant by change of origin*),

Lemma 1385 (*affine subspace plus vector subspace is affine subspace*),

Lemma 1386 (*closed under barycenter is affine subspace*),

Lemma 1387 (*barycenter closure is affine subspace*),

Lemma 1390 (*equivalent definition of affine map*),

Lemma 1391 (*change of origin in affine map*),

Lemma 1393 (*range of affine map is affine subspace*),

Lemma 1398 (*affine submap*),

Lemma 1407 (*surjective affine submap is full linear range*),

Lemma 1409 (*inverse of affine submap is affine submap*).

**Lemma 1380** (*origin is in affine subspace*)

is not yet used.

**Lemma 1381** (*equivalent definition of affine subspace*)

is explicitly cited in the proof of:

Lemma 1382 (*affine subspace is invariant by change of origin*),

Lemma 1383 (*vector subspace is affine subspace*),

Lemma 1385 (*affine subspace plus vector subspace is affine subspace*),  
 Lemma 1386 (*closed under barycenter is affine subspace*),  
 Lemma 1390 (*equivalent definition of affine map*),  
 Lemma 1391 (*change of origin in affine map*),  
 Lemma 1407 (*surjective affine submap is full linear range*),  
 Lemma 1409 (*inverse of affine submap is affine submap*).

**Lemma 1382** (*affine subspace is invariant by change of origin*)

is explicitly cited in the proof of:

Lemma 1397 (*output restriction of affine map*),  
 Lemma 1407 (*surjective affine submap is full linear range*).

**Lemma 1383** (*vector subspace is affine subspace*)

is not yet used.

**Lemma 1385** (*affine subspace plus vector subspace is affine subspace*)

is not yet used.

**Lemma 1386** (*closed under barycenter is affine subspace*)

is not yet used.

**Lemma 1387** (*barycenter closure is affine subspace*)

is explicitly cited in the proof of:

Lemma 1563 (*equivalent definition of face hyperplane*),  
 Lemma 1569 (*equivalent definition of l-face affine space*).

**Definition 1388** (*affine map*)

is explicitly cited in the proof of:

Lemma 1390 (*equivalent definition of affine map*),  
 Lemma 1391 (*change of origin in affine map*),  
 Lemma 1393 (*range of affine map is affine subspace*),  
 Lemma 1398 (*affine submap*),  
 Lemma 1399 (*equivalent definition of affine map (finite dimension)*),  
 Lemma 1400 (*affine map preserves barycenter*),  
 Lemma 1403 (*affine maps are closed by composition*),  
 Lemma 1407 (*surjective affine submap is full linear range*),  
 Lemma 1409 (*inverse of affine submap is affine submap*),  
 Lemma 1550 (*properties of geometric mapping*),  
 Lemma 1581 (*properties of geometric l-face mapping*).

**Lemma 1390** (*equivalent definition of affine map*)

is explicitly cited in the proof of:

Lemma 1403 (*affine maps are closed by composition*),  
 Lemma 1404 (*continuous affine map is continuous linear map*),  
 Lemma 1407 (*surjective affine submap is full linear range*),  
 Lemma 1409 (*inverse of affine submap is affine submap*).

**Lemma 1391** (*change of origin in affine map*)

is explicitly cited in the proof of:

Lemma 1398 (*affine submap*).

**Lemma 1393** (*range of affine map is affine subspace*)

is explicitly cited in the proof of:

Lemma 1397 (*output restriction of affine map*).

**Definition 1395** (*set of affine maps*)

is explicitly cited in the proof of:

Lemma 1396 (*space of affine maps*),  
 Lemma 1397 (*output restriction of affine map*),  
 Lemma 1398 (*affine submap*),  
 Lemma 1403 (*affine maps are closed by composition*).

**Lemma 1396 (*space of affine maps*)**

is explicitly cited in the proof of:  
 Lemma 1418 ( $\mathbb{P}_k^1$  is space of degree at most  $k$ ).

**Lemma 1397 (*output restriction of affine map*)**

is explicitly cited in the proof of:  
 Lemma 1398 (*affine submap*).

**Lemma 1398 (*affine submap*)**

is explicitly cited in the proof of:  
 Lemma 1405 (*injective affine submap is zero linear kernel*).

**Lemma 1399 (*equivalent definition of affine map (finite dimension)*)**

is explicitly cited in the proof of:  
 Lemma 1418 ( $\mathbb{P}_k^1$  is space of degree at most  $k$ ),  
 Lemma 1462 (*properties of geometric mapping in dimension 1*),  
 Lemma 1509 (*constant and affine spaces  $\mathbb{P}_0^d$  and  $\mathbb{P}_1^d$* ),  
 Lemma 1539 (*affine mapping of monomials is  $\mathbb{P}_k^l$* ).

**Lemma 1400 (*affine map preserves barycenter*)**

is explicitly cited in the proof of:  
 Lemma 1402 (*affine map preserves isobarycenter*),  
 Lemma 1607 (*image of nodes by geometric hyperface mapping*).

**Definition 1401 (*isobarycenter*)**

is explicitly cited in the proof of:  
 Lemma 1402 (*affine map preserves isobarycenter*),  
 Lemma 1435 (*reference isobarycenter*),  
 Lemma 1591 (*barycentric coordinates of Lagrange nodes of  $\mathbb{P}_k^d$* ).

**Lemma 1402 (*affine map preserves isobarycenter*)**

is not yet used.

**Lemma 1403 (*affine maps are closed by composition*)**

is explicitly cited in the proof of:  
 Lemma 1554 (*Lagrange polynomials is basis of  $\mathbb{P}_1^d$* ).

**Lemma 1404 (*continuous affine map is continuous linear map*)**

is not yet used.

**Lemma 1405 (*injective affine submap is zero linear kernel*)**

is explicitly cited in the proof of:  
 Lemma 1406 (*injective affine map is zero linear kernel*),  
 Lemma 1409 (*inverse of affine submap is affine submap*).

**Lemma 1406 (*injective affine map is zero linear kernel*)**

is explicitly cited in the proof of:  
 Lemma 1550 (*properties of geometric mapping*),  
 Lemma 1581 (*properties of geometric  $l$ -face mapping*).

**Lemma 1407 (*surjective affine submap is full linear range*)**

is explicitly cited in the proof of:  
 Lemma 1408 (*surjective affine map is full linear range*),  
 Lemma 1409 (*inverse of affine submap is affine submap*).

**Lemma 1408** (*surjective affine map is full linear range*)  
is not yet used.

**Lemma 1409** (*inverse of affine submap is affine submap*)  
is explicitly cited in the proof of:  
Lemma 1410 (*inverse of affine map is affine map*),  
Lemma 1581 (*properties of geometric l-face mapping*).

**Lemma 1410** (*inverse of affine map is affine map*)  
is explicitly cited in the proof of:  
Lemma 1462 (*properties of geometric mapping in dimension 1*),  
Lemma 1550 (*properties of geometric mapping*),  
Lemma 1623 (*factorization of zero polynomial on hyperplane  $\mathbb{P}_k^d$* ).

**Definition 1411** (*affinely independent family*)  
is explicitly cited in the proof of:  
Lemma 1412 (*equivalent definition of affinely independent family*),  
Lemma 1413 (*affinely independent family of 2 elements*),  
Lemma 1414 (*affinely independent family is closed by sub-family*),  
Lemma 1436 (*family of reference points is affinely independent*),  
Lemma 1550 (*properties of geometric mapping*),  
Lemma 1559 (*barycentric coordinate*),  
Lemma 1570 (*d-face affine space is full space*),  
Lemma 1597 (*sub-vertices are affinely independent*).

**Lemma 1412** (*equivalent definition of affinely independent family*)  
is explicitly cited in the proof of:  
Lemma 1414 (*affinely independent family is closed by sub-family*),  
Lemma 1559 (*barycentric coordinate*).

**Lemma 1413** (*affinely independent family of 2 elements*)  
is explicitly cited in the proof of:  
Lemma 1625 (*Lagrange linear forms for  $\mathbb{P}_k^d$  are injective*).

**Lemma 1414** (*affinely independent family is closed by sub-family*)  
is explicitly cited in the proof of:  
Lemma 1570 (*d-face affine space is full space*),  
Lemma 1581 (*properties of geometric l-face mapping*).

**Definition 1416** (*monomial of a single variable*)  
is explicitly cited in the proof of:  
Lemma 1418 ( $\mathbb{P}_k^1$  is space of degree at most  $k$ ),  
Lemma 1504 (*monomial in  $d$  variables for  $d = 1$  is monomial of a single variable*).

**Definition 1417** (*polynomial space  $\mathbb{P}_k^1$* )  
is explicitly cited in the proof of:  
Lemma 1418 ( $\mathbb{P}_k^1$  is space of degree at most  $k$ ),  
Lemma 1420 (*dimension of  $\mathbb{P}_k^1$* ),  
Lemma 1449 (*Lagrange polynomials is basis of  $\mathbb{P}_k^1$* ),  
Lemma 1471 (*geometric mapping of  $\mathbb{P}_k^1$  is  $\mathbb{P}_k^1$* ),  
Lemma 1506 (*polynomial space  $\mathbb{P}_k^d$  for  $d = 1$  is  $\mathbb{P}_k^1$* ).

