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Abstract

This article is mainly concerned with the time-discretisation of multivalued Hamiltonian systems with multivalued dissipation, a special class of differential inclusions. Two classes of set-valued Hamiltonian systems are considered, depending on the dissipation function being position or momentum dependent. The backward-Euler discretisation is analyzed in both cases: well-posedness of the generalized equation obtained after discretisation is proved, then stability of fixed-points is tackled. The well-known sliding-mode twisting and super-twisting algorithms, as well as an example from Contact Mechanics, illustrate the theoretical developments.

Key words:
Hamiltonian systems, differential inclusions, backward Euler time-discretisation, Lyapunov stability, twisting algorithm, super-twisting algorithm.

1 Introduction

The Hamiltonian formalism encompasses a large class of systems with diverse applications such as mechanics [3], circuit theory [29], thermodynamics [14], ecology [32], and economy [36]. In the classical formulation, a Hamiltonian system is characterized by two objects: a two-form on a smooth manifold (called a symplectic form) and a smooth function (called a Hamiltonian function) [3]. The main property of Hamiltonian systems is the simultaneous preservation of the symplectic form and the Hamiltonian function along the system trajectories [17]. For electrical or mechanical systems, the Hamiltonian function corresponds to the energy stored in the system, so preservation of the Hamiltonian amounts to energy conservation.

The Hamiltonian approach has been generalised in several directions. An important one in the control literature is the port-Hamiltonian formalism, where Hamiltonian systems are endowed with dissipative elements and external ports so that they can interact with the environment and, *e.g.*, a controller [28,?]. Port-Hamiltonian systems are not necessarily energy-conserving, but satisfy the more general property of passivity and are amenable to passivity-based control [39]. Another interesting generalisation consists in allowing systems to be described by differential inclusions in place of differential equations. There are several reasons for doing so, the main one being the possibility to treat nondifferentiable Hamiltonian functions [13]. This paper considers systems with possibly nonsmooth Hamiltonian functions and nonsmooth Rayleigh dissipation potentials (systems with additional external ports will be addressed in future contributions). Our main motivation is that this class of systems contains several higher-order sliding-mode controllers, such as the twisting and super-twisting algorithms [23], which have been difficult to classify in the literature. The primary interest in such controllers is the finite-time stability of the closed-loop system.

An important issue concerning dynamical systems is time discretisation. In the context of pure Hamiltonian systems, research has focused on the development of

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discrete-time maps that approximate the Hamiltonian flow while ensuring that the symplectic form and a perturbed Hamiltonian function are invariant along discrete orbits [17]. In the context of port-Hamiltonian systems, the usual objective is to preserve either passivity [37,20,16,2,31,27] or the closely related Dirac structure [38].

In the framework of this work, multivalued Hamiltonian systems result from closing the loop with multivalued controls. The objective is to derive well-posed discretisation schemes that preserve finite-time stability. The practical incentive is that the corresponding discrete-time control law is immediately revealed once an appropriate discrete-time model is obtained.

The following section contains the minimal background. The main contributions of this paper are distributed in Sections 3 and 4. These are:

- A unified framework for proving the finite-time stability of the twisting and super-twisting controllers. The proposed framework, multivalued Hamiltonian systems with dissipation, can be used to study other systems and creates the possibility for devising new sliding-mode algorithms (Section 3).
- A criterion for establishing finite-time stability in discrete time (Proposition 2 in Section 4).
- Sufficient conditions for the backward-Euler discretisation of multivalued Hamiltonian systems to inherit finite-time convergence (Theorem 3 in Subsection 4.1 and Theorem 5 in Subsection 4.3). The twisting and super-twisting algorithms satisfy such conditions.
- A splitting method, inspired by the Douglas-Rachford method in Optimisation, for approximating the proposed discrete-time maps (and hence the corresponding discrete-time controllers) of multivalued Hamiltonian systems with intricate Hamiltonian or dissipation functions (Theorem 4 in Subsection 4.2). The method is applied to the twisting algorithm and shown to preserve finite-time stability, even though it is an approximation.

Finally, Section 5 contains some conclusions.

2 Background

In this section we provide the necessary material to recall a useful invariance theorem for differential inclusions and level sets of nonsmooth functions [4]. The following preliminary results and definitions are taken from classical references [33,34,18,6]. We consider convex functions $V : \mathbb{R}^n \to \mathbb{R}$. Note that, in contrast with the abovementioned references, we exclude the value $+\infty$ from the range of the functions we consider. Thus, convex functions are automatically continuous and proper (indeed, their domains are equal to $\mathbb{R}^n$). The convex *subdifferential* of V at the point x is given by

$$\partial V(x) := \{\eta \in \mathbb{R}^n \mid \langle \eta, \xi - x\rangle \le V(\xi) - V(x), \text{ for all } \xi \in \mathbb{R}^n\}. \quad (1)$$

By [18, Remark 4.1.7], ∂V also has domain $\mathbb{R}^n$ and is locally bounded. The *graph* of a set-valued map $\mathbf{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is the set

$$\operatorname{gph} \mathbf{F} := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in \mathbf{F}(x)\}. \quad (2)$$

The *set of zeros* of $\mathbf{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is denoted by

$$\operatorname{Zero} \mathbf{F} := \mathbf{F}^{-1}(0) = \{\xi \in \mathbb{R}^n \mid 0 \in \mathbf{F}(\xi)\}. \quad (3)$$

Let $r \in [1, +\infty)$, the r-norm of a vector $x \in \mathbb{R}^n$ is denoted as

$$\|x\|_r = \left(\sum_{i=1}^n |x_i|^r\right)^{\frac{1}{r}}. \quad (4)$$

By convention, we set $\|x\|_\infty = \max_{i \in 1,\dots,n} |x_i|$. The *open ball*, with respect to the r-norm, centered at the origin and having radius 1, is denoted by

$$\mathcal{B}_r = \{\xi \in \mathbb{R}^n \mid \|\xi\|_r < 1\}. \quad (5)$$

To avoid cumbersome notation, $\|\cdot\|_2$ and $\mathcal{B}_2$ are denoted simply as $\|\cdot\|$ and $\mathcal{B}$, respectively. The closure of a set S is denoted $\operatorname{cl} S$.

$\mathbf{F}$ is said to be *outer semicontinuous* (osc) at $x^* \in \operatorname{dom} \mathbf{F}$ if, for any $y \notin \mathbf{F}(x^*)$, there are neighborhoods of x^* and y (denoted respectively as $\mathcal{N}(x^*)$ and $\mathcal{N}(y)$) such that $\mathcal{N}(x^*) \bigcap \mathbf{F}^{-1}(\mathcal{N}(y)) = \emptyset$. When $\mathbf{F}$ is closed-valued and locally bounded, outer semicontinuity at x^* is equivalent to the following (see *e.g.*, [11, Proposition 2.5.24]): For every $\varepsilon > 0$, there exists a neighborhood $\mathcal{N}(x^*)$ such that $x \in \mathcal{N}(x^*)$ implies that

$$\mathbf{F}(x) \subset \mathbf{F}(x^*) + \varepsilon \mathcal{B}. \quad (6)$$

A set-valued operator $\mathbf{F}$ is said to be *monotone* (respectively, *strongly monotone*) if, for any two pairs (x_i, y_i), $x_i \in \mathbb{R}^n$, $y_i \in \mathbf{F}(x_i)$, $i = 1, 2$,

$$\langle x_1 - x_2, y_1 - y_2\rangle \ge 0 \quad (7)$$

(respectively, if

$$\langle x_1 - x_2, y_1 - y_2\rangle \ge \gamma \|x_1 - x_2\|^2, \quad (8)$$

for some $\gamma > 0$). The operator $\mathbf{F}$ is *maximal monotone* if it is monotone and its graph is not strictly contained inside the graph of any other monotone operator. It is well known that, in finite dimensions, maximal monotone operators are osc [11, Proposition 4.2.1].

The *proximal map* associated with a convex function $V : \mathbb{R}^n \to \mathbb{R}$ is

$$\operatorname{Prox}_V(x) = \underset{w\in\mathbb{R}^n}{\arg\min} \left\{ V(w) + \frac{1}{2}\|w - x\|^2 \right\}. \tag{9}$$

Equivalently, $p_x = \operatorname{Prox}_V(x)$ if, and only if, $x \in (\mathrm{I_d} + \partial V)(p_x)$, where $\mathrm{I_d}$ denotes the identity map. In other words, $\operatorname{Prox}_V(x) = (\mathrm{I_d} + \partial V)^{-1}(x)$.

Consider a differential inclusion

$$\dot{x}(t) \in \mathbf{F}(x(t)), \quad x(0) = x_0. \tag{10}$$

We assume that the multivalued vector field $\mathbf{F}$ is such that, for each $x_0 \in \mathbb{R}^n$, there exists a unique absolutely continuous solution $x(\cdot)$. Recall that the *set-valued derivative* of a convex function $V : \mathbb{R}^n \to \mathbb{R}$ along $\mathbf{F}$ is defined as [4]

$$\begin{aligned} \mathcal{L}_{\mathbf{F}}V(x) := \{a \in \mathbb{R} \mid \exists\, \xi \in \mathbf{F}(x) \text{ such that} \\ \langle \eta, \xi \rangle = a, \text{ for all } \eta \in \partial V(x)\}. \end{aligned} \tag{11}$$

Lemma 1. [4] *Let $x(\cdot)$ be a solution of* (10) *and let $V : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then $\frac{\mathrm{d}}{\mathrm{d}t}V(x(t))$ exists almost everywhere and $\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) \in \mathcal{L}_{\mathbf{F}}V(x(t))$ almost everywhere.*

Theorem 1. [4] *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a convex positive-definite function. If, for all $x \in \mathbb{R}^n$, one has either $\max \mathcal{L}_{\mathbf{F}}V(x) \leq 0$ or $\mathcal{L}_{\mathbf{F}}V(x) = \emptyset$, then* (10) *is Lyapunov stable at $x = 0$.*

To simplify the exposition, we follow the convention $\max \emptyset = -\infty$. A set Ω is said to be *invariant* with respect to (10) if for every initial condition $x_0 \in \Omega$, the maximal solution $x(\cdot)$ of (10) lays in Ω.

Theorem 2. [4] *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a function satisfying the conditions of Theorem 1 and let us assume that, for some $l > 0$, the set $\{\xi \in \mathbb{R}^n \mid V(\xi) \leq l\}$ is bounded. Define the set*

$$\mathcal{Z}_{\mathbf{F}}V := \operatorname{Zero} \mathcal{L}_{\mathbf{F}}V, \tag{12}$$

let $x_0 \in L_l$, and let $\mathcal{M}$ be the largest invariant subset of $\mathcal{Z}_{\mathbf{F}}V \cap L_l$. Then,

$$\operatorname{Dist}(x(t), \mathcal{M}) \to 0 \text{ as } t \to \infty. \tag{13}$$

3 Multivalued Hamiltonian systems with dissipation

Let us consider a Hamiltonian system with generalized coordinate $q \in \mathbb{R}^n$ and generalized momentum $p \in \mathbb{R}^n$. For a given convex function $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, (q, p) \mapsto H(q, p)$, the subdifferentials of H with respect to the first and second arguments are denoted as $\partial_q H$ and $\partial_p H$, respectively. We consider differential inclusions of the form

$$\begin{aligned} \dot{q} &\in \partial_p H(q, p) - \partial_q g(q, p) \\ \dot{p} &\in -\partial_q H(q, p) - \partial_p g(q, p) \end{aligned}, \tag{14}$$

where $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is also convex. Let us define the *multivalued Hamiltonian vector field* $\mathbf{H} : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n \times \mathbb{R}^n$ as

$$\mathbf{H}(q, p) = \begin{pmatrix} \partial_p H(q, p) \\ -\partial_q H(q, p) \end{pmatrix}. \tag{15}$$

Notice that, if H is continuously differentiable and g is a constant function, the multivalued Hamiltonian vector field reduces to a classic one. In analogy with single-valued Hamiltonian vector fields, multivalued Hamiltonian vector fields preserve the Hamiltonian function H.

