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A novel lagged estimation framework for sparsely observed systems supplemented with virtual measurements

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Abstract. Monitoring high-dimensional structures using model-based estimation faces challenges due to limited sensors, necessitating extensive sensor channels and high-fidelity models. This conventional approach is hampered by its laborious, expensive, and computationally intensive nature, which is often impractical due to sensor inaccessibility, especially to attain input robustness. While typically input rejection enforces conditions on the availability of sufficient measurement channels, this study introduces a cost-effective monitoring method operating in a lagged time domain that can reject the input forces, even with sparse instrumentation. To provide sufficient measurements, the sparse sensor data sampled at a certain time instance is augmented with responses from subsequent time steps, posed as responses sampled from virtual sensors. This eventually mandates employing a unique measurement model that maps state estimates to real and virtual measurements, effectively overcoming the scarcity of physical sensors. Tested on simplified linear time-invariant (LTI) systems, in terms of a lumped mass model, the proposed approach utilizes the traditional Kalman filtering framework. A numerical observability study identifies required instrumentation nodes, successfully estimating unobserved states by fusing real and virtual sensor data. This success suggests the potential exploration of linear time-variant (LTV) systems and extension to parameter estimation towards structural condition monitoring.

Keywords: Virtual Sensor · Condition Monitoring · State Estimation · Limited Data · High-dimensionality · Complex Structures.

1 Introduction

The perpetual challenge in system identification research has been the scarcity of data for inverse estimation frameworks. The insufficiency of observed states for the estimable system often renders the approach ill-posed, resulting in impractical or divergent solutions. This challenge is accentuated in studies focusing

on the health estimation of real-life infrastructures, where these structures may not always be abundantly instrumented. In the realm of model-based estimation frameworks, the intricate nature of real-life structures necessitates extensive modeling, leading to an overparameterized, high-fidelity representation that becomes challenging to manage with limited sensor measurements. A practical solution lies in approaches that either simplify the model representation or reduce parameterization to accommodate sparse instrumentation, potentially impacting estimation resolution or accuracy [3].

The proposed alternative in this study introduces a novel virtual sensor approach where estimable system states are observed through current and future sensor measurements. In the typical estimation approach, current measurements are viewed as responses sampled from real (physical) sensors, while responses from future time steps are considered as sampled from virtual sensors. The need for future responses in this estimation approach requires it to be conducted in a lagged time domain. However, this is not a concern in the context of structural health monitoring, as the responses are sampled at a high sampling rate, rendering the estimation lag insignificant.

By supplementing actual sensor measurements with virtual measurements, the inherent issues of the estimation approach can be mitigated. Recognizing the significant damping in real-world structures, it is prudent to avoid utilizing responses too distant from the current time steps. However, given the rapid sampling of the system measurements, future responses can still serve as valuable observations for estimating the system states. This methodology also necessitates the modeling of the input force, a task undertaken in this context through the utilization of a Gaussian process.

2 Methodology

A high-dimensional system can be defined as n^{th} order mechanical system with time-invariant mass, damping, and stiffness matrices. The pertinent dynamics can be defined with discrete-time state-space model as [1],

$$\mathbf{x}_k = \mathbf{F}_k \mathbf{x}_{k-1} + \mathbf{E}_k \mathbf{q}_k + \mathbf{w}_k \quad (1a)$$

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{L}_k \mathbf{q}_k + \nu_k \quad (1b)$$

Let $\mathbf{x}_k \in \mathbb{R}^n$, $\mathbf{y}_k \in \mathbb{R}^m$, and $\mathbf{q}_k \in \mathbb{R}^s$. The first equation describes the evolution of the states \mathbf{x}_k over time, governed by the state transition matrix \mathbf{F}_k , under the influence of the unknown input force (\mathbf{q}_k) through the input matrix \mathbf{E}_k . The second equation illustrates the measurement of unobservable state vectors as \mathbf{y}_k using the measurement matrix \mathbf{H}_k . The discrete-time entities, including \mathbf{F}_k , \mathbf{E}_k , \mathbf{H}_k , \mathbf{L}_k , \mathbf{x}_k , \mathbf{y}_k , \mathbf{v}_k , and \mathbf{w}_k , are obtained from their continuous-time counterparts using the zero-order-hold technique, as detailed in [4]. The process and measurement noise, denoted as \mathbf{w}_k and \mathbf{v}_k respectively, follow zero-mean Gaussian noise statistics with covariances \mathbf{Q}_k and \mathbf{R}_k .