**Lemma 1418** ( $\mathbb{P}_k^1$  is space of degree at most  $k$ )  
is explicitly cited in the proof of:  
Lemma 1449 (*Lagrange polynomials is basis of  $\mathbb{P}_k^1$* ),  
Lemma 1462 (*properties of geometric mapping in dimension 1*).

**Lemma 1419** (*monomials are free in  $\mathbb{P}_k^1$* )

is explicitly cited in the proof of:

Lemma 1420 (*dimension of  $\mathbb{P}_k^1$* ),

Lemma 1534 (*expression of  $\mathbb{P}_k^d$  as polynomial of  $x_d$* ).

**Lemma 1420** (*dimension of  $\mathbb{P}_k^1$* )

is explicitly cited in the proof of:

Lemma 1449 (*Lagrange polynomials is basis of  $\mathbb{P}_k^1$* ),

Lemma 1459 (*unisolvence of  $\mathbb{P}_k^1$  (reference)*),

Theorem 1460 ( *$\overset{\text{Log}}{\hat{\mathbb{P}}}_k^1$  reference Lagrange finite element*),

Lemma 1477 (*unisolvence of  $\mathbb{P}_k^1$* ),

Theorem 1478 ( *$\overset{\text{Log}}{\mathbb{P}}_k^1$  Lagrange finite element*).

**Lemma 1422** (*product of two univariate polynomials*)

is explicitly cited in the proof of:

Lemma 1449 (*Lagrange polynomials is basis of  $\mathbb{P}_k^1$* ),

Lemma 1537 (*product of two polynomials (alternate proof)*).

**Definition 1424** (*finite element triple*)

is explicitly cited in the proof of:

Lemma 1426 (*injectivity implies unisolvence*),

Lemma 1458 (*reference Lagrange linear forms for  $\mathbb{P}_k^1$  are injective*),

Theorem 1460 ( *$\overset{\text{Log}}{\hat{\mathbb{P}}}_k^1$  reference Lagrange finite element*),

Lemma 1476 (*Lagrange linear forms for  $\mathbb{P}_k^1$  are injective*),

Theorem 1478 ( *$\overset{\text{Log}}{\mathbb{P}}_k^1$  Lagrange finite element*),

Lemma 1615 (*Lagrange linear forms for  $\mathbb{P}_0^d$  are injective*),

Lemma 1619 (*Lagrange linear forms for  $\mathbb{P}_1^d$  are injective*),

Lemma 1625 (*Lagrange linear forms for  $\mathbb{P}_k^d$  are injective*),

Lemma 1628 (*face unisolvence of  $\mathbb{P}_k^d$* ),

Theorem 1629 ( *$\overset{\text{Log}}{\mathbb{P}}_k^d$  Lagrange finite element*).

**Lemma 1426** (*injectivity implies unisolvence*)

is explicitly cited in the proof of:

Lemma 1459 (*unisolvence of  $\mathbb{P}_k^1$  (reference)*),

Lemma 1477 (*unisolvence of  $\mathbb{P}_k^1$* ),

Lemma 1617 (*unisolvence of  $\mathbb{P}_0^d$* ),

Lemma 1620 (*unisolvence of  $\mathbb{P}_1^d$* ),

Theorem 1626 (*unisolvence of  $\mathbb{P}_k^d$* ).

**Lemma 1427** (*dimension of approximation space*)

is explicitly cited in the proof of:

Lemma 1428 (*degrees of freedom are basis*).

**Lemma 1428** (*degrees of freedom are basis*)

is not yet used.

**Definition 1430** (*shape function*)

is not yet used.

**Definition 1432** (*family of points*)

is not yet used.

**Definition 1434** (*family of reference points*)

is explicitly cited in the proof of:

Lemma 1435 (*reference isobarycenter*),

Lemma 1436 (*family of reference points is affinely independent*),

- Lemma 1443 (*simplex of reference vertices is reference simplex*),
- Lemma 1543 (*reference Lagrange polynomials is basis of  $\mathbb{P}_1^d$* ),
- Lemma 1549 (*reference geometric mapping is identity*),
- Lemma 1565 (*face hyperplane is image of reference face hyperplane*),
- Lemma 1599 (*reference Lagrange nodes of  $\mathbb{P}_k^d$* ).

**Lemma 1435** (*reference isobarycenter*)

is explicitly cited in the proof of:

- Lemma 1599 (*reference Lagrange nodes of  $\mathbb{P}_k^d$* ),
- Lemma 1601 (*equivalent definition of reference Lagrange nodes of  $\mathbb{P}_k^d$* ).

**Lemma 1436** (*family of reference points is affinely independent*)

is explicitly cited in the proof of:

- Lemma 1564 (*reference face hyperplane*),
- Lemma 1602 (*number of reference Lagrange nodes of  $\mathbb{P}_k^d$* ),
- Lemma 1625 (*Lagrange linear forms for  $\mathbb{P}_k^d$  are injective*),
- Lemma 1628 (*face unisolvence of  $\mathbb{P}_k^d$* ),
- Theorem 1631 ( *$\overset{\text{Log}}{\hat{\mathbb{P}}}_k^d$  reference Lagrange finite element*).

**Definition 1437** (*reference simplex*)

is explicitly cited in the proof of:

- Lemma 1438 (*coordinates in reference simplex are smaller than 1*),
- Lemma 1439 (*nontrivial reference simplex*),
- Lemma 1443 (*simplex of reference vertices is reference simplex*),
- Lemma 1452 (*reference simplex is nontrivial in  $\mathbb{R}$* ),
- Lemma 1550 (*properties of geometric mapping*),
- Lemma 1581 (*properties of geometric l-face mapping*).

**Lemma 1438** (*coordinates in reference simplex are smaller than 1*)

is not yet used.

**Lemma 1439** (*nontrivial reference simplex*)

is explicitly cited in the proof of:

- Lemma 1452 (*reference simplex is nontrivial in  $\mathbb{R}$* ),
- Lemma 1557 (*nontrivial simplex*).

**Definition 1440** (*simplex*)

is explicitly cited in the proof of:

- Lemma 1442 (*coordinates in simplex are smaller than 1*),
- Lemma 1443 (*simplex of reference vertices is reference simplex*),
- Lemma 1463 (*current simplex is nontrivial in  $\mathbb{R}$* ),
- Lemma 1550 (*properties of geometric mapping*),
- Lemma 1574 (*d-face is simplex*).

**Lemma 1442** (*coordinates in simplex are smaller than 1*)

is not yet used.

**Lemma 1443** (*simplex of reference vertices is reference simplex*)

is explicitly cited in the proof of:

- Theorem 1631 ( *$\overset{\text{Log}}{\hat{\mathbb{P}}}_k^d$  reference Lagrange finite element*).

**Definition 1445** (*multi-indices  $\mathcal{A}_k^1$* )

is explicitly cited in the proof of:

- Lemma 1446 (*cardinal of  $\mathcal{A}_k^1$* ),
- Lemma 1487 (*multi-indices  $\mathcal{A}_k^d$  for  $d = 1$  is  $\mathcal{A}_k^1$* ).

**Lemma 1446** (*cardinal of  $\mathcal{A}_k^1$* )

is not yet used.

**Definition 1448** (*Lagrange polynomials of  $\mathbb{P}_k^1$* )

is explicitly cited in the proof of:

Lemma 1449 (*Lagrange polynomials is basis of  $\mathbb{P}_k^1$* ),

Lemma 1470 (*Lagrange polynomials of  $\mathbb{P}_k^1$  are images of reference*),

Lemma 1542 (*reference Lagrange polynomials of  $\mathbb{P}_1^d$  for  $d = 1$  are reference Lagrange polynomials of  $\mathbb{P}_k^1$  for  $k = 1$* ).

**Lemma 1449** (*Lagrange polynomials is basis of  $\mathbb{P}_k^1$* )

is explicitly cited in the proof of:

Lemma 1450 (*decomposition of  $\mathbb{P}_k^1$  polynomial in Lagrange basis*),

Lemma 1455 (*reference Lagrange basis of  $\mathbb{P}_k^1$* ),

Lemma 1469 (*Lagrange basis of  $\mathbb{P}_k^1$* ).

**Lemma 1450** (*decomposition of  $\mathbb{P}_k^1$  polynomial in Lagrange basis*)

is explicitly cited in the proof of:

Lemma 1458 (*reference Lagrange linear forms for  $\mathbb{P}_k^1$  are injective*),

Lemma 1476 (*Lagrange linear forms for  $\mathbb{P}_k^1$  are injective*).

**Lemma 1452** (*reference simplex is nontrivial in  $\mathbb{R}$* )

is explicitly cited in the proof of:

Theorem 1460 ( *$\overset{\text{Log}}{\hat{\mathbb{P}}}_k^1$  reference Lagrange finite element*),

Lemma 1464 (*current simplex is image of reference in  $\mathbb{R}$* ).

