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# Creating a bridge between Probabilities and fuzzy sets and its impact on Drought Severity Assessment

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**Abstract.** In hydrology and water resources management problems the theoretical probability distribution functions are widely used with the aim of the empirical probability function. However, it is difficult to exploit the probability functions in case that algebraic operations between random variables are required. A solution should be the motivation from the probability functions to fuzzy sets by using the fuzzy estimators. Finally based on the possibility theory the authors conclude that based on a probability distribution, a possibility distribution with the maximum specificity can be produced, that is near to the probability measure. The Reconnaissance Drought Index (RDI) was proposed to assess meteorological drought severity based on the precipitation to potential evapotranspiration ratio (P/PET). However it is difficult to express the bivariate probability density function for this ratio. Hence based on the fuzzy estimators, the analysis can be concluded to fuzzy sets, and the extension principle of fuzzy sets can provide the required ratio as fuzzy sets.

**Keywords:** Reconnaissance Drought Index (RDI), Drought, Theoretical probability density function, normal distribution, possibility distribution, fuzzy estimators, consistency principle, maximal specificity,

## 1 Introduction

In hydrology and water resources management problems the theoretical probability distribution functions are widely used. An advantage of the probabilistic approach, as a choice to deal with the uncertainty, is the exploitation of the cumulative empirical (observed) probability distribution in order to test the goodness-of-fit for an examined theoretical probability distribution with respect to the historical sample [1].

The use of theoretical probability distribution instead of the empirical function arises from the fact that historical sample contains no many years (e.g. 40 years in Greece) whilst an event with higher return period can be included within the sample and hence, finally, probability density function are used.

However, it is difficult to exploit the probability functions in case that algebraic operations between random variables are required. This is very useful, since the probability associated with the precipitation to potential evapotranspiration ratio (P/PET) can provide a useful information about the severity of drought.

The Reconnaissance Drought Index (RDI) can be characterized as a general meteorological index for drought assessment ([2],[3], [4]) with many applications. Compared with the SPI index it incorporates both precipitation and potential evapotranspiration, which are directly affected by climate change [3]. Also a strong advantage of RDI is that it offers a rational comparison of drought conditions between areas with different climatic characteristics [4]. Vangelis et al., 2011 [4] proposed a rather probabilistic approach to characterize the drought whilst the majority of the Reconnaissance Drought Index (RDI) applications used mainly simple algebraic operations. The approach of [4] can be applied only in case that normal distribution is used.

In this work a correspondence between the fuzzy sets and theoretical probability function is proposed and furthermore, this assumption is applied in order to estimate the severity of drought based on both the precipitation and the potential evapotranspiration.

Compared with work of Papadopoulos et al., 2021 [5] instead of the estimation of the mean and the standard deviation of the examined hydrological variable, here the direct transformation from probabilistic to fuzzy sets is developed.

## 2 Fuzzy methodology

### 2.1 Fuzzy sets

In general, if  $A$  is a function from  $U$  into the interval  $[0,1]$ , then  $A$  is called a fuzzy set.

$A$  is convex if and only if, for every  $t \in [0,1]$  and  $x_1, x_2 \in X$  it holds:

$$A(tx_1 + (1-t)x_2) \geq \min \{A(x_1), A(x_2)\} \quad (1)$$

$A$  is normalized if there exists  $x \in X$ , such that  $A(x) = 1$ . If  $A$  is a fuzzy set, by  $\alpha$ -cuts  $a \in (0,1]$  we define the crisp sets:

$$A[\alpha] = \{x \in X : A(x) \geq \alpha\} \quad (2)$$

Considering the 0-cut, this can be defined as previously (Equation 2), without the equality, that is, the zero-cut contains all the elements of the general set  $X$ , which have a membership function greater than zero.