Typically, existing methods necessitate more sensors than unknown inputs \mathbf{q}_k for effective rejection, requiring $m > s$ [5]. This study, however, explores case studies where this condition is not met, demanding a sophisticated approach capable of addressing such constraints [2].

This study further considers a sequence of independent white noises $\gamma_k \in \mathbb{R}^s$ to model the input as $\mathbf{q}_{k+1} \sim \alpha \mathbf{q}_k + \gamma_k$, where α belongs to $]0, 1[$. Expanding \mathbf{x}_{k+1} and \mathbf{q}_{k+1} as $\mathbf{x}_{k+1} = \mathbf{F}_{k+1}\mathbf{x}_k + \mathbf{E}_{k+1}\mathbf{q}_{k+1} + \mathbf{w}_{k+1}$, and $\mathbf{q}_{k+1} = \alpha \mathbf{q}_k + \gamma_k$ respectively, \mathbf{y}_{k+1} be expressed as follows :

$$\begin{aligned} \mathbf{y}_{k+1} &= \mathbf{H}_{k+1}\mathbf{x}_{k+1} + \mathbf{L}_{k+1}\mathbf{q}_{k+1} + \nu_{k+1} \\ &= [\mathbf{H}_{k+1}\mathbf{F}_{k+1} \alpha[\mathbf{H}_{k+1}\mathbf{E}_{k+1} + \mathbf{L}_{k+1}]] \begin{bmatrix} \mathbf{x}_k \\ \mathbf{q}_k \end{bmatrix} \\ &\quad + [\mathbf{H}_{k+1}\mathbf{E}_{k+1} + \mathbf{L}_{k+1}]\gamma_k + \mathbf{H}_{k+1}\mathbf{w}_{k+1} + \nu_{k+1} \end{aligned} \quad (2)$$

Further, following the same approach, expanding \mathbf{x}_{k+2} and \mathbf{q}_{k+2} as $x_{k+2} = \mathbf{F}_{k+2}\mathbf{x}_{k+1} + \mathbf{E}_{k+2}\mathbf{q}_{k+2} + \mathbf{w}_{k+2}$, and $\mathbf{q}_{k+2} = \alpha \mathbf{q}_{k+1} + \gamma_{k+1}$ respectively, \mathbf{y}_{k+2} can be represented as,

$$\begin{aligned} \mathbf{y}_{k+2} &= \mathbf{H}_{k+2}\mathbf{x}_{k+2} + \mathbf{L}_{k+2}\mathbf{q}_{k+2} + \nu_{k+2} \\ &= [\mathbf{H}_{k+2}\mathbf{F}_{k+2}\mathbf{F}_{k+1} \alpha\mathbf{H}_{k+2}\mathbf{F}_{k+2}\mathbf{E}_{k+1} + \alpha^2(\mathbf{H}_{k+2}\mathbf{E}_{k+2} + \mathbf{L}_{k+2})] \begin{bmatrix} \mathbf{x}_k \\ \mathbf{q}_k \end{bmatrix} \\ &\quad + [\mathbf{H}_{k+2}\mathbf{F}_{k+2}\mathbf{E}_{k+1} + \alpha(\mathbf{H}_{k+2}\mathbf{E}_{k+2} + \mathbf{L}_{k+2})(\mathbf{H}_{k+2}\mathbf{E}_{k+2} + \mathbf{L}_{k+2})] \begin{bmatrix} \gamma_k \\ \gamma_{k+1} \end{bmatrix} \\ &\quad + [\mathbf{H}_{k+2}\mathbf{F}_{k+2} \ \mathbf{H}_{k+2}] \begin{bmatrix} \mathbf{w}_{k+1} \\ \mathbf{w}_{k+2} \end{bmatrix} + \nu_{k+2} \end{aligned} \quad (3)$$