Proposition 1. *Consider a convex function $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. The function H remains constant along the integral curves of the vector field $\mathbf{H}$.*

Proof. The set-valued derivative of H along the trajectories of $\mathbf{H}$ is

$$\begin{aligned} \mathcal{L}_{\mathbf{H}}H(q, p) = \big\{a \in \mathbb{R} \mid \exists\, (\xi_1, \xi_2) \in \partial H(q, p), \\ \text{such that } \langle \eta_1, \xi_2 \rangle + \langle \eta_2, -\xi_1 \rangle = a, \\ \text{for all } (\eta_1, \eta_2) \in \partial H(q, p)\big\}. \end{aligned} \tag{16}$$

The condition $\langle \eta_1, \xi_2 \rangle + \langle \eta_2, -\xi_1 \rangle = a$ must hold, in particular, for $\eta_1 = \xi_1$ and $\eta_2 = \xi_2$, in which case we have

$$\langle \xi_1, \xi_2 \rangle + \langle \xi_2, -\xi_1 \rangle = a = 0, \tag{17}$$

so either $\mathcal{L}_{\mathbf{H}}H(q, p) = \emptyset$ or $\mathcal{L}_{\mathbf{H}}H(q, p) = \{0\}$. By Lemma 1, H remains constant along the integral curves of $\mathbf{H}$. ☐

Assumption 1. *The vector field $-\partial g$ in* (14) *is* dissipative. *That is,*

$$\max \mathcal{L}_{-\partial g}H(q, p) \leq 0 \tag{18}$$

for all $(q, p) \in \mathbb{R}^n \times \mathbb{R}^n$.

This assumption is common, *e.g.*, in classical mechanics. Indeed, in the Lagrangian and Hamiltonian frameworks, the dissipative forces are typically included in the form of *Rayleigh dissipation potential* functions. These are typically convex functions depending on the second argument only, *i.e.*. such that $\partial_q g = \{0\}$. The *twisting* sliding-mode algorithm [23] also falls in this category, as shown below.

The dissipativity of $-\partial g$ implies that $\max \mathcal{L}_{(\mathbf{H} - \partial g)}H \leq 0$, so H is a Lyapunov function if it is also positive definite. Asymptotic stability can then be established using the Invariance Principle formulated in Theorem 2.

Example 1 (Twisting Algorithm)**.** *Consider a differential inclusion of the form* (14)*, where* $n = 1$*,*

$$H(q,p) = \gamma_1|q| + \frac{1}{2}p^2, \quad g(q,p) = \gamma_2|p|, \tag{19}$$

and

$$\gamma_1 > \gamma_2 > 0. \tag{20}$$

These parameters give the inclusion

$$\begin{aligned} &\dot{q} = p \\ &\dot{p} \in -\gamma_1 \,\mathbf{sgn}(q) - \gamma_2 \,\mathbf{sgn}(p), \end{aligned} \tag{21}$$

which is the twisting algorithm (we have used $\partial|\cdot| = \mathbf{sgn}(\cdot)$*, the set-valued signum function). The Hamiltonian* H *is positive definite and its set-valued derivative* H *along* (21) *is*

$$\begin{aligned} \mathcal{L}_{(\mathbf{H}-\partial g)}H(q,p) = \big\{a \in \mathbb{R} \mid \exists\, \xi_2 \in -\gamma_1 \,\mathbf{sgn}(q) - \gamma_2 \,\mathbf{sgn}(p) \\ \textit{such that } \langle \eta_1, p\rangle + \langle p, \xi_2\rangle = a, \\ \textit{for all } \eta_1 \in \gamma_1 \,\mathbf{sgn}(q)\big\}. \end{aligned} \tag{22}$$

Note that $\mathcal{L}_{(\mathbf{H}-\partial g)}H(q,0) = 0$ *for any* $q \in \mathbb{R}$*, that* $\mathcal{L}_{(\mathbf{H}-\partial g)}H(q,p) = -\gamma_2|p|$ *for any* $p \in \mathbb{R}$ *if* $q \neq 0$*, and that* $\mathcal{L}_{(\mathbf{H}-\partial g)}H(q,p) = \emptyset$ *otherwise. This can be compactly written as*

$$\mathcal{L}_{(\mathbf{H}-\partial g)}H(q,p) = \begin{cases} \emptyset & \textit{if } q = 0 \textit{ and } p \neq 0 \\ -\gamma_2|p| & \textit{otherwise} \end{cases}, \tag{23}$$

from which Lyapunov stability can be readily concluded. To establish asymptotic stability, we begin by computing

$$\mathcal{Z}_{(\mathbf{H}-\partial g)}H = \mathrm{Zero}\,\mathcal{L}_{(\mathbf{H}-\partial g)}H = \mathbb{R} \times \{0\}. \tag{24}$$

Clearly, any element (q,p) *of an invariant subset of* $\mathbb{R} \times \{0\}$ *must also be contained in the set*

$$S = \big\{(q,p) \in \mathbb{R}^2 \mid 0 \in -\gamma_1 \,\mathbf{sgn}(q) - \gamma_2 \,\mathbf{sgn}(p)\big\}. \tag{25}$$

In other words, the set $\mathcal{M}$ *in Theorem 2 is a subset of* $(\mathbb{R} \times \{0\}) \cap S$*. It can be readily seen that*

$$(\mathbb{R}\times\{0\})\cap S = \big\{(q,0) \in \mathbb{R}^2 \mid 0 \in -\gamma_1 \,\mathbf{sgn}(q) + [-\gamma_2, \gamma_2]\big\}, \tag{26}$$

and that the condition (20) *ensures that* $(\mathbb{R} \times \{0\}) \cap S = \{(0,0)\}$*. This establishes the asymptotic stability of the origin. Finite-time convergence can be established by verifying that* (28) *is homogeneous with weights* $(2,1)$ *and negative degree* -1 *[5, Cor. 5.4].* △

Before presenting the following example, allow us to define the *signed bracket* as $\lceil x \rfloor^\gamma = |x|^\gamma \,\mathbf{sgn}(x)$, $\gamma \geq 0$, $x \in \mathbb{R}$.

Example 2 (Super-Twisting Algorithm)**.** *Consider a differential inclusion of the form* (14)*, where* $n = 1$*,*

$$H(q,p) = \gamma_2|q| + \frac{1}{2}p^2, \quad g(q,p) = \gamma_1 \frac{2}{3}|q|^{3/2}, \tag{27}$$

$\gamma_1 > 0$*, and* $\gamma_2 > 0$*. These parameters give the differential inclusion*

$$\begin{aligned} &\dot{q} = -\gamma_1 \lceil q \rfloor^{1/2} + p \\ &\dot{p} \in -\gamma_2 \,\mathbf{sgn}(q) \end{aligned}, \tag{28}$$

which corresponds to the super-twisting algorithm [25,22,24]. The only minimum of H *is the origin. Following the same reasoning as in Example 1, we can see that the set-valued derivative of* H *along* (28) *is*

$$\mathcal{L}_{(\mathbf{H}-\partial g)}H(q,p) = \begin{cases} \emptyset & \textit{if } q = 0 \textit{ and } p \neq 0 \\ -\gamma_1|q|^{1/2} & \textit{otherwise} \end{cases}, \tag{29}$$

from which Lyapunov stability follows. Asymptotic stability of the origin can be easily confirmed by noting that $\mathcal{Z}_{(\mathbf{H}-\partial g)}H = \{(0,0)\}$*. Finite-time convergence can be established by verifying that* (28) *is homogeneous with weights* $(2,1)$ *and negative degree* -1 *[5, Corollary 5.4].* △

Example 3 (First-order sliding-mode control with actuator dynamics)**.** *Consider a simple integrator*

$$\dot{q} = p \tag{30a}$$

with p *a control input. It is well-known that the control law* $p \in -\gamma \,\mathbf{sgn}(q)$ *renders the system finite-time stable. Suppose that the actuator has parasitic dynamics, so that the control law cannot be enforced exactly. Instead, we have*

$$\dot{p} \in -\frac{1}{\tau}\left(p + \gamma \,\mathbf{sgn}(q)\right), \quad 0 < \tau \ll 1. \tag{30b}$$

The differential inclusion (30) *is of the form* (14) *with* $H(q,p) = \frac{\gamma}{\tau}|q| + \frac{1}{2}p^2$ *and* $g(q,p) = \frac{1}{\tau}p^2$*. Similar computations as those performed in Example 1 show that the system is asymptotically stable. However, the system is not homogeneous, so finite-time stability cannot be claimed.* △

Example 4 (Mechanical system with Coulomb's friction)**.** *Let us consider a multibody system with* n *degrees of freedom and* m *contact points with 3-dimensional friction. It is assumed that the Lagrange dynamics read as [8, Section 5.3.2]*

$$\begin{aligned} M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \sum_{i=1}^{m} H_{\mathrm{t},i}(q)\lambda_{\mathrm{t},i} + \nabla h_{\mathrm{n}}(q)\lambda_{\mathrm{n}} \\ \lambda_{\mathrm{t},i} \in -\mu_i|\lambda_{\mathrm{n},i}|\,\partial_{\dot{q}}\|H_{\mathrm{t},i}^\top(q)\dot{q}\| \\ h_{\mathrm{n}}(q) = 0 \end{aligned}, \tag{31}$$

where $q \in C_{\text{conf}} \subseteq \mathbb{R}^n$, C_{conf} is the configuration space of the system when all m constraints are removed, $M(q) = M(q)^\top \succ 0$ is the inertia matrix, $C(q,\dot q)\dot q$ contains Coriolis and centripetal torques, $G(q)$ is the potential force that derives from a smooth potential $P(q)$ (gravity, elasticity), $\mu_i > 0$ are the friction coefficients, $\lambda_{\text{n},i}$ are the normal components of the contact forces at contact i, $H_{\text{t},i}(q)$ and $h_{\text{n}}(q) : C_{\text{conf}} \to \mathbb{R}^m$ are defined from the local kinematics at contact points, such that the relative tangential velocity at contact i satisfies $v_{\text{t},i} = H_{\text{t},i}^\top(q)\dot q$, and the relative normal velocity is $v_{\text{n},i} = \nabla h_{\text{n},i}^\top(q)\dot q$ [8, Chapters 4, 5]. Proceeding as usual with the generalized momentum $p = M(q)\dot q$, Equation (31) is rewritten in a Hamiltonian formalism:

$$\begin{aligned}
&\dot q = \frac{\partial H}{\partial p}(q,p)\\
&\dot p \in -\frac{\partial H}{\partial q}(q,p) - \\
&\quad \sum_{i=1}^{m} \mu_i |\lambda_{\text{n},i}|\ H_{\text{t},i}(q)\ \partial_{\dot q}\|H_{\text{t},i}^\top(q)M^{-1}(q)p\| + \nabla h_{\text{n}}(q)\lambda_{\text{n}}\\
&h_{\text{n}}(q) = 0
\end{aligned}, \tag{32}$$

where $H(q,p) = \frac{1}{2}p^\top M^{-1}(q)p + P(q)$. We may rewrite the dissipative contact force as

$$\begin{aligned}
\mu_i|\lambda_{\text{n},i}|\ M(q)M^{-1}(q)H_{\text{t},i}(q)\ \partial_{\dot q}\|H_{\text{t},i}^\top(q)M^{-1}(q)p\| \\
= \mu_i|\lambda_{\text{n},i}|M(q)\ \partial_p g_i(q,p),
\end{aligned} \tag{33}$$

with $g_i(q,\cdot) = \|\cdot\| \circ H_{\text{t},i}^\top(q)M^{-1}(q)$, i.e., *$g_i(q,p) = \|H_{\text{t},i}^\top(q)M^{-1}(q)p\|$. Notice that $\dot q^\top \nabla h_{\text{n}}(q)\lambda_{\text{n}} = 0$, while $\dot q^\top M(q)\partial_p g_i(q,p) = p^\top \partial_p g_i(q,p) \geq 0$. Also, it is well-known that bilaterally constrained systems such as (31) may possess solutions with discontinuous velocities [8, section 5.6.4], [15,21]. Therefore (32) does not fit (14) in general. However in some cases it has absolutely continuous solutions and it fits with the above dissipative multivalued Hamiltonian framework. This is for instance the case for a planar disc sliding on a ground with persistent contact, or n stacked blocks with horizontal motion as in Fig. 4. In both cases $M(q) = M$ and $H_{\text{t}}(q) = H_{\text{t}}$ are constant matrices, and $\lambda_{\text{n}} = \lambda_{\text{n}}(t)$ is known, hence $g_i(q,p) = g_i(p)$. Thus, in a reduced set of coordinates Equation (32) boils down to*

$$\begin{aligned}
\dot q &= \frac{\partial H}{\partial p}(q,p)\\
\dot p &\in -\frac{\partial H}{\partial q}(q,p) - \sum_{i=1}^{m} \mu_i|\lambda_{\text{n},i}(t)|M\partial g_i(p)
\end{aligned} \tag{34}$$

where the term $\nabla h_{\text{n}}(q)\lambda_{\text{n}}$ is no longer present, because it is irrelevant for the considered set of reduced coordinates. Notice that q and p in (34) are not the same as q and p in (32), however we keep the same notation (see the next examples).

For the homogeneous disc system with one frictional contact point we have $q = (x,\theta)^\top$ (the disc center has coordinates (x,y), its rotation angle is θ, and it is acted upon by a linear spring horizontally). The kinetic energy is $T(\dot x,\dot\theta) = \frac{1}{2}m\dot x^2 + \frac{1}{2}I\dot\theta^2$ with $I = \frac{mr^2}{2}$, while the potential energy is $P(x) = \frac{1}{2}k(x-l)^2$ with $l > 0$ the unloaded linear spring's length, and $h_{\text{n}}(y) = y - r = 0$ (hence the vertical coordinate y of the center of mass can be eliminated from the dynamics). Thus $p = (m\dot x, I\dot\theta)^\top$, $H(q,p) = \frac{1}{2m}p_1^2 + \frac{1}{2I}p_2^2 + \frac{1}{2}k(x-l)^2$, $g_i(q,p) = g_i(p) = |r\dot\theta + \dot x| = |r\frac{p_2}{I} + \frac{p_1}{m}|$, $i = 1,2$, $\lambda_{\text{n}} = m\mathbf{g}$, $H_{\text{t},i} = (1,r)^\top$. The disc's dynamics is in the set-valued Hamiltonian formalism

$$\begin{aligned}
\dot x &= \frac{1}{m}p_1\\
\dot\theta &= \frac{1}{I}p_2\\
\dot p_1 &\in -k(x-l) - \mu m^2\mathbf{g}\ \partial_{p_1}\left|\left(\frac{1}{m},\frac{r}{I}\right)p\right|\\
\dot p_2 &\in -\mu m\mathbf{g}I\ \partial_{p_2}\left|\left(\frac{1}{m},\frac{r}{I}\right)p\right|.
\end{aligned} \tag{35}$$

Let us now examine the dynamics of n stacked blocks with mass m_i, $i = 1,\dots,n$, moving horizontally. The generalized coordinates and momentum are $q = (q_1,\dots,q_n)^\top$ and $p = (p_1,\dots,p_n)^\top$, respectively, with $p_i = m_i\dot q_i$. Let $H(q,p) = \frac{1}{2}p^\top M^{-1}p + P(q)$, where the potential energy is $P(q) = \frac{1}{2}(q-L)^\top K(q-L)$, $L = (l_1,\dots,l_n)^\top$, $l_i > 0$ are the springs' unloaded linear lengths, $K = K^\top \succcurlyeq 0$ is the stiffness matrix, and $M = \mathrm{diag}(m_i)$. Coulomb friction is acting between each block and between the first block and the ground. Hence $\lambda_{\text{n},i} = \sum_{j=i}^n m_j\mathbf{g}$. Then:

$$\begin{aligned}
\dot q &= M^{-1}p\\
\dot p &= -K(q-L) + H_{\text{t}}\lambda_{\text{t}}\\
\lambda_{\text{t},i} &\in -\mu_i\left(\sum_{j=i}^{n} m_j\right)\mathbf{g}\ \mathbf{sgn}\left(\frac{p_i}{m_i} - \frac{p_{i-1}}{m_{i-1}}\right)
\end{aligned} \tag{36}$$

where $p_0 = 0$ (the ground velocity is null),

$$\begin{aligned}
H_{\text{t},1} &= (1,0,\dots,0)^\top\\
H_{\text{t},i} &= (0,\dots,0,-1,1,0,\dots,0)^\top\\
H_{\text{t},n} &= (0,\dots,0,-1,1)^\top,
\end{aligned} \tag{37}$$

$i = 2,\dots,n-1$ and $H_{\text{t}} = (H_{\text{t},1},\dots,H_{\text{t},n})$. Therefore,

$$\begin{aligned}
\dot p &\in -K(q-L)\\
&- \sum_{i=1}^{n} \mu_i\left(\sum_{j=i}^{n} m_j\right)\mathbf{g}MM^{-1}H_{\text{t},i}\partial\left|H_{\text{t},i}^\top M^{-1}p\right|.
\end{aligned} \tag{38}$$

That is,

$$\dot{p} \in -K(q-L) - \sum_{i=1}^{n} \mu_i \left(\sum_{j=i}^{n} m_j \right) \mathbf{g} M \partial g_i(p), \quad (39)$$

where $g_i(p) = (|\cdot| \circ H_{\mathrm{t},i}^\top M^{-1})(p) = |H_{\mathrm{t},i}^\top M^{-1} p|$. $\triangle$

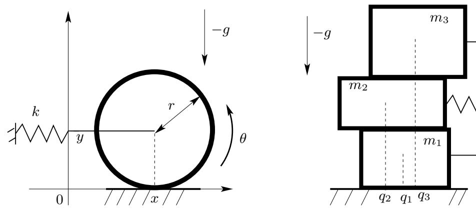

Fig. 1. Left: Sliding disc. Right: stacked blocks.

4 Time discretisation

A vast literature exists on the time discretisation of classical Hamiltonian systems, *i.e.*, single-valued and without dissipation, see, *e.g.*, [17] and references therein. It has been found that discrete-time models are well-posed and best retain the continuous-time properties when discretised using sophisticated combinations of backward–forward schemes [17]. In the multivalued scenario, however, well-posedness becomes critical, and many of such schemes are not implementable. In this section we show that, under reasonable assumptions, pure backward-Euler schemes for multivalued Hamiltonian systems are well-posed and inherit finite-time stability.

Motivated by Examples 1 and 2, we consider the time discretisation of (14) for two classes of dissipation potentials, one that depends on p only and is analogous to mechanical friction, and one that depends on q alone and has been disregarded in the literature. We will show that backward-Euler discretisation schemes preserve the dissipativity property of multivalued systems with dissipation, inasmuch as the Hamiltonian function is decreasing along the orbits of the discrete scheme. Thus, when positive definite, the Hamiltonian serves as a Lyapunov function for proving stability.

Finite-time convergence is proved using a simple argument that is particularly well suited for backward-Euler schemes.

Proposition 2. *Consider the discrete-time system* $x_{k+1} = T(x_k)$ *and let*

$$\operatorname{Fix} T := \{\xi \in \mathbb{R}^n \mid T(\xi) = \xi\} \quad (40)$$

be the set of fixed points *of* T. *Suppose that* $\operatorname{Fix} T$ *is a globally asymptotically stable set and suppose that there exist* $m \in \mathbb{N}$ *and a uniform neighborhood* U *of* $\operatorname{Fix} T$ *such that*

$$U \subseteq T^{-m}(\operatorname{Fix} T). \quad (41)$$

Then, $\operatorname{Fix} T$ *is globally finite-time stable.*

Proof. If $\operatorname{Fix} T$ is globally asymptotically stable, there exists a finite $k^* \in \mathbb{N}$ such that $x_k \in U$ for all $k \geq k^*$. Thus, the inclusion (41) implies that $\operatorname{Fix} T$ is reached after $k^* + m$ steps. $\square$

4.1 Momentum-dependent dissipation

Motivated by mechanical friction, we consider first the case in which dissipation depends on the momenta alone.

Assumption 2. *The functions* H *and* g *in* (14) *satisfy the following:*

(1) The Hamiltonian function is of the form $H(q,p) = P(q) + \frac{1}{2}\|p\|^2$ *for some convex function* $P : \mathbb{R}^n \to \mathbb{R}$.
(2) The function g *depends only on* p, *i.e.,* $g(q,p) = \tilde{g}(p)$ *for some convex function* $\tilde{g}$.