A special kind of fuzzy sets is the fuzzy numbers. The definition of fuzzy numbers can be found in Klir and Yuan, 1995 [6]. It is proved that the membership function of a fuzzy number can be expressed as:

$$A(x) = \begin{cases} 0 & \text{for } x < \omega_1 \\ A_L(x) & \text{for } \omega_1 \leq x \leq \alpha_1 \\ 1 & \text{for } \alpha_1 \leq x \leq \alpha_2 \\ A_R(x) & \text{for } \alpha_2 \leq x \leq \omega_3 \\ 0 & \text{for } x > \omega_3 \end{cases} \quad (3)$$

where  $A_L : [\omega_1, \alpha_1] \rightarrow [0, 1]$  and  $A_R : [\alpha_2, \omega_3] \rightarrow [0, 1]$  are the left and right membership functions of the fuzzy number  $A$ . In addition,  $A_L$  is increasing and continuous from the right, and  $A_R$  is decreasing and continuous from the left [6].

The interval  $[\alpha_1, \alpha_2]$  can be an interval or a point but it can not be an empty set .

Let now  $A$  and  $B$  denote fuzzy numbers and let  $*$  denote any of the four basic arithmetic operations. Then, we define a fuzzy set on  $\mathfrak{R}$ ,  $A * B$ , by defining its  $\alpha$ -cut,  $(A * B)[a]$  as:

$$(A * B)[a] = A[a] * B[a] \quad \text{for any } a \in [0, 1]. \quad (4)$$

Among the binary arithmetic operations between the  $\alpha$ -cuts, the interval arithmetic is applied. Here, from the fuzzy algebra we use the division and the subtraction operations

$$[\alpha, \beta] - [c, d] = [\alpha - d, \beta - c] \quad (5)$$

$$[\alpha, \beta] / [c, d] = [\alpha, \beta] \cdot \left[ \frac{1}{d}, \frac{1}{c} \right] \quad (6)$$

and

$$[\alpha, \beta] \cdot [c', d'] = [\min(\alpha c', \alpha d', \beta c', \beta d'), \max(\alpha c', \alpha d', \beta c', \beta d')] \quad (7)$$

Finally, in conjunction with the fuzzy decomposition theorem, the following equation holds for all the fuzzy sets of the fuzzy operation (e.g. [6], [7]):

$$A * B = \cup_{\alpha} (A * B) \quad (8)$$

In fact, we select a significant discrete number of  $\alpha$ -cuts, and thus, the Equation (4) can be effectively approximated.

The question between the fuzzy sets and its relation with the conventional probability theory can be found in the field of possibility theory.

## 2.2 Fuzzy estimators

Let  $X$  be a variable which takes values in a universe  $U$  and  $N$  a fuzzy set of  $U$ . Then the truth value of the fuzzy proposition "X is N" when  $X=u, u \in U$  is defined as the value  $\mu_N(u)$  of the membership value of the fuzzy set  $N$  (see [8]),

$$T("X \text{ is } N|X=u") = \mu_N(u).$$

Therefore, the fuzzy proposition "X is N" associates the variable  $X$  with a possibility distribution. The possibility distribution function associated with  $X$  is denoted by  $\pi_X$  and is defined to be the membership function  $\mu_N$  of  $N$ .

$$\pi_X \equiv \mu_N$$

So, the possibility  $\pi_X(u)$  that  $X=u$  is postulated to be equal to the value  $\mu_N(u)$  of the membership function of  $N$  at  $u$ ,

$$\pi_X(u) = \mu_N(u)$$

**Definition 1** The possibility measure or simply the possibility of a subset  $A \subset U$  for a possibility distribution  $\Pi_X$  associated with the variable  $X$  with universe  $U$  is defined as the supremum of the possibilities of its elements (see [8], [9])

$$\pi_X(A) = \sup\{\pi_X(u), u \in A\} \quad (9)$$

In the case of finite sets, the possibility measure is the maximum of the possibilities of its elements

$$\pi_X(A) = \max\{\pi_X(u), u \in A\} \quad (10)$$

The degree of necessity of  $A$  for the possibility distribution  $\Pi_X$  is defined as (see [9]),

$$Ness_X(A) = 1 - \pi_X(A') \quad (11)$$

According to Zadeh [10] from this definition follows that  $Ness_X(A)$  is a measure of its "certainty". According to [9]:

