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HDG approximation of time-harmonic waves in elastic heterogeneous media with application to Full-Waveform Inversion

Hélène Barucq¹, Florian Faucher¹ and Ha Pham¹.

Contemporary Challenges in Trefftz Methods, from Theory to Applications

CMO-BIRS workshop, Oaxaca, 13–17 May 2024.



PROJECT

Exa-MA : Methods and Algorithms for Exascale

¹Project-Team Makutu, Inria–TotalEnergies–UPPA, LMAP, UMR CNRS 5142.

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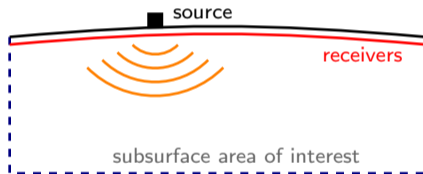
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- 2 Elastic wave propagation with HDG
 - Mathematical formulation and HDG algorithm
 - Illustration of gains
- 3 Inverse problem
 - Iterative minimization algorithm
 - Numerical experiment
- 4 Conclusion

Plan

1 Introduction

Motivation

Seismic imaging for the **Reconstruction/Monitoring** of subsurface Earth properties, where elastic-wave problems remain computationally expensive.



- 1 Accurate simulation of wave propagation in large-scale complex media,
- 2 Efficient procedure for nonlinear reconstruction of properties.

Plan

- 2 Elastic wave propagation with HDG
 - Mathematical formulation and HDG algorithm
 - Illustration of gains

Plan

- 2 Elastic wave propagation with HDG
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Elastic wave problem

New sensing devices such as fiber optic measure strain (ϵ). Therefore, we want to solve for $(\mathbf{u}, \boldsymbol{\sigma})$ to have maximum accuracy rather than replacing in terms of \mathbf{u} only.

$$\begin{cases} -\omega^2 \mathbf{u} - \nabla \cdot \boldsymbol{\sigma} = \mathbf{f}, \\ \boldsymbol{\sigma} = \frac{1}{2} \mathbf{C} : \boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} = (\nabla \mathbf{u} + \nabla^T \mathbf{u}), \end{cases} \quad (1)$$

The physical properties describing the medium are contained in the stiffness tensor \mathbf{C} .

Time-harmonic wave problems:

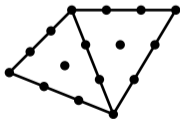
- + Easily encode the **attenuation** with complex-valued parameters,
- + Direct solvers allow for **multiple right-hand sides** once the factorization is obtained,
- **Memory cost** of the matrix factorization.

HDG discretization

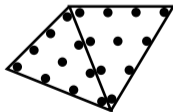
$$\begin{cases} -\omega^2 \mathbf{u} - \nabla \cdot \boldsymbol{\sigma} = \mathbf{f}, \\ \boldsymbol{\sigma} = \frac{1}{2} \mathbf{C} : (\nabla \mathbf{u} + \nabla^T \mathbf{u}), \end{cases}$$

Hybridizable Discontinuous Galerkin method

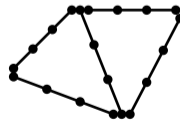
- ▶ **Static condensation** on first-order DG Formulations **without** increasing the number of unknown: The unknown of the global matrix is **only** the numerical trace $\hat{\mathbf{u}}$.
- ▶ Handle **complex geometry** (topography) with *p*-adaptivity,
- ▶ Reduces the computational cost by removing inner dofs.



Finite Element



Discontinuous Galerkin



HDG



B. Cockburn J. Gopalakrishnan and R. Lazarov

Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems

SIAM Journal on Numerical Analysis (47), 2009.

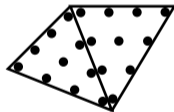
HDG discretization

Hybridizable Discontinuous Galerkin method for the discretization

- ▶ Static condensation for first-order problems **without** increasing the number of unknowns: The unknown of the global matrix is **only** the numerical trace $\hat{\mathbf{u}}$.
- ▶ Handle **complex geometry** (topography) with *p*-adaptivity,
- ▶ Reduces the computational cost by removing inner dofs.



Finite Element



Discontinuous Galerkin



HDG

HDG is more efficient with high-order polynomial (order > 4).
 High-order \Rightarrow Large cell \Rightarrow variable \mathbf{C} within cells for resolution.

