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# Mitigating ill-posedness in parameter estimation under sparse measurement for linear time-varying systems employing virtual sensor responses <sup>\*</sup>

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**Abstract.** Monitoring linear time-varying (LTV) systems using model-based approaches typically requires dense instrumentation and high-fidelity support models, resulting in significant computational and financial costs. Bayesian filtering-based methods adopt a joint state-parameter estimation approach for LTV system monitoring wherein states/parameters are observed as well as inferred from the measurements. Eventually, sparsity in measurement while dealing with high-fidelity models can aggravate the ill-posedness in the estimation diverging the estimation to impractical or no solutions. To enhance estimation resolution and precision, an alternative approach can be supporting estimation with virtual sensor measurements sampled from future time steps. Virtual measurements are in fact measurements sampled from a future time step relative to the time at which estimation is sought. To leverage virtual measurements, the estimation needs to be time-delayed, enabling the observation of states/parameters through measurements taken at both current and subsequent time steps, thereby alleviating the ill-posedness. This method has been explored for an LTV spring-mass-dashpot system, where the numerical investigation utilizes an interacting filtering environment to estimate states and parameters in the presence of sparse measurements. The study has shown how incorporating additional information can stabilize the estimation process, leading to improved estimates for both states and parameters.

**Keywords:** Virtual Sensor · Condition Monitoring · State Estimation · Limited Data · High-dimensionality · Linear Time Variant Systems.

## 1 Introduction

The inverse estimation strategies, based on existing system identification approaches, often encounter challenges due to limited data availability, resulting

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in ill-posed problems that may yield impractical or divergent solutions. This issue is particularly prominent in health estimation studies of real-life infrastructures, which are typically instrumented sparsely compared to their large size. In model-based estimation frameworks, the complex nature of real-life structures necessitates high-fidelity models for monitoring in high resolution, potentially resulting in an over-parameterized system formulation. Managing such high dimensionality in system formulation becomes challenging, especially with limited sensor measurements. These challenges have been addressed through substructuring and focusing solely on an independent subdomain for estimation, which remains robust to subdomain boundary conditions [4]. However, such an approach may still be ill-posed, especially when available instrumentation is insufficient to provide the desired robustness in estimation. To address this issue, the present study proposes supporting the estimation with virtual sensor measurements, which can mitigate ill-posedness and stabilize the estimation process. The methodology along with the aspect of virtual measurements are elaborated upon in the following sections.

## 2 Methodology

A typical governing differential equation for the dynamics of an LTV mechanical system can be defined as:

$$\mathbf{M}(t)\ddot{\mathbf{x}} + \mathbf{C}(t)\dot{\mathbf{x}} + \mathbf{K}(t)\mathbf{x} = \mathbf{F}(t) \quad (1)$$

wherein  $\mathbf{M}$ ,  $\mathbf{C}$ , and  $\mathbf{K}$  represent mass, damping, and stiffness matrices with additional time dependence denoting the time-varying nature of the system. Defining such a mechanical system with a time-varying system model of order  $n$  in discrete-time state-space as [1],

$$\begin{aligned} \text{Process equation: } \mathbf{x}_k &= \mathbf{A}_k\mathbf{x}_{k-1} + \mathbf{B}_k\mathbf{u}_k + \mathbf{w}_k \\ \text{Measurement equation: } \mathbf{y}_k &= \mathbf{H}_k\mathbf{x}_k + \mathbf{G}_k\mathbf{u}_k + \mathbf{v}_k \end{aligned} \quad (2)$$

with  $\mathbf{x}_k \in \mathbb{R}^n$ ,  $\mathbf{y}_k \in \mathbb{R}^m$ , and  $\mathbf{u}_k \in \mathbb{R}^s$  denoting state, output, and input.  $\mathbf{A}_k$ ,  $\mathbf{B}_k$ , and  $\mathbf{H}_k$  represent state transition, input, and measurement matrices for which time-dependence is attributed to the time-varying nature of the system. For the detailed formulation of these system matrices, the readers may follow the works of [5]  $\mathbf{w}_k$ , and  $\mathbf{v}_k$  represent the noise processes in the process and measurements respectively that can be idealized with stationary white Gaussian noise (SWGNN) model with covariances  $\mathbf{Q}_k$  and  $\mathbf{R}_k$ .