**Definition 1453** (*reference Lagrange nodes of  $\mathbb{P}_k^1$* )

is explicitly cited in the proof of:

Lemma 1454 (*reference Lagrange nodes are distinct*),

Lemma 1462 (*properties of geometric mapping in dimension 1*),

Lemma 1468 (*Lagrange nodes of  $\mathbb{P}_k^1$  are images of reference*),

Lemma 1542 (*reference Lagrange polynomials of  $\mathbb{P}_1^d$  for  $d = 1$  are reference Lagrange polynomials of  $\mathbb{P}_k^1$  for  $k = 1$* ),

Lemma 1600 (*reference Lagrange nodes of  $\mathbb{P}_k^d$  for  $d = 1$  are reference Lagrange nodes of  $\mathbb{P}_k^1$* ).

**Lemma 1454** (*reference Lagrange nodes are distinct*)

is explicitly cited in the proof of:

Lemma 1455 (*reference Lagrange basis of  $\mathbb{P}_k^1$* ),

Lemma 1458 (*reference Lagrange linear forms for  $\mathbb{P}_k^1$  are injective*),

Lemma 1459 (*unisolvence of  $\mathbb{P}_k^1$  (reference)*).

**Lemma 1455** (*reference Lagrange basis of  $\mathbb{P}_k^1$* )

is explicitly cited in the proof of:

Lemma 1462 (*properties of geometric mapping in dimension 1*),

Lemma 1470 (*Lagrange polynomials of  $\mathbb{P}_k^1$  are images of reference*),

Lemma 1471 (*geometric mapping of  $\mathbb{P}_k^1$  is  $\mathbb{P}_k^1$* ),

Lemma 1542 (*reference Lagrange polynomials of  $\mathbb{P}_1^d$  for  $d = 1$  are reference Lagrange polynomials of  $\mathbb{P}_k^1$  for  $k = 1$* ).

**Definition 1456** (*reference Lagrange linear forms for  $\mathbb{P}_k^1$* )

is explicitly cited in the proof of:

Lemma 1457 (*reference Lagrange linear forms for  $\mathbb{P}_k^1$  are linear*),

Lemma 1458 (*reference Lagrange linear forms for  $\mathbb{P}_k^1$  are injective*),

Lemma 1474 ( *$\mathbb{P}_k^1$  Lagrange linear forms are images of reference*),

Lemma 1613 (*reference Lagrange linear forms for  $\mathbb{P}_k^d$  for  $d = 1$  are reference Lagrange linear forms for  $\mathbb{P}_k^1$* ).



**Lemma 1457** (*reference Lagrange linear forms for  $\mathbb{P}_k^1$  are linear*)  
is not yet used.

**Lemma 1458** (*reference Lagrange linear forms for  $\mathbb{P}_k^1$  are injective*)  
is explicitly cited in the proof of:  
Lemma 1459 (*unisolvence of  $\mathbb{P}_k^1$  (reference)*).

**Lemma 1459** (*unisolvence of  $\mathbb{P}_k^1$  (reference)*)  
is explicitly cited in the proof of:  
Theorem 1460 ( *${}^{\text{Lag}}\hat{\mathbb{P}}_k^1$  reference Lagrange finite element*).

**Theorem 1460** ( *${}^{\text{Lag}}\hat{\mathbb{P}}_k^1$  reference Lagrange finite element*)  
is explicitly cited in the proof of:  
Lemma 1632 ( *${}^{\text{Lag}}\hat{\mathbb{P}}_k^d$  reference Lagrange finite element for  $d = 1$  is Lagrange finite element on the reference segment*).

**Definition 1461** (*geometric mapping in dimension 1*)  
is explicitly cited in the proof of:  
Lemma 1462 (*properties of geometric mapping in dimension 1*),  
Lemma 1464 (*current simplex is image of reference in  $\mathbb{R}$* ),  
Lemma 1468 (*Lagrange nodes of  $\mathbb{P}_k^1$  are images of reference*),  
Lemma 1470 (*Lagrange polynomials of  $\mathbb{P}_k^1$  are images of reference*),  
Lemma 1471 (*geometric mapping of  $\mathbb{P}_k^1$  is  $\mathbb{P}_k^1$* ),  
Lemma 1548 (*geometric mapping for  $d = 1$  is geometric mapping in dimension 1*).

**Lemma 1462** (*properties of geometric mapping in dimension 1*)  
is not yet used.

**Lemma 1463** (*current simplex is nontrivial in  $\mathbb{R}$* )  
is explicitly cited in the proof of:  
Lemma 1464 (*current simplex is image of reference in  $\mathbb{R}$* ),  
Theorem 1478 ( *${}^{\text{Lag}}\mathbb{P}_k^1$  Lagrange finite element*).

**Lemma 1464** (*current simplex is image of reference in  $\mathbb{R}$* )  
is not yet used.

**Definition 1465** (*Lagrange nodes of  $\mathbb{P}_k^1$* )  
is explicitly cited in the proof of:  
Lemma 1467 (*Lagrange nodes are distinct*),  
Lemma 1468 (*Lagrange nodes of  $\mathbb{P}_k^1$  are images of reference*),  
Lemma 1589 (*Lagrange nodes of  $\mathbb{P}_k^d$  for  $d = 1$  are Lagrange nodes of  $\mathbb{P}_k^1$* ).

**Lemma 1467** (*Lagrange nodes are distinct*)  
is explicitly cited in the proof of:  
Lemma 1469 (*Lagrange basis of  $\mathbb{P}_k^1$* ),  
Lemma 1476 (*Lagrange linear forms for  $\mathbb{P}_k^1$  are injective*),  
Lemma 1477 (*unisolvence of  $\mathbb{P}_k^1$* ).

**Lemma 1468** (*Lagrange nodes of  $\mathbb{P}_k^1$  are images of reference*)  
is explicitly cited in the proof of:  
Lemma 1470 (*Lagrange polynomials of  $\mathbb{P}_k^1$  are images of reference*),  
Lemma 1474 ( *$\mathbb{P}_k^1$  Lagrange linear forms are images of reference*).

**Lemma 1469** (*Lagrange basis of  $\mathbb{P}_k^1$* )  
is explicitly cited in the proof of:  
Lemma 1470 (*Lagrange polynomials of  $\mathbb{P}_k^1$  are images of reference*),  
Lemma 1471 (*geometric mapping of  $\mathbb{P}_k^1$  is  $\mathbb{P}_k^1$* ).

**Lemma 1470** (*Lagrange polynomials of  $\mathbb{P}_k^1$  are images of reference*)

is explicitly cited in the proof of:

Lemma 1471 (*geometric mapping of  $\mathbb{P}_k^1$  is  $\mathbb{P}_k^1$* ).

**Lemma 1471** (*geometric mapping of  $\mathbb{P}_k^1$  is  $\mathbb{P}_k^1$* )

is not yet used.

**Definition 1472** (*Lagrange linear forms for  $\mathbb{P}_k^1$* )

is explicitly cited in the proof of:

Lemma 1473 ( *$\mathbb{P}_k^1$  Lagrange linear forms are linear*),

Lemma 1474 ( *$\mathbb{P}_k^1$  Lagrange linear forms are images of reference*),

Lemma 1476 (*Lagrange linear forms for  $\mathbb{P}_k^1$  are injective*),

Lemma 1609 (*Lagrange linear forms for  $\mathbb{P}_k^d$  for  $d=1$  are Lagrange linear forms for  $\mathbb{P}_k^1$* ).

**Lemma 1473** ( *$\mathbb{P}_k^1$  Lagrange linear forms are linear*)

is not yet used.

**Lemma 1474** ( *$\mathbb{P}_k^1$  Lagrange linear forms are images of reference*)

is not yet used.

**Lemma 1476** (*Lagrange linear forms for  $\mathbb{P}_k^1$  are injective*)

is explicitly cited in the proof of:

Lemma 1477 (*unisolvence of  $\mathbb{P}_k^1$* ),

Lemma 1625 (*Lagrange linear forms for  $\mathbb{P}_k^d$  are injective*).

**Lemma 1477** (*unisolvence of  $\mathbb{P}_k^1$* )

is explicitly cited in the proof of:

Theorem 1478 ( *${}^{\text{Log}}\mathbb{P}_k^1$  Lagrange finite element*).

**Theorem 1478** ( *${}^{\text{Log}}\mathbb{P}_k^1$  Lagrange finite element*)

is explicitly cited in the proof of:

Lemma 1630 ( *${}^{\text{Log}}\mathbb{P}_k^d$  Lagrange finite element for  $d = 1$  is Lagrange finite element on a current segment*).

**Definition 1480** (*length of multi-indices*)

is explicitly cited in the proof of:

Lemma 1481 (*length of multi-indices is additive*),

Lemma 1487 (*multi-indices  $\mathcal{A}_k^d$  for  $d = 1$  is  $\mathcal{A}_k^1$* ),

Lemma 1488 (*indices are smaller than maximal length*),

Lemma 1500 (*cardinal of  $\mathcal{C}_k^d$  and  $\mathcal{A}_k^{d-1}$* ),

Lemma 1501 (*cardinal of  $\mathcal{A}_{k,i}^d$  and  $\mathcal{A}_k^{d-1}$* ),

Lemma 1539 (*affine mapping of monomials is  $\mathbb{P}_k^l$* ),

Lemma 1591 (*barycentric coordinates of Lagrange nodes of  $\mathbb{P}_k^d$* ),

Lemma 1592 (*vertices are Lagrange nodes of  $\mathbb{P}_k^d$* ),

Lemma 1595 (*equivalent definition of sub-vertices of Lagrange nodes of  $\mathbb{P}_k^d$* ),

Lemma 1598 ( *$\mathbb{P}_{k-1}^d$  sub-nodes of sub-vertices are some nodes of  $\mathbb{P}_{k-1}^d$* ).

