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# An exact robust high-order differentiator with hyperexponential convergence

Jian Wang, Konstantin Zimenko, Andrey Polyakov, Denis Efimov

**Abstract**—A linear time-varying state observer is presented for a chain of integrators having bounded disturbances in the last equation. It is demonstrated that in the noise-free setting, for the continuous-time realization, the estimation error converges to zero with a hyperexponential rate (faster than any exponential) uniformly in the disturbance. An implicit discretization scheme of the observer is proposed, which in the discrete time preserves all main properties of the continuous-time counterpart. In addition, the discrete-time estimation error is robustly stable with respect to the measurement noise. The efficiency of the suggested observer is illustrated through comparison with a linear high-gain observer and a sliding mode high-order differentiator.

## I. INTRODUCTION

For dynamical systems, the state estimation problem through its noisy partial measurements in real time is a mature and well-known challenge, which has numerous observer solutions for different classes of models and performance requirements [1], [2], [3]. An example of such a problem is estimation of the derivatives of a signal, where popular solutions are presented by differentiators [3], [4], [5], [6], [7], [8], while the recent advances can be found in [9].

There are several principal estimation quality requirements for an observer including the time of convergence of the estimate to the true value of the state, asymptotic precision (static error) in the noise-free case, noise sensitivity, and the implementation complexity, to mention the most relevant ones. In different applications, the order of importance for these characteristics is varying. The existence of many different observers in the literature is related with optimization of distinct criteria, and with the fact that it is frequently difficult to design a unique algorithm outperforming others by all existing indicators. For example, the famous Kalman filter is optimal in the presence of Gaussian noises, but its augmented computational complexity (the observer gain has its own dynamics) and a static error in the presence of biased disturbances become shortages in the case if the distributions

of perturbations are not known or they are not normal. High-gain Luenberger observer [3] has a rapid convergence and requires a lower calculation power, but it becomes sensitive to noises also possessing the static error in the presence of disturbances. High-order sliding mode observer [10] is exact and converges in a finite time, but its discrete-time implementation is tricky [11], [12], [13]; moreover, tuning is sophisticated, while noise gains are qualitative.

In this paper we are going to propose a new state observer for linear time-invariant systems in a canonical form of integrator chain having an accelerated (faster than any exponential) rate of error convergence, providing asymptotically exact estimates in the noise-free scenario under bounded disturbances (no knowledge of the upper bound is required for the tuning), robust in the presence of measurement perturbations, and admitting a simple digital implementation yielding all these benefits. The gain selection rules are formulated using linear matrix inequalities (LMIs), and it is demonstrated in simulations that the new observer has advantageous performance qualities even in the case of a slow sampling. The preliminary conference version [14] deals only with the second order differentiator case.

The paper is organized as follows. The brief preliminaries are given in Section II. The problem statement is introduced in Section III. The properties of the proposed observer in continuous time are investigated in Section IV. The characteristics of its implicit Euler discretization are considered in Section V. The results of numeric comparison with a linear high-gain algorithm and a high-order sliding mode exact differentiator are shown in Section VI.

## Notation

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ , where  $\mathbb{R}$  is the set of real numbers,  $\mathbb{Z}$  is the set of integer numbers,  $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$ .
- $|\cdot|$  denotes the absolute value in  $\mathbb{R}$ ,  $\|\cdot\|$  is used for the Euclidean norm on  $\mathbb{R}^n$ .
- For a (Lebesgue) measurable function  $d : \mathbb{R}_+ \rightarrow \mathbb{R}^m$  we define the norm  $\|d\|_\infty = \text{ess sup}_{t \in \mathbb{R}_+} \|d(t)\|$  and the set of  $d$  with the property  $\|d\|_\infty < +\infty$  we further denote as  $\mathcal{L}_\infty^m$  (the set of essentially bounded functions).
- For a sequence  $d_k \in \mathbb{R}^m$  with  $k \in \mathbb{Z}_+$  define its norm by  $|d|_\infty = \sup_{k \in \mathbb{Z}_+} \|d_k\|$  and the set of  $d$  with  $|d|_\infty < +\infty$  we denote by  $l_\infty^m$ .
- A continuous function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{K}$  if  $\alpha(0) = 0$  and it is strictly increasing. The

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function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{K}_\infty$  if  $\alpha \in \mathcal{K}$  and it is increasing to infinity. A continuous function  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{KL}$  if  $\beta(\cdot, t) \in \mathcal{K}$  for each fixed  $t \in \mathbb{R}_+$  and  $\beta(s, \cdot)$  is decreasing to zero for each fixed  $s > 0$ .

- The matrix composed by zeros of dimension  $n \times m$  is denoted by  $0_{n \times m}$ ;  $\text{diag}\{g\}$  represents a diagonal matrix of dimension  $n \times n$  with a vector  $g \in \mathbb{R}^n$  on the main diagonal, and  $\mathbb{D}^n$  is the set of such diagonal matrices. Denote the identity matrix of dimension  $n \times n$  by  $I_n$ .
- The relation  $P \prec 0$  ( $P \succeq 0$ ) means that a symmetric matrix  $P \in \mathbb{R}^{n \times n}$  is negative (positive semi-) definite,  $\lambda_{\min}(P)$  denotes the minimal eigenvalue of  $P$ .
- $\exp(1) = e$ .

## II. PRELIMINARIES

The used standard stability notions and their definitions can be found in [15].

### A. Uniform hyperexponential stability

1) *Continuous-time case*: Consider a non-autonomous differential equation:

$$dx(t)/dt = f(t, x(t), d(t)), \quad t \geq t_0, \quad t_0 \in \mathbb{R}_+, \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $d(t) \in \mathbb{R}^m$  is the vector of external disturbances,  $d \in \mathcal{L}_\infty^m$ ;  $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is a continuous function with respect to  $x, d$  and piecewise continuous with respect to  $t$ ,  $f(t, 0, 0) = 0$  for all  $t \in \mathbb{R}_+$ . A solution of the system (1) for an initial condition  $x_0 \in \mathbb{R}^n$  at time instant  $t_0 \in \mathbb{R}_+$  and some  $d \in \mathcal{L}_\infty^m$  is denoted as  $X(t, t_0, x_0, d)$ , and we assume that  $f$  ensures existence of solutions  $X(t, t_0, x_0, d)$  at least locally in forward time.

**Definition 1.** For a given set  $\mathbb{D} \subset \mathcal{L}_\infty^m$ , the system (1) is called uniformly hyperexponentially stable if there exist  $\kappa_0 \in \mathbb{R}_+$ ,  $\rho, \kappa \in \mathcal{K}_\infty$  and  $\beta \in \mathcal{KL}$  such that

$$\|X(t, t_0, x_0, d)\| \leq e^{-(\kappa(t-t_0)+\kappa_0)(t-t_0)} \rho(\|x_0\|) + \beta(\|d\|_\infty, t-t_0), \quad \forall t \geq t_0$$

for all  $x_0 \in \mathbb{R}^n$ ,  $t_0 \in \mathbb{R}_+$  and  $d \in \mathbb{D}$ .