Typically, existing methods necessitate more sensors than unknown inputs  $\mathbf{u}_k$  for effective rejection, requiring  $m > s$  [6]. Nevertheless, the estimation of systems wherein such conditionality is not met requires a sophisticated approach capable of addressing such constraints [3]. This study accordingly adopts an approach wherein states are observed with current as well as future measurements with future measurements acting as virtual measurements to alleviate the estimation ill-posedness. The term ‘‘future’’ here is used in a relative sense, wherein

the system estimation is done in a delayed time to avail the measurements of subsequent steps as future measurements. This study further defines the sequence of unknown forces  $\{\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_k\}$  with a discrete-time random walk model as  $\mathbf{u}_{k+1} \sim \alpha \mathbf{u}_k + \gamma_k$ , where  $\alpha$  belongs to  $]0, 1[$  and  $\gamma_k \in \mathbb{R}^s$  is SWGN process.

Next, with the system model defined in Equation 2 and the force model defined earlier,  $\mathbf{y}_{k+1}$  can be expressed as functions of an extended state vector  $\{\mathbf{x}_k; \mathbf{u}_k\}^T$ .

$$\begin{aligned} \mathbf{y}_{k+1} &= \mathbf{H}_{k+1} \mathbf{x}_{k+1} + \mathbf{G}_{k+1} \mathbf{u}_{k+1} + \mathbf{v}_{k+1} \\ &= [\mathbf{H}_{k+1} \mathbf{A}_{k+1} \ \alpha [\mathbf{H}_{k+1} \mathbf{B}_{k+1} + \mathbf{G}_{k+1}]] \begin{bmatrix} \mathbf{x}_k \\ \mathbf{u}_k \end{bmatrix} \\ &\quad + [\mathbf{H}_{k+1} \mathbf{B}_{k+1} + \mathbf{G}_{k+1}] \gamma_k + \mathbf{H}_{k+1} \mathbf{w}_{k+1} + \mathbf{v}_{k+1} \end{aligned} \quad (3)$$

This in turn helps to observe the states  $\{\mathbf{x}_k; \mathbf{u}_k\}^T$  defined at time instant  $k$  with future measurements. The same can be followed to observe the states  $\{\mathbf{x}_k; \mathbf{u}_k\}^T$  through  $\mathbf{y}_{k+2}$  as,

$$\begin{aligned} \mathbf{y}_{k+2} &= \mathbf{H}_{k+2} \mathbf{x}_{k+2} + \mathbf{G}_{k+2} \mathbf{u}_{k+2} + \mathbf{v}_{k+2} \\ &= [\mathbf{H}_{k+2} \mathbf{A}_{k+2} \mathbf{A}_{k+1} \ \alpha \mathbf{H}_{k+2} \mathbf{A}_{k+2} \mathbf{B}_{k+1} + \alpha^2 (\mathbf{H}_{k+2} \mathbf{B}_{k+2} + \mathbf{G}_{k+2})] \begin{bmatrix} \mathbf{x}_k \\ \mathbf{u}_k \end{bmatrix} \\ &\quad + [\mathbf{H}_{k+2} \mathbf{A}_{k+2} \mathbf{B}_{k+1} + \alpha (\mathbf{H}_{k+2} \mathbf{B}_{k+2} + \mathbf{G}_{k+2}) (\mathbf{H}_{k+2} \mathbf{B}_{k+2} + \mathbf{G}_{k+2})] \begin{bmatrix} \gamma_k \\ \gamma_{k+1} \end{bmatrix} \\ &\quad + [\mathbf{H}_{k+2} \mathbf{A}_{k+2} \ \mathbf{H}_{k+2}] \begin{bmatrix} \mathbf{w}_{k+1} \\ \mathbf{w}_{k+2} \end{bmatrix} + \mathbf{v}_{k+2} \end{aligned} \quad (4)$$

Accumulating responses from  $z$  numbers of subsequent steps,  $\bar{\mathbf{y}}_k \in \mathbb{R}^{zm}$  can be constituted and represented as  $\bar{\mathbf{y}}_k = [\mathbf{y}_k^T \ \mathbf{y}_{k+1}^T \ \dots \ \mathbf{y}_{k+z-1}^T]^T$  from where  $\{\mathbf{x}_k^T; \mathbf{u}_k^T\}^T$  can be inferred. For the sake of lucid comprehension, the subsequent demonstration has adopted a value of 3 which however can be extended depending on the demand of the system.