**Lemma 1481** (*length of multi-indices is additive*)

is explicitly cited in the proof of:

Lemma 1514 (*product of monomials*).

**Definition 1482** (*factorial of multi-indices*)

is explicitly cited in the proof of:

Lemma 1483 (*factorial of multi-index is positive*),

Lemma 1522 (*partial derivative of monomials at 0*).

**Lemma 1483** (*factorial of multi-index is positive*)

is explicitly cited in the proof of:

Lemma 1523 (*monomials are free in  $\mathbb{P}_k^d$* ).

**Definition 1484** (*Kronecker delta of multi-indices*)

is explicitly cited in the proof of:

Lemma 1485 (*value of Kronecker delta of multi-indices*),

Lemma 1522 (*partial derivative of monomials at 0*).

**Lemma 1485** (*value of Kronecker delta of multi-indices*)

is not yet used.

**Definition 1486** (*sets of multi-indices  $\mathcal{A}_k^d$  and  $\mathcal{C}_k^d$* )

is explicitly cited in the proof of:

Lemma 1487 (*multi-indices  $\mathcal{A}_k^d$  for  $d = 1$  is  $\mathcal{A}_k^1$* ),

Lemma 1488 (*indices are smaller than maximal length*),

Lemma 1489 (*first  $\mathcal{C}_k^d$* ),

Lemma 1493 (*slices of  $\mathcal{C}_k^d$* ),

Lemma 1496 ( *$\mathcal{C}_k^d$  are layers of  $\mathcal{A}_k^d$* ),

Lemma 1500 (*cardinal of  $\mathcal{C}_k^d$  and  $\mathcal{A}_k^{d-1}$* ),

Lemma 1501 (*cardinal of  $\mathcal{A}_{k,i}^d$  and  $\mathcal{A}_k^{d-1}$* ),

Lemma 1512 (*monomials of  $\mathcal{C}_k^d$  have degree  $k$* ),

Lemma 1515 (*product of monomial and polynomial*),

Lemma 1521 (*derivating more than degree is 0*),

Lemma 1539 (*affine mapping of monomials is  $\mathbb{P}_k^l$* ),

Lemma 1591 (*barycentric coordinates of Lagrange nodes of  $\mathbb{P}_k^d$* ),

Lemma 1592 (*vertices are Lagrange nodes of  $\mathbb{P}_k^d$* ),

Lemma 1595 (*equivalent definition of sub-vertices of Lagrange nodes of  $\mathbb{P}_k^d$* ),

Lemma 1605 (*face hyperplanes of Lagrange nodes of  $\mathbb{P}_k^d$* ),

Lemma 1607 (*image of nodes by geometric hyperface mapping*).

**Lemma 1487** (*multi-indices  $\mathcal{A}_k^d$  for  $d = 1$  is  $\mathcal{A}_k^1$* )

is not yet used.

**Lemma 1488** (*indices are smaller than maximal length*)

is explicitly cited in the proof of:

Lemma 1489 (*first  $\mathcal{C}_k^d$* ),

Lemma 1493 (*slices of  $\mathcal{C}_k^d$* ),

Lemma 1591 (*barycentric coordinates of Lagrange nodes of  $\mathbb{P}_k^d$* ).

**Lemma 1489** (*first  $\mathcal{C}_k^d$* )

is explicitly cited in the proof of:

Lemma 1495 (*cardinal of  $\mathcal{C}_k^d$* ),

Lemma 1497 (*first multi-indices  $\mathcal{A}_k^d$* ),

Lemma 1504 (*monomial in  $d$  variables for  $d = 1$  is monomial of a single variable*).

**Definition 1491** (*slices  $\check{S}_{k,i}^d$  and  $\tilde{S}_{k,i}^d$* )

is explicitly cited in the proof of:

Lemma 1493 (*slices of  $\mathcal{C}_k^d$* ),

Lemma 1494 (*cardinal of slices of  $\mathcal{C}_k^d$* ),

Lemma 1529 (*decomposition of  $\mathbb{P}_k^d$* ).

**Lemma 1493** (*slices of  $\mathcal{C}_k^d$* )

is explicitly cited in the proof of:

Lemma 1495 (*cardinal of  $\mathcal{C}_k^d$* ),

Lemma 1529 (*decomposition of  $\mathbb{P}_k^d$* ).

**Lemma 1494** (*cardinal of slices of  $\mathcal{C}_k^d$* )

is explicitly cited in the proof of:

Lemma 1495 (*cardinal of  $\mathcal{C}_k^d$* ),

Lemma 1529 (*decomposition of  $\mathbb{P}_k^d$* ).

**Lemma 1495** (*cardinal of  $\mathcal{C}_k^d$* )

is explicitly cited in the proof of:

Lemma 1498 (*cardinal of  $\mathcal{A}_k^d$* ),

Lemma 1500 (*cardinal of  $\mathcal{C}_k^d$  and  $\mathcal{A}_k^{d-1}$* ),

Lemma 1605 (*face hyperplanes of Lagrange nodes of  $\mathbb{P}_k^d$* ).

**Lemma 1496** ( *$\mathcal{C}_k^d$  are layers of  $\mathcal{A}_k^d$* )

is explicitly cited in the proof of:

Lemma 1497 (*first multi-indices  $\mathcal{A}_k^d$* ),

Lemma 1498 (*cardinal of  $\mathcal{A}_k^d$* ),

Lemma 1508 ( *$\mathbb{P}_k^d$  is nondecreasing sequence in  $k$* ),

Lemma 1513 ( *$\mathbb{P}_k^d$  is space of degree at most  $k$* ),

Lemma 1529 (*decomposition of  $\mathbb{P}_k^d$* ),

Lemma 1540 (*affine mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^l$* ),

Lemma 1598 ( *$\mathbb{P}_{k-1}^d$  sub-nodes of sub-vertices are some nodes of  $\mathbb{P}_{k-1}^d$* ),

Lemma 1625 (*Lagrange linear forms for  $\mathbb{P}_k^d$  are injective*).

**Lemma 1497** (*first multi-indices  $\mathcal{A}_k^d$* )

is explicitly cited in the proof of:

Lemma 1506 (*polynomial space  $\mathbb{P}_k^d$  for  $d = 1$  is  $\mathbb{P}_k^1$* ),

Lemma 1509 (*constant and affine spaces  $\mathbb{P}_0^d$  and  $\mathbb{P}_1^d$* ),

Lemma 1589 (*Lagrange nodes of  $\mathbb{P}_k^d$  for  $d = 1$  are Lagrange nodes of  $\mathbb{P}_k^1$* ),

Lemma 1593 (*Lagrange nodes of  $\mathbb{P}_1^d$  are vertices*),

Lemma 1600 (*reference Lagrange nodes of  $\mathbb{P}_k^d$  for  $d = 1$  are reference Lagrange nodes of  $\mathbb{P}_k^1$* ),

Lemma 1609 (*Lagrange linear forms for  $\mathbb{P}_k^d$  for  $d=1$  are Lagrange linear forms for  $\mathbb{P}_k^1$* ),

Lemma 1613 (*reference Lagrange linear forms for  $\mathbb{P}_k^d$  for  $d = 1$  are reference Lagrange linear forms for  $\mathbb{P}_k^1$* ),

Lemma 1619 (*Lagrange linear forms for  $\mathbb{P}_1^d$  are injective*).

**Lemma 1498** (*cardinal of  $\mathcal{A}_k^d$* )

is explicitly cited in the proof of:

Lemma 1500 (*cardinal of  $\mathcal{C}_k^d$  and  $\mathcal{A}_k^{d-1}$* ),

Lemma 1525 (*dimension of  $\mathbb{P}_k^d$* ),

Lemma 1590 (*number of Lagrange nodes of  $\mathbb{P}_k^d$* ),

Lemma 1602 (*number of reference Lagrange nodes of  $\mathbb{P}_k^d$* ),

Lemma 1605 (*face hyperplanes of Lagrange nodes of  $\mathbb{P}_k^d$* ).

**Lemma 1500** (*cardinal of  $\mathcal{C}_k^d$  and  $\mathcal{A}_k^{d-1}$* )

is explicitly cited in the proof of:

Lemma 1605 (*face hyperplanes of Lagrange nodes of  $\mathbb{P}_k^d$* ),

Lemma 1607 (*image of nodes by geometric hyperface mapping*),

Lemma 1625 (*Lagrange linear forms for  $\mathbb{P}_k^d$  are injective*),

Lemma 1628 (*face unisolvence of  $\mathbb{P}_k^d$* ).

**Lemma 1501** (*cardinal of  $\mathcal{A}_{k,i}^d$  and  $\mathcal{A}_k^{d-1}$* )

is explicitly cited in the proof of:

Lemma 1605 (*face hyperplanes of Lagrange nodes of  $\mathbb{P}_k^d$* ),

Lemma 1607 (*image of nodes by geometric hyperface mapping*),

Lemma 1628 (*face unisolvence of  $\mathbb{P}_k^d$* ).