**Proposition 1** For a continuous possibility distribution  $\Pi_X$  for which

$$\pi_X(u) = 1 \leftrightarrow u = u_0, u_0 \in U$$

(the possibility distribution function  $\pi_X$  is a triangular shaped fuzzy number), the degree of necessity of the  $\alpha$ -cut  $\pi_X[\alpha]$  of the possibility distribution function  $\pi_X$ , is  $1 - \alpha$ ,

$$Ness_X(\pi_X[\alpha]) = 1 - \alpha, \alpha \in [0,1] \quad (12)$$

According to [9]:

**Definition 2** A possibility distribution  $\Pi_X$  for a variable  $X$  is defined as consistent with the probability distribution  $\Pi_X$  of  $X$ , if and only if the possibility  $\pi_X(A)$  of any subset  $A$  of the universe  $U$  of  $X$  is greater or equal to its probability  $p_X(A)$ ,

$$\pi_X(A) \geq p_X(A), \forall A \subseteq U \quad (13)$$

This inequality is referred as consistency principle.

**Definition 3** A possibility distribution  $\Pi_X^*$  consistent with the probability distribution  $P_X$  is defined as maximally specific if it is more specific than any possibility distribution  $\Pi_X$  consistent with the probability distribution  $P_X$ , that is, if

$$\pi_X^*(x) < \pi_X(x), \quad \forall x \in U \quad (14)$$

If the possibility distribution  $\Pi_X$  is consistent with the probability distribution  $P_X$ , then according to (13) the possibility  $\pi_X(A')$  of the complement  $A'$  of any subset  $A$  of  $U$  is greater or equal to the probability of  $A'$ , so because of (11)

$$p_X(A') \leq \pi_X(A') \leftrightarrow 1 - p_X(A) \leq 1 - \text{Ness}_X(A) \leftrightarrow p_X(A) \geq \text{Ness}_X(A) \quad (15)$$

Therefore because of (13):

**Proposition 2** For a possibility distribution  $\Pi_X$  consistent with the probability distribution  $P_X$  of a variable  $X$ , the probability  $p_X(A)$  of any subset  $A$  of the universe of  $X$  is greater or equal to its necessity  $\text{Ness}_X(A)$  and less or equal to its possibility,

$$\text{Ness}_X(A) \leq p_X(A) \leq \pi_X(A) \quad (16)$$

From Proposition 1 and (16) follows that:

**Proposition 3** If the possibility distribution function  $\pi_X$  of a possibility distribution  $\Pi_X$  consistent with the probability distribution  $P_X$  is a triangular shaped fuzzy number, then the probability of its  $\alpha$ -cut is greater or equal to  $1 - \alpha$ ,

$$P(\pi_X[\alpha]) \geq 1 - \alpha \quad (17)$$

so the  $\alpha$ -cuts of  $\pi_X$  are confidence intervals of  $X$  of degree of confidence greater or equal to  $1 - \alpha$ .

Let  $X$  a continuous random variable with universe  $U$ , unique mode  $m$ , probability density  $p_X(u)$ , symmetric about  $m$  and distribution function  $F_X$  and  $\tilde{X}^* \subseteq U$  a fuzzy subset of  $U$  with membership function  $\mu_{\tilde{X}^*}(u)$ ,  $u \in U$ , the  $\alpha$ -cuts of which are intervals in which the probability of a value of  $X$  is  $1 - \alpha$ . If  $F_X^{-1}(\alpha)$ ,  $0 \leq \alpha \leq 1$  the inverse distribution function of  $X$ , then it holds that [11]:

$$P\left(F_X^{-1}\left(\frac{\alpha}{2}\right) < X < F_X^{-1}\left(1 - \frac{\alpha}{2}\right)\right) = F_X\left(F_X^{-1}\left(1 - \frac{\alpha}{2}\right)\right) - F_X\left(F_X^{-1}\left(\frac{\alpha}{2}\right)\right) = 1 - \frac{\alpha}{2} - \frac{\alpha}{2} = 1 - \alpha \quad (18)$$