$$\begin{aligned} \bar{\mathbf{y}}_k &= \begin{bmatrix} \mathbf{H}_k & \vdots & \mathbf{G}_k \\ \mathbf{H}_{k+1} \mathbf{A}_{k+1} & \vdots & \alpha (\mathbf{H}_{k+1} \mathbf{B}_{k+1} + \mathbf{G}_{k+1}) \\ \mathbf{H}_{k+2} \mathbf{A}_{k+2} \mathbf{A}_{k+1} & \vdots & \alpha \mathbf{H}_{k+2} [\mathbf{A}_{k+2} \mathbf{B}_{k+1} + \alpha \mathbf{B}_{k+2}] + \alpha^2 \mathbf{G}_{k+2} \end{bmatrix} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{u}_k \end{bmatrix} \\ &\quad + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ (\mathbf{H}_{k+1} \mathbf{B}_{k+1} + \mathbf{G}_{k+1}) & \mathbf{0} \\ \mathbf{H}_{k+2} \mathbf{A}_{k+2} \mathbf{B}_{k+1} + \alpha (\mathbf{H}_{k+2} \mathbf{B}_{k+2} + \mathbf{G}_{k+2}) (\mathbf{H}_{k+2} \mathbf{B}_{k+2} + \mathbf{G}_{k+2}) & \mathbf{0} \end{bmatrix} \begin{bmatrix} \gamma_k \\ \gamma_{k+1} \end{bmatrix} \\ &\quad + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{k+1} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{k+2} \mathbf{A}_{k+2} & \mathbf{H}_{k+2} \end{bmatrix} \begin{bmatrix} \mathbf{w}_k \\ \mathbf{w}_{k+1} \\ \mathbf{w}_{k+2} \end{bmatrix} + \begin{bmatrix} \mathbf{v}_k \\ \mathbf{v}_{k+1} \\ \mathbf{v}_{k+2} \end{bmatrix} \end{aligned} \quad (5)$$

The underlying assumption is that  $zm > s$ , signifying a scenario where the number of measurements (real and virtual combined) available is greater than the dimensionality of the rejected inputs. To reject  $q_k$ , let us proceed as follows:

$$\mathbf{C}_{1,\mathbf{k}} = \begin{bmatrix} \mathbf{H}_k \\ \mathbf{H}_{k+1}\mathbf{A}_{k+1} \\ \mathbf{H}_{k+2}\mathbf{A}_{k+2}\mathbf{A}_{k+1} \end{bmatrix}, \quad \mathbf{C}_{2,\mathbf{k}} = \begin{bmatrix} \mathbf{G}_k \\ \alpha(\mathbf{H}_{k+1}\mathbf{B}_{k+1} + \mathbf{G}_{k+1}) \\ \alpha(\mathbf{H}_{k+2}[\mathbf{A}_{k+2}\mathbf{B}_{k+1} + \alpha\mathbf{B}_{k+2}] + \alpha\mathbf{G}_{k+2}) \end{bmatrix},$$

$$\mathbf{C}_{3,\mathbf{k}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ (\mathbf{H}_{k+1}\mathbf{B}_{k+1} + \mathbf{G}_{k+1}) & \mathbf{0} \\ \mathbf{H}_{k+2}\mathbf{A}_{k+2}\mathbf{B}_{k+1} + \alpha(\mathbf{H}_{k+2}\mathbf{B}_{k+2} + \mathbf{G}_{k+2}) & (\mathbf{H}_{k+2}\mathbf{B}_{k+2} + \mathbf{G}_{k+2}) \end{bmatrix},$$

$$\mathbf{C}_{4,\mathbf{k}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{k+1} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{k+2}\mathbf{A}_{k+2} & \mathbf{H}_{k+2} \end{bmatrix}.$$

The compact form for the measurement equation can be defined as,

$$\bar{y}_k = [\mathbf{C}_{1,\mathbf{k}} : \mathbf{C}_{2,\mathbf{k}}] \begin{bmatrix} \mathbf{x}_k \\ \mathbf{u}_k \end{bmatrix} + \mathbf{C}_{3,\mathbf{k}}\bar{\gamma}_k + \mathbf{C}_{4,\mathbf{k}}\bar{w}_k + \bar{\mathbf{v}}_k \quad (6)$$

