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# Spatial pattern analysis of a $A\beta$ -monomer model with inflammation processes for Alzheimer's disease.

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## Abstract

We study the emergence of spatial patterns for a system of reaction-diffusion equations, modeling the progression of Alzheimer's disease through the interaction of  $A\beta$ -monomers, oligomers, microglial cells, and interleukins with neurons. In our work, these spatial patterns stand for inert amyloid plaques, which are extracellular deposits of  $A\beta$ -proteins and a characteristic feature of this neurodegenerative disease. Using linear analysis and numerical simulations, we show the existence of spatially heterogeneous solutions and exhibit a wide variety of possible spatially-dependent solutions: time-oscillating, low-amplitude, and high-amplitude patterns. Moreover, we carry out an extensive analysis of high-amplitude patterns in the one- and two-dimensional domains. In particular, we study the stability of branches of heterogeneous steady states through bifurcation diagrams and their selection. From these numerical simulations, we develop some conjectures concerning the influence of inflammation and microglial cells in the formation of amyloid plaques. These findings offer insights into potential anti-inflammatory treatments that might be used to mitigate the progression of Alzheimer's disease and the emergence of inert amyloid plaques.

# 1 Introduction

Alzheimer's disease (AD) is a progressive neurodegenerative disorder that affects memory, thinking, and behavior. The pathogenesis of AD is multifactorial and involves several risk factors such as age, genetics, head injuries, vascular diseases, infections, and environment. One of the main hypotheses proposed to explain the disease's pathogenesis is the misfolding and aggregation of amyloid  $\beta$ -proteins in the nervous system.

Amyloid-beta ( $A\beta$ ) is a peptide that forms the primary component of the amyloid plaques found in the brains of Alzheimer's disease patients (see figure 1).  $A\beta$  peptides are intrinsically disordered in their monomeric form and assemble into stable structures such as oligomers and fibrils via a nucleation-dependent pathway. Many studies point to a key role for these  $A\beta$ -aggregates in neurotoxicity, leading to progressive, irreversible neuronal damage (see, for example, [1, 2, 3, 4, 5]). Understanding the role of  $A\beta$  aggregates and the mechanisms of their formation and degradation could provide valuable insights into the pathogenesis of Alzheimer's disease and potential therapeutic strategies.

Neurons produce  $A\beta$ -monomers that almost instantaneously start to polymerize into proto-oligomers. In this aggregation process, proto-oligomers can polymerize or depolymerize, and once they reach a critical size, they become stable in the form of  $A\beta$ -oligomers. These latter are assumed to be completely stable in the sense that neither polymerization nor depolymerization is possible for  $A\beta$ -oligomers in equilibrium [6, 7]. This mechanism involving  $A\beta$ -oligomers is known as the amyloid cascade hypothesis, and there is a consensus that it is a key factor in AD progression.

The role of microglia, the primary immune cells in the brain, in the progression of Alzheimer's disease is also significant. They induce an inflammatory reaction through a chemical cascade in microglial cells, releasing interleukins [8, 9, 10, 11]. These interleukins then activate an increase in  $A\beta$ -monomer production from neurons. However, if the concentration of  $A\beta$ -oligomers is high enough, a stress reaction called the unfolded protein response (UPR) [4] is triggered, which leads to a decrease in  $A\beta$ -monomer production. At the same time, the remaining oligomers diffuse into the neuronal environment. In this context, two opposing mechanisms of stimulation and inhibition determine whether AD persists. Moreover, microglial cells also play a role in

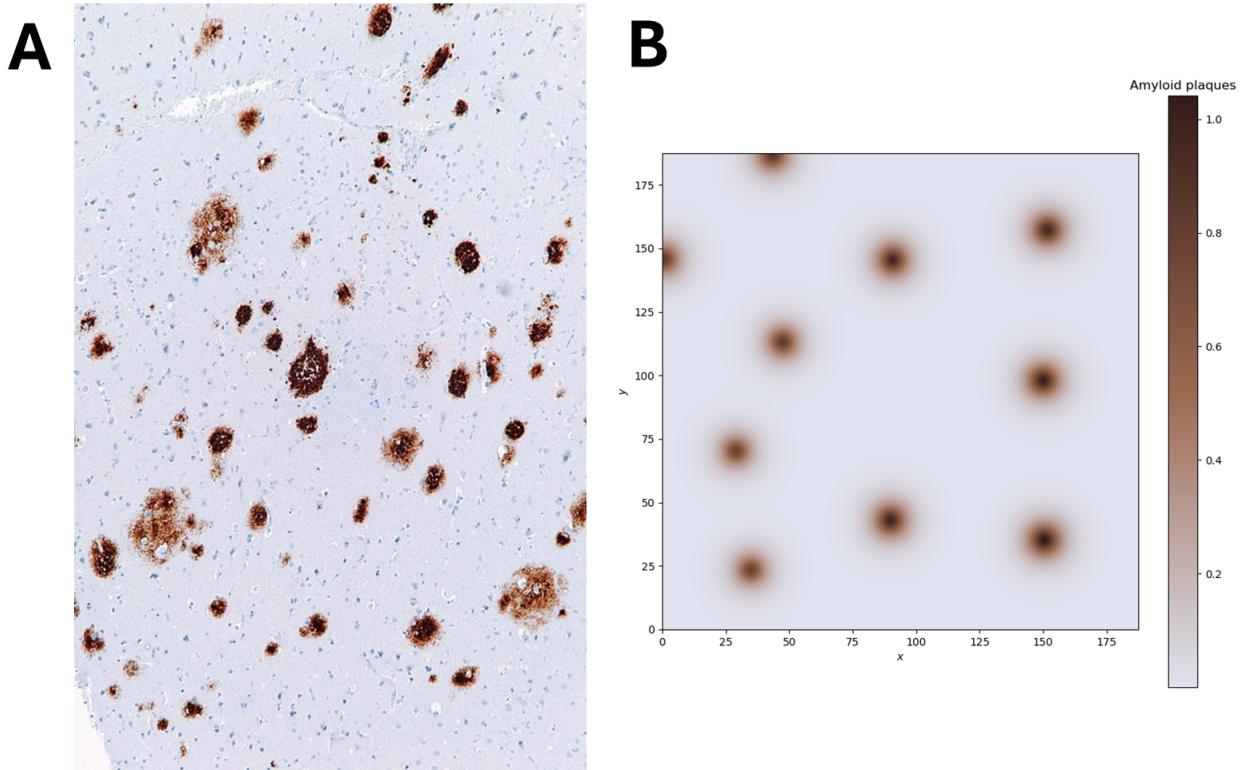


Figure 1: **A.** A picture of  $A\beta$  amyloid plaques in the brain.<sup>1</sup> **B.** Stationary patterns of our mathematical model.

the transfer of oligomers to amyloid plaques, where these oligomers become inert elements (no diffusion, polymerization, or de-polymerization) [12, 13, 14].

From a mathematical point of view, previous works on partial differential equations on AD include the works of Andrade *et al.* [15] on a space-dependent model for the polymerization of  $A\beta$ -proteins, Ciuperca *et al.* [16] which takes into account the formation of  $A\beta$ -oligomers and fibrils through a continuous size model based on Lifshitz–Slyozov equations and Hao *et al.* [17] on a multi-component model taking into account astrocytes, microglia cells and peripheral cell population, besides the  $A\beta$ -oligomers (for other works on reaction-diffusion models see also [18, 19, 20]).

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In this article, we study the bi-monomeric model introduced in the work of [21], which is given by the following reaction-diffusion system in a bounded domain  $\Omega$ ,

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} = \nu_2 \Delta u + r_1 m^2 - \gamma_0 u, \\ \frac{\partial u_p}{\partial t} = \gamma_0 u - \tau_p u_p, \\ \frac{\partial m}{\partial t} = \nu_1 \Delta m + \frac{\tau_S}{1 + Cu^n} I - dm - r_2 um - r_1 m^2, \\ \frac{\partial M}{\partial t} = D_1 \Delta M - \alpha \nabla \cdot [M \nabla u] + \frac{\alpha_1 u}{1 + \alpha_2 u} (\hat{M} - M) M - \sigma M + \lambda_M, \\ \frac{\partial I}{\partial t} = D_I \Delta I + \frac{\tau_1 u}{1 + \tau_2 u} M - \tau_3 I. \end{array} \right. \quad (1)$$

The term  $u$  corresponds to the concentration of the oligomers, while the term  $m$  represents the concentration of the monomers. In this model, we simplify the assumption that oligomers correspond to the aggregate of two monomers, as in [21]. The term  $u_p$  corresponds to the concentration of oligomers in amyloid plaques and is the main focus of our study. Finally, the terms  $M$  and  $I$  correspond to the concentration of microglial cells and interleukins respectively.

To simplify notation, we define the vector  $\mathbf{V}(x, t)$  as follows

$$\mathbf{V}(t, x) = (u(t, x), u_p(t, x), m(t, x), M(t, x), I(t, x)).$$

We study this problem in the one-dimensional case, *i.e.*  $\Omega = [0, L]$ , and also in the two-dimensional case, *i.e.*  $\Omega = [0, L] \times [0, L]$ , where  $L > 0$  represents the size of the domain. Concerning the boundary conditions, we assume Neumann boundary conditions, *i.e.*

$$\partial_{\mathbf{n}} \mathbf{V}(y) = 0, \quad \text{for all } y \in \partial\Omega, \text{ where } \mathbf{n} \text{ denotes the normal to the boundary } \partial\Omega.$$

In the article [21], the authors studied the spatially homogeneous steady states of the model (1). In particular, the point  $(0, 0, 0, \lambda_M/\sigma, 0)$  is always an equilibrium point for the model (1), corresponding to the disease-free equilibrium, which we denote as  $\tilde{0}$ . Moreover, when the following condition is satisfied

$$\sigma \gamma_0 \tau_3 < \tau_1 \tau_S \lambda_M, \quad (2)$$

there is a critical threshold for the monomer degradation, denoted  $d_m$ , so that for  $d < d_m$  the model (1) admits two other positive spatially-homogeneous steady states, denoted  $\mathbf{V}_u$  and  $\mathbf{V}_s$ . In the absence of chemotaxis and diffusion, the first positive equilibrium  $\mathbf{V}_u$  is unstable and the second one  $\mathbf{V}_s$  is stable.

In this article, we focus on the emergence of spatially periodic patterns corresponding biologically to the emergence of amyloid plaques. To achieve this goal, we study the possible instability of the positive equilibrium  $\mathbf{V}_s$  when the chemotaxis parameter  $\alpha$  is strictly positive. First, we use a linear analysis to study the impact of the chemotaxis parameter on the stability of the equilibria in our problem. In particular, we prove in section 2 that the disease-free equilibrium is linearly asymptotically stable. Conversely, the positive equilibrium  $\mathbf{V}_s$  becomes unstable when the chemotaxis parameter is large enough. Secondly, we use numerical simulations to study the spatial patterns of the model (1). These patterns correspond to successions of zones of high and low oligomer concentration in the plaques.

We focus on the bistable regime of the model (1). Even if the analysis of the patterns of bistable systems is less studied, this type of numerical analysis has already been carried out for several bistable reaction-diffusion systems, notably for predatory prey systems or for systems describing chemical processes (see for example [22, 23, 24, 25]).

In the bistable regime, stable pattern branches compete with the stable homogeneous state. Therefore, it is not surprising to observe a convergence towards the stable equilibrium point. In the article by Krause *et al.* [26], the authors prove, through numerous examples, that multistability can significantly alter the predictions made by linear analysis. Indeed, they observe that the solution may converge towards a stable homogeneous state, even though the parameter values within the Turing space suggest the emergence of patterns. This phenomenon is also illustrated in [27], where Al-Karkhi *et al.* study the patterns of a predator-prey model with multiple stable homogeneous equilibria without diffusion. By conducting a bifurcation diagram, they show that the pattern branches can all lose their stability, thus explaining the absence of convergence towards a spatially heterogeneous pattern. In this article, we also observe and numerically analyze this phenomenon.

In the one-dimensional case, we find several possible scenarios: low-amplitude or high-amplitude stationary patterns, time-oscillating patterns, and convergence to the disease-free equilibrium. In the two-dimensional case, numerical simulations revealed the following phenomena: the emergence of stripe patterns, dots, complex stationary patterns between stripes and dots, and convergence to the disease-free equilibrium.