Therefore the  $\alpha$ -cuts of the fuzzy set  $\tilde{X}^*$  are

$$X^*[\alpha] = \left[F_X^{-1}\left(\frac{\alpha}{2}\right), F_X^{-1}\left(1 - \frac{\alpha}{2}\right)\right], \quad 0 \leq \alpha \leq 1 \quad (19)$$

According to [9]:

**Proposition 4** *The possibility distribution  $\Pi_{\tilde{X}^*}$  induced by the fuzzy proposition "X is  $X^*$ ", the possibility distribution function of which  $\pi_{X^*}(x)$  is the membership function  $\mu_{X^*}(x)$  of  $\tilde{X}^*$  with  $\alpha$ -cuts given in (19), is consistent with the probability distribution  $P_X$ , that is, it satisfies the consistency principle*

$$\pi_{X^*}(A) > p_X(A), \quad \forall x \in U \quad (20)$$

*The fuzzy set  $\tilde{X}^*$  is called fuzzy estimator of  $X$ .*

Also, if the probability density  $p_X(x)$  of  $X$  is symmetric about the mode, then the intervals of (18) are the shortest intervals in which the probability to find a value of  $X$  is  $1 - \alpha$ . Therefore, for any possibility distribution  $\Pi_X$  consistent with the probability distribution  $P_X$  it is true that

$$\pi_X(x) > \pi_{X^*}(x), \quad \forall x \in U \quad (21)$$

In this case,  $\Pi_{\tilde{X}^*}$  is the most specific possibility distribution consistent with the probability distribution  $P_X$  and the fuzzy set  $\tilde{X}^*$  is called *fuzzy estimator of maximal specificity* of  $X$ . Therefore, the triangular shaped fuzzy number  $\tilde{X}^*$  which is produced putting one above the other the confidence intervals in which a value of  $X$  is found with a given probability is estimator (of maximal specificity) of  $X$ .

It is true that ( $F_X(x)$  the distribution function of  $X$ )  
for  $x \leq m$ ,

$$x = F_X^{-1}\left(\frac{\alpha}{2}\right) \leftrightarrow \frac{\alpha}{2} = F_X(x) \leftrightarrow \alpha = 2F_X(x) \quad (22)$$

for  $x > m$ ,

$$x = F_X^{-1}\left(1 - \frac{\alpha}{2}\right) \leftrightarrow 1 - \frac{\alpha}{2} = F_X(x) \leftrightarrow \alpha = 2(1 - F_X(x)) \quad (23)$$

so the membership function of the fuzzy set  $\tilde{X}^*$  is

$$\mu_{\tilde{X}^*}(x) = \begin{cases} 2F_X(x), & x \leq m \\ 2(1 - F_X(x)), & x > m \end{cases} \quad (24)$$

where  $F_X(x)$  the distribution function and  $m$  the mode of  $X$ . Therefore:

**Proposition 5** *The membership function of the fuzzy estimator  $\tilde{X}^*$  of a random variable  $X$ , the  $\alpha$ -cuts of which are given in (19), is*

$$\pi_{X^*}(x) = \mu_{\tilde{X}^*}(x) = \begin{cases} 2F_X(x), & x \leq m \\ 2(1 - F_X(x)), & x > m \end{cases} \quad (25)$$

From Proposition 1 follows that the degree of necessity of the  $\alpha$ -cuts of the fuzzy number  $\tilde{X}^*$  of (19) is  $1 - \alpha$ ,

$$\text{Ness}(\tilde{X}^*[\alpha]) = 1 - \alpha \quad (26)$$



Also because of (18) and (19), the probability of finding  $X$  in the  $\alpha$  - cut  $\tilde{X}^*[\alpha]$  of  $\tilde{X}^*$  is

$$P(\tilde{X}^*[\alpha]) = 1 - \alpha \quad (27)$$

so:

**Proposition 6** *The degree of necessity of the  $\alpha$  - cuts of the fuzzy estimator  $\tilde{X}^*$  defined by (19) is equal to its probability,*

$$Ness(\tilde{X}^*[\alpha]) = P(\tilde{X}^*[\alpha]) = 1 - \alpha \quad (28)$$

so that the  $\alpha$ - cuts of  $\tilde{X}^*$  are confidence intervals of  $X$  of degree of confidence  $1 - \alpha$ .