Further, by combining Eq. (2) and Eq. (6), and subsequently reorganizing the equation, we arrive at the following expression:

$$\begin{bmatrix} \mathbf{x}_k \\ \mathbf{C}_{1,\mathbf{k}}\mathbf{x}_k \end{bmatrix} = \begin{bmatrix} \mathbf{A}_k\mathbf{x}_{k-1} + \mathbf{B}_k\mathbf{u}_k + [\mathbf{I} \mathbf{0} \mathbf{0}] \bar{w}_k \\ \bar{y}_k - \mathbf{C}_{2,\mathbf{k}}\mathbf{u}_k - \mathbf{C}_{3,\mathbf{k}}\bar{\gamma}_k - \mathbf{C}_{4,\mathbf{k}}\bar{w}_k - \bar{\mathbf{v}}_k \end{bmatrix} \quad (7)$$

The equation can further be restructured to represent a process equation for  $\mathbf{x}_k$  with  $\bar{\mathbf{y}}_k$  being an additional input injected into the process, as,

$$\begin{bmatrix} \mathbf{I} & -\mathbf{A}_k \\ \mathbf{C}_{1,\mathbf{k}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{x}_{k-1} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \bar{\mathbf{y}}_k + \begin{bmatrix} \mathbf{B}_k \\ -\mathbf{C}_{2,\mathbf{k}} \end{bmatrix} \mathbf{u}_k + \begin{bmatrix} \mathbf{0} \\ -\mathbf{C}_{3,\mathbf{k}} \end{bmatrix} \bar{\gamma}_k + \begin{bmatrix} \mathcal{I} \\ -\mathbf{C}_{4,\mathbf{k}} \end{bmatrix} \bar{w}_k + \begin{bmatrix} \mathbf{0} \\ -\mathbf{I} \end{bmatrix} \bar{\mathbf{v}}_k \quad (8)$$

With

$$\mathcal{B}_k = \begin{Bmatrix} \mathbf{B}_k \\ -\mathbf{C}_{2,\mathbf{k}} \end{Bmatrix} \in \mathbb{R}^{(n+zm) \times s}$$

Please note that the matrix  $\mathcal{B}_k$  has full column rank, guaranteeing unique solutions. Assuming the existence of  $\mathbf{P}_k \in \mathbb{R}^{(n+zm-s) \times (n+zm)}$  such that  $\mathbf{P}_k\mathcal{B}_k = \mathbf{0}$ , pre-multiplying both sides of the equation by  $\mathbf{P}_k$  removes the dependency of the process equation on  $\mathbf{u}_k$ , resulting in:

$$\mathbf{P}_k \begin{bmatrix} \mathbf{I} & -\mathbf{A}_k \\ \mathbf{C}_{1,\mathbf{k}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{x}_{k-1} \end{bmatrix} = \mathbf{P}_k \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \bar{\mathbf{y}}_k + \mathbf{P}_k \begin{bmatrix} \mathbf{0} \\ -\mathbf{C}_{3,\mathbf{k}} \end{bmatrix} \bar{\gamma}_k + \mathbf{P}_k \begin{bmatrix} \mathcal{I} \\ -\mathbf{C}_{4,\mathbf{k}} \end{bmatrix} \bar{w}_k + \mathbf{P}_k \begin{bmatrix} \mathbf{0} \\ -\mathbf{I} \end{bmatrix} \bar{\mathbf{v}}_k \quad (9)$$

Further assigning

$\mathcal{M}_k = \mathbf{P}_k \begin{bmatrix} \mathbf{I} & -\mathbf{A}_k \\ \mathbf{C}_{1,\mathbf{k}} & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{(n+zm-s) \times 2n}$ , and considering a  $QR$  decomposition of  $\mathcal{M}_k$  as  $M = \mathcal{Q}_k\mathcal{R}_k$ , with  $\mathcal{Q}_k \in \mathbb{R}^{(n+zm-s) \times (n+zm-s)}$  and  $\mathcal{R}_k \in$

$\mathbb{R}^{(n+zm-s) \times 2n}$  such that  $\mathcal{Q}_k^T \mathcal{Q}_k = I$  and  $\mathcal{R}_k$  is upper triangular, the following expression can be arrived at,

$$\mathcal{R}_k \begin{bmatrix} \mathbf{x}_k \\ \mathbf{x}_{k-1} \end{bmatrix} = \mathcal{Q}_k^T \mathbf{P}_k \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \bar{y}_k + \mathcal{Q}_k^T \mathbf{P}_k \begin{bmatrix} \mathbf{0} \\ -\mathbf{C}_{3,k} \end{bmatrix} \bar{\gamma}_k + \mathcal{Q}_k^T \mathbf{P}_k \begin{bmatrix} \mathcal{I} \\ -\mathbf{C}_{4,k} \end{bmatrix} \bar{w}_k + \mathcal{Q}_k^T \mathbf{P}_k \begin{bmatrix} \mathbf{0} \\ -\mathbf{I} \end{bmatrix} \bar{\mathbf{v}}_k \quad (10)$$