A simple example is

$$\dot{x}(t) = -(1+t)x(t) + d(t), \quad t \geq 0$$

with  $x(t), d(t) \in \mathbb{R}$ , whose solutions admit an estimate:

$$|x(t)| \leq e^{-(\frac{t+t_0}{2}+1)(t-t_0)} |x(0)| + \frac{2\|d\|_\infty}{1 + (\frac{t+t_0}{2} + 1)(t-t_0)}$$

for all  $t \geq t_0 \geq 0$ , for any  $x(0) \in \mathbb{R}$  and  $d \in \mathcal{L}_\infty^1$ . Consequently, for  $t_0 \in \mathbb{R}_+$  we can define  $\rho(s) = s$ ,  $\kappa(s) = \frac{s}{2}$ ,  $\kappa_0 = 1$  and  $\beta(s, t) = \frac{2s}{1+(\frac{t}{2}+1)t}$ .

In this definition the hyperexponential rate of convergence is demanded only in the initial conditions, while uniformity is understood in double sense: as independence in the initial time  $t_0$  and in the exogenous input  $d \in \mathbb{D}$ . Despite it is assumed that  $\mathbb{D} \subset \mathcal{L}_\infty^m$ , any other suitable class of inputs can be considered.

2) *Discrete-time case*: Consider a non-autonomous difference equation:

$$x_{k+1} = f(k, x_k, d_k), \quad k \geq k_0, \quad k_0 \in \mathbb{Z}_+,$$

where  $x_k \in \mathbb{R}^n$  and  $d_k \in \mathbb{R}^m$  are the state and the disturbance vectors,  $d \in l_\infty^m$ ;  $f : \mathbb{Z}_+ \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  and  $f(k, 0, 0) = 0$  for all  $k \in \mathbb{Z}_+$ . A solution of the system for an initial condition  $x_0 \in \mathbb{R}^n$  at time instant  $k_0 \in \mathbb{Z}_+$  and some  $d \in l_\infty^m$  is denoted as  $X_k(k_0, x_0, d)$ .

**Definition 2.** For a given set  $\mathbb{D} \subset l_\infty^m$ , this system is called uniformly hyperexponentially stable if there exist  $\rho, \kappa \in \mathcal{K}_\infty$ ,  $\kappa_0 \in (0, \kappa^{-1}(1))$  and  $\beta \in \mathcal{KL}$  such that

$$\|X_k(k_0, x_0, d)\| \leq \kappa(\kappa_0 + k - k_0)^{k_0 - k} \rho(\|x_0\|) + \beta(\|d\|_\infty, k - k_0)$$

for all  $k \geq k_0$ , for any  $x_0 \in \mathbb{R}^n$ ,  $k_0 \in \mathbb{Z}_+$  and  $d \in \mathbb{D}$ .

A similar example of uniformly hyperexponentially stable system can be given for the discrete-time setting:

$$x_{k+1} = \frac{x_k + d_k}{1+k}, \quad k \in \mathbb{Z}_+$$

for  $x_k, d_k \in \mathbb{R}$ , then

$$|x_k| \leq \frac{|x_0| + |d_0|}{k!} + \frac{1}{k} \sum_{i=1}^{k-1} |d_i| \prod_{j=i+1}^{k-1} \frac{1}{j} \leq \frac{|x_0|}{k!} + \frac{|d|_\infty}{k} \sum_{i=1}^k \frac{1}{2^{i-2}}$$

for all  $k \geq 1$ , for any  $x_0 \in \mathbb{R}$  and  $d \in l_\infty^1$ . Hence,  $\rho(s) = s$ ,  $\kappa(s) = \frac{s}{2}$ ,  $\kappa_0 = 1$  (recall that  $k!$  can be lower bounded by  $\sqrt{2\pi k} (\frac{k}{e})^k$ ) and  $\beta(s, k) = \frac{8s}{1+k^2}$  for  $k_0 = 0$ .

### B. Auxiliary properties

We will use the following estimate:

**Lemma 1.** For all  $\tau \geq 0$  and all integer  $i \geq 1$  it holds:

$$\int_0^\tau \frac{e^{s-\tau}}{(2s+1)^i} ds \leq \frac{r_i}{(2\tau+1)^i},$$

$$r_i = (2i)^i \int_0^{\frac{2i-1}{2}} \frac{e^{s-\frac{2i-1}{2}}}{(2s+1)^i} ds + \left(\frac{2i+1}{2i}\right)^i (1 - e^{-0.5}) + (2i+1).$$

*Proof.* The integral above can be calculated using the exponential integral special function, but there is no analytical expression for it. The property to prove can be rewritten as

$$g_i(\tau) = (2\tau+1)^i \int_0^\tau \frac{e^{s-\tau}}{(2s+1)^i} ds \leq r_i, \quad \forall \tau \geq 0, \quad \forall i \geq 1.$$

Note that

$$\frac{\partial g_i(\tau)}{\partial \tau} = p_i(\tau) g_i(\tau) + 1, \quad p_i(\tau) = \frac{2i}{2\tau+1} - 1,$$

which is a first order linear time-varying differential equation with respect to  $g_i(\tau)$  for  $\tau \geq 0$  and  $g_i(0) = 0$ . In such a case, using the standard formula for solutions of linear differential equations we get:

$$g_i(\tau) = \int_0^\tau e^{\int_s^\tau p_i(\sigma) d\sigma} ds$$

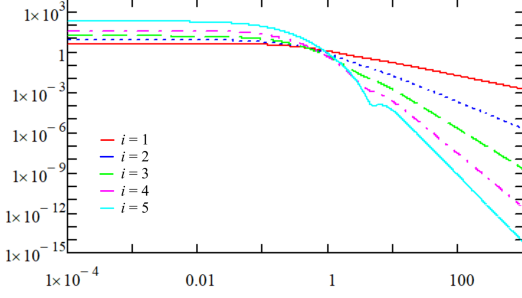


Figure 1:  $\delta_i(\tau)$  versus  $\tau \in [10^{-4}, 10^3]$ ,  $i = 1, \dots, 5$

for any  $\tau \geq 0$ . Define  $\tau_* = \frac{2i-1}{2}$ , then

$$p_i(\tau) \begin{cases} \geq 0 & \text{if } \tau \in [0, \tau_*], \\ \leq 0 & \text{if } \tau \in (\tau_*, \tau_* + 0.5], \\ \leq -\frac{1}{2i+1} & \text{if } \tau > \tau_* + 0.5. \end{cases}$$