To study the existence and stability of these different possible patterns, we perform several

bifurcation diagrams for the chemotaxis parameter  $\alpha$ , using the package PDEPATH [28, 29]. We notice a difference between these bifurcation diagrams and the typical ones, which are generally obtained for a monostable reaction-diffusion system. Generally, close enough to the Turing bifurcation, the only stable pattern is the one associated with the admissible critical frequency. Then moving away from this bifurcation, several other frequencies gradually become admissible and thus several stable branches emerge and coexist. In general, linear analysis shows that close to the Turing bifurcation, only a single mode can be destabilized, thereby predicting a stationary pattern with a spatial period corresponding to the critical frequency. Moving further away from this bifurcation, the study of the dispersion relation shows that several other modes progressively become admissible, and thus, several stable branches associated with these modes emerge and coexist. For our model (1), we notice an inverse trend: initially, several branches of stable patterns coexist, and the further we move away from the critical Turing threshold, the more the branches lose their stability, to the point where they all lose their stability. The result predicted by our numerical simulations is presented in the following conjecture:

**Conjecture 1** (Selection of disease-free equilibrium for  $\alpha$  large enough). *We assume that the condition (2) is satisfied.*

*Then there exists  $\alpha_0 > 0$  such that for all  $\alpha > \alpha_0$ ,*

$$\lim_{t \rightarrow \infty} \mathbf{V}(x, t) \rightarrow \tilde{\mathbf{0}}, \text{ uniformly for all } x \in \Omega.$$

*From a modeling point of view, this means counterintuitively that when the chemotaxis coefficient is large enough the system converges to the disease-free equilibrium without any amyloid plaques or inflammation.*

In the case where  $d > d_m$ , the disease-free equilibrium is the only stable steady state and it is a global attractor for all initial data. Therefore, the value of  $\alpha$  does not influence the selection of the final stationary state and we have  $\alpha_0 = 0$ . In the case where  $d \leq d_m$ , the spatially homogeneous states have bistable dynamics and there is a critical threshold  $\alpha_c$  for which the equilibrium  $\mathbf{V}_s^*$  loses its stability. We observe numerically that there exists a  $\alpha_0 \geq \alpha_c$  such that for  $\alpha > \alpha_0$  the solution converges, independently of the slight noise, to the disease-free equilibrium. This threshold  $\alpha_0$  depends on the values of the rest of the parameters in the model



but also depends on the size of the domain  $\Omega$ . In particular, we also note that  $\alpha_0 \rightarrow \alpha_c$  when  $d \rightarrow d_m$ .

Therefore, in the context of modeling, the parameter  $d$ , representing the natural degradation rate of monomers, is a reliable indicator of disease progression. When this parameter is sufficiently large ( $d > d_m$ ), natural degradation is sufficient to prevent the formation of amyloid plaques. In this case, the individual can be considered healthy or in a pre-symptomatic stage where the disease has not yet manifested. Conversely, when natural degradation is insufficient ( $d < d_m$ ), plaque formation becomes possible, with a probability that depends on its value: the lower the value of  $d$ , the wider the range of parameter values for which plaques form and persist.

Our numerical simulations reveal that the stability range of pattern branches seems to be correlated with the frequencies of these patterns. For the examples given, we notice that the greater the number of peaks in a pattern, the faster its associated branch tends to lose stability as  $\alpha$  increases. Based on these numerical results, we propose the following conjecture:

**Conjecture 2** (Decrease in the average number of amyloid plaques when  $\alpha$  increases). *We assume that the condition (2) is satisfied. Let  $\mathbf{V}_0$  be an initial data corresponding to a slight Gaussian noise from the positive equilibrium  $\mathbf{V}_s$ .*

*We assume that  $\alpha > \alpha_c$ , we denote  $\# \max(\alpha)$  the number of local maxima of the final stationary solution of the system (1), with  $\# \max$  defined as 0 in the spatially homogeneous case. The average of  $\# \max(\alpha)$  over several independent realizations for different noisy initial conditions is denoted with some abuse of notation  $\mathbb{E}\# \max(\alpha)$ . In this setting, we find the following property:*

*the function  $\mathbb{E}\# \max(\alpha)$  is non-increasing with respect to  $\alpha$ .*

*From a modeling perspective, this implies that as the chemotaxis coefficient increases, the number of amyloid plaques decreases.*

The plan of the paper is the following: in Section 2, we begin with a linear analysis of the model (1) to obtain the conditions for pattern emergence, and in Section 3 we numerically study the stationary patterns generated by the model (1) in the one-dimensional case and the two-dimensional case.

## 2 Stability Analysis

### 2.1 Linear stability analysis of the disease-free equilibrium

We start by studying the stability of the disease-free equilibrium. The linearization around this steady state can be rewritten in a system of the form

$$\frac{\partial \tilde{\omega}}{\partial t} = D\Delta \tilde{\omega} + J\tilde{\omega}, \quad (3)$$

where  $\tilde{\omega} = (\tilde{u}, \tilde{u}_p, \tilde{m}, \tilde{M}, \tilde{I})^\top$  is a vector for a small perturbation of the steady state  $(0, 0, 0, \frac{\lambda}{\sigma}, 0)$  with Neumann boundary conditions and

$$D = \begin{pmatrix} \nu_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \nu_1 & 0 & 0 \\ -\alpha \frac{\lambda}{\sigma} & 0 & 0 & D_1 & 0 \\ 0 & 0 & 0 & 0 & D_I \end{pmatrix}, \quad J = \begin{pmatrix} -\gamma_0 & 0 & 0 & 0 & 0 \\ \gamma_0 & -\tau_p & 0 & 0 & 0 \\ 0 & 0 & -d & 0 & \tau_S \\ \left(\hat{M} - \frac{\lambda}{\sigma}\right) \frac{\lambda}{\sigma} & 0 & 0 & -\sigma & 0 \\ \tau_1 \frac{\lambda}{\sigma} & 0 & 0 & 0 & -\tau_3 \end{pmatrix}.$$

In this setting, the corresponding eigenvalue problem is given by

$$\lambda \tilde{\omega} = D\Delta \tilde{\omega} + J\tilde{\omega}. \quad (4)$$

Concerning the stability, we have the following result.

**Proposition 1** (Disease-free equilibrium is asymptotically-stable). *The eigenvalue problem (4) has a countable set of eigenvalues that are all strictly negative. Therefore the disease-free equilibrium of the system (1) is linearly asymptotically stable.*

*Proof.* Consider the following eigenvalue problem

$$\begin{aligned} \Delta \varphi &= \lambda \varphi & \text{in } \Omega, \\ \nabla \varphi \cdot \hat{n} &= 0 & \text{on } \partial\Omega, \end{aligned} \quad (5)$$

which has a set of eigenvalues  $\{\lambda_k\}_{k \in \mathbb{N}}$  with  $\lambda_k \leq 0$  and  $\lambda_k \rightarrow -\infty$  when  $k \rightarrow \infty$ . Let  $\varphi_k$  be the corresponding eigenfunction.

We prove that the eigenvalues of problem (4) are strictly negative.

From Equation (5), we deduce that the solution of the first equation of the eigenvalue problem (4) given by

$$\begin{cases} \nu_2 \Delta \tilde{u} = (\lambda + \gamma_0) \tilde{u} & \text{in } \Omega, \\ \nabla \tilde{u} \cdot \hat{n} = 0 & \text{on } \partial\Omega, \end{cases} \quad (6)$$

is of the form  $\tilde{u} = \varphi_k$  with  $\lambda = \nu_2\lambda_k - \gamma_0 < 0$  and thus the system (4) has a solution for the corresponding eigenvector (take for instance  $(\varphi_k, 0, 0, 0, 0)^\top$ ).

Assume now that  $\lambda \neq \nu_2\lambda_k - \gamma_0$ . From the theory of elliptic operators [30], we get that  $\tilde{u} \equiv 0$  and the system (4) is reduced as follows.

$$\begin{cases} (\lambda + \tau_p)\tilde{u}_p = 0, \\ \nu_1\Delta\tilde{m} = (\lambda + d)\tilde{m} - \tau_S\tilde{I}, \\ D_1\Delta\tilde{M} = (\lambda + \sigma)\tilde{M}, \\ D_I\Delta\tilde{I} = (\lambda + \tau_3)\tilde{I}. \end{cases} \quad (7)$$

By applying the same argument used for  $\tilde{u}$  to rest of components of  $\tilde{\omega}$  (first for  $\tilde{u}_p, \tilde{M}, \tilde{I}$  and then for  $\tilde{m}$ ), we conclude that set of eigenvalues  $\mathcal{E}$  is given by

$$\mathcal{E} = \{-\tau_p\} \cup \bigcup_{k \in \mathbb{N}} \{\nu_1\lambda_k - d, \nu_2\lambda_k - \gamma_0, D_1\lambda_k - \sigma, D_I\lambda_k - \tau_3\} \subset (-\infty, 0).$$

This proves the desired result. From the theory of elliptic equations [30], we remark that the spectrum of the linear operator determined by the system (4) consists only of eigenvalues.  $\square$

## 2.2 Linear stability analysis of the positive equilibrium $\mathbf{V}_s$

The corresponding linearization around the point  $\mathbf{V}_s$  gives the following equation

$$\frac{\partial \tilde{\omega}}{\partial t} = D\Delta\tilde{\omega} + J\tilde{\omega}, \quad (8)$$

where  $\tilde{\omega} = (\tilde{u}, \tilde{u}_p, \tilde{m}, \tilde{M}, \tilde{I})^\top$  is a vector that corresponds to a small perturbation of the equilibrium  $\mathbf{V}_s$  with

$$D = \begin{pmatrix} \nu_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \nu_1 & 0 & 0 \\ -\alpha M_0 & 0 & 0 & D_1 & 0 \\ 0 & 0 & 0 & 0 & D_I \end{pmatrix},$$

$$\text{and, } J = \begin{pmatrix} -\gamma_0 & 0 & 2r_1m_0 & 0 & 0 \\ \gamma_0 & -\tau_p & 0 & 0 & 0 \\ -r_2m_0 - I_0\tau_S \frac{nCu_0^{n-1}}{(1+Cu_0^n)^2} & 0 & -d - r_2u_0 - 2r_1m_0 & 0 & \frac{\tau_S}{1+Cu_0^n} \\ (\hat{M} - M_0)M_0 \frac{\alpha_1}{(1+\alpha_2u_0)^2} & 0 & 0 & \frac{\alpha_1u_0}{1+\alpha_2u_0}(\hat{M} - 2M_0) - \sigma & 0 \\ M_0 \frac{\tau_1}{(1+\tau_2u_0)^2} & 0 & 0 & \frac{\tau_1u_0}{1+\tau_2u_0} & -\tau_3 \end{pmatrix}.$$

The solution of (8) with the form  $\tilde{\omega} = e^{ikx+\lambda t}$  leads to the following equation,

$$\lambda\tilde{\omega} = (J - k^2D)\tilde{\omega}, \quad (9)$$

where  $\lambda \in \mathbb{C}$  is the eigenvalue and  $k \in \mathbb{R}$  corresponds to the wavenumber. We denote  $P_{J-k^2D}$ , the characteristic polynomial associated with the problem (9). We obtain the following factorization,  $P_{J-k^2D}(\lambda) = (-\tau_p - \lambda)Q_{J-k^2D}(\lambda)$ . From this, we derive the first eigenvalue of the problem, equals to  $\lambda_0 = -\tau_p < 0$  and the polynomial  $Q_{J-k^2D}$  is given by the following relation

$$Q_{J-k^2D}(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4,$$

with

$$\begin{cases} a_1 = A + F + I + P, \\ a_2 = -BE + A(F + I + P) + F(I + P) + IP, \\ a_3 = -BGW + BE(I + P) - AFI - AFP - AIP - FIP, \\ a_4 = AFIP - BEIP + BGIW - BGKH(\alpha), \end{cases}$$

and where the constants are given by

$$A = -\gamma_0 - k^2\nu_2, \quad B = 2r_1m_0, \quad E = -r_2m_0 - I_0\tau_S \frac{nCu_0^{n-1}}{(1 + Cu_0^n)^2}, \quad F = -d - r_2u_0 - 2r_1m_0 - k^2\nu_1,$$

$$G = \frac{\tau_S}{1 + Cu_0^n}, \quad H(\alpha) = (\hat{M} - M_0)M_0 \frac{\alpha_1}{(1 + \alpha_2u_0)^2} + \alpha M_0k^2, \quad I = \frac{\alpha_1u_0}{1 + \alpha_2u_0} (\hat{M} - 2M_0) - \sigma - k^2D_1,$$

$$K = \frac{\tau_1u_0}{1 + \tau_2u_0}, \quad P = -\tau_3 - k^2D_I, \quad W = M_0 \frac{\tau_1}{(1 + \tau_2u_0)^2}.$$

To study Turing bifurcation we first prove the following lemma.

