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# On the minimum number of arcs in 4-dicritical oriented graphs \*

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## Abstract

The dichromatic number  $\vec{\chi}(D)$  of a digraph  $D$  is the minimum number of colours needed to colour the vertices of a digraph such that each colour class induces an acyclic subdigraph. A digraph  $D$  is  $k$ -dicritical if  $\vec{\chi}(D) = k$  and each proper subdigraph  $H$  of  $D$  satisfies  $\vec{\chi}(H) < k$ .

For integers  $k$  and  $n$ , we define  $d_k(n)$  (respectively  $o_k(n)$ ) as the minimum number of arcs possible in a  $k$ -dicritical digraph (respectively oriented graph). Kostochka and Stiebitz have shown [10] that  $d_4(n) \geq \frac{10}{3}n - \frac{4}{3}$ . They also conjectured that there is a constant  $c$  such that  $o_k(n) \geq cd_k(n)$  for  $k \geq 3$  and  $n$  large enough. This conjecture is known to be true for  $k = 3$  (Aboulker et al. [1]).

In this work, we prove that every 4-dicritical oriented graph on  $n$  vertices has at least  $(\frac{10}{3} + \frac{1}{51})n - 1$  arcs, showing the conjecture for  $k = 4$ . We also characterise exactly the 4-dicritical digraphs on  $n$  vertices with exactly  $\frac{10}{3}n - \frac{4}{3}$  arcs.

## 1 Introduction

Let  $G$  be a graph. We denote by  $V(G)$  its vertex set and by  $E(G)$  its edge set; we set  $n(G) = |V(G)|$  and  $m(G) = |E(G)|$ . A  $k$ -**colouring** of  $G$  is a function  $\varphi : V(G) \rightarrow [k]$ . It is **proper** if for every edge  $uv \in E(G)$ ,  $\varphi(u) \neq \varphi(v)$ . The smallest integer  $k$  such that  $G$  has a proper  $k$ -colouring is the **chromatic number**, and is denoted by  $\chi(G)$ . Since  $\chi$  is non-decreasing with respect to the subgraph relation, it is natural to consider the minimal graphs (for this relation) which are not  $(k - 1)$ -colourable. Following this idea, Dirac defined  $k$ -**critical** graphs as the graphs  $G$  with  $\chi(G) = k$  and  $\chi(H) < k$  for every proper subgraph

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$H$  of  $G$ . A first property of  $k$ -critical graph is that their minimum degree is at least  $k - 1$ . Indeed, if a vertex  $v$  has degree at most  $k - 2$ , then a  $(k - 1)$ -colouring of  $G - v$  can be easily extended to  $G$ , contradicting the fact that  $\chi(G) = k$ . As a consequence, the number of edges in a  $k$ -critical graph is at least  $\frac{k-1}{2}n$ . This bound is tight for complete graphs and odd cycles, but Dirac [3] proved an inequality of the form  $m \geq \frac{k-1+\varepsilon_k}{2}n - c_k$  for every  $n$ -vertex  $k$ -critical graph with  $m$  edges, for some  $c_k$  and  $\varepsilon_k > 0$ . This shows that, for  $n$  sufficiently large, the average degree of a  $k$ -critical graph is at least  $k - 1 + \varepsilon_k$ . This initiated the quest after the best lower bound on the number of edges in  $n$ -vertex  $k$ -critical graphs. This problem was almost completely solved by Kostochka and Yancey in 2014 [11].

**Theorem 1** (Kostochka and Yancey [11]).

*Every  $k$ -critical graph on  $n$  vertices has at least  $\frac{1}{2}(k - \frac{2}{k-1})n - \frac{k(k-3)}{2(k-1)}$  edges. For every  $k$ , this bound is tight for infinitely many values of  $n$ .*

Kostochka and Yancey [12] also characterised  $k$ -critical graphs for which this inequality is an equality, and all of them contain a copy of  $K_{k-2}$ , the complete graph on  $k - 2$  vertices. This motivated the following conjecture of Postle [13].

**Conjecture 2** (Postle [13]). For every integer  $k \geq 4$ , there exists  $\varepsilon_k > 0$  such that every  $k$ -critical  $K_{k-2}$ -free graph  $G$  on  $n$  vertices has at least  $\frac{1}{2}(k - \frac{2}{k-1} + \varepsilon_k)n - \frac{k(k-3)}{2(k-1)}$  edges.

For  $k = 4$ , the conjecture trivially holds as there is no  $K_2$ -free 4-critical graph. Moreover, this conjecture has been confirmed for  $k = 5$  by Postle [13], for  $k = 6$  by Gao and Postle [5], and for  $k \geq 33$  by Gould, Larsen, and Postle [6].

Let  $D$  be a digraph. We denote by  $V(D)$  its vertex set and by  $A(D)$  its arc set; we set  $n(D) = |V(D)|$  and  $m(D) = |A(D)|$ . A  **$k$ -colouring** of  $D$  is a function  $\varphi : V(D) \rightarrow [k]$ . It is a  **$k$ -dicolouring** if every directed cycle  $C$  in  $D$  is not monochromatic for  $\varphi$  (that is  $|\varphi(V(C))| > 1$ ). Equivalently, it is a  $k$ -dicolouring if every colour class induces an acyclic subdigraph. The smallest integer  $k$  such that  $D$  has a  $k$ -dicolouring is the **dichromatic number** of  $D$  and is denoted by  $\vec{\chi}(D)$ .

A **digon** in  $D$  is a pair of opposite arcs between two vertices. Such a pair of arcs  $\{uv, vu\}$  is denoted by  $[u, v]$ . We say that  $D$  is a **bidirected graph** if every pair of adjacent vertices forms a digon. In this case,  $D$  can be viewed as obtained from an undirected graph  $G$  by replacing each edge  $\{u, v\}$  of  $G$  by the digon  $[u, v]$ . We say that  $D$  is a bidirected  $G$ , and we denote it by  $\overleftrightarrow{G}$ . Observe that  $\chi(G) = \vec{\chi}(\overleftrightarrow{G})$ . Thus every statement on proper colouring of undirected graphs can be seen as a statement on dicolouring of bidirected graphs.

Exactly as in the undirected case, one can define  **$k$ -dicritical** digraphs to be digraphs  $D$  with  $\vec{\chi}(D) = k$  and  $\vec{\chi}(H) < k$  for every proper subdigraph  $H$  of  $D$ . It is easy to check that if  $G$  is a  $k$ -critical graph, then  $\overleftrightarrow{G}$  is  $k$ -dicritical. Kostochka and Stiebitz [10] conjectured that the  $k$ -dicritical digraphs with the minimum number of arcs are bidirected graphs. Thus they conjectured the following generalisation of Theorem 1 to digraphs.

**Conjecture 3** (Kostochka and Stiebitz [10]). Let  $k \geq 2$ . Every  $k$ -dicritical digraph on  $n$  vertices has at least  $(k - \frac{2}{k-1})n - \frac{k(k-3)}{k-1}$  arcs. Moreover, equality holds only if  $D$  is bidirected.

In the case  $k = 2$ , this conjecture is easy and weak as it states that a 2-dicritical digraph on  $n$  vertices has at least two arcs, while, for all  $n \geq 2$ , the unique 2-dicritical digraph of order  $n$  is the directed  $n$ -cycle which has  $n$  arcs. The case  $k = 3$  of the conjecture has been confirmed by Kostochka and Stiebitz [10]. Using a Brooks-type result for digraphs due to Harutyunyan and Mohar [7], they proved the following: if  $D$  is a 3-dicritical digraph of order  $n \geq 3$ , then  $m(D) \geq 2n$  and equality holds if and only if  $n$  is odd and  $D$  is a bidirected odd cycle. The conjecture has also been proved for  $k = 4$  by Kostochka and Stiebitz [10]. However, the conjecture is open for every  $k \geq 5$ . Recently, this problem has been investigated by Aboulker and Vermande [2] who proved the weaker bound  $(k - \frac{1}{2} - \frac{2}{k-1})n - \frac{k(k-3)}{k-1}$  for the number of arcs in an  $n$ -vertex  $k$ -dicritical digraph.

For integers  $k$  and  $n$ , let  $d_k(n)$  denote the minimum number of arcs in a  $k$ -dicritical digraph of order  $n$ . By the above observations,  $d_2(n) = n$  for all  $n \geq 2$ , and  $d_3(n) \geq 2n$  for all possible  $n$ , and equality holds if and only if  $n$  is odd and  $n \geq 3$ . Moreover, if  $n$  is even then  $d_3(n) = 2n + 1$  (see [1]).

Kostochka and Stiebitz [9] showed that if a  $k$ -critical graph  $G$  is triangle-free (that is has no cycle of length 3), then  $m(G)/n(G) \geq k - o(k)$  as  $k \rightarrow +\infty$ . Informally, this means that the minimum average degree of a  $k$ -critical triangle-free graph is (asymptotically) twice the minimum average degree of a  $k$ -critical graph. Similarly to this undirected case, it is expected that the minimum number of arcs in a  $k$ -dicritical digraph of order  $n$  is larger than  $d_k(n)$  if we impose this digraph to have no short directed cycles, and in particular if the digraph is an **oriented graph**, that is a digraph with no digon. Let  $o_k(n)$  denote the minimum number of arcs in a  $k$ -dicritical oriented graph of order  $n$  (with the convention  $o_k(n) = +\infty$  if there is no  $k$ -dicritical oriented graph of order  $n$ ). Clearly  $o_k(n) \geq d_k(n)$ .

**Conjecture 4** (Kostochka and Stiebitz [10]). For any  $k \geq 3$ , there is a constant  $\alpha_k > 0$  such that  $o_k(n) > (1 + \alpha_k)d_k(n)$  for  $n$  sufficiently large.

For  $k = 3$ , this conjecture has been recently confirmed by Aboulker, Bellitto, Havet, and Rambaud [1] who proved that  $o_3(n) \geq (2 + \frac{1}{3})n + \frac{2}{3}$ .

In view of Conjecture 2, Conjecture 4 can be generalized to  $\overleftrightarrow{K}_{k-2}$ -free digraphs.

**Conjecture 5.** For any  $k \geq 4$ , there is a constant  $\beta_k > 0$  such that every  $k$ -dicritical  $\overleftrightarrow{K}_{k-2}$ -free digraph  $D$  on  $n$  vertices has at least  $(1 + \beta_k)d_k(n)$  arcs.

Together with Conjecture 3, this conjecture would imply the following generalisation of Conjecture 2.

**Conjecture 6.** For every integer  $k \geq 4$ , there exists  $\varepsilon_k > 0$  such that every  $k$ -dicritical  $\overleftrightarrow{K}_{k-2}$ -free digraph  $D$  on  $n$  vertices has at least  $(k - \frac{2}{k-1} + \varepsilon_k)n - \frac{k(k-3)}{k-1}$  arcs.

A  $\overleftrightarrow{K}_2$ -free digraph is an oriented graph, and there are infinitely many 4-dicritical oriented graphs. Thus, while Conjecture 2 holds vacuously for  $k = 4$ , this is not the case for Conjecture 6. In this paper, we prove that Conjectures 4, 5, and 6 hold for  $k = 4$ .

**Theorem 7.** *If  $\vec{G}$  is a 4-dicritical oriented graph, then*

$$m(\vec{G}) \geq \left( \frac{10}{3} + \frac{1}{51} \right) n(\vec{G}) - 1.$$

To prove Theorem 7, we use an approach similar to the proof of the case  $k = 5$  of Conjecture 2 by Postle [13]. This proof is based on the potential method, which was first popularised by Kostochka and Yancey [11] when they proved Theorem 1. The idea is to prove a more general result on every 4-dicritical digraphs that takes into account the digons.

With a slight abuse, we call **digon** a subdigraph isomorphic to  $\overleftrightarrow{K_2}$ , the bidirected complete graph on two vertices. We also call **bidirected triangle** a subdigraph isomorphic to  $\overleftrightarrow{K_3}$ , the bidirected complete graph on three vertices. A **packing** of digons and bidirected triangles is a set of vertex-disjoint digons and bidirected triangles. To take into account the digons, we define a parameter  $T(D)$  as follows.

$$T(D) = \max\{d + 2t \mid \text{there exists a packing of } d \text{ digons and } t \text{ bidirected triangles}\}$$

Clearly,  $T(D) = 0$  if and only if  $D$  is an oriented graph.

Let  $\varepsilon, \delta$  be fixed non-negative real numbers. We define the **potential** (with respect to  $\varepsilon$  and  $\delta$ ) of a digraph  $D$  to be

$$\rho(D) = \left( \frac{10}{3} + \varepsilon \right) n(D) - m(D) - \delta T(D).$$

Thus Theorem 7 can be rephrased as follows.

**Theorem 7.** *Set  $\varepsilon = \frac{1}{51}$  and  $\delta = 6\varepsilon = \frac{2}{17}$ . If  $\vec{G}$  is a 4-dicritical oriented graph, then  $\rho(\vec{G}) \leq 1$ .*

In fact, we prove a more general statement which holds for every 4-dicritical digraph (with or without digons), except for some exceptions called the **4-Ore digraphs**. Those digraphs, which are formally defined in Section 2, are the bidirected graphs whose underlying graph is one of the 4-critical graphs reaching equality in Theorem 1. In particular, every 4-Ore digraph  $D$  has  $\frac{10}{3}n(D) - \frac{4}{3}$  arcs. Moreover, the statement holds for all non-negative constants  $\varepsilon$  and  $\delta$  satisfying the following inequalities:

- $\delta \geq 6\varepsilon$ ;
- $3\delta - \varepsilon \leq \frac{1}{3}$ ;

**Theorem 8.** *Let  $\varepsilon, \delta \geq 0$  be constants satisfying the aforementioned inequalities. If  $D$  is a 4-dicritical digraph with  $n$  vertices, then*

(i)  $\rho(D) \leq \frac{4}{3} + \varepsilon n - \delta \frac{2(n-1)}{3}$  if  $D$  is 4-Ore, and

(ii)  $\rho(D) \leq 1$  otherwise.

In order to provide some intuition to the reader, let us briefly describe the main ideas of our proof. We will consider a minimum counterexample  $D$  to Theorem 8, and show that every subdigraph of  $D$  must have large potential. To do so, we need to construct some smaller 4-dicritical digraphs to leverage the minimality of  $D$ . These smaller 4-dicritical digraphs will be constructed by identifying some vertices of  $D$ . This is why, in the definition of the potential, we consider  $T(D)$  instead of the number of digons: when identifying a set of vertices, the number of digons may be arbitrary larger in the resulting digraph, but  $T(D)$  increases at most by 1. Using the fact that every subdigraph of  $D$  has large potential, we will prove that some subdigraphs are forbidden in  $D$ . Using this, we get the final contradiction by a discharging argument.

In addition to Theorem 7, Theorem 8 has also the following consequence when we take  $\varepsilon = \delta = 0$ .

