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Existence and Uniqueness of Strong Solutions to a Bi-dimensional Fluid-Structure Interaction System

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Abstract

We study an unsteady non-linear fluid-structure interaction problem where a two-dimensional viscous incompressible Newtonian fluid and a mono-dimensionnal elastic structure, located on one part of the fluid domain boundary, interacts. The fluid motion is modelled by the bi-dimensional incompressible Navier-Stokes equations set in an unknown domain which depends on the structure's displacement. We consider longitudinal as well as transversal structure displacement. We assume that the longitudinal displacement of the structure satisfies a wave equation whereas the transversal displacement follows a beam equation with inertia of rotation. The fluid and structure systems are coupled through a kinematic condition which corresponds to a no-slip condition at the fluid-structure interface and the fluid exerts a force on the elastic structure. We prove the existence and uniqueness of strong solution to the considered problem with no gap between the initial conditions regularity and the ones obtained in positive time. To our knowledge, this is the first result regarding existence and uniqueness of strong solutions for fluid-beam interaction problem in the unsteady case taking into account both the transversal displacement and the longitudinal structure displacement.

Keywords. Fluid-structure interaction, Navier-Stokes system, beam equations, strong solutions.

1 Introduction

In this paper, we study a fluid-structure interaction problem involving a two-dimensional viscous incompressible Newtonian homogeneous fluid and a moving deformable structure located on one part of the fluid domain boundary. The model that we consider herein can be viewed as a toy model to describe blood flow in arteries [11, 36]. We suppose that the fluid is occupying a 2D container whose reference configuration is $\bar{\mathcal{F}}_0 = \cup_{k \in \mathbb{Z}} (\bar{\Omega}_0 + kLe_1)$ where $\Omega_0 = (0, L) \times (0, 1)$ denotes the reference periodic cell as we are looking for L -periodic in x solutions. The lower boundary $\mathbb{R} \times \{0\} = \cup_{k \in \mathbb{Z}} (\bar{\Gamma}_0 + kLe_1)$, with $\Gamma_0 = (0, L) \times \{0\}$ is assumed to be fixed and the reference upper boundary $\cup_{k \in \mathbb{Z}} (\bar{\Gamma}_e + kLe_1)$, with $\Gamma_e = (0, L) \times \{1\}$ is an elastic mono-dimensionnal structure, that can move in the longitudinal direction and in the transversal one. Let us denote by $(x, t) \mapsto (\zeta(x, t), \eta(x, t))^T$ the displacement of the structure at time t . The displacement of the structure modifies the fluid domain and hence the fluid domain is a time

dependent unknown in the system. Let us denote by $\Gamma_{\zeta(t),\eta(t)} = \{(x + \zeta(x, t), 1 + \eta(x, t)), x \in [0, L] + kLe_1, k \in \mathbb{Z}\}$ the deformed upper boundary. We then denote by $\mathcal{F}_{\zeta(t),\eta(t)} \subset \mathbb{R}^2$ the fluid domain at time $t > 0$ enclosed between $\Gamma_{\zeta(t),\eta(t)}$ and $\mathbb{R} \times \{0\}$. We assume that the motion of the fluid is described by the incompressible Navier-Stokes equations in $F_T := \bigcup_{t \in (0, T)} \mathcal{F}_{\zeta(t),\eta(t)} \times \{t\}$, with $T > 0$. Thus the fluid velocity u and the fluid pressure p satisfy in F_T

$$\rho_f(\partial_t u + (u \cdot \nabla)u) - \operatorname{div} \sigma(u, p) = 0, \quad (1.1)$$

$$\nabla \cdot u = 0, \quad (1.2)$$

with $\rho_f > 0$ the fluid density, and where σ denotes the stress tensor of the fluid which is defined by

$$\sigma(u, p) = -pI_2 + 2\mu D(u),$$

where μ denotes the viscosity of the fluid, I_2 is the identity matrix and $D(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ is deformation tensor (A^T denotes the transpose of the matrix A).

Furthermore the longitudinal structure displacement ζ solves a wave equation in the reference structure cell $(0, L)$, that writes

$$\rho_s \partial_{tt} \zeta - \beta \partial_{xx} \zeta = \phi(u, p, \zeta, \eta) \cdot e_1, \quad \text{in } (0, L) \times (0, T), \quad (1.3)$$

whereas the transversal displacement η satisfies a linear beam equation with inertia of rotation

$$\rho_s \partial_{tt} \eta - \gamma \partial_{xxt} \eta + \alpha \partial_{xxx} \eta = \phi(u, p, \zeta, \eta) \cdot e_2, \quad \text{in } (0, L) \times (0, T), \quad (1.4)$$

where $\rho_s, \alpha, \beta,$ and γ are positive real constants, and (e_1, e_2) is the canonical basis of \mathbb{R}^2 . These beam equations can be derived from three dimensional elasticity thanks to an asymptotic analysis and, in particular, in such analysis the term $-\gamma \partial_{xxt} \eta$, which models the inertia of rotation, appears (see [37], in the case of a plate). We emphasize that the equations modelling the displacement of structure are written in a reference configuration whereas the fluid equations are written in Eulerian coordinates and consequently in an unknown non cylindrical domain F_T .

The fluid and structure equations are coupled through the forcing term $\phi(u, p, h)$ in the right hand side of (1.3) and (1.4). It corresponds to the trace of normal component of the fluid stress tensor, multiplied by the element length, $\sigma(u, p) \mathbf{n} dl$ (where \mathbf{n} is the unit normal vector directed outward to $\Gamma_{\zeta(t),\eta(t)}$) that is further transported in the reference configuration. Precisely, ϕ reads as

$$\phi(u, p, \zeta, \eta)(x, t) = -\sigma(u, p)(x + \zeta(x, t), 1 + \eta(x, t), t) \cdot (-\partial_x \eta(x, t)e_1 + (1 + \partial_x \zeta(x, t))e_2), \quad \text{on } (0, L) \times (0, T). \quad (1.5)$$

We underline that the vector $-\partial_x \eta(x, t)e_1 + (1 + \partial_x \zeta(x, t))e_2$ is a non unitary normal vector to the elastic the interface $\Gamma_{\zeta(t),\eta(t)}$. The fluid and the structure are also coupled by the no-slip boundary condition at the interface:

$$u(x + \zeta(x, t), 1 + \eta(x, t), t) = \partial_t \zeta(x, t)e_1 + \partial_t \eta(x, t)e_2, \quad (x, t) \in (0, L) \times (0, T). \quad (1.6)$$

We complement our system with

- i. L-periodicity w.r.t x for the fluid velocity and pressure and for the structure displacement;

ii. no-slip boundary condition on the bottom of the fluid container:

$$u(\cdot, 0, t) = 0. \quad (1.7)$$

To close the problem herein, we impose initial conditions at $t = 0$:

$$\zeta(x, 0) = \zeta_0(x), \quad x \in (0, L), \quad (1.8)$$

$$\partial_t \zeta(x, 0) = \zeta_1(x), \quad x \in (0, L), \quad (1.9)$$

$$\eta(x, 0) = \eta_0(x), \quad x \in (0, L), \quad (1.10)$$

$$\partial_t \eta(x, 0) = \eta_1(x), \quad x \in (0, L), \quad (1.11)$$

$$u(x, y, 0) = u_0(x, y), \quad \text{in } \mathcal{F}_{\zeta_0, \eta_0}. \quad (1.12)$$

The initial displacement and velocities are assumed to be L -periodic in x . The purpose of this work is to show the existence of a unique local-in-time strong solution to the system (1.1)-(1.7) completed with initial conditions (1.8)-(1.12).

Most of the literature on the well-posedness of 2D/1D or 3D/2D fluid-structure coupled problems is done ignoring the longitudinal displacement of the structure. Indeed, these studies tackle the fluid-structure interaction problem consisting of the incompressible Navier-Stokes equations (1.1)-(1.2) describing the fluid behaviour and an elastic beam, plate or shell in flexion. In the two-dimensional case either the wave equation (1.3) or the beam equation (1.4) or the damped beam equation have been considered to describe the transversal structure displacement as the structure is only allowed to move along the normal direction of the reference configuration. In dimension three, the case of plate or shell in flexion have also been considered. In this respect, existence of weak solutions as long as the structure does not touch the bottom of the fluid cavity is established in [14] in the case the transversal structure displacement satisfies a beam equation with an additional damping term. In this case existence beyond contact has also been derived in [13]. We also mention [25, 32] in 2D and [26] in 3D for the case of a plate equation without any additional damping or inertia of rotation terms. The case of Koiter shell is also considered in [33] and in [18] for generalized Newtonian flow. The uniqueness of weak solution is not an easy issue, even with small initial data and it is still an open question. However, a weak-strong uniqueness result is established in [38] in case where the structure satisfies a wave equation.

Several works have also been devoted to study of the existence and uniqueness of strong solutions to similar problems but still neglecting the longitudinal displacement of the structure. In particular, existence and uniqueness of strong solutions in two-dimension was established in the case the transversal structure displacement satisfies a wave equation with additional damping term in [31] and without damping in [4, 28]. The method used in [28] is based on a regularization arguments and *a priori* estimates whereas the strategy of proof in [4] consists on showing that the semi-group associated to the linearized problem is analytic. In both strategy the damping coming from the fluid is crucial. Moreover when considering the beam equation (1.4) the presence of a damping or inertia rotation term seems to be important to obtain existence and uniqueness of strong solutions in the case where the initial fluid velocity is in H^1 (see [12, 28, 30]), in particular with no gap between the regularity in space of the initial conditions and the one obtained for the solution [28]. Note however a recent preprint [34] requiring more regular initial data.

All these results are local-in-time except [27] where the authors proved that the solution of the fluid-damped beam problem is global, by showing that no contact occurs between the structure and the bottom of the fluid cavity. Without any additional term (damping or inertia of rotation), existence and uniqueness of strong solutions to the fluid-beam interaction problem is studied in [2,3] by studying the underlying semi-group associated to the linearized coupled problem and [34] by energy estimates for more regular initial data. In particular, it has been proven that the associated semi-group operator is not analytic in this case, but only of Gevrey class unlike the damped beam or wave equation case. Existence and uniqueness of strong solutions of the fluid-beam problem in the $L^p - L^q$ setting is obtained in [19].

All the results mentioned up to now deals with the fluid-structure problem by ignoring the longitudinal displacement of the structure. An existence result of strong solution considering longitudinal displacement of the fluid-beam problem can be found in [23] in the steady state.

One of the main difficulties when the longitudinal displacement is taken into account is that the incompressibility condition (1.2) together with the boundary conditions (1.6)-(1.7) and the L -periodicity of the fluid velocity u with respect to first space variable x imply that the following volume preserving constraint should be satisfied

$$\int_0^L (-\partial_t \zeta(x, t) \partial_x \eta(x, t) + \partial_t \eta(x, t) (1 + \partial_x \zeta(x, t))) dx = 0. \quad (1.13)$$

We emphasize that this constraint is non-linear contrary to the case where only the transversal displacement of the structure is taken into account [28]. In the latter case, (1.13) reduces to

$$\frac{d}{dt} \int_0^L \eta(x, t) dx = 0. \quad (1.14)$$

Thus, with null mean value initial deformation η_0 , (1.14) implies that $\int_0^L \eta(x, t) dx = 0, \forall t \geq 0$, which can easily be satisfied. In our case the volume preserving constraint (1.13) couples both beam equations (1.3)-(1.4). This volume preserving constraint fixes the pressure of the fluid p in a unique way. So the pressure of the fluid p in problem (1.1)-(1.7) is not defined up to a constant, but is uniquely defined thanks to the constraint (1.13). Precisely we can split the pressure into

$$p = p_0 + c, \quad (1.15)$$

where one imposes

$$\int_{\Omega_{\zeta(t), \eta(t)}} p_0 dx dy = 0, \quad (1.16)$$

for any deformed unit cell $\Omega_{\zeta(t), \eta(t)}$, such that $\overline{\mathcal{F}}_{\zeta(t), \eta(t)} = \cup_{k \in \mathbb{Z}} (\overline{\Omega}_{\zeta(t), \eta(t)} + kLe_1)$. The average pressure, $t \mapsto c(t)$, is the Lagrange multiplier associated to the constraint (1.13). To find the expression of c , remembering the expression of the force applied by the fluid on the structure (1.5), we may multiply equations (1.3) and (1.4) by $-\partial_x \eta$ and $(1 + \partial_x \zeta)$ respectively, then the resulting equations are integrated with respect to x on $(0, L)$, and the resulting identities are added. Doing so, we obtain by using the periodic boundary conditions of the structure that

$$\begin{aligned} c(t) = & -\frac{1}{\|(-\partial_x \eta, 1 + \partial_x \zeta)\|_{L^2(0, L)}^2} \left(\rho_s \int_0^L \left(\partial_{tt} \zeta \partial_x \eta - \partial_{tt} \eta (1 + \partial_x \zeta) \right) dx + \beta \int_0^L \partial_x \zeta \partial_{xx} \eta dx \right. \\ & \left. + \alpha \int_0^L \partial_{xxx} \eta \partial_{xx} \zeta dx + \gamma \int_0^L \partial_{xtt} \eta \partial_{xx} \zeta dx + \int_0^L \phi(u, p_0, \zeta, \eta) \cdot (-\partial_x \eta, 1 + \partial_x \zeta) dx \right), \end{aligned}$$

and hence

$$c(t) = -\frac{1}{\|(-\partial_x \eta, 1 + \partial_x \zeta)\|_{L^2(0,L)}^2} \left(\rho_s \frac{d}{dt} \int_0^L \left(\partial_t \zeta \partial_x \eta - \partial_t \eta (1 + \partial_x \zeta) \right) dx + \rho_s \int_0^L \left(-\partial_t \zeta \partial_{tx} \eta + \partial_t \eta \partial_{tx} \zeta \right) dx \right. \\ \left. + \beta \int_0^L \partial_x \zeta \partial_{xx} \eta dx + \alpha \int_0^L \partial_{xxx} \eta \partial_{xx} \zeta dx + \gamma \int_0^L \partial_{xtt} \eta \partial_{xx} \zeta dx + \int_0^L \phi(u, p_0, \zeta, \eta) \cdot (-\partial_x \eta, 1 + \partial_x \zeta) dx \right).$$

Thanks to the constraint (1.13), we get that

$$c(t) = -\frac{1}{\|(-\partial_x \eta, 1 + \partial_x \zeta)\|_{L^2(0,L)}^2} \left(\rho_s \int_0^L \left(-\partial_t \zeta \partial_{tx} \eta + \partial_t \eta \partial_{tx} \zeta \right) dx + \beta \int_0^L \partial_x \zeta \partial_{xx} \eta dx \right. \\ \left. + \alpha \int_0^L \partial_{xxx} \eta \partial_{xx} \zeta dx + \gamma \int_0^L \partial_{xtt} \eta \partial_{xx} \zeta dx + \int_0^L \phi(u, p_0, \zeta, \eta) \cdot (-\partial_x \eta, 1 + \partial_x \zeta) dx \right). \quad (1.17)$$

Reciprocally, given c as in (1.17) we can recover

$$\frac{d}{dt} \int_0^L \left(-\partial_t \zeta \partial_x \eta + \partial_t \eta (1 + \partial_x \zeta) \right) dx = 0,$$

and thus we recover the volume preserving constraint (1.13) if it is satisfied at the initial time.

Note that this additional constraint is not required when Neumann boundary conditions on inlet and outlet lateral boundaries are imposed. These conditions have been considered in [32] for the existence of weak solutions (by considering the total pressure instead of the pressure to obtain energy estimates) or in [12] in the 2D case, [16] in the 3D case both for existence of a unique strong solution for a damped beam or shell in flexion respectively. Studying existence of strong solutions of the problem considered here by replacing the periodic boundary conditions with more realistic Neumann boundary conditions presents additional difficulties since, in particular, one needs to derive elliptic estimates for the Stokes problem in a domain with corners, low regularity and mixed boundary conditions. In [12], and [16] such result are obtained for damped beam or shell in flexion (whose dynamic involves a fourth order operator in space and an additional damping term).

The paper follows now standard steps when dealing with fluid-structure interaction problems. In Section 2, we introduce some functional spaces and we construct a suitable change of variables which enables us to write the considered free boundary problem into a problem set in the reference configuration that can be reformulated as a linear problem involving variable coefficients (as the initial displacement of structure is non-zero), and whose source terms are perturbations of the initial structure deformations. Note that, by doing so, we also linearized the non linear volume preserving constraint (1.13) and the right hand side of the divergence constraint has no zero mean, contrary to the case when only the transversal displacement is considered (in the present study this average is equal to the right hand side coming from the linearization of (1.13)). Next, we state our main result. We end this section by proving an elliptic result for a steady Stokes-like problem set in the reference configuration. In Section 3, we tackle the linear problem and we prove existence and uniqueness of strong solutions. We first study the linear problem with homogeneous divergence constraint and then we study the non-homogeneous case by lifting the divergence term using the elliptic result of the steady Stokes-like problem established in the previous section. Note that, here, in order to lift the divergence constraint one needs to both change the fluid unknowns and the structure unknowns, contrary to the case of only transversal motions. In the last section, we introduce a mapping for which a solution of the nonlinear coupled problem can be defined as a fixed point of this mapping. By applying

Picard fixed point Theorem, we obtain the existence of a unique fixed point. Compared to the only transversal motion case additional terms appears. We emphasize that the main novelty is to take into account the longitudinal displacement of the elastic beam. Moreover we do not add any damping to the equations but take into account the inertia of rotation effect.

2 Notations, Change of Variables, and Main Result

2.1 Notations

Let us fix the notations and introduce some functional spaces. Let $\Omega \subset \mathbb{R}^2$ be an open and connected set denoting the reference cell that can be reproduced by periodicity in x to generate the domain \mathcal{O} such that $\overline{\mathcal{O}} = \cup_{k \in \mathbb{Z}} (\overline{\Omega} + kLe_1)$. We denote by $C_{\sharp}^{\infty}(\Omega)$ the restriction to Ω of functions of $C_{\sharp}^{\infty}(\mathcal{O})$ of infinitely differentiable functions which are L -periodic with respect to the first space variable in \mathcal{O} . We denote by $H_{\sharp}^s(\Omega)$ with $s > 0$ and $L_{\sharp}^p(\Omega)$ with $0 \leq p \leq \infty$ the closure of $C_{\sharp}^{\infty}(\Omega)$ in the usual Sobolev space $H^s(\Omega)$ and the Lebesgue space $L^p(\Omega)$ respectively. In a similar way, we define $H_{\sharp}^s(0, L)$ with $s \geq 0$, $L_{\sharp}^p(0, L)$ with $0 \leq p \leq \infty$, $C_{\sharp}(0, L)$. We note that $H_{\sharp}^0(0, L) = L_{\sharp}^2(0, L)$. Note that any function of $H_{\sharp}^s(\Omega)$ can be extended by periodicity as a function of $H_{\sharp}^s(\mathcal{O})$. The same notations shall be used for the restriction or the extension. We emphasize that contrary to the usual convention, we consider that the time variable is the last variable of a function so that we may use unified definition for the periodic functions whether they have only one space variable or two space variables. Moreover in all what follows we will not differentiate spaces of vector valued functions from scalar valued functions.

2.2 Construction of a Change of Variables

One of the classical difficulties in fluid-structure interaction problems is that we are dealing with a free boundary problems. To overcome this difficulty, we introduce in this subsection a change of variables which maps the reference configuration $\overline{\mathcal{F}}_0$ into a deformed one and we state some useful properties of this change of variables.

Let us introduce a general change of variables and consider a given structure displacement $d = (a, b)^T$, where $(a, b) \in H_{\sharp}^{\frac{3}{2}+s}(0, L)$ with $s > 0$ and such that

$$\min_{x \in (0, L)} 1 + b(x) > 0; \tag{2.1}$$

$$\text{the mapping } \mathcal{S}_{a,b} : x \mapsto (x + a(x), 1 + b(x)) \text{ is injective as a function in } \mathbb{R} \tag{2.2}$$

Thanks to the periodicity of a and b , one has $\mathcal{S}_{a,b}(x + kL) = \mathcal{S}_{a,b}(x) + kLe_1$. Let $\Gamma_{a,b} = \cup_{k \in \mathbb{Z}} (\mathcal{S}_{a,b}([0, L]) + kLe_1)$. Let $\mathcal{F}_{a,b}$ the subset of \mathbb{R}_+^2 that has a rigid lower boundary $\mathbb{R} \times \{0\}$ and its upper boundary is $\Gamma_{a,b}$. We are going to built a change of variables that maps \mathcal{F}_0 onto $\mathcal{F}_{a,b}$. We consider $\chi_{a,b}$ a L -periodic in x mapping, such that $\chi_{a,b}(x, 1) = (x + a(x), 1 + b(x))$ and $\chi_{a,b}(x, 0) = (x, 0)$ for any $x \in [0, L]$. One can define $\chi_{a,b}$ as follows

$$\chi_{a,b} = I_2 + (\mathcal{R}a, \mathcal{R}b),$$

where \mathcal{R} is a continuous linear lifting operator from $H_{\sharp}^{\theta}(0, L) \rightarrow H_{\sharp, \Gamma_0}^{\theta+\frac{1}{2}}(\Omega_0)$, $\theta \geq \frac{1}{2}$, where $H_{\sharp, \Gamma_0}^{\theta+\frac{1}{2}}(\Omega_0)$ denotes the subspace of $H_{\sharp}^{\theta+\frac{1}{2}}(\Omega_0)$ of functions that have a zero trace on Γ_0 . With a slight abuse of notation we set

$\mathcal{R}(a, b) = (\mathcal{R}a, \mathcal{R}b)$. Given $(a, b) \in H_{\sharp}^{\frac{3}{2}+s}(0, L)$ with $s > 0$, we have $\mathcal{R}(a, b) \in H_{\sharp}^{2+s}(\Omega_0)$ with

$$\|\mathcal{R}(a, b)\|_{H_{\sharp}^{2+s}(\Omega_0)} \leq C \|(a, b)\|_{H_{\sharp}^{\frac{3}{2}+s}(0, L)},$$

where $C > 0$ depending on Ω_0 .

Setting $\Omega_{a,b} = \chi_{a,b}(\Omega_0)$, if $\chi_{a,b}$ is injective on $\overline{\mathcal{F}_0}$, we have $\overline{\mathcal{F}_{a,b}} = \chi_{a,b}(\overline{\mathcal{F}_0}) = \cup_{k \in \mathbb{Z}} (\overline{\Omega}_{a,b} + kLe_1)$ (see [15]).

Consequently we have built a mapping from the reference configuration onto a given deformed one.

One can show easily the following proposition

Proposition 2.1 *Let $(a, b) \in H_{\sharp}^{\frac{3}{2}+s}(0, L)$ with $s > 0$. There exists a positive constant \mathcal{M} such that, for all $0 < R \leq \mathcal{M}$, if $\|(a, b)\|_{H_{\sharp}^{\frac{3}{2}+s}(0, L)} + \|(\det \nabla \chi_{a,b})^{-1}\|_{C_{\sharp}(\overline{\Omega}_0)} \leq R$, then*

- *the mapping $\chi_{a,b}$ is a C^1 -diffeomorphism from $\overline{\mathcal{F}_0}$ onto $\overline{\mathcal{F}_{a,b}}$;*
- *for any $m \leq 2$, the mapping $f \mapsto f \circ \chi_{a,b}$ realizes a linear homeomorphism from $H_{\sharp}^m(\Omega_{a,b})$ into $H_{\sharp}^m(\Omega_0)$;*
- *for any $m \leq 2$, there exists a non-decreasing constant $K_m : [0, \infty) \rightarrow (0, \infty)$ such that*

$$\|f \circ \chi_{a,b}\|_{H_{\sharp}^m(\Omega_0)} \leq K_m(R) \|f\|_{H_{\sharp}^m(\Omega_{a,b})}, \quad \|f\|_{H_{\sharp}^m(\Omega_{a,b})} \leq K_m(R) \|f \circ \chi_{a,b}\|_{H_{\sharp}^m(\Omega_0)};$$

The first point can be obtained adapting results in [15] and the second and third point can be proved as in [27]. Thus we have sufficient conditions ensuring that $\chi_{a,b}$ is a change of variables. This type of change of variables will enable us to write system (1.1)-(1.12) in the reference configuration and show the existence of a unique strong solution as long as we are able to built such an injective mapping whose jacobian remains strictly positive.

Remark 2.1 *Proposition 2.1 gives sufficient conditions to obtain a C^1 -diffeomorphism from $\overline{\mathcal{F}_0}$ onto $\overline{\mathcal{F}_{a,b}} = \chi_{a,b}(\overline{\mathcal{F}_0})$. One can build such a change of variables as long as $\det \nabla \chi_{a,b} > 0$ on $\overline{\Omega}_0$ and $\chi_{a,b}$ is injective on $\overline{\mathcal{F}_0}$.*

2.3 Main Result

Before stating our main result, we introduce the notion of strong solutions to the periodic problem (1.1)-(1.11).

Definition 2.1 *Let $T > 0$. Consider $(u_0, \zeta_0, \zeta_1, \eta_0, \eta_1) \in H_{\sharp}^1(\mathcal{F}_{\zeta_0, \eta_0}) \times H_{\sharp}^2(0, L) \times H_{\sharp}^1(0, L) \times H_{\sharp}^3(0, L) \times H_{\sharp}^1(0, L)$ satisfying the following compatibility conditions*

$$\begin{cases} u_0(x + \zeta_0(x), 1 + \eta_0(x)) = (\zeta_1(x), \eta_1(x)), & x \in (0, L), \\ u_0(x, 0) = 0, & x \in (0, L), \\ \operatorname{div} u_0 = 0, & \text{in } \mathcal{F}_{\zeta_0, \eta_0}. \end{cases} \quad (2.3)$$

Assume that the initial conditions are such that there exists a mapping χ_{ζ_0, η_0} satisfying $\det \nabla \chi_{\zeta_0, \eta_0} > 0$ on $\overline{\Omega}_0$ and χ_{ζ_0, η_0} is injective on $\overline{\mathcal{F}_0}$.

A strong solution to Problem (1.1)-(1.12) on $(0, T)$ associated with the initial conditions $(u_0, \zeta_0, \zeta_1, \eta_0, \eta_1)$ as above is a quadruplet (u, p, ζ, η) such that

- it satisfies the following regularities

$$\begin{cases} \partial_t u \in L_{\sharp}^2(F_T), \nabla^2 u \in L_{\sharp}^2(F_T), p \in L_{\sharp}^2(F_T), \nabla p \in L_{\sharp}^2(F_T), \\ \zeta \in H^2(0, T; L_{\sharp}^2(0, L)) \cap W^{1, \infty}(0, T; H_{\sharp}^1(0, L)) \cap L^{\infty}(0, T; H_{\sharp}^2(0, L)) \cap H^1(0, T; H_{\sharp}^{\frac{3}{2}}(0, L)), \\ \eta \in H^2(0, L; H_{\sharp}^1(0, L)) \cap W^{1, \infty}(0, T; H_{\sharp}^2(0, L)) \cap L^{\infty}(0, T; H_{\sharp}^3(0, L)); \end{cases}$$

- equations (1.1)-(1.2) are satisfied a.e. in \mathcal{F}_T ;
- equation (1.3) is satisfied a.e. in $(0, L) \times (0, T)$;
- equation (1.4) is satisfied in $L^2(0, T; H_{\sharp}^{-1}(0, L))$;
- equations (1.6)-(1.7) are satisfied a.e. in $(0, T) \times (0, L)$;
- equation (1.12) is satisfied a.e. in $\mathcal{F}_{\zeta_0, \eta_0}$ and equations (1.8)-(1.11) are satisfied a.e. in $(0, L)$.

We underline that the compatibility conditions on the initial data (2.3) (the equality of the velocities at the interface together with the incompressibility constraint) imply

$$\int_0^L (-\zeta_1(x) \partial_x \eta_0(x) + \eta_1(x)(1 + \partial_x \zeta_0(x))) dx = 0. \quad (2.4)$$

This is a non linear constraint that has to be verified by the initial conditions. Our main result now reads as follows:

Theorem 2.1 *Consider initial data $(u_0, \zeta_0, \zeta_1, \eta_0, \eta_1) \in H_{\sharp}^1(\mathcal{F}_{\zeta_0, \eta_0}) \times H_{\sharp}^2(0, L) \times H_{\sharp}^1(0, L) \times H_{\sharp}^3(0, L) \times H_{\sharp}^1(0, L)$ such that (2.3) is satisfied and such that there exists a mapping χ_{ζ_0, η_0} satisfying $\det \nabla \chi_{\zeta_0, \eta_0} > 0$ on $\overline{\Omega}_0$ and χ_{ζ_0, η_0} is injective on $\overline{\mathcal{F}}_0$. Then there exists a time T_0 depending decreasingly on*

$$\|u_0\|_{H_{\sharp}^1(\mathcal{F}_{\zeta_0, \eta_0})} + \|\zeta_0\|_{H_{\sharp}^2(0, L)} + \|\zeta_1\|_{H_{\sharp}^1(0, L)} + \|\eta_0\|_{H_{\sharp}^3(0, L)} + \|\eta_1\|_{H_{\sharp}^1(0, L)} + \|(\det \nabla \chi_{\zeta_0, \eta_0})^{-1}\|_{C_{\sharp}(\overline{\Omega}_0)}$$

such that Problem (1.1)-(1.11) admits a unique strong solution on $(0, T_0)$.

Moreover, the solution exists as long as

- i. One can build a mapping $\chi_{\zeta, \eta} \in \mathcal{C}^0([0, T]; H_{\sharp}^{2+s}(\Omega_0))$, for $s > 0$ and such that for $t \leq T$

$$\chi_{\zeta(t), \eta(t)}(x, 1) = (x + \zeta(x, t), 1 + \eta(x, t)) \text{ and } \chi_{\zeta(t), \eta(t)}(x, 0) = (x, 0), \quad \forall x \in [0, L], \quad (2.5)$$

$$\det \nabla \chi_{\zeta(t), \eta(t)} > 0, \text{ on } \overline{\Omega}_0, \quad (2.6)$$

$$\chi_{\zeta(t), \eta(t)} \text{ is injective on } \overline{\mathcal{F}}_0. \quad (2.7)$$

- ii. $\|u(t)\|_{H^1(\mathcal{F}(t))} + \|\zeta(t)\|_{H_{\sharp}^2(0, L)} + \|\partial_t \zeta(t)\|_{H_{\sharp}^1(0, L)} + \|\eta(t)\|_{H_{\sharp}^3(0, L)} + \|\partial_t \eta(t)\|_{H_{\sharp}^1(0, L)} < +\infty$.

Remark 2.2 *Note that one can not extend the solution if we have self-contact of the elastic beam or contact with the bottom of the fluid cavity in particular if for some T^* we have either $1 + \eta(T^*) = 0$ or the mapping $x \mapsto (x + \zeta(x, T^*), 1 + \eta(x, T^*))$ is not injective on \mathbb{R} .*

Remark 2.3 The constraint (1.13) is satisfied by any solution and is indeed a volume preserving constraint. Indeed thanks to the identity (see [15])

$$\frac{d}{dt} \int_{\Omega_0} \det \nabla \chi_{\zeta(t), \eta(t)} dx dy = \int_0^L \partial_t \chi_{\zeta(t), \eta(t)} \operatorname{cof} \nabla \chi_{\zeta(t), \eta(t)} \cdot e_2 dx = \int_0^L (-\partial_t \zeta(x, t) \partial_x \eta(x, t) + \partial_t \eta(x, t) (1 + \partial_x \zeta(x, t))) dx = 0,$$

we get

$$\int_{\Omega_0} \det \nabla \chi_{\zeta(t), \eta(t)} dx dy = \int_{\Omega_0} \det \nabla \chi_{\zeta_0, \eta_0} dx dy,$$

and implying that the volume of the initial reference cell is preserved by $\chi_{\zeta(t), \eta(t)}$ as it is injective, namely we have $|\Omega_{\zeta(t), \eta(t)}| = |\Omega_{\zeta_0, \eta_0}|$.