Therefore,  $\frac{\partial g_i(\tau)}{\partial \tau} > 0$  for  $\tau \in [0, \tau_*]$ , hence,  $g_i(\tau) \leq g_i(\tau_*)$ . For  $\tau \in (\tau_*, \tau_* + 0.5]$  we obtain:

$$\begin{aligned} g_i(\tau) &= g_i(\tau_*) + \int_{\tau_*}^{\tau} e^{\int_s^{\tau} p_i(\sigma) d\sigma} ds \\ &= g_i(\tau_*) + \int_{\tau_*}^{\tau} e^{s-\tau} \left( \frac{2\tau+1}{2s+1} \right)^i ds \\ &\leq g_i(\tau_*) + \int_{\tau_*}^{\tau} e^{s-\tau} \left( \frac{2\tau+1}{2\tau_*+1} \right)^i ds \\ &= g_i(\tau_*) + \left( \frac{2\tau+1}{2\tau_*+1} \right)^i (1 - e^{\tau_*-\tau}), \end{aligned}$$

which implies that

$$g_i(\tau) \leq g_i^* = g_i(\tau_*) + \left( \frac{2\tau_*+2}{2\tau_*+1} \right)^i (1 - e^{-0.5})$$

for any  $\tau \in [0, \tau_* + 0.5]$ . Finally, for  $\tau > \tau_* + 0.5$ :

$$\begin{aligned} g_i(\tau) &= g_i(\tau_* + 0.5) + \int_{\tau_*+0.5}^{\tau} e^{\int_s^{\tau} p_i(\sigma) d\sigma} ds \\ &\leq g_i^* + \int_{\tau_*+0.5}^{\tau} e^{-\int_s^{\tau} \frac{1}{2i+1} d\sigma} ds \\ &= g_i^* + (2i+1) \left( 1 - e^{-\frac{\tau-\tau_*+0.5}{2i+1}} \right) \leq g_i^* + (2i+1). \end{aligned}$$

Consequently,  $g_i(\tau) \leq g_i^* + (2i+1) = r_i$  for any  $\tau \geq 0$ .  $\square$

To illustrate the result of Lemma 1, the discrepancy  $\delta_i(\tau) = \frac{r_i}{(2\tau+1)^i} - \int_0^{\tau} \frac{e^{s-\tau}}{(2s+1)^i} ds$  is shown in Fig. 1 in logarithmic scales for  $i = 1, \dots, 5$ .

The following formulas are used in the sequel [16]:

$$\begin{aligned} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} &= \begin{bmatrix} A^{-1} + A^{-1}BS_1CA^{-1} & -A^{-1}BS_1 \\ -S_1CA^{-1} & S_1 \end{bmatrix} \\ &= \begin{bmatrix} S_2 & -S_2BD^{-1} \\ -D^{-1}CS_2 & D^{-1} + D^{-1}CS_2BD^{-1} \end{bmatrix}, \\ S_1 &= (D - CA^{-1}B)^{-1}, \quad S_2 = (A - BD^{-1}C)^{-1}, \end{aligned}$$

where  $A, B, C$  and  $D$  are matrices of appropriate dimensions ( $A, D, S_1$  and  $S_2$  should be nonsingular).

### III. PROBLEM STATEMENT

Assume that a linear system is given:

$$\dot{x}(t) = Ax(t) + Bu(t) + bd(t), \quad y(t) = Cx(t) + v(t), \quad (2)$$

where  $x(t) \in \mathbb{R}^n$  is the state;  $u(t) \in \mathbb{R}^m$  is the known input,  $u \in \mathcal{L}_{\infty}^m$ ;  $y(t) \in \mathbb{R}$  is the measured output with the noise  $v(t) \in \mathbb{R}$ ,  $v \in \mathcal{L}_{\infty}^1$ ;  $d(t) \in \mathbb{R}$  is the disturbance appearing in the last equation,  $d \in \mathcal{L}_{\infty}^1$ ; the matrices

$$A = \begin{bmatrix} 0_{n-1 \times 1} & I_{n-1} \\ 0 & 0_{1 \times n-1} \end{bmatrix}, \quad b = \begin{bmatrix} 0_{n-1 \times 1} \\ 1 \end{bmatrix}, \\ C = [1 \quad 0_{1 \times n-1}]$$

are in the canonical form and  $B \in \mathbb{R}^{n \times m}$  is arbitrary. Further in this work an observer for this system will be designed estimating  $x(t)$  with a hyperexponential rate of convergence uniformly in  $d \in \mathbb{D} = \mathcal{L}_{\infty}^1$  while  $\|v\|_{\infty} = 0$ , and having a bounded estimation error for  $v \in \mathcal{L}_{\infty}^1$ .

The problem can be equivalently formulated as estimation of derivatives for a sufficiently smooth signal  $\phi(t) \in \mathbb{R}$  measured with a noise  $v(t) \in \mathbb{R}$ :

$$y(t) = \phi(t) + v(t),$$

where  $y(t) \in \mathbb{R}$ ,  $v \in \mathcal{L}_{\infty}^1$ , and  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}$  has the  $n$ th derivative  $d(t) = \frac{d^n \phi(t)}{dt^n}$  with  $d \in \mathcal{L}_{\infty}^1$  (without a known constant upper bound). If it is required to estimate the derivatives  $\phi^{(i)}(t) = \frac{d^i \phi(t)}{dt^i}$ ,  $i = 1, \dots, n-1$  of the signal  $\phi$  with accelerated time of convergence and robustly with respect to the perturbations  $d$  and  $v$ , then it is the observer design problem studied in this work.

### IV. DIFFERENTIATOR IN CONTINUOUS TIME

Let the state observer for (2) be chosen in the form:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + D(t)L(y(t) - C\hat{x}(t)), \quad (3)$$

$$D(t) = \varrho(t)\Gamma(t), \quad \varrho(t) = 1 + t,$$

$$\Gamma(t) = \text{diag}\{\rho(t)\}, \quad \rho(t) = [1 \quad \varrho(t) \quad \dots \quad \varrho^{n-1}(t)]^{\top},$$

where  $\hat{x}(t) \in \mathbb{R}^n$  is the estimate of the state  $x(t)$ ,  $L \in \mathbb{R}^n$  is the observer gain that will be selected later,  $\varrho(t)$  is a strictly growing function of time.

*Remark 1.* Any continuous strictly growing function  $\varrho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with  $\varrho(0) > 0$  and unbounded integral can be used in (3), e.g.,  $\varrho(t) = ae^{\alpha t}$  or  $\varrho(t) = (b + at)^{\alpha}$  for  $a > 0$ ,  $b > 0$  and  $\alpha > 0$  (by a change of time all of them can be transformed to one used for the proof in this work).

Define the estimation error as  $e(t) = x(t) - \hat{x}(t)$ , whose dynamics can be written as follows:

$$\dot{e}(t) = (A - D(t)LC)e(t) + bd(t) - D(t)Lv(t).$$

We are in position to formulate the main result of this section:

**Theorem 1.** Let  $P = P^\top \in \mathbb{R}^{n \times n}$ ,  $X \in \mathbb{D}^n$ ,  $U \in \mathbb{R}^n$ ,  $\gamma_1 > 0$  and  $\gamma_2 > 0$  be a solution of LMIs:

$$P \succ 0, X \succ 0, Q \preceq 0, Q = \begin{bmatrix} Q_{11} & Pb & -U \\ b^\top P & -\gamma_1 & 0 \\ -U^\top & 0 & -\gamma_2 \end{bmatrix},$$

$$PH + HP + X \succeq 0, H = \text{diag}\{[0 \ 1 \ 2 \ \dots \ n - 1]^\top\},$$

$$Q_{11} = A^\top P + PA - UC - C^\top U^\top + X + P.$$

Then in (2), (3) for  $L = P^{-1}U$ :

$$\begin{bmatrix} |e_1(t)| \\ \vdots \\ |e_n(t)| \end{bmatrix} \leq \frac{\rho(t)}{\sqrt{\lambda_{\min}(P)}} (e^{-\frac{t+2}{4}t} \sqrt{e(0)^\top P e(0)} + \sqrt{\gamma_1 r_n} \frac{\|d\|_\infty}{\varrho^n(t)} + \sqrt{\gamma_2} \|v\|_\infty)$$

for all  $t \geq 0$  and any  $e(0) \in \mathbb{R}^n$ ,  $d, v \in \mathcal{L}_\infty^1$  (the latter inequality is understood elementwise).