Accumulating responses from z numbers of subsequent steps, $\bar{\mathbf{y}}_k \in \mathbb{R}^{zm}$ can be constituted and represented as $\bar{\mathbf{y}}_k = [\mathbf{y}_k^T \ \mathbf{y}_{k+1}^T \ \mathbf{y}_{k+2}^T]^T$ observation for unknown augmented system states $\{\mathbf{x}_k^T; \mathbf{q}_k^T\}^T$ as,

$$\begin{aligned} \bar{\mathbf{y}}_k &= \begin{bmatrix} \mathbf{H}_k & \vdots & \mathbf{L}_k \\ \mathbf{H}_{k+1}\mathbf{F}_{k+1} & \vdots & \alpha(\mathbf{H}_{k+1}\mathbf{E}_{k+1} + \mathbf{L}_{k+1}) \\ \mathbf{H}_{k+2}\mathbf{F}_{k+2}\mathbf{F}_{k+1} & \vdots & \alpha\mathbf{H}_{k+2}[\mathbf{F}_{k+2}\mathbf{E}_{k+1} + \alpha\mathbf{E}_{k+2}] + \alpha^2\mathbf{L}_{k+2} \end{bmatrix} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{q}_k \end{bmatrix} \\ &\quad + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ (\mathbf{H}_{k+1}\mathbf{E}_{k+1} + \mathbf{L}_{k+1}) & \mathbf{0} \\ \mathbf{H}_{k+2}\mathbf{F}_{k+2}\mathbf{E}_{k+1} + \alpha(\mathbf{H}_{k+2}\mathbf{E}_{k+2} + \mathbf{L}_{k+2})(\mathbf{H}_{k+2}\mathbf{E}_{k+2} + \mathbf{L}_{k+2}) & \mathbf{0} \end{bmatrix} \begin{bmatrix} \gamma_k \\ \gamma_{k+1} \end{bmatrix} \\ &\quad + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{k+1} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{k+2}\mathbf{F}_{k+2} & \mathbf{H}_{k+2} \end{bmatrix} \begin{bmatrix} \mathbf{w}_k \\ \mathbf{w}_{k+1} \\ \mathbf{w}_{k+2} \end{bmatrix} + \begin{bmatrix} \nu_k \\ \nu_{k+1} \\ \nu_{k+2} \end{bmatrix} \end{aligned} \quad (4)$$

assuming $z = 3$ for the sake of lucid apprehension. The underlying assumption is that $zm > s$, signifying a scenario where the number of measurements available is greater than the dimensionality of the rejected inputs. To reject q_k , let us proceed as follows:

$$\mathbf{C}_{1,\mathbf{k}} = \begin{bmatrix} \mathbf{H}_k \\ \mathbf{H}_{k+1}\mathbf{F}_{k+1} \\ \mathbf{H}_{k+2}\mathbf{F}_{k+2}\mathbf{F}_{k+1} \end{bmatrix}, \quad \mathbf{C}_{2,\mathbf{k}} = \begin{bmatrix} \mathbf{L}_k \\ \alpha(\mathbf{H}_{k+1}\mathbf{E}_{k+1} + \mathbf{L}_{k+1}) \\ \alpha(\mathbf{H}_{k+2}[\mathbf{F}_{k+2}\mathbf{E}_{k+1} + \alpha\mathbf{E}_{k+2}] + \alpha\mathbf{L}_{k+2}) \end{bmatrix},$$

$$\mathbf{C}_{3,\mathbf{k}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ (\mathbf{H}_{k+1}\mathbf{E}_{k+1} + \mathbf{L}_{k+1}) & \mathbf{0} \\ \mathbf{H}_{k+2}\mathbf{F}_{k+2}\mathbf{E}_{k+1} + \alpha(\mathbf{H}_{k+2}\mathbf{E}_{k+2} + \mathbf{L}_{k+2}) & (\mathbf{H}_{k+2}\mathbf{E}_{k+2} + \mathbf{L}_{k+2}) \end{bmatrix},$$

$$\mathbf{C}_{4,\mathbf{k}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{k+1} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{k+2}\mathbf{F}_{k+2} & \mathbf{H}_{k+2} \end{bmatrix}.$$

