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# Bounded Extremum Seeking for Single-Variable Static Map using State Transformation

Frédéric Mazenc, Michael Malisoff, and Emilia Fridman

**Abstract**—We solve a gradient based bounded extremum seeking problem for single-variable static maps in the presence of time-varying piecewise continuous measurement uncertainty. Instead of using previously reported averaging-based methods, we introduce a new state transformation, allowing us to use new comparison function and generalized Lyapunov function approaches to obtain our ultimate bounds on the parameter estimation error. We illustrate significant advantages of our new method, including less restrictive conditions on the extremum seeking parameters, as compared with previous methods.

## I. INTRODUCTION

Extremum seeking is an online, real time, model free optimization method, whose goal is to find extrema of objective functions that contain uncertainties. Although basic ideas of extremum seeking were studied by Leblanc in 1922 [8], the first stability analysis of extremum seeking algorithms appears to be in the averaging and singular perturbation approaches by Miroslav Krstic and his collaborators, e.g., in [6]; see [15] for a detailed history of the development of extremum seeking. This led to widespread use of extremum seeking in significant engineering applications, including in aerospace models and source seeking [11], [17]. For more recent theoretical studies of extremum seeking, see, e.g., [2], [4], [7], [16]–[20]. Particularly, [16]–[18] included bounded extremum seeking schemes where the unknown maps only occur in arguments of a sine or cosine, leading to guaranteed bounds on update rates.

When applying extremum seeking, one frequently encounters measurement uncertainties and measurement delays [11]. Multiple approaches to delayed extremum seeking compensate for arbitrarily long measurement delays by combining predictor approaches with averaging ideas from works such as [5]; see, e.g., the pioneering work [12], work [13] and [14] for time-varying and uncertain delays using analogs of the approach from [12], and a chain predictor approach in [9] for multivariable extremum seeking with arbitrarily long delays that eliminates the integral terms arising in [12].

Another approach to addressing such challenges uses a time delay approach to averaging [3], [26], including sampled-data extremum seeking under small constant delays

and a Lyapunov-Krasovskii approach under small time-varying delays [24]. The works [21] and [22] built on [24] and [26], by studying the effects of large measurement delays, using an averaged model with disturbances and the variation of constants formula. Then [23] provided bounded extremum seeking (as introduced in [16]) under constant measurement delays  $D = \ell\epsilon$  for integers  $\ell > 0$  when  $\omega = 2\pi/\epsilon$  is the frequency of the dither, by transforming the original delayed dynamics into a neutral time delay system whose stability properties implied practical stability for the original dynamics, using bounds on fundamental solutions of linear delay systems from [1]. This led to upper bounds on the dither period under which practical stability occurs, and ultimate bounds on estimation errors. Since [23] required that  $\epsilon = D/\ell$ , it only applies when the constant  $D$  is known.

Here, we pursue a different goal, under measurement uncertainty. We introduce a new state transformation, allowing us to use a time-varying quadratic Lyapunov function, which is reminiscent of Lyapunov functions arising from ‘strictification’ [10], and to apply a comparison lemma to a differential inequality with a square root of the state on its right side. Then we achieve much larger bounds on the parameter  $\epsilon$  in the dither period  $\omega$  (and so a usefully smaller  $\omega = 2\pi/\epsilon$ ) without significantly increasing the ultimate bounds on the estimation error, compared with averaging-based works. We also provide an example showing how for arbitrarily small  $\epsilon$ ’s and time-varying delays of maximal size proportional to  $\epsilon$ , one cannot prove a corresponding stability property. This is a notable difference between (a) the work [23], which allows arbitrarily long known constant delays and (b) nonconstant delays, which can have a significant destabilizing influence (which is consistent with [24], which considered delays whose bounds were  $O(\epsilon)$ ). Also, similar to [24], [25], we provide a robustness result under time-varying delays that does not rely on exact knowledge of delays, but only on knowledge of bounds on the delays.

## II. EXTREMUM SEEKING PROBLEM AND THEOREM

Following [23], we consider a single input real valued function  $Q(\theta(t))$  in the quadratic form

$$Q(\theta(t)) = Q^* + \frac{H}{2}(\theta(t) - \theta^*)^2 \quad (1)$$

for a real valued input  $\theta$ , where  $Q^*$  and  $\theta^*$  are unknown constants, and the unknown real value  $H$  is known to be nonzero. For simplicity, we assume that  $H > 0$ , but this is without loss of generality because otherwise we can replace  $Q$  by  $-Q$  in what follows. Also, following [23], we assume that bounds on  $\theta^*$  and  $H$  are known:

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*Assumption 1:* An interval  $[\underline{\theta}^*, \bar{\theta}^* + \sigma_0]$  containing the unknown parameter  $\theta^*$  is known. Also, an interval  $[\underline{H}_*, \bar{H}^*]$  containing the unknown constant  $H$  is known.  $\square$

We further assume that the available measurement is

$$y(t) = Q(\theta(t) + d_0(t)) \quad (2)$$

where the unknown piecewise continuous function  $d_0$  represents measurement uncertainty (and our placement of  $d_0$  inside  $Q$  in (2) is instrumental for our analysis of robustness with respect to measurement delays below, but we can prove analogs when we have added uncertainty on  $y$ ) and satisfies:

*Assumption 2:* There is a known constant  $\bar{d}_0$  such that

$$d_0(t) \in [0, \bar{d}_0] \quad (3)$$

for all  $t \geq 0$ .  $\square$

In terms of the function (1) and the estimation error

$$\tilde{\theta}(t) = \theta(t) - \theta^*, \quad (4)$$

we therefore have the measurement

$$y(t) = Q^* + \frac{H}{2}(\tilde{\theta}(t) + d_0(t))^2. \quad (5)$$

Following [17] and [23], and for a constant  $\alpha > 0$ , we then use the gradient-based bounded extremum seeking dynamics

$$\dot{\tilde{\theta}}(t) = \sqrt{\alpha\omega} \cos\left(\omega t + kQ^* + \frac{kH}{2}(\tilde{\theta}(t) + d_0(t))^2\right), \quad (6)$$

where  $\omega$  is the frequency of the dither signal having a magnitude that is proportional to  $\alpha$ , and the adaptation gain  $k > 0$  is selected to have the same sign as  $H$ , but see Section IV below where  $d_0$  is used to represent the effects of delayed measurements. As in [23], we find it convenient to write

$$\omega = \frac{2\pi}{\epsilon}, \quad (7)$$

and then we find appropriate conditions on the constants  $\epsilon > 0$  and  $\bar{d}_0$  such that the dynamics for  $\tilde{\theta}$  satisfies a practical stability estimate. To this end, we use the constants

$$g = \frac{kH\alpha}{2}, \quad \bar{d} = \frac{\bar{d}_0}{\sqrt{\alpha}}, \quad \text{and} \quad (8)$$

$$B_* = \frac{2g\sqrt{2\epsilon}}{3\sqrt{3\pi}} + \frac{\bar{d}^2 g^2 \sqrt{2\epsilon}}{\sqrt{\pi}} + g\bar{d}^2 \sqrt{\frac{2\pi}{\epsilon}} + 2g\bar{d} + \frac{4g^2 \bar{d}\epsilon}{\pi}.$$

Using the constant  $\sigma_0$  from Assumption 1 and the other notation that we defined above, we will prove the following:

*Theorem 1:* Let Assumptions 1-2 hold. With the preceding notation, assume that the three conditions

$$\frac{4g\epsilon}{\pi} + \frac{9\bar{d}}{2} \left(\bar{d}g + \sqrt{\frac{2\pi}{\epsilon}}\right) \leq \frac{1}{8}, \quad (9)$$

$$B_* \left(9\bar{d} + \sqrt{\frac{2\epsilon}{\pi}}\right) < \frac{1225}{10368}, \quad \text{and} \quad (10)$$

$$\left(\frac{\sigma_0^2}{\alpha} + \frac{\epsilon}{2\pi}\right) \left(18\bar{d} + 2\sqrt{\frac{2\epsilon}{\pi}}\right)^2 g^2 < \frac{8575}{41472} \quad (11)$$

all hold. Then, for each initial value  $\tilde{\theta}(0)$  in  $[-\sigma_0, \sigma_0]$ , the corresponding solution  $\tilde{\theta}(t)$  of (6) is such that

$$\limsup_{t \rightarrow +\infty} |\tilde{\theta}(t)| \leq \sqrt{\alpha} \left(\frac{108B_*}{35g} + \sqrt{\frac{\epsilon}{2\pi}}\right) \quad (12)$$

is satisfied.  $\square$

*Remark 1:* For each value of the constant  $g$  from (8), we can find positive constants  $\bar{\epsilon}$  and  $\ell_d$  (both depending on  $g$ ) such that (9) holds if  $\epsilon \in (0, \bar{\epsilon})$  and  $0 \leq \bar{d} \leq \ell_d \sqrt{\epsilon}$ .  $\square$

*Remark 2:* Our proof of Theorem 1 will construct constants  $c_* > 0$  and  $r_* > 0$  such that

$$|\tilde{\theta}(t)| \leq c_* e^{-r_* t} + \sqrt{\alpha} \left(\frac{108B_*}{35g} + \sqrt{\frac{\epsilon}{2\pi}}\right) \quad (13)$$

for all  $t \geq 0$ ; see Remark 3 below for the construction of  $c_*$  and  $r_*$  formulas. Theorem 1 is new, even in the special case where  $\bar{d}_0 = 0$ . In this special case, (9)-(11) hold if

$$g\epsilon \leq \frac{\pi}{32} \quad \text{and} \quad \left(\frac{\sigma_0^2}{\alpha} + \frac{\epsilon}{2\pi}\right) \epsilon g^2 < \frac{8575}{331776} \pi \quad (14)$$

and in this special case, we conclude that

$$\limsup_{t \rightarrow +\infty} |\tilde{\theta}(t)| \leq \sqrt{\alpha} \left(\frac{48\sqrt{3}}{35} + 1\right) \sqrt{\frac{\epsilon}{2\pi}}, \quad (15)$$

instead of the ultimate bound in (12). See Section V below to see how (14) can allow larger upper bounds on  $\epsilon$  and small enough ultimate bounds, compared with earlier results.  $\square$

### III. PROOF OF THEOREM 1

The proof has four parts. In the first part, we introduce a new state transformation that makes the estimation error dynamics amenable to a new quadratic Lyapunov function analysis. In the second part, we use a quadratic Lyapunov function  $V$  to build a new function  $W$  that satisfies a differential inequality with a  $\sqrt{W}$  on its right side. In the third part, we use a comparison lemma to prove an ultimate boundedness condition on  $W$ . In the final part, we use upper and lower bounds for  $W$  to get the final ultimate bound.

*First Part: State Transformation.* Without loss of generality and to simplify, we assume that  $Q^* = 0$ . Indeed, if  $Q^* \neq 0$ , then we can remove the  $kQ^*$  by using the fact that

$$\tilde{\theta}_{\text{new}}(t) = \tilde{\theta}\left(t - \frac{kQ^*}{\omega}\right) \quad \text{and} \quad d_{0,\text{new}}(t) = d_0\left(t - \frac{kQ^*}{\omega}\right) \quad (16)$$

satisfy

$$\dot{\tilde{\theta}}_{\text{new}}(t) = \sqrt{\alpha\omega} \cos\left(\omega t + \frac{kH}{2} \left(\tilde{\theta}_{\text{new}}(t) + d_{0,\text{new}}(t)\right)^2\right). \quad (17)$$

Here in the sequel, all equalities, inclusions, and inequalities should be understood to hold for all  $t \geq 0$ , unless otherwise indicated. We next perform the change of variable

$$\hat{\theta}(t) = \frac{1}{\sqrt{\alpha}} \tilde{\theta}(t) \quad \text{and} \quad d(t) = \frac{1}{\sqrt{\alpha}} d_0(t). \quad (18)$$

In terms of the constant  $g$  defined in (8), they give

$$\dot{\hat{\theta}}(t) = \sqrt{\omega} \cos\left(\omega t + g \left(\hat{\theta}(t) + d(t)\right)^2\right). \quad (19)$$

In terms of the bound  $\bar{d}_0$  from Assumption 2 and the constant  $\bar{d}$  from (8), we then have  $d(t) \in [0, \bar{d}]$ . To simplify the analysis, we rewrite (19) as

$$\dot{\hat{\theta}}(t) = \sqrt{\omega} \cos\left(\omega t + g\hat{\theta}^2(t) + \phi(t)\right), \quad (20)$$

where  $\phi(t) = gd^2(t) + 2gd(t)\hat{\theta}(t)$ .