**Proposition 2.** *Assume that there is a Turing bifurcation. Then the positive equilibrium  $\mathbf{V}_s$  becomes unstable when  $\det(J - k^2D) > 0$  holds for at least one  $k > 0$  and it remains stable when  $\det(J - k^2D) < 0$  holds for all  $k \geq 0$ .*

*Proof.* Using the Routh-Hurwitz criterion, we state that all eigenvalues have strictly negative real parts if and only if these five conditions hold

$$a_i > 0 \text{ for all } i \in \{1, 2, 3, 4\} \text{ and } a_1a_2a_3 > a_3^2 + a_1^2a_4.$$

We assume that  $(\lambda_i)_{i \in \{1, 2, 3, 4\}}$  represent the roots of the characteristic equation and accordingly to the characteristic equation, we obtain

$$-a_1 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \quad a_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4,$$

$$-a_3 = \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4, \quad a_4 = \lambda_1\lambda_2\lambda_3\lambda_4.$$

Turing instability emerges when the real part of one eigenvalue passes through zero and the remaining eigenvalues have negative real parts. We denote  $k_c$  as the critical wavenumber. Thus, at this critical wavenumber  $k = k_c$  we can assume without any loss of generality that  $\lambda_1(k_c) = 0$  and  $\lambda_2, \lambda_3, \lambda_4 < 0$ .

Under this assumption, it follows that  $a_1(k_c) > 0$ ,  $a_2(k_c) > 0$ ,  $a_3(k_c) > 0$  and  $a_1(k_c)a_2(k_c) > a_3(k_c)$ . However, for  $k = k_c$ , we obtain  $a_4(k_c) = 0$ . Thus, if there is a Turing bifurcation, that comes from the nullity of the term  $a_4$  which also corresponds to the nullity of the determinant of the matrix  $J - k^2D$ .  $\square$

The condition  $a_4 < 0$  is equivalent to the following inequality

$$\alpha > \frac{AFIP - BEIP + BGIW}{BGKM_0k^2} - \frac{(\hat{M} - M_0)\frac{\alpha_1}{(1+\alpha_2u_0)^2}}{k^2}, \quad (10)$$

by positivity of terms  $B$ ,  $G$ ,  $K$  and  $M_0$ . Therefore, a spatial pattern appears on a discrete domain and for a large enough chemotaxis parameter.

The term  $a_4$  can be rewritten as a polynomial function of degree 4 in terms of  $k^2$ ,

$$a_4(k^2) = b_1(k^2)^4 + b_2(k^2)^3 + b_3(k^2)^2 + b_4(\alpha)k^2 + b_5,$$

with functions  $b_i$ ,  $i = 1, 2, 3, 4$ , which have cumbersome expressions not reported in this work. We determine  $k_{min}$  the frequency for which the  $a_4$  function reaches its minimum. This term depends on the system parameters, particularly the chemotaxis parameter  $\alpha$ . The critical frequency  $k_c$  is given by  $a_4(k_c) = \min_{k'} a_4(k') = 0$ . Thus, Turing bifurcation can occur at the critical value  $\alpha_c$ , defined by a root of the function

$$\alpha \mapsto b_1(k_{min}^2(\alpha))^4 + b_2(k_{min}^2(\alpha))^3 + b_3(k_{min}^2(\alpha))^2 + b_4(\alpha)k_{min}^2(\alpha) + b_5. \quad (11)$$

We compute this root using numerical tools. The results of these calculations are consistent with the numerical simulations and bifurcation diagrams performed in the next section.

### 3 Pattern analysis

In this section, we use numerical tools to study the patterns induced by the loss of stability of the equilibrium  $\mathbf{V}_s$ . The parameter values used in these simulations are similar to those

presented in the article [21], whose orders of magnitude respect biological laws. These parameter values are given in Table 1. In this study, we only investigate the influence of the following variable parameters: chemotaxis coefficient,  $\alpha$ , and monomer degradation,  $d$ . First, we work on the domain  $\Omega = [0, L]$  and then we perform a numerical study of the patterns for the two-dimensional square domain,  $\Omega = [0, L] \times [0, L]$ ,  $L > 0$ .

The initial conditions chosen for our numerical simulations play a crucial role in pattern selection. Moreover, adding noise in these initial conditions is necessary for consistent modeling. Throughout this article, except for the Section 3.4, we choose a weak perturbation of the positive equilibrium state  $\mathbf{V}_s$ , described in the remark below.

**Remark 1** (Choice of the noisy initial condition).

- For a one-dimensional domain, we discretize the space into the points  $(x_i)_{i \in I}$  and for each grid point we add a slight Gaussian noise,

$$\mathbf{V}_0(x_i) = \mathbf{V}_s + \epsilon \mathcal{N}^i,$$

where  $\mathcal{N}^i$  corresponds to a Gaussian vector of dimension 5 and  $\epsilon$  corresponds to a positive constant (e.g.  $\epsilon = 10^{-5}$ ) small enough to preserve positivity.

- For the two-dimensional domain, we use the same initial conditions, but this time on the  $(x_i, y_j)_{i \in I, j \in J}$  grid.

### 3.1 Numerical simulation in the one-dimensional case

#### Low and high amplitude patterns

For the one-dimensional domain, we numerically observe the existence of two different stationary pattern profiles. When the Turing bifurcation is supercritical, we notice the presence of *low-amplitude* patterns. In this case, the pattern corresponds to a sinusoidal perturbation of the positive equilibrium state with the critical frequency  $k_c$ . The amplitude of this perturbation increases with distance from the bifurcation. However, the stable branch associated with this pattern rapidly loses its stability. An example of a low-amplitude pattern is shown in figure 2 (left). When the bifurcation is subcritical, we notice the emergence of an unstable branch as expected. For our system and the parameter values tested, this branch remains unstable.

Table 1: **Parameter values for the numerical simulations of the system (1).**

Parameter	Value	Units	Description
$r_1$	$10^{-1}$	$L (\text{mol})^{-1} (\text{months})^{-1}$	Bi-monomeric polymerization rate
$r_2$	$10^{-1}$	$L (\text{mol})^{-1} (\text{months})^{-1}$	Polymerization rate of monomers attaching to oligomers
$d$	Variable	$(\text{month})^{-1}$	Degradation rate of monomers
$\gamma_0$	$5 \times 10^{-2}$	$(\text{month})^{-1}$	Recruitment rate of oligomers to the amyloid plaques
$\tau_1$	1	$L (\text{mol})^{-1} (\text{months})^{-1}$	Production coefficient of interleukins interacting with oligomers
$\tau_2$	1	$L (\text{mol})^{-1}$	Production coefficient of interleukins interacting with oligomers
$\tau_3$	1	$(\text{months})^{-1}$	Degradation rate of interleukins
$\tau_p$	$3 \times 10^{-2}$	$(\text{months})^{-1}$	Degradation rate of oligomers in the amyloid plaques
$\tau_S$	1	$(\text{months})^{-1}$	Coefficient of neural stress
$C$	1	$L^n (\text{mol})^{-n}$	Coefficient of neural stress function
$n$	2	-	Power coefficient of neural stress function
$\alpha_1$	1	$L^2 (\text{mol})^{-2} (\text{months})^{-1}$	Proliferation coefficient of microglial cells interacting with oligomers
$\alpha_2$	1	$L (\text{mol})^{-1}$	Proliferation coefficient of microglial cells interacting with oligomers
$\lambda_M$	$10^{-3}$	$\text{mol } L^{-1} (\text{months})^{-1}$	Source term of microglial cells
$\bar{M}$	1	$\text{mol } L^{-1}$	Carrying capacity of microglial cells
$\sigma$	$10^{-3}$	$(\text{months})^{-1}$	Degradation rate of microglial cells
$D_1$	1	$m^2 (\text{months})^{-1}$	Diffusion coefficient of microglial cells
$D_I$	1	$m^2 (\text{months})^{-1}$	Diffusion coefficient of interleukins
$\nu_1$	1	$m^2 (\text{months})^{-1}$	Diffusion coefficient of monomers
$\nu_2$	1	$m^2 (\text{months})^{-1}$	Diffusion coefficient of oligomers
$\alpha$	Variable	$m^2 L (\text{mol})^{-1}$	Coefficient of chemotaxis

Note that a weakly nonlinear analysis could be applied to approximate the stationary pattern in the supercritical case. This method can also determine the parameter values for which the bifurcation is sub- or supercritical (see, for example, [31, 32, 33]).

The nonlinear interactions of the system (1) can cause the emergence of large amplitude patterns. These correspond to higher peaks, distributed periodically in space, according to a certain frequency which is not the expected one and it is not even necessarily an admissible frequency according to the linear study. In figure 2 (right) we represent an example of high amplitude patterns. These high amplitude patterns are similar to biological images of amyloid plaques in the human brain, with the sequence of areas of high concentration and areas with the absence of beta-amyloid proteins. Note that these two types of low and high amplitude patterns can mathematically coexist.

### Time-oscillating patterns

We also notice the presence of heterogeneous patterns in space oscillating in time. Time-oscillating patterns have already been studied for classical reaction-diffusion systems [34, 35],

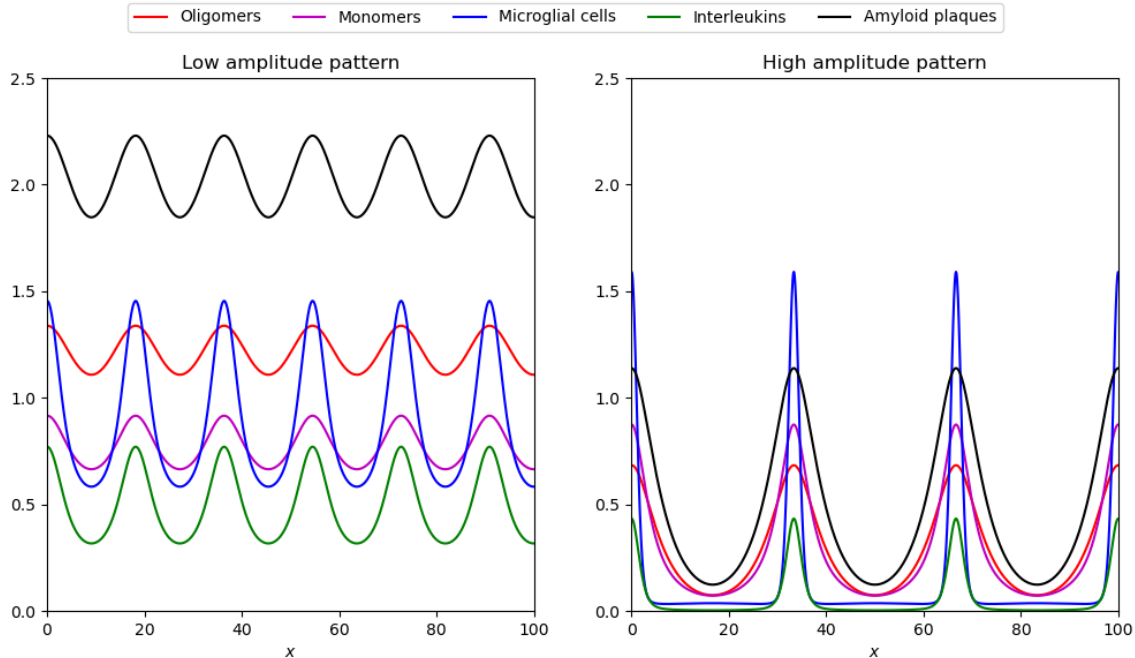


Figure 2: **Examples of final stationary patterns of the system (1).** **Left.** Example of a low-amplitude pattern. **Right** Example of a high-amplitude pattern. The concentration of oligomers is shown in red, the concentration of monomers in magenta, the concentration of microglial cells in blue, the concentration of interleukins in green, and the concentration of oligomers in amyloid plaques in black. Parameter values are those given in Table 1, with  $d = 0.05$  and  $\alpha = 21.7$  for the figure on the left and  $\alpha = 28$  for the figure on the right.

for a prey-predator model [36], with a cross-diffusion term [37, 38, 39] and as well as for bistable reaction-diffusion models [23, 24].