The compact form for the measurement equation can be defined as,

$$\bar{y}_k = [\mathbf{C}_{1,\mathbf{k}} \vdots \mathbf{C}_{2,\mathbf{k}}] \begin{bmatrix} \mathbf{x}_k \\ \mathbf{q}_k \end{bmatrix} + \mathbf{C}_{3,\mathbf{k}}\bar{\gamma}_k + \mathbf{C}_{4,\mathbf{k}}\bar{w}_k + \bar{v}_k \quad (5)$$

Further, combining Eq: (1a) and Eq: (5), and further reorganizing the equation, leads to the following equation,

$$\begin{bmatrix} \mathbf{x}_k \\ \mathbf{C}_{1,\mathbf{k}}\mathbf{x}_k \end{bmatrix} = \begin{bmatrix} \mathbf{F}_k\mathbf{x}_{k-1} + \mathbf{E}_k\mathbf{q}_k + [\mathbf{I} \mathbf{0} \mathbf{0}] \bar{w}_k \\ \bar{y}_k - \mathbf{C}_{2,\mathbf{k}}\mathbf{q}_k - \mathbf{C}_{3,\mathbf{k}}\bar{\gamma}_k - \mathbf{C}_{4,\mathbf{k}}\bar{w}_k - \bar{v}_k \end{bmatrix} \quad (6)$$

The equation can further be restructured to represent a process equation for \mathbf{x}_k with \bar{y}_k being an additional input injected into the process, as,

$$\begin{bmatrix} \mathbf{I} & -\mathbf{F}_k \\ \mathbf{C}_{1,\mathbf{k}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{x}_{k-1} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \bar{y}_k + \begin{bmatrix} \mathbf{E}_k \\ -\mathbf{C}_{2,\mathbf{k}} \end{bmatrix} \mathbf{q}_k + \begin{bmatrix} \mathbf{0} \\ -\mathbf{C}_{3,\mathbf{k}} \end{bmatrix} \bar{\gamma}_k + \begin{bmatrix} \mathcal{I} \\ -\mathbf{C}_{4,\mathbf{k}} \end{bmatrix} \bar{w}_k + \begin{bmatrix} \mathbf{0} \\ -\mathbf{I} \end{bmatrix} \bar{v}_k \quad (7)$$

With

$$\mathcal{E}_k = \left\{ \begin{bmatrix} \mathbf{E}_k \\ -\mathbf{C}_{2,\mathbf{k}} \end{bmatrix} \right\} \in \mathbb{R}^{(n+zm) \times s}$$

Note that the matrix \mathcal{E}_k possesses full column rank, implying unique solutions. Assuming $\exists \mathbf{P}_k \in \mathbb{R}^{(n+zm-s) \times (n+zm)} : \mathbf{P}_k \mathcal{E}_k = \mathbf{0}$, pre-multiplying both sides of the equation by \mathbf{P}_k , eliminates the dependency of the process equation from \mathbf{q}_k , yielding:

$$\mathbf{P}_k \begin{bmatrix} \mathbf{I} & -\mathbf{F}_k \\ \mathbf{C}_{1,\mathbf{k}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{x}_{k-1} \end{bmatrix} = \mathbf{P}_k \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \bar{y}_k + \mathbf{P}_k \begin{bmatrix} \mathbf{0} \\ -\mathbf{C}_{3,\mathbf{k}} \end{bmatrix} \bar{\gamma}_k + \mathbf{P}_k \begin{bmatrix} \mathcal{I} \\ -\mathbf{C}_{4,\mathbf{k}} \end{bmatrix} \bar{w}_k + \mathbf{P}_k \begin{bmatrix} \mathbf{0} \\ -\mathbf{I} \end{bmatrix} \bar{v}_k \quad (8)$$