We are going to prove the well-posedness of Problem (1.1)-(1.12) by following quite standard steps: write the system in the reference configuration, study the associated non homogeneous linear problem, prove existence and uniqueness of solution by perturbation for small time. One main difficulty comes from the coupling between both longitudinal and transversal components of the elastic displacement that are coupled through the flow and the non linear volume preserving constraint (1.13).

2.4 Equivalent system in a reference configuration

The first step to prove Theorem 2.1 is standard and consists in rewriting the free boundary problem (1.1)-(1.12) in a reference configuration, here \mathcal{F}_0 (or equivalently in the reference cell Ω_0). We do this by using a change of variable $\chi_{\zeta(t), \eta(t)}$ which maps the \mathcal{F}_0 onto $\mathcal{F}_{\zeta(t), \eta(t)}$ by mapping the reference cell Ω_0 onto $\Omega_{\zeta(t), \eta(t)} = \chi_{\zeta(t), \eta(t)}(\Omega_0)$. To this end, we introduce the new unknowns

$$v(x, y, t) = u(\chi_{\zeta(t), \eta(t)}(x, y), t) \text{ and } q(x, y, t) = p(\chi_{\zeta(t), \eta(t)}(x, y), t), \quad (x, y, t) \in \Omega_0 \times (0, T).$$

Then the Navier-Stokes equations on the reference cell configuration $\Omega_0 \times (0, T)$ write as follows:

$$\rho_{f, \zeta, \eta} \partial_t v + \rho_f ((v - \partial_t \chi_{\zeta, \eta}) \cdot (B_{\zeta, \eta} \nabla)) v - \mu \operatorname{div}((A_{\zeta, \eta} \nabla) v) + (B_{\zeta, \eta} \nabla) q = 0, \quad (2.8)$$

$$\operatorname{div}(B_{\zeta, \eta}^T v) = 0, \quad (2.9)$$

where

$$\rho_{f, \zeta, \eta} = \rho_f \det \nabla \chi_{\zeta, \eta}, \quad (2.10)$$

$$B_{\zeta, \eta} = \operatorname{cof} \nabla \chi_{\zeta, \eta}, \quad (2.11)$$

$$A_{\zeta, \eta} = \frac{1}{\det \nabla \chi_{\zeta, \eta}} B_{\zeta, \eta}^T B_{\zeta, \eta}. \quad (2.12)$$

With these new unknowns, the structure equations on $(0, L)$ write

$$\rho_s \partial_{tt} \zeta - \beta \partial_{xx} \zeta = - \left(\mu (A_{\zeta, \eta} \nabla) v + \frac{\mu}{\det \nabla \chi_{\zeta, \eta}} B_{\zeta, \eta} \nabla v^T B_{\zeta, \eta} - q B_{\zeta, \eta} \right) \Big|_{y=1} \cdot e_2 \cdot e_1, \quad (2.13)$$

$$\rho_s \partial_{tt} \eta - \gamma \partial_{xxtt} \eta + \alpha \partial_{xxxx} \eta = - \left(\mu (A_{\zeta, \eta} \nabla) v + \frac{\mu}{\det \nabla \chi_{\zeta, \eta}} B_{\zeta, \eta} \nabla v^T B_{\zeta, \eta} - q B_{\zeta, \eta} \right) \Big|_{y=1} \cdot e_2 \cdot e_2. \quad (2.14)$$

Remark 2.4 An important remark here is that since $\operatorname{div}(\nabla u^T) = 0$ (the divergence operator in this equality is the one associated to the variables in the deformed domain $\mathcal{F}_{\zeta(t), \eta(t)}$), then we have

$$\operatorname{div}\left(\frac{1}{\det \nabla \chi_{\zeta, \eta}} B_{\zeta, \eta} \nabla v^T B_{\zeta, \eta}\right) = 0,$$

where the divergence operator in the above equality is the one which is associated to the variables in the reference configuration \mathcal{F}_0 . Note however that contrary to the case where the longitudinal displacement is neglected and the beam moves in the transversal direction only, this term gives a non zero contribution in the right hand side of the structure equations (2.13), (2.14).

Moreover, the boundary conditions satisfied by (v, ζ, η) are

- $v|_{y=1} = \partial_t \zeta e_1 + \partial_t \eta e_2$;
- L-periodicity w.r.t x for the velocity field v , the pressure field q and structure displacement (ζ, η) ;
- no-slip boundary conditions on the bottom of the fluid container: $v|_{y=0} = 0$.

Concerning the initial conditions we define

$$v_0(x, y) := v(x, y, 0) = u_0(\chi_{\zeta_0, \eta_0}(x, y)), \quad (2.15)$$

and, in view of (2.3),

$$\begin{cases} \operatorname{div}(B_{\zeta_0, \eta_0}^T v_0) = 0, & \text{in } \Omega_0, \\ v_0|_{y=1} = \zeta_1 e_1 + \eta_1 e_2, \\ v_0|_{y=0} = 0, \end{cases} \quad (2.16)$$

whereas the initial conditions for the structure (1.8)-(1.11) are still valid.

For $T > 0$ such that there exists a mapping $\chi_{\zeta, \eta} \in \mathcal{C}^0([0, T]; H_{\sharp}^{2+s}(\Omega_0))$, with $s > 0$ verifying

$$\begin{aligned} \chi_{\zeta(t), \eta(t)}(x, 1) &= (x + \zeta(x, t), 1 + \eta(x, t)) \text{ and } \chi_{\zeta(t), \eta(t)}(x, 0) = (x, 0), \forall (x, t) \in [0, L] \times [0, T], \\ \det \nabla \chi_{\zeta(t), \eta(t)} &> 0, \text{ on } \overline{\Omega}_0, \forall t \in [0, T], \\ \chi_{\zeta(t), \eta(t)} &\text{ is injective on } \overline{\mathcal{F}}_0, \forall t \in [0, T], \\ \partial_t \chi_{\zeta, \eta} &\in L^\infty(0, T; H^1(\Omega_0)), \end{aligned}$$

the existence of a unique strong solution (u, p, ζ, η) to Problem (1.1)-(1.12) with initial conditions $(u_0, \zeta_0, \zeta_1, \eta_0, \eta_1)$ satisfying (2.3) is equivalent to the existence and uniqueness of strong solution (v, q, ζ, η) of Problem (2.8)-(2.14) on $\Omega_0 \times (0, T)$ together with the above boundary conditions and initial conditions $(v_0, \zeta_0, \zeta_1, \eta_0, \eta_1)$ defined through (2.15), (1.8)-(1.12) and satisfying (2.16). For the sake of simplicity the problem in the reference configuration $\Omega_0 \times (0, T)$ is called $(FS)_{ref}$ problem.

Next, we apply a standard perturbation method to linearize the whole problem $(FS)_{ref}$ around the initial configuration. We note that the constraint (1.13) can be rewritten as follows:

$$\int_0^L (-\partial_t \zeta(x, t) \partial_x \eta_0(x) + \partial_t \eta(x, t) (1 + \partial_x \zeta_0(x))) dx = V(t), \quad (2.17)$$

with

$$V(t) = \int_0^L (\partial_t \zeta(x, t) \partial_x (\eta(x, t) - \eta_0(x)) - \partial_t \eta(x, t) \partial_x (\zeta(x, t) - \zeta_0(x))) dx. \quad (2.18)$$

Similarly Problem $(FS)_{ref}$ can be written as follows:

$$\left\{ \begin{array}{l} \rho_f \partial_t v - \mu \operatorname{div} \left((A_{\zeta_0, \eta_0} \nabla) v + \frac{1}{\det \nabla \chi_{\zeta_0, \eta_0}} B_{\zeta_0, \eta_0} \nabla v^T B_{\zeta_0, \eta_0} - q B_{\zeta_0, \eta_0} \right) \\ \quad = f_1 + f_2 + \operatorname{div} h[v, q, \zeta, \eta], \quad \text{in } \Omega_0 \times (0, T), \\ \quad \operatorname{div} (B_{\zeta_0, \eta_0}^T v) = g[v, \zeta, \eta], \quad \text{in } \Omega_0 \times (0, T), \\ \quad v(x, 1, t) = (\partial_t \zeta(x, t), \partial_t \eta(x, t)), \quad \text{on } (0, L) \times (0, T), \\ \quad v(x, 0, t) = 0, \quad \text{on } (0, L) \times (0, T), \\ \quad \rho_s \partial_{tt} \zeta - \beta \partial_{xx} \zeta = -T_{\zeta_0, \eta_0}[v, q] \cdot e_1 - h[v, q, \zeta, \eta] e_2 \cdot e_1, \quad \text{in } (0, L) \times (0, T), \\ \quad \rho_s \partial_{tt} \eta + \alpha \partial_{xxxx} \eta - \gamma \partial_{ttx} \eta = -T_{\zeta_0, \eta_0}[v, q] \cdot e_2 - h[v, q, \zeta, \eta] e_2 \cdot e_2, \quad \text{in } (0, L) \times (0, T), \end{array} \right. \quad (2.19)$$

with

$$T_{\zeta_0, \eta_0}[v, q] = \left(\mu (A_{\zeta_0, \eta_0} \nabla) v e_2 + \frac{\mu}{\det \nabla \chi_{\zeta_0, \eta_0}} B_{\zeta_0, \eta_0} \nabla v^T B_{\zeta_0, \eta_0} e_2 - q B_{\zeta_0, \eta_0} e_2 \right) \Big|_{y=1}, \quad (2.20)$$

$$f_1[v, \zeta, \eta] = \rho_f (\det \nabla \chi_{\zeta_0, \eta_0} - \det \nabla \chi_{\zeta, \eta}) \partial_t v, \quad (2.21)$$

$$f_2[v, \zeta, \eta] = -\rho_f ((v - \partial_t \chi_{\zeta, \eta}) \cdot (B_{\zeta, \eta} \nabla)) v, \quad (2.22)$$

$$g[v, \zeta, \eta] = \operatorname{div} ((B_{\zeta_0, \eta_0}^T - B_{\zeta, \eta}^T) v), \quad (2.23)$$

$$h[v, q, \zeta, \eta] = -\mu [(A_{\zeta_0, \eta_0} - A_{\zeta, \eta}) \nabla] v - \mu \left(\frac{1}{\det \nabla \chi_{\zeta_0, \eta_0}} B_{\zeta_0, \eta_0} \nabla v^T B_{\zeta_0, \eta_0} - \frac{1}{\det \nabla \chi_{\zeta, \eta}} B_{\zeta, \eta} \nabla v^T B_{\zeta, \eta} \right) + q (B_{\zeta_0, \eta_0} - B_{\zeta, \eta}). \quad (2.24)$$

Consequently, we shall consider at first f_1, f_2, h, g as a source terms of the above linear problem with initial data $(v_0, \zeta_0, \zeta_1, \eta_0, \eta_1)$ and unknowns (v, q, ζ, η) . We underline that the right hand side of the structure equations contains terms that are the counter part of $\operatorname{div} h$ in the fluid equation. Moreover, we remark that by integrating the divergence constraint over Ω_0 and integrating by parts, we obtain

$$\begin{aligned} \int_{\Omega_0} g \, dx dy &= \int_{\Omega_0} \operatorname{div} (B_{\zeta_0, \eta_0}^T v) \, dx dy \\ &= \int_{\partial \Omega_0} B_{\zeta_0, \eta_0}^T v \cdot n \, d\gamma \\ &= \int_0^L B_{\zeta_0, \eta_0}^T|_{y=1} (\partial_t \zeta, \partial_t \eta) \cdot e_2 \, dx. \end{aligned}$$

Therefore, by (2.17)

$$\int_{\Omega_0} g \, dx dy = V(t). \quad (2.25)$$

In particular g does not have a zero mean as it is the case where the longitudinal displacement ζ is neglected. As a consequence one can not lift this divergence constraint with a velocity satisfying homogeneous boundary conditions on the upper and lower boundaries. Moreover, the decomposition of the pressure in (1.15) yields to

$$q(x, y, t) = p_0(\chi_{\zeta(t), \eta(t)}(x, y), t) + c(t) := q_0(x, y, t) + c(t).$$

Due to (1.16), q_0 shall satisfy the following constraint:

$$\int_{\Omega_0} q_0 \det \nabla \chi_{\zeta, \eta} dx dy = 0. \quad (2.26)$$

However, the constraint in (2.26) is not easy to deal with as it is a non linear combination of unknowns q_0, ζ and η . To this end, we split again the pressure as follows:

$$q(x, y, t) = r_0(x, y, t) + d(t), \quad (2.27)$$

with

$$\int_{\Omega_0} r_0 dx dy = 0, \quad (2.28)$$

and

$$\begin{aligned} d(t) = \frac{1}{\|n_0\|_{L^2(0,L)}^2} & \left(\rho_s \frac{d}{dt} \int_0^L \left(-\partial_t \zeta \partial_x \eta_0 + \partial_t \eta (1 + \partial_x \zeta_0) \right) dx - \beta \int_0^L \zeta \partial_{xxx} \eta_0 dx - \alpha \int_0^L \partial_{xxx} \eta \partial_{xx} \zeta_0 dx \right. \\ & \left. + \gamma \int_0^L \partial_{xtt} \eta \partial_{xx} \zeta_0 dx + \int_0^L T[v, r_0] \cdot n_0 dx + \int_0^L h_{|y=1} e_2 \cdot n_0 dx \right), \end{aligned} \quad (2.29)$$

where $n_0 = -\partial_x \eta_0 e_1 + (1 + \partial_x \zeta_0) e_2$ is a non unitary normal vector to Γ_{ζ_0, η_0} .

Thus, in view of (2.17), $d(t)$ can be rewritten as

$$\begin{aligned} d(t) = \frac{1}{\|n_0\|_{L^2(0,L)}^2} & \left(\rho_s \dot{V}(t) - \beta \int_0^L \zeta \partial_{xxx} \eta_0 dx - \alpha \int_0^L \partial_{xx} \eta \partial_{xx} \zeta_0 dx + \gamma \int_0^L \partial_{xtt} \eta \partial_{xx} \zeta_0 dx \right. \\ & \left. + \int_0^L T[v, r_0] \cdot n_0 dx + \int_0^L h_{|y=1} e_2 \cdot n_0 dx \right), \end{aligned} \quad (2.30)$$

where \dot{V} denotes the time derivative of V . We underline that from the knowledge of (r_0, d) , we can recover (q_0, c) .

In fact, we may rewrite

$$q = (r_0 + d(t) - c(t)) + c(t) = q_0 + c(t). \quad (2.31)$$

In view of (2.26), we obtain

$$\int_{\Omega_0} (r_0 + d(t) - c(t)) \det \nabla \chi_{\zeta(t), \eta(t)} dx dy = 0.$$

It follows that

$$\int_{\Omega_0} r_0 \det \nabla \chi_{\zeta(t), \eta(t)} dx dy + (d(t) - c(t)) |\Omega_{\zeta(t), \eta(t)}| = 0.$$

Therefore,

$$c(t) = d(t) + \frac{1}{|\Omega_{\zeta(t), \eta(t)}|} \int_{\Omega_0} r_0 \det \nabla \chi_{\zeta(t), \eta(t)} dx dy. \quad (2.32)$$

By substituting $c(t)$ in (2.31), we get

$$q_0 = r_0 - \frac{1}{|\Omega_{\zeta(t), \eta(t)}|} \int_{\Omega_0} r_0 \det \nabla \chi_{\zeta(t), \eta(t)} dx dy. \quad (2.33)$$

2.5 Useful propositions and Elliptic Result

We first state a useful uniform coercivity property and a uniform Korn inequality.

Proposition 2.2 *Let $s > 0$. For any $(a, b) \in H_{\#}^{\frac{3}{2}+s}(0, L)$ such that $\det \nabla \chi_{a,b} > 0$ on $\bar{\Omega}_0$, then the symmetric matrix $A_{a,b} \in H_{\#}^{1+s}(\Omega_0) \subset C_{\#}(\bar{\Omega}_0)$. Moreover, there exists two non negative functions α_1, α_2 , resp. non increasing and non decreasing with respect to R such that for any $(a, b) \in H_{\#}^{\frac{3}{2}+s}(0, L)$ satisfying $\|(a, b)\|_{H_{\#}^{\frac{3}{2}+s}(0, L)} + \|(\det \nabla \chi_{a,b})^{-1}\|_{C_{\#}(\bar{\Omega}_0)} \leq R$*

$$\alpha_1(R)I_2 \leq A_{a,b}(x, y) \leq \alpha_2(R)I_2, \quad \text{for any } (x, y) \in \bar{\Omega}_0. \quad (2.34)$$

In the same way, we have a uniform Korn like inequality: there exists a constant $\lambda(R) > 0$, non increasing with respect to R , such that, for any $(a, b) \in H_{\#}^{\frac{3}{2}+s}(0, L)$ such that $\chi_{a,b}$ is injective on $\bar{\Omega}_0$ and $\|(a, b)\|_{H_{\#}^{\frac{3}{2}+s}(0, L)} + \|(\det \nabla \chi_{a,b})^{-1}\|_{C_{\#}(\bar{\Omega}_0)} \leq R$,

$$\int_{\Omega_0} \frac{1}{\det \nabla \chi_{a,b}} |((B_{a,b} \nabla)v + ((B_{a,b} \nabla)v)^T)|^2 dx dy \geq \lambda(R) \int_{\Omega_0} |\nabla v|^2 dx dy, \quad \forall v \in H_{\#}^1(\Omega_0), v|_{y=0} = 0. \quad (2.35)$$

Proof. Since $H_{\#}^{1+s}(\Omega_0)$ is an algebra, if $\det \nabla \chi_{a,b} > 0$ on $\bar{\Omega}_0$, then $A_{a,b} \in H_{\#}^{1+s}(\Omega_0)$. The proof of (2.34) relies on the fact that $H_{\#}^{1+s}(\Omega_0)$ compactly embedded in $C_{\#}(\bar{\Omega}_0)$. For the Korn inequality, since $\chi_{a,b}$ is injective on $\bar{\Omega}_0$ and $\det \nabla \chi_{a,b} > 0$ on $\bar{\Omega}_0$, it is a C^1 -diffeomorphism from Ω_0 onto $\Omega_{a,b} = \chi_{a,b}(\Omega_0)$. Applying this change of variables, we observe

$$\begin{aligned} \int_{\Omega_0} \frac{1}{\det \nabla \chi_{a,b}} |((B_{a,b} \nabla)v + ((B_{a,b} \nabla)v)^T)|^2 dx dy &= 2 \int_{\Omega_0} \left((A_{a,b} \nabla)v + \frac{1}{\det \nabla \chi_{a,b}} B_{a,b} \nabla v^T B_{a,b} \right) : \nabla v dx dy \\ &= \int_{\Omega_{a,b}} |\nabla \tilde{v} + (\nabla \tilde{v})^T|^2 dx' dy'. \end{aligned}$$

where $\tilde{v}(x', y') = v(\chi_{a,b}^{-1}(x', y'))$ for any vector field v .

Hence, by applying Korn inequality in $\Omega_{a,b}$ and there exists $\tilde{\lambda}_{a,b} > 0$ such that, $\forall v \in H_{\#}^1(\Omega_0)$ with $v|_{y=0} = 0$,

$$\int_{\Omega_0} \frac{1}{\det \nabla \chi_{a,b}} |((B_{a,b} \nabla)v + ((B_{a,b} \nabla)v)^T)|^2 dx dy \geq \tilde{\lambda}_{a,b} \int_{\Omega_{a,b}} |\nabla \tilde{v}|^2 dx' dy'.$$

Changing of variables we deduce that there exists $\lambda_{a,b} > 0$ such that, $\forall v \in H_{\#}^1(\Omega_0)$, $v|_{y=0} = 0$,

$$\int_{\Omega_0} \frac{1}{\det \nabla \chi_{a,b}} |((B_{a,b} \nabla)v + ((B_{a,b} \nabla)v)^T)|^2 dx dy \geq \lambda_{a,b} \int_{\Omega_0} |\nabla v|^2 dx dy.$$

The next step is to prove that we can choose a uniform constant for any $(a, b) \in H_{\#}^{\frac{3}{2}+s}(0, L)$ with $\chi_{a,b}$ is injective on $\bar{\Omega}_0$ and $\|(a, b)\|_{H_{\#}^{\frac{3}{2}+s}(0, L)} + \|(\det \nabla \chi_{a,b})^{-1}\|_{C_{\#}(\bar{\Omega}_0)} \leq R$, depending only on R . By choosing $\lambda_{a,b}$ as the best constant and defining $\lambda(R)$ as the infimum over the set of $(a, b) \in H_{\#}^{\frac{3}{2}+s}(0, L)$ with $\chi_{a,b}$ is injective on $\bar{\Omega}_0$ and $\|(a, b)\|_{H_{\#}^{\frac{3}{2}+s}(0, L)} + \|(\det \nabla \chi_{a,b})^{-1}\|_{C_{\#}(\bar{\Omega}_0)} \leq R$, it is easy to show by contradiction that $\lambda(R)$ is not equal to zero following the same steps as in [35]. It is clear that the mapping $R \mapsto \lambda(R)$ is a non increasing function.

Remark 2.5 *Note that in the only transversal case this uniform Korn inequality is not needed for the study of the full coupled problem as, in this case, the contribution of $((B_{a,b} \nabla)v)^T$ disappears (see [14], and Remark 2.4).*

Next to achieve the desired regularity of the fluid velocity and pressure solution to Problem (2.19), we need an elliptic regularity result of the following periodic steady state Stokes-like problem, for $(a, b) \in H_{\sharp}^{\frac{3}{2}+s}(0, L)$ with $s > 0$

$$-\mu \operatorname{div}\left((A_{a,b}\nabla)\hat{u} + \frac{1}{\det \nabla \chi_{a,b}} B_{a,b} \nabla \hat{u}^T B_{a,b}\right) + (B_{a,b}\nabla)\hat{p} = \hat{f}, \quad \text{in } \Omega_0 \times (0, T), \quad (2.36)$$

$$\operatorname{div}(B_{a,b}^T \hat{u}) = \hat{g}, \quad \text{in } \Omega_0 \times (0, T), \quad (2.37)$$

completed with the following boundary conditions

$$\hat{u}|_{y=1} = \psi, \quad (2.38)$$

$$\hat{u}|_{y=0} = 0, \quad (2.39)$$

where $\hat{f} \in L_{\sharp}^2(\Omega_0)$, $\hat{g} \in H_{\sharp}^1(\Omega_0)$ and $\psi = (\psi_1, \psi_2) \in H_{\sharp}^{\frac{3}{2}}(0, L)$. From Remark 2.4, equation (2.36) writes

$$-\mu \operatorname{div}((A_{a,b}\nabla)\hat{u}) + (B_{a,b}\nabla)\hat{p} = \hat{f}, \quad \text{in } \Omega_0 \times (0, T), \quad (2.40)$$

and even if $\frac{1}{\det \nabla \chi_{a,b}} B_{a,b} \nabla \hat{u}^T B_{a,b} \cdot n$ is not equal to zero on the fluid-structure interface, since in this subsection we are interested in the elliptic regularity of a Dirichlet boundary problem, we can work here with (2.40). Here we underline that the couple (a, b) does not depend on time and that, moreover, due to (2.37) and to the boundary conditions (2.38)-(2.39), the data should satisfy the following compatibility condition

$$\int_{\Omega_0} \hat{g} \, dx dy = \int_0^L (-\psi_1(x) \partial_x b(x) + \psi_2(x) (1 + \partial_x a(x))) \, dx. \quad (2.41)$$

We remark that the pressure \hat{p} in (2.40) is defined up to a constant. Consequently, we impose, for instance, the additional condition

$$\int_{\Omega_0} \hat{p} \, dx dy = 0,$$

to ensure the uniqueness of pressure. To this end, we introduce the following space

$$L_{\sharp,0}^2(\Omega_0) = \{p \in L_{\sharp}^2(\Omega_0) : \int_{\Omega_0} p \, dx dy = 0\}.$$

We underline that Problem (2.40)-(2.39) is equivalent to the stationary Stokes problem written in the domain $\Omega_{a,b}$ whose boundary is only $H^{\frac{3}{2}+s}$ regular, which does not embed into $C^{1,1}$ or $W^{2,\infty}$ as in [9]. Thus, standard regularity results of the stationary Stokes problem does not apply here because it requires that the boundary of the domain to be smooth enough. Moreover a key issue is to control the different constants uniformly with respect to (a, b) . In [27] an elliptic result has been established for Problem (2.40)-(2.39) in the case $a = 0$ and $b \in H_{\sharp}^2(0, L)$. In [9] the regularity of the Stokes problem is derived for domains with $W^{2,\infty}$ boundary and in [10] the regularity of the solution of a divergence free Stokes system in non smooth domains is derived.

A key argument in all these $H^2 \times H^1$ regularity results is that the matrices $A_{a,b}$, and $B_{a,b}$ belongs to a multiplier space of $H^1(\Omega_0)$, and, in our case, we moreover rely on the following lemma

Lemma 2.1 *Given a function $k \in H_{\sharp}^s(\Omega_0)$ with $0 < s < 1$, a function $\varphi \in H_{\sharp}^1(\Omega_0)$. Then there exists a positive constant C which depends on the domain Ω_0 and for each $\delta > 0$ there exists a constant $C_{\delta} > 0$ such that*

$$\|k\varphi\|_{L_{\sharp}^2(\Omega_0)} \leq C \|k\|_{H_{\sharp}^s(\Omega_0)} \|\varphi\|_{H_{\sharp}^{1-s}(\Omega_0)} \leq C \|k\|_{H_{\sharp}^s(\Omega_0)} \left(C_{\delta} \|\varphi\|_{L_{\sharp}^2(\Omega_0)} + \delta \|\nabla \varphi\|_{L_{\sharp}^2(\Omega_0)} \right).$$

Proof. Let $k \in H_{\sharp}^s(\Omega_0)$ with $0 < s < 1$ and consider a function $\varphi \in H_{\sharp}^1(\Omega_0)$, we have

$$\begin{aligned} \|k\varphi\|_{L_{\sharp}^2(\Omega_0)} &\leq \|k\|_{L_{\sharp}^{\frac{2}{1-s}}(\Omega_0)} \|\varphi\|_{L_{\sharp}^{\frac{2}{s}}(\Omega_0)} \\ &\leq C \|k\|_{H_{\sharp}^s(\Omega_0)} \|\varphi\|_{H_{\sharp}^{1-s}(\Omega_0)} \\ &\leq C \|k\|_{H_{\sharp}^s(\Omega_0)} \|\varphi\|_{L_{\sharp}^2(\Omega_0)}^s \|\varphi\|_{H_{\sharp}^1(\Omega_0)}^{1-s} \end{aligned}$$

where $C > 0$ depends on Ω_0 .

By Young's inequality, we get

$$\|k\varphi\|_{L_{\sharp}^2(\Omega_0)} \leq C \|k\|_{H_{\sharp}^s(\Omega_0)} \left(C_{\delta} \|\varphi\|_{L_{\sharp}^2(\Omega_0)} + \delta \|\nabla\varphi\|_{L_{\sharp}^2(\Omega_0)} \right). \quad (2.42)$$

We note that C_{δ} is a decreasing function of δ .

We now state an elliptic result of the Stokes-like problem (2.40)-(2.39).

Proposition 2.3 *Let $s > 0$ and suppose that $(a, b) \in H_{\sharp}^{\frac{3}{2}+s}(0, L)$ such that $\chi_{a,b}$ is injective on $\overline{\Omega_0}$. Then for any source terms and boundary condition*

$$(\hat{f}, \hat{g}) \in H_{\sharp}^{-1}(\Omega_0) \times L_{\sharp}^2(\Omega_0), \quad \psi \in H_{\sharp}^{\frac{1}{2}}(0, L),$$

such that

$$\int_{\Omega_0} \hat{g} \, dx dy = \int_0^L (-\psi_1(x) \partial_x b(x) + \psi_2(x) (1 + \partial_x a(x))) \, dx, \quad (2.43)$$

there exists a unique solution $(\hat{u}, \hat{p}) \in H_{\sharp}^1(\Omega_0) \times L_{\sharp,0}^2(\Omega_0)$ to the Stokes-like problem (2.40)-(2.39) such that, if we have $\|(a, b)\|_{H_{\sharp}^{\frac{3}{2}+s}(0,L)} + \|(\det \nabla \chi_{a,b})^{-1}\|_{C_{\sharp}(\overline{\Omega_0})} \leq R$, then

$$\|\hat{u}\|_{H_{\sharp}^1(\Omega_0)} + \|\hat{p}\|_{L_{\sharp,0}^2(\Omega_0)} \leq K(R) \left(\|\hat{f}\|_{H_{\sharp}^{-1}(\Omega_0)} + \|\hat{g}\|_{L_{\sharp}^2(\Omega_0)} + \|\psi\|_{H_{\sharp}^{\frac{1}{2}}(0,L)} \right), \quad (2.44)$$

where $K : [0, \infty) \mapsto (0, \infty)$ is a non decreasing function.

Moreover, if

$$(\hat{f}, \hat{g}) \in L_{\sharp}^2(\Omega_0) \times H_{\sharp}^1(\Omega_0), \quad \psi \in H_{\sharp}^{\frac{3}{2}}(0, L),$$

then $(\hat{u}, \hat{p}) \in H_{\sharp}^2(\Omega_0) \times H_{\sharp}^1(\Omega_0)$ and satisfies

$$\|\hat{u}\|_{H_{\sharp}^2(\Omega_0)} + \|\hat{p}\|_{H_{\sharp}^1(\Omega_0)} \leq K(R) \left(\|\hat{f}\|_{L_{\sharp}^2(\Omega_0)} + \|\hat{g}\|_{H_{\sharp}^1(\Omega_0)} + \|\psi\|_{H_{\sharp}^{\frac{3}{2}}(0,L)} \right). \quad (2.45)$$

Proof. We split the proof into two steps.

Step 1. Let $\hat{f} \in H_{\sharp}^{-1}(\Omega_0)$, $\hat{g} \in L_{\sharp}^2(\Omega_0)$ and $\psi \in H_{\sharp}^{\frac{1}{2}}(0, L)$. Then there exists $u_{\psi} \in H_{\sharp}^1(\Omega_0)$ such that

$$u_{\psi}|_{y=1} = \psi_1 e_1 + \psi_2 e_2 \quad \text{and} \quad u_{\psi}|_{y=0} = 0,$$

and satisfying

$$\|u_{\psi}\|_{H_{\sharp}^1(\Omega_0)} \leq C \|\psi\|_{H_{\sharp}^{\frac{1}{2}}(0,L)}, \quad (2.46)$$

where C is a positive constant depending on the domain Ω_0 .