An example of a time-periodic oscillating pattern is shown in figure 3A. This figure illustrates the evolution of oligomer concentration in amyloid plaques in space and time. The solution becomes periodic and oscillates between low-amplitude and high-amplitude patterns. In figure 3B, we illustrate the phase portrait of the concentration of oligomers at  $x = 20$  as a function of the concentration of oligomers in the amyloid plaques. At the same time, we notice the emergence of a limit cycle in agreement with the periodicity in time of the solution. The dots shown in black and red correspond to the phase portrait at time  $t = t_1$  and  $t = t_2$  respectively and these two-time points are accordingly represented by two vertical lines of the same color in figure 3A. In figure 3C, we illustrate an example of the high amplitude in the oscillating pattern.



This corresponds to the solution shown in figure 3A at time  $t = t_1$ . The solid curve shows the spatial distribution of oligomer concentration in amyloid plaques, while the dotted curve shows the evolution of microglial cell concentration. Finally, we also illustrate a low-amplitude pattern in figure 3D, taking a solution at time  $t = t_2$ . We compare it with the high-amplitude pattern, shown here in red.

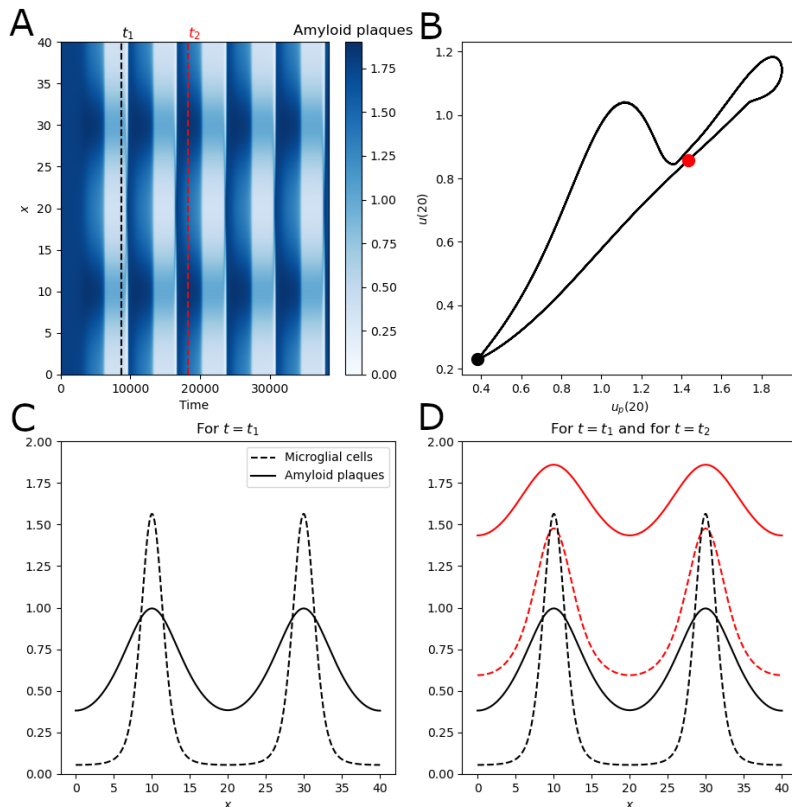


Figure 3: **Spatially heterogeneous patterns that also exhibit regular oscillations in time.** **A.** Concentration of oligomers in amyloid plaques over time and space. **B.** Evolution over time of the concentration of oligomers at  $x = 20$  compared to the concentration of oligomers in the amyloid plaques also at  $x = 20$ . The black dot corresponds to time  $t = t_1$ , while the red dot corresponds to time  $t = t_2$ . **C.** Concentration of microglial cells and oligomers in amyloid plaques at time  $t = t_1$ . (dotted black line in sub-figure A.). **D.** Concentration of microglial cells (dotted line) and oligomers in amyloid plaques (solid line) at time  $t = t_1$  in black and at time  $t = t_2$  in red (vertical dotted red line in sub-figure A.).

Like in other articles (e.g. [37, 34]), we also notice the presence of quasi-periodicity and chaotic oscillations. For example, for a slightly larger value of  $\alpha$  we notice the emergence of

a torus in the phase portrait shown in figure 15 in Appendix A. Biologically speaking, to our knowledge, there is no evidence of temporal oscillation in amyloid plaque formation. Thus, we do not develop the study of time-oscillating patterns in this work.

According to our numerical simulations, time-oscillating and low-amplitude exist only very close to the Turing bifurcation, which occurs at  $\alpha = \alpha_c$ . Therefore we mainly study the patterns for  $\alpha$  assumed to be large enough to obtain only high amplitude. In the following section, we study in particular the influence of the parameter  $d$  on stationary patterns.

### **Influence of monomer degradation on high-amplitude patterns**

We successively study the formation of high-amplitude for the following parameter values:  $d = 0.15$ ,  $d = 0.35$  and  $d = 0.45$ .

- **For  $d = 0.15$ .** To study the stationary patterns of the system (1), we perform a bifurcation shown in figure 4 diagram by using the PDEPATH package [28, 29] and more specifically the code presented in the article [40] for a reaction-diffusion system including a chemotaxis term. Stable branches are represented by a thick line, while unstable branches are represented by a thin line and the black horizontal branch corresponds to the positive steady state  $\mathbf{V}_s$ . As predicted in the section above, this branch loses its stability for  $\alpha \geq \alpha_c$ . At this Turing bifurcation, a branch is created which is associated with a periodic pattern in space. For  $d = 0.15$ , we notice that the Turing bifurcation is supercritical, indeed the branch is initially stable and associated with low amplitude. This branch rapidly loses its stability as the  $\alpha$  parameter increases. Nevertheless, following several other bifurcations of this branch, we obtain new stable branches associated with high amplitude. First of all, we note the stability, over a short interval, of the branch associated with the pattern *One peak in the middle and one peak on the edge* represented by the blue line in figure 4. Then we notice two similar branches for the  $L^1$  norm, associated respectively with the patterns *Middle peak* and *Two peaks at the edges*. The pattern with a middle peak can be seen as a reorganization of the pattern with two half-peaks at the edges, where each half-peak is shifted towards the center of the domain to form a complete, centered peak. Finally, we also notice the stability of the branch in magenta, corresponding to the pattern *One peak at the edge*. This diagram shows that for a wide range of  $\alpha$ , the system admits multi-stability of different

branches, which is expected for models with a Turing-type bifurcation.

We notice that the size of the stability domain of the different branches of high amplitude decreases when the number of peaks or half-peaks (peaks on the edge) increases. Moreover, we notice that these domains of stability are finite and that for a large enough  $\alpha$ , none of the branches that we have detected is stable. For this bifurcation diagram, we have assumed a relatively small domain for the spatial variable,  $\Omega = [0, 40]$ , to have fewer possible stable patterns. For a larger domain, we notice again this negative correlation between the number of peaks and the length of the stability domain. Note that in addition to the stable branches shown in figure 4, the disease-free equilibrium is also stable and can be an attractor.

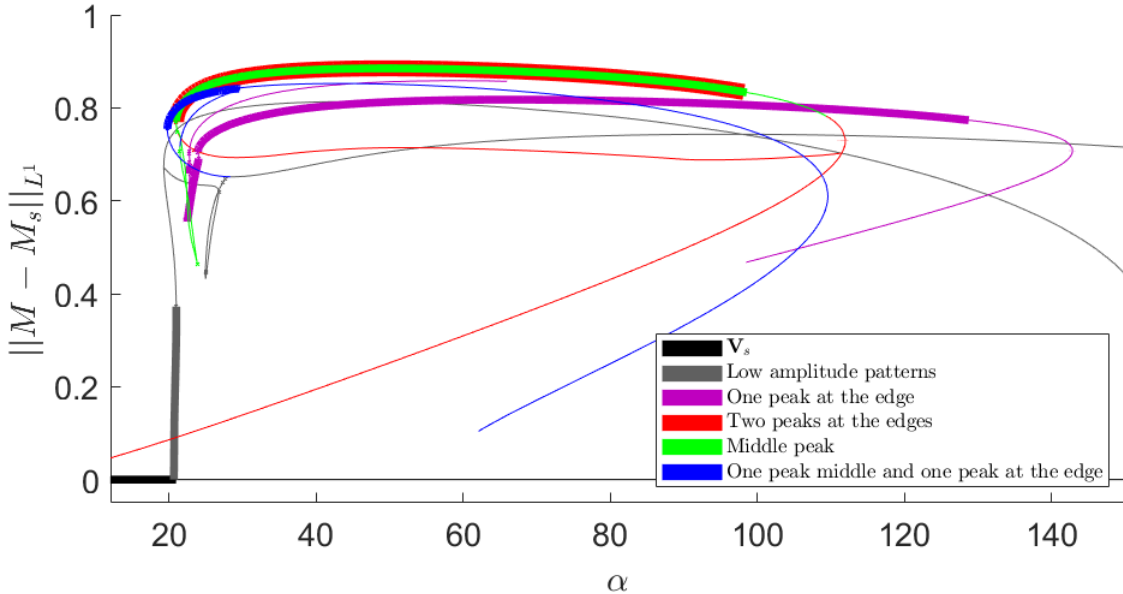


Figure 4: **Bifurcation diagram for  $d = 0.15$ .** This bifurcation diagram is obtained using the package PDEPATH [28, 29]. Thick lines indicate stable branches, while thin lines indicate unstable branches. The horizontal branch in black corresponds to the stationary state  $\mathbf{V}_s$ , which loses its stability in  $\alpha_c$ . For  $d = 0.15$ , this bifurcation point is supercritical, giving rise to a stable branch associated with a low-amplitude pattern. Following several bifurcations, we obtain the presence of other stable branches, associated with high-amplitude patterns described in the legend on the right. The term  $M_s$  corresponds to the microglial cell concentration component of the positive equilibrium  $\mathbf{V}_s$ . Parameter values are given in Table 1.

In addition to the bifurcation diagram, which exhibits a multi-stability of different branches

of possible patterns, we illustrate the probability of selection of these different stable structures in figure 5 by performing 200 independent simulations for different values of parameter  $\alpha$ . Pattern selection is highly dependent on the energy of each branch and the choice of initial condition. In our case, we choose an initial condition presented in Remark 1 with a very slight noise with an order of magnitude  $10^{-5}$ . Despite this relatively low noise level, we find at least once, a wide disparity. Indeed for several values of parameter  $\alpha$ , we notice three patterns out of the four determined using the bifurcation diagram. When  $\alpha$  is relatively close to the critical threshold  $\alpha_c$  the pattern *Two peaks at the edges* seems to be strongly selected. Then when  $\alpha$  is between  $\alpha = 40$  and  $\alpha = 90$  the probabilities of selection of the *Two peaks at the edges*, *One peak at the edge* and *Middle peak* are in the same order of magnitude. Finally for  $\alpha > 120$ , in agreement with the Conjecture 1, the only possible stationary solution is the disease-free equilibrium.

In the top panel of figure 5, we also illustrate the domains where stationary solutions are stable. These domains corroborate the results shown at the bottom panel, obtained independently. Indeed, no pattern was selected outside its stability zone determined by the bifurcation diagram.

In figure 6, we plot the average number of peaks in the final pattern associated with the numerical simulations in figure 5. We note that this average number of peaks tends to decrease as the chemotaxis coefficient,  $\alpha$ , increases. In particular, we note a strong discontinuity linked to the loss of stability of the *Two peaks at the edges* and *Middle peak* branches. These results are in agreement with the Conjecture 2.