HDG Variational formulation (1/2): stiffness tensor version

The local problem is written on each cell K_e of the mesh, and the HDG problem is written in terms of $(\mathbf{u}, \boldsymbol{\sigma}, \hat{u})$. Using test-functions (ψ, ϕ) ,

$$\left\{ \begin{array}{l} \int_{K_e} -\omega^2 \mathbf{u} \bar{\psi} - \int_{K_e} \nabla \cdot \boldsymbol{\sigma} \bar{\psi} = \int_{K_e} \mathbf{f} \bar{\psi}, \end{array} \right. \quad (2a)$$

$$\left\{ \begin{array}{l} \int_{K_e} \frac{1}{2} \mathbf{C} : \nabla \mathbf{u} \bar{\phi} + \int_{K_e} \frac{1}{2} \mathbf{C} : \nabla^T \mathbf{u} \bar{\phi} - \int_{K_e} \boldsymbol{\sigma} \bar{\phi} = 0. \end{array} \right. \quad (2b)$$

To make appear the numerical trace \hat{u} , we need to integrate by parts leading to derivative of \mathbf{C} .
 \Rightarrow **when \mathbf{C} is not constant per cell, one would need to provide its derivative...**

Variational formulation (2/2): compliance tensor version

$$\text{Using } \mathbf{S} = \mathbf{C}^{-1}, \quad \begin{cases} -\omega^2 \mathbf{u} - \nabla \cdot \boldsymbol{\sigma} = \mathbf{f}, \\ \mathbf{S} : \boldsymbol{\sigma} = \frac{1}{2}(\nabla \mathbf{u} + \nabla^T \mathbf{u}), \end{cases}$$

On each cell K_e of the mesh,

$$\begin{cases} \int_{K_e} -\omega^2 \mathbf{u} \bar{\psi} - \int_{K_e} \nabla \cdot \boldsymbol{\sigma} \bar{\psi} = \int_{K_e} \mathbf{f} \bar{\psi}, \end{cases} \quad (3a)$$

$$\begin{cases} \int_{K_e} \frac{1}{2} \nabla \mathbf{u} \bar{\phi} + \int_{K_e} \frac{1}{2} \nabla^T \mathbf{u} \bar{\phi} - \int_{K_e} \mathbf{S} : \boldsymbol{\sigma} \bar{\phi} = 0. \end{cases} \quad (3b)$$

⇒ using quadrature formula, \mathbf{S} can easily vary within cell.

Note that under isotropy, we have an explicit formulation of \mathbf{S} from Lamé parameters.

HDG workflow in a nutshell

Unknowns are the discretized variables: $\mathbf{X}^h = (\mathbf{u}^h, \boldsymbol{\sigma}^h)$ and numerical trace $\hat{\mathbf{u}}^h$.

$$\left\{ \begin{array}{l} \text{local problem on each cell } K_e : \quad \mathbb{A}_e \mathbf{X}_e^h + \mathbb{C}_e \mathcal{R}_e \hat{\mathbf{u}}^h = \mathbb{F}, \\ \text{relation for numerical trace :} \quad \sum_e \mathcal{R}_e^T (\mathbb{B}_e \mathbf{X}_e^h + \mathbb{L}_e \mathcal{R}_e \hat{\mathbf{u}}^h) = 0. \end{array} \right.$$

Reorder to write **global problem** in terms of numerical trace only

$$\sum_e \mathcal{R}_e^T (\mathbb{L}_e - \mathbb{B}_e \mathbb{A}_e^{-1} \mathbb{C}_e) \mathcal{R}_e \hat{\mathbf{u}}_h = - \sum_e \mathcal{R}_e^T \mathbb{B}_e \mathbb{A}_e^{-1} \mathbb{F}_e \quad \Leftrightarrow \quad \mathcal{A} \hat{\mathbf{u}}_h = \mathcal{B}.$$

- ① Create local matrices on each cell (embarrassingly parallel) with p -adaptivity.
- ② Assemble global matrix \mathcal{A} .
- ③ Solve the large linear system $\mathcal{A} \hat{\mathbf{u}}^h = \mathcal{B}$ with MUMPS (solve multiple rhs at limited cost).
- ④ Solve the local linear systems to obtain the volume solution $(\mathbf{u}^h, \boldsymbol{\sigma})$: (small matrices, embarrassingly parallel, $< 2\%$ of run time).