Let $\bar{u} = \hat{u} - u_{\psi}$. Then (\bar{u}, \hat{p}) satisfies

$$-\mu \operatorname{div} \left((A_{a,b} \nabla) \bar{u} \right) + (B_{a,b} \nabla) \hat{p} = \bar{f}, \quad \text{in } \Omega_0 \times (0, T), \quad (2.47)$$

$$\operatorname{div} (B_{a,b}^T \bar{u}) = \bar{g}, \quad \text{in } \Omega_0 \times (0, T), \quad (2.48)$$

with

$$\bar{f} := \hat{f} + \mu \operatorname{div} \left((A_{a,b} \nabla) u_\psi \right), \quad \bar{g} := \hat{g} - \operatorname{div} (B_{a,b}^T u_\psi), \quad (2.49)$$

completed with the following boundary conditions

$$\bar{u}|_{y=1} = \bar{u}|_{y=0} = 0. \quad (2.50)$$

Since $\hat{f} \in H_{\sharp}^{-1}(\Omega_0)$, $A_{a,b} \in H_{\sharp}^{1+s}(\Omega_0)$, $B_{a,b} \in H_{\sharp}^{1+s}(\Omega_0)$, and $u_\psi \in H_{\sharp}^1(\Omega_0)$ and thanks to the Sobolev injection of $H_{\sharp}^{1+s}(\Omega_0)$ into $L_{\sharp}^{\infty}(\Omega_0)$ we get that $\bar{f} \in H_{\sharp}^{-1}(\Omega_0)$. Similarly since $\hat{g} \in L_{\sharp}^2(\Omega_0)$ and thanks to the fact that $H_{\sharp}^{1+s}(\Omega_0)$ is a multiplier space of $H_{\sharp}^1(\Omega_0)$, we have $\bar{g} \in L_{\sharp}^2(\Omega_0)$.

Moreover, equation (2.43) implies

$$\int_{\Omega_0} \bar{g} \, dx dy = 0.$$

Hence, there exists unique $v \in H_{\sharp}^1(\Omega_0)$ such that $\operatorname{div} v = \bar{g}$, in Ω_0 , $v|_{y=1} = 0$, $v|_{y=0} = 0$, and satisfying

$$\|v\|_{H_{\sharp}^1(\Omega_0)} \leq C \|\bar{g}\|_{L_{\sharp}^2(\Omega_0)},$$

where $C > 0$ depends on Ω_0 .

Let us take $w = B_{a,b}^{-T} v$, then $w \in H_{\sharp}^1(\Omega_0)$ and satisfies $\operatorname{div} (B_{a,b}^T w) = \bar{g}$, on Ω_0 , $w|_{y=1} = 0$, $w|_{y=0} = 0$. Moreover, as $B_{a,b}$ is invertible in $H_{\sharp}^{1+s}(\Omega_0)$, we obtain

$$\|w\|_{H_{\sharp}^1(\Omega_0)} \leq K(R) \|\bar{g}\|_{L_{\sharp}^2(\Omega_0)},$$

where K is a non decreasing positive function of R .

Set $U = \bar{u} - w$. Then U satisfies

$$-\mu \operatorname{div} \left((A_{a,b} \nabla) U \right) + (B_{a,b} \nabla) \hat{p} = F, \quad \text{in } \Omega_0 \times (0, T), \quad (2.51)$$

$$\operatorname{div} (B_{a,b}^T U) = 0, \quad \text{in } \Omega_0 \times (0, T), \quad (2.52)$$

with

$$F = \bar{f} + \mu \operatorname{div} \left((A_{a,b} \nabla) w \right),$$

completed with the following boundary conditions

$$U|_{y=1} = 0, \quad (2.53)$$

$$U|_{y=0} = 0. \quad (2.54)$$

We underline that due to (2.52), equation (2.51) is equivalent to

$$-\mu \operatorname{div} \left((A_{a,b} \nabla) U \right) + (B_{a,b} \nabla) \hat{p} = F, \quad \text{in } \Omega_0 \times (0, T). \quad (2.55)$$

Next, by assumption, we can apply Proposition 2.2 to $A_{a,b}$ which is consequently uniformly coercive. Moreover, since $B_{a,b}$ is invertible in $H_{\sharp}^{1+s}(\Omega_0)$, we find, by standard arguments (see for instance [23, 25, 27] for similar arguments in the context of transversal displacement only) a weak solution $(U, \hat{p}) \in H_{\sharp}^1(\Omega_0) \times L_{\sharp}^2(\Omega_0)$ to Problem (2.51)-(2.54) depending continuously on F and satisfying

$$\|U\|_{H_{\sharp}^1(\Omega_0)} + \|\hat{p}\|_{L_{\sharp,0}^2(\Omega_0)} \leq K(R) \|F\|_{H_{\sharp}^{-1}(\Omega_0)}.$$

It follows that

$$\|\bar{u}\|_{H_{\sharp}^1(\Omega_0)} + \|\hat{p}\|_{L_{\sharp,0}^2(\Omega_0)} \leq K(R) \left(\|\bar{f}\|_{H_{\sharp}^{-1}(\Omega_0)} + \|\bar{g}\|_{L_{\sharp,0}^2(\Omega_0)} \right). \quad (2.56)$$

By combining the above inequality with (2.46), we get

$$\|\hat{u}\|_{H_{\sharp}^1(\Omega_0)} + \|\hat{p}\|_{L_{\sharp,0}^2(\Omega_0)} \leq K(R) \left(\|\hat{f}\|_{H_{\sharp}^{-1}(\Omega_0)} + \|\hat{g}\|_{L_{\sharp}^2(\Omega_0)} + \|\psi\|_{H_{\sharp}^{\frac{1}{2}}(0,L)} \right). \quad (2.57)$$

Step 2. Let $\hat{f} \in L_{\sharp}^2(\Omega_0)$, $\hat{g} \in H_{\sharp}^1(\Omega_0)$ and $\psi \in H_{\sharp}^{\frac{3}{2}}(0,L)$. At this step we prove that the solution (\hat{u}, \hat{p}) of System (2.40)-(2.39) belongs to $H_{\sharp}^2(\Omega_0) \times \left(H_{\sharp}^1(\Omega_0) \cap L_{\sharp,0}^2(\Omega_0) \right)$ and satisfies the estimate (2.45). We start by lifting the boundary conditions but this time the lifting vector $u_{\psi} \in H_{\sharp}^2(\Omega_0)$ as $\psi \in H_{\sharp}^{\frac{3}{2}}(0,L)$ and satisfies

$$\|u_{\psi}\|_{H_{\sharp}^2(\Omega_0)} \leq C \|\psi\|_{H_{\sharp}^{\frac{3}{2}}(0,L)}, \quad (2.58)$$

where C depends only on Ω_0 . We already proved that there exists $(\bar{u}, \hat{p}) \in H_{\sharp}^1(\Omega_0) \times L_{\sharp,0}^2(\Omega_0)$, where $\bar{u} = \hat{u} - u_{\psi}$ unique solution of (2.47)-(2.50) and satisfying (2.56). Hence, to prove the estimate (2.45), it remains to bound the second order derivatives of \bar{u} and first order derivatives of \hat{p} in $L_{\sharp}^2(\Omega_0)$.

Due to the regularity of (a,b) , $A_{a,b} \in H_{\sharp}^{1+s}(\Omega_0)$, $B_{a,b} \in H_{\sharp}^{1+s}(\Omega_0)$ which is a multiplier space of $H_{\sharp}^1(\Omega_0)$. Remembering Piola identity (see [15]) that states $\operatorname{div}(B_{a,b}) = 0$, it implies that $\bar{g} = \hat{g} - B_{a,b} : \nabla u_{\psi}$ and thus the new sources terms (\bar{f}, \bar{g}) belong to $L_{\sharp}^2(\Omega_0) \times H_{\sharp}^1(\Omega_0)$. We now differentiate equations (2.47)-(2.48) satisfied by (\bar{u}, \hat{p}) with respect to x . Formally, we obtain

$$-\mu \operatorname{div}((A_{a,b} \nabla) \partial_x \bar{u}) + (B_{a,b} \nabla) \partial_x \hat{p} = \tilde{f}, \quad \text{in } \Omega_0 \times (0, T), \quad (2.59)$$

$$\operatorname{div}(B_{a,b}^T \partial_x \bar{u}) = \partial_x \bar{g} - \partial_x B_{a,b}^T : \nabla \bar{u}, \quad \text{in } \Omega_0 \times (0, T), \quad (2.60)$$

with

$$\tilde{f} := \mu \operatorname{div}((\partial_x A_{a,b} \nabla) \bar{u}) - (\partial_x B_{a,b} \nabla) \hat{p} + \partial_x \bar{f}. \quad (2.61)$$

We note that $\partial_x \bar{u}$ is L -periodic with respect to x and it vanishes on $y = 1$ and $y = 0$. Consequently, thanks to boundary conditions and Piola identity, since $\partial_x B_{a,b} = \operatorname{cof}(\partial_x \nabla \chi_{a,b})$, the right hand side of the divergence constraint satisfies

$$\int_{\Omega_0} (\partial_x \bar{g} - \partial_x B_{a,b}^T : \nabla \bar{u}) \, dx dy = \int_{\Omega_0} \operatorname{div}(\partial_x B_{a,b}^T \nabla \bar{u}) \, dx dy = 0.$$

Therefore Step 1 implies

$$\|\partial_x \bar{u}\|_{H_{\sharp}^1(\Omega_0)} + \|\partial_x \hat{p}\|_{L_{\sharp,0}^2(\Omega_0)} \leq K(R) \left(\|\tilde{f}\|_{H_{\sharp}^{-1}(\Omega_0)} + \|\partial_x \bar{g} - \partial_x B_{a,b}^T : \nabla \bar{u}\|_{L_{\sharp}^2(\Omega_0)} \right). \quad (2.62)$$

Hence to bound the left hand side of the above equation, we have to bound \tilde{f} in $H_{\sharp}^{-1}(\Omega_0)$ and the term $\partial_x \bar{g} - \partial_x B_{a,b}^T : \nabla \bar{u}$ in $L_{\sharp}^2(\Omega_0)$. By performing integration by parts, for any $w \in H_{\sharp}^1(\Omega_0)$ such that $w|_{y=0} = w|_{y=1} = 0$, we have

$$\int_{\Omega_0} \operatorname{div}((\partial_x A_{a,b} \nabla) \bar{u}) \cdot w \, dx dy = - \int_{\Omega_0} ((\partial_x A_{a,b} \nabla) \bar{u}) : \nabla w \, dx dy,$$

and hence

$$\|\operatorname{div}((\partial_x A_{a,b} \nabla) \bar{u})\|_{H_{\sharp}^{-1}(\Omega_0)} \leq \|(\partial_x A_{a,b} \nabla) \bar{u}\|_{L_{\sharp}^2(\Omega_0)}. \quad (2.63)$$

By using Lemma 2.1 for $\partial_x A_{a,b} \in H_{\sharp}^s(\Omega_0)$ and $\nabla \bar{u} \in H_{\sharp}^1(\Omega_0)$, we get, for any $\delta > 0$,

$$\|(\partial_x A_{a,b} \nabla) \bar{u}\|_{L_{\sharp}^2(\Omega_0)} \leq K(R) \left(C_{\delta} \|\nabla \bar{u}\|_{L_{\sharp}^2(\Omega_0)} + \delta \|\nabla^2 \bar{u}\|_{L_{\sharp}^2(\Omega_0)} \right). \quad (2.64)$$

Combining the above inequality with (2.63) and (2.57), we obtain, for any $\delta > 0$,

$$\|\operatorname{div}((\partial_x A_{a,b} \nabla) \bar{u})\|_{H_{\sharp}^{-1}(\Omega_0)} \leq K(R) \left(C_{\delta} \left(\|\hat{f}\|_{H_{\sharp}^{-1}(\Omega_0)} + \|\hat{g}\|_{L_{\sharp}^2(\Omega_0)} + \|\psi\|_{H_{\sharp}^{\frac{1}{2}}(0,L)} \right) + \delta \|\nabla^2 \bar{u}\|_{L_{\sharp}^2(\Omega_0)} \right). \quad (2.65)$$

By performing again integration by parts and using Piola identity, we get that for any $w \in H_{\sharp}^1(\Omega_0)$ such that $w|_{y=0} = w|_{y=1} = 0$

$$\int_{\Omega_0} (\partial_x B_{a,b} \nabla) \hat{p} \cdot w \, dx dy = - \int_{\Omega_0} \partial_x B_{a,b} \hat{p} : \nabla w \, dx dy,$$

and thus

$$\|(\partial_x B_{a,b} \nabla) \hat{p}\|_{H_{\sharp}^{-1}(\Omega_0)} \leq \|\partial_x B_{a,b} \hat{p}\|_{L_{\sharp}^2(\Omega_0)}. \quad (2.66)$$

Hence applying again Lemma 2.1 (for $\partial_x B_{a,b} \in H_{\sharp}^s(\Omega_0)$ and $\hat{p} \in H_{\sharp}^1(\Omega_0)$) leads to

$$\|\partial_x B_{a,b} \hat{p}\|_{L_{\sharp}^2(\Omega_0)} \leq K(R) \left(C_{\delta} \|\hat{p}\|_{L_{\sharp}^2(\Omega_0)} + \delta \|\nabla \hat{p}\|_{L_{\sharp}^2(\Omega_0)} \right), \quad (2.67)$$

for any $\delta > 0$. Consequently, gathering the above estimate (2.67) with (2.57) and (2.66), we obtain, for any $\delta > 0$

$$\|(\partial_x B_{a,b} \nabla) \hat{p}\|_{H_{\sharp}^{-1}(\Omega_0)} \leq K(R) \left(C_{\delta} \left(\|\hat{f}\|_{L_{\sharp}^2(\Omega_0)} + \|\hat{g}\|_{L_{\sharp,0}^2(\Omega_0)} + \|\psi\|_{H_{\sharp}^{\frac{1}{2}}(0,L)} \right) + \delta \|\nabla \hat{p}\|_{L_{\sharp}^2(\Omega_0)} \right). \quad (2.68)$$

Hence, it remains to bound $\partial_x \bar{f}$ in $H_{\sharp}^{-1}(\Omega_0)$ and $\partial_x \bar{g}$ in $L_{\sharp}^2(\Omega_0)$. From the definition of \bar{f} and \bar{g} in (2.49), we have

$$\|\partial_x \bar{f}\|_{H_{\sharp}^{-1}(\Omega_0)} \leq \|\partial_x \hat{f}\|_{H_{\sharp}^{-1}(\Omega_0)} + \|(\partial_x A_{a,b} \nabla) u_{\psi}\|_{L^2(\Omega_0)} + \|(A_{a,b} \nabla) \partial_x u_{\psi}\|_{L^2(\Omega_0)}, \quad (2.69)$$

and

$$\|\partial_x \bar{g}\|_{L_{\sharp}^2(\Omega_0)} \leq \|\partial_x \hat{g}\|_{L_{\sharp}^2(\Omega_0)} + \|\partial_x B_{a,b}^T : \nabla u_{\psi}\|_{L_{\sharp}^2(\Omega_0)} + \|B_{a,b}^T : \partial_x \nabla u_{\psi}\|_{L_{\sharp}^2(\Omega_0)}. \quad (2.70)$$

Repeating the same arguments we easily get

$$\|\partial_x \bar{f}\|_{H_{\sharp}^{-1}(\Omega_0)} \leq \|\partial_x \hat{f}\|_{H_{\sharp}^{-1}(\Omega_0)} + K(R) \|u_{\psi}\|_{H_{\sharp}^2(\Omega_0)},$$

$$\|\partial_x \bar{g}\|_{L_{\sharp}^2(\Omega_0)} \leq \|\partial_x \hat{g}\|_{L_{\sharp}^2(\Omega_0)} + K(R) \|u_{\psi}\|_{H_{\sharp}^2(\Omega_0)}.$$

Since $u_{\psi} \in H_{\sharp}^2(\Omega_0)$ and satisfies (2.58), we finally obtain

$$\|\partial_x \bar{f}\|_{H_{\sharp}^{-1}(\Omega_0)} \leq \|\partial_x \hat{f}\|_{H_{\sharp}^{-1}(\Omega_0)} + K(R) \|\psi\|_{H_{\sharp}^{\frac{3}{2}}(0,L)}, \quad (2.71)$$

$$\|\partial_x \bar{g}\|_{L^2(\Omega_0)} \leq \|\partial_x \hat{g}\|_{L^2(\Omega_0)} + K(R) \|\psi\|_{H_{\sharp}^{\frac{3}{2}}(0,L)}. \quad (2.72)$$

Combining the above estimates (2.71) and (2.72) with (2.65) and (2.68), we deduce

$$\begin{aligned} \|\tilde{f}\|_{H_{\sharp}^{-1}(\Omega_0)} &\leq \|\partial_x \hat{f}\|_{H_{\sharp}^{-1}(\Omega_0)} + K(R) \|\psi\|_{H_{\sharp}^{\frac{3}{2}}(0,L)} \\ &\quad + K(R) \left(C_{\delta} \left(\|\hat{f}\|_{H_{\sharp}^{-1}(\Omega_0)} + \|\hat{g}\|_{L_{\sharp}^2(\Omega_0)} + \|\psi\|_{H_{\sharp}^{\frac{1}{2}}(0,L)} \right) + \delta \|\nabla^2 \bar{u}\|_{L_{\sharp}^2(\Omega_0)} + \delta \|\nabla \hat{p}\|_{L_{\sharp}^2(\Omega_0)} \right), \end{aligned} \quad (2.73)$$

and

$$\begin{aligned} \|\partial_x \bar{g} - \partial_x B_{a,b}^T : \nabla \bar{u}\|_{H_{\sharp}^{-1}(\Omega_0)} &\leq \|\partial_x \hat{g}\|_{L^2(\Omega_0)} + K(R) \|\psi\|_{H_{\sharp}^{\frac{3}{2}}(0,L)} \\ &\quad + K(R) \left(C_{\delta} \left(\|\hat{f}\|_{H_{\sharp}^{-1}(\Omega_0)} + \|\hat{g}\|_{L_{\sharp}^2(\Omega_0)} + \|\psi\|_{H_{\sharp}^{\frac{1}{2}}(0,L)} \right) + \delta \|\nabla^2 \bar{u}\|_{L_{\sharp}^2(\Omega_0)} \right). \end{aligned} \quad (2.74)$$

Therefore by combining (2.73) and (2.74) with (2.62), we get

$$\|\partial_x \bar{u}\|_{H_{\sharp}^1(\Omega_0)} + \|\partial_x \hat{p}\|_{L_{\sharp,0}^2(\Omega_0)} \leq K_{\delta}(R) \left(\|\hat{f}\|_{L_{\sharp}^2(\Omega_0)} + \|\hat{g}\|_{H_{\sharp}^1(\Omega_0)} + \|\psi\|_{H_{\sharp}^{\frac{3}{2}}(0,L)} \right) + K(R) \left(\delta \|\nabla^2 \bar{u}\|_{L_{\sharp}^2(\Omega_0)} + \delta \|\nabla \hat{p}\|_{L_{\sharp}^2(\Omega_0)} \right).$$

For δ small enough, we obtain

$$\|\partial_x \bar{u}\|_{H_{\sharp}^1(\Omega_0)} + \|\partial_x \hat{p}\|_{L_{\sharp,0}^2(\Omega_0)} \leq K_{\delta}(R) \left(\|\hat{f}\|_{L_{\sharp}^2(\Omega_0)} + \|\hat{g}\|_{H_{\sharp}^1(\Omega_0)} + \|\psi\|_{H_{\sharp}^{\frac{3}{2}}(0,L)} \right) + K(R) \left(\delta \|\partial_{yy} \bar{u}\|_{L_{\sharp}^2(\Omega_0)} + \delta \|\partial_y \hat{p}\|_{L_{\sharp}^2(\Omega_0)} \right). \quad (2.75)$$

It remains to estimate $\|\partial_{yy} \bar{u}\|_{L_{\sharp}^2(\Omega_0)}$ and $\|\partial_y \hat{p}\|_{L_{\sharp}^2(\Omega_0)}$. To obtain a similar estimate on the full second gradient of \bar{u} and the full gradient of \hat{p} , we differentiate the divergence equation (2.48) with respect to y . This yields

$$\begin{aligned} \sum_{j=1}^2 (B_{a,b})_{j2} \partial_{yy} \bar{u}_j &= \partial_y \bar{g} - \sum_{j=1}^2 \left(\partial_y \left(\partial_x (B_{a,b})_{j1} + \partial_y (B_{a,b})_{j2} \right) \bar{u}_j - 2 \partial_y (B_{a,b})_{j2} \partial_y \bar{u}_j - \partial_y (B_{a,b})_{j1} \partial_x \bar{u}_j \right. \\ &\quad \left. - \partial_x (B_{a,b})_{j1} \partial_y \bar{u}_j + (B_{a,b})_{j1} \partial_{xy} \bar{u}_j \right) \end{aligned} \quad (2.76)$$

However, by Piola's identity we have $\operatorname{div}(B_{a,b}) = 0$ and hence we get

$$\partial_x (B_{a,b})_{j1} + \partial_y (B_{a,b})_{j2} = \left(\operatorname{div}(B_{a,b}) \right)_j = 0.$$

It then follows

$$\sum_{j=1}^2 (B_{a,b})_{j2} \partial_{yy} \bar{u}_j = \partial_y \bar{g} - \sum_{j=1}^2 \left(2 \partial_y (B_{a,b})_{j2} \partial_y \bar{u}_j - \partial_y (B_{a,b})_{j1} \partial_x \bar{u}_j - \partial_x (B_{a,b})_{j1} \partial_y \bar{u}_j + (B_{a,b})_{j1} \partial_{xy} \bar{u}_j \right) \quad (2.77)$$

By applying Lemma 2.1 with $\varphi = \partial_x \bar{u}$ or $\varphi = \partial_y \bar{u}$ and by noting that the estimate (2.72) remains true for partial derivatives with respect to y , we obtain

$$\begin{aligned} \left\| \sum_{j=1}^2 (B_{a,b})_{j2} \partial_{yy} \bar{u}_j \right\|_{L_{\sharp}^2(\Omega_0)} &\leq \|\hat{g}\|_{H_{\sharp}^1(\Omega_0)} + K(R) \left(\|\psi\|_{H_{\sharp}^{\frac{3}{2}}(0,L)} + C_{\delta} \|\nabla \bar{u}\|_{L_{\sharp}^2(\Omega_0)} + \delta \|\partial_x \nabla \bar{u}\|_{L_{\sharp}^2(\Omega_0)} \right. \\ &\quad \left. + \delta \|\partial_{yy} \bar{u}\|_{L_{\sharp}^2(\Omega_0)} + \|\partial_{xy} \bar{u}\|_{L_{\sharp}^2(\Omega_0)} \right). \end{aligned} \quad (2.78)$$

Combining the above inequality with (2.56) and (2.75) implies

$$\left\| \sum_{j=1}^2 (B_{a,b})_{j2} \partial_{yy} \bar{u}_j \right\|_{L_{\sharp}^2(\Omega_0)} \leq K_{\delta}(R) \left(\|\hat{f}\|_{L_{\sharp}^2(\Omega_0)} + \|\hat{g}\|_{H_{\sharp}^1(\Omega_0)} + \|\psi\|_{H_{\sharp}^{\frac{3}{2}}(0,L)} \right) + K(R) \left(\delta \|\partial_{yy} \bar{u}\|_{L_{\sharp}^2(\Omega_0)} + \delta \|\partial_y \hat{p}\|_{L_{\sharp}^2(\Omega_0)} \right). \quad (2.79)$$

We note that j -th row (2.47) reads as follows

$$\begin{aligned} -\mu (A_{a,b})_{22} \partial_{yy} \bar{u}_j + (B_{a,b})_{j2} \partial_y \hat{p} &= \bar{f}_j + \mu \partial_x \left((A_{a,b} \nabla) \bar{u} \right)_{j1} + \mu \left((\partial_y A_{a,b} \nabla) \bar{u} \right)_{j2} \\ &\quad + \mu (A_{a,b})_{12} \partial_{xy} \bar{u}_j - (B_{a,b})_{j1} \partial_x \hat{p}. \end{aligned} \quad (2.80)$$

Next, we multiply the above equation by $(B_{a,b})_{j2} = (B_{a,b})_{2j}^T$, then we sum the resulting equations over $j = 1$ and $j = 2$. Noting that

$$\sum_{j=1}^2 (B_{a,b})_{2j}^T (B_{a,b})_{j2} = (\det \nabla \chi_{a,b})(A_{a,b})_{22},$$

we get

$$(\det \nabla \chi_{a,b})(A_{a,b})_{22} \partial_y \hat{p} = \sum_{j=1}^2 (B_{a,b})_{j2} f_j^* + 2\mu(A_{a,b})_{22} \sum_{j=1}^2 (B_{a,b})_{j1} \partial_{yy} \bar{u}_j,$$

where f_j^* is the right hand side of (2.80).

By performing similar arguments as before to bound f^* in $L_{\sharp}^2(\Omega_0)$, we obtain

$$\begin{aligned} \|\partial_y \hat{p}\|_{L_{\sharp}^2(\Omega_0)} \leq K(R) & \left(\|\bar{f}\|_{L_{\sharp}^2(\Omega_0)} + C_{\delta} \|\nabla \bar{u}\|_{L_{\sharp}^2(\Omega_0)} + \delta \|\nabla^2 \bar{u}\|_{L_{\sharp}^2(\Omega_0)} + \|\nabla \partial_x \bar{u}\|_{L_{\sharp}^2(\Omega_0)} \right. \\ & \left. + \|\partial_x \hat{p}\|_{L_{\sharp}^2(\Omega_0)} + \left\| \sum_{j=1}^2 (B_{a,b})_{j1} \partial_{yy} \bar{u}_j \right\|_{L^2(\Omega_0)} \right). \end{aligned}$$

From (2.46), (2.57), (2.75) and (2.79), it follows that

$$\|\partial_y \hat{p}\|_{L_{\sharp}^2(\Omega_0)} \leq K_{\delta}(R) \left(\|\hat{f}\|_{L_{\sharp}^2(\Omega_0)} + \|\hat{g}\|_{H_{\sharp}^1(\Omega_0)} + \|\psi\|_{H_{\sharp}^{\frac{3}{2}}(0,L)} \right) + K(R) \left(\delta \|\partial_{yy} \bar{u}\|_{L_{\sharp}^2(\Omega_0)} + \delta \|\partial_y \hat{p}\|_{L_{\sharp}^2(\Omega_0)} \right).$$

Hence, for δ small enough, we obtain

$$\|\partial_y \hat{p}\|_{L_{\sharp}^2(\Omega_0)} \leq K_{\delta}(R) \left(\|\hat{f}\|_{L_{\sharp}^2(\Omega_0)} + \|\hat{g}\|_{H_{\sharp}^1(\Omega_0)} + \|\psi\|_{H_{\sharp}^{\frac{3}{2}}(0,L)} \right) + \delta K(R) \|\partial_{yy} \bar{u}\|_{L_{\sharp}^2(\Omega_0)}. \quad (2.81)$$

Going back to (2.80) and using (2.75), we get

$$\|\partial_{yy} \bar{u}\|_{L_{\sharp}^2(\Omega_0)} \leq K_{\delta}(R) \left(\|\hat{f}\|_{L_{\sharp}^2(\Omega_0)} + \|\hat{g}\|_{H_{\sharp}^1(\Omega_0)} + \|\psi\|_{H_{\sharp}^{\frac{3}{2}}(0,L)} \right) + \delta K(R) \|\partial_{yy} \bar{u}\|_{L_{\sharp}^2(\Omega_0)}.$$

Hence, for δ small enough, we have

$$\|\partial_{yy} \bar{u}\|_{L_{\sharp}^2(\Omega_0)} \leq K(R) \left(\|\hat{f}\|_{L_{\sharp}^2(\Omega_0)} + \|\hat{g}\|_{H_{\sharp}^1(\Omega_0)} + \|\psi\|_{H_{\sharp}^{\frac{3}{2}}(0,L)} \right). \quad (2.82)$$

and thus in view with (2.81), we get

$$\|\partial_y \hat{p}\|_{L_{\sharp}^2(\Omega_0)} \leq K(R) \left(\|\hat{f}\|_{L_{\sharp}^2(\Omega_0)} + \|\hat{g}\|_{H_{\sharp}^1(\Omega_0)} + \|\psi\|_{H_{\sharp}^{\frac{3}{2}}(0,L)} \right). \quad (2.83)$$

Combining (2.83) and (2.82) with (2.75), we obtain

$$\|\nabla \bar{u}\|_{H_{\sharp}^1(\Omega_0)} + \|\nabla \hat{p}\|_{L_{\sharp}^2(\Omega_0)} \leq K(R) \left(\|\hat{f}\|_{L_{\sharp}^2(\Omega_0)} + \|\hat{g}\|_{H_{\sharp}^1(\Omega_0)} + \|\psi\|_{H_{\sharp}^{\frac{3}{2}}(0,L)} \right). \quad (2.84)$$

By gathering (2.57) with (2.84), we finally deduce

$$\|\hat{u}\|_{H_{\sharp}^2(\Omega_0)} + \|\hat{p}\|_{H_{\sharp}^1(\Omega_0) \cap L_{\sharp,0}^2(\Omega_0)} \leq K(R) \left(\|\hat{f}\|_{L_{\sharp}^2(\Omega_0)} + \|\hat{g}\|_{H_{\sharp}^1(\Omega_0)} + \|\psi\|_{H_{\sharp}^{\frac{3}{2}}(0,L)} \right).$$

This ends the proof of the elliptic regularity result. Note that the same could be done in the case where the whole boundary of the fluid domain is elastic by localisation strategy and quotient method. The result relies mainly on the fact that the matrices $A_{a,b}$, $B_{a,b}$ belong to multiplier spaces of H^1 , on the Piola identity and on an interpolation result similar to Lemma 2.1 (see [10, 23, 24] for a perturbative method and [9, 25, 27, 39] for a similar method as the one used here).