Further assigning

$\mathcal{M}_k = \mathbf{P}_k \begin{bmatrix} \mathbf{I} & -\mathbf{F}_k \\ \mathbf{C}_{1,\mathbf{k}} & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{(n+zm-s) \times 2n}$, and considering a QR decomposition of \mathcal{M}_k as $M = \mathcal{Q}_k \mathcal{R}_k$, with $\mathcal{Q}_k \in \mathbb{R}^{(n+zm-s) \times (n+zm-s)}$ and $\mathcal{R}_k \in$

$\mathbb{R}^{(n+zm-s) \times 2n}$ such that $\mathcal{Q}_k^T \mathcal{Q}_k = I$ and \mathcal{R}_k is upper triangular, the following expression can be arrived at,

$$\mathcal{R}_k \begin{bmatrix} \mathbf{x}_k \\ \mathbf{x}_{k-1} \end{bmatrix} = \mathcal{Q}_k^T \mathbf{P}_k \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \bar{y}_k + \mathcal{Q}_k^T \mathbf{P}_k \begin{bmatrix} \mathbf{0} \\ -\mathbf{C}_{\mathbf{3},\mathbf{k}} \end{bmatrix} \bar{\gamma}_k + \mathcal{Q}_k^T \mathbf{P}_k \begin{bmatrix} \mathcal{I} \\ -\mathbf{C}_{\mathbf{4},\mathbf{k}} \end{bmatrix} \bar{w}_k + \mathcal{Q}_k^T \mathbf{P}_k \begin{bmatrix} \mathbf{0} \\ -\mathbf{I} \end{bmatrix} \bar{v}_k \quad (9)$$

With,

$$G_k = \mathcal{Q}_k^T \mathbf{P}_k \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}; \Gamma_k = \mathcal{Q}_k^T \mathbf{P}_k \begin{bmatrix} \mathbf{0} \\ -\mathbf{C}_{\mathbf{3},\mathbf{k}} \end{bmatrix}; \mathcal{W}_k = \mathcal{Q}_k^T \mathbf{P}_k \begin{bmatrix} \mathcal{I} \\ -\mathbf{C}_{\mathbf{4},\mathbf{k}} \end{bmatrix}; \mathcal{V}_k = \mathcal{Q}_k^T \mathbf{P}_k \begin{bmatrix} \mathbf{0} \\ -\mathbf{I} \end{bmatrix}$$

The process equation can be compacted as,

$$\mathcal{R}_k \begin{bmatrix} \mathbf{x}_k \\ \mathbf{x}_{k-1} \end{bmatrix} = G_k \bar{y}_k + \Gamma_k \bar{\gamma}_k + \mathcal{W}_k \bar{w}_k + \mathcal{V}_k \bar{v}_k \quad (10)$$

Segmenting the upper triangular matrix \mathcal{R}_k into 4 blocks as:

$$\mathcal{R}_k = \begin{bmatrix} R_k^{11} & R_k^{12} \\ \mathbf{0} & R_k^{22} \end{bmatrix} \quad (11)$$

where $R_k^{11} \in \mathbb{R}^{n \times n}$, $R_k^{12} \in \mathbb{R}^{n \times n}$, and $R_k^{22} \in \mathbb{R}^{(zm-s) \times n}$ and accordingly segmenting G_k , Γ_k , \mathcal{W}_k , and \mathcal{V}_k two blocks of dimension $\mathbb{R}^{n \times zm}$ and $\mathbb{R}^{(zm-s) \times zm}$, the following expression is attained.