Efficient stabilization for anisotropy

The efficiency of the HDG method relies onto the **stabilization matrix** τ that appears in the definition of the numerical trace:

$$\widehat{\sigma}_h \nu = \sigma_h \nu - \tau(\mathbf{u}_h - \widehat{\mathbf{u}}_h).$$

A naive one (multiple of the identity) is **not robust**.

Efficient stabilization for anisotropy

The efficiency of the HDG method relies onto the **stabilization matrix** τ that appears in the definition of the numerical trace:

$$\widehat{\sigma}_h \nu = \sigma_h \nu - \tau(\mathbf{u}_h - \widehat{\mathbf{u}}_h).$$

A naive one (multiple of the identity) is **not robust**.

We instead construct a hybridized Godunov-upwind flux: Assuming three distinct wavespeeds in anisotropy (c_p , c_{s1} , c_{s2}),

$$\tau = -i\omega \rho (c_p + c_{s1} + c_{s2}) \left(1 + \frac{\gamma}{\rho} (\mathbf{\Gamma} + \rho \mathbf{p}) \right),$$

with $\mathbf{\Gamma}$ the Kelvin–Christoffel matrix, γ and \mathbf{p} auxiliary quantities depending on the wavespeed.



H. Pham, F. Faucher and H. Barucq

Numerical investigation of stabilization in the Hybridizable Discontinuous Galerkin method for linear anisotropic elastic equation
arXiv preprint 2403.02862, 2024.

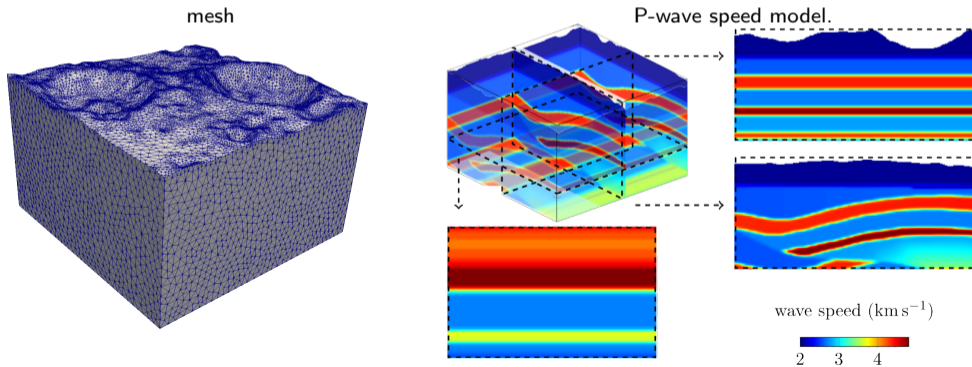
Plan

- 2 Elastic wave propagation with HDG
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Elastic-wave propagation with HDG

3D Model with topography, size $20 \times 20 \times 10$ km³ with topography.

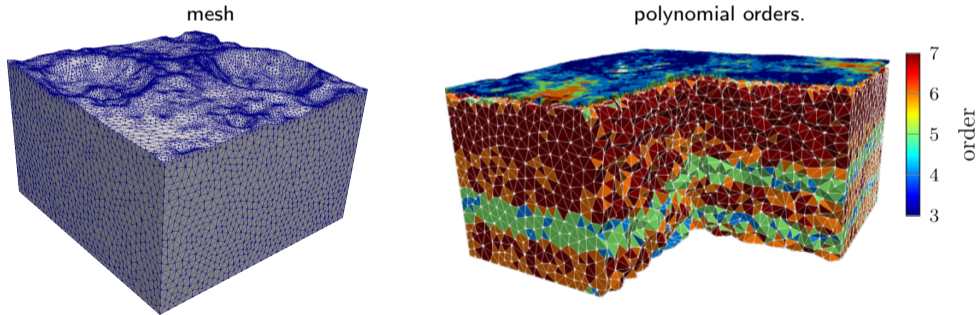
- ▶ Small cells required to accurately describe the topography, *p*-adaptivity
- ▶ Large cells elsewhere to benefit from HDG,
- ▶ models properties vary within the cell, here we use Lagrange basis per cell.