**Definition 1503** (*monomial in  $d$  variables*)

is explicitly cited in the proof of:

- Lemma 1504 (*monomial in  $d$  variables for  $d = 1$  is monomial of a single variable*),
- Lemma 1512 (*monomials of  $\mathcal{C}_k^d$  have degree  $k$* ),
- Lemma 1514 (*product of monomials*),
- Lemma 1517 (*partial derivative of monomials*),
- Lemma 1539 (*affine mapping of monomials is  $\mathbb{P}_k^l$* ).

**Lemma 1504** (*monomial in  $d$  variables for  $d = 1$  is monomial of a single variable*)  
is not yet used.

**Definition 1505** (*polynomial space  $\mathbb{P}_k^d$* )

is explicitly cited in the proof of:

- Lemma 1506 (*polynomial space  $\mathbb{P}_k^d$  for  $d = 1$  is  $\mathbb{P}_k^1$* ),
- Lemma 1507 ( *$\mathbb{P}_k^d$  is vector space*),
- Lemma 1508 ( *$\mathbb{P}_k^d$  is nondecreasing sequence in  $k$* ),
- Lemma 1509 (*constant and affine spaces  $\mathbb{P}_0^d$  and  $\mathbb{P}_1^d$* ),
- Lemma 1513 ( *$\mathbb{P}_k^d$  is space of degree at most  $k$* ),
- Lemma 1515 (*product of monomial and polynomial*),
- Lemma 1516 (*product of two polynomials*),
- Lemma 1519 (*partial derivative is linear*),
- Lemma 1523 (*monomials are free in  $\mathbb{P}_k^d$* ),
- Lemma 1524 (*monomials are a basis of  $\mathbb{P}_k^d$* ),
- Lemma 1529 (*decomposition of  $\mathbb{P}_k^d$* ),
- Lemma 1533 ( *$\mathbb{P}_k^d$  is nondecreasing sequence in  $d$* ),
- Lemma 1540 (*affine mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^l$* ).

**Lemma 1506** (*polynomial space  $\mathbb{P}_k^d$  for  $d = 1$  is  $\mathbb{P}_k^1$* )  
is not yet used.

**Lemma 1507** ( *$\mathbb{P}_k^d$  is vector space*)

is explicitly cited in the proof of:

- Lemma 1516 (*product of two polynomials*),
- Lemma 1529 (*decomposition of  $\mathbb{P}_k^d$* ),
- Lemma 1531 (*isomorphism between  $\mathbb{P}_k^d$  and  $\mathbb{P}_k^{d-1} \times \mathbb{P}_{k-1}^d$* ),
- Lemma 1533 ( *$\mathbb{P}_k^d$  is nondecreasing sequence in  $d$* ),
- Lemma 1537 (*product of two polynomials (alternate proof)*),
- Lemma 1540 (*affine mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^l$* ).

**Lemma 1508** ( *$\mathbb{P}_k^d$  is nondecreasing sequence in  $k$* )

is explicitly cited in the proof of:

- Lemma 1529 (*decomposition of  $\mathbb{P}_k^d$* ),
- Lemma 1537 (*product of two polynomials (alternate proof)*),
- Lemma 1540 (*affine mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^l$* ).

**Lemma 1509** (*constant and affine spaces  $\mathbb{P}_0^d$  and  $\mathbb{P}_1^d$* )

is explicitly cited in the proof of:

- Lemma 1526 (*isomorphism between  $\mathbb{P}_0^d$  and  $\mathbb{P}_0^{d-1}$* ),
- Lemma 1529 (*decomposition of  $\mathbb{P}_k^d$* ),
- Lemma 1533 ( *$\mathbb{P}_k^d$  is nondecreasing sequence in  $d$* ),
- Lemma 1537 (*product of two polynomials (alternate proof)*),
- Lemma 1539 (*affine mapping of monomials is  $\mathbb{P}_k^l$* ),
- Lemma 1543 (*reference Lagrange polynomials is basis of  $\mathbb{P}_1^d$* ),
- Lemma 1545 (*differential of reference Lagrange polynomials*),
- Lemma 1554 (*Lagrange polynomials is basis of  $\mathbb{P}_1^d$* ),
- Lemma 1556 (*differential of Lagrange polynomials  $\mathbb{P}_1^d$* ),
- Lemma 1615 (*Lagrange linear forms for  $\mathbb{P}_0^d$  are injective*).

**Definition 1510** (*degree of polynomial*)

is explicitly cited in the proof of:

Lemma 1511 (*values of degree of polynomial*),

Lemma 1512 (*monomials of  $\mathcal{C}_k^d$  have degree  $k$* ),

Lemma 1513 ( *$\mathbb{P}_k^d$  is space of degree at most  $k$* ),

Lemma 1543 (*reference Lagrange polynomials is basis of  $\mathbb{P}_1^d$* ).

**Lemma 1511** (*values of degree of polynomial*)

is not yet used.

**Lemma 1512** (*monomials of  $\mathcal{C}_k^d$  have degree  $k$* )

is explicitly cited in the proof of:

Lemma 1514 (*product of monomials*),

Lemma 1517 (*partial derivative of monomials*),

Lemma 1529 (*decomposition of  $\mathbb{P}_k^d$* ).

**Lemma 1513** ( *$\mathbb{P}_k^d$  is space of degree at most  $k$* )

is not yet used.

**Lemma 1514** (*product of monomials*)

is explicitly cited in the proof of:

Lemma 1515 (*product of monomial and polynomial*).

**Lemma 1515** (*product of monomial and polynomial*)

is explicitly cited in the proof of:

Lemma 1516 (*product of two polynomials*),

Lemma 1537 (*product of two polynomials (alternate proof)*).

**Lemma 1516** (*product of two polynomials*)

is explicitly cited in the proof of:

Lemma 1538 (*product of polynomials*).

**Lemma 1517** (*partial derivative of monomials*)

is explicitly cited in the proof of:

Lemma 1519 (*partial derivative is linear*),

Lemma 1521 (*derivating more than degree is 0*),

Lemma 1522 (*partial derivative of monomials at 0*).

**Lemma 1519** (*partial derivative is linear*)

is explicitly cited in the proof of:

Lemma 1520 (*partial derivative of 0*),

Lemma 1523 (*monomials are free in  $\mathbb{P}_k^d$* ).

**Lemma 1520** (*partial derivative of 0*)

is explicitly cited in the proof of:

Lemma 1523 (*monomials are free in  $\mathbb{P}_k^d$* ).

**Lemma 1521** (*derivating more than degree is 0*)

is not yet used.

**Lemma 1522** (*partial derivative of monomials at 0*)

is explicitly cited in the proof of:

Lemma 1523 (*monomials are free in  $\mathbb{P}_k^d$* ).

**Lemma 1523** (*monomials are free in  $\mathbb{P}_k^d$* )

is explicitly cited in the proof of:

Lemma 1524 (*monomials are a basis of  $\mathbb{P}_k^d$* ),

Lemma 1529 (*decomposition of  $\mathbb{P}_k^d$* ).

**Lemma 1524** (*monomials are a basis of  $\mathbb{P}_k^d$* )

is explicitly cited in the proof of:

Lemma 1525 (*dimension of  $\mathbb{P}_k^d$* ).

**Lemma 1525** (*dimension of  $\mathbb{P}_k^d$* )

is explicitly cited in the proof of:

Lemma 1526 (*isomorphism between  $\mathbb{P}_0^d$  and  $\mathbb{P}_0^{d-1}$* ),

Lemma 1531 (*isomorphism between  $\mathbb{P}_k^d$  and  $\mathbb{P}_k^{d-1} \times \mathbb{P}_{k-1}^d$* ),

Lemma 1543 (*reference Lagrange polynomials is basis of  $\mathbb{P}_1^d$* ),

Lemma 1554 (*Lagrange polynomials is basis of  $\mathbb{P}_1^d$* ),

Lemma 1617 (*unisolvence of  $\mathbb{P}_0^d$* ),

Lemma 1620 (*unisolvence of  $\mathbb{P}_1^d$* ),

Theorem 1626 (*unisolvence of  $\mathbb{P}_k^d$* ),

Theorem 1629 ( *$\text{Log } \mathbb{P}_k^d$  Lagrange finite element*).

**Lemma 1526** (*isomorphism between  $\mathbb{P}_0^d$  and  $\mathbb{P}_0^{d-1}$* )

is explicitly cited in the proof of:

Lemma 1534 (*expression of  $\mathbb{P}_k^d$  as polynomial of  $x_d$* ).

**Lemma 1529** (*decomposition of  $\mathbb{P}_k^d$* )

is explicitly cited in the proof of:

Lemma 1531 (*isomorphism between  $\mathbb{P}_k^d$  and  $\mathbb{P}_k^{d-1} \times \mathbb{P}_{k-1}^d$* ),

Lemma 1534 (*expression of  $\mathbb{P}_k^d$  as polynomial of  $x_d$* ),

Lemma 1537 (*product of two polynomials (alternate proof)*),

Lemma 1621 (*factorization of zero polynomial on last reference hyperplane*).