**Corollary 9.** *If  $D$  is a 4-dicritical digraph, then  $m(D) \geq \frac{10}{3}n(D) - \frac{4}{3}$ . Moreover, equality holds if and only if  $D$  is 4-Ore, otherwise  $m(D) \geq \frac{10}{3}n(D) - 1$ .*

This is a slight improvement on a result of Kostochka and Stiebitz [10] who proved the inequality  $m(D) \geq \frac{10}{3}n(D) - \frac{4}{3}$  without characterising the equality case.

Another interesting consequence of our result is the following bound on the number of vertices in a 4-dicritical oriented graph embedded on a fixed surface. Since a graph on  $n$  vertices embedded on a surface of Euler characteristic  $c$  has at most  $3n - 3c$  edges, we immediately deduce the following from Theorem 7.

**Corollary 10.** *If  $\vec{G}$  is a 4-dicritical oriented graph embedded on a surface of Euler characteristic  $c$ , then  $n(\vec{G}) \leq \frac{17}{6}(1 - 3c)$ .*

The previous best upper bound was  $n(\vec{G}) \leq 4 - 9c$  [10].

In Section 2 we prove some first preliminary results on 4-Ore digraphs, before proving Theorem 8 in Section 3. In Section 4, we show that  $o_k(n) \leq (2k - \frac{7}{2})n$  for every fixed  $k$  and infinitely many values of  $n$ . The proof is strongly based on the proof of [1, Theorem 4.4], which shows  $o_k(n) \leq (2k - 3)n$  for every fixed  $k, n$  (with  $n$  large enough). For  $k = 4$ , the construction implies in particular that there is a 4-dicritical oriented graph with 76 vertices and 330 arcs, and there are infinitely many 4-dicritical oriented graphs with  $m/n \leq 9/2$ .

## 2 The 4-Ore digraphs and their properties

We start with a few notations. We denote by  $\llbracket x_1, \dots, x_n \rrbracket$  the bidirected path with vertex set  $\{x_1, \dots, x_n\}$  in this order. If  $x_1 = x_n$ ,  $\llbracket x_1, \dots, x_n \rrbracket$  denotes the bidirected cycle of order  $n$  with cyclic order  $x_1, \dots, x_n$ . If  $D$  is a digraph, for any  $X \subseteq V(D)$ ,  $D - X$  is the subdigraph induced by  $V(D) \setminus X$ . We abbreviate  $D - \{x\}$  into  $D - x$ . Moreover, for any  $F \subseteq V(D) \times V(D)$ ,  $D \setminus F$  is the subdigraph  $(V(D), A(D) \setminus F)$  and  $D \cup F$  is the digraph  $(V(D), A(D) \cup F)$ .

Let  $D_1, D_2$  be two bidirected graphs,  $[x, y] \subseteq A(D_1)$ , and  $z \in V(D_2)$ . An **Ore-composition**  $D$  of  $D_1$  and  $D_2$  with **replaced digon**  $[x, y]$  and **split vertex**  $z$  is a digraph obtained by removing  $[x, y]$  of  $D_1$  and  $z$  of  $D_2$ , and adding the set of arcs  $\{xz_1 \mid zz_1 \in A(D_2) \text{ and } z_1 \in Z_1\}$ ,

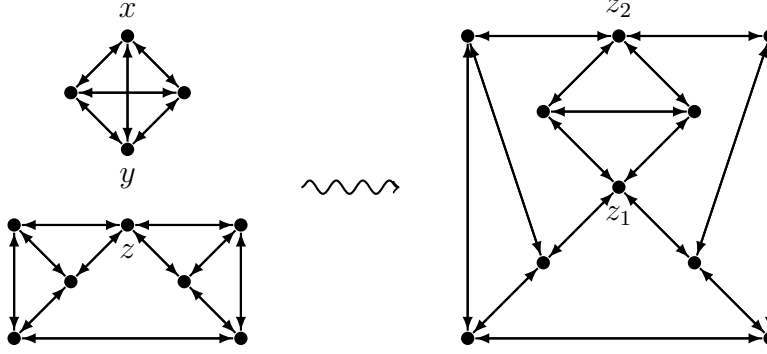


Figure 1: An example of a 4-Ore digraph obtained by an Ore-composition of two smaller 4-Ore digraphs, with replaced digon  $[x, y]$  and split vertex  $z$ .

$\{z_1x \mid z_1z \in A(D_2) \text{ and } z_1 \in Z_1\}$ ,  $\{yz_2 \mid zz_2 \in A(D_2) \text{ and } z_2 \in Z_2\}$ ,  $\{z_2y \mid z_2z \in A(D_2) \text{ and } z_2 \in Z_2\}$ , where  $(Z_1, Z_2)$  is a partition of  $N_{D_2}(z)$  into non-empty sets. We call  $D_1$  the **digon side** and  $D_2$  the **split side** of the Ore-composition. The class of the **4-Ore digraphs** is the smallest class containing  $\overleftrightarrow{K}_4$  which is stable under Ore-composition. See Figure 1 for an example of a 4-Ore digraph. Observe that all the 4-Ore-digraphs are bidirected.

**Proposition 11** (Dirac [4], see also [12]). *4-Ore digraphs are 4-dicritical.*

*Proof.* One can easily show that a bidirected digraph is 4-dicritical if and only if its undirected underlying graph is 4-critical. Then the result follows from the undirected analogous proved by [4].  $\square$

**Lemma 12.** *Let  $D$  be a 4-dicritical bidirected digraph and  $v \in V(D)$ . Let  $(N_1^+, N_2^+)$  and  $(N_1^-, N_2^-)$  be two partitions of  $N(v)$ . Consider  $D'$  the digraph with vertex set  $V(D) \setminus \{v\} \cup \{v_1, v_2\}$  with  $N^+(v_i) = N_i^+$ ,  $N^-(v_i) = N_i^-$  for  $i = 1, 2$  and  $D' \setminus \{v\} = D - v$ . Then  $D'$  has a 3-dicolouring with  $v_1$  and  $v_2$  coloured the same except if  $N_1^+ = N_1^-$  (that is  $D'$  is bidirected).*

*Proof.* Suppose that  $D'$  is not bidirected. Consider a vertex  $u \in N_D(v)$  such that  $v_1u, uv_2 \in A(D')$  or  $v_2u, uv_1 \in A(D')$ . Without loss of generality, suppose  $v_1u, uv_2 \in A(D')$ . As  $D$  is 4-dicritical,  $D \setminus [u, v]$  has a proper 3-dicolouring  $\varphi$ . We set  $\varphi(v_1) = \varphi(v_2) = \varphi(v)$  and claim that  $\varphi$  is a 3-dicolouring of  $D'$ . To show that, observe that  $\varphi$  is a proper 3-colouring of the underlying undirected graph of  $D' \setminus \{v_1u, uv_2\}$ , and so  $\varphi$  is a 3-dicolouring of  $D'$  as wanted.  $\square$

**Lemma 13.** *Let  $D$  be a digraph. If  $v$  is a vertex of  $D$ , then  $T(D - v) \geq T(D) - 1$ .*

*Proof.* Let  $M$  be a packing of  $d$  digons and  $t$  bidirected triangles in  $H$  such that  $d+2t = T(D)$ . If  $v$  belongs to a digon  $[u, v]$  in  $M$ , then  $M \setminus \{[u, v]\}$  witnesses the fact that  $T(D - v) \geq T(D) - 1$ . If  $v$  belongs to a bidirected triangle  $\llbracket u, v, w, u \rrbracket$ , then  $M \setminus \{\llbracket u, v, w, u \rrbracket\} \cup [u, w]$  witnesses the fact that  $T(D - v) \geq T(D) - 2 + 1$ . Otherwise  $T(D - v) \geq T(D)$ .  $\square$

**Lemma 14.** *If  $D_1, D_2$  are two digraphs, and  $D$  is an Ore-composition of  $D_1$  and  $D_2$ , then  $T(D) \geq T(D_1) + T(D_2) - 2$ . Moreover, if  $D_1$  or  $D_2$  is isomorphic to  $\overleftrightarrow{K}_4$ , then  $T(D) \geq T(D_1) + T(D_2) - 1$ .*

*Proof.* Let  $D$  be the Ore-composition of  $D_1$  (the digon side with replaced digon  $[x, y]$ ) and  $D_2$  (the split side with split vertex  $z$ ). One can easily see that  $T(D) \geq T(D_1 - x) + T(D - z) \geq T(D_1) + T(D_2) - 2$  by Lemma 13. Moreover, if  $D_1$  (resp.  $D_2$ ) is a copy of  $\overleftrightarrow{K}_4$ , then  $T(D_1 - x) = 2 = T(D_1)$  (resp.  $T(D_2 - z) = 2 = T(D_2)$ ) and therefore  $T(D) \geq T(D_1) + T(D_2) - 1$ .  $\square$

**Lemma 15.** *If  $D$  is 4-Ore, then  $T(D) \geq \frac{2}{3}(n(D) - 1)$ .*

*Proof.* If  $D$  is  $\overleftrightarrow{K}_4$ , then the result is clear. Suppose now that  $D$  is an Ore-composition of  $D_1$  and  $D_2$ . Then  $n(D) = n(D_1) + n(D_2) - 1$  and, by Lemma 14,  $T(D) \geq T(D_1) + T(D_2) - 2$ . By induction,  $T(D_1) \geq \frac{2}{3}(n(D_1) - 1)$  and  $T(D_2) \geq \frac{2}{3}(n(D_2) - 1)$ , and so  $T(D) \geq \frac{2}{3}(n(D_1) + n(D_2) - 1 - 1) = \frac{2}{3}(n(D) - 1)$ .  $\square$

Let  $D$  be a digraph. A **diamond** in  $D$  is a subdigraph isomorphic to  $\overleftrightarrow{K}_4$  minus a digon  $[u, v]$ , with vertices different from  $u$  and  $v$  having degree 6 in  $D$ . An **emerald** in  $D$  is a subdigraph isomorphic to  $\overleftrightarrow{K}_3$  whose vertices have degree 6 in  $D$ .

Let  $R$  be an induced subdigraph of  $D$  with  $n(R) < n(D)$ . The **boundary** of  $R$  in  $D$ , denoted by  $\partial_D(R)$ , or simply  $\partial(R)$  when  $D$  is clear from the context, is the set of vertices of  $R$  having a neighbour in  $V(D) \setminus R$ . We say that  $R$  is **Ore-collapsible** if the boundary of  $R$  contains exactly two vertices  $u$  and  $v$  and  $R \cup [u, v]$  is 4-Ore.

**Lemma 16.** *If  $D$  is 4-Ore and  $v \in V(D)$ , then there exists either an Ore-collapsible subdigraph of  $D$  disjoint from  $v$  or an emerald of  $D$  disjoint from  $v$ .*

*Proof.* If  $D$  is a copy of  $\overleftrightarrow{K}_4$ , then  $D - v$  is an emerald. Otherwise,  $D$  is the Ore-composition of two 4-Ore digraphs:  $D_1$  the digon side with replaced digon  $[x, y]$ , and  $D_2$  the split side with split vertex  $z$ . If  $v \in V(D_2 - z)$ , then  $D_1$  is an Ore-collapsible subdigraph with boundary  $\{x, y\}$ . Otherwise  $v \in V(D_1)$  and we apply induction on  $D_2$  to find an emerald or an Ore-collapsible subdigraph in  $D_2$  disjoint from  $z$ .  $\square$

**Lemma 17.** *If  $D \neq \overleftrightarrow{K}_4$  is 4-Ore and  $T$  is a copy of  $\overleftrightarrow{K}_3$  in  $D$ , then there exists either an Ore-collapsible subdigraph of  $D$  disjoint from  $T$  or an emerald of  $D$  disjoint from  $T$ .*

*Proof.* As  $D$  is not  $\overleftrightarrow{K}_4$ , it is an Ore-composition of two 4-Ore digraphs:  $D_1$  the digon side with replaced digon  $[x, y]$ , and  $D_2$  the split side with split vertex  $z$ . As  $x$  and  $y$  are not adjacent, we have either  $T \subseteq D_1$ ,  $T \subseteq D_2 - z$ , or  $T$  contains a vertex  $w \in \{x, y\}$  and two vertices in  $V(D_2 - z)$ .

If  $T \subseteq D_1$ , then by Lemma 16, in  $D_2$  there exists either an Ore-collapsible subdigraph  $O$  or an emerald  $E$  disjoint from  $z$ . In the former case  $O$  is an Ore-collapsible subdigraph of  $D$  disjoint from  $T$ , and in the later one  $E$  is an emerald in  $D$  disjoint from  $T$ .



If  $T \subseteq D_2 - z$ , then  $D_1 \setminus \{x, y\}$  is an Ore-collapsible subdigraph disjoint from  $T$ .

Assume now that  $T$  contains a vertex  $w \in \{x, y\}$  and two vertices in  $V(D_2 - z)$ . Without loss of generality, we may assume that  $y \notin T$ . Let  $z_1$  and  $z_2$  be the two vertices of  $T$  disjoint from  $w$ . Then  $\{z, z_1, z_2\}$  induces a bidirected triangle  $T'$  in  $D_2$ . If  $D_2 \neq \overleftrightarrow{K}_4$ , then by induction in  $D_2$ , there exists either an Ore-collapsible subdigraph  $O$  or an emerald  $E$  disjoint from  $T'$ . In the former case  $O$  is an Ore-collapsible subdigraph of  $D$  disjoint from  $T$ , and in the later one  $E$  is an emerald in  $D$  disjoint from  $T$ .

Henceforth we may assume that  $D_2 = \overleftrightarrow{K}_4$ . This implies that  $y$  has exactly one neighbour in  $D_2 - z$  and so its degree is the same in  $D_1$  and  $D$ . By Lemma 16, in  $D_1$  there exists either an Ore-collapsible subdigraph  $O$  or an emerald  $E$  disjoint from  $x$ . In the former case  $O$  is an Ore-collapsible subdigraph of  $D$  disjoint from  $T$ , and in the later one  $E$  is an emerald in  $D$  disjoint from  $T$  even if  $y \in V(E)$  because  $y$  has the same degree in  $D_1$  and  $D$ .  $\square$

**Lemma 18.** *If  $R$  is an Ore-collapsible induced subdigraph of a 4-Ore digraph  $D$ , then there exists a diamond or an emerald of  $D$  whose vertices lie in  $V(R)$ .*

*Proof.* Let  $D$  be a digraph. Let  $R$  be a minimal counterexample to this lemma, and let  $\partial(R) = \{u, v\}$  and  $H = D \langle R \rangle \cup [u, v]$ . If  $H = \overleftrightarrow{K}_4$ , then  $R$  is a diamond in  $D$ . Suppose now that  $H$  is the Ore-composition of two 4-Ore digraphs  $H_1$  (the digon side with replaced digon  $[x, y]$ ) and  $H_2$  (the split side with split vertex  $z$ ). If  $\{u, v\} \not\subset V(H_2)$ , then by Lemma 16 there exists an Ore-collapsible subdigraph in  $H_2$  disjoint from  $z$ . As it is smaller than  $H$ , it contains an emerald or a diamond as desired, a contradiction.