We next introduce the state transformation

$$X(t) = \hat{\theta}(t) - \frac{1}{\sqrt{\omega}} \sin\left(\omega t + g\hat{\theta}^2(t)\right) \quad (21)$$

and the functions

$$\begin{aligned}\Delta(t) &= \cos\left(\omega t + g\hat{\theta}^2(t)\right) \text{ and} \\ \Delta_\phi(t) &= \cos\left(\omega t + g\hat{\theta}^2(t) + \phi(t)\right).\end{aligned}\quad (22)$$

Then (20) gives  $\dot{\hat{\theta}}(t) = \sqrt{\omega}\Delta_\phi(t)$ , and so also

$$\dot{X}(t) = \sqrt{\omega}\Delta_\phi(t) - \frac{1}{\sqrt{\omega}}\Delta(t)\left(\omega + 2g\hat{\theta}(t)\dot{\hat{\theta}}(t)\right). \quad (23)$$

By subtracting and adding  $\sqrt{\omega}\Delta(t)$  on the right side of the first equality (20) and then substituting the result into (23), we get

$$\begin{aligned}\dot{X}(t) &= -2g\Delta^2(t)\hat{\theta}(t) - 2g\Delta(t)\hat{\theta}(t)[\Delta_\phi(t) - \Delta(t)] \\ &\quad + \sqrt{\omega}[\Delta_\phi(t) - \Delta(t)].\end{aligned}\quad (24)$$

By grouping terms in (24), we obtain

$$\dot{X}(t) = -2g\Delta^2(t)\hat{\theta}(t) + \left[\sqrt{\omega} - 2g\Delta(t)\hat{\theta}(t)\right][\Delta_\phi(t) - \Delta(t)]$$

so the definition (21) of  $X$  gives

$$\begin{aligned}\dot{X}(t) &= -2g\Delta^2(t)\left[X(t) + \frac{1}{\sqrt{\omega}}\sin\left(\omega t + g\hat{\theta}^2(t)\right)\right] \\ &\quad + \left[\sqrt{\omega} - 2g\Delta(t)\hat{\theta}(t)\right][\Delta_\phi(t) - \Delta(t)] \\ &= -2g\Delta^2(t)X(t) + \gamma(t), \text{ where} \\ \gamma(t) &= -\frac{2g}{\sqrt{\omega}}\Delta^2(t)\sin\left(\omega t + g\hat{\theta}^2(t)\right) \\ &\quad + \left[\sqrt{\omega} - 2g\Delta(t)\hat{\theta}(t)\right][\Delta_\phi(t) - \Delta(t)].\end{aligned}\quad (25)$$

Then the double angle formula for cosine gives

$$\dot{X}(t) = -g\left[1 + \cos\left(2\omega t + 2g\hat{\theta}^2(t)\right)\right]X(t) + \gamma(t). \quad (26)$$

*Second Part: Lyapunov Approach.* We use the functions

$$\begin{aligned}V(X) &= \frac{1}{2}X^2, \quad \mathcal{C}(t) = \cos\left(2\omega t + 2g\hat{\theta}^2(t)\right), \text{ and} \\ W(t, \hat{\theta}) &= \\ &\left(1 + \frac{g}{\omega}\sin\left(2\omega t + 2g\hat{\theta}^2(t)\right)\right)V\left(\hat{\theta} - \frac{1}{\sqrt{\omega}}\sin\left(\omega t + g\hat{\theta}^2(t)\right)\right),\end{aligned}\quad (27)$$

where  $W$  will be our Lyapunov function for the dynamics for  $X$  in (26); see also (44)-(45) below. The function  $V$  satisfies

$$\dot{V}(t) = -2gV(X(t)) + \gamma(t)X(t) - 2g\mathcal{C}(t)V(X(t)) \quad (28)$$

by (26). Also, our choice (21) of  $X$  gives

$$W(t, \hat{\theta}(t)) = \left(1 + \frac{g}{\omega}\sin\left(2\omega t + 2g\hat{\theta}^2(t)\right)\right)V(X(t)). \quad (29)$$

The Lyapunov function  $W$  was found by applying a variant of the ‘strictification’ technique of [10] to  $V$ , except that the function  $\mathcal{C}$  from (27) does not depend only on  $t$ , but also on  $\hat{\theta}$ . The reason why this approach works is that the term  $2g\hat{\theta}^2(t)$  is small when  $g$  is small. We deduce that

$$\begin{aligned}\dot{W}(t) &= \dot{V}(t) + \frac{g}{\omega}\sin\left(2\omega t + 2g\hat{\theta}^2(t)\right)\dot{V}(t) \\ &\quad + \frac{g}{\omega}\mathcal{C}(t)\left(2\omega + 4g\hat{\theta}(t)\dot{\hat{\theta}}(t)\right)V(X(t)) \\ &= -2gV(X(t)) + \gamma(t)X(t) \\ &\quad + \frac{g}{\omega}\sin\left(2\omega t + 2g\hat{\theta}^2(t)\right)\dot{V}(t) \\ &\quad + \frac{4g^2}{\omega}\mathcal{C}(t)\hat{\theta}(t)\dot{\hat{\theta}}(t)V(X(t))\end{aligned}\quad (30)$$

where the last equality is a consequence of (28).