**Lemma 1531** (*isomorphism between  $\mathbb{P}_k^d$  and  $\mathbb{P}_k^{d-1} \times \mathbb{P}_{k-1}^d$* )

is explicitly cited in the proof of:

Lemma 1533 ( *$\mathbb{P}_k^d$  is nondecreasing sequence in  $d$* ).

**Lemma 1533** ( *$\mathbb{P}_k^d$  is nondecreasing sequence in  $d$* )

is explicitly cited in the proof of:

Lemma 1537 (*product of two polynomials (alternate proof)*).

**Lemma 1534** (*expression of  $\mathbb{P}_k^d$  as polynomial of  $x_d$* )

is not yet used.

**Lemma 1537** (*product of two polynomials (alternate proof)*)

is explicitly cited in the proof of:

Lemma 1538 (*product of polynomials*).

**Lemma 1538** (*product of polynomials*)

is explicitly cited in the proof of:

Lemma 1539 (*affine mapping of monomials is  $\mathbb{P}_k^l$* ).

**Lemma 1539** (*affine mapping of monomials is  $\mathbb{P}_k^l$* )

is explicitly cited in the proof of:

Lemma 1540 (*affine mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^l$* ).

**Lemma 1540** (*affine mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^l$* )

is explicitly cited in the proof of:

Lemma 1582 (*geometric  $l$ -face mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^l$* ),

Lemma 1583 (*geometric mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^l$* ),

Lemma 1623 (*factorization of zero polynomial on hyperplane  $\mathbb{P}_k^d$* ).

**Definition 1541** (*reference Lagrange polynomials of  $\mathbb{P}_1^d$* )

is explicitly cited in the proof of:

- Lemma 1542 (*reference Lagrange polynomials of  $\mathbb{P}_1^d$  for  $d = 1$  are reference Lagrange polynomials of  $\mathbb{P}_k^1$  for  $k = 1$* ),  
 Lemma 1543 (*reference Lagrange polynomials is basis of  $\mathbb{P}_1^d$* ),  
 Lemma 1548 (*geometric mapping for  $d = 1$  is geometric mapping in dimension 1*),  
 Lemma 1549 (*reference geometric mapping is identity*),  
 Lemma 1550 (*properties of geometric mapping*),  
 Lemma 1564 (*reference face hyperplane*),  
 Lemma 1565 (*face hyperplane is image of reference face hyperplane*),  
 Lemma 1581 (*properties of geometric l-face mapping*),  
 Lemma 1621 (*factorization of zero polynomial on last reference hyperplane*).

**Lemma 1542** (*reference Lagrange polynomials of  $\mathbb{P}_1^d$  for  $d = 1$  are reference Lagrange polynomials of  $\mathbb{P}_k^1$  for  $k = 1$* )  
 is not yet used.

**Lemma 1543** (*reference Lagrange polynomials is basis of  $\mathbb{P}_1^d$* )  
 is explicitly cited in the proof of:  
 Lemma 1545 (*differential of reference Lagrange polynomials*),  
 Lemma 1550 (*properties of geometric mapping*),  
 Lemma 1554 (*Lagrange polynomials is basis of  $\mathbb{P}_1^d$* ),  
 Lemma 1581 (*properties of geometric l-face mapping*),  
 Lemma 1586 (*geometric mapping with permutation*).

**Lemma 1545** (*differential of reference Lagrange polynomials*)  
 is explicitly cited in the proof of:  
 Lemma 1556 (*differential of Lagrange polynomials  $\mathbb{P}_1^d$* ).

**Definition 1547** (*geometric mapping*)  
 is explicitly cited in the proof of:  
 Lemma 1548 (*geometric mapping for  $d = 1$  is geometric mapping in dimension 1*),  
 Lemma 1549 (*reference geometric mapping is identity*),  
 Lemma 1550 (*properties of geometric mapping*),  
 Lemma 1560 (*Lagrange polynomials of  $\mathbb{P}_1^d$  are barycentric coordinate*),  
 Lemma 1565 (*face hyperplane is image of reference face hyperplane*),  
 Lemma 1579 (*geometric d-face mapping is geometric mapping*).

**Lemma 1548** (*geometric mapping for  $d = 1$  is geometric mapping in dimension 1*)  
 is not yet used.

**Lemma 1549** (*reference geometric mapping is identity*)  
 is explicitly cited in the proof of:  
 Lemma 1553 (*Lagrange polynomials of reference vertices are reference Lagrange polynomials of  $\mathbb{P}_1^d$* ).

**Lemma 1550** (*properties of geometric mapping*)  
 is explicitly cited in the proof of:  
 Lemma 1551 (*differential of geometric mapping*),  
 Lemma 1552 (*Lagrange polynomials of  $\mathbb{P}_1^d$* ),  
 Lemma 1554 (*Lagrange polynomials is basis of  $\mathbb{P}_1^d$* ),  
 Lemma 1557 (*nontrivial simplex*),  
 Lemma 1560 (*Lagrange polynomials of  $\mathbb{P}_1^d$  are barycentric coordinate*),  
 Lemma 1583 (*geometric mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^d$* ),  
 Lemma 1604 (*Lagrange nodes of  $\mathbb{P}_k^d$  are image of reference*).

**Lemma 1551** (*differential of geometric mapping*)  
 is explicitly cited in the proof of:



Lemma 1556 (*differential of Lagrange polynomials  $\mathbb{P}_1^d$* ),  
 Lemma 1557 (*nontrivial simplex*).

**Lemma 1552** (*Lagrange polynomials of  $\mathbb{P}_1^d$* )

is explicitly cited in the proof of:

Lemma 1553 (*Lagrange polynomials of reference vertices are reference Lagrange polynomials of  $\mathbb{P}_1^d$* ),  
 Lemma 1554 (*Lagrange polynomials is basis of  $\mathbb{P}_1^d$* ),  
 Lemma 1560 (*Lagrange polynomials of  $\mathbb{P}_1^d$  are barycentric coordinate*).

**Lemma 1553** (*Lagrange polynomials of reference vertices are reference Lagrange polynomials of  $\mathbb{P}_1^d$* )

is explicitly cited in the proof of:

Lemma 1564 (*reference face hyperplane*).

**Lemma 1554** (*Lagrange polynomials is basis of  $\mathbb{P}_1^d$* )

is explicitly cited in the proof of:

Lemma 1555 (*decomposition of  $\mathbb{P}_1^d$  polynomial in Lagrange basis*),  
 Lemma 1556 (*differential of Lagrange polynomials  $\mathbb{P}_1^d$* ).

**Lemma 1555** (*decomposition of  $\mathbb{P}_1^d$  polynomial in Lagrange basis*)

is explicitly cited in the proof of:

Lemma 1561 (*decomposition of  $\mathbb{P}_1^d$  polynomial with barycentric coordinates*),  
 Lemma 1618 (*decomposition of  $\mathbb{P}_1^d$  polynomial with  $\sigma_\alpha$* ).

**Lemma 1556** (*differential of Lagrange polynomials  $\mathbb{P}_1^d$* )

is not yet used.

**Lemma 1557** (*nontrivial simplex*)

is explicitly cited in the proof of:

Theorem 1629 ( *${}^{\text{Lag}}\mathbb{P}_k^d$  Lagrange finite element*).

**Lemma 1559** (*barycentric coordinate*)

is explicitly cited in the proof of:

Lemma 1560 (*Lagrange polynomials of  $\mathbb{P}_1^d$  are barycentric coordinate*),  
 Lemma 1563 (*equivalent definition of face hyperplane*),  
 Lemma 1586 (*geometric mapping with permutation*),  
 Lemma 1590 (*number of Lagrange nodes of  $\mathbb{P}_k^d$* ),  
 Lemma 1591 (*barycentric coordinates of Lagrange nodes of  $\mathbb{P}_k^d$* ),  
 Lemma 1602 (*number of reference Lagrange nodes of  $\mathbb{P}_k^d$* ).

**Lemma 1560** (*Lagrange polynomials of  $\mathbb{P}_1^d$  are barycentric coordinate*)

is explicitly cited in the proof of:

Lemma 1561 (*decomposition of  $\mathbb{P}_1^d$  polynomial with barycentric coordinates*),  
 Lemma 1564 (*reference face hyperplane*),  
 Lemma 1586 (*geometric mapping with permutation*).

**Lemma 1561** (*decomposition of  $\mathbb{P}_1^d$  polynomial with barycentric coordinates*)

is explicitly cited in the proof of:

Lemma 1618 (*decomposition of  $\mathbb{P}_1^d$  polynomial with  $\sigma_\alpha$* ).

**Definition 1562** (*face hyperplane*)

is explicitly cited in the proof of:

Lemma 1563 (*equivalent definition of face hyperplane*),  
 Lemma 1565 (*face hyperplane is image of reference face hyperplane*).

**Lemma 1563** (*equivalent definition of face hyperplane*)

is explicitly cited in the proof of:

- Lemma 1564 (*reference face hyperplane*),
- Lemma 1567 (*hyperface is included in face hyperplane*),
- Lemma 1576 ( *$(d - 1)$ -face is hyperface*),
- Lemma 1586 (*geometric mapping with permutation*),
- Lemma 1605 (*face hyperplanes of Lagrange nodes of  $\mathbb{P}_k^d$* ),
- Lemma 1623 (*factorization of zero polynomial on hyperplane  $\mathbb{P}_k^d$* ),
- Lemma 1625 (*Lagrange linear forms for  $\mathbb{P}_k^d$  are injective*).