For  $\|v\|_\infty = 0$  the result of the theorem implies uniform hyperexponential stability of the estimation error  $e(t)$  in (2), (3) with  $d \in \mathcal{L}_\infty^1$ . This property means that (3) is an asymptotically exact observer (as the high-order sliding mode differentiation algorithm [4] in finite time). Moreover, the class of higher-order derivatives  $d \in \mathcal{L}_\infty^1$ , for which the uniformity of the estimates is kept, can be enlarged by ones growing not faster than  $\varrho^{n-1}(t)$  in time (in [4], the upper bound on the disturbance amplitude  $\|d\|_\infty$  should be known for a better tuning, while (3) does not need it).

Note also that the given LMIs are always feasible for the matrices  $A$  and  $C$  in this canonical form.

*Proof.* Define new auxiliary variable  $\epsilon(t) = \Gamma^{-1}(t)e(t)$ , then  $\epsilon(0) = e(0)$ , whose dynamics takes the form (note that  $\Gamma^{-1}(t)A\Gamma(t) = \varrho(t)A$  and  $\Gamma^{-1}(t)D(t)LC\Gamma(t) = \varrho(t)LC$ ):

$$\begin{aligned} \dot{\epsilon}(t) &= -\text{diag}\{[0 \ \varrho^{-2}(t) \ 2\varrho^{-3}(t) \ \dots \ (n-1)\varrho^{-n}(t)]^\top\} \epsilon(t) \\ &\quad + \Gamma^{-1}(t)\dot{e}(t) \\ &= \varrho(t) \left( (A - LC - \Delta(t))\epsilon(t) + \varrho^{-n}(t)bd(t) - Lv(t) \right), \\ \Delta(t) &= \varrho^{-2}(t)H. \end{aligned}$$

Next, let us define a new time variable from  $d\tau = (t+1)dt$ :

$$\tau = \varphi(t) = \frac{t(t+2)}{2}, \quad t = \varphi^{-1}(\tau) = \sqrt{2\tau+1} - 1,$$

then after the change of the time, the dynamics of auxiliary error  $\epsilon(\tau) = \epsilon(\varphi^{-1}(\tau))$  (with a slight ambiguity in the notation we keep the same name of the variable) is reduced to an equivalent useful representation (note that  $\tau \geq 0$ ):

$$\begin{aligned} \frac{d\epsilon(\tau)}{d\tau} &= \frac{d\epsilon(t)}{dt} \Big|_{t=\varphi^{-1}(\tau)} \frac{d\varphi^{-1}(\tau)}{d\tau} = (A - LC - \Delta(\tau))\epsilon(\tau) \\ &\quad + \frac{1}{(2\tau+1)^{n/2}} bd(\tau) - Lv(\tau), \end{aligned} \quad (4)$$

where  $d(\tau) = d(\varphi^{-1}(\tau))$  and  $v(\tau) = v(\varphi^{-1}(\tau))$  are the external signals in the new time,

$$\Delta(\tau) = \Delta(\varphi^{-1}(\tau)) = (2\tau+1)^{-1}H.$$

For analysis of stability properties in (4), let us choose a candidate Lyapunov function  $V(\epsilon) = \epsilon^\top P \epsilon$  (that is positive definite since  $P \succ 0$ ), whose derivative in the new time  $\tau$  for the dynamics (4) can be written as follows:

$$\begin{aligned} \frac{dV(\epsilon(\tau))}{d\tau} &= \epsilon(\tau)^\top \left( (A - LC)^\top P + P(A - LC) \right) \epsilon(\tau) \\ &\quad - 2\epsilon(\tau)^\top P \left( \Delta(\tau)\epsilon(\tau) - \frac{1}{(2\tau+1)^{n/2}} bd(\tau) + Lv(\tau) \right). \end{aligned}$$

Due to the form of  $\Delta(\tau)$  and the introduced LMIs, we have:

$$-P\Delta(\tau) - \Delta(\tau)P - \frac{X - X}{2\tau+1} \preceq \frac{X}{2\tau+1} \preceq X$$

for all  $\tau \geq 0$ . Therefore,

$$\begin{aligned} \frac{dV(\epsilon(\tau))}{d\tau} &\leq \begin{bmatrix} \epsilon(\tau) \\ \frac{d(\tau)}{(2\tau+1)^{n/2}} \\ v(\tau) \end{bmatrix}^\top \tilde{Q} \begin{bmatrix} \epsilon(\tau) \\ \frac{d(\tau)}{(2\tau+1)^{n/2}} \\ v(\tau) \end{bmatrix} - V(\epsilon(\tau)) \\ &\quad + \gamma_1 \frac{d^2(\tau)}{(2\tau+1)^n} + \gamma_2 v^2(\tau), \\ \tilde{Q} &= \begin{bmatrix} \tilde{Q}_{11} & Pb & -PL \\ b^\top P & -\gamma_1 & 0 \\ -L^\top P & 0 & -\gamma_2 \end{bmatrix}, \\ \tilde{Q}_{11} &= (A - LC)^\top P + P(A - LC) + X + P. \end{aligned}$$

It is easy to see that  $\tilde{Q}$  equals to  $Q$  under substitution  $U = PL$ , then  $\tilde{Q} \preceq 0$  and

$$\begin{aligned} \frac{dV(\epsilon(\tau))}{d\tau} &\leq -V(\epsilon(\tau)) + \gamma_1 \frac{d^2(\tau)}{(2\tau+1)^n} + \gamma_2 v^2(\tau) \\ &\leq -V(\epsilon(\tau)) + \gamma_1 \frac{\|d\|_\infty^2}{(2\tau+1)^n} + \gamma_2 \|v\|_\infty^2. \end{aligned}$$

Passing to the time domain we get an estimate:

$$\begin{aligned} V(\epsilon(\tau)) &\leq e^{-\tau} V(\epsilon(0)) + \gamma_1 \int_0^\tau \frac{e^{s-\tau} \|d\|_\infty^2}{(2s+1)^n} ds \\ &\quad + \gamma_2 \int_0^\tau e^{s-\tau} \|v\|_\infty^2 ds \end{aligned}$$

for all  $\tau \geq 0$ . According to Lemma 1:

$$\gamma_1 \int_0^\tau \frac{e^{s-\tau} \|d\|_\infty^2}{(2s+1)^n} ds \leq \gamma_1 \frac{r_n \|d\|_\infty^2}{(2\tau+1)^n},$$

and  $\gamma_2 \int_0^\tau e^{s-\tau} \|v\|_\infty^2 ds \leq \gamma_2 \|v\|_\infty^2$  for all  $\tau \geq 0$ , then

$$V(\epsilon(\tau)) \leq e^{-\tau} V(\epsilon(0)) + \gamma_1 \frac{r_n \|d\|_\infty^2}{(2\tau+1)^n} + \gamma_2 \|v\|_\infty^2, \quad \tau \geq 0,$$

and in the original time

$$V(\epsilon(t)) \leq e^{-\frac{t(t+2)}{2}} V(\epsilon(0)) + \gamma_1 r_n \left( \frac{\|d\|_\infty}{\varrho^n(t)} \right)^2 + \gamma_2 \|v\|_\infty^2$$

for all  $t \geq 0$ . Since  $e(t) = \Gamma(t)\epsilon(t)$ , we get  $V(\epsilon(t)) \geq \lambda_{\min}(P) \|\epsilon(t)\|^2 \geq \lambda_{\min}(P) |\varrho^{1-i}(t) e_i(t)|^2$  for all  $i = \overline{1, n}$ , and the desired estimate on the behavior of  $e(t)$  can be derived (note that  $\rho(t)$  is a vector function).  $\square$