Elastic-wave propagation with HDG

3D Model with topography, size $20 \times 20 \times 10$ km³ with topography.

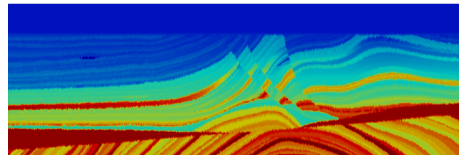
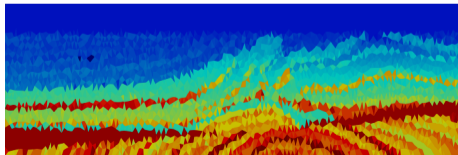
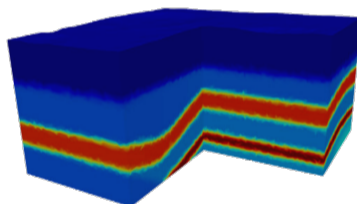
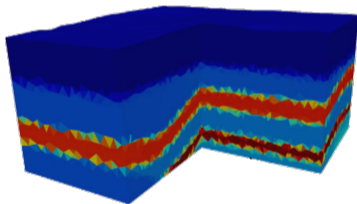
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Elastic-wave propagation with HDG

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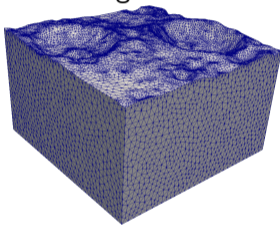
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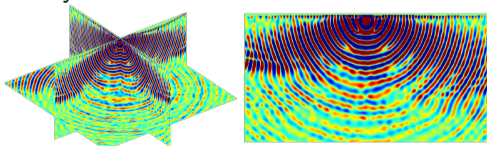
Matrix size reduction with HDG+MUMPS

3D elastic wave propagation, size $20 \times 20 \times 10 \text{ km}^3$ with topography.

Mesh using 120 000 cells

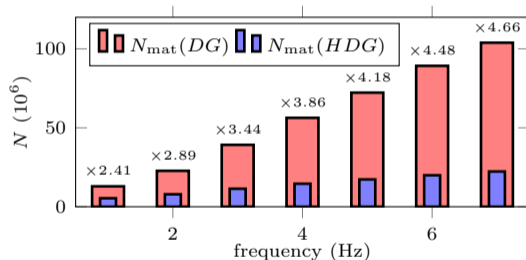


Polynomial order between 2 and 10



displacement field u_z at 4 Hz

Global matrix size with frequency



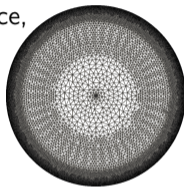
elastic simulation 7 Hz with 1600 cores HDG+MUMPS:

- ▶ **matrix size** $N = 22.3 \times 10^6$,
- ▶ **analysis time** 2 min 30 sec,
- ▶ **factorization time** 34 min,
- ▶ **factorization memory** 3589 GiB,
- ▶ **solve time (19 rhs)** 1 min 40 s.

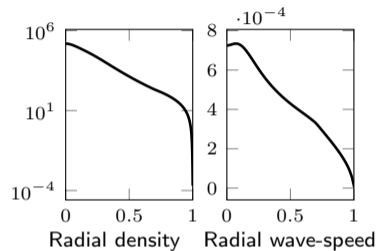
Anisotropic modeling with highly-varying model parameters

2D anisotropic simulation with solar-like model background:

- ▶ Density is exponentially decreasing near surface,
- ▶ Velocity is collapsing near surface,
- ▶ Anisotropy encoded with Thomsen's parameters $\epsilon = 0.25$, $\delta = 0.15$, $\theta = 45^\circ$.