3 Study of a linear system

Throughout this section, we fix a couple $(a, b) \in H_{\sharp}^2(0, L)$ such that $\chi_{a,b}$ is injective on $\overline{\mathcal{F}}_0$. We recall that this implies that $\det \nabla \chi_{a,b} \in H_{\sharp}^{\frac{3}{2}}(\Omega_0)$ and hence $\det \nabla \chi_{a,b} \in C_{\sharp}(\overline{\Omega}_0)$. We moreover assume that $\|(a, b)\|_{H_{\sharp}^2(0, L)} + \|(\det \nabla \chi_{a,b})^{-1}\|_{C_{\sharp}(\overline{\Omega}_0)} \leq R$. We consider the following linear problem:

$$\left\{ \begin{array}{l} \rho_{f,a,b} \partial_t v - \mu \operatorname{div} \left((A_{a,b} \nabla) v + \frac{1}{\det \nabla \chi_{a,b}} B_{a,b} \nabla v^T B_{a,b} - q B_{a,b} \right) \\ \qquad \qquad \qquad = f + \operatorname{div} h, \quad \text{in } \Omega_0 \times (0, T), \\ \operatorname{div} (B_{a,b}^T v) = g, \quad \text{in } \Omega_0 \times (0, T), \\ v(x, 1, t) = (\partial_t \zeta(x, t), \partial_t \eta(x, t)), \quad \text{on } (0, L) \times (0, T), \\ v(x, 0, t) = 0, \quad \text{on } (0, L) \times (0, T); \\ \rho_s \partial_{tt} \zeta - \beta \partial_{xx} \zeta = -T_{a,b}[v, q] \cdot e_1 - h_{|_{y=1}} e_2 \cdot e_1 + \ell, \quad \text{in } (0, L), \\ \rho_s \partial_{tt} \eta - \gamma \partial_{xxtt} \eta + \alpha \partial_{xxxx} \eta = -T_{a,b}[v, q] \cdot e_2 - h_{|_{y=1}} e_2 \cdot e_2 + m, \quad \text{in } (0, L), \end{array} \right. \quad (3.1)$$

where

$$T_{a,b}[v, q] = (\mu(A_{a,b} \nabla) v e_2 + \frac{\mu}{\det \nabla \chi_{a,b}} B_{a,b} \nabla v^T B_{a,b} e_2 - q B_{a,b} e_2)_{|_{y=1}}.$$

L -periodic boundary conditions w.r.t. x for the fluid velocity and the structure displacement are considered together with the following initial conditions at $t = 0$:

$$v(\cdot, 0) = v_0(\cdot), \quad \text{in } \Omega_0, \quad (3.2)$$

$$\zeta(\cdot, 0) = \zeta_0(\cdot), \quad \text{on } (0, L), \quad (3.3)$$

$$\partial_t \zeta(\cdot, 0) = \zeta_1(\cdot), \quad \text{on } (0, L), \quad (3.4)$$

$$\eta(\cdot, 0) = \eta_0(\cdot), \quad \text{on } (0, L), \quad (3.5)$$

$$\partial_t \eta(\cdot, 0) = \eta_1(\cdot), \quad \text{on } (0, L), \quad (3.6)$$

satisfying the following compatibility conditions

$$\left\{ \begin{array}{l} \operatorname{div} (B_{a,b}^T v_0) = 0, \text{ in } \Omega_0, \\ v_0|_{y=1} = (\zeta_1, \eta_1), \\ v_0|_{y=0} = 0. \end{array} \right. \quad (3.7)$$

The equality of the initial velocities and the divergence free constraint implies that the initial conditions have to satisfy

$$\int_0^L (-\zeta_1(x) \partial_x b(x) + \eta_1(x) (1 + \partial_x a(x))) dx = 0. \quad (3.8)$$

Moreover we emphasize that any solution satisfies

$$\delta_{a,b}(\partial_t \zeta, \partial_t \eta)(t) = \int_{\Omega_0} g \, dx dy, \quad (3.9)$$

with $\delta_{a,b}(\xi, \varphi)$ defined by

$$\delta_{a,b}(\xi, \varphi) = \int_0^L (-\xi \partial_x b + \varphi (1 + \partial_x a)) dx. \quad (3.10)$$

We note that, at initial time, necessarily $g(0) = 0$ and $\delta_{a,b}(\partial_t \zeta, \partial_t \eta)(0) = \delta_{a,b}(\zeta_1, \eta_1) = 0$.

As mentioned in the previous section, the pressure of this system is uniquely defined and can be decomposed as follows

$$q(x, y, t) = s_0(x, y, t) + k(t), \quad (3.11)$$

with

$$\int_{\Omega_0} s_0 \, dx dy = 0, \quad (3.12)$$

and

$$\begin{aligned} k(t) := & \frac{1}{\|(-\partial_x b, 1 + \partial_x a)\|_{L^2(0,L)}^2} (\rho_s \frac{d}{dt} (\delta_{a,b}(\partial_t \zeta, \partial_t \eta)))(t) + \beta \int_0^L \partial_{xx} \zeta \partial_x b \, dx - \alpha \int_0^L \partial_{xxx} \eta \partial_{xx} a \, dx + \gamma \int_0^L \partial_{xtt} \eta \partial_{xx} a \, dx \\ & + \int_0^L (T_{a,b}[v, s_0] + h)(x, 1) e_2 \cdot (-\partial_x b, 1 + \partial_x a) \, dx - \int_0^L (\ell, m) \cdot (-\partial_x b, 1 + \partial_x a) \, dx. \end{aligned} \quad (3.13)$$

In order to prove existence of a unique strong solution for the linear system (3.1) we first consider the case of a divergence free constraint ($g = 0$), then the case of a non homogeneous divergence constraint. Note that the average of the right hand side g is not equal to zero preventing to lift this non homogeneous divergence with a velocity satisfying homogeneous Dirichlet boundary conditions on Γ_0 and on Γ_e and thus preventing to modify only the fluid unknowns in the lifting process. This one of the major difference compared to the case where only the transversal displacement is considered.

3.1 Functional spaces

Let us consider the following functional spaces respectively for the initial data, solution and source terms in which we solve the linear problem associated to (2.19). Concerning initial data, we set

$$X^0 := \left\{ (v_0, \zeta_0, \zeta_1, \eta_0, \eta_1) \in X_f^0 \times X_s^0 : v_0|_{y=1} = (\zeta_1, \eta_1) \right\},$$

where

$$\begin{aligned} X_f^0 &:= \{ v_0 \in H_{\sharp}^1(\Omega_0) : \operatorname{div}(B_{a,b}^T v_0) = 0 \text{ in } \Omega_0 \text{ and } v_0|_{y=0} = 0 \}, \\ X_s^0 &:= H_{\sharp}^2(0, L) \times H_{\sharp}^1(0, L) \times H_{\sharp}^3(0, L) \times H_{\sharp}^2(0, L). \end{aligned}$$

The space X^0 is endowed with the product-norm:

$$\|(v_0, \zeta_0, \zeta_1, \eta_0, \eta_1)\|_{X^0} = \|v_0\|_{H_{\sharp}^1(\Omega_0)} + \|\zeta_0\|_{H_{\sharp}^2(\Omega_0)} + \|\zeta_1\|_{H_{\sharp}^1(\Omega_0)} + \|\eta_0\|_{H_{\sharp}^3(\Omega_0)} + \|\eta_1\|_{H_{\sharp}^2(\Omega_0)}.$$

For any $(v_0, \zeta_0, \zeta_1, \eta_0, \eta_1) \in X^0$, the compatibility condition (3.8) is satisfied. Given $T > 0$, the source term (f, h, g, ℓ, m) of the problem under consideration belongs to the following space:

$$S_T = \left\{ (f, h, g, \ell, m) : \begin{aligned} & f \in L^2(0, T; L_{\sharp}^2(\Omega_0)), \quad h \in L^2(0, T; H_{\sharp}^1(\Omega_0)), \\ & (\ell, m) \in (L^2(0, T; H_{\sharp}^{\frac{1}{2}}(0, L)))^2, \quad g \in L^2(0, T; H_{\sharp}^1(\Omega_0)) \cap H_{0,0}^1(0, T; (H_{\sharp}^1(\Omega_0))'), \end{aligned} \right\},$$

where

$$H_{0,0}^1(0, T; (H_{\sharp}^1(\Omega_0))') = \{g \in H^1(0, T; (H_{\sharp}^1(\Omega_0))') : g(\cdot, 0) = 0\}.$$

We underline that we enforce the vanishing condition at initial time so that (3.9) is satisfied at the initial time since $\delta_{a,b}(0) = 0$ because of (3.8). The space S_T is endowed by the product norm.

Now, we introduce the spaces in which, respectively, the fluid and structure unknowns belong:

$$X_{f,T} := (H^1(0, T; L_{\sharp}^2(\Omega_0)) \cap L^\infty(0, T; H_{\sharp}^1(\Omega_0)) \cap L^2(0, T; H_{\sharp}^2(\Omega_0))) \times L^2(0, T; H_{\sharp}^1(\Omega_0)),$$

$$X_{s,T} := \left\{ (\zeta, \eta) : \begin{array}{l} \zeta \in H^2(0, T; L_{\sharp}^2(0, L)) \cap W^{1,\infty}(0, T; H_{\sharp}^1(0, L)) \cap L^\infty(0, T; H_{\sharp}^2(0, L)), \\ \eta \in H^2(0, L; H_{\sharp}^1(0, L)) \cap W^{1,\infty}(0, T; H_{\sharp}^2(0, L)) \cap L^\infty(0, T; H_{\sharp}^3(0, L)) \end{array} \right\}$$

Again, these spaces are endowed with the obvious product-intersection norms denoted $\|\cdot\|_{X_{f,T}}$ and $\|\cdot\|_{X_{s,T}}$.

3.2 Homogeneous divergence constraint

We first focuss on the case $g = 0$. This subsection is devoted to the proof of the following proposition:

Proposition 3.1 *Consider $(a, b) \in H_{\sharp}^2(0, L)$ such that $\chi_{a,b}$ is a injective on $\overline{\mathcal{F}}_0$. Given $0 < T \leq T^*$, initial data $(v_0, \zeta_0, \zeta_1, \eta_0, \eta_1) \in X^0$ and a source term $(f, h, 0, \ell, m) \in S_T$. Then there exists a unique solution $(v, q, \zeta, \eta) \in X_{f,T} \times X_{s,T}$ to Problem (3.1)-(3.7) and for any $T \leq T_0$ a non decreasing function $K_{T^*} : [0, \infty) \rightarrow [0, \infty)$ such that assuming $\|(a, b)\|_{H_{\sharp}^2(0, L)} + \|(\det \nabla \chi_{a,b})^{-1}\|_{C_{\sharp}(\overline{\Omega}_0)} \leq R$, the solution (v, q, ζ, η) satisfies*

$$\|(v, q)\|_{X_{f,T}} + \|(\zeta, \eta)\|_{X_{s,T}} \leq K_{T_0}(R) (\|v_0, \zeta_0, \zeta_1, \eta_0, \eta_1\|_{X^0} + \|(f, h, 0, \ell, m)\|_{S_T}), \quad \forall T \leq T^*. \quad (3.14)$$

The proof of the above proposition is divided into three steps similar to the ones followed in [28]. In the first step, we prove the existence of strong solution for a regularized Problem of (3.1)-(3.7) by adding a dissipative term $\varepsilon \partial_{xxxxt} \zeta$ to the wave equation and $\varepsilon \partial_{xxxxt} \eta$ to the beam equation with $\varepsilon > 0$. Next, we derive, for the solution of the regularized problem, estimates independent of the regularizing parameter ε . These uniform estimates enable us to pass into the limit in the final step. Since some steps are similar to [28] we emphasize at which are the main differences. Note that we assume that $(a, b) \in H_{\sharp}^2(0, L)$ which is more than $(a, b) \in H_{\sharp}^{\frac{3}{2}+s}(0, L)$ that was sufficient for the elliptic regularity result. Requiring this extra regularity seems to be crucial to estimate pressure boundary terms appearing in the regularity estimate. These terms comes from the fact that, contrary to the Stokes steady state case studied at Section 2.5, we do not deal only with Dirichlet boundary conditions but with a coupled fluid-structure system. Note that these extra terms do not appears in the case where the longitudinal displacement is neglected ($a = \zeta = 0$).

3.2.1 Regularized problem

The regularized problem of Problem (3.1) reads as follows: for $\varepsilon > 0$, $T > 0$ and a source term $(f_1, f_2, h, 0, \ell, m) \in S_T$, find $(v_\varepsilon, q_\varepsilon, \zeta_\varepsilon, \eta_\varepsilon) \in X_{f,T} \times X_{s,T}$ such that

$$\rho_{f,a,b} \partial_t v_\varepsilon - \mu \operatorname{div} \left((A_{a,b} \nabla) v_\varepsilon + \frac{1}{\det \nabla \chi_{a,b}} B_{a,b} \nabla v_\varepsilon^T B_{a,b} - q_\varepsilon B_{a,b} \right) = f + \operatorname{div} h, \quad \text{in } \Omega_0 \times (0, T), \quad (3.15)$$

$$\operatorname{div}(B_{a,b}^T v_\varepsilon) = 0, \quad \text{in } \Omega_0 \times (0, T), \quad (3.16)$$

$$v_\varepsilon(x, 1, t) = (\partial_t \zeta_\varepsilon(x, t), \partial_t \eta_\varepsilon(x, t)), \quad \text{on } (0, L) \times (0, T), \quad (3.17)$$

$$v_\varepsilon(x, 0, t) = 0, \quad \text{on } (0, L) \times (0, T), \quad (3.18)$$

$$\rho_s \partial_{tt} \zeta_\varepsilon - \beta \partial_{xx} \zeta_\varepsilon + \varepsilon \partial_{xxxxt} \zeta_\varepsilon = -T_{a,b}[v_\varepsilon, q_\varepsilon] \cdot e_1 - h_{|y=1} e_2 \cdot e_1 + \ell, \quad \text{in } (0, L), \quad (3.19)$$

$$\rho_s \partial_{tt} \eta_\varepsilon - \gamma \partial_{xxtt} \eta_\varepsilon + \alpha \partial_{xxxx} \eta_\varepsilon + \varepsilon \partial_{xxxxt} \eta_\varepsilon = -T_{a,b}[v_\varepsilon, q_\varepsilon] \cdot e_2 - h_{|y=1} e_2 \cdot e_2 + m, \quad \text{in } (0, L). \quad (3.20)$$

As we have regularized the solution of the linear problem (3.1), one has also to regularize the initial data as well. There exists $(\zeta_0^\varepsilon, \zeta_1^\varepsilon, \eta_0^\varepsilon, \eta_1^\varepsilon) \in (H_\#^4(0, L) \times H_\#^3(0, L))^2$ such that

$$\begin{aligned} & (\zeta_0^\varepsilon, \zeta_1^\varepsilon, \eta_0^\varepsilon, \eta_1^\varepsilon) \rightarrow (\zeta_0, \zeta_1, \eta_0, \eta_1) \text{ in } X_s^0 \text{ when } \varepsilon \rightarrow 0, \\ & \varepsilon \left(\|(\zeta_0^\varepsilon, \eta_0^\varepsilon)\|_{H_\#^4(0, L)} + \|(\zeta_1^\varepsilon, \eta_1^\varepsilon)\|_{H_\#^3(0, L)} \right) \text{ is bounded independently of } \varepsilon > 0. \end{aligned}$$

We observe that $\int_0^L (-\zeta_1^\varepsilon(x) \partial_x b(x) + \eta_1^\varepsilon(x) (1 + \partial_x a(x))) dx = I_\varepsilon$, with $I_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. Thus we modify the vertical displacement so that the regularized initial beam velocities satisfy the constraint (3.8). Let us define $\eta_1^\varepsilon = \bar{\eta}_1^\varepsilon - \frac{I_\varepsilon}{L}$, we get that $\eta_1^\varepsilon \in H_\#^3(0, L)$ and $(\zeta_1^\varepsilon, \eta_1^\varepsilon)$ satisfies (3.8). Moreover, since $I_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$, we have

$$\begin{aligned} & (\zeta_0^\varepsilon, \zeta_1^\varepsilon, \eta_0^\varepsilon, \eta_1^\varepsilon) \rightarrow (\zeta_0, \zeta_1, \eta_0, \eta_1) \text{ in } X_s^0 \text{ when } \varepsilon \rightarrow 0, \\ & \varepsilon \left(\|(\zeta_0^\varepsilon, \eta_0^\varepsilon)\|_{H_\#^4(0, L)} + \|(\zeta_1^\varepsilon, \eta_1^\varepsilon)\|_{H_\#^3(0, L)} \right) \text{ is bounded independently of } \varepsilon > 0. \end{aligned}$$

Due to the compatibility condition (3.7) that should be satisfied by the initial data of the linear problem, one has to approximate the initial velocity v_0 by v_0^ε such that

$$\begin{cases} \operatorname{div}(B_{a,b}^T v_0^\varepsilon) = 0, & \text{in } \Omega_0, \\ v_0^\varepsilon|_{y=1} = (\zeta_1^\varepsilon, \eta_1^\varepsilon), \\ v_0^\varepsilon|_{y=0} = 0. \end{cases} \quad (3.21)$$

To this end, we define

$$v_0^\varepsilon = v_0 - U(0, 0, \zeta_1^\varepsilon - \zeta_1, \eta_1^\varepsilon - \eta_1), \quad (3.22)$$

where $U(0, 0, \zeta_1^\varepsilon - \zeta_1, \eta_1^\varepsilon - \eta_1) \in H_\#^2(\Omega_0)$ is the velocity solution of the periodic Stokes-like Problem (2.40)-(2.39) with $\hat{f} = 0$, $\hat{g} = 0$ and $\psi = (\zeta_1^\varepsilon - \zeta_1, \eta_1^\varepsilon - \eta_1)$. Thanks to (3.8) that is verified by both (ζ_1, η_1) and $(\zeta_1^\varepsilon, \eta_1^\varepsilon)$ the compatibility condition

$$\int_0^L \left(-(\zeta_1^\varepsilon(x) - \zeta_1(x)) \partial_x b(x) + (\eta_1^\varepsilon(x) - \eta_1(x)) (1 + \partial_x a(x)) \right) dx = 0$$

holds true. With this construction, we have $v_0^\varepsilon \in H_\#^1(\Omega_0)$ as $v_0 \in H_\#^1(\Omega_0)$ and v_0^ε satisfies (3.21). Moreover, we have

$$v_0^\varepsilon \rightarrow v_0, \text{ in } H_\#^1(\Omega_0) \text{ as } \varepsilon \rightarrow 0. \quad (3.23)$$

Variational formulation of the problem. Let us consider the following space

$$V = \{(w, \xi, \varphi) \in H_\#^1(\Omega_0) \times H_\#^2(0, L) \times H_\#^2(0, L), \operatorname{div}(B_{a,b}^T w) = 0, w|_{y=1} = (\xi, \varphi), w|_{y=0} = 0\}.$$

Note that for any $(w, \xi, \varphi) \in V$, then $\delta_{a,b}(\xi, \varphi) = 0$, where $\delta_{a,b}(\xi, \varphi)$ is defined by (3.10). Let $(w, \xi, \varphi) \in V$ and let us suppose that $(v_\varepsilon, q_\varepsilon, \zeta_\varepsilon, \eta_\varepsilon)$ is a classical solution of (3.15)-(3.20). By multiplying the fluid equation (3.15) by w ,

(3.19) by ξ and (3.20) by φ and performing integration by parts, we obtain, by using (3.16)-(3.18) and the periodic boundary conditions, that

$$\begin{aligned}
& \int_{\Omega_0} \rho_{f,a,b} \partial_t v_\varepsilon \cdot w \, dx dy + \rho_s \int_0^L \partial_{tt} \zeta_\varepsilon \xi \, dx + \rho_s \int_0^L \partial_{tt} \eta_\varepsilon \varphi \, dx + \gamma \int_0^L \partial_{xtt} \eta_\varepsilon \partial_x \varphi \, dx \\
& + \beta \int_0^L \partial_x \zeta_\varepsilon \partial_x \xi \, dx + \alpha \int_0^L \partial_{xx} \eta_\varepsilon \partial_{xx} \varphi \, dx + \int_{\Omega_0} \left(\mu (A_{a,b} \nabla) v_\varepsilon + \frac{\mu}{\det \nabla \chi_{a,b}} B_{a,b} \nabla v_\varepsilon^T B_{a,b} \right) : \nabla w \, dx dy \\
& + \varepsilon \left(\int_0^L \partial_{xxt} \zeta_\varepsilon \partial_{xx} \xi \, dx + \int_0^L \partial_{xxt} \eta_\varepsilon \partial_{xx} \varphi \, dx \right) \\
& = \int_{\Omega_0} f \cdot w \, dx dy - \int_{\Omega_0} h : \nabla w \, dx dy + \int_0^L (\ell, m) \cdot (\xi, \varphi) \, dx. \quad (3.24)
\end{aligned}$$

This leads to the following weak formulation of the regularized problem (3.15)-(3.20): For given $(f, h, 0, \ell, m) \in S_T$ and $(v_0^\varepsilon, \zeta_0^\varepsilon, \zeta_1^\varepsilon, \eta_0^\varepsilon, \eta_1^\varepsilon)$ as above, find

$$(v_\varepsilon, \zeta_\varepsilon, \eta_\varepsilon) \in L^2(0, T; H_{\#}^1(\Omega_0)) \cap L^\infty(0, T; L_{\#}^2(\Omega_0)) \times W^{1,\infty}(0, T; H_{\#}^2(0, L)) \times W^{1,\infty}(0, T; H_{\#}^2(\Omega_0)),$$

such that (3.24) holds for all $(w, \xi, \varphi) \in V$ with

$$(v_\varepsilon(0), \zeta_\varepsilon(0), \partial_t \zeta_\varepsilon(0), \eta_\varepsilon(0), \partial_t \eta_\varepsilon(0)) = (v_0^\varepsilon, \zeta_0^\varepsilon, \zeta_1^\varepsilon, \eta_0^\varepsilon, \eta_1^\varepsilon). \quad (3.25)$$

In order to obtain the existence of a solution and *a priori* estimates we make use of the uniform Korn-like inequality derived at Proposition 2.2. Let us mention that in the case where the longitudinal displacement is neglected and the beam only moves in the transversal direction, the Korn inequality is trivially satisfied since the symmetric part of the gradient of the fluid velocity has no contribution in the applied force on the structure. Here, even if $\operatorname{div} \left(\frac{1}{\det \nabla \chi_{a,b}} B_{a,b} \nabla v_\varepsilon^T B_{a,b} \right) = 0$ due the divergence free constraint, this term has a non zero contribution on the boundary when integrated by parts and thus its normal trace acts on the structure.

We use Faedo-Galerkin method to prove the existence of weak solutions of Problem (3.24)-(3.25). To this end, we approximate the space V by a space V_m by following the same lines in [14] and we prove existence and uniqueness of solutions to the approximated problem. We remark that the estimates that will be derived later on for the solution $(v_\varepsilon, \zeta_\varepsilon, \eta_\varepsilon)$ can be justified by passing over the limit over $m \rightarrow \infty$ of the estimates satisfied by the approximated solution of the approximated problem as in the approximated problem any time derivative of the solution is a valid test function.

3.2.2 Energy estimates

We take $(v_\varepsilon, \partial_t \zeta_\varepsilon, \partial_t \eta_\varepsilon)$ as test function in the weak (3.24). Thus, setting $(w, \xi, \varphi) = (v_\varepsilon, \partial_t \zeta_\varepsilon, \partial_t \eta_\varepsilon)$ in (3.24), we get

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left[\int_{\Omega_0} \rho_{f,a,b} |v_\varepsilon|^2 \, dx dy + \rho_s \int_0^L |\partial_t \zeta_\varepsilon(x)|^2 \, dx + \rho_s \int_0^L |\partial_t \eta_\varepsilon(x)|^2 \, dx + \beta \int_0^L L |\partial_x \zeta_\varepsilon(x)|^2 \, dx \right. \\
& \left. + \alpha \int_0^L |\partial_{xx} \eta_\varepsilon(x)|^2 \, dx \right] + \int_{\Omega_0} \left(\mu (A_{a,b} \nabla) v_\varepsilon + \frac{\mu}{\det \nabla \chi_{a,b}} B_{a,b} \nabla v_\varepsilon^T B_{a,b} \right) : \nabla v_\varepsilon \, dx dy \\
& + \frac{\gamma}{2} \frac{d}{dt} \left(\int_0^L |\partial_{tx} \eta_\varepsilon(x)|^2 \, dx \right) + \varepsilon \left(\int_0^L |\partial_{xxt} \zeta_\varepsilon(x)|^2 \, dx + \int_0^L |\partial_{xxt} \eta_\varepsilon(x)|^2 \, dx \right) \\
& = \int_{\Omega_0} f \cdot v_\varepsilon \, dx dy - \int_{\Omega_0} h : \nabla v_\varepsilon \, dx dy + \int_0^L (\ell, m) \cdot (\partial_t \zeta_\varepsilon, \partial_t \eta_\varepsilon) \, dx.
\end{aligned}$$

Since

$$\int_{\Omega_0} \left(\mu(A_{a,b}\nabla)v_\varepsilon + \frac{\mu}{\det \nabla \chi_{a,b}} B_{a,b} \nabla v_\varepsilon^T B_{a,b} \right) : \nabla v_\varepsilon \, dx dy = \frac{\mu}{2} \int_{\Omega_0} \frac{1}{\det \nabla \chi_{a,b}} |((B_{a,b}\nabla)v + ((B_{a,b}\nabla)v_\varepsilon)^T)|^2 \, dx dy$$

By using (2.35), Cauchy-Schwartz, Poincaré and Young's inequalities, there exists $C > 0$ such that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\int_{\Omega_0} \rho_{f,a,b} |v_\varepsilon|^2 \, dx dy + \rho_s \int_0^L |\partial_t \zeta_\varepsilon(x)|^2 \, dx + \rho_s \int_0^L |\partial_t \eta_\varepsilon(x)|^2 \, dx + \beta \int_0^L |\partial_x \zeta_\varepsilon(x)|^2 \, dx \right. \\ & \quad + \alpha \int_0^L |\partial_{xx} \eta_\varepsilon(x)|^2 \, dx + \frac{\mu \lambda(R)}{2} \int_{\Omega_0} |\nabla v_\varepsilon|^2 \, dx dy + \frac{\gamma}{2} \frac{d}{dt} \left(\int_0^L |\partial_{tx} \eta_\varepsilon(x)|^2 \, dx \right) \\ & \quad \left. + \varepsilon \left(\int_0^L |\partial_{xxt} \zeta_\varepsilon(x)|^2 \, dx + \int_0^L |\partial_{xxt} \eta_\varepsilon(x)|^2 \, dx \right) \leq \frac{C}{\lambda(R)} \left(\|f\|_{L^2_\#(\Omega_0)}^2 + \|h\|_{L^2_\#(\Omega_0)}^2 \right) \right. \\ & \quad \left. + \frac{\rho_s}{4} \left(\|\partial_t \zeta_\varepsilon\|_{L^2_\#(0,L)}^2 + \|\partial_t \eta_\varepsilon\|_{L^2_\#(0,L)}^2 \right) + \frac{1}{\rho_s} \left(\|\ell\|_{L^2_\#(0,L)}^2 + \|\partial_t m\|_{L^2_\#(0,L)}^2 \right). \end{aligned}$$

By noting that $\rho_{f,a,b} \geq \frac{\rho_f}{R}$ and integrating with respect to time on $[0, t] \subset [0, T]$, we obtain

$$\begin{aligned} & \frac{\rho_f}{2R} \|v_\varepsilon\|_{L^\infty(0,t;L^2_\#(\Omega_0))}^2 + \frac{\rho_s}{4} \|\partial_t \zeta_\varepsilon\|_{L^\infty(0,t;L^2_\#(0,L))}^2 + \frac{\rho_s}{4} \|\partial_t \eta_\varepsilon\|_{L^\infty(0,t;L^2_\#(0,L))}^2 + \frac{\beta}{2} \|\partial_x \zeta_\varepsilon\|_{L^\infty(0,t;L^2_\#(0,L))}^2 \\ & \quad + \frac{\alpha}{2} \|\partial_{xx} \eta_\varepsilon\|_{L^\infty(0,t;L^2_\#(0,L))}^2 + \frac{\mu \lambda(R)}{2} \|\nabla v_\varepsilon\|_{L^2(0,t;L^2(\Omega_0))}^2 + \frac{\gamma}{2} \|\partial_{tx} \eta_\varepsilon\|_{L^\infty(0,t;L^2_\#(0,L))}^2 \\ & \quad + \varepsilon \left(\|\partial_{xxt} \zeta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 + \|\partial_{xxt} \eta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 \right) \leq \frac{C}{\lambda(R)} \left(\|f\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 + \|h\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 \right) \\ & \quad + \frac{1}{\rho_s} \left(\|\ell\|_{L^2(0,t;L^2_\#(0,L))}^2 + \|m\|_{L^2(0,t;L^2_\#(0,L))}^2 \right) + C_0. \quad (3.26) \end{aligned}$$

where $C_0 > 0$ depends only on the initial data, and $C > 0$ independent of $T > 0$.

Thus, we obtain uniform bounds independent of ε in the following spaces:

- v_ε in $L^\infty(0, T; L^2_\#(\Omega_0)) \cap L^2(0, T; H^1_\#(\Omega_0))$;
- ζ_ε in $L^\infty(0, T; H^1_\#(0, L)) \cap W^{1,\infty}(0, T; L^2_\#(0, L))$;
- η_ε in $L^\infty(0, T; H^2_\#(0, L)) \cap W^{1,\infty}(0, T; H^1_\#(0, L))$;
- $\sqrt{\varepsilon} \zeta_\varepsilon$ and $\sqrt{\varepsilon} \eta_\varepsilon$ in $H^1(0, T; H^2_\#(0, L))$.

