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Correction to “Analysis of the implicit Euler time-discretization of  
a class of descriptor-variable linear cone complementarity systems”,  
Journal of Convex Analysis, 29(2):481-517, 2022

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**Abstract**

This note corrects a mistake in [3, section 3.2.1], concerning the characterization of the domain of an operator of the form  $\Phi(t, \cdot) := (\partial\sigma_{\Gamma(t)} + D)^{-1}(\cdot)$ ,  $D = D^\top$  a constant positive semi-definite matrix,  $\Gamma(t)$  a nonempty closed convex set for each  $t$ ,  $\sigma_{\Gamma(t)}(\cdot)$  its support function.

The results in [2, Theorems 3, 4] are wrongly used in [3] to conclude that  $\text{dom}(\Phi(t, \cdot)) = \Gamma(t) + \text{Im}(D)$ , while it can only be inferred that  $\text{dom}(\Phi(t, \cdot)) \simeq \Gamma(t) + \text{Im}(D)$ , where  $\simeq$  has the meaning as stated in [2]. Actually, the equality holds if in addition  $\Gamma(t)$  is polyhedral. The set  $\Gamma(t)$  is defined as  $\Gamma(t) = K^* - F(t)$  in [3], where  $F : \mathbb{R}_+ \rightarrow \mathbb{R}^m$  is a function of time. Assume that  $K \subseteq \mathbb{R}^m$  is a closed convex nonempty polyhedral cone defined as  $K := \{v \in \mathbb{R}^m \mid v = G^\top \xi, \xi \in \mathbb{R}_+^j\}$ ,  $G \in \mathbb{R}^{j \times m}$ . Thus  $K^* = \{z \in \mathbb{R}^m \mid Gz \geq 0\} \subseteq \mathbb{R}^m$  (this is assumed in the subsequent developments in [3] hence it has no consequence on the results which are stated after section 3.2.1 “Calculation of sets” in that article). We will also use the following. If  $S := \{\gamma \in \mathbb{R}^m : A\gamma \geq b\}$  for some  $A \in \mathbb{R}^{n \times m}$  and  $b \in \mathbb{R}^n$ , then for any  $x \in S$ , the normal cone to  $S$  at  $x$  is given by:

$$\mathcal{N}_S(x) = \{p \in \mathbb{R}^m : p = -A^\top \lambda, 0 \leq \lambda \perp Ax - b \geq 0, \lambda \in \mathbb{R}^n\}. \quad (1)$$

Also  $\partial\sigma_S(\cdot) = \mathcal{N}_S^{-1}(\cdot)$ , where  $\partial$  denotes the subdifferential of convex analysis. Let  $D \succcurlyeq 0$  (not necessarily symmetric), then  $\text{SOL}(0, GDG^\top)$  is a polyhedral convex cone [5, p.1054], where  $\text{SOL}(0, GDG^\top)$  is the set of solutions to the homogeneous Linear Complementarity Problem (LCP):  $0 \leq \lambda \perp GDG^\top \lambda \geq 0$ . Therefore, there exists a positive number  $k$  and a matrix  $\hat{G} \in \mathbb{R}^{k \times j}$  such that  $\text{SOL}(0, GDG^\top) = \{v \in \mathbb{R}^j \mid v = \hat{G}^\top \xi, \xi \in \mathbb{R}_+^k\}$  (for instance, if  $GDG^\top \succ 0$ , then  $k$  can be chosen as any positive number, and  $\hat{G} \in \mathbb{R}^{k \times j}$  is a zero matrix since the only solution is null, while if  $GDG^\top = 0$  then  $k = j$  can be chosen and  $\hat{G}$  is an identity matrix since all vectors in  $\mathbb{R}_+^j$  are solutions). Let us remind that  $\Phi(t, \cdot) := (\partial\sigma_{\Gamma(t)} + D)^{-1}(\cdot)$ . The following result holds:

**Lemma 1** *Let  $\mathcal{H} := \{h \in \mathbb{R}^m : \hat{G}Gh \geq 0\}$ . Then,  $\text{dom}(\Phi(t, \cdot)) = \mathcal{H} - F(t)$  for all  $t \geq 0$ .*

**Proof:** For each  $t \geq 0$ , we have  $\text{dom}(\Phi(t, \cdot)) = \text{Im}\left(\mathcal{N}_{K^* - F(t)}^{-1} + D\right)$ . Let us prove that

$$\text{Im}\left(\mathcal{N}_{K^* - F(t)}^{-1} + D\right) = \mathcal{H} - F(t).$$

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Take any  $p \in \text{Im} \left( \mathcal{N}_{K^* - F(t)}^{-1} + D \right)$ , then there exists  $x \in \mathbb{R}^m$  such that  $p \in \left( \mathcal{N}_{K^* - F(t)}^{-1} + D \right) (x)$ , *i.e.*,  $x \in \mathcal{N}_{K^* - F(t)}(p - Dx)$ . Since  $K^* - F(t) = \{z \in \mathbb{R}^m : Gz + GF(t) \geq 0\}$ , using (1), then there exists  $\lambda \in \mathbb{R}^j$  such that

$$\begin{cases} x = -G^\top \lambda \\ w = G(p - Dx) + GF(t) \\ 0 \leq \lambda \perp w \geq 0 \end{cases} \Leftrightarrow \begin{cases} w = GDG^\top \lambda + Gp + GF(t) \\ 0 \leq \lambda \perp w \geq 0. \end{cases}$$