$$\begin{bmatrix} R_k^{11} & R_k^{12} \\ \mathbf{0} & R_k^{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{x}_{k-1} \end{bmatrix} = \begin{bmatrix} G_k^1 \\ G_k^2 \end{bmatrix} \bar{y}_k + \begin{bmatrix} \Gamma_k^1 \\ \Gamma_k^2 \end{bmatrix} \bar{\gamma}_k + \begin{bmatrix} \mathcal{W}_k^1 \\ \mathcal{W}_k^2 \end{bmatrix} \bar{w}_k + \begin{bmatrix} \mathcal{V}_k^1 \\ \mathcal{V}_k^2 \end{bmatrix} \bar{v}_k \quad (12)$$

and further simplified to be integrated within the filtering framework by shifting the time index for the measurement equation as,

$$\begin{aligned} \mathbf{x}_k &= -[R_k^{11-1} \mathcal{R}_k^{12}] \mathbf{x}_{k-1} + [R_k^{11-1} \cdot G_k^1] \bar{y}_k + \tilde{\mathbf{w}}_k \quad \text{with } \tilde{\mathbf{w}}_k = \Gamma_k^1 \bar{\gamma}_k + \mathcal{W}_k^1 \bar{w}_k + \mathcal{V}_k^1 \bar{v}_k \\ \bar{z}_k &= R_{k+1}^{22} \mathbf{x}_k + \tilde{\mathbf{v}}_k \quad \text{with } \tilde{\mathbf{v}}_k = \Gamma_k^2 \bar{\gamma}_k + \mathcal{W}_k^2 \bar{w}_k + \mathcal{V}_k^2 \bar{v}_k \end{aligned} \quad (13)$$

wherein the system state \mathbf{x}_k is being observed with a transformed measurement as $\bar{z}_k = G_{k+1}^2 \bar{y}_{k+1}$. Finally, the filtering equations are derived as,

$$\begin{aligned} \mathbf{x}_k &= \mathcal{F}_k \mathbf{x}_{k-1} + \mathcal{G}_k \bar{y}_k + \tilde{\mathbf{w}}_k \\ \bar{z}_k &= \mathcal{H}_k \mathbf{x}_k + \tilde{\mathbf{v}}_k \end{aligned} \quad (14)$$

with $\mathcal{F}_k = -[R_k^{11-1} \cdot \mathcal{R}_k^{12}]$, $\mathcal{G}_k = [R_k^{11-1} \cdot G_k^1]$, and $\mathcal{H}_k = R_{k+1}^{22}$. $\tilde{\mathbf{w}}_k$ and $\tilde{\mathbf{v}}_k$ represent the process and measurement uncertainty of covariance $\tilde{\mathbf{Q}}_k$ and $\tilde{\mathbf{R}}_k$ respectively. The covariances can be obtained from their respective expressions given in this derivation, which are not explicitly detailed here for the sake of brevity. Next, the traditional Kalman Filter (KF) framework for state estimation is employed to estimate the system states with sparse measurement data.

Discussion regarding the KF is also avoided here, and the readers can refer to any pertinent literature for the same.

In the following, the proposed methodology has been experimentally validated using a lumped mass representation of a system. The numerical validation is detailed in the following.

3 Numerical examples

The numerical demonstration of the virtual sensor-based state estimation approach involves two distinct case studies: (a) the analysis of a 6-DOF lumped mass system and (b) the examination of a cantilever beam. Both systems are subjected to ambient forcing represented with stationary white Gaussian noise (SWGN). During a simulated duration of 100 seconds, acceleration data is generated and consistently sampled. To emulate real-world conditions, the simulated response is intentionally tainted with 1% SWGN noise.

