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(2022) 481–517**

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► **To cite this version:**

Quang-Hung Pham, Bernard Brogliato. Correction to "Analysis of the Implicit Euler Time-Discretization of a Class of Descriptor-Variable Linear Cone Complementarity Systems", J. Convex Analysis 29/2 (2022) 481–517. Journal of Convex Analysis, 2024, 31 (3), pp.1035-1037. <hal-04704041>

HAL Id: hal-04704041

<https://inria.hal.science/hal-04704041v1>

Submitted on 20 Sep 2024

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Correction to “Analysis of the implicit Euler time-discretization of
a class of descriptor-variable linear cone complementarity systems”,
Journal of Convex Analysis, 29(2):481-517, 2022

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July 23, 2024

Abstract

This note corrects a mistake in [3, section 3.2.1], concerning the characterization of the domain of an operator of the form $\Phi(t, \cdot) := (\partial\sigma_{\Gamma(t)} + D)^{-1}(\cdot)$, $D = D^\top$ a constant positive semi-definite matrix, $\Gamma(t)$ a nonempty closed convex set for each t , $\sigma_{\Gamma(t)}(\cdot)$ its support function.

The results in [2, Theorems 3, 4] are wrongly used in [3] to conclude that $\text{dom}(\Phi(t, \cdot)) = \Gamma(t) + \text{Im}(D)$, while it can only be inferred that $\text{dom}(\Phi(t, \cdot)) \simeq \Gamma(t) + \text{Im}(D)$, where \simeq has the meaning as stated in [2]. Actually, the equality holds if in addition $\Gamma(t)$ is polyhedral. The set $\Gamma(t)$ is defined as $\Gamma(t) = K^* - F(t)$ in [3], where $F : \mathbb{R}_+ \rightarrow \mathbb{R}^m$ is a function of time. Assume that $K \subseteq \mathbb{R}^m$ is a closed convex nonempty polyhedral cone defined as $K := \{v \in \mathbb{R}^m \mid v = G^\top \xi, \xi \in \mathbb{R}_+^j\}$, $G \in \mathbb{R}^{j \times m}$. Thus $K^* = \{z \in \mathbb{R}^m \mid Gz \geq 0\} \subseteq \mathbb{R}^m$ (this is assumed in the subsequent developments in [3] hence it has no consequence on the results which are stated after section 3.2.1 “Calculation of sets” in that article). We will also use the following. If $S := \{\gamma \in \mathbb{R}^m : A\gamma \geq b\}$ for some $A \in \mathbb{R}^{n \times m}$ and $b \in \mathbb{R}^n$, then for any $x \in S$, the normal cone to S at x is given by:

$$\mathcal{N}_S(x) = \{p \in \mathbb{R}^m : p = -A^\top \lambda, 0 \leq \lambda \perp Ax - b \geq 0, \lambda \in \mathbb{R}^n\}. \quad (1)$$

Also $\partial\sigma_S(\cdot) = \mathcal{N}_S^{-1}(\cdot)$, where ∂ denotes the subdifferential of convex analysis. Let $D \succcurlyeq 0$ (not necessarily symmetric), then $\text{SOL}(0, GDG^\top)$ is a polyhedral convex cone [5, p.1054], where $\text{SOL}(0, GDG^\top)$ is the set of solutions to the homogeneous Linear Complementarity Problem (LCP): $0 \leq \lambda \perp GDG^\top \lambda \geq 0$. Therefore, there exists a positive number k and a matrix $\hat{G} \in \mathbb{R}^{k \times j}$ such that $\text{SOL}(0, GDG^\top) = \{v \in \mathbb{R}^j \mid v = \hat{G}^\top \xi, \xi \in \mathbb{R}_+^k\}$ (for instance, if $GDG^\top \succ 0$, then k can be chosen as any positive number, and $\hat{G} \in \mathbb{R}^{k \times j}$ is a zero matrix since the only solution is null, while if $GDG^\top = 0$ then $k = j$ can be chosen and \hat{G} is an identity matrix since all vectors in \mathbb{R}_+^j are solutions). Let us remind that $\Phi(t, \cdot) := (\partial\sigma_{\Gamma(t)} + D)^{-1}(\cdot)$. The following result holds:

Lemma 1 *Let $\mathcal{H} := \{h \in \mathbb{R}^m : \hat{G}Gh \geq 0\}$. Then, $\text{dom}(\Phi(t, \cdot)) = \mathcal{H} - F(t)$ for all $t \geq 0$.*

Proof: For each $t \geq 0$, we have $\text{dom}(\Phi(t, \cdot)) = \text{Im}\left(\mathcal{N}_{K^* - F(t)}^{-1} + D\right)$. Let us prove that

$$\text{Im}\left(\mathcal{N}_{K^* - F(t)}^{-1} + D\right) = \mathcal{H} - F(t).$$