With,

$$G_k = \mathcal{Q}_k^T \mathbf{P}_k \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}; \Gamma_k = \mathcal{Q}_k^T \mathbf{P}_k \begin{bmatrix} \mathbf{0} \\ -\mathbf{C}_{3,k} \end{bmatrix}; \mathcal{W}_k = \mathcal{Q}_k^T \mathbf{P}_k \begin{bmatrix} \mathcal{I} \\ -\mathbf{C}_{4,k} \end{bmatrix}; \mathcal{V}_k = \mathcal{Q}_k^T \mathbf{P}_k \begin{bmatrix} \mathbf{0} \\ -\mathbf{I} \end{bmatrix}$$

The process equation can be compacted as,

$$\mathcal{R}_k \begin{bmatrix} \mathbf{x}_k \\ \mathbf{x}_{k-1} \end{bmatrix} = G_k \bar{y}_k + \Gamma_k \bar{\gamma}_k + \mathcal{W}_k \bar{w}_k + \mathcal{V}_k \bar{\mathbf{v}}_k \quad (11)$$

Segmenting the upper triangular matrix  $\mathcal{R}_k$  into 4 blocks as:

$$\mathcal{R}_k = \begin{bmatrix} R_k^{11} & R_k^{12} \\ \mathbf{0} & R_k^{22} \end{bmatrix} \quad (12)$$

where  $R_k^{11} \in \mathbb{R}^{n \times n}$ ,  $R_k^{12} \in \mathbb{R}^{n \times n}$ , and  $R_k^{22} \in \mathbb{R}^{(zm-s) \times n}$  and accordingly segmenting  $G_k$ ,  $\Gamma_k$ ,  $\mathcal{W}_k$ , and  $\mathcal{V}_k$  two blocks of dimension  $\mathbb{R}^{n \times zm}$  and  $\mathbb{R}^{(zm-s) \times zm}$ , the following expression is arrived at.

$$\begin{bmatrix} R_k^{11} & R_k^{12} \\ \mathbf{0} & R_k^{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{x}_{k-1} \end{bmatrix} = \begin{bmatrix} G_k^1 \\ G_k^2 \end{bmatrix} \bar{y}_k + \begin{bmatrix} \Gamma_k^1 \\ \Gamma_k^2 \end{bmatrix} \bar{\gamma}_k + \begin{bmatrix} \mathcal{W}_k^1 \\ \mathcal{W}_k^2 \end{bmatrix} \bar{w}_k + \begin{bmatrix} \mathcal{V}_k^1 \\ \mathcal{V}_k^2 \end{bmatrix} \bar{\mathbf{v}}_k \quad (13)$$

and further simplified to be integrated within the filtering framework by shifting the time index for the measurement equation as,

$$\begin{aligned} \mathbf{x}_k &= -[R_k^{11-1} \mathcal{R}_k^{12}] \mathbf{x}_{k-1} + [R_k^{11-1} \cdot G_k^1] \bar{y}_k + \tilde{\mathbf{w}}_k \quad \text{with } \tilde{\mathbf{w}}_k = \Gamma_k^1 \bar{\gamma}_k + \mathcal{W}_k^1 \bar{w}_k + \mathcal{V}_k^1 \bar{\mathbf{v}}_k \\ \bar{z}_k &= R_{k+1}^{22} \mathbf{x}_k + \tilde{\mathbf{v}}_k \quad \text{with } \tilde{\mathbf{v}}_k = \Gamma_k^2 \bar{\gamma}_k + \mathcal{W}_k^2 \bar{w}_k + \mathcal{V}_k^2 \bar{\mathbf{v}}_k \end{aligned} \quad (14)$$

wherein the system state  $\mathbf{x}_k$  is being observed with a transformed measurement as  $\bar{z}_k = G_{k+1}^2 \bar{y}_{k+1}$ . Finally, the filtering equations are derived as,

