



HAL
open science

**Correction to "Analysis of the Implicit Euler
Time-Discretization of a Class of Descriptor-Variable
Linear Cone Complementarity Systems", J. Convex
Analysis 29/2 (2022) 481–517**

Quang-Hung Pham, Bernard Brogliato

► **To cite this version:**

Quang-Hung Pham, Bernard Brogliato. Correction to "Analysis of the Implicit Euler Time-Discretization of a Class of Descriptor-Variable Linear Cone Complementarity Systems", J. Convex Analysis 29/2 (2022) 481–517. Journal of Convex Analysis, 2024, 31 (3), pp.1035-1037. hal-04704041

HAL Id: hal-04704041

<https://inria.hal.science/hal-04704041v1>

Submitted on 20 Sep 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

Correction to “Analysis of the implicit Euler time-discretization of
a class of descriptor-variable linear cone complementarity systems”,
Journal of Convex Analysis, 29(2):481-517, 2022

Quang-Hung Pham and Bernard Brogliato*

July 23, 2024

Abstract

This note corrects a mistake in [3, section 3.2.1], concerning the characterization of the domain of an operator of the form $\Phi(t, \cdot) := (\partial\sigma_{\Gamma(t)} + D)^{-1}(\cdot)$, $D = D^\top$ a constant positive semi-definite matrix, $\Gamma(t)$ a nonempty closed convex set for each t , $\sigma_{\Gamma(t)}(\cdot)$ its support function.

The results in [2, Theorems 3, 4] are wrongly used in [3] to conclude that $\text{dom}(\Phi(t, \cdot)) = \Gamma(t) + \text{Im}(D)$, while it can only be inferred that $\text{dom}(\Phi(t, \cdot)) \simeq \Gamma(t) + \text{Im}(D)$, where \simeq has the meaning as stated in [2]. Actually, the equality holds if in addition $\Gamma(t)$ is polyhedral. The set $\Gamma(t)$ is defined as $\Gamma(t) = K^* - F(t)$ in [3], where $F : \mathbb{R}_+ \rightarrow \mathbb{R}^m$ is a function of time. Assume that $K \subseteq \mathbb{R}^m$ is a closed convex nonempty polyhedral cone defined as $K := \{v \in \mathbb{R}^m \mid v = G^\top \xi, \xi \in \mathbb{R}_+^j\}$, $G \in \mathbb{R}^{j \times m}$. Thus $K^* = \{z \in \mathbb{R}^m \mid Gz \geq 0\} \subseteq \mathbb{R}^m$ (this is assumed in the subsequent developments in [3] hence it has no consequence on the results which are stated after section 3.2.1 “Calculation of sets” in that article). We will also use the following. If $S := \{\gamma \in \mathbb{R}^m : A\gamma \geq b\}$ for some $A \in \mathbb{R}^{n \times m}$ and $b \in \mathbb{R}^n$, then for any $x \in S$, the normal cone to S at x is given by:

$$\mathcal{N}_S(x) = \{p \in \mathbb{R}^m : p = -A^\top \lambda, 0 \leq \lambda \perp Ax - b \geq 0, \lambda \in \mathbb{R}^n\}. \quad (1)$$

Also $\partial\sigma_S(\cdot) = \mathcal{N}_S^{-1}(\cdot)$, where ∂ denotes the subdifferential of convex analysis. Let $D \succcurlyeq 0$ (not necessarily symmetric), then $\text{SOL}(0, GDG^\top)$ is a polyhedral convex cone [5, p.1054], where $\text{SOL}(0, GDG^\top)$ is the set of solutions to the homogeneous Linear Complementarity Problem (LCP): $0 \leq \lambda \perp GDG^\top \lambda \geq 0$. Therefore, there exists a positive number k and a matrix $\hat{G} \in \mathbb{R}^{k \times j}$ such that $\text{SOL}(0, GDG^\top) = \{v \in \mathbb{R}^j \mid v = \hat{G}^\top \xi, \xi \in \mathbb{R}_+^k\}$ (for instance, if $GDG^\top \succ 0$, then k can be chosen as any positive number, and $\hat{G} \in \mathbb{R}^{k \times j}$ is a zero matrix since the only solution is null, while if $GDG^\top = 0$ then $k = j$ can be chosen and \hat{G} is an identity matrix since all vectors in \mathbb{R}_+^j are solutions). Let us remind that $\Phi(t, \cdot) := (\partial\sigma_{\Gamma(t)} + D)^{-1}(\cdot)$. The following result holds:

Lemma 1 *Let $\mathcal{H} := \{h \in \mathbb{R}^m : \hat{G}Gh \geq 0\}$. Then, $\text{dom}(\Phi(t, \cdot)) = \mathcal{H} - F(t)$ for all $t \geq 0$.*

Proof: For each $t \geq 0$, we have $\text{dom}(\Phi(t, \cdot)) = \text{Im}\left(\mathcal{N}_{K^* - F(t)}^{-1} + D\right)$. Let us prove that

$$\text{Im}\left(\mathcal{N}_{K^* - F(t)}^{-1} + D\right) = \mathcal{H} - F(t).$$

*Univ. Grenoble Alpes, INRIA, CNRS, Grenoble INP, LJK, 38000 Grenoble, France. quang-hung.pham@inria.fr, bernard.brogliato@inria.fr

Take any $p \in \text{Im} \left(\mathcal{N}_{K^* - F(t)}^{-1} + D \right)$, then there exists $x \in \mathbb{R}^m$ such that $p \in \left(\mathcal{N}_{K^* - F(t)}^{-1} + D \right) (x)$, *i.e.*, $x \in \mathcal{N}_{K^* - F(t)}(p - Dx)$. Since $K^* - F(t) = \{z \in \mathbb{R}^m : Gz + GF(t) \geq 0\}$, using (1), then there exists $\lambda \in \mathbb{R}^j$ such that

$$\begin{cases} x = -G^\top \lambda \\ w = G(p - Dx) + GF(t) \\ 0 \leq \lambda \perp w \geq 0 \end{cases} \Leftrightarrow \begin{cases} w = GDG^\top \lambda + Gp + GF(t) \\ 0 \leq \lambda \perp w \geq 0. \end{cases}$$

This implies that LCP $(Gp + GF(t), GDG^\top)$ is solvable, where LCP (q, M) denotes the LCP of the form: $0 \leq \lambda \perp M\lambda + q \geq 0$, for some matrix M and vector q . Since $GDG^\top \succcurlyeq 0$ due to $D \succcurlyeq 0$, it is copositive-plus [6, Exercise 3.12.1]. Then using [6, Corollary 3.8.10], we get $Gp + GF(t) \in (\text{SOL}(0, GDG^\top))^* = \{z \in \mathbb{R}^j : \hat{G}z \geq 0\}$. Therefore, $p \in \mathcal{H} - F(t)$. Reversing the above process, we also obtain $p \in \text{Im} \left(\mathcal{N}_{K^* - F(t)}^{-1} + D \right)$ for any $p \in \mathcal{H} - F(t)$. The proof is complete. \square

Let us now prove the second part of the statement, *i.e.*, $\text{dom}(\Phi(t, \cdot)) = \Gamma(t) + \text{Im}(D)$ if $D = D^\top \succcurlyeq 0$. Firstly, since $\text{Im}(D) = (\ker D^\top)^\perp$ [1, equation (2.4.14)], using the symmetry of D and [7, Example 6.23], we get $\text{Im}(D) = (\ker D)^*$. Therefore, $K^* + \text{Im}(D) = (K \cap \ker D)^*$ [7, Corollary 11.25]. Thus, let us prove that $\mathcal{H} = (K \cap \ker D)^*$.

[\subseteq] Let any $y \in \mathcal{H}$, then $\hat{G}Gy \geq 0$, that is, $Gy \in (\text{SOL}(0, GDG^\top))^*$. This implies that

$$\langle Gy, z \rangle \geq 0, \quad \forall z \in \text{SOL}(0, GDG^\top). \quad (2)$$

Take any $\alpha \in K \cap \ker D$, then there exists $\xi \in \mathbb{R}_+^j$ such that $\alpha = G^\top \xi$ and $D\alpha = 0$. Therefore, we obtain $GDG^\top \xi = 0$, and thus $\xi \in \text{SOL}(0, GDG^\top)$. It follows from (2) that $\langle y, \alpha \rangle = \langle y, G^\top \xi \rangle = \langle Gy, \xi \rangle \geq 0$. Because α is chosen arbitrarily, $\langle y, \alpha \rangle \geq 0$ for all $\alpha \in K \cap \ker D$. This means that $y \in (K \cap \ker D)^*$.