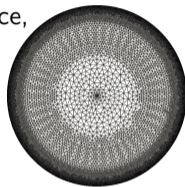
2D disk mesh used



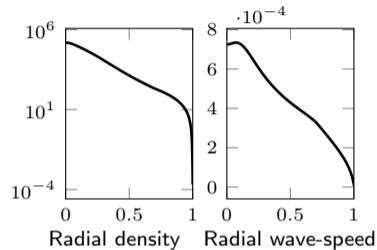
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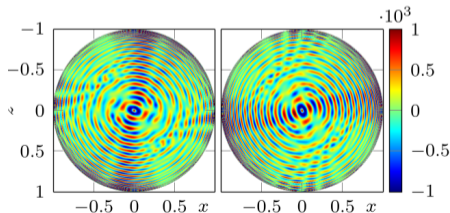


2D disk mesh used



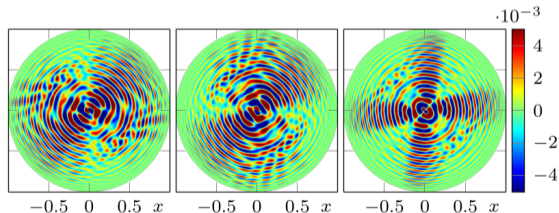
Radial density

Radial wave-speed



(a) $\sqrt{\rho} \operatorname{Re}(u_x)$.

(b) $\sqrt{\rho} \operatorname{Re}(u_z)$.



(c) $\operatorname{Re}(\sigma_{xx}) / \sqrt{\rho}$.

(d) $\operatorname{Re}(\sigma_{zz}) / \sqrt{\rho}$.

(e) $\operatorname{Re}(\sigma_{xz}) / \sqrt{\rho}$.

Plan

- 3 Inverse problem
 - Iterative minimization algorithm
 - Numerical experiment

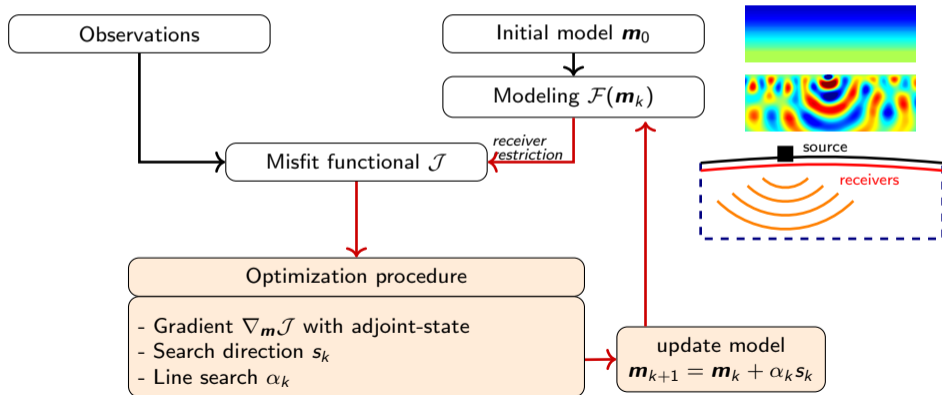
Plan

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Quantitative reconstruction algorithm: FWI

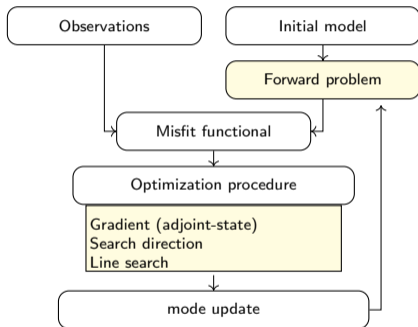
Quantitative reconstruction of properties $\mathbf{m} = (\lambda, \mu, \rho)$ solving **iterative minimization problem**.

$$\min_{\mathbf{m}} \mathcal{J}(\mathbf{m}) \quad \text{with} \quad \mathcal{J}(\mathbf{m}) = \text{dist}(\mathcal{F}(\mathbf{m}), \mathbf{d}), \quad \mathcal{F}: \text{simulations}, \quad \mathbf{d}: \text{observations}.$$



Quantitative reconstruction algorithm: FWI

minimize $\mathcal{J}(\mathbf{m}) = \text{dist}(\mathcal{F}(\mathbf{m}), \mathbf{d})$



- 1 Repeated use of forward wave propagation solver,

$$\begin{cases} \mathbb{A}_e \mathbf{X}_e^h + \mathbb{C}_e \mathcal{R}_e \hat{\mathbf{u}}^h = \mathbb{F}, & (4a) \end{cases}$$

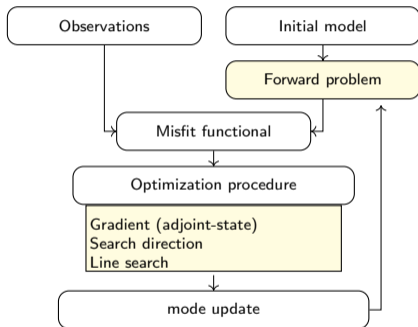
$$\begin{cases} \sum_e \mathcal{R}_e^T (\mathbb{B}_e \mathbf{X}_e^h + \mathbb{L}_e \mathcal{R}_e \hat{\mathbf{u}}^h) = 0. & (4b) \end{cases}$$

- 2 Adjoint-state method for gradient adapted to HDG
Lagrangian written from \mathcal{J} subject to (4a), (4b).