The upper bound on the estimation error of (2), (3) calculated in Theorem 1 implies that the gain of  $|e_i(t)|$ ,  $i = 1, \dots, n$  in  $v(t)$  is proportional to  $\varrho^{i-1}(t)$ . For a small  $\|v\|_\infty$  and having the computations on a bounded interval of time it can be not a blocking problem in application. Otherwise, if the noise is sufficiently smooth and its  $n-1$ th derivative is growing not faster than  $\varrho^{n-1}(t)$ , then (3) will estimate the derivatives of the signal  $y(t) = x_1(t) + v(t)$ . However, for simulation or implementation of (3) we need its discrete-time counterpart, let us check that is happening after discretization of (3). It is a known issue in numeric methods that after a proper discretization the resulted algorithms may possess better characteristics [17].

## V. DIFFERENTIATOR IN DISCRETE TIME

Note that (3) is modeled by a linear time-varying system with external known inputs  $u(t)$  and  $y(t)$ . Since the time-varying gain  $D(t)$  is strictly growing, the explicit Euler discretization cannot be used for all  $t \geq 0$  (it becomes unstable with time for any sampling), however, the implicit one can be effectively applied [17]. Let  $h > 0$  be a constant discretization step, denote by  $t_k = hk$  for  $k \in \mathbb{Z}_+$  the discretization time instants, then application of the implicit Euler method to (3) gives for  $k \in \mathbb{Z}_+$ :

$$\begin{aligned} \hat{x}_{k+1} &= Z(t_{k+1})(\hat{x}_k + hBu_{k+1} + hD(t_{k+1})Ly_{k+1}), \quad (5) \\ Z(t) &= (I_n - h(A - D(t)LC))^{-1} = \frac{K(t)}{O(t)}, \\ K(t) &= \begin{bmatrix} 1 & -\mathcal{B}\mathcal{D} \\ -\mathcal{D}\mathcal{C}(t) & \mathcal{D}O(t) + \mathcal{D}\mathcal{C}(t)\mathcal{B}\mathcal{D} \end{bmatrix}, \\ O(t) &= \sum_{i=1}^n L_i h^i \varrho^i(t) + 1, \quad \mathcal{B} = [-h \ 0 \ \dots \ 0], \\ \mathcal{D} &= \left( I_{n-1} - h \begin{bmatrix} 0_{n-2 \times 1} & I_{n-2} \\ 0 & 0_{1 \times n-2} \end{bmatrix} \right)^{-1} = \\ &= \begin{bmatrix} 1 & h & \dots & h^{n-2} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & h \\ 0 & \dots & 0 & 1 \end{bmatrix}, \quad \mathcal{C}(t) = h \begin{bmatrix} L_2 \varrho^2(t) \\ L_3 \varrho^3(t) \\ \vdots \\ L_n \varrho^n(t) \end{bmatrix}, \end{aligned}$$

where  $\hat{x}_k$  is an approximation of  $\hat{x}(t_k)$ ,  $u_k = u(t_k)$ ,  $y_k = y(t_k)$ , and we have used the block matrix inversion formula given above with  $O = S_2^{-1}$  (these computations are omitted for brevity). Note that  $O(t) \geq \sum_{i=1}^n L_i h^i + 1 > 0$  for all  $t \geq 0$  provided that the matrix  $A - LC$  is Hurwitz (i.e., the conditions of Theorem 1 are satisfied). Since

$$\begin{aligned} \mathcal{B}\mathcal{D} &= -[h \ \dots \ h^{n-1}], \\ \mathcal{D}\mathcal{C}(t) &= [(\mathcal{D}\mathcal{C}(t))_1 \ \dots \ (\mathcal{D}\mathcal{C}(t))_{n-1}]^\top, \\ (\mathcal{D}\mathcal{C}(t))_j &= \sum_{i=j+1}^n L_i h^{i-j} \varrho^i(t), \quad j = 1, \dots, n-1, \end{aligned}$$

the discrete state transition matrix  $Z(t)$  is nonsingular and elementwise bounded for all  $t \geq 0$ . The known input  $u$  gain

$Z(t)B$  has obviously the same property; moreover, the output  $y$  (the noise  $v$ ) gain is also bounded for all  $t \geq 0$ :

$$\begin{aligned} F(t) &= Z(t)D(t)L = \frac{1}{O(t)}[\tilde{F}_1(t) \ \dots \ \tilde{F}_n(t)]^\top, \\ \tilde{F}_j(t) &= \sum_{i=j}^n L_i h^{i-j} \varrho^i(t), \quad j = 1, \dots, n, \end{aligned}$$

where the derivations are done using the expression for  $K(t)$  and following the fact that  $F_1(t) = L_1 \varrho(t) - \mathcal{B}\mathcal{D}\Lambda(t)$  with

$$\Lambda(t) = [(D(t)L)_2 \ \dots \ (D(t)L)_n]^\top = h^{-1}\mathcal{C}(t),$$

then recalling that  $O(t) = L_1 h \varrho(t) + 1 - \mathcal{B}\mathcal{D}\mathcal{C}(t)$  we get:

$$\begin{aligned} \begin{bmatrix} \tilde{F}_2(t) \\ \vdots \\ \tilde{F}_n(t) \end{bmatrix} &= -\mathcal{D}\mathcal{C}(t)L_1 \varrho(t) + (\mathcal{D}O(t) + \mathcal{D}\mathcal{C}(t)\mathcal{B}\mathcal{D})\Lambda(t) \\ &= \mathcal{D}((O(t) - L_1 h \varrho(t))I_{n-1} + \mathcal{C}(t)\mathcal{B}\mathcal{D})h^{-1}\mathcal{C}(t) \\ &= \mathcal{D}((1 - \mathcal{B}\mathcal{D}\mathcal{C}(t))I_{n-1} + \mathcal{C}(t)\mathcal{B}\mathcal{D})h^{-1}\mathcal{C}(t) = h^{-1}\mathcal{D}\mathcal{C}(t). \end{aligned}$$

Therefore,

$$\begin{aligned} &h \lim_{t \rightarrow +\infty} F(t) \\ &= \lim_{t \rightarrow +\infty} \frac{1}{\sum_{i=1}^n L_i h^i \varrho^i(t) + 1} \begin{bmatrix} \sum_{i=1}^n L_i h^i \varrho^i(t) \\ \vdots \\ \sum_{i=n}^n L_i h^{i-n+1} \varrho^i(t) \end{bmatrix} \\ &= [1 \ h^{-1} \ \dots \ h^{-n+1}]^\top, \end{aligned}$$

which evaluates the asymptotic noise sensitivity.