**Lemma 1564** (*reference face hyperplane*)

is explicitly cited in the proof of:

- Lemma 1586 (*geometric mapping with permutation*),
- Lemma 1621 (*factorization of zero polynomial on last reference hyperplane*).

**Lemma 1565** (*face hyperplane is image of reference face hyperplane*)

is not yet used.

**Definition 1566** (*hyperface*)

is explicitly cited in the proof of:

- Lemma 1567 (*hyperface is included in face hyperplane*),
- Lemma 1576 ( *$(d - 1)$ -face is hyperface*).

**Lemma 1567** (*hyperface is included in face hyperplane*)

is not yet used.

**Definition 1568** ( *$l$ -face affine space*)

is explicitly cited in the proof of:

- Lemma 1569 (*equivalent definition of  $l$ -face affine space*),
- Lemma 1573 ( *$l$ -face is included in  $l$ -face affine space*),
- Lemma 1576 ( *$(d - 1)$ -face is hyperface*),
- Lemma 1581 (*properties of geometric  $l$ -face mapping*).

**Lemma 1569** (*equivalent definition of  $l$ -face affine space*)

is explicitly cited in the proof of:

- Lemma 1570 ( *$d$ -face affine space is full space*),
- Lemma 1571 ( *$0$ -face affine space is vertex*),
- Lemma 1581 (*properties of geometric  $l$ -face mapping*).

**Lemma 1570** ( *$d$ -face affine space is full space*)

is explicitly cited in the proof of:

- Lemma 1586 (*geometric mapping with permutation*).

**Lemma 1571** ( *$0$ -face affine space is vertex*)

is not yet used.

**Definition 1572** ( *$l$ -face*)

is explicitly cited in the proof of:

- Lemma 1573 ( *$l$ -face is included in  $l$ -face affine space*),
- Lemma 1574 ( *$d$ -face is simplex*),
- Lemma 1576 ( *$(d - 1)$ -face is hyperface*),
- Lemma 1581 (*properties of geometric  $l$ -face mapping*).

**Lemma 1573** ( *$l$ -face is included in  $l$ -face affine space*)

is not yet used.

**Lemma 1574** (*d-face is simplex*)

is explicitly cited in the proof of:

Lemma 1586 (*geometric mapping with permutation*).

**Lemma 1576** (*(d-1)-face is hyperface*)

is explicitly cited in the proof of:

Lemma 1584 (*geometric hyperface mapping*),

Lemma 1585 (*hyperface geometric mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^{d-1}$* ).

**Definition 1578** (*geometric l-face mapping*)

is explicitly cited in the proof of:

Lemma 1579 (*geometric d-face mapping is geometric mapping*),

Lemma 1581 (*properties of geometric l-face mapping*),

Lemma 1586 (*geometric mapping with permutation*).

**Lemma 1579** (*geometric d-face mapping is geometric mapping*)

is explicitly cited in the proof of:

Lemma 1583 (*geometric mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^d$* ).

**Lemma 1581** (*properties of geometric l-face mapping*)

is explicitly cited in the proof of:

Lemma 1582 (*geometric l-face mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^l$* ),

Lemma 1584 (*geometric hyperface mapping*),

Lemma 1586 (*geometric mapping with permutation*).

**Lemma 1582** (*geometric l-face mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^l$* )

is explicitly cited in the proof of:

Lemma 1583 (*geometric mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^d$* ),

Lemma 1585 (*hyperface geometric mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^{d-1}$* ).

**Lemma 1583** (*geometric mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^d$* )

is not yet used.

**Lemma 1584** (*geometric hyperface mapping*)

is explicitly cited in the proof of:

Lemma 1607 (*image of nodes by geometric hyperface mapping*),

Lemma 1625 (*Lagrange linear forms for  $\mathbb{P}_k^d$  are injective*),

Lemma 1628 (*face unisolvence of  $\mathbb{P}_k^d$* ).

**Lemma 1585** (*hyperface geometric mapping of  $\mathbb{P}_k^d$  is  $\mathbb{P}_k^{d-1}$* )

is explicitly cited in the proof of:

Lemma 1625 (*Lagrange linear forms for  $\mathbb{P}_k^d$  are injective*),

Lemma 1628 (*face unisolvence of  $\mathbb{P}_k^d$* ).

**Lemma 1586** (*geometric mapping with permutation*)

is explicitly cited in the proof of:

Lemma 1623 (*factorization of zero polynomial on hyperplane  $\mathbb{P}_k^d$* ).

**Definition 1588** (*Lagrange nodes of  $\mathbb{P}_k^d$* )

is explicitly cited in the proof of:

Lemma 1589 (*Lagrange nodes of  $\mathbb{P}_k^d$  for  $d = 1$  are Lagrange nodes of  $\mathbb{P}_k^1$* ),

Lemma 1590 (*number of Lagrange nodes of  $\mathbb{P}_k^d$* ),

Lemma 1591 (*barycentric coordinates of Lagrange nodes of  $\mathbb{P}_k^d$* ),

Lemma 1592 (*vertices are Lagrange nodes of  $\mathbb{P}_k^d$* ),

Lemma 1599 (*reference Lagrange nodes of  $\mathbb{P}_k^d$* ),

Lemma 1604 (*Lagrange nodes of  $\mathbb{P}_k^d$  are image of reference*),

Lemma 1625 (*Lagrange linear forms for  $\mathbb{P}_k^d$  are injective*).

**Lemma 1589** (*Lagrange nodes of  $\mathbb{P}_k^d$  for  $d = 1$  are Lagrange nodes of  $\mathbb{P}_k^1$* )

is explicitly cited in the proof of:

Lemma 1609 (*Lagrange linear forms for  $\mathbb{P}_k^d$  for  $d=1$  are Lagrange linear forms for  $\mathbb{P}_k^1$* ).

**Lemma 1590** (*number of Lagrange nodes of  $\mathbb{P}_k^d$* )

is explicitly cited in the proof of:

Lemma 1605 (*face hyperplanes of Lagrange nodes of  $\mathbb{P}_k^d$* ),

Lemma 1611 (*cardinal of Lagrange linear forms of  $\mathbb{P}_k^d$* ).

**Lemma 1591** (*barycentric coordinates of Lagrange nodes of  $\mathbb{P}_k^d$* )

is explicitly cited in the proof of:

Lemma 1595 (*equivalent definition of sub-vertices of Lagrange nodes of  $\mathbb{P}_k^d$* ),

Lemma 1598 ( *$\mathbb{P}_{k-1}^d$  sub-nodes of sub-vertices are some nodes of  $\mathbb{P}_{k-1}^d$* ),

Lemma 1599 (*reference Lagrange nodes of  $\mathbb{P}_k^d$* ),

Lemma 1605 (*face hyperplanes of Lagrange nodes of  $\mathbb{P}_k^d$* ),

Lemma 1607 (*image of nodes by geometric hyperface mapping*).

**Lemma 1592** (*vertices are Lagrange nodes of  $\mathbb{P}_k^d$* )

is explicitly cited in the proof of:

Lemma 1593 (*Lagrange nodes of  $\mathbb{P}_1^d$  are vertices*).

**Lemma 1593** (*Lagrange nodes of  $\mathbb{P}_1^d$  are vertices*)

is explicitly cited in the proof of:

Lemma 1618 (*decomposition of  $\mathbb{P}_1^d$  polynomial with  $\sigma_\alpha$* ).

**Definition 1594** (*sub-vertices of Lagrange nodes of  $\mathbb{P}_k^d$* )

is explicitly cited in the proof of:

Lemma 1595 (*equivalent definition of sub-vertices of Lagrange nodes of  $\mathbb{P}_k^d$* ).

**Lemma 1595** (*equivalent definition of sub-vertices of Lagrange nodes of  $\mathbb{P}_k^d$* )

is explicitly cited in the proof of:

Lemma 1597 (*sub-vertices are affinely independent*),

Lemma 1598 ( *$\mathbb{P}_{k-1}^d$  sub-nodes of sub-vertices are some nodes of  $\mathbb{P}_{k-1}^d$* ).

**Lemma 1597** (*sub-vertices are affinely independent*)

is explicitly cited in the proof of:

Lemma 1598 ( *$\mathbb{P}_{k-1}^d$  sub-nodes of sub-vertices are some nodes of  $\mathbb{P}_{k-1}^d$* ),

Lemma 1625 (*Lagrange linear forms for  $\mathbb{P}_k^d$  are injective*).

**Lemma 1598** ( *$\mathbb{P}_{k-1}^d$  sub-nodes of sub-vertices are some nodes of  $\mathbb{P}_{k-1}^d$* )

is explicitly cited in the proof of:

Lemma 1625 (*Lagrange linear forms for  $\mathbb{P}_k^d$  are injective*).