**Definition 4** *As fuzzy estimator  $\tilde{X}$  of a random variable  $X$  is defined any fuzzy number such, that the possibility distribution  $\Pi_X$  induced by the fuzzy proposition " $X$  is  $\tilde{X}$ " to be consistent with the probability distribution  $P_X$  of  $X$ .*

The membership function  $\mu_{\tilde{X}^*}(x)$  of the fuzzy estimator  $\tilde{X}^*$  of  $X$  (or the possibility distribution function  $\pi_{\tilde{X}^*}(x)$ ) is below the membership function  $\mu_{\tilde{X}}(x)$  of any other fuzzy estimator  $\tilde{X}$  of  $X$  (any possibility distribution function  $\pi_{\tilde{X}}$  consistent with the probability distribution  $P_X$ ), i.e.

$$\mu_{\tilde{X}^*}(x) \leq \mu_{\tilde{X}}(x), \quad \forall x \in U \quad (29)$$

or equivalently the  $\alpha$ - cuts of the fuzzy estimator  $\tilde{X}^*$  are subsets of the  $\alpha$  - cuts of any other fuzzy estimator  $\tilde{X}$ , i.e.

$$\tilde{X}^*[\alpha] \subseteq \tilde{X}[\alpha], \quad \forall \alpha \in [0,1] \quad (30)$$

Consequently, the intervals  $\tilde{X}[\alpha]$  are wider than  $\tilde{X}^*[\alpha]$ , so that according to Propositions 3 and 6 it is true that:

**Proposition 7** *The probability of the  $\alpha$ - cuts  $\tilde{X}[\alpha]$  of any fuzzy estimator  $\tilde{X}$  of  $X$  is greater or equal to  $1 - \alpha$ .*

So, the  $\alpha$  - cuts  $\tilde{X}[\alpha]$  are confidence intervals of  $X$  of degree of confidence greater or equal to  $1 - \alpha$ . Especially, the  $\alpha$  - cuts  $\tilde{X}^*[\alpha]$  of the fuzzy estimator  $\tilde{X}^*$  are confidence intervals of  $X$  of degree of confidence  $1 - \alpha$ .

The Definition 4 and the Proposition 7 have no value for a random variable with known probability distribution, since for this there is the fuzzy estimator of maximal specificity  $\tilde{X}^*$  with  $\alpha$  - cuts given in (19), but they are useful in cases of random variables for which is not easy to find the probability distribution, as presented in next section.

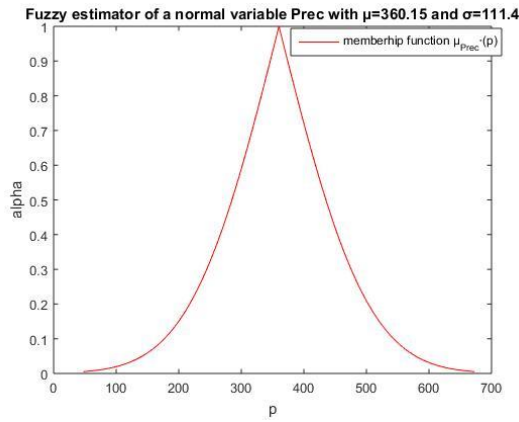
**Example 1** *We plot the membership function of the fuzzy estimator of maximal specificity  $\widetilde{Prec}^*$  of a normal variable  $Prec$  (precipitation) with mean*