Now assume that  $\{u, v\} \subset V(H_2)$ , then  $H_1$  is an Ore-collapsible subdigraph of  $D$  smaller than  $H$ , and by induction,  $H_1$  contains a diamond or an emerald in  $D$ .  $\square$

**Lemma 19.** *If  $D$  is a 4-Ore digraph and  $v$  is a vertex in  $D$ , then  $D$  contains a diamond or an emerald disjoint from  $v$ .*

*Proof.* Follows from Lemmas 16 and 18.  $\square$

**Lemma 20.** *If  $D$  is a 4-Ore digraph and  $T$  is a bidirected triangle in  $D$ , then either  $D = \overleftrightarrow{K}_4$  or  $D$  contains a diamond or an emerald disjoint from  $T$ .*

*Proof.* Follows from Lemmas 17 and 18.  $\square$

The following theorem was formulated for undirected graphs, but by replacing every edge by a digon, it can be restated as follows:

**Theorem 21** (Kostochka and Yancey [12], Theorem 6). *Let  $D$  be a 4-dicritical bidirected digraph.*

*If  $\frac{10}{3}n(D) - m(D) > 1$ , then  $D$  is 4-Ore and  $\frac{10}{3}n(D) - m(D) = \frac{4}{3}$ .*

**Lemma 22.** *If  $D$  is a 4-Ore digraph with  $n$  vertices, then  $\rho(D) \leq \frac{4}{3} + \varepsilon n - \delta \frac{2(n-1)}{3}$ .*

*Proof.* Follows from Theorem 21 and Lemma 15.  $\square$

**Lemma 23** (Kostochka and Yancey [12], Claim 16). *Let  $D$  be a 4-Ore digraph. If  $R \subseteq D$  and  $0 < n(R) < n(D)$ , then  $\frac{10}{3}n(R) - m(R) \geq \frac{10}{3}$ .*

**Lemma 24.** *Let  $D$  be a 4-Ore digraph obtained from a copy  $J$  of  $\overleftrightarrow{K}_4$  by successive Ore-compositions with 4-Ore digraphs, vertices and digons in  $J$  being always on the digon side. Let  $[u, v]$  be a digon in  $D \setminus V(J)$ . For every 3-dicolouring  $\varphi$  of  $D \setminus [u, v]$ , vertices in  $V(J)$  receive distinct colours except  $u$  and  $v$ .*

*Proof.* We proceed by induction on  $n(D)$ , the result holding trivially when  $D$  is  $\overleftrightarrow{K}_4$ . Now assume that  $D$  is the Ore-composition of  $D_1$ , the digon side containing  $J$ , and  $D_2$ , with  $D_1$  and  $D_2$  being 4-Ore digraphs. Let  $[x, y] \subseteq A(D_1)$  be the replaced digon in this Ore-composition, and let  $z \in V(D_2)$  be the split vertex. Let  $\varphi$  be a 3-dicolouring of  $D \setminus [u, v]$ . Then  $\varphi$  induces a 3-dicolouring of  $D \setminus (V(D_2 - z) \cup \{x, y\})$ . Necessarily  $\varphi(x) \neq \varphi(y)$ , for otherwise  $\varphi_2$  defined by  $\varphi_2(w) = \varphi(w)$  if  $w \in V(D_2 - z)$  and  $\varphi_2(z) = \varphi(x)$  is a 3-dicolouring of  $D_2$ , contradicting the fact that 4-Ore digraphs have dichromatic number 4 by Lemma 11. Hence  $\varphi$  induces a 3-dicolouring of  $D_1 \setminus [u, v]$ . So, by the induction hypothesis, vertices in  $V(J)$  have distinct colours in  $\varphi$ , except  $u$  and  $v$ .  $\square$

**Lemma 25.** *Let  $D$  be a 4-Ore digraph obtained from a copy  $J$  of  $\overleftrightarrow{K}_4$  by successive Ore-compositions with 4-Ore digraphs, vertices and digons in  $J$  being always on the digon side. Let  $v$  be a vertex in  $V(J)$ . For every 3-dicolouring  $\varphi$  of  $D - v$ , vertices in  $J$  receive distinct colours.*

*Proof.* We proceed by induction on  $n(D)$ , the result holding trivially when  $D$  is  $\overleftrightarrow{K}_4$ . Now assume that  $D$  is the Ore-composition of  $D_1$ , the digon side containing  $J$ , and  $D_2$ , with  $D_1$  and  $D_2$  being 4-Ore digraphs. Let  $[x, y] \subseteq A(D_1)$  be the replaced digon in this Ore-composition, and let  $z \in V(D_2)$  be the split vertex. Let  $\varphi$  be a 3-dicolouring of  $D - v$ . If  $v \in \{x, y\}$ , then  $\varphi$  is a 3-dicolouring of  $D_1 - v$  and the result follows by induction. Now assume  $v \notin \{x, y\}$ . Then  $\varphi$  induces a 3-dicolouring of  $D \setminus (V(D_2 - z) \cup \{x, y\})$ . Necessarily  $\varphi(x) \neq \varphi(y)$ , for otherwise  $\varphi_2$  defined by  $\varphi_2(w) = \varphi(w)$  if  $w \in V(D_2 - z)$  and  $\varphi_2(z) = \varphi(x)$  is a 3-dicolouring of  $D_2$ , contradicting the fact that 4-Ore digraphs have dichromatic number 4 by Lemma 11. Hence  $\varphi$  induces a 3-dicolouring of  $D_1 - v$ . So, by the induction hypothesis, vertices in  $V(J)$  have distinct colours in  $\varphi$ .  $\square$

### 3 Proof of Theorem 8

Let  $D$  be a 4-dicritical digraph,  $R$  be an induced subdigraph of  $D$  with  $4 \leq n(R) < n(D)$  and  $\varphi$  a 3-dicolouring of  $R$ . The  $\varphi$ -**identification** of  $R$  in  $D$ , denoted by  $D_\varphi(R)$  is the digraph obtained from  $D$  by identifying for each  $i \in [3]$  the vertices coloured  $i$  in  $V(R)$  to a vertex  $x_i$ , adding the digons  $[x_i, x_j]$  for all  $1 \leq i < j \leq 3$ . Observe that  $D_\varphi(R)$  is not 3-dicolourable. Indeed, assume for a contradiction that  $D_\varphi(R)$  has a 3-dicolouring  $\varphi'$ . Since  $\{x_1, x_2, x_3\}$  induces a  $\overleftrightarrow{K}_3$ , we may assume without loss of generality that  $\varphi'(x_i) = i$  for  $i \in [3]$ . Consider

the 3-colouring  $\varphi''$  of  $D$  defined by  $\varphi''(v) = \varphi'(v)$  if  $v \notin R$  and  $\varphi''(v) = \varphi(v)$  if  $v \in R$ . One easily checks that  $\varphi''$  is a 3-dicolouring of  $D$ , a contradiction to the fact that  $\vec{\chi}(D) = 4$ .

Now let  $W$  be a 4-dicritical subdigraph of  $D_\varphi(R)$  and  $X = \{x_1, x_2, x_3\}$ . Then we say that  $R' = D \setminus ((V(W) \setminus X) \cup R)$  is the **dicritical extension** of  $R$  with **extender**  $W$ . We call  $X_W = X \cap V(W)$  the **core** of the extension. Note that  $X_W$  is not empty, because  $W$  is not a subdigraph of  $D$ . Thus  $1 \leq |X_W| \leq 3$ . See Figure 2 for an example of a  $\varphi$ -identification and a dicritical extension.

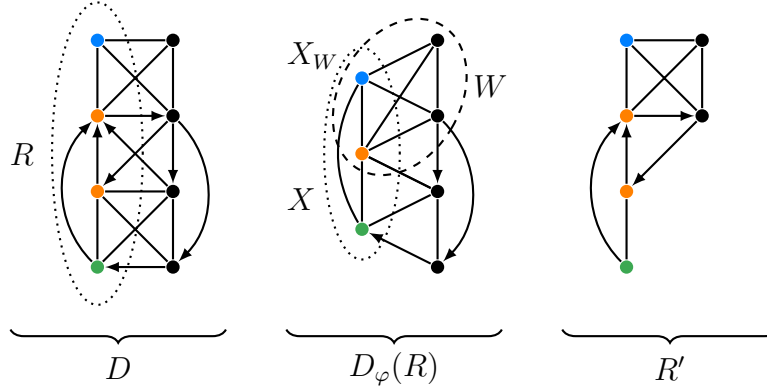


Figure 2: A 4-dicritical digraph  $D$  together with an induced subdigraph  $R$  of  $D$  and  $\varphi$  a 3-dicolouring of  $R$ , the  $\varphi$ -identification  $D_\varphi(R)$  of  $R$  in  $D$  and the dicritical extension  $R'$  of  $R$  with extender  $W$  and core  $X_W$ . For clarity, the digons are represented by undirected edges.

Let  $D$  be a counterexample to Theorem 8 with minimum number of vertices. By Lemma 22,  $D$  is not 4-Ore. Thus  $\rho(D) > 1$ .

**Claim 1.** *If  $\tilde{D}$  is a 4-dicritical digraph with  $n(\tilde{D}) < n(D)$ , then  $\rho(\tilde{D}) \leq \frac{4}{3} + 4\varepsilon - 2\delta$ .*

*Proof of claim.* If  $\tilde{D}$  is not 4-Ore, then  $\rho(\tilde{D}) \leq 1$  by minimality of  $D$ . Thus  $\rho(\tilde{D}) \leq \frac{4}{3} + 4\varepsilon - 2\delta$  because  $4\varepsilon - 2\delta \geq \frac{-1}{3}$ . Otherwise, by Lemma 22,  $\rho(\tilde{D}) \leq \frac{4}{3} + \varepsilon n(\tilde{D}) - \delta \frac{2(n(\tilde{D})-1)}{3} \leq \frac{4}{3} + 4\varepsilon - 2\delta$  because  $\delta \geq \frac{3}{2}\varepsilon$  and  $n(\tilde{D}) \geq 4$ .  $\diamond$

**Claim 2.** *Let  $R$  be a subdigraph of  $D$  with  $4 \leq n(R) < n(D)$ . If  $R'$  is a dicritical extension of  $R$  with extender  $W$  and core  $X_W$ , then*

$$\rho(R') \leq \rho(W) + \rho(R) - \left( \rho(\overleftarrow{K}_{|X_W|}) + \delta \cdot T(\overleftarrow{K}_{|X_W|}) \right) + \delta \cdot (T(W) - T(W - X_W))$$

and in particular

$$\rho(R') \leq \rho(W) + \rho(R) - \frac{10}{3} - \varepsilon + \delta.$$

*Proof of claim.* We have

- $n(R') = n(W) - |X_W| + n(R)$ ,

- $m(R') \geq m(W) + m(R) - m(\overleftarrow{K}_{|X_W|})$ ,
- $T(R') \geq T(W - X_W) + T(R)$

and by summing these inequalities, we get the first result.

Now observe that  $T(W) - T(W - X_W) \leq |X_W|$  by Lemma 13, and that the maximum of  $-\left(\rho(\overleftarrow{K}_{|X_W|}) + \delta T(\overleftarrow{K}_{|X_W|})\right) + \delta|X_W|$  is reached when  $|X_W| = 1$ , in which case it is equal to  $-\frac{10}{3} - \varepsilon + \delta$ . The second inequality follows.  $\diamond$

**Claim 3.** *If  $R$  is a subdigraph of  $D$  with  $4 \leq n(R) < n(D)$ , then  $\rho(R) \geq \rho(D) + 2 - 3\varepsilon + \delta > 3 - 3\varepsilon + \delta$ .*

*Proof of claim.* We proceed by induction on  $n - n(R)$ . Let  $R'$  be a dicritical extension of  $R$  with extender  $W$  and core  $X_W$ . By Claim 2, we have

$$\rho(R') \leq \rho(W) + \rho(R) - \frac{10}{3} - \varepsilon + \delta.$$

Either  $V(R') = V(D)$  and so  $\rho(R') \geq \rho(D)$  or  $V(R')$  is a proper subset of  $V(D)$  and, since  $R$  is a proper subdigraph of  $R'$ , by induction  $\rho(R') \geq \rho(D) + 2 - 3\varepsilon + \delta \geq \rho(D)$ . In both cases,  $\rho(R') \geq \rho(D)$ . Now  $W$  is smaller than  $D$  so  $\rho(W) \leq \frac{4}{3} + 4\varepsilon - 2\delta$  by Claim 1. Thus

$$\rho(D) \leq \rho(R') \leq \frac{4}{3} + 4\varepsilon - 2\delta + \rho(R) - \frac{10}{3} - \varepsilon + \delta.$$

This gives  $\rho(R) \geq \rho(D) + 2 - 3\varepsilon + \delta > 3 - 3\varepsilon + \delta$ , because  $\rho(D) > 1$ .  $\diamond$

As a consequence of Claim 3, any subdigraph (proper or not) of size at least 4 has potential at least  $\rho(D)$ .

We say that an induced subdigraph  $R$  of  $D$  is **collapsible** if, for every 3-dicolouring  $\varphi$  of  $R$ , its dicritical extension  $R'$  (with extender  $W$  and core  $X_W$ ) is  $D$ , has core of size 1 (i.e.  $|X_W| = 1$ ), and the border  $\partial_D(R)$  of  $R$  is monochromatic in  $\varphi$ .

**Claim 4.** *Let  $R$  be an induced subdigraph of  $D$  and  $\varphi$  a 3-dicolouring of  $R$  such that  $\partial(R)$  is not monochromatic in  $\varphi$ . If  $D$  is a dicritical extension of  $R$  dicoloured by  $\varphi$  with extender  $W$  and core  $X_W$  with  $|X_W| = 1$ , then*

$$\rho(R) \geq \rho(D) + 3 - 3\varepsilon + \delta.$$

*Proof of claim.* Assume  $D$  is a dicritical extension of  $R$  dicoloured by  $\varphi$  with extender  $W$  and core  $X_W$  with  $|X_W| = 1$ . Observe that each of the following inequalities holds:

- $n(D) = n(W) - |X_W| + n(R) = n(W) + n(R) - 1$ ,
- $m(D) \geq m(W) + m(R) - m(\overleftarrow{K}_{|X_W|}) + 1 = m(W) + m(R) + 1$  because  $\partial_D(R)$  is not monochromatic in  $\varphi$ , and

- $T(D) \geq T(W - X_W) + T(R) \geq T(W) + T(R) - 1$  by Lemma 13.

By Claim 1, we have

$$\rho(D) \leq \rho(W) + \rho(R) - \left(\frac{10}{3} + \varepsilon\right) - 1 + \delta \leq \left(\frac{4}{3} + 4\varepsilon - 2\delta\right) + \rho(R) - \frac{13}{3} - \varepsilon + \delta$$

and so  $\rho(R) \geq \rho(D) + 3 - 3\varepsilon + \delta$ .  $\diamond$

**Claim 5.** *If  $R$  is a subdigraph of  $D$  with  $4 \leq n(R) < n(D)$  and  $R$  is not collapsible, then  $\rho(R) \geq \rho(D) + \frac{8}{3} - \varepsilon - \delta > \frac{11}{3} - \varepsilon - \delta$ .*

*Proof of claim.* Let  $R'$  be a dicritical extension of  $R$  dicoloured by  $\varphi$  with extender  $W$  and core  $X_W$ .