3.2.3 Additional estimates for $\varepsilon > 0$

Next one takes $(w, \xi, \varphi) = (\partial_t v_\varepsilon, \partial_{tt} \zeta_\varepsilon, \partial_{tt} \eta_\varepsilon)$ as a test function in the weak formulation (3.24) and performing integration by parts to the terms involving h that

$$\begin{aligned} & \int_{\Omega_0} \rho_{f,a,b} |\partial_t v_\varepsilon|^2 \, dx dy + \rho_s \int_0^L |\partial_{tt} \zeta_\varepsilon|^2 \, dx + \rho_s \int_0^L |\partial_{tt} \eta_\varepsilon|^2 \, dx + \gamma \int_0^L |\partial_{xxt} \eta_\varepsilon|^2 \, dx \\ & \quad + \frac{\varepsilon}{2} \frac{d}{dt} \int_0^L |\partial_{xxt} \zeta_\varepsilon|^2 \, dx + \frac{\varepsilon}{2} \frac{d}{dt} \int_0^L |\partial_{xxt} \eta_\varepsilon|^2 \, dx + \int_{\Omega_0} \left(\mu(A_{a,b}\nabla)v_\varepsilon + \frac{\mu}{\det \nabla \chi_{a,b}} B_{a,b} \nabla v_\varepsilon^T B_{a,b} \right) : \nabla \partial_t v_\varepsilon \, dx dy \\ & \quad + \beta \int_0^L \partial_x \zeta_\varepsilon \partial_{xxt} \zeta_\varepsilon \, dx + \alpha \int_0^L \partial_{xx} \eta_\varepsilon \partial_{xxt} \eta_\varepsilon \, dx = \int_{\Omega_0} f \cdot \partial_t v_\varepsilon \, dx dy \\ & \quad + \int_{\Omega_0} \operatorname{div} h \cdot \partial_t v_\varepsilon \, dx dy - \int_0^L (he_2)|_{y=1} \cdot (\partial_{tt} \zeta_\varepsilon, \partial_{tt} \eta_\varepsilon) \, dx + \int_0^L (\ell, m) \cdot (\partial_{tt} \zeta_\varepsilon, \partial_{tt} \eta_\varepsilon) \, dx. \end{aligned}$$

Next, integrating the above equation with respect to time on any time interval $[0, t] \subset [0, T]$ and using the following equality

$$\int_{\Omega_0} \left(\mu(A_{a,b}\nabla)v_\varepsilon + \frac{\mu}{\det \nabla \chi_{a,b}} B_{a,b} \nabla v_\varepsilon^T B_{a,b} \right) : \nabla \partial_t v_\varepsilon \, dx dy = \frac{1}{4} \frac{d}{dt} \int_{\Omega_0} \frac{1}{\det \nabla \chi_{a,b}} |((B_{a,b}\nabla)v_\varepsilon + ((B_{a,b}\nabla)v_\varepsilon)^T)|^2 \, dx dy.$$

lead to

$$\begin{aligned} & \int_0^t \int_{\Omega_0} \rho_{f,a,b} |\partial_t v_\varepsilon|^2 \, dx dy ds + \rho_s \int_0^t \int_0^L |\partial_{tt} \zeta_\varepsilon|^2 \, dx ds + \rho_s \int_0^t \int_0^L |\partial_{tt} \eta_\varepsilon|^2 v \, dx ds + \gamma \int_0^t \int_0^L |\partial_{xtt} \eta_\varepsilon(t)|^2 \, dx ds \\ & + \frac{1}{2} \left(\int_{\Omega_0} \frac{\mu}{2 \det \nabla \chi_{a,b}} |((B_{a,b}\nabla)v_\varepsilon(t) + ((B_{a,b}\nabla)v_\varepsilon(t))^T)|^2 \, dx dy + \varepsilon \int_0^L |\partial_{xxt} \zeta_\varepsilon(t)|^2 \, dx + \varepsilon \int_0^L |\partial_{xxt} \eta_\varepsilon(t)|^2 \, dx \right) \\ & \leq -\beta \int_0^t \int_0^L \partial_x \zeta_\varepsilon \partial_{xtt} \zeta_\varepsilon \, dx ds - \alpha \int_0^t \int_0^L \partial_{xx} \eta_\varepsilon \partial_{xxtt} \eta_\varepsilon \, dx ds + \int_0^t \int_{\Omega_0} f \cdot \partial_t v_\varepsilon \, dx dy ds \\ & + \int_0^t \int_{\Omega_0} \operatorname{div} h \cdot \partial_t v_\varepsilon \, dx dy ds - \int_0^t \int_0^L (he_2)|_{y=1} \cdot (\partial_{tt} \zeta_\varepsilon, \partial_{tt} \eta_\varepsilon) \, dx ds + \int_0^t \int_0^L (\ell, m) \cdot (\partial_{tt} \zeta_\varepsilon, \partial_{tt} \eta_\varepsilon) \, dx ds \\ & + \frac{1}{2} \left(\int_{\Omega_0} \frac{\mu}{2 \det \nabla \chi_{a,b}} |((B_{a,b}\nabla)v_0^\varepsilon + ((B_{a,b}\nabla)v_0^\varepsilon)^T)|^2 \, dx dy + \varepsilon \int_0^L |\partial_{xx} \zeta_1^\varepsilon|^2 \, dx + \varepsilon \int_0^L |\partial_{xx} \eta_1^\varepsilon|^2 \, dx \right). \end{aligned}$$

It follows then by using the uniform Korn's inequality (2.35) and the assumption on (a, b) that there exists $K(R)$ such that

$$\begin{aligned} & \int_0^t \int_{\Omega_0} \rho_{f,a,b} |\partial_t v_\varepsilon|^2 \, dx dy ds + \rho_s \int_0^t \int_0^L |\partial_{tt} \zeta_\varepsilon|^2 \, dx ds + \rho_s \int_0^t \int_0^L |\partial_{tt} \eta_\varepsilon|^2 v \, dx ds + \gamma \int_0^t \int_0^L |\partial_{xtt} \eta_\varepsilon(t)|^2 \, dx ds \\ & + \frac{1}{2} \left(\frac{\mu \lambda(R)}{2} \int_{\Omega_0} |\nabla v_\varepsilon(t)|^2 \, dx dy + \varepsilon \int_0^L |\partial_{xxt} \zeta_\varepsilon(t)|^2 \, dx + \varepsilon \int_0^L |\partial_{xxt} \eta_\varepsilon(t)|^2 \, dx \right) \\ & \leq -\beta \int_0^t \int_0^L \partial_x \zeta_\varepsilon \partial_{xtt} \zeta_\varepsilon \, dx ds - \alpha \int_0^t \int_0^L \partial_{xx} \eta_\varepsilon \partial_{xxtt} \eta_\varepsilon \, dx ds + \int_0^t \int_{\Omega_0} f \cdot \partial_t v_\varepsilon \, dx dy ds \\ & + \int_0^t \int_{\Omega_0} \operatorname{div} h \cdot \partial_t v_\varepsilon \, dx dy ds - \int_0^t \int_0^L (he_2)|_{y=1} \cdot (\partial_{tt} \zeta_\varepsilon, \partial_{tt} \eta_\varepsilon) \, dx ds + \int_0^t \int_0^L (\ell, m) \cdot (\partial_{tt} \zeta_\varepsilon, \partial_{tt} \eta_\varepsilon) \, dx ds \\ & + \frac{1}{2} \left(\frac{\mu K(R)}{2} \int_{\Omega_0} |\nabla v_0^\varepsilon|^2 \, dx dy + \varepsilon \int_0^L |\partial_{xx} \zeta_1^\varepsilon|^2 \, dx + \varepsilon \int_0^L |\partial_{xx} \eta_1^\varepsilon|^2 \, dx \right). \end{aligned}$$

Since $\rho_{f,a,b} \geq \frac{\rho f}{R}$, and by performing integration by parts in time in the first two terms on the right hand side of the above inequality, we further obtain

$$\begin{aligned} & \frac{\rho f}{R} \int_0^t \int_{\Omega_0} |\partial_t v_\varepsilon|^2 \, dx dy ds + \rho_s \int_0^t \int_0^L |\partial_{tt} \zeta_\varepsilon|^2 \, dx ds + \rho_s \int_0^t \int_0^L |\partial_{tt} \eta_\varepsilon|^2 \, dx ds + \gamma \int_0^t \int_0^L |\partial_{xtt} \eta_\varepsilon(t)|^2 \, dx ds \\ & + \frac{1}{2} \left(\frac{\mu \lambda(R)}{2} \int_{\Omega_0} |\nabla v_\varepsilon(t)|^2 \, dx dy + \varepsilon \int_0^L |\partial_{xxt} \zeta_\varepsilon(t)|^2 \, dx + \varepsilon \int_0^L |\partial_{xxt} \eta_\varepsilon(t)|^2 \, dx \right) \\ & \leq \beta \int_0^t \int_0^L |\partial_{xt} \zeta_\varepsilon|^2 \, dx ds + \alpha \int_0^t \int_0^L |\partial_{xxt} \eta_\varepsilon|^2 \, dx ds + \int_0^t \int_{\Omega_0} f \cdot \partial_t v_\varepsilon \, dx dy ds \\ & + \int_0^t \int_{\Omega_0} \operatorname{div} h \cdot \partial_t v_\varepsilon \, dx dy ds - \int_0^t \int_0^L (he_2)|_{y=1} \cdot (\partial_{tt} \zeta_\varepsilon, \partial_{tt} \eta_\varepsilon) \, dx ds + \int_0^t \int_0^L (\ell, m) \cdot (\partial_{tt} \zeta_\varepsilon, \partial_{tt} \eta_\varepsilon) \, dx ds \\ & + \frac{1}{2} \left(\frac{\mu K(R)}{2} \int_{\Omega_0} |\nabla v_0^\varepsilon|^2 \, dx dy + \varepsilon \int_0^L |\partial_{xx} \zeta_1^\varepsilon|^2 \, dx + \varepsilon \int_0^L |\partial_{xx} \eta_1^\varepsilon|^2 \, dx \right) \\ & - \beta \int_0^L \partial_x \zeta_\varepsilon(t) \partial_{xt} \zeta_\varepsilon(t) \, dx - \alpha \int_0^L \partial_{xx} \eta_\varepsilon(t) \partial_{xxt} \eta_\varepsilon(t) \, dx + \beta \int_0^L \partial_x \zeta_0^\varepsilon \partial_x \zeta_1^\varepsilon \, dx + \alpha \int_0^L \partial_{xx} \eta_0^\varepsilon \partial_{xx} \eta_1^\varepsilon \, dx. \end{aligned}$$

After integration by parts of the last two terms in the above inequality with respect to x , we obtain

$$\begin{aligned}
& \frac{\rho_f}{R} \int_0^t \int_{\Omega_0} |\partial_t v_\varepsilon|^2 dx dy ds + \rho_s \int_0^t \int_0^L |\partial_{tt} \zeta_\varepsilon|^2 dx ds + \rho_s \int_0^t \int_0^L |\partial_{tt} \eta_\varepsilon|^2 dx ds + \gamma \int_0^t \int_0^L |\partial_{xtt} \eta_\varepsilon(t)|^2 dx ds \\
& \quad + \frac{1}{2} \left(\frac{\mu \lambda(R)}{2} \int_{\Omega_0} |\nabla v_\varepsilon(t)|^2 dx dy + \varepsilon \int_0^L |\partial_{xxt} \zeta_\varepsilon(t)|^2 dx + \varepsilon \int_0^L |\partial_{xxt} \eta_\varepsilon(t)|^2 dx \right) \\
& \leq \beta \int_0^t \int_0^L |\partial_{xt} \zeta_\varepsilon|^2 dx ds + \alpha \int_0^t \int_0^L |\partial_{xxt} \eta_\varepsilon|^2 dx ds + \int_0^t \int_{\Omega_0} f \cdot \partial_t v_\varepsilon dx dy ds \\
& + \int_0^t \int_{\Omega_0} \operatorname{div} h \cdot \partial_t v_\varepsilon dx dy ds - \int_0^t \int_0^L (h e_2)|_{y=1} \cdot (\partial_{tt} \zeta_\varepsilon, \partial_{tt} \eta_\varepsilon) dx ds + \int_0^t \int_0^L (\ell, m) \cdot (\partial_{tt} \zeta_\varepsilon, \partial_{tt} \eta_\varepsilon) dx ds \\
& \quad + \frac{1}{2} \left(\mu K(R) \int_{\Omega_0} |\nabla v_0^\varepsilon|^2 dx dy + \varepsilon \int_0^L |\partial_{xx} \zeta_1^\varepsilon|^2 dx + \varepsilon \int_0^L |\partial_{xx} \eta_1^\varepsilon|^2 dx \right) \\
& - \beta \int_0^L \partial_x \zeta_\varepsilon(t) \partial_{xt} \zeta_\varepsilon(t) dx - \alpha \int_0^L \partial_{xx} \eta_\varepsilon(t) \partial_{xxt} \eta_\varepsilon(t) dx - \beta \int_0^L \partial_{xx} \zeta_0^\varepsilon \zeta_1^\varepsilon dx - \alpha \int_0^L \partial_{xxx} \eta_0^\varepsilon \partial_x \eta_1^\varepsilon dx.
\end{aligned}$$

Thanks to the way we have built $(v_0^\varepsilon, \zeta_0^\varepsilon, \zeta_1^\varepsilon, \eta_0^\varepsilon, \eta_1^\varepsilon)$, by Cauchy Schwartz, Young's and trace inequalities there exists C_0 which depends on the initial data and R such that

$$\begin{aligned}
& \frac{\rho_f}{2R} \int_0^t \int_{\Omega_0} |\partial_t v_\varepsilon|^2 dx dy ds + \frac{\rho_s}{2} \int_0^t \int_0^L |\partial_{tt} \zeta_\varepsilon|^2 dx ds + \frac{\rho_s}{2} \int_0^t \int_0^L |\partial_{tt} \eta_\varepsilon|^2 dx ds \\
& \quad + \gamma \int_0^t \int_0^L |\partial_{xtt} \eta_\varepsilon(t)|^2 dx ds + \frac{\mu \lambda(R)}{4} \int_{\Omega_0} |\nabla v_\varepsilon(t)|^2 dx dy \\
& \quad + \frac{\varepsilon}{2} \int_0^L |\partial_{xxt} \zeta_\varepsilon(t)|^2 dx + \frac{\varepsilon}{4} \int_0^L |\partial_{xxt} \eta_\varepsilon(t)|^2 dx \\
& \leq \beta \int_0^t \int_0^L |\partial_{xt} \zeta_\varepsilon|^2 dx ds + \alpha \int_0^t \int_0^L |\partial_{xxt} \eta_\varepsilon|^2 dx ds + \frac{\beta^2}{\varepsilon} \int_0^L |\partial_x \zeta_\varepsilon(t)|^2 dx \\
& + \frac{\varepsilon}{4} \int_0^L |\partial_{xt} \zeta_\varepsilon(t)|^2 dx + \frac{\alpha^2}{\varepsilon} \int_0^L |\partial_{xx} \eta_\varepsilon(t)|^2 dx + C(R) \left(\|\operatorname{div} h\|_{L^2(0,t;L^2_\sharp(\Omega_0))}^2 + \|f\|_{L^2(0,t;L^2_\sharp(\Omega_0))}^2 \right) \\
& \quad + C(\rho_s, \Omega_0) \left(\|h\|_{L^2(0,t;H^1_\sharp(\Omega_0))}^2 + \|\ell\|_{L^2(0,t;L^2_\sharp(0,L))}^2 + \|m\|_{L^2(0,t;L^2_\sharp(0,L))}^2 \right) + C_0.
\end{aligned}$$

Thanks to the energy estimates (3.26), we obtain

$$\begin{aligned}
& \frac{\rho_f}{2R} \|\partial_t v_\varepsilon\|_{L^2(0,t;L^2_\sharp(\Omega_0))}^2 + \frac{\rho_s}{2} \|\partial_{tt} \zeta_\varepsilon\|_{L^2(0,t;L^2_\sharp(\Omega_0))}^2 + \frac{\rho_s}{2} \|\partial_{tt} \eta_\varepsilon\|_{L^2(0,t;L^2_\sharp(\Omega_0))}^2 \\
& \quad + \gamma \|\partial_{xtt} \eta_\varepsilon(t)\|_{L^2(0,t;L^2_\sharp(\Omega_0))}^2 + \frac{\mu \lambda(R)}{4} \|\nabla v_\varepsilon(t)\|_{L^2_\sharp(\Omega_0)}^2 \\
& + \frac{\varepsilon}{4} \|\partial_{xxt} \zeta_\varepsilon(t)\|_{L^2_\sharp(0,L)}^2 + \frac{\varepsilon}{4} \|\partial_{xxt} \eta_\varepsilon(t)\|_{L^2_\sharp(0,L)}^2 \leq C(R) \left(\|\operatorname{div} h\|_{L^2(0,t;L^2_\sharp(\Omega_0))}^2 + \|f\|_{L^2(0,t;L^2_\sharp(\Omega_0))}^2 \right) \\
& \quad + C(\rho_s, \Omega_0) \left(\|h\|_{L^2(0,t;H^1_\sharp(\Omega_0))}^2 + \|\ell\|_{L^2(0,t;L^2_\sharp(0,L))}^2 + \|m\|_{L^2(0,t;L^2_\sharp(0,L))}^2 \right) + C_0 + C_\varepsilon, \quad (3.27)
\end{aligned}$$

where C_ε depends on initial data, source terms and $1/\varepsilon$.

Consequently by applying Gronwall Lemma, we obtain bounds on the solution in the following spaces:

- v_ε in $H^1(0, T; L^2_\sharp(\Omega_0)) \cap L^\infty(0, T; H^1_\sharp(\Omega_0))$;
- ζ_ε in $H^2(0, T; L^2_\sharp(0, L))$, η_ε in $H^2(0, T; H^1_\sharp(0, L))$;
- $\sqrt{\varepsilon} \zeta_\varepsilon$ in $W^{1,\infty}(0, T; H^2_\sharp(0, L))$, $\sqrt{\varepsilon} \eta_\varepsilon$ in $W^{1,\infty}(0, T; H^2_\sharp(0, L))$;

These estimates furnish uniform bounds satisfied by approximate solutions that are sufficient to pass to the limit in the Galerkin approximated problems and obtain a solution of the weak formulation. We emphasize that (3.26) and (3.27) can be then obtained by passing to the limit in a Galerkin approximation. Note that the right hand side of (3.26) is independent of ε , whereas the constant C_ε appearing in (3.27) depends on $1/\varepsilon$. Thus, the energy estimate (3.26) is still valid when passing to the limit as $\varepsilon \rightarrow 0$ whereas the bounds in (3.27) blow up when passing to the limit as $\varepsilon \rightarrow 0$. However, we will see later that we can control the left hand side of (3.27) independently of ε thanks to additional estimates we derive in the last part of this subsection.

Further regularity of the regularized solution With the derived estimates we can further recover elliptic regularities and the strong formulations satisfied by the solution of the regularized problem. The key point here is that the equations of the longitudinal displacement ζ and the transversal one η are coupled through the constraint $\delta_{a,b}(\partial_t \zeta, \partial_t \eta) = 0$, and that further fixes the pressure average. Indeed we just obtained that there exists a triplet $(v_\varepsilon, \zeta_\varepsilon, \eta_\varepsilon)$ such that

$$v_\varepsilon \in H^1(0, T; L^2_{\sharp}(\Omega_0)) \cap L^\infty(0, T; H^1_{\sharp}(\Omega_0)), \quad (\zeta_\varepsilon, \eta_\varepsilon) \in (H^2(0, T; L^2_{\sharp}(0, L)) \cap W^{1,\infty}(0, T; H^2_{\sharp}(0, L)))^2;$$

satisfying (3.24) for any test function in V and the initial conditions (3.25).

Then by considering $(\xi, \varphi) = (0, 0)$ in (3.24), we obtain for any $w \in H^1_{\sharp}(\Omega_0)$ such that $\operatorname{div}(B_{a,b}^T w) = 0$ and $w|_{y=1} = w|_{y=0} = 0$

$$\begin{aligned} \int_{\Omega_0} \rho_{f,a,b} \partial_t v_\varepsilon \cdot w \, dx dy + \int_{\Omega_0} \left(\mu(A_{a,b} \nabla) v_\varepsilon + \frac{\mu}{\det \nabla \chi_{a,b}} B_{a,b} \nabla v_\varepsilon^T B_{a,b} \right) : \nabla w \, dx dy \\ = \int_{\Omega_0} f \cdot w \, dx dy - \int_{\Omega_0} h : \nabla w \, dx dy. \end{aligned}$$

Since $B_{a,b}$ is invertible and belong to $H_{\sharp}^{1+s}(\Omega_0)$ which a multiplier space of $H_{\sharp}^1(\Omega_0)$ the standard de Rham Lemma can easily be adapted to the operators $(B_{a,b} \nabla) \cdot$ and $\operatorname{div}(B_{a,b}^T \cdot)$ (see for instance [23], [25]), to get that there exists $s_{0,\varepsilon} \in L^2_{\sharp,0}(\Omega_0)$ such that

$$(B_{a,b} \nabla) s_{0,\varepsilon} = -\rho_{f,a,b} \partial_t v_\varepsilon + \operatorname{div} \left(\mu(A_{a,b} \nabla) v_\varepsilon + \frac{\mu}{\det \nabla \chi_{a,b}} B_{a,b} \nabla v_\varepsilon^T B_{a,b} \right) + f + \operatorname{div} h. \quad (3.28)$$

As f and $\operatorname{div} h$ belong to $L^2(0, T; L^2_{\sharp}(\Omega_0))$ and by using the fact that $\partial_t v_\varepsilon \in L^2(0, T; L^2_{\sharp}(\Omega_0))$, $(\partial_t \zeta_\varepsilon, \partial_t \eta_\varepsilon) \in L^2(0, T; H^2_{\sharp}(\Omega_0))$, Proposition 2.3 implies

$$(v_\varepsilon, s_{0,\varepsilon}) \in L^2(0, T; H^2_{\sharp}(\Omega_0)) \times L^2(0, T; H^1_{\sharp}(\Omega_0) \cap L^2_{\sharp,0}(\Omega_0))$$

and consequently $(v_\varepsilon, s_{0,\varepsilon})$ satisfies (3.15) a.e on $(0, T) \times \Omega_0$. Hence, $(v_\varepsilon, s_{0,\varepsilon})$ satisfies (3.15)-(3.18) with periodic boundary conditions in x .

Next we recover the regularized beam equations. For any $(\xi, \varphi) \in H^2_{\sharp}(0, L)$, such that $\delta_{a,b}(\xi, \varphi) = 0$, where $\delta_{a,b}(\xi, \varphi)$ is defined by (3.10), we construct an admissible fluid test function $w = U(0, 0, \xi, \varphi) \in H^2_{\sharp}(\Omega_0)$ thanks again to Proposition 2.3. Thus, using (w, ξ, φ) as a test function in (3.24), we obtain, after integration by parts in

space,

$$\begin{aligned}
& \int_{\Omega_0} (\rho_{f,a,b} \partial_t v_\varepsilon - \operatorname{div} \left(\mu(A_{a,b} \nabla) v_\varepsilon + \frac{\mu}{\det \nabla \chi_{a,b}} B_{a,b} \nabla v_\varepsilon^T B_{a,b} - \operatorname{div} h - f \right)) \cdot w \, dx dy \\
& \quad + \int_0^L \left(\mu(A_{a,b} \nabla) v_\varepsilon + \frac{\mu}{\det \nabla \chi_{a,b}} B_{a,b} \nabla v_\varepsilon^T B_{a,b} \right) \Big|_{y=1} e_2 \cdot (\xi e_1 + \varphi e_2) dx \\
& \quad + \int_0^L (\rho_s \partial_{tt} \zeta_\varepsilon - \beta \partial_{xx} \zeta_\varepsilon + h|_{y=1} e_2 \cdot e_1 - \ell) \xi \, dx \\
& \quad + \int_0^L (\rho_s \partial_{tt} \eta_\varepsilon + h|_{y=1} e_2 \cdot e_2 - m) \varphi \, dx + \gamma \int_0^L \partial_{ttx} \eta_\varepsilon \partial_x \varphi \, dx + \alpha \int_0^L \partial_{xx} \eta_\varepsilon \partial_{xx} \varphi \, dx \\
& \quad + \varepsilon \int_0^L \partial_{xxt} \zeta_\varepsilon(x) \partial_{xx} \xi \, dx + \varepsilon \int_0^L \partial_{xxt} \eta_\varepsilon \partial_{xx} \varphi \, dx = 0.
\end{aligned}$$

This implies that in view of (3.28) that

$$\begin{aligned}
& \int_{\Omega_0} -(B_{a,b} \nabla) s_{0,\varepsilon} \cdot w \, dx dy + \int_0^L \left(\mu(A_{a,b} \nabla) v_\varepsilon + \frac{\mu}{\det \nabla \chi_{a,b}} B_{a,b} \nabla v_\varepsilon^T B_{a,b} \right) \Big|_{y=1} e_2 \cdot (\xi e_1 + \varphi e_2) dx \\
& \quad + \int_0^L (\rho_s \partial_{tt} \zeta_\varepsilon - \beta \partial_{xx} \zeta_\varepsilon + h|_{y=1} e_2 \cdot e_1 - \ell) \xi \, dx \\
& \quad + \int_0^L (\rho_s \partial_{tt} \eta_\varepsilon + h|_{y=1} e_2 \cdot e_2 - m) \varphi \, dx + \gamma \int_0^L \partial_{ttx} \eta_\varepsilon \partial_x \varphi \, dx + \alpha \int_0^L \partial_{xx} \eta_\varepsilon \partial_{xx} \varphi \, dx \\
& \quad + \varepsilon \int_0^L \partial_{xxt} \zeta_\varepsilon(x) \partial_{xx} \xi \, dx + \varepsilon \int_0^L \partial_{xxt} \eta_\varepsilon \partial_{xx} \varphi \, dx = 0.
\end{aligned}$$

Remembering that $\operatorname{div}(B_{a,b}^T w) = 0$, an integration by parts on the pressure term yields

$$\begin{aligned}
& \int_0^L (\rho_s \partial_{tt} \zeta_\varepsilon - \beta \partial_{xx} \zeta_\varepsilon + h|_{y=1} e_2 \cdot e_1 + T_{a,b}[v_\varepsilon, s_{0,\varepsilon}] \cdot e_1 - \ell) \xi \, dx + \varepsilon \int_0^L \partial_{xxt} \zeta_\varepsilon(x) \partial_{xx} \xi \, dx \\
& + \int_0^L (\rho_s \partial_{tt} \eta_\varepsilon + h|_{y=1} e_2 \cdot e_2 + T_{a,b}[v_\varepsilon, s_{0,\varepsilon}] \cdot e_2 - m) \varphi \, dx + \gamma \int_0^L \partial_{ttx} \eta_\varepsilon \partial_x \varphi \, dx + \alpha \int_0^L \partial_{xx} \eta_\varepsilon \partial_{xx} \varphi \, dx + \varepsilon \int_0^L \partial_{xxt} \eta_\varepsilon \partial_{xx} \varphi \, dx = 0.
\end{aligned}$$

This identity is valid for any test function $(\xi, \varphi) \in H_{\sharp}^2(0, L)$, such that $\delta_{a,b}(\xi, \varphi) = 0$. Thus for any $(\xi, \varphi) \in H_{\sharp}^2(0, L)$, such that $\delta_{a,b}(\xi, \varphi) = 0$, we end up, a. e. in t , with an identity that writes

$$\langle (F_1^\varepsilon, F_2^\varepsilon)(t), (\xi, \varphi) \rangle_{H_{\sharp}^{-2}(0,L), H_{\sharp}^2(0,L)} = 0.$$

For any $(\xi, \bar{\varphi}) \in H_{\sharp}^2(0, L)$ we observe that $(\xi, \varphi) = (\xi, \bar{\varphi} - \frac{\delta_{a,b}(\xi, \bar{\varphi})}{L})$ satisfies the constraint $\delta_{a,b}(\xi, \varphi) = 0$ and we can easily deduce that there exists a unique $k_\varepsilon : t \mapsto k_\varepsilon(t)$ such that, a. e. in t

$$\langle (F_1^\varepsilon, F_2^\varepsilon)(t), (\xi, \bar{\varphi}) \rangle_{H_{\sharp}^{-2}(0,L), H_{\sharp}^2(0,L)} = k_\varepsilon(t) \delta_{a,b}(\xi, \bar{\varphi}), \quad \forall (\xi, \bar{\varphi}) \in H_{\sharp}^2(0, L).$$

In a straightforward way $k_\varepsilon(t) = \frac{1}{L} \int_0^L F_2^\varepsilon(x, t) dx$. The fluid pressure q_ε is then given by $q_\varepsilon(x, y, t) = s_{0,\varepsilon}(x, y, t) + k_\varepsilon(t)$ with $s_{0,\varepsilon} \in L^2(0, T; H_{\sharp}^1(\Omega_0) \cap L_{\sharp,0}^2(\Omega_0))$ and

$$k_\varepsilon(t) = \frac{1}{L} \int_0^L (\rho_s \partial_{tt} \eta_\varepsilon + h|_{y=1} e_2 \cdot e_2 + T_{a,b}[v_\varepsilon, s_{0,\varepsilon}] \cdot e_2 - m) \, dx. \quad (3.29)$$

Hence, we have $k_\varepsilon \in L^2(0, T)$ and $q_\varepsilon \in L^2(0, T; H_{\sharp}^1(\Omega_0))$. This gives a meaning to $T_{a,b}[v_\varepsilon, q_\varepsilon]$ in $L^2(0, T; H_{\sharp}^{\frac{1}{2}}(0, L))$

and we obtain the following weak formulation for any test function $(\xi, \varphi) \in H_{\sharp}^2(0, L)$

$$\begin{aligned} & \int_0^L (\rho_s \partial_{tt} \zeta_\varepsilon - \beta \partial_{xx} \zeta_\varepsilon + h_{|y=1} e_2 \cdot e_1 + T_{a,b}[v_\varepsilon, q_\varepsilon] \cdot e_1 - \ell) \xi \, dx + \varepsilon \int_0^L \partial_{xxt} \zeta_\varepsilon(x) \partial_{xx} \xi \, dx \\ & + \int_0^L (\rho_s \partial_{tt} \eta_\varepsilon + h_{|y=1} e_2 \cdot e_2 + T_{a,b}[v_\varepsilon, q_\varepsilon] \cdot e_2 - m) \varphi \, dx + \gamma \int_0^L \partial_{ttx} \eta_\varepsilon \partial_x \varphi \, dx + \alpha \int_0^L \partial_{xx} \eta_\varepsilon \partial_{xx} \varphi \, dx + \varepsilon \int_0^L \partial_{xxt} \eta_\varepsilon \partial_{xx} \varphi \, dx = 0. \end{aligned} \quad (3.30)$$

Thus we now can decouple the longitudinal and transversal equations of the beam. Finally by taking as test functions the projection of $\partial_{xxxxt} \zeta_\varepsilon$ and $\partial_{xxxxt} \eta_\varepsilon$ on the first Fourier modes on $(0, L)$, we obtain an identity which implies that $\zeta_\varepsilon \in W^{1,\infty}(0, T; H_{\sharp}^2(0, L)) \cap H^1(0, T; H_{\sharp}^4(0, L))$ and $\eta_\varepsilon \in W^{1,\infty}(0, T; H_{\sharp}^3(0, L)) \cap H^1(0, T; H_{\sharp}^4(0, L))$ (by letting the number of Fourier modes tends to infinity). Moreover we already know that $\zeta_\varepsilon \in H^2(0, T; L_{\sharp}^2(0, L))$ and $\eta_\varepsilon \in H^2(0, T; H_{\sharp}^1(0, L))$. It follows then, in view of (3.30), that equation (3.19) is satisfied *a.e* and (3.20) in $L^2(0, T; H_{\sharp}^{-1}(\Omega_0))$. From the regularities derived on η_ε we further obtain that $\partial_{tt} \eta_\varepsilon \in L^2(0, T; H_{\sharp}^2(0, L))$ and consequently equation (3.20) is satisfied *a.e*.

Remark 3.1 *Note that, since $\delta_{ab}(\partial_t \zeta_\varepsilon, \partial_t \eta_\varepsilon) = 0$, by multiplying (3.19) by $-\partial_x b$ and (3.20) by $1 + \partial_x a$, k_ε is also given by*

$$\begin{aligned} k_\varepsilon(t) &= \frac{1}{\|(-\partial_x b, 1 + \partial_x a)\|_{L^2(0, L)}^2} \left(\beta \int_0^L \partial_x \zeta_\varepsilon \partial_{xx} b \, dx - \alpha \int_0^L \partial_{xxx} \eta_\varepsilon \partial_{xx} a \, dx + \gamma \int_0^L \partial_{xxt} \eta_\varepsilon \partial_{xx} a \, dx \right. \\ & \left. + \int_0^L T[v_\varepsilon, s_{0,\varepsilon}] \cdot (-\partial_x b e_1 + (1 + \partial_x a) e_2) \, dx + \int_0^L h_{|y=1} e_2 \cdot (-\partial_x b e_1 + (1 + \partial_x a) e_2) \, dx - \int_0^L (\ell, m) \cdot (-\partial_x b e_1 + (1 + \partial_x a) e_2) \, dx \right). \end{aligned} \quad (3.31)$$

3.2.4 Estimates independent of $\varepsilon > 0$

Except for the energy estimate (3.26), the already obtained regularity estimates depend on the regularization parameter ε . Thus in this subsection we derive regularity estimates independent of ε . We multiply (3.15) by $\partial_t v_\varepsilon$ and integrate the resulting equality over $\Omega_0 \times (0, t)$. It follows that

$$\begin{aligned} & \int_0^t \int_{\Omega_0} \rho_{f,a,b} |\partial_t v_\varepsilon|^2 \, dx dy ds - \int_0^t \int_{\Omega_0} \operatorname{div} \left(\mu(A_{a,b} \nabla) v_\varepsilon + \frac{\mu}{\det \nabla \chi_{a,b}} B_{a,b} \nabla v_\varepsilon^T B_{a,b} - q_\varepsilon B_{a,b} \right) \cdot \partial_t v_\varepsilon \, dx dy ds \\ & = \int_0^t \int_{\Omega_0} f \cdot \partial_t v_\varepsilon \, dx dy ds + \int_0^t \int_{\Omega_0} \operatorname{div} h \cdot \partial_t v_\varepsilon \, dx dy ds. \end{aligned} \quad (3.32)$$

To integrate by parts the second term in the left hand side of the above equality, we need the following lemma as $\partial_t v_\varepsilon \in L^2(0, T; L_{\sharp}^2(\Omega_0))$ only and its trace is not well defined on the fluid-structure interface.