As an immediate consequence of (19) and (30), we get

$$\begin{aligned}\dot{W}(t) &\leq -2gV(X(t)) + \sqrt{2}|\gamma(t)|\sqrt{V(X(t))} \\ &\quad + \frac{g}{\omega}|\dot{V}(t)| + \frac{4g^2}{\omega}|\hat{\theta}(t)||\dot{\hat{\theta}}(t)|V(X(t)) \\ &\leq -2gV(X(t)) + \sqrt{2}|\gamma(t)|\sqrt{V(X(t))} \\ &\quad + \frac{g}{\omega}|\dot{V}(t)| + \frac{4g^2}{\omega}|\hat{\theta}(t)|V(X(t)).\end{aligned}\quad (31)$$

Using (28) again, we obtain  $|\dot{V}(t)| \leq 4gV(X(t)) + \sqrt{2}|\gamma(t)|\sqrt{V(X(t))}$ . It follows from (31) that

$$\begin{aligned}\dot{W}(t) &\leq -2gV(X(t)) + \sqrt{2}|\gamma(t)|\sqrt{V(X(t))} \\ &\quad + \frac{4g^2}{\omega}V(X(t)) + \frac{\sqrt{2}g}{\omega}|\gamma(t)|\sqrt{V(X(t))} \\ &\quad + \frac{4g^2}{\omega}|\hat{\theta}(t)|V(X(t)) \\ &= -2gV(X(t)) \\ &\quad + \left(\sqrt{2} + \frac{\sqrt{2}g}{\omega}\right)|\gamma(t)|\sqrt{V(X(t))} \\ &\quad + \frac{4g^2}{\omega}V(X(t)) + \frac{4g^2}{\omega}|\hat{\theta}(t)|V(X(t)).\end{aligned}\quad (32)$$

Also, the definitions of  $\phi$  and  $\gamma$  in (20) and (25) give

$$\begin{aligned}|\gamma(t)| &\leq \frac{4g}{3\sqrt{3}\sqrt{\omega}} + 4g^2\bar{d}|\hat{\theta}(t)|^2 \\ &\quad + g\bar{d}^2\sqrt{\omega} + (2\bar{d}^2g^2 + 2g\bar{d}\sqrt{\omega})|\hat{\theta}(t)|,\end{aligned}\quad (33)$$

by noting that  $f(x) = \cos^2(x)\sin(x)$  is maximized when  $\tan^2(x) = 1/2$ , hence when  $\cos^2(x) = 1/(\tan^2(x) + 1) = 2/3$  and  $|\sin(x)| = 1/\sqrt{3}$ , and also using the fact that cosine has the global Lipschitz constant 1. Since (21) and (27) give

$$\begin{aligned}|\hat{\theta}(t)| &\leq \sqrt{2}\sqrt{V(X(t))} + \frac{1}{\sqrt{\omega}} \text{ and} \\ \hat{\theta}^2(t) &\leq 4V(X(t)) + \frac{2}{\omega},\end{aligned}\quad (34)$$

we can use (33) to obtain

$$\begin{aligned}|\gamma(t)| &\leq \frac{4g}{3\sqrt{3}\sqrt{\omega}} + g\bar{d}^2\sqrt{\omega} \\ &\quad + (2\bar{d}^2g^2 + 2g\bar{d}\sqrt{\omega})\left(\sqrt{2V(X(t))} + \frac{1}{\sqrt{\omega}}\right) \\ &\quad + 4g^2\bar{d}\left(4V(X(t)) + \frac{2}{\omega}\right) \\ &= \bar{\gamma}_1 + \bar{\gamma}_2\sqrt{V(X(t))} + 16g^2\bar{d}V(X(t))\end{aligned}\quad (35)$$

where

$$\begin{aligned}\bar{\gamma}_1 &= \frac{4g}{3\sqrt{3}\sqrt{\omega}} + g\bar{d}^2\sqrt{\omega} + \frac{2\bar{d}^2g^2}{\sqrt{\omega}} + 2g\bar{d} + \frac{8g^2\bar{d}}{\omega} \text{ and} \\ \bar{\gamma}_2 &= 2\sqrt{2}\left(\bar{d}^2g^2 + g\bar{d}\sqrt{\omega}\right).\end{aligned}\quad (36)$$

Then, we can combine (32) and (34)-(35) to obtain

$$\begin{aligned}\dot{W}(t) &\leq -2gV(X(t)) \\ &\quad + \left(\sqrt{2} + \frac{\sqrt{2}g}{\omega}\right)\left(\bar{\gamma}_1 + \bar{\gamma}_2\sqrt{V(X(t))}\right) \\ &\quad + 16g^2\bar{d}V(X(t))\sqrt{V(X(t))} + \frac{4g^2}{\omega}V(X(t)) \\ &\quad + \frac{4g^2}{\sqrt{\omega}}\left(\sqrt{2}\sqrt{V(X(t))} + \frac{1}{\sqrt{\omega}}\right)V(X(t)) \\ &= -2gV(X(t)) \\ &\quad + \left(\sqrt{2} + \frac{\sqrt{2}g}{\omega}\right)\left(\bar{\gamma}_1\sqrt{V(X(t))} + \bar{\gamma}_2V(X(t))\right) \\ &\quad + 16g^2\bar{d}V(X(t))\sqrt{V(X(t))} \\ &\quad + \frac{8g^2}{\omega}V(X(t)) + \frac{4\sqrt{2}g^2}{\sqrt{\omega}}V^{3/2}(X(t))\end{aligned}\quad (37)$$

and therefore also

$$\begin{aligned} \dot{W}(t) \leq & \left[ -2g + \frac{8g^2}{\omega} + \left( \sqrt{2} + \frac{\sqrt{2}g}{\omega} \right) \bar{\gamma}_2 \right] V(X(t)) \\ & + \left( \sqrt{2} + \frac{\sqrt{2}g}{\omega} \right) \bar{\gamma}_1 \sqrt{V(X(t))} \\ & + \left[ \left( \sqrt{2} + \frac{\sqrt{2}g}{\omega} \right) 16g^2 \bar{d} + \frac{4\sqrt{2}g^2}{\sqrt{\omega}} \right] V^{3/2}(X(t)). \end{aligned} \quad (38)$$