(i) If  $R'$  is not  $D$ , then it has a dicritical extension  $R''$  with extender  $W'$ . By (the consequence of) Claim 3, we have  $\rho(D) \leq \rho(R'')$ ; by Claim 2 (applied twice),  $\rho(R'') \leq \rho(R) + \rho(W') + \rho(W) + 2\left(-\frac{10}{3} - \varepsilon + \delta\right)$ ; both  $W$  and  $W'$  are smaller than  $D$ , so, by Claim 1,  $\rho(W), \rho(W') \leq \frac{4}{3} + 4\varepsilon - 2\delta$ . Those three inequalities imply

$$\rho(D) \leq \rho(R'') \leq \rho(R) + 2\left(\frac{4}{3} + 4\varepsilon - 2\delta\right) + 2\left(-\frac{10}{3} - \varepsilon + \delta\right) = \rho(R) - 4 + 6\varepsilon - 2\delta$$

and so  $\rho(R) \geq \rho(D) + 4 - 6\varepsilon + 2\delta \geq \rho(D) + \frac{8}{3} - \varepsilon - \delta$ .

(ii) If  $R' = D$  and  $|X_W| = 2$ , then  $\rho(\overleftarrow{K}_{|X_W|}) + \delta T(\overleftarrow{K}_{|X_W|}) = \frac{14}{3} + 2\varepsilon$ , and, by Lemma 13,  $T(W) - T(W - X_W) \leq |X_w| = 2$ . Thus, by Claim 2,

$$\rho(D) \leq \rho(W) + \rho(R) - \frac{14}{3} - 2\varepsilon + 2\delta$$

Now, since  $W$  is smaller than  $D$ ,  $\rho(W) \leq \frac{4}{3} + 4\varepsilon - 2\delta$  by Claim 1. Thus

$$\rho(D) \leq \rho(R) + \frac{4}{3} + 4\varepsilon - 2\delta - \frac{14}{3} - 2\varepsilon + 2\delta = \rho(R) - \frac{10}{3} + 2\varepsilon$$

and so  $\rho(R) \geq \rho(D) + \frac{10}{3} - 2\varepsilon \geq \rho(D) + \frac{8}{3} - \varepsilon - \delta$ .

(iii) If  $R' = D$  and  $|X_W| = 3$ , then  $\rho(\overleftarrow{K}_{|X_W|}) + \delta T(\overleftarrow{K}_{|X_W|}) = 4 + 3\varepsilon$ , and, by Lemma 13,  $T(W) - T(W - X_W) \leq |X_w| = 3$ . Thus, by Claim 2,

$$\rho(D) \leq \rho(W) + \rho(R) - 4 - 3\varepsilon + 3\delta.$$

Now, since  $W$  is smaller than  $D$ ,  $\rho(W) \leq \frac{4}{3} + 4\varepsilon - 2\delta$  by Claim 1. Thus

$$\rho(D) \leq \rho(R) + \frac{4}{3} + 4\varepsilon - 2\delta - 4 - 3\varepsilon + 3\delta = \rho(R) - \frac{8}{3} + \varepsilon + \delta$$

and so  $\rho(R) \geq \rho(D) + \frac{8}{3} - \varepsilon - \delta$ .

(iv) If  $R' = D$ ,  $|X_W| = 1$  and  $\partial(R)$  is not monochromatic in  $\varphi$ , then, by Claim 4, we have  $\rho(R) \geq \rho(D) + 3 - 3\varepsilon + \delta \geq \rho(D) + \frac{8}{3} - \varepsilon - \delta$ .

If  $R$  is not collapsible, then, by definition, it has a dicritical extension  $R'$  satisfying the hypothesis of one of the cases (i)–(iv). In any case,  $\rho(R) \geq \rho(D) + \frac{8}{3} - \varepsilon - \delta$ .  $\diamond$

Recall that a  **$k$ -cutset** in a graph  $G$  is a set  $S$  of  $k$  vertices such that  $G - S$  is not connected. A graph is  **$k$ -connected** if it has more than  $k$  vertices and has no  $(k - 1)$ -cutset. A  **$k$ -cutset** in a digraph is a  $k$ -cutset in its underlying graph, and a digraph is  **$k$ -connected** if its underlying graph is  $k$ -connected.

**Claim 6.**  *$D$  is 2-connected.*

*Proof of claim.* Suppose for contradiction that  $\{x\}$  is a 1-cutset in  $D$ . Let  $(A_0, B_0)$  be a partition of  $V(D - x)$  into non-empty sets such that there is no edge between  $A_0$  and  $B_0$ , and set  $A = A_0 \cup \{x\}$  and  $B = B_0 \cup \{x\}$ .

Since  $D$  is 4-dicritical, there exist a 3-dicolouring  $\varphi_A$  of  $D\langle A \rangle$  and a 3-dicolouring  $\varphi_B$  of  $D\langle B \rangle$ . Free to swap the colours, we may assume  $\varphi_A(x) = \varphi_B(x)$ . Let  $\varphi$  be defined by  $\varphi(v) = \varphi_A(v)$  if  $v \in A$  and  $\varphi(v) = \varphi_B(v)$  if  $v \in B$ . Since  $\vec{\chi}(D) = 4$ ,  $D$ , coloured with  $\varphi$ , must contain a monochromatic directed cycle. Such a directed cycle must be contained in  $D\langle A \rangle$  or  $D\langle B \rangle$ , a contradiction.  $\diamond$

**Claim 7.**  *$D$  is 3-connected. In particular,  $D$  contains no diamond.*

*Proof of claim.* Suppose for contradiction that  $\{x, y\}$  is a 2-cutset of  $D$ . Let  $(A_0, B_0)$  be a partition of  $V(D) \setminus \{x, y\}$  into non-empty sets such that there is no edge between  $A_0$  and  $B_0$ , and set  $A = A_0 \cup \{x, y\}$  and  $B = B_0 \cup \{x, y\}$ .

Assume for a contradiction that there exists a 3-dicolouring  $\varphi_A$  of  $D\langle A \rangle$  and a 3-dicolouring  $\varphi_B$  of  $D\langle B \rangle$  such that  $\varphi_A(x) \neq \varphi_A(y)$  and  $\varphi_B(x) \neq \varphi_B(y)$ . Free to swap the colours, we may assume  $\varphi_A(x) = \varphi_B(x)$  and  $\varphi_A(y) = \varphi_B(y)$ . Let  $\varphi$  be defined by  $\varphi(v) = \varphi_A(v)$  if  $v \in A$  and  $\varphi(v) = \varphi_B(v)$  if  $v \in B$ . Every directed cycle either is in  $D\langle A \rangle$ , or is in  $D\langle B \rangle$  or contains both  $x$  and  $y$ . Therefore it cannot be monochromatic with  $\varphi$  because  $\varphi_A$  and  $\varphi_B$  are 3-dicolourings of  $D\langle A \rangle$  and  $D\langle B \rangle$  respectively, and  $\varphi(x) \neq \varphi(y)$ . Thus  $\varphi$  is a 3-dicolouring of  $D$ , a contradiction. Henceforth either  $D\langle A \rangle$  or  $D\langle B \rangle$  has no 3-dicolouring  $\varphi$  such that  $\varphi(x) \neq \varphi(y)$ . Suppose without loss of generality that it is  $D\langle A \rangle$ .

Let  $D_A = D\langle A \rangle \cup [x, y]$ .  $D_A$  is not 3-dicolourable because in every 3-dicolouring of  $D\langle A \rangle$ ,  $x$  and  $y$  are coloured the same. Let  $D_B$  be the digraph obtained from  $D\langle B \rangle$  by identifying  $x$  and  $y$  into a vertex  $z$ . Assume for a contradiction that  $D_B$  has a 3-dicolouring  $\psi_B$ . Set  $\psi(x) = \psi(y) = \psi_B(z)$ , and  $\psi(u) = \psi_B(u)$  for every  $u \in B \setminus \{x, y\}$ . Then consider a 3-dicolouring  $\psi_A$  of  $D\langle A \rangle$  such that  $\psi_A(x) = \psi(x) = \psi_A(y) = \psi(y)$  (such a colouring exists because  $A$  is a proper subdigraph of  $D$ ) and we set  $\psi(u) = \psi_A(u)$  for every  $u \in V(A) \setminus \{x, y\}$ . As  $D$  is not 3-dicolourable, it contains a monochromatic directed cycle  $C$  (with respect to  $\psi$ ). The cycle  $C$  is not included in  $D\langle A \rangle$  nor in  $D_B$ . As a consequence, there is a monochromatic

directed path from  $\{x, y\}$  to  $\{x, y\}$  in  $B$ , and so there is a monochromatic directed cycle in  $D_B$  for  $\psi_B$ , a contradiction. Therefore  $D_B$  is not 3-dicolourable

Now  $D_A$  has a 4-dicritical subdigraph  $W_A$  which necessarily contains  $\{x, y\}$ , and  $D_B$  has a 4-dicritical subdigraph  $W_B$  which necessarily contains  $z$ . As  $W_A$  and  $W_B$  are 4-dicritical digraphs smaller than  $D$ , we have  $\rho(W_A), \rho(W_B) \leq \frac{4}{3} + 4\varepsilon - 2\delta$  by Claim 1. Let  $H$  be the subdigraph of  $D$  induced by  $V(W_A) \cup V(W_B - z)$ .

Note that  $n(H) = n(W_A) + n(W_B) - 1$  and  $m(H) \geq m(W_A) + m(W_B) - 2$ . Moreover  $T(H) \geq T(W_A - x) + T(W_B - z) \geq T(W_A) + T(W_B) - 2$ , by Lemma 13. Hence we have

$$\begin{aligned}
\rho(H) &\leq \rho(W_A) + \rho(W_B) - \left(\frac{10}{3} + \varepsilon\right) + (m(W_A) + m(W_B) - m(H)) + 2\delta \\
&\leq \rho(W_A) + \rho(W_B) - \frac{10}{3} - \varepsilon + 2 + 2\delta \\
&= \rho(W_A) + \rho(W_B) - \frac{4}{3} - \varepsilon + 2\delta \\
&\leq 2\left(\frac{4}{3} + 4\varepsilon - 2\delta\right) - \frac{4}{3} - \varepsilon + 2\delta \\
&= \frac{4}{3} + 7\varepsilon - 2\delta
\end{aligned} \tag{1}$$

By Claim 3, if  $n(H) < n(D)$  then  $\rho(H) > 3 - 3\varepsilon + \delta$ . As  $10\varepsilon - 3\delta \leq \frac{5}{3}$ , we deduce that  $H = D$ . In the above chain of inequalities, we upper bounded  $m(W_A) + m(W_B) - m(H)$  by 2, doing the same computation with  $2 + m(W_A) + m(W_B) - m(H) - 2$  instead of 2, we get  $1 < \rho(D) = \rho(H) \leq \frac{4}{3} + 7\varepsilon - 2\delta + (m(W_A) + m(W_B) - m(H) - 2)$  and so  $m(H) = m(W_A) + m(W_B) - 2$  because  $2\delta - 7\varepsilon \leq \frac{2}{3}$ . In particular, there is no arc between  $x$  and  $y$  in  $D$ . Moreover, no arc was suppressed when identifying  $x$  and  $y$  into  $z$  to obtain  $D_B$ , so  $x$  and  $y$  have no common out-neighbour (resp. in-neighbour) in  $B_0$ .

We first show that either  $W_A$  or  $W_B$  is not 4-Ore. Assume for contradiction that both  $W_A$  and  $W_B$  are 4-Ore. If  $H = D$  is not bidirected, then by Lemma 12,  $D\langle B \rangle$  admits a 3-dicolouring  $\varphi_B$  such that  $\varphi_B(x) = \varphi_B(y)$ . Now let  $\varphi_A$  be a 3-dicolouring of  $D\langle A \rangle$ . We have  $\varphi_A(x) = \varphi_A(y)$ . Free to exchange colours, we may assume  $\varphi_A(x) = \varphi_A(y) = \varphi_B(x) = \varphi_B(y)$ . Hence we can define the 3-colouring  $\varphi$  of  $D$  by  $\varphi(v) = \varphi_A(v)$  if  $v \in A$ , and  $\varphi(v) = \varphi_B(v)$  if  $v \in B$ . Observe that, since  $A$  is bidirected, all neighbours of  $x$  and  $y$  in  $D\langle A \rangle$  have a colour distinct from  $\varphi(x)$ . Therefore there is no monochromatic directed cycle in  $D$  coloured by  $\varphi$ . Thus  $\varphi$  is a 3-dicolouring of  $D$ , a contradiction. Therefore,  $H = D$  is bidirected, and so  $H$  is an Ore-composition of  $W_A$  and  $W_B$  (because  $D$  is 2-connected by Claim 6), and so  $D$  is 4-Ore, a contradiction. Henceforth, we may assume that either  $W_A$  or  $W_B$  is not 4-Ore.

If none of  $W_A$  and  $W_B$  are a 4-Ore, then by minimality of  $D$ ,  $\rho(W_A) \leq 1$  and  $\rho(W_B) \leq 1$ . Together with Equation (1), this yields

$$\rho(H) \leq \frac{2}{3} - \varepsilon + 2\delta \leq 1$$

because  $2\delta - \varepsilon \leq \frac{1}{3}$ , a contradiction.

If none of  $W_A$  and  $W_B$  is  $\overleftrightarrow{K}_4$ , then  $\rho(W_A) + \rho(W_B) \leq 1 + (\frac{4}{3} + 7\varepsilon - 4\delta)$  (recall that if a digraph is 4-Ore but not  $\overleftrightarrow{K}_4$ , then it has potential at most  $\frac{4}{3} + 7\varepsilon - 4\delta$  by Lemma 22). Thus, with Equation (1), we get

$$\rho(H) \leq 1 + \left(\frac{4}{3} + 7\varepsilon - 4\delta\right) - \frac{4}{3} - \varepsilon + 2\delta = 1 + 6\varepsilon - 2\delta \leq 1$$

because  $\delta \geq 3\varepsilon$ .

Finally, if exactly one of  $W_A$  or  $W_B$  is isomorphic to  $\overleftrightarrow{K}_4$ , then either  $T(W_A - x) = T(W_A) = 2$  (if  $W_A = \overleftrightarrow{K}_4$ ) or  $T(W_B - z) = T(W_B) = 2$  (if  $W_B = \overleftrightarrow{K}_4$ ). Therefore  $T(H) \geq T(W_A - x) + T(W_B - z) \geq T(W_A) + T(W_B) - 1$  by Lemma 13, and so

$$\rho(H) \leq \rho(W_A) + \rho(W_B) - \left(\frac{10}{3} + \varepsilon\right) + 2 + \delta.$$

Now the non 4-Ore digraph among  $W_A, W_B$  has potential at most 1 and the other has potential  $\rho(\overleftrightarrow{K}_4) = \frac{4}{3} + 4\varepsilon - 2\delta$ . Thus

$$\rho(H) \leq 1 + \left(\frac{4}{3} + 4\varepsilon - 2\delta\right) - \left(\frac{10}{3} + \varepsilon\right) + 2 + \delta = 1 + 3\varepsilon - \delta \leq 1$$

because  $\delta \geq 3\varepsilon$ .