Lemma 2. *Consider a set-valued Hamiltonian system* (14) *satisfying Assumption 2. For each current state* (q_k, p_k) *there exists a unique future state*

$$(q_{k+1}, p_{k+1}) = T(q_k, p_k) \quad (42)$$

satisfying the backward-Euler discretisation scheme

$$q_{k+1} \in q_k + h\partial_p H(q_{k+1}, p_{k+1}) \quad (43a)$$
$$p_{k+1} \in p_k - h\partial_q H(q_{k+1}, p_{k+1}) - h\partial\tilde{g}(p_{k+1}). \quad (43b)$$

Proof. Taking into account Assumption 2, system (43) takes the particular form

$$q_{k+1} = q_k + hp_{k+1} \quad (44a)$$
$$p_{k+1} \in p_k - h\partial P(q_{k+1}) - h\partial\tilde{g}(p_{k+1}) \quad (44b)$$

The substitution of (44a) into (44b) yields

$$p_{k+1} \in p_k - h\partial\tilde{G}_k(p_{k+1}), \quad (45)$$

where $\tilde{G}_k$ is the step-dependent map $p \mapsto \frac{1}{h}P(q_k + hp) + \tilde{g}(p)$. Since $\tilde{G}_k$ is maximal monotone for each k, the future momentum p_{k+1} is uniquely determined by the expression

$$p_{k+1} = \operatorname{Prox}_{h\tilde{G}_k}(p_k). \quad (46)$$

The future position q_{k+1} is now uniquely determined via (44a), and the conclusion follows. $\square$

A usual strategy for showing the asymptotic stability of continuous-time dissipative Hamiltonian systems consists of three steps: i) Use H as a Lyapunov function to establish stability, ii) use Barbalat's lemma to show that the dissipation function $\mathcal{L}_{\mathbf{H}-\partial\tilde{g}}H$ converges to zero, and iii) show (or assume) that $\mathcal{L}_{\mathbf{H}-\partial\tilde{g}}H = 0$ implies that the state converges to zero. We will follow a similar approach in the discrete-time setting. The following lemma will be useful in accomplishing the discrete-time counterparts of steps i) and ii).

Lemma 3. *Consider a convex function $V : \mathbb{R}^n \to \mathbb{R}$ and a sequence $\{(v_k, w_k)\}_{k\in\mathbb{N}}$ such that $(v_k, w_k) \subset \operatorname{gph} \partial V$ for all $k \in \mathbb{N}$. If $\lim_{k\to\infty} v_k = 0$ then*

$$\lim_{k\to\infty} \operatorname{Dist}(\partial V(0), w_k) = 0. \tag{47}$$

Proof. Assume, for the sake of contradiction, that $\operatorname{Dist}(\partial V(0), w_k) \nrightarrow 0$ as $k \to \infty$. That is, that there exists $\varepsilon > 0$ such that, for any $k^* \in \mathbb{N}$, there is a $k > k^*$ such that $\operatorname{Dist}(\partial V(0), w_k) > \varepsilon$. In particular, $w_k \notin \partial V(0)$ for some $k > k^*$. Thus, as ∂V is outer semicontinuous at zero [11, Proposition 4.2.1-ii)], it follows that there exist neighborhoods of w_k and zero (denoted, respectively, as $\mathcal{N}(w_k)$ and $\mathcal{N}(0)$) such that

$$\mathcal{N}(0) \cap (\partial V)^{-1}(\mathcal{N}(w_k)) = \emptyset, \tag{48}$$

a contradiction, since $v_k \to 0$ as $k \to \infty$ and $v_k \in (\partial V^{-1})(\mathcal{N}(w_k))$. Therefore, (47) holds. ☐

The following assumption is used to establish the finite-time convergence to $(q, p) = (0, 0)$ for the scheme (44). It is a generalization of inequality (20).

Assumption 3. *The subdifferentials ∂P and $\partial\tilde{g}$ in Assumption 2 satisfy*

$$0 \in -\partial\tilde{g}(0) \subset \operatorname{int} \partial P(0), \tag{49}$$

where $\operatorname{int} \partial P(0)$ is the interior of $\partial P(0)$. In addition, for any pair $(p, \xi) \in \operatorname{gph} \partial\tilde{g}$ there exists $m \geq 1$ and $\alpha > 0$ such that

$$\langle p, \xi\rangle \geq \alpha\|p\|^m. \tag{50}$$

Since $0 \in \partial\tilde{g}(0)$, condition (50) holds under diverse situations. For instance, if $\tilde{g}$ is μ-strongly convex [18], then it holds with $m = 2$ and $\alpha = \mu$. Alternatively, if $\tilde{g}$ is convex and $0 \in \operatorname{int} \partial\tilde{g}(0)$, then there is $\alpha > 0$ such that (50) holds with $m = 1$. Indeed, if $0 \in \operatorname{int} \partial\tilde{g}(0)$, then there exists $\alpha > 0$, sufficiently small, such that $\alpha\mathcal{B} \in \partial\tilde{g}(0)$. Thus, it follows from the maximal monotonicity of $\partial\tilde{g}$ that, for any $(p, \xi) \in \operatorname{gph} \partial\tilde{g}$,

$$\langle p - 0, \xi - \alpha b\rangle \geq 0 \text{ for all } b \in \mathcal{B}. \tag{51}$$

Consequently,

$$\langle p, \xi\rangle \geq \alpha \sup_{b\in\mathcal{B}}\langle b, p\rangle = \alpha\|p\|. \tag{52}$$

Also functions of the form $\tilde{g}(p) = \|p\|_r^{1+\beta}$ where $\beta > 0$ satisfy condition (50). Notice that in this case $\tilde{g}$ is neither strongly monotone for all cases of $\beta > 0$ nor $0 \in \operatorname{int} \partial\tilde{g}(0)$, since $\tilde{g}$ is continuously differentiable. Nevertheless, the inequality (50) holds since, for any $(p, \xi) \in \operatorname{gph} \partial\tilde{g}$,

$$\begin{aligned}\langle p, \xi\rangle &= (1+\beta)\|p\|_r^\beta\langle p, \tilde{\xi}\rangle \\ &= (1+\beta)\|p\|_r^\beta(\|p\|_r + \Psi_{\mathcal{B}_s}(\xi)) \\ &= (1+\beta)\|p\|_r^{1+\beta} \\ &\geq \alpha\|p\|^{1+\beta},\end{aligned} \tag{53}$$

where we have used the fact that $\xi = (1+\beta)\|p\|_r^\beta\tilde{\xi}$, $\tilde{\xi} \in \partial\|p\|_r$ in the first equality and in the second equality the fact that, for any $(p, \xi) \in \operatorname{gph} \partial V$, $\langle p, \xi\rangle = V(p) + V^\star(\xi)$, see *e.g.*, [6, Proposition 16.9].

Note that, by maximal monotonicity, the condition $0 \in \operatorname{int} \partial P(0)$ implies that

$$\operatorname{Zero} \partial P = \{0\}. \tag{54}$$

Indeed, for the sake of contradiction, let us assume there is $v \neq 0$ such that $0 \in \partial P(v)$. Thus, using Assumption 3, there exists $\varepsilon > 0$ such that $\varepsilon b \in \partial P(0)$ for all $b \in \mathcal{B}$. Now, the maximal monotonicity of ∂P implies that

$$0 \leq \langle 0 - \varepsilon b, v - 0\rangle = -\varepsilon\langle b, v\rangle \text{ for all } b \in \mathcal{B}, \tag{55}$$

which leads to $v = 0$, a contradiction. Therefore, (54) holds. This in turn ensures that the origin is the only equilibrium and that P is positive definite.

Theorem 3. *Consider the backward-Euler discretisation (44) of an inclusion (14) satisfying Assumptions 2 and 3. Then, the origin is globally finite-time stable.*

Proof. Take H as a Lyapunov function candidate. By the definition of the convex subdifferential we have

$$\begin{aligned}H(q_{k+1}, p_{k+1}) - H(q_k, p_k) \leq \langle\eta_{1,k+1}, q_{k+1} - q_k\rangle \\ + \langle\eta_{2,k+1}, p_{k+1} - p_k\rangle\end{aligned} \tag{56}$$

for any $\eta_{1,k} \in \partial P(q_k)$ and $\eta_{2,k} = p_k$. It follows from the discrete dynamics (44) that

$$\begin{aligned}H(q_{k+1}, p_{k+1}) - H(q_k, p_k) \leq \langle\eta_{1,k+1}, hp_{k+1}\rangle \\ + \langle p_{k+1}, -h\xi_{1,k+1} - h\xi_{2,k+1}\rangle,\end{aligned} \tag{57}$$

where $\xi_{1,k} \in \partial P(q_k)$ and $\xi_{2,k} \in \partial\tilde{g}(p_k)$. Inequality (57) holds in particular for $\eta_{1,k} = \xi_{1,k}$, which implies that

$$H(q_{k+1}, p_{k+1}) - H(q_k, p_k) \leq -h\langle p_{k+1}, \xi_{2,k+1}\rangle. \tag{58}$$

Because of Assumption 3, we have

$$\langle p_k, \xi_{2,k}\rangle \geq \gamma\alpha\|p_k\|^m, \tag{59}$$

for all $k \in \mathbb{N}$ and for some $\gamma > 0$, $\alpha > 0$, $m \geq 1$, so that $H(q_{k+1}, p_{k+1}) \leq H(q_k, p_k)$. This establishes the stability of the origin. By summing both sides of (58) we obtain

$$\begin{aligned}\sum_{k=0}^{N}&(H(q_{k+1}, p_{k+1}) - H(q_k, p_k)) \\ &= H(q_{N+1}, p_{N+1}) - H(q_0, p_0) \\ &\leq -h \sum_{k=0}^{N} \langle p_{k+1}, \xi_{2,k+1} \rangle. \quad (60)\end{aligned}$$

Since the sequence $\{H(q_N, p_N)\}_{N \in \mathbb{N}}$ is decreasing and bounded from below, we conclude that it converges to a limit $\bar{H} \geq \min_{q,p} H(q, p)$. Thus,

$$h \sum_{k=0}^{\infty} \langle p_{k+1}, \xi_{2,k+1} \rangle \leq H(q_0, p_0) - \bar{H}. \quad (61)$$

The fact that $\langle p_k, \xi_{2,k} \rangle$ is non-negative implies that $\lim_{k\to\infty} \langle p_k, \xi_{2,k} \rangle = 0$, while the bound (59) ensures that $\lim_{k\to\infty} p_k = 0$. It follows from Lemma 3 that $\lim_{k\to\infty} \mathrm{Dist}(\partial \tilde{g}(0), \xi_{2,k}) = 0$ but, according to (44b), we have

$$\xi_{1,k} + \xi_{2,k} \to 0 \quad \text{as} \quad k \to \infty, \quad (62)$$

so $\lim_{k\to\infty} \mathrm{Dist}(\partial \tilde{g}(0), -\xi_{1,k}) = 0$ and, by (49), there exist $\varepsilon > 0$ and $k^* \in \mathbb{N}$ such that, for all $k \geq k^*$, $\xi_{1,k} + b_k \in \partial P(0)$ for any $b_k \in \varepsilon\mathcal{B}$. From $\xi_{1,k} \in \partial P(q_{k+1})$ and the monotonicity property of subdifferentials we know that

$$\langle b_k, q_k \rangle \geq 0, \quad (63)$$

for any $b_k \in \varepsilon\mathcal{B}$ and all $k \geq k^*$. Therefore, $q_k = 0$ for all $k \geq k^*$, and hence $p_{k+1} = 0$ for all $k \geq k^*$ in view of (44).