**Lemma 1599** (*reference Lagrange nodes of  $\mathbb{P}_k^d$* )

is explicitly cited in the proof of:

Lemma 1600 (*reference Lagrange nodes of  $\mathbb{P}_k^d$  for  $d = 1$  are reference Lagrange nodes of  $\mathbb{P}_k^1$* ),

Lemma 1601 (*equivalent definition of reference Lagrange nodes of  $\mathbb{P}_k^d$* ),

Lemma 1602 (*number of reference Lagrange nodes of  $\mathbb{P}_k^d$* ),

Lemma 1607 (*image of nodes by geometric hyperface mapping*),

Theorem 1631 ( *$\overset{\text{Lag}}{\hat{\mathbb{P}}}_k^d$  reference Lagrange finite element*).

**Lemma 1600** (*reference Lagrange nodes of  $\mathbb{P}_k^d$  for  $d = 1$  are reference Lagrange nodes of  $\mathbb{P}_k^1$* )

is explicitly cited in the proof of:

Lemma 1613 (*reference Lagrange linear forms for  $\mathbb{P}_k^d$  for  $d = 1$  are reference Lagrange linear forms for  $\mathbb{P}_k^1$* ).

**Lemma 1601** (*equivalent definition of reference Lagrange nodes of  $\mathbb{P}_k^d$* )

is explicitly cited in the proof of:

Lemma 1604 (*Lagrange nodes of  $\mathbb{P}_k^d$  are image of reference*).

**Lemma 1602** (*number of reference Lagrange nodes of  $\mathbb{P}_k^d$* )

is not yet used.

**Lemma 1604** (*Lagrange nodes of  $\mathbb{P}_k^d$  are image of reference*)

is explicitly cited in the proof of:

Lemma 1614 (*Lagrange linear forms of  $\mathbb{P}_k^d$  are images of reference*).

**Lemma 1605** (*face hyperplanes of Lagrange nodes of  $\mathbb{P}_k^d$* )

is explicitly cited in the proof of:

Lemma 1625 (*Lagrange linear forms for  $\mathbb{P}_k^d$  are injective*),

Lemma 1628 (*face unisolvence of  $\mathbb{P}_k^d$* ).

**Lemma 1607** (*image of nodes by geometric hyperface mapping*)

is explicitly cited in the proof of:

Lemma 1625 (*Lagrange linear forms for  $\mathbb{P}_k^d$  are injective*),

Lemma 1628 (*face unisolvence of  $\mathbb{P}_k^d$* ).

**Definition 1608** (*Lagrange linear forms for  $\mathbb{P}_k^d$* )

is explicitly cited in the proof of:

Lemma 1609 (*Lagrange linear forms for  $\mathbb{P}_k^d$  for  $d=1$  are Lagrange linear forms for  $\mathbb{P}_k^1$* ),

Lemma 1610 (*Lagrange linear forms of  $\mathbb{P}_k^d$  are linear*),

Lemma 1611 (*cardinal of Lagrange linear forms of  $\mathbb{P}_k^d$* ),

Lemma 1614 (*Lagrange linear forms of  $\mathbb{P}_k^d$  are images of reference*),

Lemma 1615 (*Lagrange linear forms for  $\mathbb{P}_0^d$  are injective*),

Lemma 1618 (*decomposition of  $\mathbb{P}_1^d$  polynomial with  $\sigma_\alpha$* ),

Lemma 1625 (*Lagrange linear forms for  $\mathbb{P}_k^d$  are injective*),

Theorem 1626 (*unisolvence of  $\mathbb{P}_k^d$* ),

Lemma 1628 (*face unisolvence of  $\mathbb{P}_k^d$* ).

**Lemma 1609** (*Lagrange linear forms for  $\mathbb{P}_k^d$  for  $d=1$  are Lagrange linear forms for  $\mathbb{P}_k^1$* )

is explicitly cited in the proof of:

Lemma 1625 (*Lagrange linear forms for  $\mathbb{P}_k^d$  are injective*),

Lemma 1630 ( *$\text{Lag} \mathbb{P}_k^d$  Lagrange finite element for  $d = 1$  is Lagrange finite element on a current segment*).

**Lemma 1610** (*Lagrange linear forms of  $\mathbb{P}_k^d$  are linear*)

is not yet used.

**Lemma 1611** (*cardinal of Lagrange linear forms of  $\mathbb{P}_k^d$* )

is explicitly cited in the proof of:

Lemma 1617 (*unisolvence of  $\mathbb{P}_0^d$* ),

Lemma 1620 (*unisolvence of  $\mathbb{P}_1^d$* ),

Theorem 1626 (*unisolvence of  $\mathbb{P}_k^d$* ).

**Definition 1612** (*reference Lagrange linear forms for  $\mathbb{P}_k^d$* )

is explicitly cited in the proof of:

Lemma 1613 (*reference Lagrange linear forms for  $\mathbb{P}_k^d$  for  $d = 1$  are reference Lagrange linear forms for  $\mathbb{P}_k^1$* ),

Lemma 1614 (*Lagrange linear forms of  $\mathbb{P}_k^d$  are images of reference*),

Theorem 1631 ( *$\text{Lag} \hat{\mathbb{P}}_k^d$  reference Lagrange finite element*).

**Lemma 1613** (*reference Lagrange linear forms for  $\mathbb{P}_k^d$  for  $d = 1$  are reference Lagrange linear forms for  $\mathbb{P}_k^1$* )

is explicitly cited in the proof of:

Lemma 1632 ( *$\overset{\text{Lag}}{\hat{\mathbb{P}}}_k^d$  reference Lagrange finite element for  $d = 1$  is Lagrange finite element on the reference segment*).

**Lemma 1614** (*Lagrange linear forms of  $\mathbb{P}_k^d$  are images of reference*)

is not yet used.

**Lemma 1615** (*Lagrange linear forms for  $\mathbb{P}_0^d$  are injective*)

is explicitly cited in the proof of:

Lemma 1617 (*unisolvence of  $\mathbb{P}_0^d$* ).

**Lemma 1617** (*unisolvence of  $\mathbb{P}_0^d$* )

is explicitly cited in the proof of:

Theorem 1629 ( *$\overset{\text{Lag}}{\mathbb{P}}_k^d$  Lagrange finite element*).

**Lemma 1618** (*decomposition of  $\mathbb{P}_1^d$  polynomial with  $\sigma_\alpha$* )

is explicitly cited in the proof of:

Lemma 1619 (*Lagrange linear forms for  $\mathbb{P}_1^d$  are injective*).

**Lemma 1619** (*Lagrange linear forms for  $\mathbb{P}_1^d$  are injective*)

is explicitly cited in the proof of:

Lemma 1620 (*unisolvence of  $\mathbb{P}_1^d$* ),

Lemma 1625 (*Lagrange linear forms for  $\mathbb{P}_k^d$  are injective*).

**Lemma 1620** (*unisolvence of  $\mathbb{P}_1^d$* )

is not yet used.

**Lemma 1621** (*factorization of zero polynomial on last reference hyperplane*)

is explicitly cited in the proof of:

Lemma 1623 (*factorization of zero polynomial on hyperplane  $\mathbb{P}_k^d$* ).

**Lemma 1623** (*factorization of zero polynomial on hyperplane  $\mathbb{P}_k^d$* )

is explicitly cited in the proof of:

Lemma 1625 (*Lagrange linear forms for  $\mathbb{P}_k^d$  are injective*).

**Lemma 1625** (*Lagrange linear forms for  $\mathbb{P}_k^d$  are injective*)

is explicitly cited in the proof of:

Theorem 1626 (*unisolvence of  $\mathbb{P}_k^d$* ),

Lemma 1628 (*face unisolvence of  $\mathbb{P}_k^d$* ).

**Theorem 1626** (*unisolvence of  $\mathbb{P}_k^d$* )

is explicitly cited in the proof of:

Theorem 1629 ( *$\overset{\text{Lag}}{\mathbb{P}}_k^d$  Lagrange finite element*).

**Lemma 1628** (*face unisolvence of  $\mathbb{P}_k^d$* )

is not yet used.

**Theorem 1629** ( *$\overset{\text{Lag}}{\mathbb{P}}_k^d$  Lagrange finite element*)

is explicitly cited in the proof of:

Lemma 1630 ( *$\overset{\text{Lag}}{\mathbb{P}}_k^d$  Lagrange finite element for  $d = 1$  is Lagrange finite element on a current segment*),

Theorem 1631 ( *$\overset{\text{Lag}}{\hat{\mathbb{P}}}_k^d$  reference Lagrange finite element*).

**Lemma 1630** ( *$\overset{\text{Lag}}{\mathbb{P}}_k^d$  Lagrange finite element for  $d = 1$  is Lagrange finite element on a current segment*)

is not yet used.

**Theorem 1631** ( ${}^{\text{Lag}}\hat{\mathbb{P}}_k^d$  *reference Lagrange finite element*)

is explicitly cited in the proof of:

Lemma 1632 ( ${}^{\text{Lag}}\hat{\mathbb{P}}_k^d$  *reference Lagrange finite element for  $d = 1$  is Lagrange finite element on the reference segment*).

**Lemma 1632** ( ${}^{\text{Lag}}\hat{\mathbb{P}}_k^d$  *reference Lagrange finite element for  $d = 1$  is Lagrange finite element on the reference segment*)

is not yet used.

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