We prove that for HDG, the gradient is still computed from the **adjoint of the direct problem**, fundamental to avoid refactorization of the global matrix.

Quantitative reconstruction algorithm: FWI

minimize $\mathcal{J}(\mathbf{m}) = \text{dist}(\mathcal{F}(\mathbf{m}), \mathbf{d})$



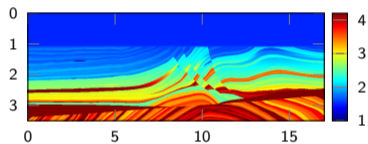
- 1 Repeated use of forward wave propagation solver,
- 2 Adjoint-state method for gradient adapted to HDG,
- 3 To alleviate mesh limitations, the model parameters are represented with Lagrange basis functions.
- 4 Inversion is carried out with respect to the weight of the Lagrange basis functions.

Plan

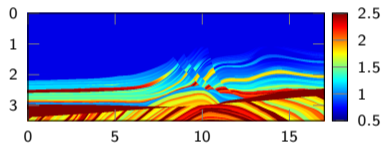
- 3 Inverse problem
 - Iterative minimization algorithm
 - Numerical experiment

2D experiments: Marmousi II setup

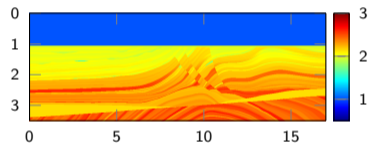
We consider the elastic isotropic Marmousi II experiment, of size $17 \times 3.5 \text{ km}^2$.
Free-surface boundary condition on top and absorbing boundary conditions elsewhere.



True P-wave speed



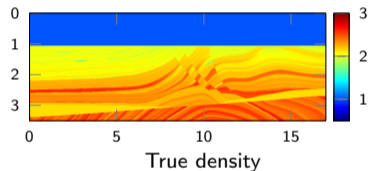
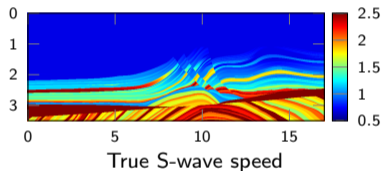
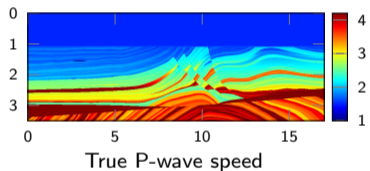
True S-wave speed



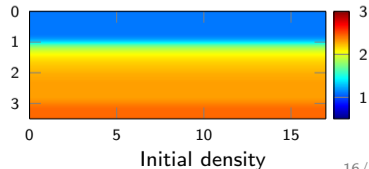
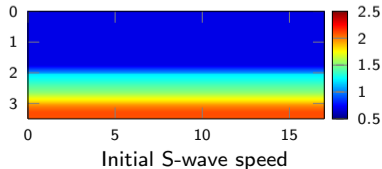
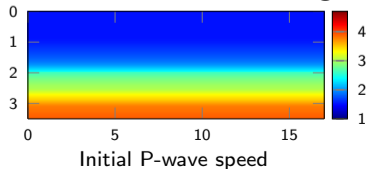
True density

2D experiments: Marmousi II setup

We consider the elastic isotropic Marmousi II experiment, of size $17 \times 3.5 \text{ km}^2$. Free-surface boundary condition on top and absorbing boundary conditions elsewhere.

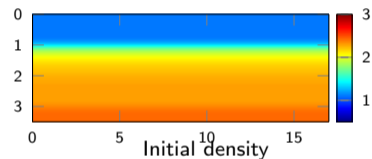
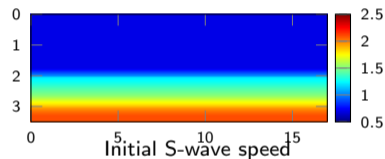
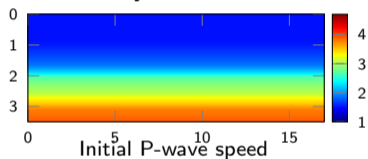


Initial models are 1D-background variation.