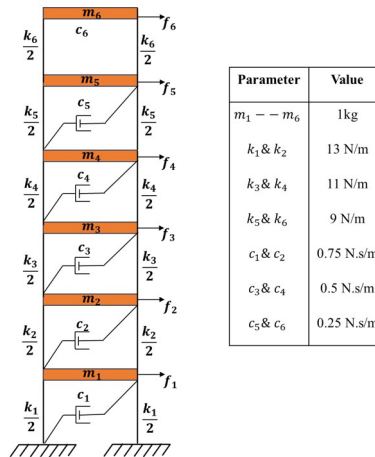


Fig. 1. Schematic diagram of 6-DOF lumped mass model

The study utilizes a 6-DOF lumped mass model, as illustrated in Fig. 1, with specific details provided in the schematic (cf. Figure 1). The chosen sampling frequency of 10 Hz is deemed adequate for capturing the system dynamics, as evidenced by the dominant natural frequencies. A SWGN forcing, represented by the same force model adopted in this study with α being 0.05, $\mathbf{q}_0 = 0$ and γ assumed to be a SWGN process of statistics $\mathcal{N}(0 \text{ N}, 10^{-2} \text{ N})$, is adopted for the simulation.

The proposed approach enables state estimation without requiring explicit or statistical information about the input forces (as indicated in Eq. (14)). The efficacy of the estimation is demonstrated by comparing the predicted measurements to the actual ones. Given that the primary objective of this study is to

devise an approach suitable for wind turbine monitoring, where displacement is considered a more proficient sensor measurement, the study focuses on comparing displacement responses.

Several experiments involving different combinations of sensor locations and density are conducted. Initially, all nodes are assumed to be instrumented (case 1), eliminating the need for virtual sensors. Still, to satisfy the constraint, an additional layer of virtual sensors is added. However, as the number of physical sensors is reduced (3 sensors in Case 2 or 2 sensors in Case 3), the employment of virtual sensors becomes necessary. The selection of the temporal depth for virtual sensors ($k + 1, k + 2, \dots$) is dependent on the condition that a non-empty \mathbf{G}_k^2 matrix exists. This condition stems from the construction of the \mathbf{P}_k matrix, derived from the left null space of \mathcal{E}_k . In cases where measurements are sparse compared to the dimension of the rejected input, the resulting \mathbf{P}_k matrix fails to guarantee a non-empty \mathbf{G}_k^2 , which is essential for effective input rejection.

The estimation results for Case 1 are depicted in Fig. 2, revealing precise estimations of displacements. While other state estimates exhibit similar high-quality precision, they are not included here for the sake of brevity.

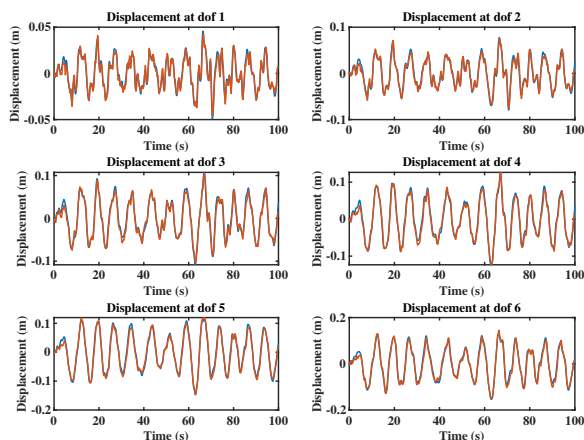


Fig. 2. Displacement estimates for all the dofs instrumented, 1% noisy data. — Actual
— Estimated

In the second case study, only the 1st, 3rd, and 6th nodes are instrumented, introducing an artificial scarcity in the available data. Due to the condition on the \mathbf{G}_k^2 matrix, it is observed that the typical Kalman Filter (KF) approach is not applicable, necessitating the use of virtual sensors. Two levels of virtual sensors ($k + 1$ and $k + 2$) are employed, resulting in an estimation lag of two time instances. The estimation results for displacement time histories are presented

in Fig. 3. Despite this lag, only negligible deterioration is observed in the time histories corresponding to the unmeasured degrees of freedom.