- **For  $d = 0.35$ .** We now assume that the degradation coefficient of the monomers is equal to  $d = 0.35$  and we perform the same analysis as before. The bifurcation diagram is shown in figure 7. This diagram has strong similarities with the previous diagram in figure 4. Again, close to the Turing-type bifurcation, we notice a strong multi-stability of different branches associated with different spatially heterogeneous patterns. The red branch corresponds to a pattern with two peaks on the edges, the green branch with one peak in the middle, and finally the blue branch corresponds to a single peak on the edge. As before, we notice that the length of the domains of stability is correlated with the number of half-peaks.

The main difference between the bifurcation diagram in figure 7 and the previous bifurcation diagram shown in figure 4, concerns the length of the stable domains of the branches. In partic-

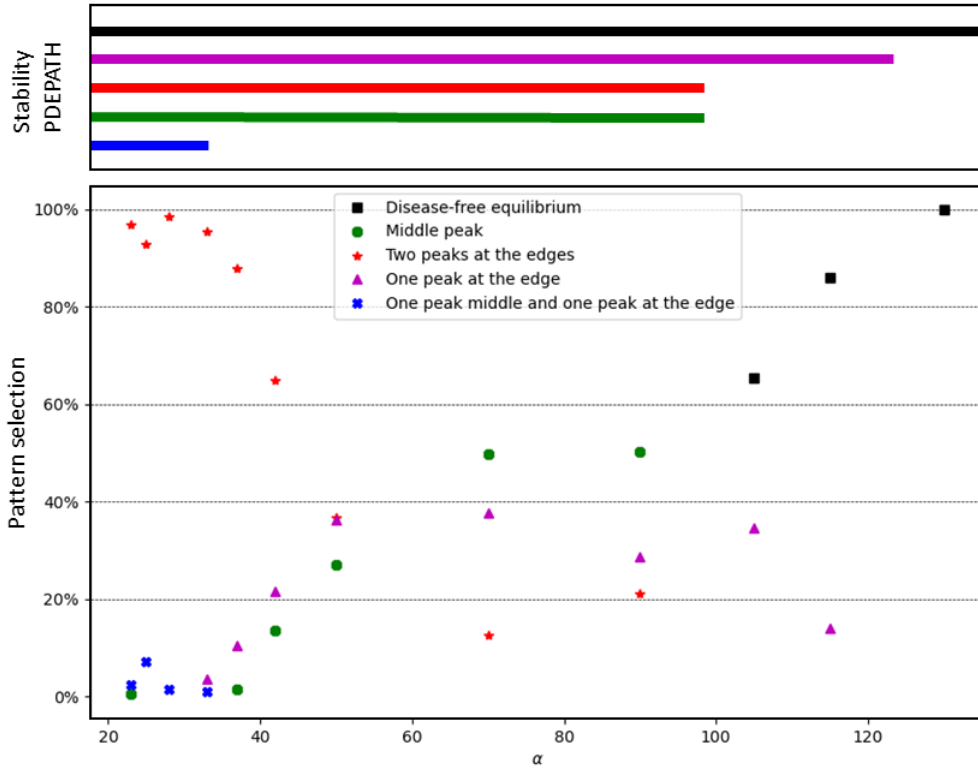


Figure 5: **Pattern Selection for  $d = 0.15$ .** **Top.** Stability of the different branches of stationary patterns obtained using the package PDEPATH for different values of parameter  $\alpha$  (See figure 4), for  $d = 0.15$ . The black branch corresponds to the disease-free equilibrium, this branch is stable for all  $\alpha \in \mathbb{R}$ . **Bottom.** Percentage of obtaining the different stationary patterns following 200 numerical simulations independent of the (1) system. The noisy initial conditions used are described in Remark 1. This process was repeated for different values of parameter  $\alpha$ . The bifurcation diagram shown in figure 4 is consistent with our independent numerical simulations. Other parameter values are given in the Table 1.

ular, we notice that the last branch to lose its stability, loses its stability close to  $\alpha = 22$ , much less than in the previous case.

To determine the probability of selection of these different possible patterns, we perform 50 simulations with a slightly noisy initial condition (see Remark 1), and the results are shown in figure 8. When the branch *Two peaks at the edges* is stable, we always select it even if other branches are also stable. As soon as the red branch loses stability, the blue branch *One peak at the edge*, becomes predominant in the selection of the stationary pattern. But in this range of value of  $\alpha$  the solution can also, with low probability, converge towards the disease-free

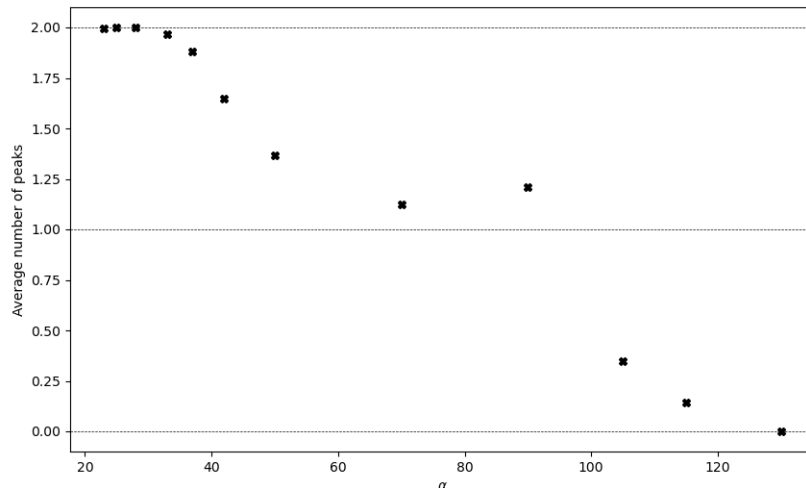


Figure 6: **Average Number of peaks for  $d = 0.15$ .** Average number of amyloid plaques for the final stationary pattern for 200 independent numerical simulations of the system (1), on the domain  $\Omega = [0,40]$ , for different values of the parameter  $\alpha$ . The noisy initial conditions used are described in Remark 1. The other parameter values are given in the Table 1.

equilibrium. According to our bifurcation diagram, once the blue branch loses its stability, no branch associated with a spatially heterogeneous pattern is stable and this result agrees with our numerical simulations of figure 8. For parameter values  $\alpha = 22, 22.5,$  and  $23,$  we systematically select the disease-free equilibrium.

The *Middle peak* structure was never selected in our simulations, despite its stability. Note that the noise is again chosen to be very low. For initial values with a larger variance, the results could differ greatly.

In the top panel of figure 8, we illustrate the average of the peaks of the selected stationary patterns presented in figure 7. As expected, we obtain a decreasing function as a function of  $\alpha,$  which is close to a step function, where the discontinuity comes from the loss of stability of a stable branch. For a larger domain, for example,  $\Omega = [0, 100],$  more patterns are theoretically eligible. In this case, we notice that the decrease in the average of the peaks is less discontinuous than before (see the bottom panel of the figure 8).

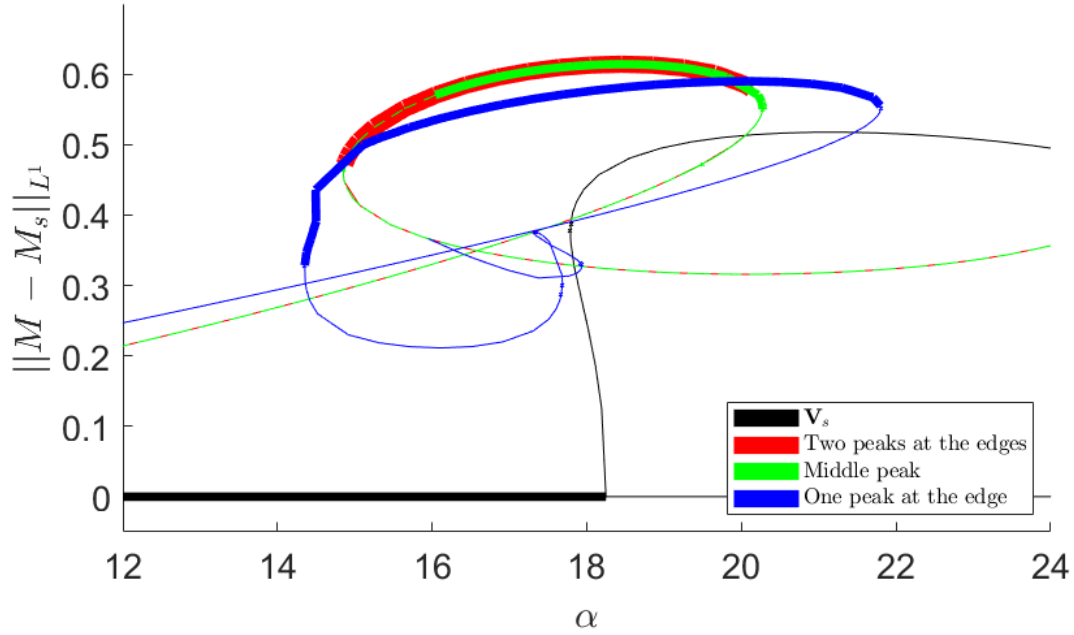


Figure 7: **Bifurcation diagram for  $d = 0.35$ .** This bifurcation diagram is obtained using the package PDEPATH [28, 29]. Thick lines indicate stable branches, while thin lines indicate unstable branches. The horizontal branch in black corresponds to the stationary state  $\mathbf{V}_s$ , which loses its stability in  $\alpha_c$ . For  $d = 0.35$ , this bifurcation point is subcritical. Following several bifurcations, we obtain the presence of other branches, which are stable, and associated with high-amplitude patterns described in the legend on the right. The term  $M_s$  corresponds to the microglial cell concentration component of the positive equilibrium  $\mathbf{V}_s$ . Parameter values are given in Table 1.

- **For  $d = 0.45$ .** We finally study the case  $d = 0.45$ . For this parameter value, we are still in the bistable domain of the model 1, so there is a Turing type bifurcation such that for  $\alpha > \alpha_c$  the stable equilibrium without chemotaxis  $\mathbf{V}_s$ , becomes unstable. This bifurcation gives rise to a branch of spatially heterogeneous patterns. However, according to our bifurcation diagram and independent numerical simulations, this branch is always unstable. So when the equilibrium  $\mathbf{V}_s$  loses its stability, the problem becomes monostable, and the solution of the model (1) converges towards the disease-free equilibrium.

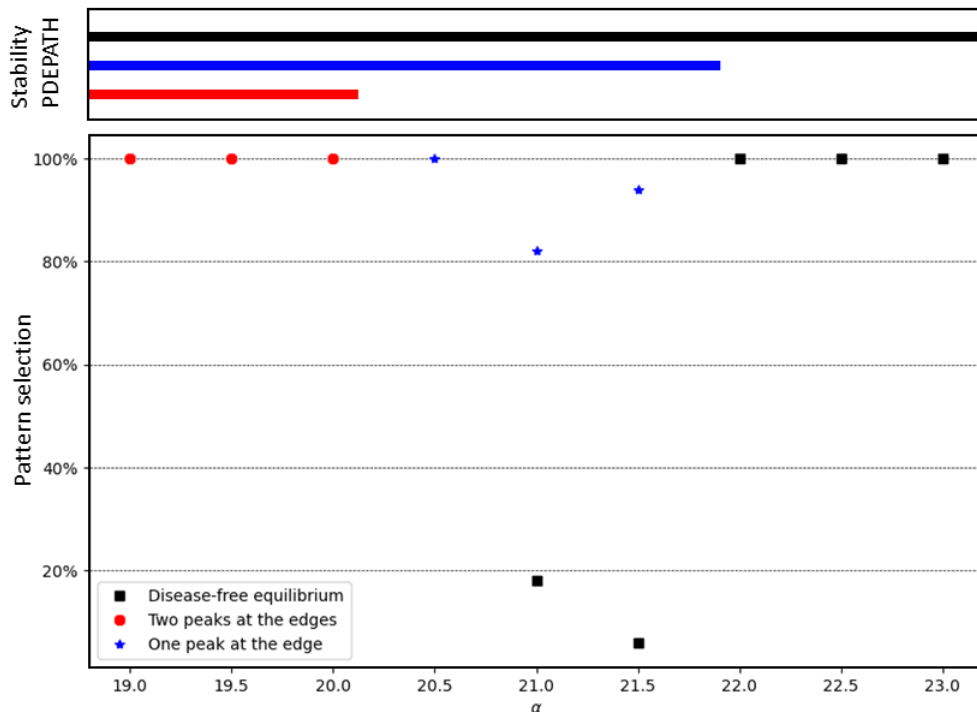


Figure 8: **Pattern Selection for  $d = 0.35$ .** **Top.** Stability of the different branches of stationary patterns obtained using the package PDEPATH for different values of the parameter  $\alpha$  (See figure 7), for  $d = 0.35$ . The black branch corresponds to the disease-free equilibrium, this branch is stable for all  $\alpha \in \mathbb{R}$ . **Bottom.** Percentage of obtaining the different stationary patterns following 50 independent numerical simulations of the system (1). The noisy initial conditions used are described in Remark 1. This process was repeated for different values of parameter  $\alpha$ . The bifurcation diagram shown in figure 4 is consistent with our independent numerical simulations. Other parameter values are given in the Table 1.