$$m = 360.15 \text{ mm and standard deviation } s = 111.4 \text{ mm}.$$

Since the probability density of the normal distribution is symmetric about the mode, according to Proposition 4, the  $\alpha$  - cuts of the fuzzy estimator of maximal specificity  $\widehat{Prec}^*$  of Precipitation are given by (19)

$$Prec^*[\alpha] = [F_{Prec}^{-1}(\frac{\alpha}{2}; m, s), F_{Prec}^{-1}(1 - \frac{\alpha}{2}; m, s)], \quad 0 \leq \alpha \leq 1 \quad (31)$$

where  $F_{Prec}^{-1}(\alpha; m, s)$ ,  $0 \leq \alpha \leq 1$  the inverse distribution function of  $\underline{Prec}$  (normal distribution with mean  $m$  and standard deviation  $s$ ). Implementing the  $\alpha$  - cuts of (21), in Figure 1 the membership function of the fuzzy estimator  $\widehat{Prec}^*$  is plotted, where  $F_{Prec}$  the distribution function of  $Prec$ .



**Fig. 1.** The fuzzy estimator of maximal specificity random variable  $Prec^*$ , in case that the precipitation is normally distributed.

Hence, with the use of the *fuzzy estimator of maximal specificity* random variable  $X^*$ , a bridge between the probabilities and the fuzzy sets can be achieved.

If a random variable  $Y$  is a function of the random variables  $X_1, X_2, \dots, X_n$ , which take values in the universe  $U$  ( $Y = g(X_1, X_2, \dots, X_n)$ ), then from the fuzzy proposition ( $\tilde{Y}^*$  is a fuzzy number with  $\alpha$  - cuts the intervals of (16) for the inverse distribution function of  $Y$ )

$\hat{Y}$  is  $\tilde{Y}^*$ "

is induced the possibility distribution  $\Pi_{\tilde{Y}^*}$  for which according to Proposition 3 is true that:

**Proposition 8** Let  $\Pi_{\tilde{Y}^*}$  the possibility distribution induced by the fuzzy proposition " $\hat{Y}$  is  $\tilde{Y}^*$ " which has as possibility distribution function  $\pi_{\tilde{Y}^*}(y)$  the membership function  $\mu_{\tilde{Y}^*}(y)$  of  $\tilde{Y}^*$ , the  $\alpha$  - cuts of which (according to (19) with  $F_Y$  the distribution function of  $Y$ ) are

$$\pi_{\tilde{Y}^*}[\alpha] = \tilde{Y}^*[\alpha] = [F_Y^{-1}(\frac{\alpha}{2}), F_Y^{-1}(1 - \frac{\alpha}{2})], \quad 0 \leq \alpha \leq 1, \quad (32)$$

$\Pi_{\tilde{Y}^*}$  is consistent with the probability distribution  $P_Y$ , i.e. it satisfies the consistency principle (20), so  $\tilde{Y}^*$  is a fuzzy estimator of  $Y$ .

Also, if the probability density  $p_Y(y)$  of  $Y$  is symmetric about the mode, then according to Proposition 3  $\Pi_{\tilde{Y}^*}$  is the most specific possibility distribution consistent with the probability distribution  $P_Y$  and the fuzzy set  $\tilde{Y}^*$  is called fuzzy estimator of maximal specificity of  $Y$ .

If the probability density  $p_Y(y)$  of  $Y$  is not known, then the membership function of the fuzzy estimator of maximal specificity  $\tilde{Y}^*$  of  $Y$  can not be found.