[\supseteq] Let any $h \in (K \cap \ker D)^*$, then $\langle h, z \rangle \geq 0$ for all $z \in K \cap \ker D$. To obtain $h \in \mathcal{H}$, we shall prove that $Gh \in (\text{SOL}(0, GDG^\top))^*$. Indeed, take any $\alpha \in \text{SOL}(0, GDG^\top)$, then $\alpha \in \mathbb{R}_+^j$ and $\langle \alpha, GDG^\top \alpha \rangle = 0$. Since $D = D^\top \succcurlyeq 0$, there exists $\tilde{D} = \tilde{D}^\top \succcurlyeq 0$ such that $\tilde{D}^2 = D$. Moreover, it follows from the symmetry of \tilde{D} that $\ker D = \ker \tilde{D}$. Therefore, we get $0 = \langle \alpha, GDG^\top \alpha \rangle = \langle \tilde{D}G^\top \alpha, \tilde{D}G^\top \alpha \rangle = \|\tilde{D}G^\top \alpha\|^2$. Hence, $G^\top \alpha \in \ker \tilde{D} = \ker D$. Alternatively, $G^\top \alpha \in K$. Therefore, $G^\top \alpha \in K \cap \ker D$ and thus $\langle h, G^\top \alpha \rangle \geq 0$, *i.e.*, $\langle Gh, \alpha \rangle \geq 0$.

Therefore it is proved that $\text{dom}(\Phi(t, \cdot)) = \mathcal{H} - F(t) = K^* + \text{Im}(D) - F(t) = \Gamma(t) + \text{Im}(D)$.

Remark 1 *If $D \succ 0$ (not necessarily symmetric) then $\text{rint}(\text{dom}(\Phi(t, \cdot))) = \text{rint}(\Gamma(t)) + \text{Im}(D)$ for all $t \geq 0$ [8, Proposition 3]. Moreover, in this case, $\text{Im}(D) = \mathbb{R}^m$ and thus $\text{rint}(\text{dom}(\Phi(t, \cdot))) = \text{dom}(\Phi(t, \cdot)) = \mathcal{H} = \mathbb{R}^m$ for all $t \geq 0$. Another characterization is in [4, Theorem 4.5].*

References

- [1] D.S. Bernstein. *Matrix Mathematics. Theory, Facts and Formulas*. Princeton University Press, 2nd edition, 2009.

- [2] H. Brézis and A. Haraux. Image d'une somme d'opérateurs monotones et applications. *Israel Journal of Mathematics*, 23(2):165–186, 1976.
- [3] B. Brogliato. Analysis of the implicit Euler time-discretization of a class of descriptor-variable linear cone complementarity systems. *Journal of Convex Analysis*, 29(2):481–517, 2022.
- [4] B. Brogliato and D. Goeleven. Existence, uniqueness of solutions and stability of nonsmooth multivalued Lur'e dynamical systems. *Journal of Convex Analysis*, 20(3):881–900, 2013.
- [5] M.K. Camlibel, L. Iannelli, and A. Tanwani. Convergence of proximal solutions for evolution inclusions with time-dependent maximal monotone operators. *Mathematical Programming A*, 194:1017–1059, 2022.
- [6] R.W. Cottle, J.S. Pang, and R.E. Stone. *The Linear Complementarity Problem*. Computer Science and Scientific Computing. Academic Press, 1992.
- [7] R.T. Rockafellar and R.J.B. Wets. *Variational Analysis*. Springer, 1998. 3rd printing, 2009.
- [8] A. Tanwani, B. Brogliato, and C. Prieur. Well-posedness and output regulation for implicit time-varying evolution variational inequalities. *SIAM Journal on Control and Optimization*, 56(2):751–781, 2018.