### 3.2 Numerical simulation in the two-dimensional case

#### Stripes and dots

We now study spatially heterogeneous patterns in a two-dimensional square domain,  $\Omega = [0, L] \times [0, L]$ . In this case, we can obtain many families such as stripes, dots, and hexagons, or even more complex ones, such as mazes.

Again we notice the presence of low-amplitude and high-amplitude. These low-amplitude,



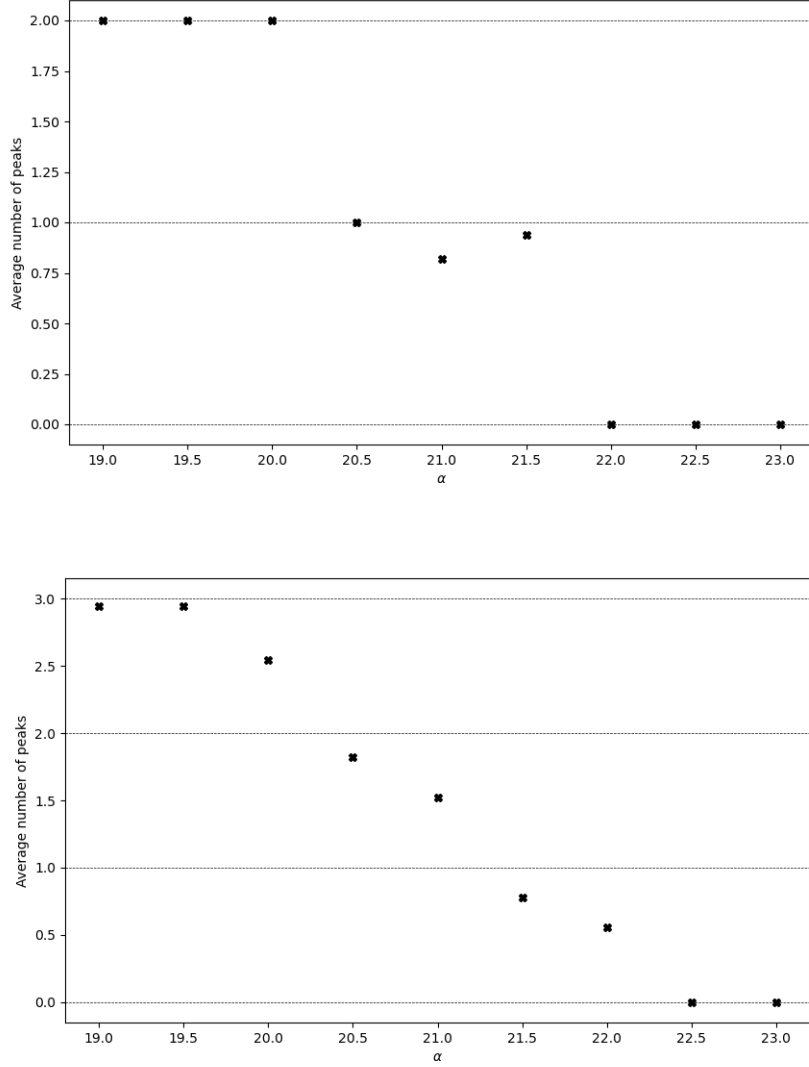


Figure 9: **Average Number of peaks for  $d = 0.35$ .** Average number of amyloid plaques for the final stationary pattern for 50 independent numerical simulations of the system (1), for different values of the parameter  $\alpha$  and for the following domain: **Top.**  $\Omega = [0, 40]$  and **Bottom.**  $\Omega = [0, 100]$ . The noisy initial conditions used are described in Remark 1. The other parameter values are given in the Table 1.

correspond to weak perturbations of the positive steady state  $\mathbf{V}_s$  and exist only for  $\alpha$  very close to the critical threshold  $\alpha_c$ . Note that the weakly nonlinear analysis is a tool to theoretically determine the family of low-amplitude [41, 42]. However, as in the previous section, we focus only

on high amplitude. Our numerical simulations reveal several possible high amplitudes, including stripes and dots. These two types are illustrated in figure 10 and can coexist for certain ranges of parameter values. Note also that we obtain complex stationary patterns corresponding to a structure between dots and stripes. For a large enough  $\alpha$ , we only notice the presence of a dot. Subsequently, we focus on those which are more reasonable in terms of modeling. Indeed, biologically, amyloid plaques form spots distributed heterogeneously in space (see figure 1).

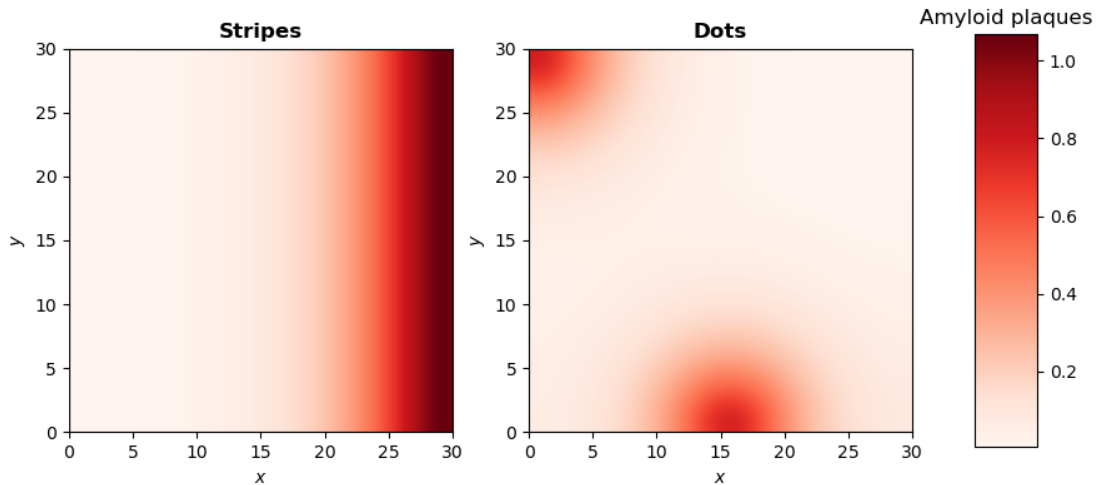


Figure 10: **Examples of high-amplitude final stationary patterns for the system (1).** **Top.** Stripe pattern. **Bottom.** Dot pattern. The parameter values are those shown in the Table 1, with  $d = 0.15$  and  $\alpha = 24$ .

### Pattern selection

In the two-dimensional case, there are many more possible structures. Therefore, it is difficult to detect all stable branches using a continuation method. In particular, there is a good chance of forgetting branches that are difficult to select but theoretically stable. Thus for the square domain  $\Omega = [0, L] \times [0, L]$  we only work with successive independent numerical simulations.

In figure 11 we plot the evolution of the average number of peaks for 50 independent simulations of the model (1). The choice of the initial conditions is detailed in Remark 1. Since it is impossible to quantify the number of peaks for stripe patterns, we study the average only for  $\alpha > 25$ . For these values of  $\alpha$  we always obtained stationary dot patterns. As for the study in

the one-dimensional case, presented above, the average number of peaks tends to decrease when the chemotaxis parameter increases. From a certain stage, this average number of peaks is equal to 0. These results are in agreement with the Conjecture 2.

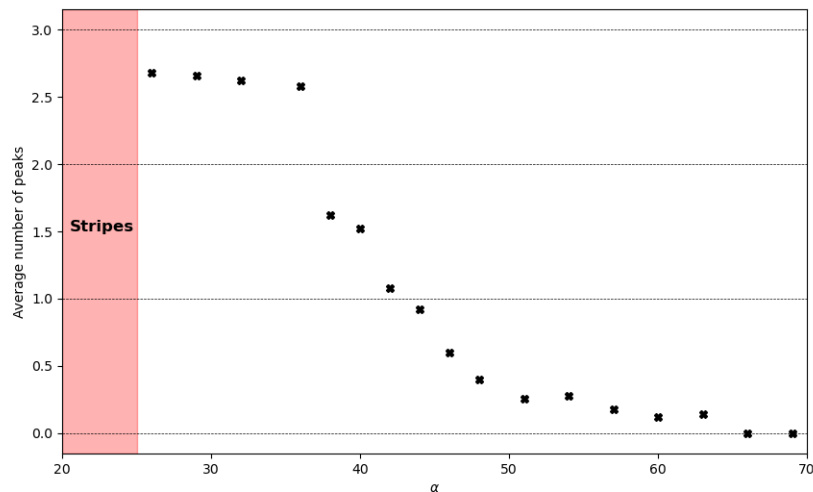


Figure 11: **Average Number of peaks for  $d = 0.15$  and  $\Omega = [0, 40] \times [0, 40]$ .** Average number of amyloid plaques for the final stationary pattern for 50 independent numerical simulations of the system (1), for different values of the parameter  $\alpha$ . For  $\alpha > 25$ , we did not obtain any stripes pattern for our various numerical simulations. The noisy initial conditions used are described in Remark 1. The other parameter values are given in the Table 1.

### 3.3 Parameter value ranges without amyloid plaque

For modeling reasons, it is interesting to determine the zones of parameters for which it is *a priori* impossible to observe the emergence and persistence of amyloid plaques. In other words, for the initial conditions presented in Remark 1, we want to determine a threshold  $\alpha_0$  such that for any  $\alpha > \alpha_0$ , the solution of the system (1) converges to the disease-free equilibrium independently of the noise of the initial condition.

We approach this threshold  $\alpha_0$  using numerical simulations. We discretize the possible values of  $\alpha$  and look for the smallest possible point such that for any  $\alpha$  greater than this point the  $n$  independent simulations have led to convergence to the disease-free equilibrium. Our algorithm first explores a possible candidate and then refines this candidate to obtain a precise approxima-

tion of the threshold  $\alpha_0$ . We apply this algorithm for different values of parameter  $d$  and obtain an approximation of the threshold  $\alpha_0^{\text{exp}}(d)$ . For  $d < d_m$ , and if we assume that  $d$  is large enough, the threshold  $\alpha_0$  is equal to the critical threshold  $\alpha_c$ . Conversely, for  $d$  small enough, we observe a strong increase in the  $\alpha_0$  threshold.

In the one-dimensional case, we represent the approximation obtained using our algorithm,  $\alpha_0^{\text{exp}}(d)$ , by the dotted red line marked by the red arrow in figure 12. In the bistable domain, we note that this curve separates two distinct parts. The first part corresponds to parameter values for which the solutions of our  $n$  consecutive independent simulations have converged toward the disease-free equilibrium. We therefore assume that for these parameter values, we always converge to the disease-free equilibrium. Note that in theory, this assumption could turn out to be wrong, especially for small parameter values  $d$  where the threshold seems very high. We denote  $\mathbf{P}_d(\alpha)$  the probability that the solution does not converge to the equilibrium without disease. Even if this probability is positive for any  $\alpha > 0$ , we theoretically get the existence of  $\alpha_0^{\text{exp}}$ . Nevertheless, the bifurcation diagrams presented in section 3.1 show that pattern branches lose their stability for a large enough  $\alpha$ ; this motivates and justifies the choice of such an assumption. From a modeling point of view, this zone of parameter values corresponds to a zone where, after a certain time  $t$ , no amyloid plaques are present.