□

Example 5. *Examples 1 and 3 satisfy Assumptions 2 and 3. Consider Example 4. It satisfies Assumption 2 if $M = M^\top \succ 0$ is constant (at the price of changing the norm accordingly). The linear elasticity potential $P(q)$ considered both in (35) and (36) does not satisfy (49) (in both cases it is possible to change q to $q' := q - L$). It is however possible to shape the potential by introducing a set-valued controller of the form $u = -G\ \partial \sum_{j=1}^{n} |q_j|$, $G \in \mathbb{R}^{n \times n}$ a suitable control gain (thus the disc has one input for x and one for θ, while each mass is controlled in the system of stacked blocks). The potential $P(q)$ is changed to $P(q) + G \sum_{j=1}^{n} |q_j|$. Then (50) can be satisfied also. The obtained closed-loop dynamics is similar to a twisting scheme and may be seen as an extension of [1].* △

4.2 Douglas-Rachford-like Splitting Discretisation

Lemma 2 and Theorem 3 show that, when the dissipation function depends only on the momenta p, the discrete-time system (44) is well-posed and finite-time stable. However, an explicit expression for the proximal map (46) is, in general, hard to obtain. Nevertheless, approximations can be computed via splitting schemes as follows. Let us start by considering the following system:

$$q_{k+1} = q_k + h p_{k+1} \quad (64a)$$
$$p_{k+1} = p_k - h\xi_{1,k+1} - h\xi_{2,k+1} \quad (64b)$$
$$\xi_{1,k+1} \in \partial P\left(q_{k+1} + h^2(\xi_{2,k+1} - \xi_{2,k})\right) \quad (64c)$$
$$\xi_{2,k+1} \in \partial \tilde{g}(p_{k+1}). \quad (64d)$$

It can be seen as a dynamic extension of (44). Indeed, the particular value of the $\xi_{2,k+1} \in \partial \tilde{g}(p_{k+1})$ now depends implicitly on $\xi_{2,k}$, effectively making ξ_2 a new state variable. By the outer semicontinuity of ∂P we know that, for any $\varepsilon > 0$, there exists $h > 0$ sufficiently small such that

$$\partial P(q_{k+1} + h^2(\xi_{2,k+1} - \xi_{2,k})) \subset \partial P(q_{k+1}) + \varepsilon\mathcal{B}. \quad (65)$$

If $\partial \tilde{g}$ is uniformly bounded, then for all $k \in \mathbb{N}$, $\|\xi_{2,k}\| \leq M$ for some $M > 0$. Thus, $q_{k+1} + h^2(\xi_{2,k+1} - \xi_{2,k}) \in \{q_{k+1}\} + 2h^2 M \cdot \mathcal{B} = \{q_{k+1}\} + \mathcal{O}(h^2) \cdot \mathcal{B}$. In this sense, the scheme (64) approximates (44). We now address its well-posedness.

Lemma 4. *Consider a difference equation (64) satisfying Assumption 2. For each current state $(q_k, p_k, \xi_{2,k})$ there exists a unique future state*

$$(q_{k+1}, p_{k+1}, \xi_{2,k+1}) = T(q_k, p_k, \xi_{2,k}) \quad (66)$$

Furthermore, the selections $\xi_{1,k+1}$, $\xi_{2,k+1}$ are determined as

$$\xi_{1,k+1} = \frac{1}{h^2}\left(\mathrm{I_d} - \mathrm{Prox}_{h^2 P}\right)\left(q_k + h p_k - h^2 \xi_{2,k}\right) \quad (67a)$$
$$\xi_{2,k+1} = \frac{1}{h}\left(\mathrm{I_d} - \mathrm{Prox}_{h\tilde{g}}\right)\left(p_k - h\xi_{1,k+1}\right). \quad (67b)$$

Proof. Define $\theta_{k+1} := q_{k+1} + h^2(\xi_{2,k+1} - \xi_{2,k})$. It follows from (64a)-(64c) that

$$\begin{aligned}\theta_{k+1} + h^2\xi_{1,k+1} &= q_k + h p_{k+1} + h^2(\xi_{2,k+1} - \xi_{2,k}) + h^2\xi_{1,k+1} \\ &= q_k + h p_k - h^2 \xi_{2,k} \quad (68)\end{aligned}$$

Since $\xi_{1,k+1} \in \partial P(\theta_{k+1})$, we have

$$\theta_{k+1} = \mathrm{Prox}_{h^2 P}\left(q_k + h p_k - h^2 \xi_{2,k}\right). \quad (69)$$

Substituting (69) into (68) and solving for $\xi_{1,k+1}$ gives (67a), while (64b) gives

$$p_{k+1} + h\xi_{2,k+1} = p_k - h\xi_{1,k+1}, \quad (70)$$

Hence, (64d) implies that

$$p_{k+1} = \mathrm{Prox}_{h\tilde{g}}(p_k - h\xi_{1,k+1}). \quad (71)$$

Substituting (71) into (70) and solving for $\xi_{2,k+1}$ gives (67b). Thus, p_{k+1}, and hence q_{k+1}, depend on data available at time t_k so that (64) is well-defined. This concludes the proof. $\square$

Lemma 4 shows the explicit expression for the selection values $\xi_{1,k+1}$ and $\xi_{2,k+1}$. Notice that $\xi_{2,k+1}$ depends on its past values, so that the full state of the proposed approximation (64) lies in $\mathbb{R}^{3n}$.

Remark 1. *As introduced in [30] for the case of multivariable super-twisting-like schemes, the approximation of the selection values in* (67) *is reminiscent of the celebrated Douglas-Rachford splitting [26,?]. Indeed, set*

$$\vartheta_{k+1} = p_{k+1} + h\xi_{2,k+1} \tag{72}$$

and note that it follows from (64d) *that* $p_{k+1} = \operatorname{Prox}_{h\tilde{g}}(\vartheta_{k+1})$, *so that after simple computations we arrive at the expressions*

$$\begin{aligned} \vartheta_{k+1} &= \frac{1}{h}\Big(\operatorname{Prox}_{h^2 P}\Big(q_k + h(2\operatorname{Prox}_{h\tilde{g}} - \mathrm{I_d})(\vartheta_k)\Big) - q_k\Big) \\ &\quad + (\mathrm{I_d} - \operatorname{Prox}_{h\tilde{g}})(\vartheta_k) && \text{(73a)} \\ p_{k+1} &= \operatorname{Prox}_{h\tilde{g}}(\vartheta_{k+1}), && \text{(73b)} \end{aligned}$$

which define the Douglas-Rachford splitting associated with the generalized equation

$$0 \in h\partial P(q + hp) + h\partial\tilde{g}(p), \tag{74}$$

see e.g., [35, Section 2.7].

In what follows the stability properties of the set

$$\Gamma := \left\{ \begin{bmatrix} q \\ p \\ \xi_2 \end{bmatrix} \in \mathbb{R}^{3n} \;\middle|\; q = p = 0,\ \xi_2 \in \mathbb{R}^n \right\} \tag{75}$$

are studied.

Theorem 4. *Consider a discrete-time system* (64) *satisfying Assumption 2. Strengthen Assumption 3 to*

$$\mu\mathcal{B} \subset -\partial\tilde{g}(0) \subset \operatorname{int}\partial P(0) \tag{76}$$

for some $\mu > 0$. *In addition, assume that* $\partial\tilde{g}$ *is uniformly bounded. Then, there exists* $h^* > 0$, *sufficiently small, such that for all* $0 < h < h^*$ *the set* Γ *in* (75) *is globally asymptotically stable.*

Proof. Let us consider the function H introduced in Assumption 2. Using again the definition of convex subdifferential, we can readily show that

$$\begin{aligned} H(q_{k+1}, p_{k+1}) - H(q_k, p_k) &\le \langle \eta_{1,k+1}, q_{k+1} - q_k\rangle \\ &\quad + \langle \eta_{2,k+1}, p_{k+1} - p_k\rangle \end{aligned} \tag{77}$$

for any $\eta_{1,k} \in \partial P(q_k)$ and $\eta_{2,k} = p_k$. It follows from (64c), (65), and the outer semicontinuity of ∂P that, for any $0 < \varepsilon_1 < \mu$, there exists $h > 0$ sufficiently small such that

$$\xi_{1,k+1} \in \partial P(q_{k+1} + h^2(\xi_{2,k+1} - \xi_{2,k})) \subset \partial P(q_{k+1}) + \varepsilon_1\mathcal{B}. \tag{78}$$

It is inferred that, for any $k \in \mathbb{N}$, there always exists $b_k \in \mathcal{B}$ such that

$$\xi_{1,k} + \varepsilon_1 b_k \in \partial P(q_k). \tag{79}$$

Hence, taking $\eta_{1,k} = \xi_{1,k} + \varepsilon_1 b_k$ and $\eta_{2,k} = p_k$ in (77) leads us to

$$\begin{aligned} H(q_{k+1}, p_{k+1}) - H(q_k, p_k) &\le \langle \xi_{1,k+1} + \varepsilon_1 b_{k+1}, q_{k+1} - q_k\rangle \\ &\quad + \langle p_{k+1}, p_{k+1} - p_k\rangle \\ &= h\langle \xi_{1,k+1} + \varepsilon_1 b_{k+1}, p_{k+1}\rangle - h\langle p_{k+1}, \xi_{1,k+1} + \xi_{2,k+1}\rangle \\ &\le h\varepsilon_1\|p_{k+1}\| - h\langle p_{k+1}, \xi_{2,k+1}\rangle. \end{aligned} \tag{80}$$

It follows from (76) that

$$\langle p_k, \xi_{2,k}\rangle > \mu\|p_k\|. \tag{81}$$

To see this, note that $-\mu b_k \in \partial\tilde{g}(0)$ for any $b_k \in \mathcal{B}$ and that, by the maximal monotonicity of $\partial\tilde{g}$, we have

$$\langle p_k, \xi_{2,k} - \mu b_k\rangle \ge 0. \tag{82}$$

In other words, $\langle p_k, \xi_{2,k}\rangle \ge \mu\langle p_k, b_k\rangle$ for all $b_k \in \mathcal{B}$. Therefore,

$$\langle p_k, \xi_{2,k}\rangle \ge \mu \sup_{b_k\in\mathcal{B}} \langle p_k, b_k\rangle, \tag{83}$$

which implies (81). Hence, (80) can be further bounded as

$$H(q_{k+1}, p_{k+1}) - H(q_k, p_k) \le -h(\mu - \varepsilon_1)\|p_{k+1}\| \tag{84}$$

Since by assumption $0 < \varepsilon_1 < \mu$, it follows that $h(\mu - \varepsilon_1)\|p_k\| \ge 0$ for all $k \in \mathbb{N}$ and stability of the set Γ is concluded. To show asymptotic convergence, we apply the same arguments as in the proof of Theorem 3, leading us to

$$\lim_{k\to\infty} p_k = 0 \quad \text{and} \quad \lim_{k\to\infty} \operatorname{Dist}(\partial\tilde{g}(0), -\xi_{1,k}) = 0. \tag{85}$$

By following the same steps as in the proof of Theorem 3, we can conclude from (64c), (76), and (85) that there exists $k^* \in \mathbb{N}$ such that for all $k > k^*$,

$$q_{k+1} + h^2(\xi_{2,k+1} - \xi_{2,k}) = 0. \tag{86}$$

Now, the substitution of (64a) into (86) yields

$$\liminf_{k\to\infty}(q_k + h^2\xi_{2,k+1}) = \liminf_{k\to\infty}(-hp_{k+1} + h^2\xi_{2,k}) \tag{87}$$

Notice that both $\liminf$ in (87) are well-defined, as their arguments are bounded. Moreover, it follows from (85) and (64a) that $\lim_{k\to\infty} q_k$ exists, as $\{q_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence. Thus, (87) becomes

$$\begin{aligned}\lim_{k\to\infty} q_k + h^2 \liminf_{k\to\infty} \xi_{2,k+1} &= \lim_{k\to\infty} -hp_{k+1} + h^2 \liminf_{k\to\infty} \xi_{2,k} \\ &= h^2 \liminf_{k\to\infty} \xi_{2,k} \qquad (88)\end{aligned}$$

and the convergence of $\{q_k\}_{k\in\mathbb{N}}$ towards zero follows. The proof is complete. $\square$

Let us illustrate the above with the twisting algorithm. In addition it will be proved for this particular (yet important) case, that the finite-time stability is obtained.