Lemma 3.1 *For any $w \in L^2(0, T; H_{\sharp}^2(\Omega_0)) \cap H^1(0, T; L_{\sharp}^2(\Omega_0))$ such that $\operatorname{div}(B_{a,b}^T w) = 0$ on Ω_0 , $w(x, 1, t) = (\partial_t \zeta(x, t), \partial_t \eta(x, t))$ for some $(\zeta, \eta) \in H^2(0, T; L_{\sharp}^2(0, L))$ and $w_{|y=0} = 0$ and for any $q \in L^2(0, T; H_{\sharp}^1(\Omega_0))$, we have*

$$\begin{aligned} & - \int_0^t \int_{\Omega_0} \operatorname{div} \left(\mu(A_{a,b} \nabla) w + \frac{\mu}{\det \nabla \chi_{a,b}} B_{a,b} \nabla w^T B_{a,b} \right) \cdot \partial_t w \, dx dy ds + \int_0^t \int_{\Omega_0} (B_{a,b} \nabla) q \cdot \partial_t w \, dx dy ds \\ & = \frac{1}{4} \int_{\Omega_0} \frac{\mu}{\det \nabla \chi_{a,b}} |((B_{a,b} \nabla) v_\varepsilon(t) + ((B_{a,b} \nabla) v_\varepsilon(t))^T)|^2 \, dx dy - \frac{1}{4} \int_{\Omega_0} \frac{\mu}{\det \nabla \chi_{a,b}} |((B_{a,b} \nabla) v_0^\varepsilon + ((B_{a,b} \nabla) v_0^\varepsilon)^T)|^2 \, dx dy \\ & \quad - \int_0^t \int_0^L T_{a,b}[w, q] \cdot (\partial_{tt} \zeta e_1 + \partial_{tt} \eta e_2) \, dx ds. \end{aligned} \quad (3.33)$$

The proof of this lemma is done by an approximation process in time, the equality is first proved for smooth functions and obtained by passing to the limit.

By using the above Lemma, (3.32) becomes

$$\begin{aligned} & \int_0^t \int_{\Omega_0} \rho_{f,a,b} |\partial_t v_\varepsilon|^2 dx dy ds + \frac{\mu}{4} \int_{\Omega_0} \frac{1}{\det \nabla \chi_{a,b}} |((B_{a,b} \nabla) v_\varepsilon(t) + ((B_{a,b} \nabla) v_\varepsilon(t))^T)|^2 dx dy ds \\ &= \int_0^t \int_{\Omega_0} f \cdot \partial_t v_\varepsilon dx dy ds + \int_0^t \int_{\Omega_0} \operatorname{div} h \cdot \partial_t v_\varepsilon dx dy ds \\ &+ \frac{\mu}{4} \int_{\Omega_0} \frac{1}{\det \nabla \chi_{a,b}} |((B_{a,b} \nabla) v_0^\varepsilon + ((B_{a,b} \nabla) v_0^\varepsilon)^T)|^2 dx dy + \int_0^t \int_0^L T[v_\varepsilon, q_\varepsilon] \cdot (\partial_{tt} \zeta e_1 + \partial_{tt} \eta e_2) dx ds. \end{aligned} \quad (3.34)$$

Next we multiply (3.19) and (3.20) respectively by $\partial_{tt} \zeta_\varepsilon$ and $\partial_{tt} \eta_\varepsilon$ which belong respectively to $L^2(0, T; L_\#^2(0, L))$ and $L^2(0, T; H_\#^1(0, L))$ then we integrate the resulting equations over $(0, t) \times (0, L)$. This yields to:

$$\begin{aligned} & \rho_s \int_0^t \int_0^L |\partial_{tt} \zeta_\varepsilon|^2 dx ds - \beta \int_0^t \int_0^L \partial_{xx} \zeta_\varepsilon \partial_{tt} \zeta_\varepsilon dx ds + \varepsilon \int_0^t \int_0^L \partial_{xxxxt} \zeta_\varepsilon \partial_{tt} \zeta_\varepsilon dx ds \\ &= - \int_0^t \int_0^L T_{a,b}[v_\varepsilon, q_\varepsilon] \cdot e_1 \partial_{tt} \zeta_\varepsilon dx ds - \int_0^t \int_0^L h_{|y=1} e_2 \cdot e_1 \partial_{tt} \zeta_\varepsilon dx ds + \int_0^t \int_0^L \ell \partial_{tt} \zeta_\varepsilon dx ds, \end{aligned} \quad (3.35)$$

$$\begin{aligned} & \rho_s \int_0^t \int_0^L |\partial_{tt} \eta_\varepsilon|^2 dx ds + \gamma \int_0^t \int_0^L |\partial_{xtt} \eta_\varepsilon|^2 dx ds + \alpha \int_0^t \int_0^L \partial_{xxxx} \eta_\varepsilon \partial_{tt} \eta_\varepsilon dx ds + \varepsilon \int_0^t \int_0^L \partial_{xxxxt} \eta_\varepsilon \partial_{tt} \eta_\varepsilon dx ds \\ &= - \int_0^t \int_0^L T_{a,b}[v_\varepsilon, q_\varepsilon] \cdot e_2 \partial_{tt} \eta_\varepsilon dx ds - \int_0^t \int_0^L h_{|y=1} e_2 \cdot e_2 \partial_{tt} \eta_\varepsilon dx ds + \int_0^t \int_0^L m \partial_{tt} \eta_\varepsilon dx ds. \end{aligned} \quad (3.36)$$

Since $\partial_{tt} \zeta_\varepsilon$ and $\partial_{tt} \eta_\varepsilon$ are not regular enough in space (only in $L^2(0, T; L_\#^2(0, L))$ and $L^2(0, T; H_\#^1(0, L))$ respectively), so integration by parts in space can not be performed directly. However, we can show by using a regularization argument that a.e. in t , the following equalities hold true

$$\begin{aligned} & \beta \int_0^t \int_0^L \partial_{xx} \zeta_\varepsilon \partial_{tt} \zeta_\varepsilon dx ds = \beta \int_0^t \int_0^L |\partial_{xt} \zeta_\varepsilon|^2 dx ds + \beta \int_0^L \partial_{xx} \zeta_\varepsilon(t) \partial_t \zeta_\varepsilon(t) dx - \beta \int_0^L \partial_{xx} \zeta_0^\varepsilon \zeta_1^\varepsilon dx, \\ & \varepsilon \int_0^t \int_0^L \partial_{xxxxt} \zeta_\varepsilon \partial_{tt} \zeta_\varepsilon dx ds = \frac{\varepsilon}{2} \int_0^L |\partial_{xxt} \zeta_\varepsilon(t)|^2 dx - \frac{\varepsilon}{2} \int_0^L |\partial_{xx} \zeta_1^\varepsilon|^2 dx, \\ & \alpha \int_0^t \int_0^L \partial_{xxxx} \eta_\varepsilon \partial_{tt} \eta_\varepsilon dx ds = -\alpha \int_0^t \int_0^L |\partial_{txx} \eta_\varepsilon|^2 dx ds - \alpha \int_0^L \partial_{xxx} \eta_\varepsilon(t) \partial_{tx} \eta_\varepsilon(t) dx + \alpha \int_0^L \partial_{xxx} \eta_0^\varepsilon \partial_x \eta_1^\varepsilon dx, \\ & \varepsilon \int_0^t \int_0^L \partial_{xxxxt} \eta_\varepsilon \partial_{tt} \eta_\varepsilon dx ds = \frac{\varepsilon}{2} \int_0^L |\partial_{xxt} \eta_\varepsilon(t)|^2 dx - \frac{\varepsilon}{2} \int_0^L |\partial_{xx} \eta_1^\varepsilon|^2 dx. \end{aligned}$$

By combining the above equalities with (3.35) and (3.36) leads to

$$\begin{aligned} & \rho_s \int_0^t \int_0^L |\partial_{tt} \zeta_\varepsilon|^2 dx ds + \frac{\varepsilon}{2} \int_0^L |\partial_{xxt} \zeta_\varepsilon(t)|^2 dx = \beta \int_0^t \int_0^L |\partial_{xt} \zeta_\varepsilon|^2 dx ds + \beta \int_0^L \partial_{xx} \zeta_\varepsilon(t) \partial_t \zeta_\varepsilon(t) dx - \beta \int_0^L \partial_{xx} \zeta_0^\varepsilon \zeta_1^\varepsilon dx \\ & - \int_0^t \int_0^L T_{a,b}[v_\varepsilon, q_\varepsilon] \cdot e_1 \partial_{tt} \zeta_\varepsilon dx ds - \int_0^t \int_0^L h_{|y=1} e_2 \cdot e_1 \partial_{tt} \zeta_\varepsilon dx ds + \int_0^t \int_0^L \ell \partial_{tt} \zeta_\varepsilon dx ds + \frac{\varepsilon}{2} \int_0^L |\partial_{xx} \zeta_1^\varepsilon|^2 dx, \end{aligned} \quad (3.37)$$

and for the vertical displacement

$$\begin{aligned}
& \rho_s \int_0^t \int_0^L |\partial_{tt}\eta_\varepsilon|^2 dx ds + \gamma \int_0^t \int_0^L |\partial_{xtt}\eta_\varepsilon|^2 dx ds + \frac{\varepsilon}{2} \int_0^L |\partial_{xxt}\eta_\varepsilon(t)|^2 dx \\
& = \alpha \int_0^t \int_0^L |\partial_{txx}\eta_\varepsilon|^2 dx ds + \alpha \int_0^L \partial_{xxx}\eta_\varepsilon(t) \partial_{tx}\eta_\varepsilon(t) dx - \alpha \int_0^L \partial_{xxx}\eta_0^\varepsilon \partial_x \eta_1^\varepsilon dx \\
& - \int_0^t \int_0^L T_{a,b}[v_\varepsilon, q_\varepsilon] \cdot e_2 \partial_{tt}\eta_\varepsilon dx ds - \int_0^t \int_0^L h_{|y=1} e_2 \cdot e_2 \partial_{tt}\eta_\varepsilon dx ds + \int_0^t \int_0^L m \partial_{tt}\eta_\varepsilon dx ds + \frac{\varepsilon}{2} \int_0^L |\partial_{xx}\eta_1^\varepsilon|^2. \quad (3.38)
\end{aligned}$$

Now by adding (3.34), (3.37) and (3.38), we obtain

$$\begin{aligned}
& \int_0^t \int_{\Omega_0} \rho_{f,a,b} |\partial_t v_\varepsilon|^2 dx dy ds + \rho_s \int_0^t \int_0^L |\partial_{tt}\zeta_\varepsilon|^2 dx ds + \rho_s \int_0^t \int_0^L |\partial_{tt}\eta_\varepsilon|^2 dx ds + \gamma \int_0^t \int_0^L |\partial_{xtt}\eta_\varepsilon|^2 dx ds \\
& + \mu \int_{\Omega_0} \frac{1}{\det \nabla \chi_{a,b}} |((B_{a,b}\nabla)v_\varepsilon(t) + ((B_{a,b}\nabla)v_\varepsilon(t))^T)|^2 dx dy + \frac{\varepsilon}{2} \left(\int_0^L |\partial_{xxt}\zeta_\varepsilon(t)|^2 dx + \int_0^L |\partial_{xxt}\eta_\varepsilon(t)|^2 dx \right) \\
& = \beta \int_0^t \int_0^L |\partial_{xt}\zeta_\varepsilon|^2 dx ds + \alpha \int_0^t \int_0^L |\partial_{txx}\eta_\varepsilon|^2 dx ds \\
& + \beta \int_0^L \partial_{xx}\zeta_\varepsilon(t) \partial_t \zeta_\varepsilon(t) dx + \alpha \int_0^L \partial_{xxx}\eta_\varepsilon(t) \partial_{tx}\eta_\varepsilon(t) dx - \beta \int_0^L \partial_{xx}\zeta_0^\varepsilon \zeta_1^\varepsilon dx - \alpha \int_0^L \partial_{xxx}\eta_0^\varepsilon \partial_x \eta_1^\varepsilon dx \\
& + \int_0^t \int_{\Omega_0} f \cdot \partial_t v_\varepsilon dx dy ds + \int_0^t \int_{\Omega_0} \operatorname{div} h \cdot \partial_t v_\varepsilon dx dy ds - \int_0^t \int_0^L h_{|y=1} e_2 \cdot (\partial_{tt}\zeta_\varepsilon, \partial_{tt}\eta_\varepsilon) dx ds + \int_0^t \int_0^L (\ell, m) \cdot (\partial_{tt}\zeta_\varepsilon, \partial_{tt}\eta_\varepsilon) dx ds \\
& + \mu \int_{\Omega_0} \frac{1}{\det \nabla \chi_{a,b}} |((B_{a,b}\nabla)v_0^\varepsilon + ((B_{a,b}\nabla)v_0^\varepsilon)^T)|^2 dx dy + \frac{\varepsilon}{2} \left(\int_0^L |\partial_{xx}\zeta_1^\varepsilon|^2 dx + \int_0^L |\partial_{xx}\eta_1^\varepsilon|^2 dx \right).
\end{aligned}$$

Then by using uniform Korn's inequality (2.35) we have

$$\begin{aligned}
& \frac{\rho_f}{2R} \|\partial_t v_\varepsilon\|_{L^2(0,t;L_\#^2(\Omega_0))}^2 + \frac{\rho_s}{2} \|\partial_{tt}\zeta_\varepsilon\|_{L^2(0,t;L_\#^2(0,L))}^2 + \frac{\rho_s}{2} \|\partial_{tt}\eta_\varepsilon\|_{L^2(0,t;L_\#^2(0,L))}^2 \\
& + \frac{\mu\lambda(R)}{4} \|\nabla v_\varepsilon\|_{L^2(0,t;L_\#^2(\Omega_0))}^2 + \gamma \|\partial_{xtt}\eta_\varepsilon\|_{L^2(0,t;L_\#^2(0,L))}^2 \\
& + \frac{\varepsilon}{2} \left(\|\partial_{xxt}\zeta_\varepsilon(t)\|_{L_\#^2(0,L)}^2 + \|\partial_{xxt}\eta_\varepsilon(t)\|_{L_\#^2(0,L)}^2 \right) \leq C \left(\frac{R}{\rho_f} \left(\|f\|_{L^2(0,t;L_\#^2(\Omega_0))}^2 + \|h\|_{L^2(0,t;H_\#^1(\Omega_0))}^2 \right) \right. \\
& \left. + \frac{1}{2\rho_s} \left(C(\Omega_0) \|h\|_{L^2(0,t;H_\#^1(\Omega_0))}^2 + \|\ell\|_{L^2(0,t;L_\#^2(0,L))}^2 + \|m\|_{L^2(0,t;L_\#^2(0,L))}^2 \right) + C_0(R) \right) \\
& + \beta \|\partial_{xt}\zeta_\varepsilon\|_{L^2(0,t;L_\#^2(\Omega_0))}^2 + \alpha \|\partial_{txx}\eta_\varepsilon\|_{L^2(0,t;L_\#^2(\Omega_0))}^2 + \beta \int_0^L \partial_{xx}\zeta_\varepsilon(t) \partial_t \zeta_\varepsilon(t) dx + \alpha \int_0^L \partial_{xxx}\eta_\varepsilon(t) \partial_{tx}\eta_\varepsilon(t) dx, \quad (3.39)
\end{aligned}$$

where $C_0(R)$ is a positive constant which depends on R and on initial data and is independent of ε .

It remains to bound the last four terms in the right hand side of the above inequality. To get a bound independent of ε , we multiply the fluid equation (3.15) by $-\partial_{xx}v_\varepsilon$, the longitudinal and transversal displacement of the structure equations (3.19) and (3.20) respectively by $-\partial_{xxt}\zeta_\varepsilon$ and $-\partial_{xxt}\eta_\varepsilon$. We underline that though $-\partial_{xx}v_\varepsilon$ matches $(-\partial_{xxt}\zeta_\varepsilon, -\partial_{xxt}\eta_\varepsilon)$ at the fluid structure interface, but $(-\partial_{xx}v_\varepsilon, -\partial_{xxt}\zeta_\varepsilon, -\partial_{xxt}\eta_\varepsilon)$ is not an admissible triplet of test functions as $\operatorname{div}(B_{a,b}^T \partial_{xx}v_\varepsilon) \neq 0$. This will lead to the appearance of pressure terms that we have to carefully control, in particular boundary terms that do not appear when considering only the transversal case. At this step we need to assume that $(a, b) \in H_\#^2(0, L)$ to take care of these extra terms. By integrating the resulting equations

on $(0, t) \times (0, L)$ and after integration by parts in space, we obtain

$$\begin{aligned}
& - \int_0^t \int_{\Omega_0} \rho_{f,a,b} \partial_t v_\varepsilon \cdot \partial_{xx} v_\varepsilon dx dy ds + \int_0^t \int_{\Omega_0} \operatorname{div} \left(\mu(A_{a,b} \nabla) v_\varepsilon + \frac{\mu}{\det \nabla \chi_{a,b}} B_{a,b} \nabla v_\varepsilon^T B_{a,b} - q_\varepsilon B_{a,b} \right) \cdot \partial_{xx} v_\varepsilon dx dy ds \\
& + \frac{\rho_s}{2} \int_0^L |\partial_{tx} \zeta_\varepsilon(t)|^2 dx + \frac{\rho_s}{2} \int_0^L |\partial_{tx} \eta_\varepsilon(t)|^2 dx + \frac{\beta}{2} \int_0^L |\partial_{xx} \zeta_\varepsilon(t)|^2 dx + \frac{\alpha}{2} \int_0^L |\partial_{xxx} \eta_\varepsilon(t)|^2 dx + \frac{\gamma}{2} \int_0^L |\partial_{txx} \eta_\varepsilon(t)|^2 dx \\
& + \varepsilon \left(\int_0^t \int_0^L |\partial_{xxx} \zeta_\varepsilon|^2 dx ds + \int_0^t \int_0^L |\partial_{xxx} \eta_\varepsilon|^2 dx ds \right) = - \int_0^t \int_{\Omega_0} f \cdot \partial_{xx} v_\varepsilon dx dy ds - \int_0^t \int_{\Omega_0} \partial_x h : \nabla \partial_x v_\varepsilon dx dy ds \\
& \quad + \int_0^t \int_0^L T[v_\varepsilon, q_\varepsilon] \cdot (\partial_{xxt} \zeta_\varepsilon e_1 + \partial_{xxt} \eta_\varepsilon e_2) dx ds - \int_0^t \int_0^L (\ell, m) \cdot (\partial_{xxt} \zeta_\varepsilon e_1 + \partial_{xxt} \eta_\varepsilon e_2) dx ds \\
& \quad + \frac{\rho_s}{2} \int_0^L |\partial_x \zeta_1^\varepsilon|^2 dx + \frac{\rho_s}{2} \int_0^L |\partial_x \eta_1^\varepsilon|^2 dx + \frac{\beta}{2} \int_0^L |\partial_{xx} \zeta_0^\varepsilon|^2 dx + \frac{\alpha}{2} \int_0^L |\partial_{xxx} \eta_0^\varepsilon|^2 dx + \frac{\gamma}{2} \int_0^L |\partial_{xxx} \eta_1^\varepsilon|^2 dx. \quad (3.40)
\end{aligned}$$

Since $\partial_{xx} v_\varepsilon \in L^2(0, T; L^2_\#(\Omega_0))$, it is not regular enough to perform integration by parts in space directly. To proceed we state the following lemma and we refer the reader to [28] for a similar proof.

Lemma 3.2 *For $w \in L^2(0, T; H^2_\#(\Omega_0)) \cap H^1(0, T; L^2_\#(\Omega_0))$, such that $w|_{y=1} = (\partial_t \zeta, \partial_t \eta)$ with $(\zeta, \eta) \in H^1(0, T; H^2_\#(0, L))$ and $w|_{y=0} = 0$, and for $q \in L^2(0, T; H^1_\#(\Omega_0))$, we have*

$$\begin{aligned}
& - \int_0^t \int_{\Omega_0} \rho_{f,a,b} \partial_t w \cdot \partial_{xx} w dx dy ds + \int_0^t \int_{\Omega_0} \operatorname{div} \left(\mu(A_{a,b} \nabla) w + \frac{\mu}{\det \nabla \chi_{a,b}} B_{a,b} \nabla w^T B_{a,b} - q B_{a,b} \right) \cdot \partial_{xx} w dx dy ds \\
& \quad = \frac{1}{2} \int_{\Omega_0} \rho_{f,a,b} |\partial_x w(t)|^2 dx dy - \frac{1}{2} \int_{\Omega_0} \rho_{f,a,b} |\partial_x w(0)|^2 dx dy + \int_0^t \int_{\Omega_0} \partial_x \rho_{f,a,b} \partial_t w \cdot \partial_x w dx dy ds \\
& \quad + \mu \int_0^t \int_{\Omega_0} ((A_{a,b} \nabla) \partial_x w + \frac{1}{\det \nabla \chi_{a,b}} B_{a,b} \nabla \partial_x w^T B_{a,b}) : \nabla \partial_x w dx dy ds + \mu \int_0^t \int_{\Omega_0} (\partial_x A_{a,b} \nabla) w : \nabla \partial_x w dx dy ds \\
& \quad + \mu \int_0^t \int_{\Omega_0} \partial_x \left(\frac{1}{\det \nabla \chi_{a,b}} B_{a,b} \right) \nabla w^T B_{a,b} : \nabla \partial_x w dx dy ds + \mu \int_0^t \int_{\Omega_0} \frac{1}{\det \nabla \chi_{a,b}} B_{a,b} \nabla w^T \partial_x B_{a,b} : \nabla \partial_x w dx dy ds \\
& \quad \quad - \int_0^t \int_{\Omega_0} \partial_x q B_{a,b} : \nabla \partial_x w dx dy ds - \int_0^t \int_{\Omega_0} q \partial_x B_{a,b} : \nabla \partial_x w dx dy ds \\
& \quad \quad + \int_0^t \langle T_{a,b}[w, q] \cdot e_1, \partial_{xxt} \zeta \rangle_{H^{\frac{1}{2}}_\#(0, L), H^{\frac{1}{2}}_\#(0, L)} ds + \int_0^t \langle T_{a,b}[w, q] \cdot e_2, \partial_{xxt} \eta \rangle_{H^{\frac{1}{2}}_\#(0, L), H^{\frac{1}{2}}_\#(0, L)} ds
\end{aligned}$$

By applying the above lemma with $(w, q) = (v_\varepsilon, q_\varepsilon)$ and combining the resulting equation with (3.40), taking into account uniform Korn inequality (2.35), we obtain

$$\begin{aligned}
& \frac{\rho_f}{2R} \int_{\Omega_0} |\partial_x v_\varepsilon(t)|^2 dx dy + \frac{\rho_s}{2} \int_0^L |\partial_{tx} \zeta_\varepsilon(t)|^2 dx + \frac{\rho_s}{2} \int_0^L |\partial_{tx} \eta_\varepsilon(t)|^2 dx dy + \frac{\mu \lambda(R)}{2} \int_0^t \int_{\Omega_0} |\nabla \partial_x v_\varepsilon|^2 dx dy ds \\
& + \frac{\beta}{2} \int_0^L |\partial_{xx} \zeta_\varepsilon(t)|^2 dx + \frac{\alpha}{2} \int_0^L |\partial_{xxx} \eta_\varepsilon(t)|^2 dx + \frac{\gamma}{2} \int_0^L |\partial_{txx} \eta_\varepsilon(t)|^2 dx + \varepsilon \left(\int_0^t \int_0^L |\partial_{xxx} \zeta_\varepsilon|^2 dx ds + \int_0^t \int_0^L |\partial_{xxx} \eta_\varepsilon|^2 dx ds \right) \\
& \leq - \int_0^t \int_{\Omega_0} f \cdot \partial_{xx} v_\varepsilon dx dy ds - \int_0^t \int_{\Omega_0} \partial_x h : \nabla \partial_x v_\varepsilon dx dy ds - \int_0^t \int_0^L (\ell, m) \cdot (\partial_{xxt} \zeta_\varepsilon, \partial_{xxt} \eta_\varepsilon) dx ds - \int_0^t \int_{\Omega_0} \partial_x \rho_{f,a,b} \partial_t v_\varepsilon \cdot \partial_x v_\varepsilon dx dy ds \\
& + \frac{1}{2} \int_{\Omega_0} \rho_{f,a,b} |\partial_x v_0^\varepsilon|^2 dx dy + \frac{\rho_s}{2} \int_0^L |\partial_x \zeta_1^\varepsilon|^2 dx + \frac{\rho_s}{2} \int_0^L |\partial_x \eta_1^\varepsilon|^2 dx + \frac{\beta}{2} \int_0^L |\partial_{xx} \zeta_0^\varepsilon|^2 dx + \frac{\alpha}{2} \int_0^L |\partial_{xxx} \eta_0^\varepsilon|^2 dx + \frac{\gamma}{2} \int_0^L |\partial_{xxx} \eta_1^\varepsilon|^2 dx \\
& \quad - \mu \int_0^t \int_{\Omega_0} (\partial_x A_{a,b} \nabla) v_\varepsilon : \nabla \partial_x v_\varepsilon dx dy ds - \mu \int_0^t \int_{\Omega_0} \partial_x \left(\frac{1}{\det \nabla \chi_{a,b}} B_{a,b} \right) \nabla v_\varepsilon^T B_{a,b} : \nabla \partial_x v_\varepsilon dx dy ds \\
& - \mu \int_0^t \int_{\Omega_0} \frac{1}{\det \nabla \chi_{a,b}} B_{a,b} \nabla v_\varepsilon^T \partial_x B_{a,b} : \nabla \partial_x v_\varepsilon dx dy ds + \int_0^t \int_{\Omega_0} \partial_x q_\varepsilon B_{a,b} : \nabla \partial_x v_\varepsilon dx dy ds + \int_0^t \int_{\Omega_0} q_\varepsilon \partial_x B_{a,b} : \nabla \partial_x v_\varepsilon dx dy ds. \quad (3.41)
\end{aligned}$$

We start by bounding each term in the right hand side of the above inequality. By using Holder's and Young's inequality, we bound the terms including the external body forces:

$$\begin{aligned} \left| \int_0^t \int_{\Omega_0} f \cdot \partial_{xx} v_\varepsilon \, dx dy ds + \int_0^t \int_{\Omega_0} \partial_x h : \nabla \partial_x v_\varepsilon \, dx dy ds \right| &\leq \int_0^t \left(\|f\|_{L^2_\#(\Omega_0)} + \|h\|_{H^1_\#(\Omega_0)} \right) \|\nabla \partial_x v_\varepsilon\|_{L^2_\#(\Omega_0)} \\ &\leq C(R) \left(\|f\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 + \|h\|_{L^2(0,t;H^1_\#(\Omega_0))}^2 \right) + \frac{\mu\lambda(R)}{N} \|\nabla \partial_x v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2, \end{aligned} \quad (3.42)$$

where N is a positive integer to be fixed later on.

Next, thanks to the equality of the fluid structure velocities at the interface (3.17) and by using Poincaré inequality for $\partial_x v_\varepsilon$ that is valid since $\partial_x v_\varepsilon|_{y=0} = 0$, we have

$$\begin{aligned} \left| \int_0^t \int_0^L (\ell, m) \cdot (\partial_{xxt} \zeta_\varepsilon, \partial_{xxt} \eta_\varepsilon) \, dx ds \right| &\leq \|\partial_{xt} \zeta_\varepsilon\|_{L^2(0,t;H^{\frac{1}{2}}_\#(0,L))} \|\ell\|_{L^2(0,t;H^{\frac{1}{2}}_\#(0,L))} \\ &\quad + \|\partial_{tx} \eta_\varepsilon\|_{L^2(0,t;H^{\frac{1}{2}}_\#(0,L))} \|m\|_{L^2(0,t;H^{\frac{1}{2}}_\#(0,L))}, \\ &\leq \|\partial_x v_\varepsilon\|_{L^2(0,t;H^{\frac{1}{2}}_\#(0,L))} \|\ell\|_{L^2(0,t;H^{\frac{1}{2}}_\#(0,L))} + \|\partial_x v_\varepsilon\|_{L^2(0,t;H^{\frac{1}{2}}_\#(0,L))} \|m\|_{L^2(0,t;H^{\frac{1}{2}}_\#(0,L))}, \\ &\leq \frac{\mu\lambda(R)}{N} \|\partial_x \nabla v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 + C(R) \|\ell\|_{L^2(0,t;H^{\frac{1}{2}}_\#(0,L))}^2 + C(R) \|m\|_{L^2(0,t;H^{\frac{1}{2}}_\#(0,L))}^2. \end{aligned} \quad (3.43)$$

Next, since $\partial_x \rho_{f,a,b} = \rho_f \partial_x (\det \nabla \chi_{a,b}) \in H^s_\#(\Omega_0)$, then by using Holder's and Young's inequalities and Lemma 2.1, we obtain

$$\begin{aligned} \left| \int_0^t \int_{\Omega_0} \partial_x \rho_{f,a,b} \partial_t v_\varepsilon \cdot \partial_x v_\varepsilon \, dx dy ds \right| &\leq \int_0^t \|\partial_x \rho_{f,a,b} \partial_x v_\varepsilon\|_{L^2_\#(\Omega_0)} \|\partial_t v_\varepsilon\|_{L^2_\#(\Omega_0)} \, ds \\ &\leq K(R) \int_0^t \left(C_\delta \|\partial_x v_\varepsilon\|_{L^2_\#(\Omega_0)} + \delta \|\nabla \partial_x v_\varepsilon\|_{L^2_\#(\Omega_0)} \right) \|\partial_t v_\varepsilon\|_{L^2_\#(\Omega_0)} \, ds \\ &\leq K(R, \frac{1}{\kappa}) \int_0^t \left(C_\delta \|\partial_x v_\varepsilon\|_{L^2_\#(\Omega_0)} + \delta \|\nabla \partial_x v_\varepsilon\|_{L^2_\#(\Omega_0)} \right)^2 \, ds + \kappa \int_0^t \|\partial_t v_\varepsilon\|_{L^2_\#(\Omega_0)}^2 \, ds. \end{aligned}$$

It yields

$$\begin{aligned} \left| \int_0^t \int_{\Omega_0} \partial_x \rho_{f,a,b} \partial_t v_\varepsilon \cdot \partial_x v_\varepsilon \, dx dy ds \right| &\leq K_\delta(R, \frac{1}{\kappa}) \|\partial_x v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 \\ &\quad + \delta^2 K(R, \frac{1}{\kappa}) \|\nabla \partial_x v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 + \kappa \|\partial_t v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2. \end{aligned} \quad (3.44)$$

Thanks to the way we construct the approximated initial conditions, we have

$$\frac{1}{2} \int_{\Omega_0} \rho_{f,a,b} |\partial_x v_0^\varepsilon|^2 + \frac{\rho_s}{2} \int_0^L |\partial_x \zeta_1^\varepsilon|^2 + \frac{\rho_s}{2} \int_0^L |\partial_x \eta_1^\varepsilon|^2 + \frac{\beta}{2} \int_0^L |\partial_{xx} \zeta_0^\varepsilon|^2 + \frac{\alpha}{2} \int_0^L |\partial_{xxx} \eta_0^\varepsilon|^2 + \frac{\gamma}{2} \int_0^L |\partial_{xx} \eta_1^\varepsilon|^2 \, dx \leq C_0(R), \quad (3.45)$$

where C_0 is a positive constant depending on initial data and R .