In all three cases,  $\rho(D) = \rho(H) \leq 1$ , which is a contradiction. Hence  $D$  is 3-connected.  $\diamond$

**Claim 8.** *If  $R$  is a collapsible subdigraph of  $D$ ,  $u, v$  are in the boundary of  $R$  and  $D\langle R \rangle \cup [u, v]$  is 4-Ore, then there exists  $R' \subseteq R$  such that*

- (i) *either  $R'$  is an Ore-collapsible subdigraph of  $D$ , or*
- (ii)  *$R'$  is an induced subdigraph of  $R$ ,  $n(R') < n(R)$ , and there exist  $u', v'$  in  $\partial_D(R')$  such that  $R' \cup [u', v']$  is 4-Ore.*

*Proof of claim.* If  $\partial(R) = \{u, v\}$ , then  $R$  is Ore-collapsible and we are done. Suppose now that there exists  $w \in \partial(R)$  distinct from  $u$  and  $v$ . Let  $H = D\langle R \rangle \cup [u, v]$ . Observe that  $H \neq \overleftrightarrow{K}_4$  as  $u, v$  and  $w$  receive the same colour in any 3-dicolouring of  $D\langle R \rangle$  because  $R$  is collapsible. Hence  $H$  is the Ore-composition of two 4-Ore digraphs  $H_1$  (the digon side with replaced digon  $[x, y]$ ) and  $H_2$  (the split side with split vertex  $z$ ).

If  $u$  or  $v$  is in  $V(H_2)$ , then  $R' = D\langle V(H_1) \rangle$  with  $u' = x, v' = y$  satisfies (ii). Now we assume that  $u, v \in V(H_1) \setminus V(H_2)$ . By repeating this argument successively on  $H_1$ , and then on the digon-side of  $H_1$ , etc, either we find a subdigraph  $R'$  satisfying (ii) or  $u$  and  $v$  are in a copy  $J$  of  $\overleftrightarrow{K}_4$  such that  $H$  is obtained by Ore-compositions between  $J$  and some 4-Ore digraphs with  $J$  being always in the digon side.

Observe that  $w \notin V(J)$  because in any 3-dicolouring of  $H \setminus [u, v]$ , vertices in  $J$  receive different colours by Lemma 24, except  $u$  and  $v$ . Hence at one step in the succession of



Ore-compositions,  $w$  was in the split-side  $S$  when a digon  $e$  in  $J$  has been replaced. However  $e \neq [u, v]$ , so either  $u$  or  $v$  is not in  $e$ . Suppose without loss of generality that  $e$  is not incident to  $v$ .

We claim that  $H' = R - v \cup [u, w]$  is not 3-dicolourable. Otherwise, let  $\varphi$  be a 3-dicolouring of  $H'$ . Then  $\varphi$  is a 3-dicolouring of  $H - v$  with  $H$  4-Ore, so vertices in  $J - v$  must receive pairwise different colours by Lemma 25. Let  $\varphi'$  be a 3-dicolouring of  $R$ . Without loss of generality, we may assume that  $\varphi(x) = \varphi'(x)$  for every  $x \in V(J - v)$ . If  $y \in S$ , let  $\varphi''(y) = \varphi(y)$ , and let  $\varphi''(y) = \varphi'(y)$  if  $y \notin S$ . Then  $\varphi''$  is a 3-dicolouring of  $R$  but with  $\varphi''(u) \neq \varphi''(w)$ , contradicting the fact that  $R$  is collapsible. This shows that  $H' = R - v \cup [u, w]$  is not 3-dicolourable.

Hence  $R - v \cup [u, w]$  contains a 4-dicritical digraph  $K$ . By Lemma 23,  $R' = D\langle V(K) \rangle$ , as a subdigraph of  $H$  which is a 4-Ore, satisfies  $\frac{10}{3}n(R') - m(R') \geq \frac{10}{3}$ . This implies that  $\frac{10}{3}n(K) - m(K) \geq \frac{4}{3}$ . Note also that  $K$  is bidirected because  $R - v$  is bidirected. Thus, by Theorem 21,  $K$  is 4-Ore. Hence  $R'$  with  $u, w$  satisfies (ii).  $\diamond$

**Claim 9.** *If  $R$  is a subdigraph of  $D$  with  $n(R) < n(D)$  and  $u, v \in V(R)$ , then  $R \cup [u, v]$  is 3-dicolourable. As a consequence, there is no collapsible subdigraph in  $D$ .*

*Proof of claim.* Assume for a contradiction that the statement is false. Consider a smallest induced subdigraph  $R$  for which the statement does not hold. Then  $K = R \cup [u, v]$  is 4-vertex-dicritical, that is for every vertex  $v \in V(K)$ ,  $\vec{\chi}(K - v) < 4 = \vec{\chi}(K)$ . Note that 4-vertex-dicritical digraphs smaller than  $D$  satisfy the outcome of Theorem 8 since adding arcs does not increase the potential. Note that  $\rho(R) \leq \rho(K) + 2 + \delta$ .

If  $R$  is not collapsible, then, by Claim 5,  $\rho(R) \geq \rho(D) + \frac{8}{3} - \varepsilon - \delta > \frac{11}{3} - \varepsilon - \delta$ . But we also have  $\rho(R) \leq \rho(K) + 2 + \delta \leq \frac{10}{3} + 4\varepsilon - \delta$  by Claim 1, which is a contradiction because  $5\varepsilon \leq \frac{1}{3}$ . Hence  $R$  is collapsible.

Let  $\varphi$  be a 3-dicolouring of  $R$ . Observe that  $\varphi(u) = \varphi(v)$  for otherwise  $R \cup [u, v]$  would be 3-dicolourable. Let  $R'$  be the dicritical extension of  $R$  with extender  $W$  and core  $X_W$ . We have  $R' = D$  and  $|X_W| = 1$ . Since  $R$  is collapsible, for every two vertices  $u', v'$  on the boundary of  $R$ ,  $R \cup [u', v']$  is not 3-dicolourable. Hence, free to consider  $u', v'$  instead of  $u, v$ , we can suppose that  $u$  and  $v$  are on the boundary of  $R$ . If  $K$  is 4-Ore, then, by Claim 8 and by minimality of  $R$ , we have that  $R$  is Ore-collapsible, and so has boundary of size 2. This contradicts the fact that  $D$  is 3-connected. Hence  $K$  is not 4-Ore.

By Claim 2, we have

$$\begin{aligned} 1 < \rho(D) = \rho(R') &\leq \rho(W) + \rho(R) - \frac{10}{3} - \varepsilon + \delta \\ &\leq \rho(W) + (\rho(K) + 2 + \delta) - \frac{10}{3} - \varepsilon + \delta \end{aligned}$$

and as  $\rho(K) \leq 1$  (because it is not 4-Ore and by minimality of  $D$ ) we get

$$1 < 1 + \rho(W) - \left( \frac{4}{3} + \varepsilon - 2\delta \right)$$

that is  $\rho(W) > \frac{4}{3} + \varepsilon - 2\delta$ . But as  $W$  is smaller than  $D$ , it satisfies Theorem 8. Thus, since  $\varepsilon - 2\delta \geq \frac{-1}{3}$ ,  $W$  must be 4-Ore. Moreover,  $W$  must be isomorphic to  $\overleftrightarrow{K}_4$ , for otherwise  $\rho(W)$  would be at most  $\frac{4}{3} + 7\varepsilon - 4\delta$ , and  $\frac{4}{3} + 7\varepsilon - 4\delta \geq \rho(W) > \frac{4}{3} + \varepsilon - 2\delta$  would contradict  $\delta \geq 3\varepsilon$ . Hence  $\rho(W) = \rho(\overleftrightarrow{K}_4) = \frac{4}{3} + 4\varepsilon - 2\delta$  and  $T(W - X_W) = 2 = T(W)$ . Thus, by Claim 2 and because  $\delta \geq 3\varepsilon$ , we have

$$1 < \rho(D) \leq \rho(W) + \rho(K) + 2 + \delta - \frac{10}{3} - \varepsilon \leq \rho(K) + 3\varepsilon - \delta \leq \rho(K) \leq 1,$$

a contradiction.

This implies that  $D$  does not contain any collapsible subdigraph. Indeed, assume for a contradiction that  $D$  contains a collapsible subdigraph  $R$ , and let  $u, v$  be two vertices in its boundary. Then there exists a 3-dicolouring  $\varphi$  of  $R \cup [u, v]$ , for which  $\partial(R)$  is not monochromatic, a contradiction.  $\diamond$

**Claim 10.** *If  $R$  is a subdigraph of  $D$  with  $n(R) < n(D)$  and  $u, v, u', v' \in R$ , then  $R \cup \{uv, u'v'\}$  is 3-dicolourable. In particular,  $D$  contains no copy of  $\overleftrightarrow{K}_4$  minus two arcs.*

*Proof of claim.* Assume for a contradiction that the statement is false. Consider a smallest subdigraph  $R$  for which the statement does not hold. Then  $K = R \cup \{uv, u'v'\}$  is 4-dicritical and smaller than  $D$ , so  $\rho(K) \leq \frac{4}{3} + 4\varepsilon - 2\delta$  by Claim 1. By Claim 9,  $R$  is not collapsible, so, by Claim 5, we have  $\rho(R) \geq \rho(D) + \frac{8}{3} - \varepsilon - \delta > \frac{11}{3} - \varepsilon - \delta$ . But  $\rho(R) \leq \rho(K) + 2 + 2\delta \leq \frac{10}{3} + 4\varepsilon$ , which is a contradiction as  $5\varepsilon + \delta \leq \frac{1}{3}$ .  $\diamond$

For any  $v \in V(D)$ , we denote by  $n(v)$  its number of neighbours, that is  $n(v) = |N^+(v) \cup N^-(v)|$ , and by  $d(v)$  its number of incident arcs, that is  $d(v) = d^+(v) + d^-(v)$ .

**Claim 11.** *Vertices of degree 6 in  $D$  have either three or six neighbours.*

*Proof of claim.* Let  $x$  be a vertex of degree 6.

If  $n(x) = 4$ , then let  $a, b, c, d$  be its neighbours such that  $N^+(x) = \{a, b, c\}$  and  $N^-(x) = \{a, b, d\}$ . Consider  $D' = D - x \cup dc$ . By Claim 10,  $D'$  has a 3-dicolouring  $\varphi$ . If  $|\varphi(N^-(x))| < 3$ , then choosing  $\varphi(x)$  in  $\{1, 2, 3\} \setminus \varphi(N^-(x))$ , we obtain a 3-dicolouring of  $D$ , a contradiction. Hence  $\varphi(N^-(x)) = \{1, 2, 3\}$ . We set  $\varphi(x) = \varphi(d)$ . As  $D$  is not 3-dicolourable,  $D$  contains a monochromatic directed cycle  $C$ . This cycle  $C$  must contain the arc  $dx$ , and an out-neighbour  $z$  of  $x$ . Since  $\varphi(a), \varphi(b)$  and  $\varphi(d)$  are all distinct, necessarily  $z = c$ . But then  $C - x \cup dc$  is a monochromatic directed cycle in  $D'$ , a contradiction.

Similarly, if  $n(x) = 5$ , let  $N^+(x) = \{a, b, c\}$  and  $N^-(x) = \{a, d, e\}$ , and consider  $D' = D - x \cup db \cup dc$ . By Claim 10,  $D'$  has a 3-dicolouring  $\varphi$ . If  $|\varphi(N^-(x))| < 3$ , then choosing  $\varphi(x)$  in  $\{1, 2, 3\} \setminus \varphi(N^-(x))$ , we obtain a 3-dicolouring of  $D$ , a contradiction. Hence  $\varphi(N^-(x)) = \{1, 2, 3\}$ . We set  $\varphi(x) = \varphi(d)$ . As  $D$  is not 3-dicolourable, there is a monochromatic directed cycle  $C$ , which must contain the arc  $dx$  and an out-neighbour  $z$  of  $x$ . Note that  $z$  must be  $b$  or  $c$  because  $\varphi(a) \neq \varphi(d)$ . Then  $C - x \cup dz$  is a monochromatic directed cycle in  $D'$ , a contradiction.  $\diamond$

**Claim 12.** *There is no bidirected triangle containing two vertices of degree 6. In particular,  $D$  contains no emerald.*

*Proof of claim.* Suppose that  $D(\{x, y, z\}) = \overleftrightarrow{K}_3$  and  $d(x) = d(y) = 6$ . By Claim 11,  $x$  and  $y$  have exactly three neighbours, and  $N[x] \neq N[y]$  because  $D$  contains no copy of  $\overleftrightarrow{K}_4$  minus two arcs by Claim 10. Let  $u$  (resp.  $v$ ) be the unique neighbour of  $x$  distinct from  $y$  and  $z$  (resp.  $x$  and  $z$ ). Consider  $D' = D - \{x, y\} \cup [u, v]$ . By Claim 9,  $D'$  has a 3-dicolouring  $\varphi$ . Without loss of generality, suppose that  $\varphi(u) = 1$  and  $\varphi(v) = 2$ . If  $\varphi(z) = 1$  (resp.  $\varphi(z) = 2$ ,  $\varphi(z) = 3$ ), we set  $\varphi(x) = 2$  and  $\varphi(y) = 3$  (resp.  $\varphi(x) = 3$  and  $\varphi(y) = 1$ ,  $\varphi(x) = 2$  and  $\varphi(y) = 1$ ). In each case, this yields a 3-dicolouring of  $D$ , a contradiction.  $\diamond$

So now we know that  $D$  contains no emerald, and no diamond by Claim 7.

**Claim 13.** *If  $R$  is an induced subdigraph of  $D$  with  $4 \leq n(R) < n(D)$ , then  $\rho(R) \geq \rho(D) + 3 + 3\varepsilon - 3\delta$ , except if  $D - R$  contains a single vertex which has degree 6 in  $D$ .*

*Proof of claim.* Let  $R$  be an induced subdigraph of  $D$  with  $4 \leq n(R) < n(D)$ . By Claim 9,  $R$  is not collapsible. Let  $\varphi$  be a 3-dicolouring of  $R$ ,  $R'$  be a dicritical extension of  $R$  with extender  $W$  and core  $X_W$  (with respect to  $\varphi$ ). By (the consequence of) Claim 3, we know that  $\rho(R') \geq \rho(D)$ .

Assume first that  $R' \neq D$ . Then, by Claims 3 and 2,

$$\rho(D) + 2 - 3\varepsilon + \delta \leq \rho(R') \leq \rho(W) + \rho(R) - \frac{10}{3} - \varepsilon + \delta.$$

Since  $\rho(W) \leq \frac{4}{3} + 4\varepsilon - 2\delta$  by Claim 1, we have  $\rho(R) \geq \rho(D) + 4 - 6\varepsilon + 2\delta \geq \rho(D) + 3 + 3\varepsilon - 3\delta$ , because  $1 \geq 9\varepsilon - 5\delta$ . In the following we suppose that  $R' = D$ . We distinguish three cases depending on the cardinality of  $|X_W|$ .