This implies that LCP  $(Gp + GF(t), GDG^\top)$  is solvable, where LCP  $(q, M)$  denotes the LCP of the form:  $0 \leq \lambda \perp M\lambda + q \geq 0$ , for some matrix  $M$  and vector  $q$ . Since  $GDG^\top \succcurlyeq 0$  due to  $D \succcurlyeq 0$ , it is copositive-plus [6, Exercise 3.12.1]. Then using [6, Corollary 3.8.10], we get  $Gp + GF(t) \in (\text{SOL}(0, GDG^\top))^* = \{z \in \mathbb{R}^j : \hat{G}z \geq 0\}$ . Therefore,  $p \in \mathcal{H} - F(t)$ . Reversing the above process, we also obtain  $p \in \text{Im} \left( \mathcal{N}_{K^* - F(t)}^{-1} + D \right)$  for any  $p \in \mathcal{H} - F(t)$ . The proof is complete.  $\square$

Let us now prove the second part of the statement, *i.e.*,  $\text{dom}(\Phi(t, \cdot)) = \Gamma(t) + \text{Im}(D)$  if  $D = D^\top \succcurlyeq 0$ . Firstly, since  $\text{Im}(D) = (\ker D^\top)^\perp$  [1, equation (2.4.14)], using the symmetry of  $D$  and [7, Example 6.23], we get  $\text{Im}(D) = (\ker D)^*$ . Therefore,  $K^* + \text{Im}(D) = (K \cap \ker D)^*$  [7, Corollary 11.25]. Thus, let us prove that  $\mathcal{H} = (K \cap \ker D)^*$ .

[ $\subseteq$ ] Let any  $y \in \mathcal{H}$ , then  $\hat{G}Gy \geq 0$ , that is,  $Gy \in (\text{SOL}(0, GDG^\top))^*$ . This implies that

$$\langle Gy, z \rangle \geq 0, \quad \forall z \in \text{SOL}(0, GDG^\top). \quad (2)$$

Take any  $\alpha \in K \cap \ker D$ , then there exists  $\xi \in \mathbb{R}_+^j$  such that  $\alpha = G^\top \xi$  and  $D\alpha = 0$ . Therefore, we obtain  $GDG^\top \xi = 0$ , and thus  $\xi \in \text{SOL}(0, GDG^\top)$ . It follows from (2) that  $\langle y, \alpha \rangle = \langle y, G^\top \xi \rangle = \langle Gy, \xi \rangle \geq 0$ . Because  $\alpha$  is chosen arbitrarily,  $\langle y, \alpha \rangle \geq 0$  for all  $\alpha \in K \cap \ker D$ . This means that  $y \in (K \cap \ker D)^*$ .

[ $\supseteq$ ] Let any  $h \in (K \cap \ker D)^*$ , then  $\langle h, z \rangle \geq 0$  for all  $z \in K \cap \ker D$ . To obtain  $h \in \mathcal{H}$ , we shall prove that  $Gh \in (\text{SOL}(0, GDG^\top))^*$ . Indeed, take any  $\alpha \in \text{SOL}(0, GDG^\top)$ , then  $\alpha \in \mathbb{R}_+^j$  and  $\langle \alpha, GDG^\top \alpha \rangle = 0$ . Since  $D = D^\top \succcurlyeq 0$ , there exists  $\tilde{D} = \tilde{D}^\top \succcurlyeq 0$  such that  $\tilde{D}^2 = D$ . Moreover, it follows from the symmetry of  $\tilde{D}$  that  $\ker D = \ker \tilde{D}$ . Therefore, we get  $0 = \langle \alpha, GDG^\top \alpha \rangle = \langle \tilde{D}G^\top \alpha, \tilde{D}G^\top \alpha \rangle = \|\tilde{D}G^\top \alpha\|^2$ . Hence,  $G^\top \alpha \in \ker \tilde{D} = \ker D$ . Alternatively,  $G^\top \alpha \in K$ . Therefore,  $G^\top \alpha \in K \cap \ker D$  and thus  $\langle h, G^\top \alpha \rangle \geq 0$ , *i.e.*,  $\langle Gh, \alpha \rangle \geq 0$ .

Therefore it is proved that  $\text{dom}(\Phi(t, \cdot)) = \mathcal{H} - F(t) = K^* + \text{Im}(D) - F(t) = \Gamma(t) + \text{Im}(D)$ .

**Remark 1** *If  $D \succ 0$  (not necessarily symmetric) then  $\text{rint}(\text{dom}(\Phi(t, \cdot))) = \text{rint}(\Gamma(t)) + \text{Im}(D)$  for all  $t \geq 0$  [8, Proposition 3]. Moreover, in this case,  $\text{Im}(D) = \mathbb{R}^m$  and thus  $\text{rint}(\text{dom}(\Phi(t, \cdot))) = \text{dom}(\Phi(t, \cdot)) = \mathcal{H} = \mathbb{R}^m$  for all  $t \geq 0$ . Another characterization is in [4, Theorem 4.5].*

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