Also, our condition (9) and our choice of  $\bar{\gamma}_2$  in (36) give

$$\frac{8g^2}{\omega} + \frac{9\sqrt{2}}{8}\bar{\gamma}_2 \leq \frac{g}{8}, \quad (39)$$

since (7) gives  $\omega = 2\pi/\epsilon$ . It follows that

$$\frac{g}{\omega} \leq \frac{1}{8} \quad (40)$$

is satisfied. Using (40) to simplify, we obtain

$$\begin{aligned} \dot{W}(t) \leq & \left( -2g + \frac{8g^2}{\omega} + \frac{9\sqrt{2}}{8}\bar{\gamma}_2 \right) V(X(t)) \\ & + c_1 \sqrt{V(X(t))} + c_2 \sqrt{V(X(t))} V(X(t)) \end{aligned} \quad (41)$$

by upper bounding the right side of (38), where

$$c_1 = \frac{9\sqrt{2}}{8}\bar{\gamma}_1 \quad \text{and} \quad c_2 = 18\sqrt{2}g^2\bar{d} + \frac{4\sqrt{2}g^2}{\sqrt{\omega}}, \quad (42)$$

where  $\bar{\gamma}_1$  was defined in (36). Then (39) gives

$$\dot{W}(t) \leq -\frac{15g}{8}V(X(t)) + c_1\sqrt{V(X(t))} + c_2V^{3/2}(X(t)), \quad (43)$$

and (29) and (40) give

$$W(t, \hat{\theta}(t)) \leq \frac{9}{8}V(X(t)) \quad (44)$$

and

$$V(X(t)) \leq \frac{8}{7}W(t, \hat{\theta}(t)). \quad (45)$$

Using (43)-(45), we obtain

$$\begin{aligned} \dot{W}(t) \leq & -\frac{5}{3}gW(t, \hat{\theta}(t)) + \sqrt{\frac{8}{7}}c_1\sqrt{W(t, \hat{\theta}(t))} \\ & + \sqrt{\frac{8}{7}}\frac{8}{7}c_2\sqrt{W(t, \hat{\theta}(t))}W(t, \hat{\theta}(t)) \\ = & 2\sqrt{W(t, \hat{\theta}(t))} \left[ c_4W(t, \hat{\theta}(t)) \right. \\ & \left. - \frac{5}{6}g\sqrt{W(t, \hat{\theta}(t))} + c_3 \right] \end{aligned} \quad (46)$$

where

$$c_3 = \sqrt{\frac{2}{7}}c_1 \quad \text{and} \quad c_4 = \frac{8}{7}\sqrt{\frac{2}{7}}c_2. \quad (47)$$

Let us observe that, roughly speaking,  $c_3$  and  $c_4$  are small constants when  $\epsilon$  and  $\bar{d}$  are small. That is why we can use (46) to derive a practical stability property, in the next part.

*Third Part: Comparison System.* Our application of the comparison theorem will call for factoring the right side of a suitable dynamics as a polynomial in the square root of the state of the dynamics. To obtain this factorization, we first note that our condition (10) from the theorem and the preceding choices of  $c_1$ ,  $c_2$ ,  $c_3$ , and  $c_4$  in (42) and (47) give

$$\frac{25}{36}g^2 - 4c_3c_4 > 0. \quad (48)$$

Then the constants

$$\begin{aligned} \lambda_s &= \frac{5g}{12c_4} - \frac{1}{2c_4}\sqrt{\frac{25}{36}g^2 - 4c_3c_4} \quad \text{and} \\ \lambda_l &= \frac{5g}{12c_4} + \frac{1}{2c_4}\sqrt{\frac{25}{36}g^2 - 4c_3c_4} \end{aligned} \quad (49)$$

are well-defined and positive. We also set

$$\xi_s = \lambda_s^2 \quad \text{and} \quad \xi_l = \lambda_l^2. \quad (50)$$

Now, we consider a solution of (20) such that

$$W(0, \hat{\theta}(0)) < \xi_l. \quad (51)$$

For states  $\xi(t) > 0$ , we introduce the comparison system

$$\dot{\xi}(t) = 2\sqrt{\xi(t)} \left[ c_4\xi(t) - \frac{5}{6}g\sqrt{\xi(t)} + c_3 \right]. \quad (52)$$

By factoring the quantity in squared brackets in (52) as a polynomial in  $\sqrt{\xi(t)}$ , we can use (49) and (50) to get

$$\dot{\xi}(t) = 2c_4\sqrt{\xi(t)} \left( \sqrt{\xi(t)} - \sqrt{\xi_s} \right) \left( \sqrt{\xi(t)} - \sqrt{\xi_l} \right). \quad (53)$$

It follows that when  $\xi(0) \in [\xi_s, \xi_l]$ , then  $\xi(t) > 0$  for all  $t \geq 0$ . Consider the solution of (53) with the initial condition

$$\xi(0) = \max\{\xi_s, W(0, \hat{\theta}(0))\}. \quad (54)$$

Then  $\xi(0) \in [\xi_s, \xi_l]$ , so  $\xi(t) > 0$  for all  $t \geq 0$  and

$$\lambda(t) = \sqrt{\xi(t)} \quad (55)$$

is well-defined and satisfies

$$\dot{\lambda}(t) = c_4(\lambda(t) - \lambda_s)(\lambda(t) - \lambda_l). \quad (56)$$

Then (54) and (51) imply that  $\lambda(0) \in [\lambda_s, \lambda_l]$ . We deduce that  $\lambda(t) \in [\lambda_s, \lambda_l]$  for all  $t \geq 0$ , and we can apply a separation of variables argument to (56) to get

$$\lambda(t) = \frac{\lambda_s(\lambda_l - \lambda(0)) + \lambda_l(\lambda(0) - \lambda_s)e^{c_4(\lambda_s - \lambda_l)t}}{\lambda_l - \lambda(0) + (\lambda(0) - \lambda_s)e^{c_4(\lambda_s - \lambda_l)t}}. \quad (57)$$