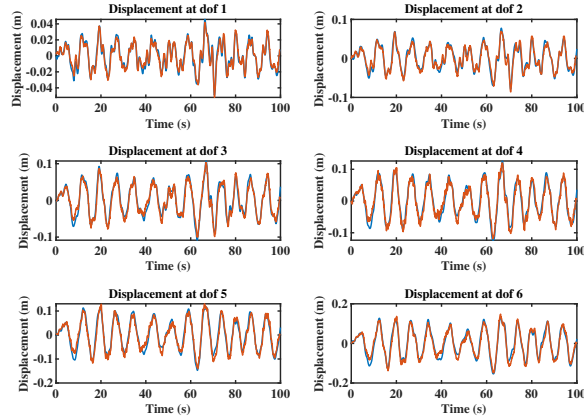


Fig. 3. Displacement estimates for half the dofs instrumented, 1% noisy data. —Actual —Estimated

In the third case study, only two sensors are positioned at the 1st and 6th nodes. The conditionality check for \mathbf{G}_k^2 indicates the necessity of three levels of virtual sensors. Consequently, the estimation is carried out for the states, and the corresponding displacements are presented in Fig. 4. The results show considerable improvement over the estimates via a conventional approach wherein the unknown inputs are modelled as a SWGN noise of known covariance (For comparison, see Fig. 5).

Due to limited page constraints, detailed analysis and investigation into the performance of the proposed approach with more complex real-life systems, under varying noise contamination levels and different sensor locations and respective stability analysis had to be omitted. These aspects will be comprehensively addressed in future publications.

4 Conclusion

The paper addresses challenges in monitoring high-dimensional structures with a limited number of sensors and intricate high-dimensional models. To overcome these challenges, it proposes an efficient estimation approach lagged in time, utilizing future measurements as responses sampled from virtual sensors. Alongside current-time measurements from physical sensors, these virtual sensors are

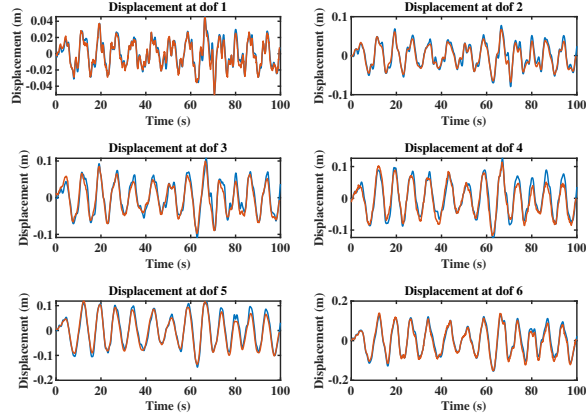


Fig. 4. Displacement estimates for one third of the dofs instrumented, 1% noise, using current input-free formulation. — Actual — Estimated

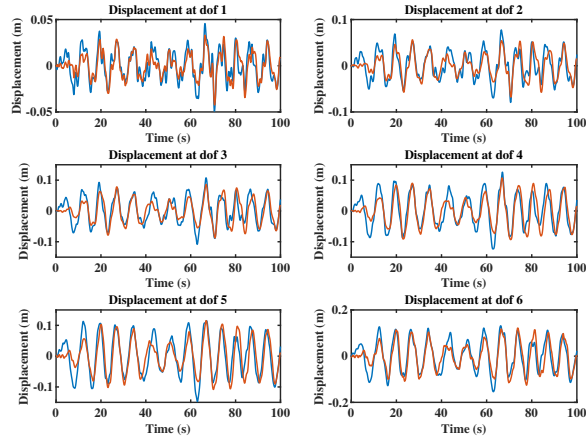


Fig. 5. Displacement estimates for one third of the dofs instrumented, 1% noise, using conventional approach. — Actual — Estimated

observed through a novel mapping, offering a robust solution to sparse instrumentation issues. The efficacy of the method is demonstrated through testing on linear time-invariant systems, successfully estimating unobserved states within the traditional Kalman filtering framework. This success encourages further exploration of the approach to more complex linear time-variant systems, with potential extensions to parameter estimation for structural condition monitoring.

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