Even if the probability distributions of these random variables are known, in general it is difficult to determine the combined probability distribution. A choice is to use the fuzzy transformation, based on the concept of *fuzzy estimator of maximal specificity* random variable  $X^*$ . Hence by exploiting the extension principle, the shape of the dependent variable  $Y$  can be determined. In such cases another fuzzy estimator of  $Y$  is constructed as follows [11]:

**Proposition 9** *The  $\alpha$ -cuts of a fuzzy estimator  $\tilde{Y}$  of the variable  $Y = g(X_1, X_2, \dots, X_n)$  are*

$$\tilde{Y}[\alpha] = g(\tilde{X}_1^*[\alpha], \tilde{X}_2^*[\alpha], \dots, \tilde{X}_n^*[\alpha]) \quad (33)$$

where

$$\tilde{X}_i^*[\alpha] = \left[ F_i^{-1}\left(\frac{\alpha}{2}\right), F_i^{-1}\left(1 - \frac{\alpha}{2}\right) \right], \quad i = 1, 2, \dots, n, \quad 0 \leq \alpha \leq 1$$

the  $\alpha$ -cuts of the fuzzy estimators  $\tilde{X}_i^*$  of  $X_i$  and  $F_i^{-1}(\alpha)$  the inverse distribution functions of  $X_i$ .

The  $\alpha$ -cuts  $\tilde{Y}[\alpha]$  are confidence intervals of  $Y$  of degree of confidence greater or equal to  $1 - \alpha$ .

### 2.3 Proposed methodology

Step1: The annual precipitation and the annual potential evapotranspiration are calculated for each meteorological station.

Step 2: The individual theoretical probability distribution function is examined for both the precipitation and the potential evapotranspiration. Statistical tests can be used to check the suitability of the used theoretical probability density function. In this work the normal probability were used.

Step 3: By using the individual *fuzzy estimators of maximal specificity* regarding random variable, we translate the information into fuzzy sets regarding the annual precipitation and the annual evapotranspiration. Practically the fuzzy sets can be achieved by using a significant number of  $\alpha$ -cuts.

Step 4: The extension principle is used in order to find the  $\alpha$ -cuts of the annual precipitation to potential evapotranspiration ratio. This will be the fuzzy estimator:

$$\tilde{Y} [a] = \frac{\widetilde{Prec}^*[a]}{\widetilde{PET}^*[a]} \quad (34)$$

$$\text{Where } \widetilde{Prec}^*[a] = \left[ F_i^{-1}\left(\frac{\alpha}{2}\right), F_i^{-1}\left(1 - \frac{\alpha}{2}\right) \right], \widetilde{PET}^*[a] = \left[ F_i^{-1}\left(\frac{\alpha}{2}\right), F_i^{-1}\left(1 - \frac{\alpha}{2}\right) \right] \quad (35)$$

### 3 Application and Discussion

For the application of the proposed methodology the data of annual precipitation and average monthly temperature from four meteorological stations in Greece were used (Helliniko (Athens), Larissa, Heraklion and Naxos were used. Monthly values of PET were then calculated using the Hargreaves method, a method based on average monthly temperatures ([4], [12]).

According to [4] the Aridity Index is calculated 0.34 for Helliniko (Athens), 0.33 for Larissa, 0.47 for Heraklion and 0.42 for Naxos. The aridity index equals to mean precipitation to potential evapotranspiration ratio. Both the precipitation and the potential evapotranspiration are normally distributed regarding the examined samples ([4]).

Let us study the Naxos meteorological station. Assuming that the random variables Precipitation (Prec) and PET follow normal distributions

$$Prec \sim N(m_1, s_1) \text{ and } PET \sim N(m_2, s_2)$$

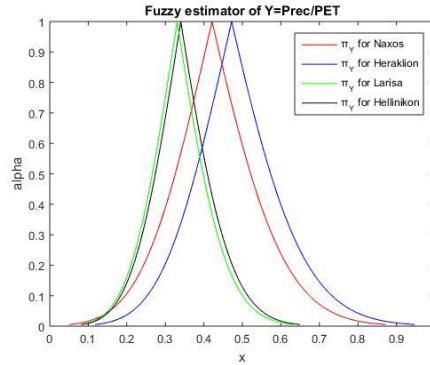
where  $m_1 = 360.15$ ,  $s_1 = 111.40$  and  $m_2 = 854.83$ ,  $s_2 = 29.85$