Consider a discrete-time model, which approximates the solutions of (2) using the same discretization method (we add and subtract the term  $D(t)Ly(t) = D(t)L(Cx(t) + v(t))$  to (2) before discretization to have a similar dynamics with (3)):

$$\begin{aligned} x_{k+1} &= Z(t_{k+1})(x_k + hBu_{k+1} + hD(t_{k+1})L(y_{k+1} \\ &\quad - v_{k+1}) + hbd_{k+1}), \quad y_k = Cx_k + v_k, \end{aligned}$$

for  $k \in \mathbb{Z}_+$ , where  $x_k$  should approach to  $x(t_k)$  as  $h$  converges to zero,  $d_k = d(t_k)$  and  $v_k = v(t_k)$  (formally, this perturbation  $v_k$  is different from the one used in (2), since it should also include the discretization error, but with a mild ambiguity in notation we will continue to use the same symbol). In this work we will assume that  $v, d \in l_\infty^1$ .

To study the properties of (5), consider the discretization error  $e_k = x_k - \hat{x}_k$ , whose stability and hyperexponential convergence rate have been analyzed in Theorem 1 for the continuous-time scenario. The direct computations show that

$$e_{k+1} = Z(t_{k+1})(e_k + hbd_{k+1} - hD(t_{k+1})Lv_{k+1}) \quad (6)$$

for all  $k \in \mathbb{Z}_+$ . The following result can be formulated:

**Theorem 2.** Let  $P = P^\top \in \mathbb{R}^{n \times n}$  and  $U \in \mathbb{R}^n$  be a solution of LMIs:

$$\begin{aligned} P &> 0, \quad A^\top P + PA - C^\top U^\top - UC < 0, \\ HP + PH &\succeq 0. \end{aligned}$$

Then for  $L = P^{-1}U$  there exist  $a > 0$ ,  $g > 0$  and  $s > 0$  such that for any  $e_0 \in \mathbb{R}^n$  in (6):

$$\begin{aligned} \lambda_{\min}(P)\|e_k\|^2 &\leq \varrho^{2(n-1)}(t_{k+1}) \left( \prod_{i=0}^{k-1} \frac{a}{h^2 \varrho^2(t_{i+1})} e_0^\top P e_0 \right. \\ &\quad + h^2 g |d|_\infty^2 \sum_{i=0}^{k-1} \frac{1}{\varrho^{2n}(t_{i+1})} \prod_{\ell=i+1}^{k-1} \frac{a}{h^2 \varrho^2(t_{\ell+1})} \\ &\quad \left. + h^2 s |v|_\infty^2 \sum_{i=0}^{k-1} \prod_{\ell=i+1}^{k-1} \frac{a}{h^2 \varrho^2(t_{\ell+1})} \right), \forall k \in \mathbb{Z}_+, \end{aligned}$$

and  $\limsup_{k \rightarrow +\infty} \|e_k\| \leq \varpi_h |v|_\infty$  for some  $\varpi_h > 0$ .

*Proof.* To investigate stability and the rate of convergence in (6), let us define a time-varying Lyapunov function candidate (the same was used before):

$$V_k = e_k^\top \Pi_k e_k, \quad \Pi_k = \Gamma^{-1}(t_{k+1}) P \Gamma^{-1}(t_{k+1}),$$

where  $P = P^\top \succ 0$  is introduced in the formulation of the theorem. For time-varying parameters  $\alpha_k > 0$ ,  $\gamma_k > 0$  and  $\sigma_k > 0$  determined later, for  $k \in \mathbb{Z}_+$  we obtain

$$V_{k+1} - \alpha_k V_k = \begin{bmatrix} e_k \\ d_{k+1} \\ v_{k+1} \end{bmatrix}^\top Q_k \begin{bmatrix} e_k \\ d_{k+1} \\ v_{k+1} \end{bmatrix} + \gamma_k d_{k+1}^2 + \sigma_k v_{k+1}^2,$$

where

$$\begin{aligned} Q_k &= \begin{bmatrix} I_n & & \\ & hb^\top & \\ -hL^\top D(t_{k+1}) & & \end{bmatrix} Z^\top(t_{k+1}) \Pi_{k+1} Z(t_{k+1}) \\ &\times \begin{bmatrix} I_n & & \\ & hb^\top & \\ -hL^\top D(t_{k+1}) & & \end{bmatrix}^\top - \begin{bmatrix} \alpha_k \Pi_k & 0 & 0 \\ 0 & \gamma_k & 0 \\ 0 & 0 & \sigma_k \end{bmatrix}. \end{aligned}$$

We want to find the restrictions on  $P$  and the gains  $\alpha_k, \gamma_k, \sigma_k$  such that  $Q_k \preceq 0$ . To this end, using the Schur complement of  $Q_k$  and multiplying the obtained matrix from both sides on  $\text{diag}\{Z(t_{k+1}), \Gamma(t_{k+1})Z^{-\top}(t_{k+1}), 1, 1\}$  and its transpose the latter property is equivalent to

$$\begin{aligned} \tilde{Q}_k \succeq 0, \quad \tilde{Q}_k &= \begin{bmatrix} \tilde{Q}_{11}^k & \tilde{Q}_{12}^k \\ (\tilde{Q}_{12}^k)^\top & \tilde{Q}_{22}^k \end{bmatrix}, \\ \tilde{Q}_{11}^k &= \begin{bmatrix} \Pi_{k+1}^{-1} & \Gamma(t_{k+1}) \\ \Gamma(t_{k+1}) & \alpha_k R_k \end{bmatrix}, \\ \tilde{Q}_{12}^k &= h \begin{bmatrix} Z(t_{k+1})b & -F(t_{k+1}) \\ 0 & 0 \end{bmatrix}, \quad \tilde{Q}_{22}^k = \begin{bmatrix} \gamma_k & 0 \\ 0 & \sigma_k \end{bmatrix}, \\ R_k &= \Gamma(t_{k+1})Z^{-\top}(t_{k+1})\Pi_k Z^{-1}(t_{k+1})\Gamma(t_{k+1}). \end{aligned}$$

Noting that

$$\begin{aligned} R_k &= \Gamma(t_{k+1}) (I_n - h(A - D(t_{k+1})LC))^\top \Gamma^{-1}(t_{k+1}) P \\ &\quad \times \Gamma^{-1}(t_{k+1}) (I_n - h(A - D(t_{k+1})LC)) \Gamma(t_{k+1}) \end{aligned}$$

and recalling the property  $\Gamma^{-1}(t)(A - D(t)LC)\Gamma(t) = \varrho(t)(A - LC)$ , we obtain

$$\begin{aligned} R_k &= P - h\varrho(t_{k+1}) ((A - LC)^\top P + P(A - LC)) \\ &\quad + h^2 \varrho^2(t_{k+1}) (A - LC)^\top P (A - LC), \end{aligned}$$

which implies that this matrix is positive definite under the restrictions of Theorem 2 for any  $k \in \mathbb{Z}_+$ .