$$\begin{aligned} \mathbf{x}_k &= \mathcal{A}_k \mathbf{x}_{k-1} + \mathcal{B}_k \bar{y}_k + \tilde{\mathbf{w}}_k \\ \bar{z}_k &= \mathcal{H}_k \mathbf{x}_k + \tilde{\mathbf{v}}_k \end{aligned} \quad (15)$$

with  $\mathcal{A}_k = -[R_k^{11-1} \cdot \mathcal{R}_k^{12}]$ ,  $\mathcal{G}_k = [R_k^{11-1} \cdot G_k^1]$ , and  $\mathcal{H}_k = R_{k+1}^{22}$ .  $\tilde{\mathbf{w}}_k$  and  $\tilde{\mathbf{v}}_k$  represent the process and measurement uncertainty described by the covariances  $\tilde{\mathbf{Q}}_k$  and  $\tilde{\mathbf{R}}_k$  respectively. The covariances can be obtained from their respective expressions given in this derivation, which are not explicitly detailed here for the sake of brevity.

Subsequently, the interacting Particle Ensemble Kalman Filter (IPEnKF) framework [2] is utilized for state parameter estimation, leveraging sparse measurement data. To incorporate the system described in Equation 15, both the

state and measurement matrices are parameterized with additional health parameters  $\theta_k$  (or Health Indices, denoted as **HI**s), representing the system's location-wise health state. These assumed health states determine the physical matrices (i.e., **K**, **M**, and **C**), consequently influencing the system matrices defined in Equation 15. As a result, these **HI**s are estimated concurrently with the states  $\mathbf{x}_k$  within the interactive filtering framework. While  $\mathbf{x}_k$  is estimated using an Ensemble Kalman Filter (EnKF),  $\theta_k$  is estimated through an enveloping Particle Filter (PF).

The pseudo-code for the same is provided herewith (cf. pseudocode 1), integrated with the virtual sensor strategy.

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**Algorithm 1:** Pseudocode 1
 