Example 6 (Twisting Algorithm)**.** *The twisting algorithm* (21) *has a dissipation function that depends only on the momenta* p*. Indeed, it satisfies Assumption 2 with* $P(q) = \gamma_1|q|$ *and* $\tilde{g}(p) = \gamma_2|p|$*. Thus, its discretised version*

$$\begin{aligned} q_{k+1} &= q_k + hp_{k+1} && (89a)\\ p_{k+1} &= p_k - h\xi_{1,k+1} - h\xi_{2,k+1} && (89b)\\ \xi_{1,k+1} &\in \gamma_1\,\mathbf{sgn}(q_{k+1}) && (89c)\\ \xi_{2,k+1} &\in \gamma_2\,\mathbf{sgn}(p_{k+1}) && (89d)\end{aligned}$$

is globally finite-time stable whenever $\gamma_1 > \gamma_2 > 0$*, in view of Theorem 3. However, an explicit expression for the proximal map* (46) *(and for the selection values* $\xi_{1,k}$ *and* $\xi_{2,k}$*) is, in general, hard to obtain. Nevertheless, the approximation proposed in* (64) *can be computed. Indeed, simple computations show that*

$$\begin{aligned}\operatorname{Prox}_{h^2P}(s) &= s - h^2\gamma_1 \operatorname{Proj}\left([-1,1]; \frac{s}{h^2\gamma_1}\right), && (90a)\\ \operatorname{Prox}_{h\tilde{g}}(s) &= s - h\gamma_2 \operatorname{Proj}\left([-1,1]; \frac{s}{h\gamma_2}\right), && (90b)\end{aligned}$$

so that (64) *becomes*

$$\begin{aligned} q_{k+1} &= q_k + hp_{k+1} && (91a)\\ p_{k+1} &= p_k - h\xi_{1,k+1} - h\xi_{2,k+1} && (91b)\\ \xi_{1,k+1} &= \gamma_1 \operatorname{Proj}\left([-1,1]; \frac{q_k + hp_k - h^2\xi_{2,k}}{h^2\gamma_1}\right) && (91c)\\ \xi_{2,k+1} &= \gamma_2 \operatorname{Proj}\left([-1,1]; \frac{p_k - h\xi_{1,k+1}}{h\gamma_2}\right) && (91d)\end{aligned}$$

Like the origin of (89)*, the origin* (91) *is also finite-time stable. Indeed, from* (64) *simple computations show that* $\Gamma = \operatorname{Fix} T$*, so that* $T^{-1}(\Gamma)$ *is computed from the implicit*

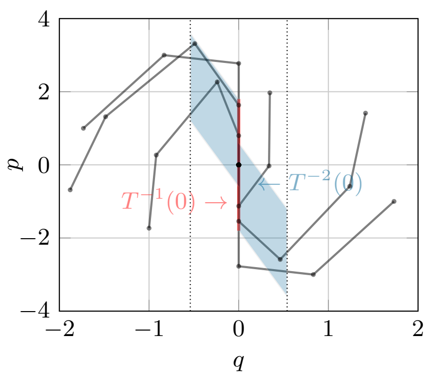

Fig. 2. Discrete orbits of backward-Euler implementation of the twisting algorithm (44) with $P(\cdot) = \gamma_1|\cdot|$, $\tilde{g} = \gamma_2|\cdot|$, $\gamma_1 = 4$, $\gamma_2 = 2$, and $h = 0.3s$. The red line denotes the set $T^{-1}(0)$, whereas the light blue region denotes the set $T^{-2}(0)$.

model (64) *as*

$$\begin{aligned} T^{-1}(\Gamma) &= \left\{ \begin{bmatrix} q\\ p\\ \xi_2\end{bmatrix} \in \mathbb{R}^3 \;\middle|\; q = 0,\ |p - h\xi_1| \le h\gamma_2, \right.\\ &\qquad\qquad \left. |\xi_1| \le \gamma_1 \right\}\\ &= \{0\} \times [-h(\gamma_1+\gamma_2), h(\gamma_1+\gamma_2)] \times \mathbb{R}, \quad (92)\end{aligned}$$

whereas the second pre-image of Γ *is given as*

$$\begin{aligned} T^{-2}(\Gamma) &= \left\{ \begin{bmatrix} q\\ p\\ \xi_2\end{bmatrix} \in \mathbb{R}^3 \;\middle|\; 0 = q + h\bar{p}, \right.\\ &\qquad \bar{p} \in p - h\xi_1 - h\gamma_2\,\mathbf{sgn}(\bar{p}),\ |\bar{p}| \le h(\gamma_1+\gamma_2),\\ &\qquad \left. \xi_1 = \gamma_1 \operatorname{Proj}\left([-1,1]; \frac{q + hp - h^2\xi_2}{h^2\gamma_1}\right) \right\}\\ &= \left\{ \begin{bmatrix} q\\ p\\ \xi_2\end{bmatrix} \in \mathbb{R}^3 \;\middle|\; q = -h \operatorname{Prox}_{h\tilde{g}}(p - h\xi_1), \right.\\ &\qquad |q| \le h^2(\gamma_1+\gamma_2),\\ &\qquad \left. \xi_1 = \gamma_1 \operatorname{Proj}\left([-1,1]; \frac{q + hp - h^2\xi_2}{h^2\gamma_1}\right) \right\} \end{aligned} \qquad (93)$$

Hence, Γ *is finite-time stable in view of Proposition 2. Notice that the above algorithm is different from the discrete-time twisting studied in [19], which is studied with a ZOH discretisation of the plant, hence yielding a closed-loop different from (89).* $\triangle$

Example 7 (Multivariable twisting algorithm)**.** *Con-*

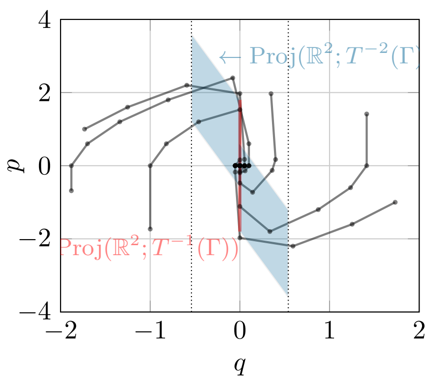

Fig. 3. Discrete orbits of the twisting algorithm with splitting (64) with $P(\cdot) = \gamma_1|\cdot|$, $\tilde{g} = \gamma_2|\cdot|$, $\gamma_1 = 4$, $\gamma_2 = 2$, and $h = 0.3s$. The red line denotes the projection onto the plane $q-p$ of the set $T^{-1}(\Gamma)$, whereas the light blue region denotes the projection of the set $T^{-2}(\Gamma)$.

sider the multivariable twisting scheme

$$q_{k+1} = q_k + hp_{k+1} \tag{94a}$$
$$p_{k+1} = p_k - h\xi_{1,k+1} - h\xi_{2,k+1} \tag{94b}$$
$$\xi_{1,k+1} \in \partial P(q_{k+1}) \tag{94c}$$
$$\xi_{2,k+1} \in \partial \tilde{g}(p_{k+1}), \tag{94d}$$

where $q_k \in \mathbb{R}^n$, $p_k \in \mathbb{R}^n$ *for all* $k \in \mathbb{N}$, $P(\cdot) = \gamma_1\|\cdot\|_{r_1}$, *and* $\tilde{g}(\cdot) = \gamma_2\|\cdot\|_{r_2}$. *It is easy to verify that, for the case of* $n = 1$, (94) *reduces to the classical twisting algorithm. The finite-time stability of* (94) *is guaranteed by Theorem 3 whenever Assumption 2 and 3 hold. Clearly, Assumption 2 holds for the choice of maps* P *and* $\tilde{g}$ *considered above, whereas for the specific case of* (94), (49) *becomes*

$$0 \in \gamma_2 \operatorname{cl} \mathcal{B}_{s_2} \subset \gamma_1 \mathcal{B}_{s_1}, \tag{95}$$

where $1/r_i + 1/s_i = 1$, *for* $i \in \{1, 2\}$. *Indeed, let* $f(x) = \|x\|_r$ *for* $r \in [1, +\infty]$. *Hence,*

$$\begin{aligned}
\partial f(0) &= \{\xi \in \mathbb{R}^n \mid \langle \xi, \eta\rangle \le \|\eta\|_r \text{ for all } \eta \in \mathbb{R}^n\} \\
&= \left\{\xi \in \mathbb{R}^n \mid \sup_{\eta\in\mathbb{R}^n} \{\langle \xi, \eta\rangle - \|\eta\|_r\} \le 0\right\} \\
&= \{\xi \in \mathbb{R}^n \mid f^\star(\xi) \le 0\} \\
&= \operatorname{cl} \mathcal{B}_s
\end{aligned} \tag{96}$$

The last equality follows from the fact that, for $f(\cdot) = \|\cdot\|_r$, $f^\star(\cdot) = \Psi_{\operatorname{cl}\mathcal{B}_s}(\cdot)$, *where* $1/r + 1/s = 1$ *and* Ψ_C *denotes the classical indicator function of the set* C. *If, for instance,* $r_1 = r_2$, *then* $s_1 = s_2$ *and* (95) *holds whenever* $\gamma_1 > \gamma_2$, *which is the same condition as* (20) *for the finite-time stability of the conventional twisting algorithm. Thus, if* (95) *holds, then the origin is finite-time stable in view of Theorem 3. Figure 4 depicts the time evolution of the state trajectories of* (94) *for the case of* $h = 0.1$, $n = 3$, $r_1 = 1$, $r_2 = 2$, $\gamma_1 = 5$ *and* $\gamma_2 = 3$. *It is shown that all trajectories converge to zero in finite time.*

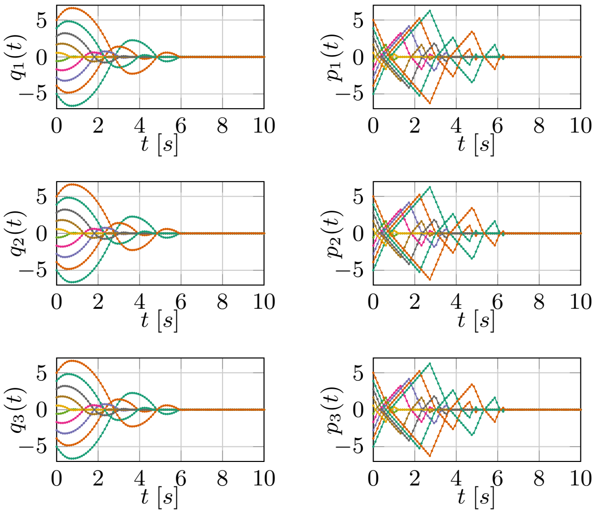

Fig. 4. Time evolution of state trajectories of (44) for several initial conditions. In this case $h = 0.1$, $n = 3$, $P(\cdot) = \gamma_1\|\cdot\|_1$, $\tilde{g}(\cdot) = \|\cdot\|_2$, $\gamma_1 = 5$ and $\gamma_2 = 3$. For this simulation the state p_{k+1} is computed via (46) using the solver CVXOPT at each iteration.