On the other hand, by applying the comparison lemma to (46) and (52), it follows from (54) that  $W(t, \hat{\theta}(t)) \leq \xi(t) = \lambda^2(t)$ . We deduce that if (51) holds, then

$$W(t, \hat{\theta}(t)) \leq \left[ \frac{\lambda_s(\lambda_l - \lambda(0)) + \lambda_l(\lambda(0) - \lambda_s)e^{c_4(\lambda_s - \lambda_l)t}}{\lambda_l - \lambda(0) + (\lambda(0) - \lambda_s)e^{c_4(\lambda_s - \lambda_l)t}} \right]^2. \quad (58)$$

*Fourth Part: Final Ultimate Bound.* We consider an initial condition such that  $|\hat{\theta}(0)| \leq \sigma_0$ . Then our change of variable (18) gives  $|\hat{\theta}(0)| \leq \frac{\sigma_0}{\sqrt{\alpha}}$ . It follows from (21) that

$$|X(0)| \leq \frac{\sigma_0}{\sqrt{\alpha}} + \frac{1}{\sqrt{\omega}}. \quad (59)$$

By our choice of  $V$  in (27), this gives the bound

$$V(X(0)) \leq \frac{\sigma_0^2}{\alpha} + \frac{1}{\omega}. \quad (60)$$

Then (44) implies that

$$W(0, \hat{\theta}(0)) \leq \frac{9}{8} \left( \frac{\sigma_0^2}{\alpha} + \frac{1}{\omega} \right). \quad (61)$$

Thus  $W(0, \hat{\theta}(0)) < \xi_l$  when

$$\frac{9}{8} \left( \frac{\sigma_0^2}{\alpha} + \frac{1}{\omega} \right) < \left( \frac{5g}{12c_4} + \frac{1}{2c_4}\sqrt{\frac{25}{36}g^2 - 4c_3c_4} \right)^2, \quad (62)$$

and (62) holds because (11) and (47) imply that

$$\frac{9}{8} \left( \frac{\sigma_0^2}{\alpha} + \frac{1}{\omega} \right) < \left( \frac{5g}{12c_4} \right)^2. \quad (63)$$

Hence, (51) holds. From (45) and (58), it follows that

$$|X(t)| \leq \frac{4}{\sqrt{7}} \left[ \frac{\lambda_s(\lambda_l - \lambda(0)) + \lambda_l(\lambda(0) - \lambda_s)e^{c_4(\lambda_s - \lambda_l)t}}{\lambda_l - \lambda(0) + (\lambda(0) - \lambda_s)e^{c_4(\lambda_s - \lambda_l)t}} \right]. \quad (64)$$

Hence, as a consequence of our formula (21) for  $X$  and (64),

$$|\hat{\theta}(t)| \leq \frac{1}{\sqrt{\omega}} + \frac{4}{\sqrt{7}} \left[ \frac{\lambda_s(\lambda_l - \lambda(0)) + \lambda_l(\lambda(0) - \lambda_s)e^{c_4(\lambda_s - \lambda_l)t}}{\lambda_l - \lambda(0) + (\lambda(0) - \lambda_s)e^{c_4(\lambda_s - \lambda_l)t}} \right], \quad (65)$$

so the ultimate bound on  $|\hat{\theta}(t)|$  is  $\hat{\theta}_U = 4\lambda_s/\sqrt{7} + 1/\sqrt{\omega}$ , so  $|\hat{\theta}(t)|$  has the ultimate bound  $\sqrt{\alpha}(4\lambda_s/\sqrt{7} + 1/\sqrt{\omega})$ . Also,

$$\begin{aligned} \hat{\theta}_U &= \frac{2}{\sqrt{7}c_4} \left( \frac{5g}{6} - \sqrt{\frac{25}{36}g^2 - 4c_3c_4} \right) + \frac{1}{\sqrt{\omega}} \\ &= \frac{2}{\sqrt{7}c_4} \frac{\frac{25g^2}{36} - \frac{25}{36}g^2 + 4c_3c_4}{\frac{5g}{6} + \sqrt{\frac{25}{36}g^2 - 4c_3c_4}} + \frac{1}{\sqrt{\omega}} \\ &= \frac{2}{\sqrt{7}c_4} \frac{4c_3c_4}{\frac{5g}{6} + \sqrt{\frac{25}{36}g^2 - 4c_3c_4}} + \frac{1}{\sqrt{\omega}} \\ &\leq \frac{48c_3}{5\sqrt{7}g} + \frac{1}{\sqrt{\omega}} = \frac{48\sqrt{2}}{35g}c_1 + \frac{1}{\sqrt{\omega}} = \frac{108}{35g}\bar{\gamma}_1 + \frac{1}{\sqrt{\omega}}, \end{aligned} \quad (66)$$

by (42). From the formula for  $\bar{\gamma}_1$  in (36), it follows that

$$\hat{\theta}_U \leq \frac{108}{35g} \left( \frac{4g}{3\sqrt{3}\sqrt{\omega}} + \frac{2\bar{d}^2g^2}{\sqrt{\omega}} + g\bar{d}^2\sqrt{\omega} + 2g\bar{d} + \frac{8g^2\bar{d}}{\omega} \right) + \frac{1}{\sqrt{\omega}} \quad (67)$$

so (12) follows from our choice (7) of  $\omega$ , (8), and (18).