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```

Input :  $\bar{y}_k, \bar{y}_{k+1}, Q, R$ 
Output: Output parameters
Procedure IPEnKF( $\bar{y}_k, \bar{y}_{k+1}, Q, R$ )
  Initialize particles  $\{\zeta_0^j\}$ , state estimates  $\{x_{0|0}^{i,j}\}$ 
  for each  $\bar{y}_k, \bar{y}_{k+1}$  do
    Procedure IP-EnKF( $\{\zeta_{k-1}^j\}, \{x_{k-1|k-1}^{i,j}\}$ )
      for each particle  $\zeta_k^j$  do
        evolve  $\zeta_k^j \leftarrow \zeta_{k-1}^j$ 
        Procedure EnKF( $\{\zeta_k^j\}, \{x_{k-1|k-1}^{i,j}\}, \bar{y}_k, \bar{y}_{k+1}$ )
          for each ensemble  $x_{k-1|k-1}^{i,j}$  do
            Prediction: Propagate state to  $x_{k|k-1}^{i,j}$ 
            Estimate transformed measurements,  $\bar{z}_{k+1|k}^{i,j}$ 
          end
          Calculate ensemble mean of  $(x_{k|k-1}^j)$  and  $\bar{z}_{k+1|k}^{i,j}$ 
          Evaluate overall innovation ( $\epsilon$ )
          Compute co-variances ( $C_k^{j,xz}$  and  $S_k^j$ ) and EnKF gain
          Correction: Correct predicted state estimate.
          Calculate ensemble mean of the corrected state, i.e.,  $x_{k|k}^j$ 
        end
      end
    Procedure Particle re-sampling( $\{\zeta_k^j\}$ )
      For each  $\zeta_k^j$ , calculate  $w(\{\zeta_k^j\})$ 
      Update: Update state,  $x_{k|k}$ , and parameter estimates  $\zeta_{k|k}$ 
    end
  end

```

---

In the following, the proposed methodology has been experimentally validated using a lumped mass representation of an LTV system. The numerical validation is detailed in the following.

### 3 Numerical examples

The virtual sensor-based state-parameter estimation approach is numerically presented using the analysis of a 6-DOF lumped mass system (cf. 1) subjected to ambient forcing represented with stationary white Gaussian noise (SWGN). Over a simulated duration of 30 seconds, acceleration data is generated and consistently sampled. To synthesize equivalent real-world conditions, the generated response is intentionally contaminated with 1% SWGN noise.

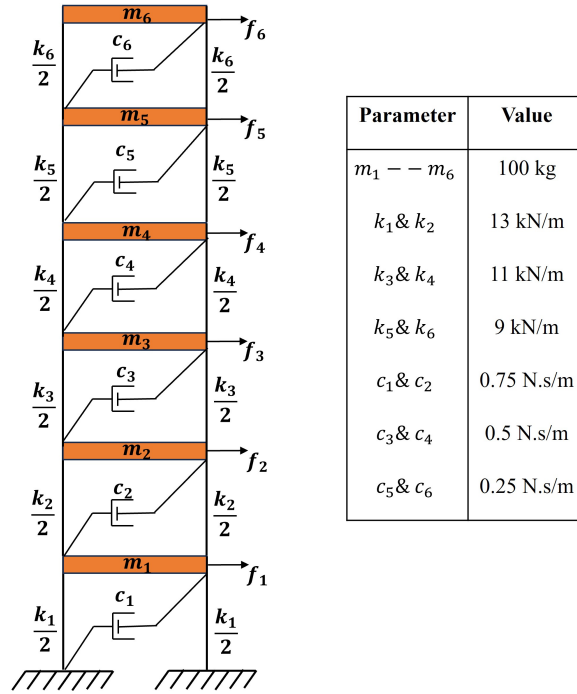


Fig. 1: Schematic diagram of 6-DOF lumped mass model

The sampling frequency is taken as 100 Hz which captures the system dynamics, without any aliasing. An SWGN forcing modeled with  $\alpha$  being 0.01,  $\mathbf{q}_0 = 0$ , and  $\gamma$  as an SWGN process of statistics  $\mathcal{N}(0 \text{ N}, 10^{-2} \text{ N})$ , is adopted for the simulation in this study. A 40% damage as characterized by a reduction in  $k_6$  is introduced in the numerical experiments, such that the stiffness at level 6 is  $(0.6k_6)$ .

The proposed approach facilitates state-parameter estimation circumventing the need for explicit or statistical information about the input forces (as indicated in Eq. (15)). The efficacy of the estimation is demonstrated by comparing the parameter estimates to their corresponding actual values. Given that the



primary objective of this study is to devise an approach suitable for large structural applications, the study focuses on estimating the segment-wise parameter estimation along the structural system.

Multiple sets of experiments based on varied sensor densities are conducted. The first set of experiments considered all nodes to be instrumented (case 1), which would ideally not require any additional virtual sensor layer. However, the constraint demands an extra layer of virtual sensors. As the number of physical sensors is lowered (3 sensors in Case 2 or 1 sensor in Case 3), the employment of virtual sensor layers becomes imperative for the parameter/health index estimation.