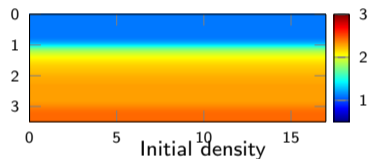
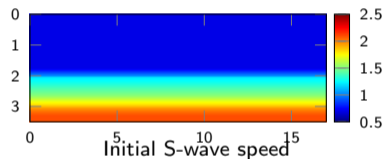
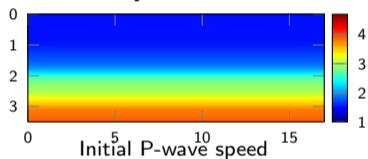
2D experiments: Marmousi II reconstructions

- ▶ Acquisition made up of 169 sources and 849 receivers near surface.
- ▶ Reconstructions using 13 frequencies from 2 to 8 Hz, 25 iterations per frequency.
- ▶ Density is not inverted and remains in its initial value.

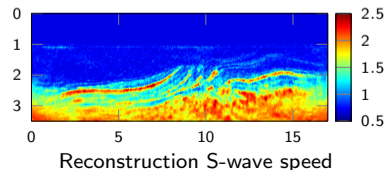
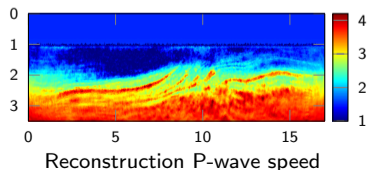


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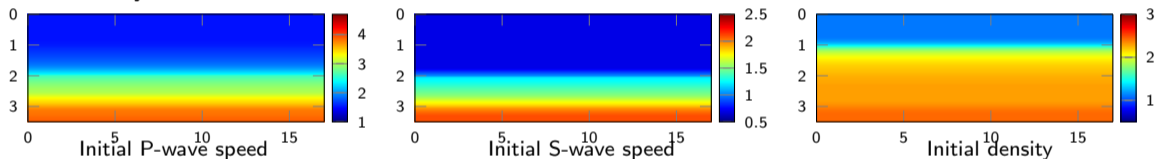


- ▶ Reconstruction using representation in order 1 Lagrange basis per cell ($\sim 20 \times 10^3$ cells)

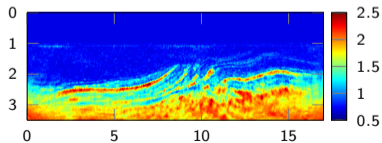


2D experiments: Marmousi II reconstructions

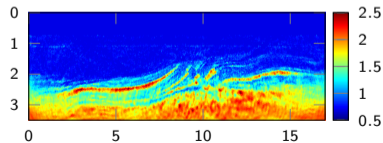
- ▶ Acquisition made up of 169 sources and 849 receivers near surface.
- ▶ Reconstructions using 13 frequencies from 2 to 8 Hz, 25 iterations per frequency.
- ▶ Density is not inverted and remains in its initial value.



- ▶ Computational time ($18\text{mpi} \times 2\text{omp}$): $2 \cdot 10^4$ cells: 3h; $5 \cdot 10^4$ cells: 4h15min, i.e. **-30%**.



S-wave speed reconstruction using mesh with 20 000 cells and Lagrange basis representation,



S-wave speed reconstruction using mesh with 50 000 cells piecewise-constant representation.

Plan

4 Conclusion

Conclusion

We propose an efficient framework for elastic inverse problem using FWI method.

① Hybridizable discontinuous Galerkin discretization

- ▶ Handle complex geometry and p -adaptivity, efficient for parallelization,
- ▶ Reduction of the size of the linear system with **high-order polynomials**,
- ▶ Model representation must allow variation within cells,

② Inversion

- ▶ Adjoint-state method for HDG re-uses forward global matrix factorization,
- ▶ Reconstruction with model representation to improve resolution.
- ▶ 3D running on toy problems, scalability analysis for regional-scale imaging is on going

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Conclusion

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THANK YOU