Again by using Holder's inequality, (2.64), Young's inequality and thanks to the energy estimate (3.26), we have:

$$\begin{aligned} \left| \int_0^t \int_{\Omega_0} \mu (\partial_x A_{a,b} \nabla) v_\varepsilon : \nabla \partial_x v_\varepsilon \, dx dy ds \right| &\leq \mu \int_0^t \|(\partial_x A_{a,b} \nabla) v_\varepsilon\|_{L^2_\#(\Omega_0)} \|\nabla \partial_x v_\varepsilon\|_{L^2_\#(\Omega_0)} \, ds \\ &\leq \mu K(R) \int_0^t \left(C_\delta \|\nabla v_\varepsilon\|_{L^2_\#(\Omega_0)} + \delta \|\nabla^2 v_\varepsilon\|_{L^2_\#(\Omega_0)} \right) \|\nabla \partial_x v_\varepsilon\|_{L^2_\#(\Omega_0)} \, ds \\ &\leq K(R) \int_0^t \left(C_\delta \|\nabla v_\varepsilon\|_{L^2_\#(\Omega_0)} + \delta \|\nabla^2 v_\varepsilon\|_{L^2_\#(\Omega_0)} \right)^2 \, ds \\ &\quad + \frac{\mu\lambda(R)}{N} \int_0^t \|\nabla \partial_x v_\varepsilon\|_{L^2_\#(\Omega_0)}^2 \, ds. \end{aligned}$$

Remembering the energy estimate (3.26) it follows

$$\begin{aligned} \int_0^t \int_{\Omega_0} \mu(\partial_x A_{a,b} \nabla) v_\varepsilon : \nabla \partial_x v_\varepsilon dx dy ds \leq C_0(R, \delta) + \left(\frac{\mu \lambda(R)}{N} + \delta^2 K(R) \right) \|\nabla \partial_x v_\varepsilon\|_{L^2(0,t;L_\#^2(\Omega_0))}^2 \\ + \delta^2 K(R) \|\partial_{yy} v_\varepsilon\|_{L^2(0,t;L_\#^2(\Omega_0))}^2, \end{aligned} \quad (3.46)$$

where C_0 depends on R , on the data (physical parameters, initial data and applied forces) and on δ . By using Proposition 2.3 and using (3.17), we have

$$\begin{aligned} \|\partial_{yy} v_\varepsilon\|_{L_\#^2(\Omega_0)} + \|\nabla s_{0,\varepsilon}\|_{L_\#^2(\Omega_0)} &\leq K(R) \left(\|f + \operatorname{div} h - \rho_{f,a,b} \partial_t v_\varepsilon\|_{L_\#^2(\Omega_0)} + \|(\partial_t \zeta_\varepsilon, \partial_t \eta_\varepsilon)\|_{H_\#^{\frac{3}{2}}(0,L)} \right) \\ &\leq K(R) \left(\|f\|_{L_\#^2(\Omega_0)} + \|h\|_{H_\#^1(\Omega_0)} + \|\partial_t v_\varepsilon\|_{L_\#^2(\Omega_0)} \right. \\ &\quad \left. + \|(\partial_t \zeta_\varepsilon, \partial_t \eta_\varepsilon)\|_{L_\#^2(0,L)} + \|(\partial_{tx} \zeta_\varepsilon, \partial_{tx} \eta_\varepsilon)\|_{H_\#^{\frac{1}{2}}(0,L)} \right) \\ &\leq K(R) \left(\|f\|_{L_\#^2(\Omega_0)} + \|h\|_{H_\#^1(\Omega_0)} + \|\partial_t v_\varepsilon\|_{L_\#^2(\Omega_0)} + \|(\partial_t \zeta_\varepsilon, \partial_t \eta_\varepsilon)\|_{L_\#^2(0,L)} + \|\partial_x v_\varepsilon\|_{H_\#^1(\Omega_0)} \right). \end{aligned}$$

This implies

$$\begin{aligned} \|\partial_{yy} v_\varepsilon\|_{L_\#^2(\Omega_0)} + \|\nabla s_{0,\varepsilon}\|_{L_\#^2(\Omega_0)} \leq K(R) \left(\|f\|_{L_\#^2(\Omega_0)} + \|h\|_{H_\#^1(\Omega_0)} + \|\partial_t v_\varepsilon\|_{L_\#^2(\Omega_0)} + \|(\partial_t \zeta_\varepsilon, \partial_t \eta_\varepsilon)\|_{L_\#^2(0,L)} \right. \\ \left. + \|\partial_x v_\varepsilon\|_{L_\#^2(\Omega_0)} + \|\nabla \partial_x v_\varepsilon\|_{L_\#^2(\Omega_0)} \right). \end{aligned} \quad (3.47)$$

By combining the above inequality with (3.46) and the energy estimate (3.26), we obtain

$$\begin{aligned} \int_0^t \int_{\Omega_0} \mu(\partial_x A_{a,b} \nabla) v_\varepsilon : \nabla \partial_x v_\varepsilon dx dy ds \leq C_0(R, \delta) + \left(\frac{\mu \lambda(R)}{N} + \delta^2 K(R, \mu) \right) \|\nabla \partial_x v_\varepsilon\|_{L^2(0,t;L_\#^2(\Omega_0))}^2 \\ + \delta^2 K(R) \|\partial_t v_\varepsilon\|_{L^2(0,T;L_\#^2(\Omega_0))}^2. \end{aligned} \quad (3.48)$$

By using the same arguments we prove as above

$$\begin{aligned} \left| \int_{\Omega_0} \partial_x \left(\frac{\mu}{\det \nabla \chi_{a,b}} B_{a,b} \right) \nabla v_\varepsilon^T B_{a,b} : \nabla \partial_x v_\varepsilon dx dy + \int_{\Omega_0} \frac{\mu}{\det \nabla \chi_{a,b}} B_{a,b} \nabla v_\varepsilon^T \partial_x B_{a,b} : \nabla \partial_x v_\varepsilon dx dy \right| \\ \leq C_0(R, \delta) + \left(\frac{\mu \lambda(R)}{N} + \delta^2 K(R, \mu) \right) \|\nabla \partial_x v_\varepsilon\|_{L^2(0,t;L_\#^2(\Omega_0))}^2 + \delta^2 K(R) \|\partial_{yy} v_\varepsilon\|_{L^2(0,t;L_\#^2(\Omega_0))}^2. \end{aligned}$$

By using (3.26) and (3.47), we obtain

$$\begin{aligned} \left| \int_{\Omega_0} \partial_x \left(\frac{\mu}{\det \nabla \chi_{a,b}} B_{a,b} \right) \nabla v_\varepsilon^T B_{a,b} : \nabla \partial_x v_\varepsilon dx dy + \int_{\Omega_0} \frac{\mu}{\det \nabla \chi_{a,b}} B_{a,b} \nabla v_\varepsilon^T \partial_x B_{a,b} : \nabla \partial_x v_\varepsilon dx dy \right| \\ \leq C_0(R, \delta) + \left(\frac{\mu \lambda(R)}{N} + \delta^2 K(R) \right) \|\nabla \partial_x v_\varepsilon\|_{L^2(0,t;L_\#^2(\Omega_0))}^2 + \delta^2 K(R) \|\partial_t v_\varepsilon\|_{L^2(0,t;L_\#^2(\Omega_0))}^2. \end{aligned} \quad (3.49)$$

It remains to bound the pressure terms in (3.41). Since $\operatorname{div} (B_{a,b}^T v_\varepsilon) = 0$, and thanks to the Piola identity,

$$\begin{aligned} B_{a,b} : \nabla \partial_x v_\varepsilon &= \partial_x (B_{a,b} : \nabla v_\varepsilon) - \partial_x B_{a,b} : \nabla v_\varepsilon \\ &= \partial_x \operatorname{div} (B_{a,b}^T v_\varepsilon) - \partial_x B_{a,b} : \nabla v_\varepsilon \\ &= -\partial_x B_{a,b} : \nabla v_\varepsilon. \end{aligned}$$

Hence,

$$\partial_x q_\varepsilon B_{a,b} : \nabla \partial_x v_\varepsilon = -\partial_x q_\varepsilon \partial_x B_{a,b} : \nabla v_\varepsilon.$$

It follows from the above equality and (2.67) that

$$\begin{aligned} \left| \int_0^t \int_{\Omega_0} \partial_x q_\varepsilon B_{a,b} : \nabla \partial_x v_\varepsilon dx dy ds \right| &= \left| \int_0^t \int_{\Omega_0} \partial_x q_\varepsilon \partial_x B_{a,b} : \nabla v_\varepsilon dx dy ds \right| \\ &\leq \int_0^t \|\partial_x q_\varepsilon\|_{L^2_\#(\Omega_0)} \|\partial_x B_{a,b} : \nabla v_\varepsilon\|_{L^2_\#(\Omega_0)} ds \\ &\leq K(R) \int_0^t \|\partial_x q_\varepsilon\|_{L^2_\#(\Omega_0)} \left(C_\delta \|\nabla v_\varepsilon\|_{L^2_\#(\Omega_0)} + \delta \|\nabla^2 v_\varepsilon\|_{L^2_\#(\Omega_0)} \right) ds. \end{aligned}$$

By using Young's inequality, it follows that

$$\begin{aligned} \left| \int_0^t \int_{\Omega_0} \partial_x q_\varepsilon B_{a,b} : \nabla \partial_x v_\varepsilon dx dy ds \right| &\leq \delta K(R) \int_0^t \left(\|\partial_x q_\varepsilon\|_{L^2_\#(\Omega_0)}^2 + \|\partial_{yy} v_\varepsilon\|_{L^2_\#(\Omega_0)}^2 + \|\nabla \partial_x v_\varepsilon\|_{L^2_\#(\Omega_0)}^2 \right) \\ &\quad + K(R, \delta) \|\nabla v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2. \end{aligned}$$

By using the energy estimate (3.26) and (3.47), and since $\partial_x q_\varepsilon = \partial_x s_{0,\varepsilon}$ we obtain

$$\left| \int_0^t \int_{\Omega_0} \partial_x q_\varepsilon B_{a,b} : \nabla \partial_x v_\varepsilon dx dy ds \right| \leq C_0(R, \delta) + \delta K(R) \left(\|\partial_t v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 + \|\nabla \partial_x v_\varepsilon\|_{L^2_\#(\Omega_0)}^2 \right). \quad (3.50)$$

Finally, to bound the last term in the right hand side of (3.41) we integrate it by parts. Precisely, we have

$$\int_0^t \int_{\Omega_0} q_\varepsilon \partial_x B_{a,b} : \nabla \partial_x v_\varepsilon dx dy ds = - \int_0^t \int_{\Omega_0} \operatorname{div} (q_\varepsilon \partial_x B_{a,b}) : \partial_x v_\varepsilon dx dy ds + \int_0^t \int_0^L (q_\varepsilon \partial_x B_{a,b}^T \partial_x v_\varepsilon)|_{y=1} \cdot e_2 dx.$$

By Piola identity, $\operatorname{div}(\partial_x B_{a,b}) = 0$. Hence,

$$\int_0^t \int_{\Omega_0} q_\varepsilon \partial_x B_{a,b} : \nabla \partial_x v_\varepsilon dx dy ds = - \int_0^t \int_{\Omega_0} \nabla q_\varepsilon \partial_x B_{a,b}^T \partial_x v_\varepsilon dx dy ds + \int_0^t \int_0^L (q_\varepsilon \partial_x B_{a,b}^T \partial_x v_\varepsilon)|_{y=1} \cdot e_2 dx ds.$$

The first term on the right hand side can be estimated once again by applying (2.67) as follows

$$\begin{aligned} \left| \int_0^t \int_{\Omega_0} \nabla q_\varepsilon \partial_x B_{a,b}^T \partial_x v_\varepsilon dx dy ds \right| &\leq \int_0^t \|\nabla q_\varepsilon\|_{L^2_\#(\Omega_0)} \|\partial_x B_{a,b}^T \partial_x v_\varepsilon\|_{L^2_\#(\Omega_0)} ds \\ &\leq K(R) \int_0^t \|\nabla q_\varepsilon\|_{L^2_\#(\Omega_0)} \left(C_\delta \|\partial_x v_\varepsilon\|_{L^2_\#(\Omega_0)} + \delta \|\nabla \partial_x v_\varepsilon\|_{L^2_\#(\Omega_0)} \right) ds. \end{aligned}$$

This yields by Young's inequality to

$$\left| \int_0^t \int_{\Omega_0} \nabla q_\varepsilon \partial_x B_{a,b}^T \partial_x v_\varepsilon dx dy ds \right| \leq \delta K(R) \int_0^t \left(\|\nabla q_\varepsilon\|_{L^2_\#(\Omega_0)}^2 + \|\nabla \partial_x v_\varepsilon\|_{L^2_\#(\Omega_0)}^2 \right) ds + K(R, \delta) \int_0^t \|\partial_x v_\varepsilon\|_{L^2_\#(\Omega_0)}^2 ds.$$

Again (3.26) and (3.47) and $\nabla q_\varepsilon = \nabla s_{0,\varepsilon}$ implies

$$\left| \int_0^t \int_{\Omega_0} \nabla q_\varepsilon \partial_x B_{a,b}^T \partial_x v_\varepsilon dx dy ds \right| \leq C_0(R, \delta) + \delta K(R) \left(\|\partial_t v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 + \|\nabla \partial_x v_\varepsilon\|_{L^2_\#(\Omega_0)}^2 \right). \quad (3.51)$$

Moreover, for the boundary term, taking into account the equality of the velocities at the interface (3.17), one can easily verifies

$$\int_0^t \int_0^L (q_\varepsilon \partial_x B_{a,b}^T \partial_x v_\varepsilon)|_{y=1} \cdot e_2 dx ds = - \int_0^t \int_0^L q_\varepsilon|_{y=1} \partial_{xx} b \partial_{tx} \zeta_\varepsilon dx ds + \int_0^t \int_0^L q_\varepsilon|_{y=1} \partial_{xx} a \partial_{tx} \eta_\varepsilon dx ds. \quad (3.52)$$

Note that this term is equal to zero in the case where $\zeta_\varepsilon = a = 0$, which is the case when the longitudinal displacement is neglected and one considers a beam in flexion only. We recall that q_ε writes $q_\varepsilon(x, y, t) = s_{0,\varepsilon}(x, y, t) + k_\varepsilon(t)$ with $s_{0,\varepsilon} \in L^2(0, T; H^1_\#(\Omega_0) \cap L^2_{\#,0}(\Omega_0))$ and k_ε is given by (3.29) that writes

$$k_\varepsilon(t) = \frac{1}{L} \int_0^L (\rho_s \partial_{tt} \eta_\varepsilon + h|_{y=1} e_2 \cdot e_2 + T_{a,b}[v_\varepsilon, s_{0,\varepsilon}] \cdot e_2 - m) dx.$$

We remark that for a. e. $t \in (0, T)$

$$|k_\varepsilon(t)| \leq C \left(\|\partial_{tt}\eta_\varepsilon\|_{L^2_\sharp(0,L)} + \|(v_\varepsilon, s_{0,\varepsilon})\|_{H^2_\sharp(\Omega_0) \times H^1_\sharp(\Omega_0)} + \|h\|_{H^1_\sharp(\Omega_0)} + \|m\|_{L^2_\sharp(0,L)} \right).$$

Thanks to the energy estimate (3.26) and (3.47), since $s_{0,\varepsilon} \in L^2_{\sharp,0}(\Omega_0)$, we have

$$\|k_\varepsilon\|_{L^2(0,t)} \leq C_0 + K(R) \left(\|\partial_{tt}\eta_\varepsilon\|_{L^2(0,t;L^2_\sharp(0,L))} + \|\partial_t v_\varepsilon\|_{L^2(0,t;L^2_\sharp(\Omega_0))} + \|\nabla \partial_x v_\varepsilon\|_{L^2(0,t;L^2_\sharp(\Omega_0))} \right) \quad (3.53)$$

By (3.52), the boundary term can be written as follows:

$$\begin{aligned} \int_0^t \int_0^L (q_\varepsilon \partial_x B_{a,b}^T \partial_x v_\varepsilon)|_{y=1} \cdot e_2 \, dx \, ds &= \int_0^t k_\varepsilon(s) \left(- \int_0^L \partial_{xx} b \partial_{tx} \zeta_\varepsilon + \int_0^L \partial_{xx} a \partial_{tx} \eta_\varepsilon \, dx \right) ds \\ &\quad - \int_0^t \int_0^L s_{0,\varepsilon}|_{y=1} \partial_{xx} b \partial_{tx} \zeta_\varepsilon \, dx \, ds + \int_0^t \int_0^L s_{0,\varepsilon}|_{y=1} \partial_{xx} a \partial_{tx} \eta_\varepsilon \, dx \, ds. \end{aligned} \quad (3.54)$$

We start by bounding the first term on the right hand side of (3.54). By Holder's and Young's inequality, and taking into account (3.53), we have

$$\begin{aligned} &\left| \int_0^t k_\varepsilon(s) \left(- \int_0^L \partial_{xx} b \partial_{tx} \zeta_\varepsilon + \int_0^L \partial_{xx} a \partial_{tx} \eta_\varepsilon \, dx \right) ds \right| \\ &\leq \int_0^t |k_\varepsilon(s)| \|(a, b)\|_{H^2_\sharp(0,L)} \left(\|\partial_{tx} \zeta_\varepsilon\|_{L^2_\sharp(0,L)} + \|\partial_{tx} \eta_\varepsilon\|_{L^2_\sharp(0,L)} \right) ds \\ &\leq \kappa \|k_\varepsilon\|_{L^2(0,T)}^2 + \frac{R^2}{4\kappa} \left(\|\partial_{tx} \eta_\varepsilon\|_{L^2(0,t;L^2_\sharp(0,L))}^2 + \|\partial_{tx} \zeta_\varepsilon\|_{L^2(0,t;L^2_\sharp(0,L))}^2 \right) \end{aligned} \quad (3.55)$$

$$\begin{aligned} &\leq C_0(R) + \kappa \left(\|\partial_{tt}\eta_\varepsilon\|_{L^2(0,t;L^2_\sharp(0,L))}^2 + \|\partial_t v_\varepsilon\|_{L^2(0,t;L^2_\sharp(\Omega_0))}^2 + \|\nabla \partial_x v_\varepsilon\|_{L^2(0,t;L^2_\sharp(\Omega_0))}^2 \right) \\ &\quad + K(R, \frac{1}{\kappa}) \left(\|\partial_{tx} \eta_\varepsilon\|_{L^2(0,t;L^2_\sharp(0,L))}^2 + \|\partial_{tx} \zeta_\varepsilon\|_{L^2(0,t;L^2_\sharp(0,L))}^2 \right) \end{aligned} \quad (3.56)$$

Finally we have to estimate the second term on the right hand side of (3.54). Since $(a, b) \in H^2_\sharp(0, L)$, it follows by using Holder's and Young's inequality that

$$\left| \int_0^t \int_0^L s_{0,\varepsilon}|_{y=1} \partial_{xx} a \partial_{tx} \eta_\varepsilon \, ds \right| \leq \int_0^t \|\partial_{xx} a\|_{L^2_\sharp(0,L)} \|s_{0,\varepsilon}\|_{L^4_\sharp(0,L)} \|\partial_{tx} \eta_\varepsilon\|_{L^4_\sharp(0,L)} \, ds.$$

Since $H^l_\sharp(0, L) \subset L^4_\sharp(0, L)$, for any $l \geq \frac{1}{4}$, using the interpolation inequality $\|\cdot\|_{H^{\frac{1}{4}}(0,L)} \leq C \|\cdot\|_{L^2(0,L)}^{\frac{1}{2}} \|\cdot\|_{H^{\frac{1}{2}}(0,L)}^{\frac{1}{2}}$, and trace inequality and Poincaré inequality on $\partial_x v_\varepsilon$ (that is valid because $\partial_x v_\varepsilon|_{y=0} = 0$), we further obtain

$$\begin{aligned} \left| \int_0^t \int_0^L s_{0,\varepsilon}|_{y=1} \partial_{xx} a \partial_{tx} \eta_\varepsilon \, ds \right| &\leq R \int_0^t \|s_{0,\varepsilon}\|_{H^{\frac{1}{2}}_\sharp(0,L)} \|\partial_{tx} \eta_\varepsilon\|_{L^2_\sharp(0,L)}^{\frac{1}{2}} \|\partial_{tx} \eta_\varepsilon\|_{H^{\frac{1}{2}}_\sharp(0,L)}^{\frac{1}{2}} \, ds \\ &\leq CR \int_0^t \|s_{0,\varepsilon}\|_{H^1_\sharp(\Omega_0)} \|\partial_{tx} \eta_\varepsilon\|_{L^2_\sharp(0,L)}^{\frac{1}{2}} \|\nabla \partial_x v_\varepsilon\|_{L^2_\sharp(\Omega_0)}^{\frac{1}{2}} \, ds \\ &\leq CR \int_0^t \|\nabla s_{0,\varepsilon}\|_{L^2_\sharp(\Omega_0)} \|\partial_{tx} \eta_\varepsilon\|_{L^2_\sharp(0,L)}^{\frac{1}{2}} \|\nabla \partial_x v_\varepsilon\|_{L^2_\sharp(\Omega_0)}^{\frac{1}{2}} \, ds. \end{aligned}$$

The energy estimate (3.26) and the bound (3.47) on the pressure $s_{0,\varepsilon}$ imply

$$\begin{aligned}
\left| \int_0^t \int_0^L s_{0,\varepsilon}|_{y=1} \partial_{xx} a \partial_{tx} \eta_\varepsilon ds \right| &\leq C_0(R) + \kappa \|\partial_t v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 + \kappa \|\nabla \partial_x v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 \\
&\quad + K(R, \frac{1}{\kappa}) \int_0^t \|\partial_{tx} \eta_\varepsilon\|_{L^2_\#(0,L)} \|\nabla \partial_x v_\varepsilon\|_{L^2_\#(\Omega_0)} ds \\
&\leq C_0(R) + \kappa \|\partial_t v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 + 2\kappa \|\nabla \partial_x v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 \\
&\quad + K(R, \frac{1}{\kappa}) \|\partial_{tx} \eta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2. \tag{3.57}
\end{aligned}$$

In a similar way, one can show

$$\begin{aligned}
\left| \int_0^t \int_0^L s_{0,\varepsilon}|_{y=1} \partial_{xx} b \partial_{tx} \zeta_\varepsilon ds \right| &\leq C_0(R) + \kappa \|\partial_t v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 + \kappa \|\nabla \partial_x v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 \\
&\quad + K(R, \frac{1}{\kappa}) \int_0^t \|\partial_{tx} \eta_\varepsilon\|_{L^2_\#(0,L)} \|\nabla \partial_x v_\varepsilon\|_{L^2_\#(\Omega_0)} ds \\
&\leq C_0(R) + \kappa \|\partial_t v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 + 2\kappa \|\nabla \partial_x v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 \\
&\quad + K(R, \frac{1}{\kappa}) \|\partial_{tx} \zeta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2. \tag{3.58}
\end{aligned}$$

Now, by combining (3.56), (3.58) and (3.57) in (3.54), we obtain

$$\begin{aligned}
\left| \int_0^t \int_0^L (q_\varepsilon \partial_x B_{\zeta_0, \eta_0}^T \partial_x v_\varepsilon)|_{y=1} \cdot e_2 dx \right| &\leq C_0(R) + 3\kappa \|\partial_t v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 + 5\kappa \|\nabla \partial_x v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 \\
&\quad + K(R, \frac{1}{\kappa}) \left(\|\partial_{tx} \zeta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 + \|\partial_{tx} \eta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 \right) \tag{3.59}
\end{aligned}$$

We are now in a position to derive the desired estimate. By combining (3.42)-(3.45), (3.48)-(3.50), and (3.59) with (3.41)

$$\begin{aligned}
&\frac{\rho_f}{2R} \|\partial_x v_\varepsilon(t)\|_{L^2_\#(\Omega_0)}^2 + \frac{\rho_s}{2} \|\partial_{tx} \zeta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 + \frac{\rho_s}{2} \|\partial_{tx} \eta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 + \frac{\mu\lambda(R)}{2} \|\nabla \partial_x v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 \\
&+ \frac{\beta}{2} \|\partial_{xx} \zeta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 + \frac{\alpha}{2} \|\partial_{xxx} \eta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 + \frac{\gamma}{2} \|\partial_{txx} \eta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 + \varepsilon \left(\|\partial_{xxx} \zeta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 + \|\partial_{xxx} \eta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 \right) \\
&\leq C_0(R) + \left(\frac{4\mu\lambda(R)}{N} + 2\delta^2 K(R) + 2\delta K(R) + 5\kappa \right) \|\nabla \partial_x v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 + (2\delta^2 K(R) + 2\delta K(R) + 3\kappa) \|\partial_t v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 \\
&\quad + K(R, \frac{1}{\kappa}) \left(\|\partial_{tx} \zeta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 + \|\partial_{tx} \eta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 \right).
\end{aligned}$$

We choose N big enough, δ and κ small enough such that

$$\frac{4\mu\lambda(R)}{N} + 2\delta^2 K(R) + 2\delta K(R) + 5\kappa < \frac{\mu\lambda(R)}{4}.$$

With this choice we obtain

$$\begin{aligned}
&\frac{\rho_f}{2R} \|\partial_x v_\varepsilon(t)\|_{L^2_\#(\Omega_0)}^2 + \frac{\rho_s}{2} \|\partial_{tx} \zeta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 + \frac{\rho_s}{2} \|\partial_{tx} \eta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 + \frac{\mu\lambda(R)}{4} \|\nabla \partial_x v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 \\
&+ \frac{\beta}{2} \|\partial_{xx} \zeta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 + \frac{\alpha}{2} \|\partial_{xxx} \eta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 + \frac{\gamma}{2} \|\partial_{txx} \eta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 + \varepsilon \left(\|\partial_{xxx} \zeta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 + \|\partial_{xxx} \eta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 \right) \\
&\leq C_0(R, T) + (2\delta^2 K(R) + 2\delta K(R) + 3\kappa) \|\partial_t v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 + K(R, \frac{1}{\kappa}) \left(\|\partial_{tx} \zeta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 + \|\partial_{tx} \eta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 \right) \tag{3.60}
\end{aligned}$$

Final regularity estimates: We are now in a position to obtain a regularity estimate independent of the parameter of regularization ε . To this end, we consider a linear combination of the previous estimates, namely $(E)_\Lambda = \Lambda(3.39) + (3.60)$, with a parameter $\Lambda > 0$ that will be chosen afterwards. Then we obtain

$$\begin{aligned}
& \frac{\Lambda\rho_f}{2R} \|\partial_t v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 + \frac{\Lambda\rho_s}{2} \|\partial_{tt}\zeta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 + \frac{\Lambda\rho_s}{2} \|\partial_{tt}\eta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 + \frac{\Lambda\mu\lambda(R)}{4} \|\nabla v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 \\
& \quad + \gamma\Lambda \|\partial_{xtt}\eta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 + \frac{\Lambda\varepsilon}{2} \left(\|\partial_{xxt}\zeta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 + \|\partial_{xxt}\eta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 \right) \\
& \quad + \frac{\rho_f}{2R} \|\partial_x v_\varepsilon(t)\|_{L^2_\#(\Omega_0)}^2 + \frac{\rho_s}{2} \|\partial_{tx}\zeta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 + \frac{\rho_s}{2} \|\partial_{tx}\eta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 + \frac{\mu\lambda(R)}{4} \|\nabla \partial_x v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 \\
& + \frac{\beta}{2} \|\partial_{xx}\zeta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 + \frac{\alpha}{2} \|\partial_{xxx}\eta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 + \frac{\gamma}{2} \|\partial_{txx}\eta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 + \varepsilon \left(\|\partial_{xxx}\zeta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 + \|\partial_{xxx}\eta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 \right) \\
& \leq C_0(R) + (2\delta^2 K(R) + 2\delta K(R) + 3\kappa) \|\partial_t v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 + K(R, \frac{1}{\kappa}) \left(\|\partial_{tx}\zeta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 + \|\partial_{tx}\eta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 \right) \\
& \quad + \Lambda\beta \|\partial_{xt}\zeta_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 + \Lambda\alpha \|\partial_{txx}\eta_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 + \Lambda\beta \int_0^L \partial_{xx}\zeta_\varepsilon(t) \partial_t \zeta_\varepsilon(t) dx + \Lambda\alpha \int_0^L \partial_{xxx}\eta_\varepsilon(t) \partial_{tx}\eta_\varepsilon(t) dx,
\end{aligned}$$

It follows by choosing $\Lambda = \min\left(\frac{\rho_s}{2\beta}, \frac{\gamma}{2\alpha}, \frac{1}{2}, \frac{\rho_s}{2\alpha}\right)$ and by using Young's inequality that

$$\begin{aligned}
& \frac{\Lambda\rho_f}{2R} \|\partial_t v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 + \frac{\Lambda\rho_s}{2} \|\partial_{tt}\zeta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 + \frac{\Lambda\rho_s}{2} \|\partial_{tt}\eta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 + \frac{\Lambda\mu\lambda(R)}{4} \|\nabla v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 \\
& \quad + \gamma\Lambda \|\partial_{xtt}\eta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 + \frac{\Lambda\varepsilon}{2} \left(\|\partial_{xxt}\zeta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 + \|\partial_{xxt}\eta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 \right) \\
& \quad + \frac{\rho_f}{2R} \|\partial_x v_\varepsilon(t)\|_{L^2_\#(\Omega_0)}^2 + \frac{\rho_s}{2} \|\partial_{tx}\zeta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 + \frac{\rho_s}{2} \|\partial_{tx}\eta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 + \frac{\mu\lambda(R)}{4} \|\nabla \partial_x v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 \\
& + \frac{\beta}{2} \|\partial_{xx}\zeta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 + \frac{\alpha}{2} \|\partial_{xxx}\eta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 + \frac{\gamma}{2} \|\partial_{txx}\eta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 + \varepsilon \left(\|\partial_{xxx}\zeta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 + \|\partial_{xxx}\eta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 \right) \\
& \leq C_0(R) + (2\delta^2 K(R) + 2\delta K(R) + 3\kappa) \|\partial_t v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 + K(R, \frac{1}{\kappa}) \left(\|\partial_{tx}\zeta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 + \|\partial_{tx}\eta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 \right) \\
& \quad + \frac{\rho_s}{2} \|\partial_{xt}\zeta_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 + \frac{\gamma}{2} \|\partial_{txx}\eta_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 + \frac{\beta}{4} \|\partial_{xx}\zeta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 + \frac{\rho_s}{4} \|\partial_t \zeta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 \\
& \quad \quad \quad + \frac{\alpha}{4} \|\partial_{xxx}\eta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 + \frac{\rho_s}{4} \|\partial_{tx}\eta_\varepsilon(t)\|_{L^2_\#(0,L)}^2
\end{aligned}$$

Hence, by choosing κ and δ small enough such that $2\delta^2 K(R) + 2\delta K(R) + 3\kappa < \frac{\Lambda\rho_f}{4R}$ remembering the energy estimate (3.26) it follows that

$$\begin{aligned}
& \frac{\Lambda\rho_f}{4R} \|\partial_t v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 + \frac{\Lambda\rho_s}{2} \|\partial_{tt}\zeta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 + \frac{\Lambda\rho_s}{2} \|\partial_{tt}\eta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 + \frac{\Lambda\mu\lambda(R)}{4} \|\nabla v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 \\
& \quad + \gamma\Lambda \|\partial_{xtt}\eta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 + \frac{\Lambda\varepsilon}{2} \left(\|\partial_{xxt}\zeta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 + \|\partial_{xxt}\eta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 \right) \\
& \quad + \frac{\rho_f}{2R} \|\partial_x v_\varepsilon(t)\|_{L^2_\#(\Omega_0)}^2 + \frac{\rho_s}{2} \|\partial_{tx}\zeta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 + \frac{\rho_s}{4} \|\partial_{tx}\eta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 + \frac{\mu\lambda(R)}{4} \|\nabla \partial_x v_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2 \\
& + \frac{\beta}{4} \|\partial_{xx}\zeta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 + \frac{\alpha}{4} \|\partial_{xxx}\eta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 + \frac{\gamma}{2} \|\partial_{txx}\eta_\varepsilon(t)\|_{L^2_\#(0,L)}^2 + \varepsilon \left(\|\partial_{xxx}\zeta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 + \|\partial_{xxx}\eta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 \right) \\
& \leq C_0(R) + \left(K(R, \frac{1}{\kappa}) + \frac{\rho_s}{2} \right) \|\partial_{tx}\zeta_\varepsilon\|_{L^2(0,t;L^2_\#(0,L))}^2 + \frac{\gamma}{2} \|\partial_{txx}\eta_\varepsilon\|_{L^2(0,t;L^2_\#(\Omega_0))}^2. \quad (3.61)
\end{aligned}$$

From the above inequality we deduce

$$\int_0^L |\partial_{xt}\zeta_\varepsilon|^2 dx ds + \int_0^L |\partial_{xxt}\eta_\varepsilon|^2 dx ds \leq \frac{2}{\min(\rho_s, \gamma)} \left(C_0(R) + M \left(\int_0^t \int_0^L |\partial_{xt}\zeta_\varepsilon|^2 dx ds + \int_0^t \int_0^L |\partial_{xxt}\eta_\varepsilon|^2 dx ds \right) \right),$$

where $M = \max\left(K(R, \frac{1}{\kappa}) + \frac{\rho_s}{2}, \frac{\gamma}{2}\right)$.