The number of additional layers for virtual sensors ( $k + 1, k + 2, \dots$ ) is chosen based on the existence of a non-empty  $\mathbf{G}_k^2$  matrix. This requirement arises from the formulation of the  $\mathbf{P}_k$  matrix, which is constructed from the left null space of  $\mathcal{E}_k$ . When the measurements are sparse compared to the dimension of the discarded input, the resultant  $\mathbf{P}_k$  matrix may result in an empty  $\mathbf{G}_k^2$ , leading to ineffective input rejection.

The performance of conventional approach with the employment of 6 sensors is shown in Fig. 2 below, and the deterioration and instability of the performance could be perceived herewith.

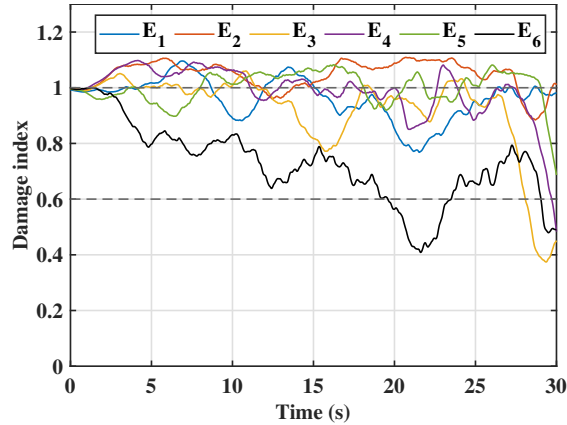


Fig. 2: Parameter estimates with 6-sensors using the conventional approach. (Dotted lines indicate actual parameter values.)

In such a scenario, the proposed approach that incorporates an additional layering with virtual sensors performs better as depicted in Fig. 3a, and Fig. 3b, revealing precise estimations of damage indices (Case 1).

In the second case study, only the 1<sup>st</sup>, 3<sup>rd</sup>, and 6<sup>th</sup> nodes are instrumented, inducing an artificial scarcity in the available data. Due to the constraint on the  $\mathbf{G}_k^2$  matrix, it is noted earlier that the conventional Kalman Filter (KF) based methods are no more effectively applicable, requiring the adoption of virtual

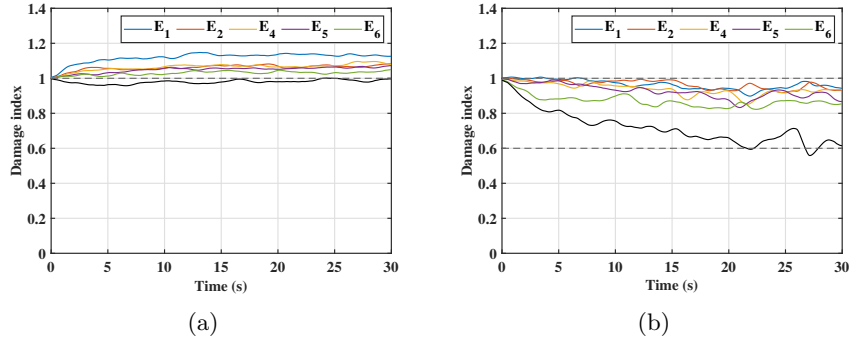


Fig. 3: Parameter estimates with 2-layer, 6-sensor measurement vector, (a) with no damage (b) with 60% damage in the 6<sup>th</sup> level (Dotted lines indicate actual parameter values.)

sensors. Further, the situation worsens with the reduced number of sensors. However, with the current approach, for improved efficiency of estimation, an additional set of two layers of virtual sensors ( $k + 1$  and  $k + 2$ ) are utilized, resulting in a delay of two time instances in estimation. The resulting parameter estimates are shown in Fig. 4a. Despite the inadequacy of observed degrees of freedom, there is only minimal degradation observed in the estimates. Further a similar case study was carried out with only the 1<sup>st</sup>, and 6<sup>th</sup> nodes instrumented, and the parameters are identified satisfactorily (cf.4b).

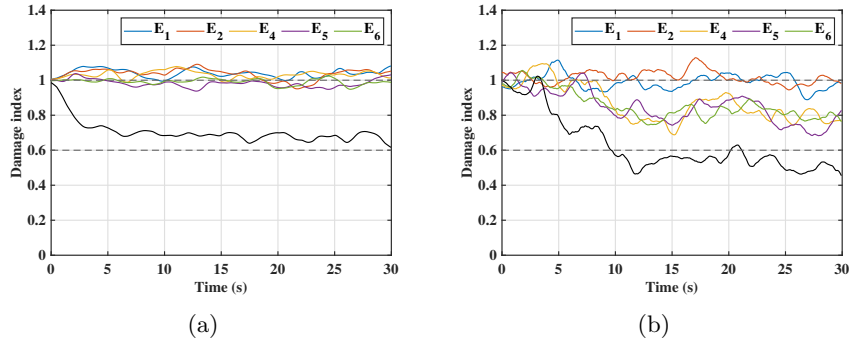


Fig. 4: Parameter estimates with z-layer, n-sensor measurement vector, with 60% damage in the 6<sup>th</sup> level (a)  $z=3, n=3$  (b)  $z=4, n=2$  (Dotted lines indicate actual parameter values.)

Due to limited page constraints, detailed analysis and investigation into the performance of the proposed approach with more complex real-life systems, under varying noise contamination levels and different sensor locations and respec-

tive stability analysis had to be omitted. These aspects will be comprehensively addressed in future publications.

## 4 Conclusion

The paper discusses the difficulties associated with monitoring complex structures using a limited number of sensors. To tackle these challenges, it introduces a novel estimation method that utilizes future measurements as responses obtained from virtual sensors, in addition to the current-time measurements from physical sensors. These virtual sensors are observed through a new mapping technique, providing a reliable solution to sparse instrumentation problems. The effectiveness of this method is demonstrated through experiments on linear time-varying systems, where it successfully estimates parameters within the interacting particle-ensemble Kalman filter framework. This success suggests the potential for further exploration of the approach in more complex linear time-variant systems with intricate computational models, including extensions to robust parameter estimation for structural condition monitoring, whether input information is available or not.

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