To formulate the conditions implying  $\tilde{Q}_k \succeq 0$ , first, let us investigate the restrictions for  $\tilde{Q}_{11}^k \succ 0$ . Note that

$$P \succeq \Gamma(t_{k+1})\Gamma^{-1}(t_{k+2})P\Gamma^{-1}(t_{k+2})\Gamma(t_{k+1})$$

for any  $k \in \mathbb{Z}_+$  provided that  $HP + PH \succeq 0$ . Indeed, let  $w \in \mathbb{R}^n$  be an arbitrary nonzero vector, denote  $q(t) = w^\top \Gamma^{-1}(t)P\Gamma^{-1}(t)w$ , then

$$\begin{aligned} \dot{q}(t) &= w^\top \frac{d\Gamma^{-1}(t)}{dt} P \Gamma^{-1}(t) w + w^\top \Gamma^{-1}(t) P \frac{d\Gamma^{-1}(t)}{dt} w \\ &= -\varrho^{-1}(t) (w^\top \Gamma^{-1}(t) H P \Gamma^{-1}(t) w \\ &\quad + w^\top \Gamma^{-1}(t) P H \Gamma^{-1}(t) w) \end{aligned}$$

$$= -\varrho^{-1}(t) w^\top \Gamma^{-1}(t) (HP + PH) \Gamma^{-1}(t) w \leq 0.$$

Hence,  $t_{k+1} < t_{k+2}$  implies

$$\Gamma^{-1}(t_{k+1})P\Gamma^{-1}(t_{k+1}) \succeq \Gamma^{-1}(t_{k+2})P\Gamma^{-1}(t_{k+2}).$$

Therefore, calculating the Schur complement of  $\tilde{Q}_{11}^k$  we get:

$$\alpha_k R_k - \Gamma(t_{k+1})\Gamma^{-1}(t_{k+2})P\Gamma^{-1}(t_{k+2})\Gamma(t_{k+1}) \succ 0,$$

and the property  $\tilde{Q}_{11}^k \succ 0$  follows the conditions of Theorem 2 for  $\alpha_k = a$  or  $\alpha_k = \frac{a}{h\varrho(t_{k+1})}$  for some  $a \geq 1$ . If  $\alpha_k = \frac{a}{h^2 \varrho^2(t_{k+1})}$  with  $a > 0$ , then an auxiliary linear matrix inequality should be verified:

$$a(A - LC)^\top P(A - LC) - P \succ 0,$$

and, obviously, always there is  $a > 0$  such that it is true for a non-singular matrix  $A - LC$ . Returning back to verification of  $\tilde{Q}_k \succeq 0$ , and having  $\tilde{Q}_{11}^k \succ 0$  for such a choice of  $\alpha_k$ , we can also use the Schur complement of  $\tilde{Q}_k$  to check the desired property:

$$\tilde{Q}_{22}^k - (\tilde{Q}_{12}^k)^\top (\tilde{Q}_{11}^k)^{-1} \tilde{Q}_{12}^k \succeq 0.$$

To simplify this expression, denote  $T_k = (\tilde{Q}_{11}^k)^{-1}$ , which can be calculated using the block matrix inversion formula given in the preliminaries with the first block element

$$\begin{aligned} T_k^{11} &= \Pi_{k+1} + \Pi_{k+1} \Gamma(t_{k+1}) S_k \Gamma(t_{k+1}) \Pi_{k+1}, \\ S_k &= (\alpha_k R_k - \Gamma(t_{k+1}) \Pi_{k+1} \Gamma(t_{k+1}))^{-1}, \end{aligned}$$

and it has been shown above that  $S_k$  is nonsingular and bounded for selected  $\alpha_k = \frac{a}{h^2 \varrho^2(t_{k+1})}$  with  $a > 0$  for any  $k \in \mathbb{Z}_+$ . This notation leads to

$$\begin{aligned} \tilde{Q}_{22}^k - (\tilde{Q}_{12}^k)^\top (\tilde{Q}_{11}^k)^{-1} \tilde{Q}_{12}^k &= \begin{bmatrix} \gamma_k & 0 \\ 0 & \sigma_k \end{bmatrix} - h^2 \times \\ &\begin{bmatrix} b^\top Z^\top(t_{k+1}) T_k^{11} Z(t_{k+1}) b & -b^\top Z^\top(t_{k+1}) T_k^{11} F(t_{k+1}) \\ -F^\top(t_{k+1}) T_k^{11} Z(t_{k+1}) b & F^\top(t_{k+1}) T_k^{11} F(t_{k+1}) \end{bmatrix}. \end{aligned}$$

Using the block form of  $Z(t)$  it is easy to check that

$$Z(t)b = \frac{1}{O(t)} \begin{bmatrix} \vdots \\ h^{n-j} \left( 1 + \sum_{i=1}^{j-1} L_i h^i \varrho^i(t) \right) \\ \vdots \\ j = 1, \dots, n \end{bmatrix},$$

is of order  $\varrho^{-1}(t)$  for any  $t \geq 0$ , while it has been discussed that  $F(t)$  is globally bounded for all  $t \geq 0$ . Note that  $T_k^{11}$  is of the order  $\Gamma^{-2}(t_{k+1})$ , hence, bounded. Therefore, there exists  $g > 0$  and  $s > 0$  such that for  $\gamma_k = \frac{h^2 g}{\varrho^{2n}(t_{k+1})}$  and  $\sigma_k = h^2 s$  the property  $\tilde{Q}_k \succeq 0$  is verified for all  $k \in \mathbb{Z}_+$ . Consequently, under the introduced restrictions, it has been shown above that

$$V_{k+1} \leq \frac{a}{h^2 \varrho^2(t_{k+1})} V_k + \frac{h^2 g |d|_\infty^2}{\varrho^{2n}(t_{k+1})} + h^2 s |v|_\infty^2$$

for all  $k \in \mathbb{Z}_+$ . Hence,

$$V_k \leq \prod_{i=0}^{k-1} \frac{a}{h^2 \varrho^2(t_{i+1})} V_0 + h^2 s |v|_\infty^2 \sum_{i=0}^{k-1} \prod_{\ell=i+1}^{k-1} \frac{a}{h^2 \varrho^2(t_{\ell+1})} \\ + h^2 g |d|_\infty^2 \sum_{i=0}^{k-1} \frac{1}{\varrho^{2n}(t_{i+1})} \prod_{\ell=i+1}^{k-1} \frac{a}{h^2 \varrho^2(t_{\ell+1})}$$

which gives the required estimate for any  $k \in \mathbb{Z}_+$ . Asymptotic validation based on the fact that  $Z_\infty = \lim_{t \rightarrow +\infty} Z(t)$  is a nilpotent matrix of order  $n$ , then there is  $0 \prec \tilde{P} = \tilde{P}^\top \in \mathbb{R}^{n \times n}$  such that  $Z_\infty^\top \tilde{P} Z_\infty \prec \tilde{P}$  and a Lyapunov function  $\tilde{V}_k = e_k^\top \tilde{P} e_k$  can be used to justify boundedness in  $v$  and canceling the influence of  $d$ : for any  $\bar{k} \in \mathbb{Z}_+$  there exist  $\lambda, \Psi_h, \Phi_h > 0$  such that  $\tilde{V}_{k+1} - \tilde{V}_k \leq -\lambda \|e_k\|^2 + \frac{\Psi_h |d|_\infty^2}{\varrho^2(t_{k+1})} + \Phi_h |v|_\infty^2$  for all  $k \geq \bar{k}$ , and the desired conclusion follows from the input-to-state stability theory [18].  $\square$

As we can see from the results obtained in Theorem 2, the convergence in the initial error is faster than any exponential (there is an index  $k^* \geq 0$  such that  $\prod_{i=0}^{k-1} \frac{a}{h^2 \varrho^2(t_{i+1})}$  starts to be strictly decreasing for all  $k \geq k^*$ , and next, there is  $k' \geq k^*$  such that the gain  $\varrho^n(t_{k+1}) \prod_{i=0}^{k-1} \frac{a}{h^2 \varrho^2(t_{i+1})}$  decays to zero) and there is no asymptotic dependence in  $d$ , while the gain in the noise  $v$  admits a static linear upper bound.