Now, let us consider the splitting scheme (64) *applied to the multivariable twisting algorithm. In this case Lemma 4 provides the explicit expressions for the control selections* $\xi_{1,k+1}$ *and* $\xi_{2,k+1}$, *which for the case of* $P(\cdot) = \gamma_1\|\cdot\|_{r_1}$ *and* $\tilde{g}(\cdot) = \gamma_2\|\cdot\|_{r_2}$, *they become*

$$\xi_{1,k+1} = \gamma_1 \operatorname{Proj}\left(\mathcal{B}_{s_1}; \frac{1}{h^2\gamma_1}(q_k + hp_k - h^2\xi_{2,k})\right) \tag{97a}$$

$$\xi_{2,k+1} = \gamma_2 \operatorname{Proj}\left(\mathcal{B}_{s_2}; \frac{1}{h\gamma_2}(p_k - h\xi_{1,k+1})\right) \tag{97b}$$

where we have used Moreau's decomposition, see e.g., [6, Theorem 14.3] and the fact that $f^\star(\cdot) = \Psi_{\mathcal{B}_s}(\cdot)$ *for* $f = \|\cdot\|_r$ *with* $1/r + 1/s = 1$. *Figure 5 shows the time evolution of the state trajectories with the same set of parameters as for the previous case. It is shown that all trajectories converge to zero in finite time.* △

4.3 Position-dependent dissipation

As mentioned before, the model (14) with $\partial_q g = \{0\}$ encompasses many mechanical systems with set-valued frictional forces. Inspired by the super-twisting example, we propose a different, somewhat complementary set of assumptions.

Assumption 4. *The functions* H *and* g *in* (14) *satisfy the following:*

(1) The Hamiltonian function is of the form $H(q, p) = P(q) + \frac{1}{2}\|p\|^2$ *for some convex function* $P : \mathbb{R}^n \to \mathbb{R}$.
(2) The subdifferential of the dissipation potential g *is single valued and* g *depends on* q *only,* i.e., $g(q, p) = \hat{g}(q)$ *for some convex function* $\hat{g}$.

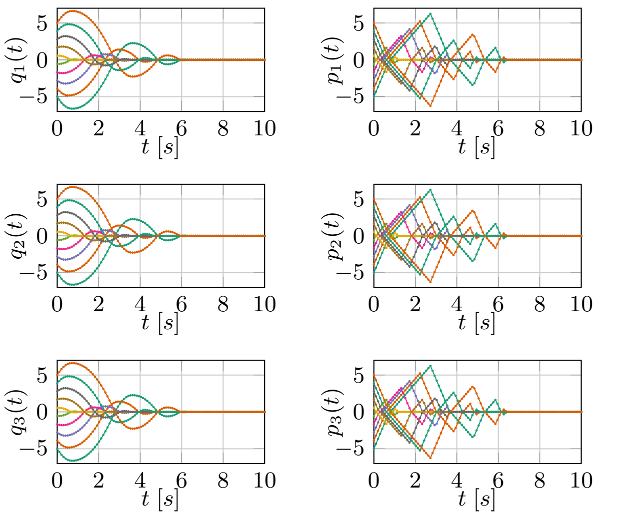

Fig. 5. Time evolution of state trajectories of (64) for several initial conditions. In this case $h = 0.1$, $n = 3$, $P(\cdot) = \gamma_1\|\cdot\|_1$, $\tilde{g}(\cdot) = \gamma_2\|\cdot\|_2$, $\gamma_1 = 5$ and $\gamma_2 = 3$. The explicit expressions for the values of selections $\xi_{1,k+1}$ and $\xi_{2,k+1}$ are given in (97).

Lemma 5. *Consider the differential inclusion* (14) *and let H and g satisfy Assumption 4. For each current state (q_k, p_k), there exists a unique future state (q_{k+1}, p_{k+1}) satisfying the backward-Euler discretisation scheme*

$$q_{k+1} \in q_k + h\partial_p H(q_{k+1}, p_{k+1}) - h\partial\hat{g}(q_{k+1}) \tag{98a}$$
$$p_{k+1} \in p_k - h\partial_q H(q_{k+1}, p_{k+1}). \tag{98b}$$

Proof. By taking into account the assumptions and substituting (98b) into (98a), we can rewrite (98) as

$$q_{k+1} \in q_k + hp_k - h\partial\hat{g}(q_{k+1}) - h^2\xi_{2,k+1} \tag{99a}$$
$$p_{k+1} = p_k - h\xi_{2,k+1} \tag{99b}$$

with $\xi_{2,k} \in \partial P(q_k)$ the potential force. Equation (99a) is equivalent to the inclusion

$$q_{k+1} \in q_k + hp_k - h\partial G(q_{k+1}), \tag{100}$$

where G is a new function defined by $G(q) = \hat{g}(q) + hP(q)$. By the maximal monotony of G, we can uniquely compute q_{k+1} as

$$q_{k+1} = \mathrm{Prox}_{hG}(q_k + hp_k). \tag{101}$$

Having q_{k+1} fixed, we can uniquely determine the potential force $\xi_{2,k+1} \in \partial P(q_{k+1})$ from (99a)

$$\xi_{2,k+1} = \frac{p_k - \partial\hat{g}(q_{k+1})}{h} - \frac{q_{k+1} - q_k}{h^2}. \tag{102}$$

Finally, we can uniquely compute p_{k+1} from (99b). $\square$

Theorem 5. *Consider the backward-Euler discretisation* (98) *of an inclusion* (14) *satisfying Assumptions 1 and 4. Suppose, further, that*

$$\mathrm{Zero}\,\partial P = 0 \tag{103}$$

and

$$\langle \eta, \partial\hat{g}(q)\rangle \geq \|q\|^m \tag{104}$$

for some $m > 0$ and all $\eta \in \partial P(q)$. Then, the origin is globally asymptotically stable.

Condition (103) implies that H has a unique minimum and that it takes place at $(0, 0)$. The latter is without loss of generality up to a simple change of coordinates. The condition (104) ensures that the dissipation is zero if and only if $q = 0$.

Proof. By the definition of the subdifferential, we have

$$H(q_{k+1}, p_{k+1}) - H(q_k, p_k) \leq \langle \eta_{1,k+1}, q_{k+1} - q_k\rangle + \langle \eta_{2,k+1}, p_{k+1} - p_k\rangle \tag{105}$$

for all $\eta_{1,k} \in \partial P(q_k)$ and $\eta_{2,k} = p_k$. We have $q_{k+1} - q_k = hp_{k+1} - h\partial\hat{g}(q_{k+1})$ and it follows from Lemma 5 that, for every $k \in \mathbb{N}$, there exists a unique vector $\xi_{1,k} \in \partial P(q_k)$ such that $p_{k+1} - p_k = h\xi_{1,k+1}$. Thus,

$$H(q_{k+1}, p_{k+1}) - H(q_k, p_k) \leq \langle \eta_{1,k+1}, hp_{k+1} - h\partial\hat{g}(q_{k+1})\rangle + \langle p_{k+1}, h\xi_{1,k+1}\rangle. \tag{106}$$

This is true, in particular, for $\eta_{1,k} = \xi_{1,k}$, which implies that

$$H(q_{k+1}, p_{k+1}) - H(q_k, p_k) \leq -h\langle \xi_{1,k+1}, \partial\hat{g}(q_{k+1})\rangle. \tag{107}$$

By (104), we have

$$H(q_{k+1}, p_{k+1}) - H(q_k, p_k) \leq 0, \tag{108}$$

which shows that H is a Lyapunov function for the set of its minima.

By summing both sides of (107) we obtain

$$\sum_{k=0}^{N}(H(q_{k+1}, p_{k+1}) - H(q_k, p_k)) = H(q_{N+1}, p_{N+1}) - H(q_0, p_0) \leq -h\sum_{k=0}^{N}\langle \xi_{1,k+1}, \partial\hat{g}(q_{k+1})\rangle, \tag{109}$$

Since the sequence $\{H(q_N, p_N)\}_{N\in\mathbb{N}}$ is decreasing and bounded from below, we conclude that it converges to a limit $\bar{H} \geq \min_{q,p} H(q,p)$. Thus,

$$h \sum_{k=0}^{\infty} \langle \xi_{1,k+1}, \partial \hat{g}(q_{k+1}) \rangle = H(q_0, p_0) - \bar{H}, \tag{110}$$

which implies that $\lim_{k\to\infty} \langle \xi_{1,k}, \hat{g}(q_k) \rangle = 0$. It follows from (104) that

$$\lim_{k\to\infty} q_k = 0. \tag{111}$$

Finally, we can see from (98a) that (111) implies that

$$\lim_{k\to\infty} p_k = 0. \tag{112}$$

□

Remark 2. *The inequality in (109) is a dissipation inequality showing that the system with input $\xi_{2k} = \partial \hat{g}(q_k)$, output $-\xi_{1,k}$, is passive with respect to the supply rate $\langle -h\xi_{1,k}, \xi_{2,k} \rangle$ and storage function $H(q_{k+1}, p_{k+1})$ [9, section 3.15].*

Example 8 (Super-Twisting Algorithm)**.** *The super-twisting algorithm* (28) *satisfies Assumption 4 with*

$$P(q) = \gamma_2 |q| \quad \text{and} \quad \hat{g}(q) = \gamma_1 \frac{2}{3} |q|^{3/2}. \tag{113}$$

Indeed, the potential function P has a unique minimum at $q = 0$, and we have $\partial \hat{g}(0) = 0$ and

$$\partial \hat{g}(q) \cdot \partial P(q) = \gamma_1 \gamma_2 |q|^{1/2}, \tag{114}$$

that is, it satisfies (104)*. The latter also implies Assumption 1. Thus, the iteration satisfying*

$$\begin{aligned} q_{k+1} &= q_k + h p_{k+1} - h\gamma_1 \lceil q_{k+1} \rfloor^{1/2} && \text{(115a)} \\ p_{k+1} &\in p_k - h\gamma_2 \, \mathbf{sgn}(q_{k+1}) && \text{(115b)} \end{aligned}$$

renders the origin globally asymptotically stable. The explicit expression associated with (115a) *was obtained in [10] as*

$$\begin{aligned} q_{k+1} &= q_k + h p_k - h\gamma_1 \xi_{1,k+1} - h^2 \gamma_2 \xi_{2,k+1} && \text{(116a)} \\ p_{k+1} &= p_k - h\gamma_2 \xi_{2,k+1}, && \text{(116b)} \end{aligned}$$

where

$$\xi_{1,k+1} = \tilde{\beta}(q_k + h p_k) \operatorname{Proj}\left([-1, 1]; \frac{q_k + h p_k}{h^2 \gamma_2} \right) = \lceil q_{k+1} \rfloor^{\frac{1}{2}} \tag{117a}$$

$$\xi_{2,k+1} = \operatorname{Proj}\left([-1, 1]; \frac{q_k + h p_k}{h^2 \gamma_2} \right) \in \mathbf{sgn}(q_{k+1}) \tag{117b}$$

and $\beta : \mathbb{R} \to \mathbb{R}_+$ is

$$\tilde{\beta}(s) = -\frac{h\gamma_1}{2} + \frac{1}{2}\sqrt{h^2 \gamma_1^2 + 4 \max\{0, |s| - h^2 \gamma_2\}}. \tag{117c}$$