In the second part, shown in red, there are several possible phenomena. In particular for  $\alpha < \alpha_c$ , for very slightly noisy initial conditions around  $\mathbf{V}_s$ , we converge to the positive homogeneous stationary state  $\mathbf{V}_s$ . When  $\alpha$  is between  $\alpha_c$  and  $\alpha_0^{\text{exp}}$ , there are plenty of possibilities, including convergence to a spatially heterogeneous stationary pattern, convergence to a time oscillating pattern and convergence to the disease-free equilibrium. Nevertheless, according to our numerical simulations, we obtained at least once a solution that is not the disease-free equilibrium.

In the monostable domain ( $d > d_m$ ), the positive equilibrium  $\mathbf{V}_s$  does not exist. For this range of parameter values, only the disease-free equilibrium is stable. Consequently, we converge, regardless of the chemotaxis parameter value and independently of the noise of the initial condition, to this disease-free equilibrium. In the two-dimensional case, *i.e.* for  $\Omega = [0, L] \times [0, L]$  we obtain a similar result, which is represented in the figure 16 in Appendix A.

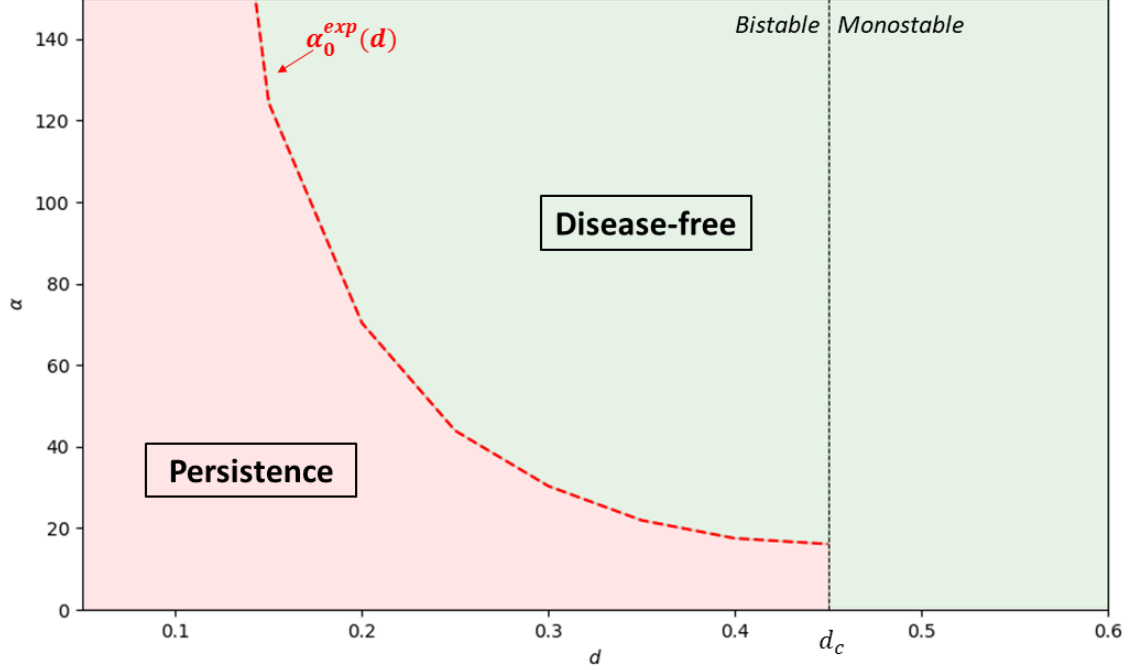


Figure 12: **Asymptotic behavior of the solutions of system (1) as a function of the degradation of monomers  $d$  and the chemotaxis parameter  $\alpha$ , for the domain  $\Omega = [0,40]$ .** For  $d > d_m$ , there is only one equilibrium which is the disease-free equilibrium. From the stability of this equilibrium, the solutions of the system converge towards it. For  $d < d_m$ , it is possible to converge to the disease-free equilibrium, the positive equilibrium  $\mathbf{V}_s$ , and a spatially heterogeneous (possibly time-oscillating) pattern. The red zone represents an area of parameter values for which we have noticed, at least once, convergence towards a stationary solution other than the disease-free equilibrium. The green zone corresponds to parameter values for which we systematically converge towards the disease-free equilibrium. This result is highly dependent on the chosen initial conditions. The choice of the noisy initial condition is described in Remark 1.

### 3.4 Influence of the initial data

We assume that the initial concentrations of monomers, oligomers, and microglial cells are initially uniformly distributed in space and there is a complete absence of amyloid plaques. In this section, we investigate the influence of initial inflammation on the final pattern. The initial data is provided in Table 2.

Table 2: **Initial data used in Section 3.4**

Parameter	Value	Units	Description
$u_0$	$10^{-4}$	$\text{mol } L^{-1}$	Concentration of free oligomers
$u_p$	0	$\text{mol } L^{-1}$	Concentration of oligomers in the amyloid plaques
$m_0$	$10^{-3}$	$\text{mol } L^{-1}$	Concentration of monomers
$M_0$	1	$\text{mol } L^{-1}$	Concentration of microglial cells
$I_0$	Variable	$\text{mol } L^{-1}$	Concentration of interleukins

### 3.4.1 Initial homogeneous inflammation in space

First, we assume that the inflammation, corresponding to the interleukin concentration, is initially uniformly distributed in space with a certain quantity  $I_0$  that we vary.

In figure 13, we follow a similar approach as previously done to identify parameter regions where the persistence of amyloid plaques is not observed. Here, we examine the influence of monomer degradation  $d$  and the initial value of interleukin concentration  $I_0$ . The algorithm used to obtain this figure is similar to the one presented in Section 3.3. In the monostable case ( $d > d_m$ ), where the only equilibrium is the disease-free equilibrium, all the solutions of the system (1) converge to this equilibrium, while in the bistable case ( $d < d_m$ ), the steady-state solution can be either homogeneous or spatially heterogeneous. The green region corresponds to parameter values for which the steady-state solution of the model (1) coincides with the disease-free equilibrium. In contrast, the red region represents other possibilities, namely a homogeneous solution equal to the other positive equilibrium or a spatially heterogeneous solution. The dichotomy observed in figure 13 is similar to the results presented in the article [21].

### 3.4.2 Initial heterogeneous inflammation in space

We now assume that the initial concentration of interleukins is heterogeneous in space. More precisely, we assume that the interleukin concentration is initially distributed in three Gaussians evenly spaced across the domain  $[0, 40]$ . Mathematically, the initial data is defined by the following family of functions,

$$I_0^\eta(x) = \frac{1}{\eta\sqrt{2\pi}}e^{-\frac{(x-10)^2}{2\eta^2}} + \frac{1}{\eta\sqrt{2\pi}}e^{-\frac{(x-20)^2}{2\eta^2}} + \frac{1}{\eta\sqrt{2\pi}}e^{-\frac{(x-30)^2}{2\eta^2}}, \quad (12)$$

dependent on the variable  $\eta > 0$ , which corresponds to the variance of the Gaussians. Note that the initial mass of the interleukin concentration,  $\int I_0$ , is independent of the variable  $\eta$ .

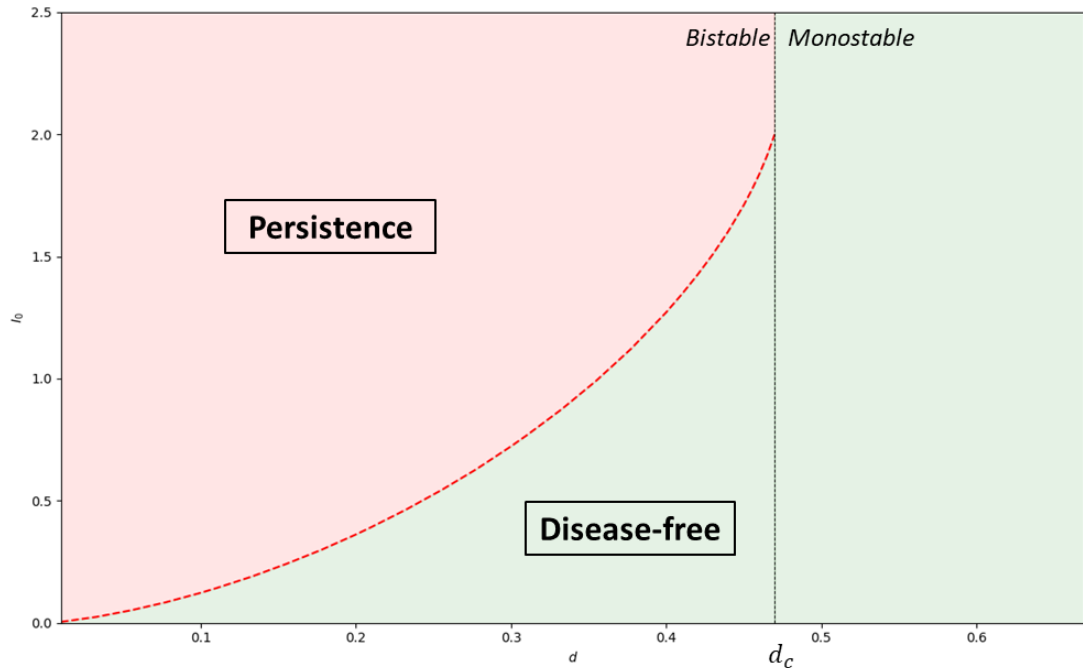


Figure 13: **Asymptotic behavior of the solutions of system (1) as a function of the degradation of the monomers  $d$  and the initial concentration of interleukin  $I_0$ .** For  $d > d_m$ , there is only one equilibrium which is the disease-free equilibrium. By stability of this equilibrium, the solutions of the system converge towards this steady state. For  $d < d_m$  it is possible to converge to the disease-free equilibrium, the positive equilibrium  $\mathbf{V}_s$ , and a spatially heterogeneous (possibly time-oscillating) pattern. The red zone represents an area of parameter values for which we have noticed at least one convergence towards a stationary solution other than the disease-free equilibrium. The green zone corresponds to parameter values for which we systematically converge towards the disease-free equilibrium. This result is very similar to the one obtained in the article [21]. The other parameter values are given in Table 1 and the initial data is provided in Table 2.

In figure 14, we investigate the impact of the variance  $\eta$  on the potential persistence of inflammation. In the left panel, we illustrate four different initial data of interleukin concentration, each associated with different variances, according to the formula (12). In the right panel, we depict the final steady-state solution associated with the corresponding initial data on the left. When the variance  $\eta$  is small, the initial data corresponds to very narrow Gaussian curves, re-

sulting in highly concentrated inflammation in small spatial regions. In this case, we observe that the solutions of the model (1) converge towards the disease-free equilibrium. Conversely, when the variance  $\eta$  is larger, the initial inflammation is more dispersed than before but with less amplitude. We notice that in this case the steady-state solutions of the model (1) correspond to spatially positive heterogeneous solutions and the initial inflammation leads to a persistent inflammation over time.