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Take any $p \in \text{Im} \left(\mathcal{N}_{K^* - F(t)}^{-1} + D \right)$, then there exists $x \in \mathbb{R}^m$ such that $p \in \left(\mathcal{N}_{K^* - F(t)}^{-1} + D \right) (x)$, *i.e.*, $x \in \mathcal{N}_{K^* - F(t)}(p - Dx)$. Since $K^* - F(t) = \{z \in \mathbb{R}^m : Gz + GF(t) \geq 0\}$, using (1), then there exists $\lambda \in \mathbb{R}^j$ such that

$$\begin{cases} x = -G^\top \lambda \\ w = G(p - Dx) + GF(t) \\ 0 \leq \lambda \perp w \geq 0 \end{cases} \Leftrightarrow \begin{cases} w = GDG^\top \lambda + Gp + GF(t) \\ 0 \leq \lambda \perp w \geq 0. \end{cases}$$

This implies that LCP $(Gp + GF(t), GDG^\top)$ is solvable, where LCP (q, M) denotes the LCP of the form: $0 \leq \lambda \perp M\lambda + q \geq 0$, for some matrix M and vector q . Since $GDG^\top \succcurlyeq 0$ due to $D \succcurlyeq 0$, it is copositive-plus [6, Exercise 3.12.1]. Then using [6, Corollary 3.8.10], we get $Gp + GF(t) \in (\text{SOL}(0, GDG^\top))^* = \{z \in \mathbb{R}^j : \hat{G}z \geq 0\}$. Therefore, $p \in \mathcal{H} - F(t)$. Reversing the above process, we also obtain $p \in \text{Im} \left(\mathcal{N}_{K^* - F(t)}^{-1} + D \right)$ for any $p \in \mathcal{H} - F(t)$. The proof is complete. \square

Let us now prove the second part of the statement, *i.e.*, $\text{dom}(\Phi(t, \cdot)) = \Gamma(t) + \text{Im}(D)$ if $D = D^\top \succcurlyeq 0$. Firstly, since $\text{Im}(D) = (\ker D^\top)^\perp$ [1, equation (2.4.14)], using the symmetry of D and [7, Example 6.23], we get $\text{Im}(D) = (\ker D)^*$. Therefore, $K^* + \text{Im}(D) = (K \cap \ker D)^*$ [7, Corollary 11.25]. Thus, let us prove that $\mathcal{H} = (K \cap \ker D)^*$.

[\subseteq] Let any $y \in \mathcal{H}$, then $\hat{G}Gy \geq 0$, that is, $Gy \in (\text{SOL}(0, GDG^\top))^*$. This implies that

$$\langle Gy, z \rangle \geq 0, \quad \forall z \in \text{SOL}(0, GDG^\top). \quad (2)$$

Take any $\alpha \in K \cap \ker D$, then there exists $\xi \in \mathbb{R}_+^j$ such that $\alpha = G^\top \xi$ and $D\alpha = 0$. Therefore, we obtain $GDG^\top \xi = 0$, and thus $\xi \in \text{SOL}(0, GDG^\top)$. It follows from (2) that $\langle y, \alpha \rangle = \langle y, G^\top \xi \rangle = \langle Gy, \xi \rangle \geq 0$. Because α is chosen arbitrarily, $\langle y, \alpha \rangle \geq 0$ for all $\alpha \in K \cap \ker D$. This means that $y \in (K \cap \ker D)^*$.

[\supseteq] Let any $h \in (K \cap \ker D)^*$, then $\langle h, z \rangle \geq 0$ for all $z \in K \cap \ker D$. To obtain $h \in \mathcal{H}$, we shall prove that $Gh \in (\text{SOL}(0, GDG^\top))^*$. Indeed, take any $\alpha \in \text{SOL}(0, GDG^\top)$, then $\alpha \in \mathbb{R}_+^j$ and $\langle \alpha, GDG^\top \alpha \rangle = 0$. Since $D = D^\top \succcurlyeq 0$, there exists $\tilde{D} = \tilde{D}^\top \succcurlyeq 0$ such that $\tilde{D}^2 = D$. Moreover, it follows from the symmetry of \tilde{D} that $\ker D = \ker \tilde{D}$. Therefore, we get $0 = \langle \alpha, GDG^\top \alpha \rangle = \langle \tilde{D}G^\top \alpha, \tilde{D}G^\top \alpha \rangle = \|\tilde{D}G^\top \alpha\|^2$. Hence, $G^\top \alpha \in \ker \tilde{D} = \ker D$. Alternatively, $G^\top \alpha \in K$. Therefore, $G^\top \alpha \in K \cap \ker D$ and thus $\langle h, G^\top \alpha \rangle \geq 0$, *i.e.*, $\langle Gh, \alpha \rangle \geq 0$.

Therefore it is proved that $\text{dom}(\Phi(t, \cdot)) = \mathcal{H} - F(t) = K^* + \text{Im}(D) - F(t) = \Gamma(t) + \text{Im}(D)$.

Remark 1 *If $D \succ 0$ (not necessarily symmetric) then $\text{rint}(\text{dom}(\Phi(t, \cdot))) = \text{rint}(\Gamma(t)) + \text{Im}(D)$ for all $t \geq 0$ [8, Proposition 3]. Moreover, in this case, $\text{Im}(D) = \mathbb{R}^m$ and thus $\text{rint}(\text{dom}(\Phi(t, \cdot))) = \text{dom}(\Phi(t, \cdot)) = \mathcal{H} = \mathbb{R}^m$ for all $t \geq 0$. Another characterization is in [4, Theorem 4.5].*

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