*Remark 3:* Using the upper bound (45) for  $V$  and the upper bound (58) for  $W(t, \hat{\theta}(t))$ , and our formulas for  $X$  and  $V$  in (21) and (27), and using (18), we obtain

$$\begin{aligned} |\tilde{\theta}(t)| &\leq \sqrt{\alpha}|\hat{\theta}(t)| \leq \sqrt{\alpha} \left( |X(t)| + \sqrt{1/\omega} \right) \\ &\leq \sqrt{2\alpha V(X(t))} + \sqrt{\alpha/\omega} \\ &\leq 4\sqrt{\frac{\alpha}{7}}W^{1/2}(t, \hat{\theta}(t)) + \sqrt{\frac{\alpha}{\omega}} \\ &\leq 4\sqrt{\frac{\alpha}{7}}\lambda_s + 4\lambda_l\sqrt{\frac{\alpha}{7}}\frac{\lambda(0) - \lambda_s}{\lambda_l - \lambda(0)}e^{c_4(\lambda_s - \lambda_l)t} + \sqrt{\frac{\alpha}{\omega}}, \end{aligned} \quad (68)$$

where  $\lambda(0)$  is defined by (54)-(55). Also, we can use (18), (21), (44), (50), and (54)-(55) to get

$$\begin{aligned} \frac{\lambda(0) - \lambda_s}{\lambda_l - \lambda(0)} &\leq \frac{1}{d_*}W^{1/2}(0, \hat{\theta}(0)) \leq \frac{3}{4d_*}|X(0)| \\ &\leq \frac{3}{4d_*} \left( |\hat{\theta}(0)| + \frac{1}{\sqrt{\omega}} \right) \leq \frac{3}{4d_*\sqrt{\alpha}}|\hat{\theta}(0)| + \frac{3}{4d_*\sqrt{\omega}} \end{aligned} \quad (69)$$

where  $d_* = \lambda_l - \max\{\lambda_s, \sqrt{W_*}\}$  and  $W_*$  is the right side of (61); the first inequality in (69) followed by noting that  $d_* \leq \lambda_l - \lambda(0)$ , and then separately considering the cases  $\xi_s \geq W(0, \hat{\theta}(0))$  and  $\xi_s < W(0, \hat{\theta}(0))$  to also get  $\lambda(0) - \lambda_s \leq W^{1/2}(0, \hat{\theta}(0))$ . Hence, we can use (68)-(69) to choose

$$c_* = \frac{3}{d_*}\lambda_l\sqrt{\frac{\alpha}{7}} \left( \frac{|\hat{\theta}(0)|}{\sqrt{\alpha}} + \frac{1}{\sqrt{\omega}} \right) \text{ and } r_* = c_4(\lambda_l - \lambda_s) \quad (70)$$

to satisfy the requirements of (13) from Remark 2.  $\square$

#### IV. USING $d_0$ TO INCLUDE MEASUREMENT DELAYS

We next consider the effects of an unknown time varying piecewise continuous measurement delay  $\tau(t) \in [0, \bar{\tau}]$  with a known bound  $\bar{\tau} \geq 0$  for all  $t \geq 0$ , which produces

$$\dot{\tilde{\theta}}(t) = \sqrt{\alpha\omega} \cos \left( \omega t + \frac{kH}{2}\tilde{\theta}^2(t - \tau(t)) \right) \quad (71)$$

which is the system (6) with  $d_0(t) = \tilde{\theta}(t - \tau(t)) - \tilde{\theta}(t)$  and  $kQ^*$  not present. Then the change of variable (18) gives

$$\dot{\hat{\theta}}(t) = \sqrt{\omega} \cos \left( \omega t + g\hat{\theta}^2(t - \tau(t)) \right). \quad (72)$$

We assume that the initial condition is constant, i.e.,  $\hat{\theta}(t) = \hat{\theta}(0)$  for all  $t \in [-\bar{\tau}, 0]$ . We now show that

$$|\hat{\theta}(t) - \hat{\theta}(t - \tau(t))| \leq \bar{\tau}\sqrt{\omega} \quad (73)$$

for all  $t \geq 0$ . If  $t - \tau(t) \leq 0$ , then  $|\hat{\theta}(t) - \hat{\theta}(t - \tau(t))| = |\hat{\theta}(t) - \hat{\theta}(0) + \hat{\theta}(0) - \hat{\theta}(t - \tau(t))| = |\hat{\theta}(t) - \hat{\theta}(0)|$ , which gives

$$|\hat{\theta}(t) - \hat{\theta}(t - \tau(t))| = \left| \int_0^t \dot{\hat{\theta}}(m) dm \right| \leq \bar{\tau}\sqrt{\omega} \quad (74)$$

because  $t \leq \bar{\tau}$ . On the other hand, if  $t - \tau(t) > 0$ , then

$$|\hat{\theta}(t) - \hat{\theta}(t - \tau(t))| = \left| \int_{t-\tau(t)}^t \dot{\hat{\theta}}(m) dm \right| \leq \bar{\tau}\sqrt{\omega}. \quad (75)$$

Thus (73) is satisfied. We therefore have

$$\dot{\hat{\theta}}(t) = \sqrt{\omega} \cos \left( \omega t + g(\hat{\theta}(t) + \mathcal{U}(t))^2 \right) \quad (76)$$

where  $\mathcal{U}(t) = \hat{\theta}(t - \tau(t)) - \hat{\theta}(t)$  is such that  $|\mathcal{U}(t)| \leq \bar{\tau}\sqrt{\omega}$  for all  $t \geq 0$ . Then, with  $\mathcal{U}$  playing the role of  $d$ , we can apply Theorem 1 from Section II above to prove attractivity of a neighborhood of the origin when  $\epsilon$  and  $\bar{\tau}$  are sufficiently small. By replacing  $\bar{d}$  by  $\bar{\tau}\sqrt{\omega}$ , this requires the condition

$$\frac{4g\epsilon}{\pi} + 9(\bar{\tau}^2g + \bar{\tau})\frac{\pi}{\epsilon} \leq \frac{1}{8}, \quad (77)$$

by our requirement (9) from Theorem 1 and our formula (7) for  $\omega$ . Notice that (77) implies that

$$\bar{\tau} \leq \frac{\epsilon}{72\pi}. \quad (78)$$

Thus, decreasing  $g$  does not make it possible to allow arbitrarily large  $\bar{\tau}$  values, which is significantly different from [23] where the known constant delay is  $D = \ell\epsilon$  for some integer  $\ell > 0$  and where adjusting the constants made it possible to compensate for arbitrarily large constant delays.