- Assume first that  $|X_W| = 2$ . Then, by Claim 2 and Lemma 13,

$$\rho(D) \leq \rho(R') \leq \rho(W) + \rho(R) - \frac{20}{3} - 2\varepsilon + 2 + 2\delta$$

and, as  $\rho(W) \leq \frac{4}{3} + 4\varepsilon - 2\delta$  by Claim 1, we have  $\rho(R) \geq \rho(D) + \frac{10}{3} - 2\varepsilon \geq \rho(D) + 3 + 3\varepsilon - 3\delta$  because  $5\varepsilon - 3\delta \leq \frac{1}{3}$ .

- Assume now that  $|X_W| = 3$ . If there is a vertex  $v \in V(D - R)$  with two out-neighbours (resp. two in-neighbours) in  $V(R)$  with the same colour for  $\varphi$ , then

- $n(R') = n(W) - |X_W| + n(R)$ ,
- $m(R') \geq m(W) + m(R) - m(\overleftrightarrow{K}_{|X_W|}) + 1$  because  $v$  has two in- or out-neighbour in  $V(R)$  with the same colour for  $\varphi$ ,
- $T(R') \geq T(W - X_W) + T(R)$ .

It follows that

$$\rho(D) \leq \rho(R') \leq \rho(W) + \rho(R) - (10 + 3\varepsilon - 6) + 3\delta - 1$$

and so  $\rho(R) \geq \rho(D) - \frac{4}{3} - 4\varepsilon + 2\delta + 5 + 3\varepsilon - 3\delta \geq \rho(D) + \frac{11}{3} - \varepsilon - \delta \geq \rho(D) + 3 + 3\varepsilon - 3\delta$  because  $4\varepsilon - 2\delta \leq \frac{2}{3}$ . Now we assume that there is no vertex with two out-neighbours (resp. two in-neighbours) in  $R$  with the same colour for  $\varphi$ . In other words, the in-degrees and out-degrees of vertices in  $D - R$  are the same in  $D$  and in  $W$ .

If  $W$  is not 4-Ore, then by Claim 2

$$\rho(D) \leq \rho(R') \leq \rho(W) + \rho(R) - (10 + 3\varepsilon - 6) + 3\delta$$

and, as  $\rho(W) \leq 1$ , we have  $\rho(R) \geq \rho(D) + 3 + 3\varepsilon - 3\delta$ .

Now suppose  $W$  is 4-Ore. If  $W \neq \overleftrightarrow{K}_4$ , then, by Lemma 20,  $W$  contains a diamond or an emerald disjoint from  $X$ , and this gives a diamond or an emerald in  $D$  because the degrees of vertices in  $D - R$  are the same in  $D$  and in  $W$ , which is a contradiction.

Now suppose that  $W = \overleftrightarrow{K}_4$ . Then  $D - R$  has a single vertex of degree 6 in  $D$ .

- Assume finally that  $|X_W| = 1$ . Since  $R$  is not collapsible by Claim 9,  $\varphi$  may have been chosen so that  $\partial(R)$  is not monochromatic in  $\varphi$ . Then, by Claim 4,  $\rho(R) \geq \rho(D) + 3 - 3\varepsilon + \delta \geq \rho(D) + 3 + 3\varepsilon - 3\delta$ , because  $6\varepsilon - 4\delta \leq 0$ .

◇

In  $D$ , we say that a vertex  $v$  is a **simple in-neighbour** (resp. **simple out-neighbour**) if  $v$  is a in-neighbour (resp. out-neighbour) of  $u$  and  $[u, v]$  is not a digon in  $D$ . If  $v$  is a simple in-neighbour or simple out-neighbour of  $u$ , we simply say that  $v$  is a **simple neighbour** of  $u$ .

**Claim 14.** *Vertices of degree 7 have seven neighbours. In other words, every vertex of degree 7 has only simple neighbours.*

*Proof of claim.* Let  $x$  be a vertex of degree 7. We suppose, without loss of generality, that  $d^-(x) = 3$  and  $d^+(x) = 4$ .

If  $n(x) = 4$ , then  $x$  has a unique simple out-neighbour  $a$ . As  $D$  is 4-dicritical,  $D \setminus xa$  has a 3-dicolouring  $\varphi$ . But then every directed cycle is either in  $D \setminus xa$  or it contains  $xa$  and thus an in-neighbour  $t$  of  $x$ . In the first case, it is not monochromatic because  $\varphi$  is a 3-dicolouring of  $D \setminus xa$ , and in the second case, it is not monochromatic because  $[t, x]$  is a digon and so  $\varphi(t) \neq \varphi(x)$ . Hence  $\varphi$  is a 3-dicolouring of  $D$ , a contradiction.

If  $n(x) = 5$ , let  $N^-(x) = \{a, b, c\}$  and  $N^+(x) = \{a, b, d, e\}$ . By Claim 10,  $D' = D - x \cup \{cd, ce\}$  has a 3-dicolouring  $\varphi$ . If  $|\varphi(N^-(x))| < 3$ , then choosing  $\varphi(x)$  in  $\{1, 2, 3\} \setminus \varphi(N^-(x))$  gives a 3-dicolouring of  $D$ , a contradiction. If  $|\varphi(N^-(x))| = 3$ , then we set  $\varphi(x) = \varphi(c)$ . Suppose for a contradiction that there is a monochromatic directed cycle  $C$  in  $D$  (with  $\varphi$ ).

Necessarily  $C$  contains  $x$  (since  $\varphi$  is a 3-dicolouring of  $D - x$ ) and so it must contain  $c$  and one vertex  $y$  in  $\{d, e\}$  because  $\varphi(a)$ ,  $\varphi(b)$ , and  $\varphi(c)$  are all distinct. Then  $C - x \cup cy$  is a monochromatic directed cycle in  $D'$ , a contradiction. Therefore  $\varphi$  is a 3-dicolouring of  $D$ , a contradiction.

If  $n(x) = 6$ , let  $N^-(x) = \{a, b, c\}$  and  $N^+(x) = \{a, d, e, f\}$ . Consider  $D' = D - x \cup \{bd, be, bf\}$ .

We first show that  $D'$  is not 3-dicolourable. Assume for a contradiction that there is a 3-dicolouring  $\varphi$  of  $D'$ . If  $|\varphi(N^-(x))| < 3$ , then choosing  $\varphi(x)$  in  $\{1, 2, 3\} \setminus \varphi(N^-(x))$  gives a 3-dicolouring of  $D$ , a contradiction. Hence  $|\varphi(N^-(x))| = 3$ . We set  $\varphi(x) = \varphi(b)$ . Since  $D$  is not 3-dicolourable, there exists a monochromatic directed cycle  $C$  in  $D$  (with  $\varphi$ ). Necessarily  $C$  contains  $x$  (since  $\varphi$  is a 3-dicolouring of  $D - x$ ) and so it must contain  $b$  and one vertex  $y$  in  $\{d, e, f\}$  because  $\varphi(a)$ ,  $\varphi(b)$ , and  $\varphi(c)$  are all distinct. Then  $C - x \cup by$  is a monochromatic directed cycle in  $D'$ , a contradiction. This gives a 3-dicolouring of  $D$ , a contradiction.

Henceforth  $D'$  is not 3-dicolourable, and so it contains a 4-dicritical digraph  $\tilde{D}$ , smaller than  $D$ . If  $\tilde{D}$  does not contain the three arcs  $bd, be, bf$ , then it can be obtained from a proper induced subdigraph of  $D$  by adding at most two arcs, and so it is 3-dicolourable by Claim 10, a contradiction.

Hence  $\{b, d, e, f\} \subseteq V(\tilde{D})$ . Now consider  $U = D \setminus (V(\tilde{D}) \cup \{x\})$ .

• Assume first that  $a \notin V(U)$  or  $c \notin V(U)$ . Then we have

- $n(U) = n(\tilde{D}) + 1$ ,
- $m(U) \geq m(\tilde{D}) + 1$  and
- $T(U) \geq T(\tilde{D} - b) \geq T(\tilde{D}) - 1$  by Lemma 13.

Hence

$$\begin{aligned}
\rho(U) &\leq \rho(\tilde{D}) + \frac{10}{3} + \varepsilon - 1 + \delta \\
&\leq \frac{4}{3} + 4\varepsilon - 2\delta + \frac{10}{3} + \varepsilon - 1 + \delta \quad \text{by Claim 1,} \\
&= 1 + \frac{8}{3} + 5\varepsilon - \delta \\
&< \rho(D) + \frac{8}{3} + 5\varepsilon - \delta \\
&\leq \rho(D) + 3 + 3\varepsilon - 3\delta \qquad \text{because } \frac{1}{3} \geq 2\delta + 2\varepsilon.
\end{aligned}$$

Hence by Claim 13,  $D - U$  has a single vertex of degree 6 (in  $D$ ), which must be either  $a$  or  $c$ . Then we have

- $n(D) = n(\tilde{D}) + 2$ ,
- $m(D) \geq m(\tilde{D}) - 3 + 11$  and
- $T(D) \geq T(\tilde{D} - b) \geq T(\tilde{D}) - 1$ .

Thus

$$\begin{aligned}
\rho(D) &\leq \rho(\tilde{D}) + 2 \left( \frac{10}{3} + \varepsilon \right) - 8 + \delta \\
&\leq \left( \frac{4}{3} + 4\varepsilon - 2\delta \right) - \frac{4}{3} + 2\varepsilon + \delta \quad \text{by Claim 1,} \\
&\leq 1 \quad \text{because } 6\varepsilon - \delta \leq 1.
\end{aligned}$$

This is a contradiction.

• Assume now that  $a, c \in V(U)$ , then we have

- $n(U) = n(\tilde{D}) + 1$ ,
- $m(U) \geq m(\tilde{D}) + 4$  and
- $T(U) \geq T(\tilde{D} - b) \geq T(\tilde{D}) - 1$  by Lemma 13.

Thus

$$\begin{aligned}
\rho(U) &\leq \rho(\tilde{D}) + \frac{10}{3} + \varepsilon - 4 + \delta \\
&\leq \left( \frac{4}{3} + 4\varepsilon - 2\delta \right) + \frac{10}{3} + \varepsilon - 4 + \delta \quad \text{by Claim 1,} \\
&\leq 1 \quad \text{because } 5\varepsilon - \delta \leq \frac{1}{3}.
\end{aligned}$$

Together with the consequence of Claim 3, we get that  $\rho(D) \leq \rho(U) \leq 1$ , a contradiction.

◇

The  $8^+$ -**valency** of a vertex  $v$ , denoted by  $\nu(v)$ , is the number of arcs incident to  $v$  and a vertex of degree at least 8.

Let  $D_6$  be the subdigraph of  $D$  induced by the vertices of degree 6 incident to digons. Let us describe the connected components of  $D_6$  and their neighbourhoods. Remember that vertices of degree 7 are incident to no digon by Claim 14, and so they do not have neighbours in  $V(D_6)$ . If  $v$  is a vertex in  $D_6$ , we define its **neighbourhood valency** to be the sum of the  $8^+$ -valency of its neighbours of degree at least 8. We denote the neighbourhood valency of  $v$  by  $\nu_N(v)$ .

**Claim 15.** *If  $[x, y]$  is a digon and both  $x$  and  $y$  have degree 6, then either*

- (i) *the two neighbours of  $y$  distinct from  $x$  have degree at least 8, or*
- (ii) *the two neighbours of  $x$  distinct from  $y$  have degree at least 8 and  $\nu_N(x) \geq 4$ .*

*Proof of claim.* Let  $[x, y]$  be a digon in  $D$  with  $d(x) = d(y) = 6$ . By Claim 11  $n(x) = n(y) = 3$ . Let  $u$  and  $v$  be the two neighbours of  $x$  different from  $y$ . By Claim 14,  $u$  and  $v$  have degree 6 or at least 8.

If  $u$  and  $v$  are linked by a digon, then by Claim 12,  $u$  and  $v$  do not have degree 6, so they have degree 8. Moreover  $\nu(u) \geq 2$  and  $\nu(v) \geq 2$ . Thus  $\nu_N(x) = \nu(u) + \nu(v) \geq 4$  and (ii) holds. Henceforth, we may assume that  $u$  and  $v$  are not linked by a digon.

Let  $D'$  the digraph obtained by removing  $x$  and  $y$  and identifying  $u$  and  $v$  into a single vertex  $u \star v$ . We claim that  $D'$  is not 3-dicolourable. To see that, suppose for contradiction that there exists a 3-dicolouring  $\varphi$  of  $D'$ . Then set  $\varphi(u) = \varphi(v) = \varphi(u \star v)$ , choose  $\varphi(y)$  in  $\{1, 2, 3\} \setminus \varphi(N(y) \setminus \{x\})$ , and finally choose  $\varphi(x)$  in  $\{1, 2, 3\} \setminus \{\varphi(u \star v), \varphi(y)\}$ . One can easily see that  $\varphi$  is now a 3-dicolouring of  $D$ , a contradiction. This proves that  $D'$  is not 3-dicolourable and so it contains a 4-dicritical digraph  $\tilde{D}$ , which must contain  $u \star v$  because every subdigraph of  $D$  is 3-dicolourable. Let  $R$  be the subdigraph of  $D$  induced by  $(V(\tilde{D}) \setminus \{u \star v\}) \cup \{u, v, x\}$ . We have

- $n(R) = n(\tilde{D}) + 2$ ,
- $m(R) \geq m(\tilde{D}) + 4$  and
- $T(R) \geq T(\tilde{D} - u \star v) + 1 \geq T(\tilde{D})$  because  $[x, u]$  is a digon, and by Lemma 13.

If  $\tilde{D}$  is not 4-Ore, then  $\rho(\tilde{D}) \leq 1$  by minimality of  $D$ , and so

$$\begin{aligned} \rho(R) &\leq \rho(\tilde{D}) + 2 \left( \frac{10}{3} + \varepsilon \right) - 4 \\ &\leq 1 + \frac{8}{3} + 2\varepsilon \\ &< \rho(D) + 3 + 3\varepsilon - 3\delta \quad \text{because } \varepsilon - 3\delta \geq -\frac{1}{3}. \end{aligned}$$

Similarly, if  $\tilde{D}$  is 4-Ore but not  $\overleftrightarrow{K}_4$ , then

$$\begin{aligned} \rho(R) &\leq \rho(\tilde{D}) + 2 \left( \frac{10}{3} + \varepsilon \right) - 4 \\ &\leq \left( \frac{4}{3} + 7\varepsilon - 4\delta \right) + \frac{8}{3} + 2\varepsilon \quad \text{by Lemma 22,} \\ &= 1 + 3 + 9\varepsilon - 4\delta \\ &< \rho(D) + 3 + 9\varepsilon - 4\delta \\ &\leq \rho(D) + 3 + 3\varepsilon - 3\delta \quad \text{because } \delta \geq 6\varepsilon. \end{aligned}$$

In both cases (that is when  $\tilde{D}$  is not  $\overleftrightarrow{K}_4$ ), by Claim 13,  $D - R$  is a single vertex of degree 6, namely  $y$ . Then every neighbour  $w$  of  $y$  different from  $x$  has degree at least 6 in  $\tilde{D}$  (because  $\tilde{D}$  is 3-dicritical) and so has degree at least 8 in  $D$  and (i) holds.