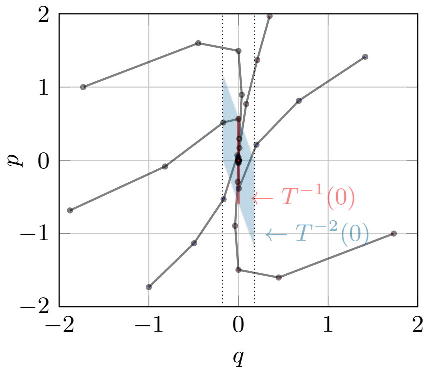

Fig. 6. Discrete orbits of the backward-Euler implementation of the super-twisting algorithm (115) with $\gamma_1 = 4$, $\gamma_2 = 2$, and $h = 0.3$. The pre-image $T^{-1}(0)$ (red) does not contain a neighborhood of the origin, however, $T^{-2}(0)$ (light blue) does. Thus, discrete-time orbits (black) reach the origin in finite time.

To establish finite-time convergence, we will begin by computing $T^{-1}(0,0)$. This is easily accomplished directly in the implicit form of T. Setting $q_{k+1} = 0$ and $p_{k+1} = 0$ in (115) *gives*

$$T^{-1}(0,0) = 0 \times [-h\gamma_2, h\gamma_2]. \tag{118}$$

The set does not contain a neighborhood of the origin. We proceed to compute $T^{-1}(0 \times [-h\gamma_2, h\gamma_2])$. Setting $q_{k+1} = 0$ in (115) *yields the conditions*

$$q_k = -h p_{k+1} \quad \text{and} \quad p_k \in p_{k+1} + [-h\gamma_2, h\gamma_2] \tag{119}$$

with $p_{k+1} \in [-h\gamma_2, h\gamma_2]$. Thus, the second pre-image of the origin is

$$T^{-2}(0,0) = \left\{ (q,p) \in \mathbb{R}^2 \mid |q + hp| \leq h^2 \gamma_2,\ |q| \leq h^2 \gamma_2 \right\}. \tag{120}$$

△

5 Conclusions

Gradient and Hamiltonian systems play important and somewhat complementary roles in Dynamical Systems Theory [7]. Hamiltonian systems with Rayleigh dissipation incorporate both classes of systems, as the dissipative vector field effectively behaves as a gradient vector field. First-order sliding-mode control systems can be framed into the generalised class of multivalued gradient systems, also called subgradient systems [12]. Unfortunately, higher-order sliding-mode control systems cannot. However, we have showed that at least two second-order sliding-mode control systems can be recast as multivalued Hamiltonian systems with dissipation, providing further insight into higher-order sliding motions.

It is well-known that the properties of continuous-time subgradient systems are best preserved under backward-Euler discretisation schemes [12]. We have shown here that the same holds for multivalued Hamiltonian systems with dissipation, as finite-time stability is also well-preserved. Finally, splitting schemes that have been traditionally applied to subgradient systems can be effectively adapted to the multivalued Hamiltonian framework.

Future research could focus on extensions towards port-Hamiltonian systems and Hamiltonian systems with disturbances and uncertainties.

References

[1] J. Alvarez, I. Orlov, and L. Acho. An invariance principle for discontinuous dynamic systems with application to a Coulomb friction oscillator. *Journal of Dynamic Systems, Measurement, and Control*, 122(4):687–690, 02 2000.

[2] S. Aoues, M. Di Loreto, D. Eberard, and W. Marquis-Favre. Hamiltonian systems discrete-time approximation: Losslessness, passivity and composability. *Systems & Control Letters*, 110:9–14, 2017.

[3] V.I. Arnold. *Mathematical Methods of Classical Mechanics*. Springer-Verlag, New York, 1989.

[4] A. Bacciotti and F. Ceragioli. Stability and stabilization of discontinuous systems and nonsmooth Lyapunov functions. *ESAIM: COCV*, 4:361 – 376, 1999.

[5] A. Bacciotti and L. Rosier. *Liapunov Functions and Stability in Control Theory*. Springer-Verlag, The Netherlands, 2005.

[6] H. H. Bauschke and P. L. Combettes. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*. CMS Books in Mathematics. Springer, 2011.

[7] H.W. Broer and F. Takens. Chapter 1 - preliminaries of dynamical systems theory. volume 3 of *Handbook of Dynamical Systems*, pages 1–42. Elsevier Science, 2010.

[8] B. Brogliato. *Nonsmooth Mechanics. Models, Dynamics, and Control*. Communications and Control Eng. Springer Int. Pub. Switzerland, 3rd edition, 2016.

[9] B. Brogliato, B. Maschke, R. Lozano, and O. Egeland. *Dissipative Systems Analysis and Control: Theory and Applications*. Communications and Control Eng. Springer, London, 3rd edition, 2020.

[10] B. Brogliato, A. Polyakov, and D. Efimov. The implicit discretization of the supertwisting sliding-mode control algorithm. *IEEE Transactions on Automatic Control*, 65(8):3707–3713, August 2020.

[11] R. S. Burachik and A. N. Iusem. *Set-valued Mappings and Enlargements of Monotone Operators*. Springer, 2008.

[12] F. Castaños, F. Miranda Villatoro, and B. Brogliato. When proximal-
point algorithms meet set-valued systems. an optimization point of view of discrete-time sliding modes. Research Report hal-04362282, INRIA, Grenoble, France, 2023. `https://dumas.ccsd.cnrs.fr/LJK-MAD-TRIPOP/hal-04362282v1`.

[13] F.H. Clarke. Periodic solutions to Hamiltonian inclusions. *Journal of Differential Equations*, 40(1):1–6, 1981.

[14] D. Eberard, B. Maschke, and A.J. van der Schaft. An extension of Hamiltonian systems to the thermodynamic phase space: Towards a geometry of nonreversible processes. *Rep. on Mathematical Physics*, 60:175 – 198, October 2007.

[15] M. Frémond. *Collisions*. Facolta di Ingegneria di Roma Tor Vergata, Roma, Italy, 2007.

[16] L. Gören-Sümer and Y. Yalçin. Gradient based discrete-time modeling and control of Hamiltonian systems. *IFAC Proceedings Volumes*, 41(2):212–217, 2008. 17th IFAC World Congress.

[17] E. Hairer, C. Lubich, and G. Wanner. *Geometric Numerical Integration: Structure-preserving algorithms for ordinary differential equations*. Springer, second edition, 2006.

[18] J.B. Hiriart-Urruty and C. Lemaréchal. *Fundamentals of Convex Analysis*. Grundlehren Text Editions. Springer-Verlag Berlin Heidelberg, 2001.

[19] O. Huber, V. Acary, and B. Brogliato. Lyapunov stability analysis of the implicit discrete-time twisting control algorithm. *IEEE Transactions on Automatic Control*, 65(6):2619–2626, June 2020.

[20] D. S. Laila and A. Astolfi. Construction of discrete-time models for port-controlled Hamiltonian systems with applications. *Systems & Control Letters*, 55(8):673–680, 2006. New Trends in Nonlinear Control.

[21] X.A. Le. *Dynamics of Mechanical Systems with Coulomb Friction*. Foundations of Engineering Mechanics. Springer, Berlin, 2003.

[22] A. Levant. Higher-order sliding modes, differentiation and output-feedback control. *Int. Journal of Control*, 76:924–941, 1993.

[23] A. Levant. Sliding order and sliding accuracy in sliding mode control. *Int. J. Control*, 58:1247–1263, 1993.

[24] A. Levant. Higher-order sliding modes, differentiation and output-feedback control. *International Journal of Control*, 76(9-10):924–941, 2003.

[25] L.V. Levantovsky. Second order sliding algorithms. Their realization. *Dynamics of Heterogeneous Systems*, pages 32–43, 1985. Moscow: Institute for System Studies (in Russian).

[26] P. L. Lions and B. Mercier. Splitting algorithms for the sum of two nonlinear operators. *SIAM Journal of Numerical Analysis*, 16(6):964–979, 1979.

[27] A. Macchelli. Control design for a class of discrete-time port-Hamiltonian systems. *IEEE Transactions on Automatic Control*, 68(12):8224–8231, 2023.

[28] B. Maschke and A.J. van der Schaft. Port-controlled Hamiltonian systems:modelling origins and system-theoretic properties. In *Proc. IFAC Symposium on Nonlinear Control Systems*, pages 359 – 365, Bordeaux, France, June 1992.

[29] B. Maschke, A.J. van der Schaft, and P.C. Breedveld. An intrinsic Hamiltonian formulation of the dynamics of LC-circuits. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 42:73–82, feb 1995.

[30] F. Miranda-Villatoro. A variational approach to the design of discrete-time super-twisting-like algorithms. `https://inria.hal.science/hal-04511213v1`, March 2024.

[31] S. Monaco, D. Normand-Cyrot, M. Mattioni, and A. Moreschini. Nonlinear Hamiltonian systems under sampling. *IEEE Transactions on Automatic Control*, 67(9):4598–4613, 2022.

[32] Y. Nutku. Hamiltonian structure of the Lotka-Volterra equations. *Physics Letters A*, 145(1):27–28, 1990.

[33] R.T. Rockafellar. *Convex Analysis*. Princeton University Press, 1970.

[34] R.T. Rockafellar and J.B. Wets. *Variational Analysis*, volume 317 of *Grundlehren der mathematischen Wissenschaften*. Springer Verlag, 3rd printing edition, 1998.

[35] E.K. Ryu and W. Yin. *Large-Scale Convex Optimization: Algorithms & Analyses via Monotone Operators*. Cambridge University Press, 2022.

[36] S.P. Sethi. *Optimal Control Theory: Applications to Management Science and Economics.* Springer New York, NY.

[37] S. Stramigioli, C. Secchi, A.J. van der Schaft, and C. Fantuzzi. Sampled data systems passivity and discrete port-hamiltonian systems. *IEEE Transactions on Robotics*, 21(4):574–587, 2005.

[38] V. Talasila, J. Clemente-Gallardo, and A.J. van der Schaft. Discrete port-hamiltonian systems. *Systems & Control Letters*, 55(6):478–486, 2006.

[39] A.J. van der Schaft. *$\mathcal{L}_2$-Gain and Passivity Techniques in Nonlinear Control*. Springer-Verlag, London, 2017.