## VI. COMPARISON IN SIMULATIONS

Let us illustrate the performance of the proposed hyperexponential observer/differentiator (5) in numeric experiments, and in comparison with the high-gain differentiator from [3] and the high-order sliding mode exact differentiator from [4], [10]. The former can be discretized also using the implicit Euler method, then its realization will take the same form as (5) by choosing a constant time value  $t_{k+1} = T$ , where  $T > 0$  is a design parameter; the latter differentiator, due to its nonlinearity and discontinuity, it is difficult to discretize using an implicit method [19], while the explicit

	SM	HG	Hyp
$\ell = 0.1$	25.176	16.759	16.757
$\ell = 0.01$	13.545	14.223	12.307

Table I: Root mean square error for  $50 \leq t \leq 100$

Euler discretization has many drawbacks [20], [21], then the approach of [22] can be used as a good compromise:

$$\hat{x}_{k+1} = \hat{x}_k + h \begin{bmatrix} -L_1^{ST} |y_k - \hat{x}_{k,1}|^{\frac{n-1}{n}} \text{sign}(y_k - \hat{x}_{k,1}) \\ + \sum_{j=1}^{n-1} \frac{h^{j-1}}{j!} \hat{x}_{k,j+1} \\ -L_2^{ST} |y_k - \hat{x}_{k,1}|^{\frac{n-2}{n}} \text{sign}(y_k - \hat{x}_{k,1}) \\ + \sum_{j=1}^{n-2} \frac{h^{j-1}}{j!} \hat{x}_{k,j+2} \\ \vdots \\ -L_{n-1}^{ST} |y_k - \hat{x}_{k,1}|^{\frac{1}{n}} \text{sign}(y_k - \hat{x}_{k,1}) \\ + \hat{x}_{k,n} \\ -L_n^{ST} \text{sign}(y_k - \hat{x}_{k,1}) \end{bmatrix},$$

where  $\hat{x}_k = [\hat{x}_{k,1} \dots \hat{x}_{k,n}]^\top \in \mathbb{R}^n$  is the state estimate as before,  $L^{ST} = [L_1^{ST} \dots L_n^{ST}]^\top \in \mathbb{R}^n$  is the observer gain to be chosen,  $\text{sign} : \mathbb{R} \rightarrow \mathbb{R}$  is the standard sign function. The results of such a comparison for the case  $n = 2$  can be found in [14], in this work  $n = 3$  is selected.

For simulations, let

$$\phi(t) = (t+1)^{1.5} + \sqrt{t+2} + \sin(5t)$$

and the noise

$$v(t) = \ell (1 + \text{rnd}(1) + \sin(15t)),$$

where  $\text{rnd}(1)$  denotes a uniformly distributed in the interval  $[0, 1]$  random number, and  $\ell > 0$  is the noise magnitude. Solution of LMIs from Theorem 1 gives:

$$L = [1.7875 \ 2.0275 \ 0.7190]^\top.$$

For  $h = 0.1$ , the results of simulations are presented in figures 2 and 3 for  $\ell = 0.1$  and  $\ell = 0.01$ , respectively. On the plots an error norm,  $\|e_k\| = |\hat{x}_{k,1} - \phi(t_k)| + |\hat{x}_{k,2} - \dot{\phi}(t_k)| + |\hat{x}_{k,3} - \ddot{\phi}(t_k)|$ , is shown in logarithmic scale versus the time,  $t_k$  (“SM” corresponds to the curve generated by sliding mode differentiator for  $L^{ST} = D(4)L$ , “HG” is for high-gain one with  $T = 40$ , and “Hyp” stands for the observer proposed here). As we can conclude from these results, asymptotically the hyperexponential differentiator outperforms its alternatives in both, noisy and less noisy, cases. The root mean square error for the final time interval  $[50, 100]$  and all three estimators is presented in Table I illustrating this observation.

For the hyperexponential observer, the error stays asymptotically bounded and not growing with time, which implies boundedness of the gain with respect to the noise  $v(t)$ . In addition, since the differentiator gain  $D(t)L$  is gradually growing, there is no peaking in the transients, which is a usual drawback in the high-gain solutions [23]. The convergence decay and the shape of transients in (5) can be optimized by the choice and tuning of  $\varrho(t)$ .



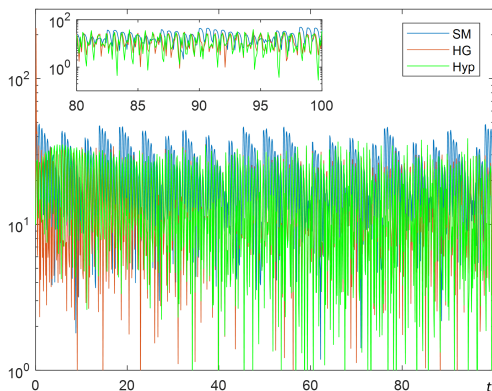


Figure 2: The results of estimation,  $\|e\|$  versus time  $t$ ,  $\ell = 0.1$

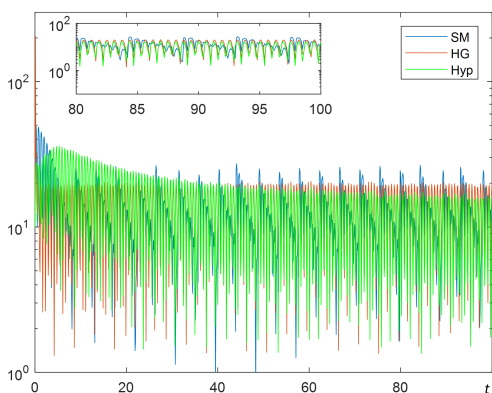


Figure 3: The results of estimation,  $\|e\|$  versus time  $t$ ,  $\ell = 0.01$

## VII. CONCLUSION

A new linear observer is proposed with time-varying gains, which has a simple discrete-time implementation, guaranteeing a hyperexponential rate of convergence of the estimation error and being asymptotically exact in the noise-free case. It has also certain robustness with respect to the measurement noise. The tuning rules are formulated using feasible linear matrix inequalities. The simulations demonstrate a good performance of the proposed differentiator, especially in the case of slow sampling. Noise filtering extension in the sense of [11] can be considered as a direction for future research.

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