Assume now that  $\tilde{D}$  is a copy of  $\overleftrightarrow{K}_4$ . Let us denote by  $a, b, c$  the vertices of  $\tilde{D}$  different from  $u \star v$ . Suppose for a contradiction that  $u$  has degree 6. Then  $u$  has exactly three

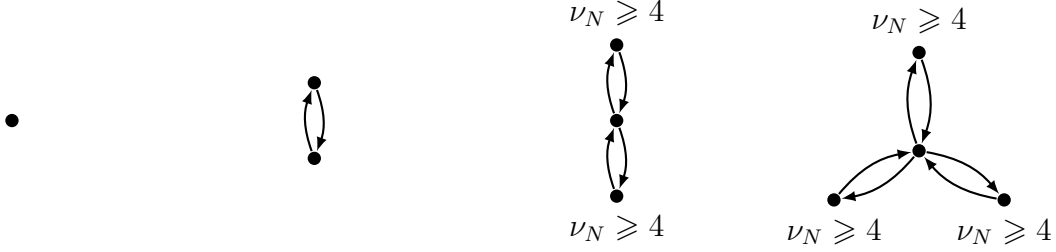


Figure 3: The possible connected components of  $D_6$ .

neighbours by Claim 11. If  $|N(u) \cap \{a, b, c\}| = 2$ , then  $D\langle\{u, a, b, c\}\rangle$  is a copy of  $\overleftrightarrow{K}_4$  minus a digon, contradicting Claim 9. If  $|N(u) \cap \{a, b, c\}| \leq 1$ , then  $v$  must be adjacent to at least two vertices of  $\{a, b, c\}$  with a digon, and so  $D\langle\{v, a, b, c\}\rangle$  contains a copy of  $\overleftrightarrow{K}_4$  minus a digon, contradicting Claim 9. Hence  $u$  has degree at least 8, and by symmetry so does  $v$ . Moreover  $D\langle\{a, b, c\}\rangle$  is a bidirected triangle, and so by Claim 12, at least two of these vertices have degree at least 8 (remember that vertices of degree 7 are in no digon by Claim 14). Hence at least four arcs between  $\{u, v\}$  and  $\{a, b, c\}$  are incident to two vertices of degree at least 8. In other word,  $\nu_N(x) = \nu(u) + \nu(v) \geq 4$ , so (ii) holds.  $\diamond$

**Claim 16.** *Let  $C$  be a connected component of  $D_6$ . Then  $C$  is one of the following (see Figure 3):*

- (i) *a single vertex, or*
- (ii) *a bidirected path on two vertices, or*
- (iii) *a bidirected path on three vertices, whose extremities have neighbourhood valency at least 4, or*
- (iv) *a star on four vertices, whose non-central vertices have neighbourhood valency at least 4.*

*Proof of claim.* First observe that  $C$  does not contain a bidirected path  $\llbracket x, y, z, w \rrbracket$  on four vertices, because otherwise, by Claim 15 applied on  $[y, z]$ , either  $y$  or  $z$  has two neighbours of degree at least 8, a contradiction. Observe also that  $C$  contains no bidirected triangle by Claim 12.

Moreover, if  $\llbracket x, y, z \rrbracket$  is a bidirected path in  $C$  on three vertices, then by Claim 15 applied both on  $[y, z]$  and  $[z, y]$ ,  $x$  and  $z$  have both neighbourhood valency at least 4. The statement of the claim follows.  $\diamond$

An arc  $xy$  is said to be **out-chelou** if

- (i)  $yx \notin A(D)$ ,
- (ii)  $d^+(x) = 3$ ,



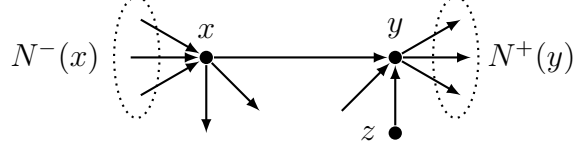


Figure 4: An example of an out-chelou arc  $xy$ .

(iii)  $d^-(y) = 3$ , and

(iv) there exists  $z \in N^-(y) \setminus N^+(y)$  distinct from  $x$ .

Symmetrically, we say that an arc  $xy$  is **in-chelou** if  $yx$  is out-chelou in the digraph obtained from  $D$  by reversing every arc. See Figure 4 for an example of an out-chelou arc.

**Claim 17.** *There is no out-chelou arc and no in-chelou arc in  $D$ .*

*Proof of claim.* By directional duality, it suffices to prove that  $D$  has no out-chelou arcs.

Let  $xy$  be an out-chelou arc with  $z \in N^-(y) \setminus (N^+(y) \cup \{x\})$ . Consider  $D' = D - \{x, y\} \cup \{zz' \mid z' \in N^+(y) \setminus N^-(y)\}$ . We claim that  $D'$  is not 3-dicolourable. To see that, suppose for contradiction that there is a 3-dicolouring  $\varphi$  of  $D'$ . As  $d^+(x) = 3$ , we can choose  $\varphi(x)$  in  $\{1, 2, 3\} \setminus \varphi(N^+(x) \setminus \{y\})$  to obtain a 3-dicolouring of  $D - y$ . If  $|\varphi(N^-(y))| < 3$ , then choosing  $\varphi(y)$  in  $\{1, 2, 3\} \setminus \varphi(N^-(y))$  gives a 3-dicolouring of  $D$ , a contradiction. Hence  $|\varphi(N^-(y))| = 3$ . Set  $\varphi(x) = \varphi(z)$ . Suppose there is a monochromatic directed cycle  $C$  in  $D$ . It must contain  $y$  and thus  $z$ , its unique in-neighbour with its colour. Let  $z'$  be the out-neighbour of  $y$  in  $C$ . It must be in  $N^+(y) \setminus N^-(y)$ , so  $zz'$  is an arc in  $D'$ . Thus  $C - y \cup zz'$  is a monochromatic directed cycle in  $D'$ , a contradiction. Therefore  $\varphi$  is a 3-dicolouring of  $D$ , a contradiction. Hence  $D'$  is not 3-dicolourable.

Consequently,  $D'$  contains a 4-dicritical digraph  $\tilde{D}$ , which is smaller than  $D$  and contains  $z$ , for otherwise  $\tilde{D}$  would be a subdigraph of  $D$ . Consider  $U = D(V(\tilde{D}) \cup \{y\})$ . We have

- $n(U) = n(\tilde{D}) + 1$ ,
- $m(U) \geq m(\tilde{D}) + 1$  and
- $T(U) \geq T(\tilde{D} - z) \geq T(\tilde{D}) - 1$  by Lemma 13.

First if  $\tilde{D}$  is not 4-Ore, then by minimality of  $D$  we have  $\rho(\tilde{D}) \leq 1$ , so

$$\rho(U) \leq \rho(\tilde{D}) + \frac{10}{3} + \varepsilon - 1 + \delta \leq \frac{10}{3} + \varepsilon + \delta \leq \frac{11}{3} - \varepsilon - \delta$$

because  $2\varepsilon + 2\delta \leq \frac{1}{3}$ .

Next if  $\tilde{D}$  is 4-Ore, but not isomorphic to  $\overleftrightarrow{K}_4$ , then  $\rho(\tilde{D}) \leq \frac{4}{3} + 7\varepsilon - 4\delta$  by Lemma 22, and

$$\rho(U) \leq \rho(\tilde{D}) + \frac{10}{3} + \varepsilon - 1 + \delta \leq \frac{11}{3} + 8\varepsilon - 3\delta \leq \frac{11}{3} - \varepsilon - \delta$$

because  $9\varepsilon - 2\delta \leq 0$ .

Finally if  $\tilde{D}$  is isomorphic to  $\overrightarrow{K_4}$ , then we have  $T(U) \geq T(\tilde{D} - z) \geq T(\tilde{D})$  and  $\rho(\tilde{D}) = \frac{4}{3} + 4\varepsilon - 2\delta$ . So the same computation yields

$$\rho(U) \leq \rho(\tilde{D}) + \frac{10}{3} + \varepsilon - 1 \leq \frac{11}{3} + 5\varepsilon - 2\delta \leq \frac{11}{3} - \varepsilon - \delta$$

because  $6\varepsilon - \delta \leq 0$ .

In all cases, we have  $\rho(U) \leq \frac{11}{3} - \varepsilon - \delta$ . This contradicts Claim 5 because  $U$  is not collapsible by Claim 9.  $\diamond$

We now use the discharging method. For every vertex  $v$ , let  $\sigma(v) = \frac{\delta}{|C|}$  if  $v$  has degree 6 and is in a component  $C$  of  $D_6$  of size at least 2, and  $\sigma(v) = 0$  otherwise. Clearly  $T(D)$  is at least the number of connected components of size at least 2 of  $D_6$  so  $\sum_{v \in V(D)} \sigma(v) \leq \delta T(D)$ . We define the **initial charge** of  $v$  to be  $w(v) = \frac{10}{3} + \varepsilon - \frac{d(v)}{2} - \sigma(v)$ . We have

$$\rho(D) \leq \sum_{v \in V(D)} w(v).$$

We now redistribute this total charge according to the following rules:

- (R1) A vertex of degree 6 incident to no digon sends  $\frac{1}{12} - \frac{\varepsilon}{8}$  to each of its neighbours.
- (R2) A vertex of degree 6 incident to digons sends  $\frac{2}{d(v)-\nu(v)}(-\frac{10}{3} + \frac{d(v)}{2} - \varepsilon)$  to each neighbour  $v$  of degree at least 8 (so  $\frac{1}{d(v)-\nu(v)}(-\frac{10}{3} + \frac{d(v)}{2} - \varepsilon)$  via each arc of the digon).
- (R3) A vertex of degree 7 with  $d^-(v) = 3$  (resp.  $d^+(v) = 3$ ) sends  $\frac{1}{12} - \frac{\varepsilon}{8}$  to each of its in-neighbours (resp. out-neighbours).

For every vertex  $v$ , let  $w^*(v)$  be the final charge of  $v$ .

**Claim 18.** *If  $v$  has degree at least 8, then  $w^*(v) \leq 0$ .*

*Proof of claim.* Let  $v$  be a vertex of degree at least 8. If  $v$  is not adjacent to a vertex of degree at most 7, then  $w^*(v) = w(v) = \frac{10}{3} + \varepsilon - \frac{d(v)}{2} \leq 0$  (because  $\varepsilon \leq \frac{2}{3}$ ). Otherwise,  $d(v) - \nu(v) \geq 1$  and

$$\begin{aligned} \frac{1}{d(v) - \nu(v)} \left( -\frac{10}{3} + \frac{d(v)}{2} - \varepsilon \right) &\geq \frac{1}{d(v)} \left( -\frac{10}{3} + \frac{d(v)}{2} - \varepsilon \right) \\ &\geq \frac{1}{12} - \frac{\varepsilon}{8}. \end{aligned}$$

Thus  $v$  receives at most  $\frac{1}{d(v)-\nu(v)}(-\frac{10}{3} + \frac{d(v)}{2} - \varepsilon)$  per arc incident with a vertex of degree 6 or 7. Since there are  $d(v) - \nu(v)$  such arcs,  $w^*(v) \leq w(v) - \frac{10}{3} - \varepsilon + \frac{d(v)}{2} = 0$ .  $\diamond$

**Claim 19.** *If  $v$  has degree 7, then  $w^*(v) \leq 0$ .*

*Proof of claim.* By Claim 14,  $v$  has seven neighbours. Without loss of generality, let us suppose that  $d^-(v) = 3$  and  $d^+(v) = 4$ . By Claim 17, the in-neighbours of  $v$  can not have out-degree 3. In particular, they do not have degree 6, and if they have degree 7, they do not send anything to  $v$  by Rule (R3). Hence  $v$  receives at most four times the charge  $\frac{1}{12} - \frac{\varepsilon}{8}$  by (R1) or (R3), and it sends three times this charge by (R3). Hence

$$\begin{aligned} w^*(v) &\leq w(v) + \frac{1}{12} - \frac{\varepsilon}{8} \\ &= -\frac{1}{12} + \frac{7}{8}\varepsilon \end{aligned}$$

and the result comes because  $\varepsilon \leq \frac{2}{21}$ . ◇

**Claim 20.** *If  $v$  is a vertex of degree 6 incident to no digon, then  $w^*(v) \leq 0$ .*

*Proof of claim.* The vertex  $v$  sends  $\frac{1}{12} - \frac{\varepsilon}{8}$  to each of its neighbours, and it receives no charge as all its in-neighbours (resp. out-neighbours) have out-degree (resp. in-degree) at least 4, by Claim 17. As a consequence,

$$w^*(v) = w(v) - 6 \left( \frac{1}{12} - \frac{\varepsilon}{8} \right) = -\frac{1}{6} + \frac{7\varepsilon}{4}$$

and the result comes because  $\varepsilon \leq \frac{2}{21}$ . ◇

**Claim 21.** *Let  $v$  be a vertex in  $D_6$  having at least two neighbours of degree at least 8. Then  $w^*(v) \leq 0$ . Moreover, if  $v$  is not an isolated vertex in  $D_6$  and  $\nu_N(v) \geq 4$ , then  $w^*(v) \leq -\frac{1}{9} + \frac{5}{3}\varepsilon - \frac{\delta}{4}$ .*

*Proof of claim.* Observe that  $v$  receives no charge and sends the following charge to each of its neighbour  $u$  with degree at least 8:

$$\begin{aligned} \frac{2}{d(u) - \nu(u)} \left( -\frac{10}{3} - \varepsilon + \frac{d(u)}{2} \right) &\geq \frac{2}{d(u)} \left( -\frac{10}{3} - \varepsilon + \frac{d(u)}{2} \right) \\ &= 1 - \frac{2}{d(u)} \left( \frac{10}{3} + \varepsilon \right) \\ &\geq \frac{2}{8} \left( -\frac{10}{3} - \varepsilon + 4 \right) \\ &= \frac{1}{6} - \frac{\varepsilon}{4}. \end{aligned}$$

Assume first that  $v$  is isolated in  $D_6$ . By Claim 14, its three neighbours do not have degree 7, and so have degree at least 8. Thus  $v$  sends three times at least  $\frac{1}{6} - \frac{\varepsilon}{4}$ , and so

$$w^*(v) \leq w(v) - 3 \left( \frac{1}{6} - \frac{\varepsilon}{4} \right) = -\frac{1}{6} + \frac{7}{4}\varepsilon$$

and the result comes because  $\varepsilon \leq \frac{2}{21}$ .