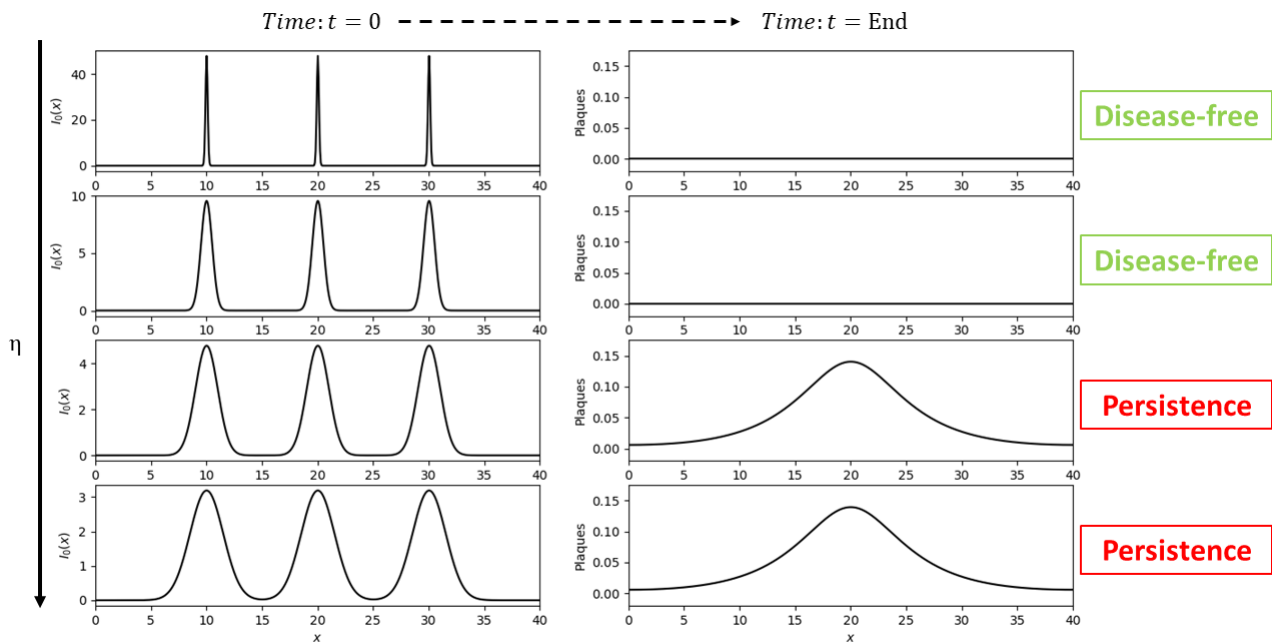


Figure 14: **Steady-state solutions of the model (1) for various initial data corresponding to sums of Gaussians.** **Left.** Different initial data for interleukin concentration given by the equation (12), associated with different variances from lower ones (Top) to higher ones (Bottom). **Right.** Steady-state solutions of the model (1) associated with the initial data from the left panel. The parameter values are listed in Table 1, and the remaining initial data is given in Table 2.

The result presented in figure 14 is in some sense quite similar to the proposition outlined in Conjecture 1. As  $\alpha$  increases, heterogeneous steady-state solutions are more prone to converge to asymptotic Dirac masses. In other words, as  $\alpha$  grows, the heterogeneous spatial concentration of interleukins becomes increasingly strong but also more concentrated in small regions of space, leaving a significant portion of space with an almost negligible interleukin density. In the previous sections, through numerical simulations, we proved that these spatially heterogeneous solutions



lose their stability when  $\alpha$  becomes excessively large ( $\alpha > \alpha_0$ ), meaning when inflammation is too concentrated in a small fraction of space.

In terms of modeling, it is, therefore, simpler to eliminate inflammation if it is highly concentrated in a small part of the space, preserving a large part of the healthy tissue, rather than to treat milder inflammation spread over a wider area.

## 4 Conclusion and perspectives

In this work, we have extended the research presented in the article by Ciuperca *et al.* [21] by studying the model (1) with its spatial components. Our linear analysis shows that the equilibrium point  $\mathbf{V}_s$  is unstable to spatially heterogeneous perturbations when  $\alpha > \alpha_c$ , with  $\alpha_c$  being a critical threshold corresponding to a root of the function defined by (11). To study pattern formation, we plotted numerous bifurcation diagrams using the PDEPATH package, revealing a wide variety of patterns. We also performed numerous independent numerical simulations to ensure that all branches were found. With this complete bifurcation diagram, we can predict interesting modeling results. For instance, according to our model, the higher the coefficient  $\alpha$ , corresponding to the intensity of the attraction of microglial cells to oligomers, the fewer plaques are expected. Additionally, when  $\alpha$  is sufficiently large, neither plaques nor inflammation are expected. This phenomenon also depends on other parameters; we specifically study the influence of the parameter  $d$ , representing the natural degradation rate of monomers.

However, to confirm or refute these predictions, it would be necessary to obtain brain images at different stages of plaque formation dynamics, as well as methods to modify parameters present in our system, such as the administration of anti-inflammatory drugs.

In our diagrams, we have highlighted that for sufficiently large values of  $\alpha$ , all pattern branches become unstable. This unusual behavior appears to be a consequence of the bistability of our model [26]. Similar results, showing a total loss of branch stability, were presented in the article by Al-Karkhi *et al.* [27]. Additionally, our bifurcation diagrams reveal that the loss of branch stability strongly depends on their associated pattern: the higher the number of peaks, the faster the loss of stability. It would be particularly interesting to theoretically study how bistability can be responsible for such phenomena.

The model (1) introduced in the article [21] is a simplified subsystem of a more complex model,

also introduced in the same article, which also models proto-oligomer structures of different sizes. Therefore, it would be interesting to study whether our results extend to the more general model. Additionally, the model was initially introduced for a study on a bounded open domain  $\Omega \subset \mathbb{R}^3$ . In this work, we performed analyses only in one and two dimensions, as these dimensions are simpler to handle. A three-dimensional analysis, although more complex, would be particularly interesting and is also made possible by the Matlab package PDEPATH (see, for example, [43]).

A direct application of our model is to study the effects of different therapeutic strategies. In this work, we have highlighted that microglial cells play a key role in plaque aggregation, particularly through the influence of the coefficient  $\alpha$ . Therefore, it would be interesting to identify factors, such as molecules, able to influence plaque structuring via microglial cells.

Anti-inflammatory treatments can also be studied using our model. As proven in the article [21], we show that initial inflammation may or may not lead to plaque formation. We also highlight that inflammation concentrated in a specific tissue area does not have the same impact as more diffuse inflammation. Given the strong multistability observed in this model, it is obvious that the administration of anti-inflammatory drugs induces transitions between different stable branches. Anti-inflammatory treatment has already been tested for Alzheimer's disease. In the article by Rivers-Auty *et al.* [44], the authors indicated that diclofenac could slow cognitive decline and offer promising prospects for the progression of Alzheimer's disease. However, rigorous clinical trials have failed to confirm the efficacy of these anti-inflammatory treatments [45, 46, 47]. In contrast, other studies suggest that the effectiveness of these treatments could be significant if administered well before the onset of clinical symptoms [48, 49].

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## A Appendix: Supplementary Figures

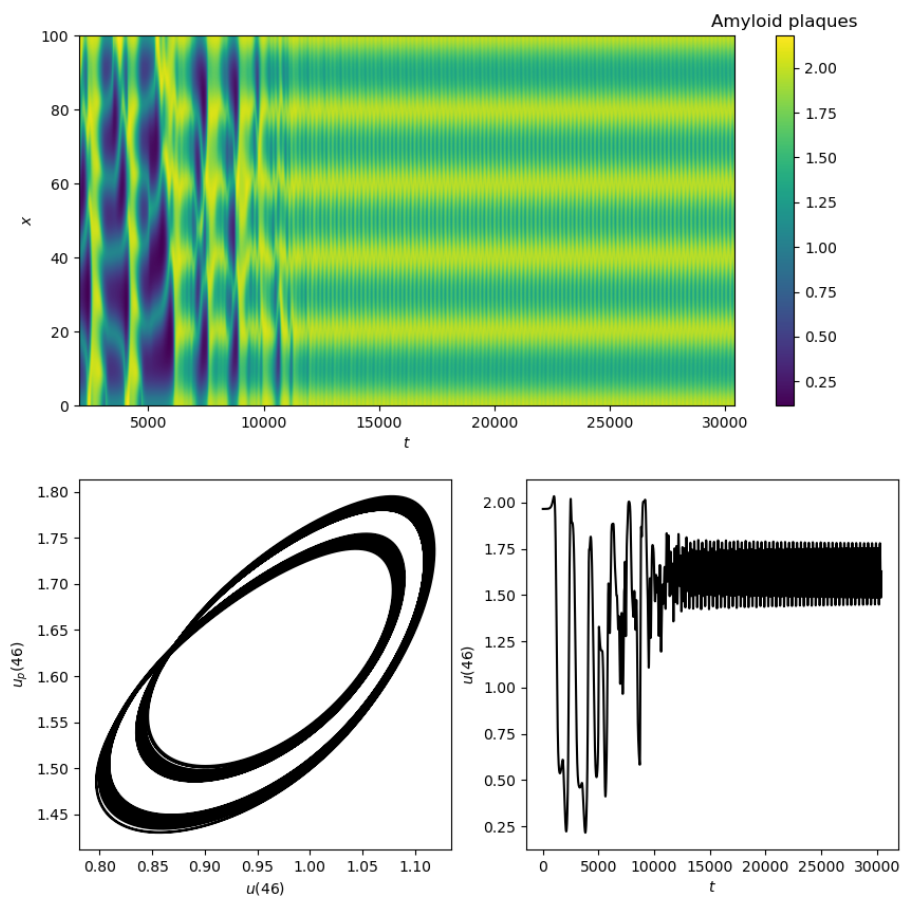


Figure 15: **Heterogeneous patterns in space with quasi-periodic oscillations in time.** **Top.** Concentration of amyloid plaques over time. **Bottom-left.** Evolution over time of the concentration of oligomers at  $x = 46$  as a function of the concentration of amyloid plaques also at  $x = 46$ . **Bottom-right.** Concentration of oligomers at  $x = 46$  over time.

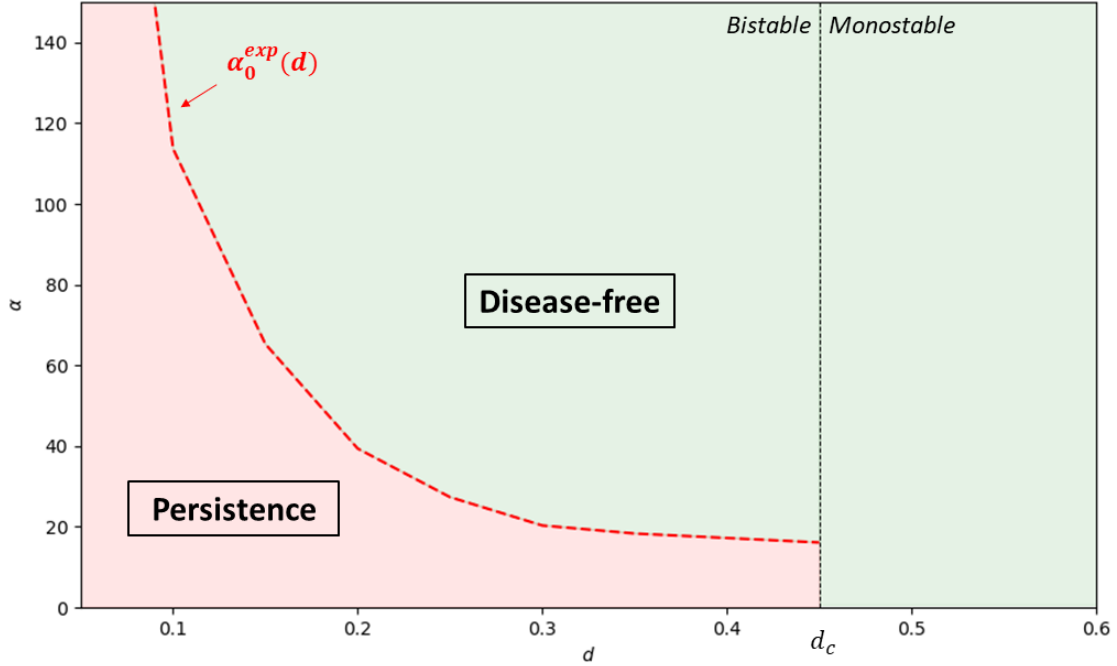


Figure 16: **Asymptotic behavior of the solutions of system (1) as a function of the degradation of the monomers  $d$  and the chemotaxis parameter  $\alpha$ , for the domain  $\Omega = [0, 30] \times [0, 30]$ .** For  $d > d_m$ , there is only one equilibrium which is the disease-free equilibrium. By stability of this equilibrium, the solutions of the system converge towards it. For  $d < d_m$ , it is possible to converge to the disease-free equilibrium, the positive equilibrium  $\mathbf{V}_s$ , and a spatially heterogeneous (possibly time-oscillating) pattern. The red zone represents an area of parameter values for which we have noticed, at least once, convergence towards a stationary solution that is not the disease-free equilibrium. The green zone corresponds to parameter values for which we systematically converge towards the disease-free equilibrium. This result is highly dependent on the chosen initial conditions. The choice of the noisy initial condition is described in Remark 1.

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