We next show that for each constant  $\bar{\epsilon} > 0$  and each constant  $\alpha > 0$ , we can find a constant  $\epsilon \in (0, \bar{\epsilon})$  and a choice of the delay  $\tau(t)$  in (71) such that (71) does not admit an attractive neighborhood of the origin. Hence, the cases of unknown time-varying and known constant delays are very different. This sheds light on why we needed a smallness condition on  $\bar{\tau}$  with respect to  $\epsilon$ . To show the impact of time-varying delays for any  $\epsilon$ 's, we consider a delay of sawtooth type. Let  $\eta > 0$  be a constant and

$$\epsilon = \frac{\eta}{j} \quad (79)$$

where  $j$  is any integer larger than 2. We use the sequence  $t_i = i\eta$  indexed by integers  $i \geq 0$  and the system defined by

$$\dot{\tilde{\theta}}(t) = \sqrt{\alpha\omega} \cos \left( \omega t + \frac{kH}{2}\tilde{\theta}^2(t_i) \right) \quad (80)$$

for all  $t \in [t_i, t_{i+1})$ . Then, for each integer  $i \geq 0$ , we get

$$\begin{aligned} \tilde{\theta}(t_{i+1}) - \tilde{\theta}(t_i) &= \sqrt{\alpha\omega} \int_{t_i}^{t_{i+1}} \cos \left( \omega m + \frac{kH}{2}\tilde{\theta}^2(t_i) \right) dm \\ &= \sqrt{\alpha\omega} \int_{t_i + \frac{kH}{2\omega}\tilde{\theta}^2(t_i)}^{t_{i+1} + \frac{kH}{2\omega}\tilde{\theta}^2(t_i)} \cos(\omega m) dm \\ &= \frac{\alpha}{\sqrt{\omega}} \left[ \sin \left( \omega\eta + \omega t_i + \frac{kH}{2}\tilde{\theta}^2(t_i) \right) \right. \\ &\quad \left. - \sin \left( \omega t_i + \frac{kH}{2}\tilde{\theta}^2(t_i) \right) \right]. \end{aligned} \quad (81)$$

By (79), we get  $\omega\eta = 2\pi\eta/\epsilon = 2\pi j$ , so  $\tilde{\theta}(t_{i+1}) - \tilde{\theta}(t_i) = 0$ , so  $\tilde{\theta}(t_i) = \tilde{\theta}(0)$  for each integer  $i > 0$ . Since  $j$  in (79) is arbitrarily large, we conclude that for any value  $\eta$ , there are arbitrarily small values of  $\epsilon$  for which the system does not admit an attractive neighborhood of the origin.

## V. ILLUSTRATIONS

We found that in many cases, our new Theorem 1 led to much larger bounds on the allowable values of  $\epsilon$  without any significant increases in the ultimate bounds on the extremum seeking error  $\tilde{\theta}(t)$ , as compared with the previous state of the art results that used averaging or other methods, including cases covered by [23], [26] where the measurement uncertainty  $d_0$  is zero. To illustrate this point, we first consider the example in [23, Section IV], which used the choices  $k = 11$ ,  $\alpha = 0.0001$ ,  $\sigma_0 = 1$ , the measurement uncertainty  $d_0 = 0$ , and values  $H \in [1, 3]$ . Following [23], we separately considered the two cases where

$$H = 2 \quad (82)$$

and where  $H$  was unknown but known to satisfy

$$H \in [1, 3]. \quad (83)$$

We summarize our comparison results in Table 1 below, where the upper bounds on  $\epsilon$  are denoted by  $\epsilon^*$  and the ultimate upper bounds are for  $|\tilde{\theta}(t)|$ , using the formulas from Theorem 1 above and [23], [26]. Following [23], we omit the  $\epsilon^*$  and ultimate bound for the case (83) when using the method from [26], since [26] did not apply when  $H$  was not known. Since our method produced much bigger  $\epsilon^*$ s without significant increases in the extremum seeking estimation error, our method from Theorem 1 above may offer significant advantages, compared with earlier extremum seeking methods.

Extremum Seeking Method	$\epsilon^*$	UB
[23, Corollary 1] with (82)	0.0754	0.017
[26] with (82)	0.013	1.52
Theorem 1 above with (82)	6.709	0.0349
[23, Corollary 1] with (83)	0.02	0.011
[26] with (83)	-	-
Theorem 1 above with (83)	2.9822	0.0232

TABLE I

COMPARISON OF  $\epsilon$  BOUNDS AND ULTIMATE BOUNDS WITH  $d_0 = 0$

Moreover, when  $d_0 = 0$  and  $\epsilon = 0.0754$ , Theorem 1 above gave the ultimate bound 0.0037 using (82); and with  $\epsilon = 0.02$ , Theorem 1 above gave the ultimate bound 0.0019 using  $H = 3$ . Finally, with a bound  $\bar{d}_0 = 0.1$  on  $|d_0|$ , and using  $\epsilon = 0.02$ ,  $\alpha = 7$ ,  $H = 3$ ,  $k = 0.00001$ , and  $\sigma_0 = 1$ , Theorem 1 above provided the ultimate bound of 1.32771 on  $\tilde{\theta}(t)$ . Although larger than the ultimate bounds that we obtained in the  $d_0 = 0$  case, we believe this is the price to pay to allow measurement uncertainties that were not allowed in earlier works. Also, our approach from Theorem 1 remains effective using the smaller  $\epsilon$ 's from [23] as well.

## VI. CONCLUSION

We advanced the state of the art for the stability analysis of bounded gradient based extremum seeking, using a new state transformation and a generalized Lyapunov function approach that allows significantly smaller dither periods, compared with previous extremum seeking results. We also analyzed the effects of measurement delays, highlighting the challenges arising when unknown time-varying delays occur.

We aim to provide analogs for higher dimensional Newton-based extremum seeking with measurement uncertainty.

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