By using the individual *fuzzy* estimator of maximal specificity and the fuzzy arithmetics according to Proposition 9, the  $\alpha$  - cut of the fuzzy estimator  $\tilde{Y}$  of the random variable:  $Y = \frac{Prec}{PET}$  is formulated as follows:

$$\begin{aligned} \tilde{Y} [a] &= \frac{\widetilde{Prec}^*[a]}{\widetilde{PET}^*[a]} = \frac{\left[ F^{-1}\left(\frac{\alpha}{2}; m_1, s_1\right), F^{-1}\left(1 - \frac{\alpha}{2}; m_1, s_1\right) \right]}{\left[ F^{-1}\left(\frac{\alpha}{2}; m_2, s_2\right), F^{-1}\left(1 - \frac{\alpha}{2}; m_2, s_2\right) \right]} \\ &= \left[ \frac{F^{-1}\left(\frac{\alpha}{2}; m_1, s_1\right)}{F^{-1}\left(1 - \frac{\alpha}{2}; m_2, s_2\right)}, \frac{F^{-1}\left(1 - \frac{\alpha}{2}; m_1, s_1\right)}{F^{-1}\left(\frac{\alpha}{2}; m_2, s_2\right)} \right] \end{aligned} \quad (36)$$

where  $F^{-1}(\alpha; m_1, s_1)$  and  $F^{-1}(\alpha; m_2, s_2)$  the inverse distribution functions of the normal random variables Prec and PET. Hence, by using a significant number of  $\alpha$ -cuts the fuzzy number can be constructed as in Figure 2.

This procedure is repeated for each meteorological station and the results are depicted in Figure 2.



**Fig. 2.** The fuzzy estimator of the ration *Prec/PET* regarding the four examined stations.

A similar behavior can be considered in case of both the Hellinikon and the Larisa station. As it is described in the theoretical part of the manuscript the transformation from probability to fuzzy is achieved by using the *fuzzy* estimator of maximal specificity. By exploiting these individual fuzzy estimator of maximum specificity a fuzzy estimator of the ration *Prec/PET* can be achieved.

Unfortunately this cannot be considered as the fuzzy estimator of maximum specificity for the examined ratio. However the proposed methodology based on Equation (36) can be applied for several probability distributions and not only for normal distributions as in [4] (as an approximation). The conventional thresholds of drought levels can be used also in the proposed possibilistic formulation based on the following approximation: the  $\alpha$ -cuts corresponds to the  $1-\alpha$  cumulative probability with the lower and upper tails of the  $1-\alpha$  confidence interval. The correspondence based on the probability threshold of the extreme hydro meteorological analysis, that is, without the dry and the wet phenomena (e.g [2], [3]) and the corresponding possibilistic approach via  $\alpha$ -cuts are shown in Table 1:

**Table 1.** Correspondence between probability levels and  $\alpha$ -cut regarding the Naxos station

Description	Probability ( $1-\alpha$ )	$\alpha$ -cuts
Without Extremely phenomena	95.4%	[0.151, 0.732]
Without severe cases	86.6%	[0.2153, 0.6500]
Without mildly extreme cases	68.3 %	[0.2819, 0.5707]

## 4 Concluding remarks

In hydrology and water resources management problems the theoretical probability distribution functions are widely used with the aim of the empirical probability function. However, it is difficult to exploit the probability functions in case that algebraic

operations between random variables are required. A solution should be the motivation from the probability functions to fuzzy sets by using the fuzzy estimators. Based on the Possibility and the Necessity measures theory the authors conclude that based on a theoretical probability distribution we can move to the possibility distribution with the maximum specificity, near to the probability measure. The Reconnaissance Drought Index (RDI) was proposed to assess meteorological drought severity based on the precipitation to potential evapotranspiration ratio (Prec/PET). However it is difficult to express the bivariate probability density function for this ratio. Hence based on the fuzzy estimators, the analysis can be concluded to fuzzy sets, and the extension principle of fuzzy sets can provide the ratio as fuzzy sets. Unfortunately this approach cannot be considered as the fuzzy estimator of maximum specificity but it can be seen as a first approximation

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