Assume now that  $v$  is in a connected component  $C$  of  $D_6$  of size at least 2. By Claim 16,  $\sigma(v) \geq \frac{\delta}{4}$ , so  $w(v) \leq \frac{1}{3} + \varepsilon - \frac{\delta}{4}$ . Moreover, it sends twice at least  $\frac{1}{6} - \frac{\varepsilon}{4}$ . Hence

$$w^*(v) \leq \left( \frac{1}{3} + \varepsilon - \frac{\delta}{4} \right) - 2 \left( \frac{1}{6} - \frac{\varepsilon}{4} \right) = \frac{3}{2}\varepsilon - \frac{\delta}{4}$$

and the result comes because  $\delta \geq 6\varepsilon$ . This shows the first part of the statement.

We will now prove the second part of the statement. Assume that  $v$  is not an isolated vertex in  $D_6$  and  $\nu_N(v) \geq 4$ . Let  $u_1$  and  $u_2$  be the two neighbours of  $v$  with degree at least 8. For every  $i \in \{1, 2\}$  we have

$$\frac{2}{d(u_i) - \nu(u_i)} \left( -\frac{10}{3} - \varepsilon + \frac{d(u_i)}{2} \right) = 1 - \frac{1}{d(u_i) - \nu(u_i)} \left( \frac{20}{3} + 2\varepsilon - \nu(u_i) \right)$$

**Case 1:**  $\nu(u_i) \geq 7$  for some  $i \in \{1, 2\}$ . Without loss of generality suppose  $i = 1$ . Then we have

$$1 - \frac{1}{d(u_1) - \nu(u_1)} \left( \frac{20}{3} + 2\varepsilon - \nu(u_1) \right) \geq 1$$

because  $\nu(u_1) \geq 7 \geq \frac{20}{3} + 2\varepsilon$  as  $\varepsilon \leq \frac{1}{6}$ . Then the total charge sent by  $v$  is at least 1, and thus

$$w^*(v) \leq w(v) - 1 \leq \left( \frac{1}{3} + \varepsilon - \frac{\delta}{4} \right) - 1 = -\frac{2}{3} + \varepsilon - \frac{\delta}{4}$$

Thus, we have  $w^*(v) \leq -\frac{1}{9} + \frac{5}{3}\varepsilon - \frac{\delta}{4}$  because  $\varepsilon, \delta \geq 0$ .

**Case 2:**  $\nu(u_1), \nu(u_2) \leq 6$ . Let  $f : [0, 6] \rightarrow \mathbb{R}$  be the function defined by

$$f(x) = \frac{2}{8-x} \left( -\frac{10}{3} - \varepsilon + \frac{8}{2} \right) = 1 - \frac{1}{8-x} \left( \frac{20}{3} - 2\varepsilon - x \right)$$

for every  $x \in [0, 6]$ . Observe that  $f$  is non-decreasing and convex on  $[0, 6]$  because  $-\frac{10}{3} - \varepsilon + \frac{8}{2} \geq 0$ . For  $i = 1, 2$ , we have

$$\frac{2}{d(u_i) - \nu(u_i)} \left( -\frac{10}{3} - \varepsilon + \frac{d(u_i)}{2} \right) \geq f(\nu(u_i))$$

because the function  $d \mapsto 1 - \frac{1}{d - \nu(u_i)} \left( \frac{20}{3} + 2\varepsilon - \nu(u_i) \right)$  is non-decreasing on  $[8, +\infty[$  as  $\nu(u_i) \leq 6 \leq \frac{20}{3} + 2\varepsilon$ . Hence the charge sent by  $v$  to  $u_i$  is at least  $f(\nu(u_i))$ . By hypothesis we have  $\nu_N(v) = \nu(u_1) + \nu(u_2) \geq 4$ . It follows that the total charge sent by  $v$  is at least

$$\begin{aligned} f(\nu(u_1)) + f(\nu(u_2)) &\geq 2f\left(\frac{\nu(u_1) + \nu(u_2)}{2}\right) \quad \text{by convexity of } f \\ &\geq 2f(2) \quad \text{because } f \text{ is non-decreasing} \\ &= \frac{4}{9} - \frac{2}{3}\varepsilon. \end{aligned}$$

Hence

$$w^*(v) \leq w(v) - \left(\frac{4}{9} - \frac{2}{3}\varepsilon\right) \leq \left(\frac{1}{3} + \varepsilon - \frac{\delta}{4}\right) - \frac{4}{9} + \frac{2}{3}\varepsilon = -\frac{1}{9} + \frac{5}{3}\varepsilon - \frac{\delta}{4}.$$

showing the second part of the statement.  $\diamond$

**Claim 22.** *If  $C$  is a connected component of  $D_6$ , then  $\sum_{v \in V(C)} w^*(v) \leq 0$ .*

*Proof of claim.* If  $C$  has a unique vertex  $v$ , then, by Claim 21, we have  $w^*(v) \leq 0$  as wanted.

If  $C$  has two vertices  $x$  and  $y$ , then, again by Claim 21,  $w^*(x), w^*(y) \leq 0$ , and so  $w^*(x) + w^*(y) \leq 0$ .

If  $C$  is a bidirected path  $[x, y, z]$ , then, by Claim 16,  $x$  and  $z$  have both neighbourhood valency at least 4 and so by Claim 21  $w^*(x), w^*(z) \leq -\frac{1}{9} - \frac{\varepsilon}{6}$ . Moreover,  $y$  sends at least  $\frac{2}{8}(-\frac{10}{3} + 4 - \varepsilon) = \frac{1}{6} - \frac{\varepsilon}{4}$  to its neighbour out of  $C$ . Hence

$$w^*(y) \leq w(y) - \left(\frac{1}{6} - \frac{\varepsilon}{4}\right) \leq \frac{1}{3} + \varepsilon - \frac{\delta}{3} - \frac{1}{6} + \frac{\varepsilon}{4} = \frac{1}{6} + \frac{5}{4}\varepsilon - \frac{\delta}{3}.$$

Altogether, we get that

$$w^*(x) + w^*(y) + w^*(z) \leq \frac{1}{6} + \frac{5}{4}\varepsilon - \frac{\delta}{3} + 2\left(-\frac{1}{9} - \frac{\varepsilon}{6}\right) = -\frac{1}{18} + \frac{11}{12}\varepsilon - \frac{\delta}{3} \leq 0$$

because  $\delta \geq 6\varepsilon$ .

Finally, if  $C$  is a bidirected star with centre  $x$  and three other vertices  $y, z, w$ , then  $w^*(x) \leq w(x) = \frac{1}{3} + \varepsilon - \frac{\delta}{4}$ . Moreover, each of  $y, z, w$  has neighbourhood valency at least 4 by Claim 16 and so has final charge at most  $-\frac{1}{9} + \frac{5}{3}\varepsilon - \frac{\delta}{4}$  by Claim 21. Hence

$$w^*(x) + w^*(y) + w^*(z) + w^*(w) \leq \frac{1}{3} + \varepsilon - \frac{\delta}{4} + 3\left(-\frac{1}{9} + \frac{5}{3}\varepsilon - \frac{\delta}{4}\right) \leq 6\varepsilon - \delta \leq 0$$

because  $\delta \geq 6\varepsilon$ .  $\diamond$

As a consequence of these last claims, we have  $\rho(D) \leq \sum_{v \in V(D)} w(v) = \sum_{v \in V(D)} w^*(v) \leq 0 \leq 1$ , a contradiction. This proves Theorem 8.  $\square$

## 4 An upper bound on $o_k(n)$

In this section, we show that, for every fixed  $k$ , there are infinitely many values of  $n$  such that  $o_k(n) \leq (2k - \frac{7}{2})n$ . The proof is strongly based on the proof of [1, Theorem 4.4], which shows  $o_k(n) \leq (2k - 3)n$  for every  $k, n$  (with  $n$  large enough). For  $k = 4$ , the construction implies in particular that there is a 4-dicritical oriented graph with 76 vertices and 330 arcs, and there are infinitely many 4-dicritical oriented graphs with  $m/n \leq 9/2$ .

**Proposition 26.** *Let  $k \geq 3$  be an integer. For infinitely many values of  $n \in \mathbb{N}$ , there exists a  $k$ -dicritical oriented graph  $\vec{G}_k$  on  $n$  vertices with at most  $(2k - \frac{7}{2})n$  arcs.*

*Proof.* Let us fix  $n_0 \in \mathbb{N}$ . We will show, by induction on  $k$ , that there exists a  $k$ -dicritical oriented graph  $\vec{G}_k$  on  $n$  vertices with at most  $(2k - \frac{7}{2})n$  arcs, such that  $n \geq n_0$ .

When  $k = 3$ , the result is known ([1, Corollary 4.3]). We briefly describe the construction for completeness. Start from any orientation of an odd cycle on  $2n_0 + 1$  vertices. Then for each arc  $xy$  in this orientation, add a directed triangle  $\vec{C}_3$  and every arc from  $y$  to  $V(\vec{C}_3)$  and every arc from  $V(\vec{C}_3)$  to  $x$  (see Figure 5). This gadget forces  $x$  and  $y$  to have different colours in every 2-dicolouring. Since we started from an orientation of an odd cycle, the result is a 3-dicritical oriented graph on  $4(2n_0 + 1)$  vertices and  $10(2n_0 + 1)$  arcs.

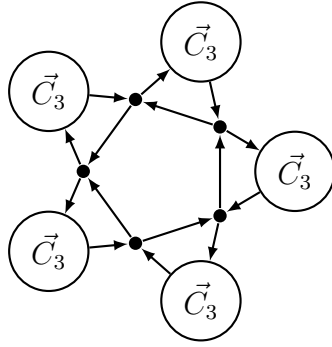


Figure 5: A 3-dicritical oriented graph with  $\frac{5}{2}n$  arcs.

Let us fix  $k \geq 4$  and assume that there exists such a  $(k - 1)$ -dicritical oriented graph  $\vec{G}_{k-1}$  on  $n_{k-1} \geq n_0$  vertices with  $m_{k-1} \leq (2(k - 1) - \frac{7}{2})n_{k-1}$  arcs. We start from any tournament  $T$  on  $k$  vertices. Then we add, for each arc  $xy$  of  $T$ , a copy  $\vec{G}_{k-1}^{xy}$  of  $\vec{G}_{k-1}$ , all arcs from  $y$  to  $\vec{G}_{k-1}^{xy}$  and all arcs from  $\vec{G}_{k-1}^{xy}$  to  $x$ . Figure 6 illustrates a possible construction of  $\vec{G}_4$ , where  $T$  is the transitive tournament on 4 vertices.

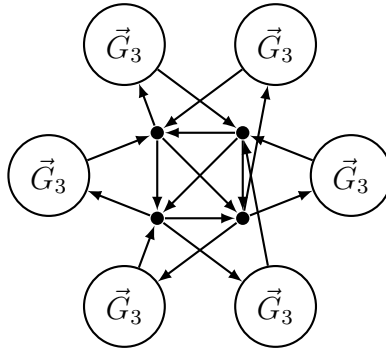


Figure 6: A 4-dicritical oriented graph with at most  $\frac{9}{2}n$  arcs.

Let  $\vec{G}_k$  be the resulting oriented graph. By construction,  $n_k = |V(\vec{G}_k)|$  and  $m_k = |A(\vec{G}_k)|$  satisfy:

$$\begin{aligned}
n_k &= k + \binom{k}{2} n_{k-1} \\
m_k &= \binom{k}{2} + \binom{k}{2} \times 2 \times n_{k-1} + \binom{k}{2} \times m_{k-1} \\
&\leq \binom{k}{2} + \binom{k}{2} \left( 2 + 2(k-1) - \frac{7}{2} \right) n_{k-1} \\
&= \binom{k}{2} + \binom{k}{2} \left( 2k - \frac{7}{2} \right) n_{k-1} \\
&= \binom{k}{2} + \left( 2k - \frac{7}{2} \right) (n_k - k) \\
&\leq \left( 2k - \frac{7}{2} \right) n_k
\end{aligned}$$

where in the last inequality we used  $k \left( 2k - \frac{7}{2} \right) \geq \binom{k}{2}$ , which holds when  $k \geq 2$ . We will now prove that  $\vec{G}_k$  is indeed  $k$ -dicritical.

We first prove that  $\vec{\chi}(\vec{G}_k) = k$ . Assume that there exists a  $(k-1)$ -dicolouring  $\alpha$  of  $\vec{G}_k$ . Then there exist  $x, y \in V(T)$  such that  $\alpha(x) = \alpha(y)$ . Since  $\vec{\chi}(\vec{G}_{k-1}) = k-1$ , there exists  $z \in V(\vec{G}_{k-1}^{xy})$  such that  $\alpha(z) = \alpha(x)$ . But then  $(x, y, z, x)$  is a monochromatic directed triangle in  $\alpha$ : a contradiction.

Let us now prove that  $\vec{\chi}(\vec{G}_k \setminus \{uv\}) \leq k-1$  for every arc  $uv \in A(\vec{G}_k)$ . This implies immediately that  $\vec{\chi}(\vec{G}_k) = k$  and shows the result.

Consider first an arc  $uv$  in  $A(T)$ . We colour each copy  $\vec{G}_{k-1}^{xy}$  of  $\vec{G}_{k-1}$  with a  $(k-1)$ -dicolouring of  $\vec{G}_{k-1}$ . We then choose a distinct colour for every vertex in  $T$ , except  $u$  and  $v$  which receive the same colour. This results in a  $(k-1)$ -dicolouring of  $\vec{G}_k \setminus \{uv\}$ .

Consider now an arc  $uv$  of  $\vec{G}_{k-1}^{xy}$  for some  $xy \in A(T)$ . Because  $\vec{G}_{k-1}$  is  $(k-1)$ -dicritical, there exists a  $(k-2)$ -dicolouring  $\xi$  of  $\vec{G}_{k-1}^{xy} \setminus \{uv\}$ . Hence we colour  $\vec{G}_{k-1}^{xy} \setminus \{uv\}$  with  $\xi$ , every other copy of  $\vec{G}_{k-1}$  a  $(k-1)$ -dicolouring of  $\vec{G}_{k-1}$ , and we choose a distinct colour for every vertex in  $T$ , except  $x$  and  $y$  which both receive colour  $k-1$ . This results in a  $(k-1)$ -dicolouring of  $\vec{G}_k \setminus \{uv\}$ .

Consider finally an arc  $uv$  arc from  $u \in V(T)$  to  $v \in V(\vec{G}_{k-1}^{uy})$  (the case of  $u \in V(\vec{G}_{k-1}^{xv})$  and  $v \in V(T)$  being symmetric). Because  $\vec{G}_{k-1}$  is dicritical, there exists a  $(k-1)$ -dicolouring  $\gamma$  of  $\vec{G}_{k-1}^{uy}$  in which  $v$  is the only vertex coloured  $k-1$ . Hence, we colour  $\vec{G}_{k-1}^{uy}$  with  $\gamma$ , every other copy of  $\vec{G}_{k-1}$  with a  $(k-1)$ -dicolouring of  $\vec{G}_{k-1}$ , and we choose a distinct colour for every vertex in  $T$ , except  $u$  and  $y$  which both receive colour  $k-1$ . This results in a  $(k-1)$ -dicolouring of  $\vec{G}_k \setminus \{uv\}$ .  $\square$

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