Gronwall's Lemma implies

$$\int_0^L |\partial_{xt}\zeta_\varepsilon|^2 dx ds + \int_0^L |\partial_{xxt}\eta_\varepsilon|^2 dx ds \leq \frac{2C_0(R)}{\min(\rho_s, \gamma)} \exp\left(\frac{2MT}{\min(\rho_s, \gamma)}\right).$$

Combining (3.61) with the above inequality implies for $T \leq T^*$

$$\begin{aligned} & \frac{\Lambda\rho_f}{4R} \|\partial_t v_\varepsilon\|_{L^2(0,t;L^2_\sharp(\Omega_0))}^2 + \frac{\Lambda\rho_s}{2} \|\partial_{tt}\zeta_\varepsilon\|_{L^2(0,t;L^2_\sharp(0,L))}^2 + \frac{\Lambda\rho_s}{2} \|\partial_{tt}\eta_\varepsilon\|_{L^2(0,t;L^2_\sharp(0,L))}^2 + \gamma\Lambda \|\partial_{xtt}\eta_\varepsilon\|_{L^2(0,t;L^2_\sharp(0,L))}^2 \\ & + \frac{\rho_f}{2R} \|\partial_x v_\varepsilon(t)\|_{L^2_\sharp(\Omega_0)}^2 + \frac{\rho_s}{2} \|\partial_{tx}\zeta_\varepsilon(t)\|_{L^2_\sharp(0,L)}^2 + \frac{\rho_s}{4} \|\partial_{tx}\eta_\varepsilon(t)\|_{L^2_\sharp(0,L)}^2 + \frac{\mu\lambda(R)}{4} \|\nabla\partial_x v_\varepsilon\|_{L^2(0,t;L^2_\sharp(\Omega_0))}^2 \\ & + \frac{\beta}{4} \|\partial_{xx}\zeta_\varepsilon(t)\|_{L^2_\sharp(0,L)}^2 + \frac{\alpha}{4} \|\partial_{xxx}\eta_\varepsilon(t)\|_{L^2_\sharp(0,L)}^2 + \frac{\gamma}{2} \|\partial_{txx}\eta_\varepsilon(t)\|_{L^2_\sharp(0,L)}^2 \leq C_{0,T^*}(R), \end{aligned} \quad (3.62)$$

with $C_{0,T^*}(R)$ non decreasing with respect to R . The elliptic estimate (3.47) further implies

$$\|\partial_{yy}v_\varepsilon\|_{L^2(0,t;L^2_\sharp(\Omega_0))} + \|\nabla q_\varepsilon\|_{L^2(0,t;L^2_\sharp(\Omega_0))} \leq C_{0,T^*}(R). \quad (3.63)$$

where C_0 depends again only on initial data, source terms, R and the different positive physical parameters.

Limit as ε goes to zero and uniqueness. By standard compactness arguments, we may extract a weakly converging subsequence in the family of ε - solutions $(v_\varepsilon, q_\varepsilon, \zeta_\varepsilon, \eta_\varepsilon)$ satisfying (3.26), (3.62) and (3.63). Then applying classical arguments, we obtain that the weak limit (v, q, ζ, η) is a solution to our linear problem with the expected regularity and satisfying (3.14). We left to the reader all these computations as they are completely standard. Finally, suppose that we have two solutions to our linear system with the same data, then the difference would satisfy (3.26) with vanishing data. We deduce the uniqueness of the solution to the homogeneous linear problem (3.1). This ends the proof of Proposition 3.1.

3.3 Non homogeneous divergence constraint.

In this subsection, we study the linear problem (3.1)-(3.7) in case when g is not equal to zero. More precisely, we establish the following proposition:

Proposition 3.2 *Let $(a, b) \in H^2_\sharp(0, L)$ such that $\chi_{a,b}$ is a injective on $\overline{\mathcal{F}}_0$. Given $T > 0$, initial data $(v_0, \zeta_0, \zeta_1, \eta_0, \eta_1) \in X^0$ and a source term $(f, h, g, 0, 0) \in S_T$. Then there exists a unique solution $(v, q, \zeta, \eta) \in X_{f,T} \times X_{s,T}$ to Problem (3.1)-(3.7) and a non decreasing function $K : [0, \infty) \rightarrow [0, \infty)$ such that assuming $\|(a, b)\|_{H^2_\sharp(0,L)} + \|(\det \nabla \chi_{a,b})^{-1}\|_{C_\sharp(\overline{\Omega}_0)} \leq R$, the solution (v, q, ζ, η) satisfies*

$$\|(v, q)\|_{X_{f,T}} + \|(\zeta, \eta)\|_{X_{s,T}} \leq K(R)(\|(v_0, \zeta_0, \zeta_1, \eta_0, \eta_1)\|_{X^0} + \|(f, h, g, 0, 0)\|_{S_T}) \quad (3.64)$$

Proof. We look for (v, q, ζ, η) solution to (3.1)-(3.7) with initial data $(v_0, \zeta_0, \zeta_1, \eta_0, \eta_1) \in X^0$ and source term $(f, h, g, 0, 0) \in S_T$. We are going to lift the right hand side of the divergence in order to apply the previous result obtained in the case of divergence free constraint. To do so we need to modify the fluid velocity unknown but also the elastic displacement one. This is due to the fact that the average of g is not equal to zero. It is a key difference compared to the case where the longitudinal displacement is neglected. Thus let us set $\delta_g = \int_{\Omega_0} g dx$. As

$g \in L^2(0, T; H_{\sharp}^1(\Omega_0)) \cap H^1(0, T; (H_{\sharp}^1(\Omega_0))')$, then $\delta_g = \langle g, 1 \rangle_{(H_{\sharp}^1(\Omega_0))', H_{\sharp}^1(\Omega_0)} \in H^1(0, T)$, and by Proposition 2.3 there exists unique couple $(w, q_1) \in L^2(0, T; H_{\sharp}^2(\Omega_0)) \times L^2(0, T; H_{\sharp}^1(\Omega_0) \cap L_{\sharp,0}^2(\Omega_0))$ such that

$$-\mu \operatorname{div} \left((A_{a,b} \nabla) w + \frac{1}{\det \nabla \chi_{a,b}} B_{a,b} \nabla w^T B_{a,b} \right) + (B_{a,b} \nabla) q_1 = 0, \quad \text{on } \Omega_0, \quad (3.65)$$

$$\operatorname{div} (B_{a,b}^T w) = g, \quad \text{in } \Omega_0, \quad (3.66)$$

$$w|_{y=1} = \left(0, \frac{\delta_g}{L} \right), \quad (3.67)$$

$$w|_{y=0} = 0. \quad (3.68)$$

that further satisfies

$$\|w\|_{L^2(0,T;H_{\sharp}^2(\Omega_0))} + \|q_1\|_{L^2(0,T;H_{\sharp}^1(\Omega_0))} \leq K(R) \left(\|g\|_{L^2(0,T;H_{\sharp}^1(\Omega_0))} + \|\delta_g\|_{L^2(0,T)} \right) \leq K(R) \|g\|_{L^2(0,T;H_{\sharp}^1(\Omega_0))}. \quad (3.69)$$

Now, we are in position to derive an estimate of $\partial_t w$ in $L^2(0, T; L^2(\Omega_0))$. Let $\varphi \in \mathcal{D}(\Omega_0 \times (0, T))$, then by integration by parts we have

$$\int_0^T \int_{\Omega_0} \partial_t w \cdot \varphi \, dx dy dt = - \int_0^T \int_{\Omega_0} w \cdot \partial_t \varphi \, dx dy dt. \quad (3.70)$$

Again by using Proposition 2.3 there exists unique couple $(z_{\varphi}, q_{\varphi}) \in C_c^1([0, T]; H_{\sharp}^2(\Omega_0)) \times C_c^1([0, T]; H_{\sharp}^1(\Omega_0) \cap L_{\sharp,0}^2(\Omega_0))$, such that

$$\begin{aligned} -\mu \operatorname{div} \left((A_{a,b} \nabla) z_{\varphi} + \frac{1}{\det \nabla \chi_{a,b}} B_{a,b} \nabla z_{\varphi}^T B_{a,b} \right) + (B_{a,b} \nabla) q_{\varphi} &= \varphi, & \text{in } \Omega_0, \\ \operatorname{div} (B_{a,b}^T z_{\varphi}) &= 0, & \text{in } \Omega_0, \\ z_{\varphi}|_{y=1} &= 0, \quad z_{\varphi}|_{y=0} = 0, \end{aligned}$$

and satisfying

$$\|z_{\varphi}\|_{L^2(0,T;H_{\sharp}^2(\Omega_0))} + \|q_{\varphi}\|_{L^2(0,T;H_{\sharp}^1(\Omega_0))} \leq K(R) \|\varphi\|_{L^2(0,T;L_{\sharp}^2(\Omega_0))}. \quad (3.71)$$

It follows that

$$\int_0^T \int_{\Omega_0} \partial_t w \cdot \varphi \, dx dy dt = - \int_0^T \int_{\Omega_0} w \cdot \left(-\mu \operatorname{div} \left((A_{a,b} \nabla) \partial_t z_{\varphi} + \frac{1}{\det \nabla \chi_{a,b}} B_{a,b} \nabla \partial_t z_{\varphi}^T B_{a,b} \right) + (B_{a,b} \nabla) \partial_t q_{\varphi} \right) \, dx dy dt. \quad (3.72)$$

By integration by parts and since $\operatorname{div} (B_{a,b}^T w) = g$, $w|_{y=0} = 0$ and $w|_{y=1} = (0, \frac{\delta_g}{L})$, we obtain

$$- \int_{\Omega_0} (B_{a,b} \nabla) \partial_t q_{\varphi} \cdot w \, dx dy = \int_{\Omega_0} g \partial_t q_{\varphi} \, dx dy - \frac{\delta_g}{L} \int_0^L \partial_t q_{\varphi}|_{y=1} (1 + \partial_x a) \, dx \quad (3.73)$$

On the other hand, by integration by parts and using again that $w|_{y=0} = 0$ and $w|_{y=1} = (0, \frac{\delta_g}{L})$, we get

$$\begin{aligned} \int_{\Omega_0} \operatorname{div} \left((A_{a,b} \nabla) \partial_t z_{\varphi} + \frac{1}{\det \nabla \chi_{a,b}} B_{a,b} \nabla \partial_t z_{\varphi}^T B_{a,b} \right) \cdot w \, dx dy \\ = - \int_{\Omega_0} \left((A_{a,b} \nabla) \partial_t z_{\varphi} + \frac{1}{\det \nabla \chi_{a,b}} B_{a,b} \nabla \partial_t z_{\varphi}^T B_{a,b} \right) : \nabla w \, dx dy \\ + \frac{\delta_g}{L} \int_0^L \left((A_{a,b} \nabla) \partial_t z_{\varphi} + \frac{1}{\det \nabla \chi_{a,b}} B_{a,b} \nabla \partial_t z_{\varphi}^T B_{a,b} \right) \Big|_{y=1} \cdot e_2 \cdot e_2 \, dx. \end{aligned}$$

Since $A_{a,b}$ is symmetric, we get

$$\begin{aligned} \int_{\Omega_0} \operatorname{div} \left((A_{a,b} \nabla) \partial_t z_\varphi + \frac{1}{\det \nabla \chi_{a,b}} B_{a,b} \nabla \partial_t z_\varphi^T B_{a,b} \right) \cdot w \, dx dy \\ = - \int_{\Omega_0} \left((A_{a,b} \nabla) w + \frac{1}{\det \nabla \chi_{a,b}} B_{a,b} \nabla w^T B_{a,b} \right) : \nabla \partial_t z_\varphi \, dx dy \\ + \frac{\delta g}{L} \int_0^L \left((A_{a,b} \nabla) \partial_t z_\varphi + \frac{1}{\det \nabla \chi_{a,b}} B_{a,b} \nabla \partial_t z_\varphi^T B_{a,b} \right) \Big|_{y=1} e_2 \cdot e_2 \, dx. \end{aligned}$$

Again by integration by parts and using that $z_\varphi|_{y=0} = z_\varphi|_{y=1} = 0$, we have

$$\begin{aligned} \int_{\Omega_0} \operatorname{div} \left((A_{a,b} \nabla) \partial_t z_\varphi + \frac{1}{\det \nabla \chi_{a,b}} B_{a,b} \nabla \partial_t z_\varphi^T B_{a,b} \right) \cdot w \, dx dy \\ = \int_{\Omega_0} \operatorname{div} \left((A_{a,b} \nabla) w + \frac{1}{\det \nabla \chi_{a,b}} B_{\zeta_0, \eta_0} \nabla w^T B_{a,b} \right) \cdot \partial_t z_\varphi \, dx dy \\ + \frac{\delta g}{L} \int_0^L \left((A_{a,b} \nabla) \partial_t z_\varphi + \frac{1}{\det \nabla \chi_{a,b}} B_{a,b} \nabla \partial_t z_\varphi^T B_{a,b} \right) \Big|_{y=1} e_2 \cdot e_2 \, dx. \end{aligned}$$

It follows thanks to (3.65) that

$$\begin{aligned} \int_{\Omega_0} \operatorname{div} \left((A_{a,b} \nabla) \partial_t z_\varphi + \frac{1}{\det \nabla \chi_{a,b}} B_{a,b} \nabla \partial_t z_\varphi^T B_{a,b} \right) \cdot w \, dx dy = - \int_{\Omega_0} ((B_{a,b} \nabla) q_1) \cdot \partial_t z_\varphi \, dx dy \\ + \frac{\delta g}{L} \int_0^L \left((A_{a,b} \nabla) \partial_t z_\varphi + \frac{1}{\det \nabla \chi_{a,b}} B_{a,b} \nabla \partial_t z_\varphi^T B_{a,b} \right) \Big|_{y=1} e_2 \cdot e_2 \, dx. \end{aligned}$$

Consequently, performing an integration by parts in the first term in the right hand side and by noting that $\operatorname{div} (B_{a,b}^T \partial_t z_\varphi) = 0$ and $z_\varphi|_{y=0} = z_\varphi|_{y=1} = 0$, we obtain

$$\begin{aligned} \int_{\Omega_0} \operatorname{div} \left((A_{a,b} \nabla) \partial_t z_\varphi + \frac{1}{\det \nabla \chi_{a,b}} B_{a,b} \nabla \partial_t z_\varphi^T B_{a,b} \right) \cdot w \, dx dy \\ = \frac{\delta g}{L} \int_0^L \left((A_{a,b} \nabla) \partial_t z_\varphi + \frac{1}{\det \nabla \chi_{a,b}} B_{a,b} \nabla \partial_t z_\varphi^T B_{a,b} \right) \Big|_{y=1} e_2 \cdot e_2 \, dx. \end{aligned}$$

Combining the above equality with (3.72) in view of (3.73), we get for any $\varphi \in \mathcal{D}(\Omega_0 \times (0, T))$

$$\begin{aligned} \int_0^T \int_{\Omega_0} \partial_t w \cdot \varphi \, dx \, dy \, dt = \int_0^T \int_{\Omega_0} g \partial_t q_\varphi \, dx \, dy \, dt + \int_0^T \frac{\delta g}{L} \left(- \int_0^L \partial_t q_\varphi|_{y=1} (1 + \partial_x a) \, dx \right. \\ \left. + \int_0^L \left((A_{a,b} \nabla) \partial_t z_\varphi + \frac{1}{\det \nabla \chi_{a,b}} B_{a,b} \nabla \partial_t z_\varphi^T B_{a,b} \right) \Big|_{y=1} e_2 \cdot e_2 \, dx \right) dt. \quad (3.74) \end{aligned}$$

By integrating the right hand side with respect to time, we obtain for any $\varphi \in \mathcal{D}(\Omega_0 \times (0, T))$

$$\begin{aligned} \int_0^T \int_{\Omega_0} \partial_t w \cdot \varphi \, dx \, dy \, dt = - \int_0^T \langle \partial_t g, q_\varphi \rangle_{(H_{\sharp}^1(\Omega_0))', H^1(\Omega_0)} \, dt - \int_0^T \frac{\delta g}{L} \left(- \int_0^L q_\varphi|_{y=1} (1 + \partial_x a) \, dx \right. \\ \left. + \int_0^L \left((A_{a,b} \nabla) z_\varphi + \frac{1}{\det \nabla \chi_{a,b}} B_{a,b} \nabla z_\varphi^T B_{a,b} \right) \Big|_{y=1} e_2 \cdot e_2 \, dx \right) dt. \quad (3.75) \end{aligned}$$

It follows

$$\begin{aligned} \left| \int_0^T \int_{\Omega_0} \partial_t w \cdot \varphi \, dx \, dy \, dt \right| \leq K(R) (\|\partial_t g\|_{L^2(0,T;H_{\sharp}^1(\Omega_0))'} \|q_\varphi\|_{L^2(0,T;H_{\sharp}^1(\Omega_0))} + \|\dot{\delta}_g\|_{L^2(0,T)} \|q_\varphi\|_{L^2(0,T;H_{\sharp}^1(\Omega_0))} \\ + \|\dot{\delta}_g\|_{L^2(0,T)} \|z_\varphi\|_{L^2(0,T;H_{\sharp}^2(\Omega_0))}). \end{aligned}$$

In view of (3.71) this implies

$$\left| \int_0^T \int_{\Omega_0} \partial_t w \cdot \varphi \, dx \, dy \, dt \right| \leq K(R) \|\partial_t g\|_{L^2(0,T;(H_{\sharp}^1(\Omega_0))')} \|\varphi\|_{L^2(0,T;L_{\sharp}^2(\Omega_0))}.$$

Hence,

$$\|\partial_t w\|_{L^2(0,T;L_{\sharp}^2(\Omega_0))} \leq K(R) \|\partial_t g\|_{L^2(0,T;H_{\sharp}^1(\Omega_0)')}. \quad (3.76)$$

Let us set $\tilde{v} = v - w$ and $\tilde{q} = q - q_1$, then (\tilde{v}, \tilde{q}) satisfies in $\Omega_0 \times (0, T)$

$$\rho_{f,a,b} \partial_t \tilde{v} - \operatorname{div} \left(\mu(A_{a,b} \nabla) \tilde{v} + \frac{\mu}{\det \nabla \chi_{a,b}} B_{a,b} \nabla \tilde{v}^T B_{a,b} - \tilde{q} B_{a,b} \right) = f + \operatorname{div} h - \rho_{f,a,b} \partial_t w, \quad (3.77)$$

$$\operatorname{div}(B_{a,b}^T \tilde{v}) = 0, \quad (3.78)$$

and the boundary conditions

$$\tilde{v}|_{y=1} = \left(\partial_t \zeta, \partial_t \eta - \frac{\delta g}{L} \right), \quad (3.79)$$

$$\tilde{v}|_{y=0} = 0. \quad (3.80)$$

Due to (3.79) we need also to change of unknown for the structure displacement. Let us denote

$$(\tilde{\zeta}, \tilde{\eta}) = \left(\zeta, \eta - \int_0^t \frac{\delta g(s)}{L} ds \right), \quad (3.81)$$

we end up with the following coupled system

$$\begin{aligned} \rho_{f,a,b} \partial_t \tilde{v} - \operatorname{div} \left(\mu(A_{a,b} \nabla) \tilde{v} + \frac{\mu}{\det \nabla \chi_{a,b}} B_{a,b} \nabla \tilde{v}^T B_{a,b} - \tilde{q} B_{a,b} \right) \\ = f + \operatorname{div} h - \rho_{f,a,b} \partial_t w, \quad \text{in } \Omega_0 \times (0, T), \end{aligned} \quad (3.82)$$

$$\operatorname{div}(B_{a,b}^T \tilde{v}) = 0, \quad \text{in } \Omega_0 \times (0, T), \quad (3.83)$$

$$\tilde{v}|_{y=1} = (\partial_t \tilde{\zeta}, \partial_t \tilde{\eta}), \quad (3.84)$$

$$\tilde{v}|_{y=0} = 0, \quad (3.85)$$

$$\rho_s \partial_{tt} \tilde{\zeta} - \beta \partial_{xx} \tilde{\zeta} = -T_{a,b}[\tilde{v}, \tilde{q}] \cdot e_1 - h|_{y=1} \cdot e_1 + \ell, \quad \text{in } (0, L), \quad (3.86)$$

$$\rho_s \partial_{tt} \tilde{\eta} + \alpha \partial_{xxxx} \tilde{\eta} - \gamma \partial_{ttxx} \tilde{\eta} = -T_{a,b}[\tilde{v}, \tilde{q}] \cdot e_2 - h|_{y=1} \cdot e_2 + m, \quad \text{in } (0, L). \quad (3.87)$$

with

$$\ell = -T_{a,b}[w, q_1] \cdot e_1, \quad (3.88)$$

$$m = -T_{a,b}[w, q_1] \cdot e_2 - \rho_s \frac{\delta g}{L}. \quad (3.89)$$

By Proposition 3.1, there exists $(\tilde{v}, \tilde{q}, \tilde{\zeta}, \tilde{\eta}) \in X_{f,T} \times X_{s,T}$ such that

$$\begin{aligned} \|(\tilde{v}, \tilde{q})\|_{X_{f,T}} + \|(\tilde{\zeta}, \tilde{\eta})\|_{X_{s,T}} \leq K(R) \left(\|(v_0, \zeta_0, \zeta_1, \eta_0, \eta_1)\|_{X^0} + \|(f, h, 0, 0, 0)\|_{S_T} \right. \\ \left. + \|\partial_t w\|_{L^2(0,T;L_{\sharp}^2(\Omega_0))} + \|w\|_{L^2(0,T;H_{\sharp}^2(\Omega_0))} + \|\delta g\|_{L^2(0,T)} \right). \end{aligned} \quad (3.90)$$

Combining the above inequality with estimates (3.69)-(3.76) implies, for $T \leq T^*$

$$\begin{aligned} \|(\tilde{v}, \tilde{q})\|_{X_{f,T}} + \|(\tilde{\zeta}, \tilde{\eta})\|_{X_{s,T}} &\leq K_{T^*}(R) \left(\|(v_0, \zeta_0, \zeta_1, \eta_0, \eta_1)\|_{X^0} + \|(f, h, 0, 0, 0)\|_{S_T} \right. \\ &\quad \left. + \|\partial_t g\|_{L^2(0,T;(H_{\sharp}^1(\Omega_0))')} + \|g\|_{L^2(0,T;H_{\sharp}^1(\Omega_0))} \right). \end{aligned}$$

Hence,

$$\|(\tilde{v}, \tilde{q})\|_{X_{f,T}} + \|(\tilde{\zeta}, \tilde{\eta})\|_{X_{s,T}} \leq K_{T^*}(R) \left(\|(v_0, \zeta_0, \zeta_1, \eta_0, \eta_1)\|_{X^0} + \|(f, h, g, 0, 0)\|_{S_T} \right).$$

By noting that $v = \tilde{v} + w$, $q = \tilde{q} + q_1$, $\zeta = \tilde{\zeta}$, $\eta = \tilde{\eta} + 1/L \int_0^t \delta_g(s) ds$ and using (3.69)-(3.76), we finally obtain the existence and uniqueness of a solution $(v, q, \zeta, \eta) \in X_{f,T} \times X_{s,T}$ of Problem (3.1)-(3.7) satisfying the following estimates

$$\|(v, q)\|_{X_{f,T}} + \|(\zeta, \eta)\|_{X_{s,T}} \leq K_{T^*}(R) \left(\|(v_0, \zeta_0, \zeta_1, \eta_0, \eta_1)\|_{X^0} + \|(f, h, g, 0, 0)\|_{S_T} \right). \quad (3.91)$$

This concludes the study of the linearized non homogeneous problem.

4 Proof of Theorem 2.1

Consider $(v_0, \zeta_0, \zeta_1, \eta_0, \eta_1) \in H_{\sharp}^1(\mathcal{F}(0)) \times H_{\sharp}^2(0, L) \times H_{\sharp}^1(0, L) \times H_{\sharp}^3(0, L) \times H_{\sharp}^1(0, L)$ such that (2.3) is satisfied and such that there exists a mapping χ_{ζ_0, η_0} satisfying $\det \nabla \chi_{\zeta_0, \eta_0} > 0$ on $\bar{\Omega}_0$ and χ_{ζ_0, η_0} is injective on $\bar{\mathcal{F}}_0$. We denote by

$$R := \|u_0\|_{H^1(\mathcal{F}(0))} + \|\zeta_0\|_{H_{\sharp}^2(0,L)} + \|\zeta_1\|_{H_{\sharp}^1(0,L)} + \|\eta_0\|_{H_{\sharp}^3(0,L)} + \|\eta_1\|_{H_{\sharp}^1(0,L)} + \|(\det \nabla \chi_{\zeta_0, \eta_0})^{-1}\|_{C_{\sharp}(\bar{\Omega}_0)}.$$

For $T > 0$ and $M > 0$, we define the set

$$\mathcal{K}_T(M) := \{(z, r, \bar{\zeta}, \bar{\eta}) \in X_{f,T} \times E_{s,T}^0, z|_{y=0} = 0, z|_{y=1} = \partial_t \zeta e_1 + \partial_t \eta e_2, \|(z, r, \bar{\zeta}, \bar{\eta})\|_{X_{f,T} \times X_{s,T}} \leq M\}, \quad (4.1)$$

where

$$E_{s,T}^0 := \{(\bar{\zeta}, \bar{\eta}) \in X_{s,T} : \bar{\zeta}|_{t=0} = \bar{\eta}|_{t=0} = 0\}.$$

We define the mapping \mathcal{L}_T^0 from the set $\mathcal{K}_T(M)$ into $X_{f,T} \times E_{s,T}^0$ as follows:

$$\mathcal{L}_T^0(z, r, \bar{\zeta}, \bar{\eta}) = (v, q, \bar{\mathcal{X}}, \bar{\mathcal{Y}}),$$

with $(v, q, \bar{\mathcal{X}}, \bar{\mathcal{Y}}) = (v, q, \mathcal{X} - \zeta_0, \mathcal{Y} - \eta_0)$ and $(v, q, \mathcal{X}, \mathcal{Y})$ is a solution of Problem (3.1)-(3.6) for $(a, b) = (\zeta_0, \eta_0)$ and source term equals to $(f, h, g, 0, 0)$, where f, g, h are defined as follows:

$$f[z, \zeta, \eta] = \rho_f(\det \nabla \chi_{\zeta_0, \eta_0} - \det \nabla \chi_{\zeta, \eta}) \partial_t z - \rho_f((z - \partial_t \chi_{\zeta, \eta}) \cdot (B_{\zeta, \eta} \nabla)) z, \quad (4.2)$$

$$\begin{aligned} h[z, r, \zeta, \eta] = -\mu[(A_{\zeta_0, \eta_0} - A_{\zeta, \eta}) \nabla] z - \mu \left(\frac{1}{\det \nabla \chi_{\zeta_0, \eta_0}} B_{\zeta_0, \eta_0} \nabla z^T B_{\zeta_0, \eta_0} \right. \\ \left. - \frac{1}{\det \nabla \chi_{\zeta, \eta}} B_{\zeta, \eta} \nabla z^T B_{\zeta, \eta} \right) + r(B_{\zeta_0, \eta_0} - B_{\zeta, \eta}). \end{aligned} \quad (4.3)$$

$$g[z, \zeta, \eta] = \operatorname{div}((B_{\zeta_0, \eta_0}^T - B_{\zeta, \eta}^T) z) = (B_{\zeta_0, \eta_0}^T - B_{\zeta, \eta}^T) : \nabla z, \quad (4.4)$$

where $(\zeta, \eta) = (\bar{\zeta} + \zeta_0, \bar{\eta} + \eta_0)$. We will prove that the source terms f, g, h defined through (4.2) -(4.4) are in the good spaces so that \mathcal{L}_T^0 is well defined. We underline that $(v, q, \bar{\zeta}, \bar{\eta})$ is a fixed point of the mapping \mathcal{L}_T^0 if and only if (v, q, ζ, η) is a solution of Problem (2.19) with initial data (3.2)-(3.6) on $\Omega_0 \times (0, T_0)$. Moreover, if the mapping \mathcal{L}_T^0 admits a unique fixed point in $\mathcal{K}_T(M)$ for any $T \leq T_0$, then Problem (2.19) with initial data (3.2)-(3.6) has a unique strong solution on $\Omega_0 \times (0, T_0)$. Hence, to prove Theorem 2.1 we will establish that for a well chosen $M \in (0, \infty)$ and T small enough, depending on R and M :

- i. $\mathcal{L}_T^0(\mathcal{K}_T(M)) \subset \mathcal{K}_T(M)$;
- ii. the mapping $\mathcal{L}_T^0 : \mathcal{K}_T(M) \rightarrow \mathcal{K}_T(M)$ is a contraction.

We start to establish the first assertion *i.e.* for T small enough and $M \geq M_0$, the mapping \mathcal{L}_T^0 maps $\mathcal{K}_T(M)$ into itself. Let $(z, r, \bar{\zeta}, \bar{\eta}) \in \mathcal{K}_T(M)$ and denote by $(v, q, \bar{\mathcal{X}}, \bar{\mathcal{Y}}) = \mathcal{L}_T^0(z, r, \bar{\zeta}, \bar{\eta})$. We recall that from the definition of \mathcal{L}_T^0 , we have $(\bar{\zeta}, \bar{\eta}) = (\zeta - \zeta_0, \eta - \eta_0)$ with $(v, q, \mathcal{X}, \mathcal{Y})$ is a solution of Problem (3.1)-(3.6) for $(a, b) = (\zeta_0, \eta_0)$ and source term $(f, h, g, 0, 0)$ defined through (4.2)-(4.4). If the source term $(f, h, g, 0, 0) \in S_T$, then by applying Proposition 3.2 for $(a, b) = (\zeta_0, \eta_0)$ we get that there exists a unique solution $(v, q, \mathcal{X}, \mathcal{Y}) \in X_{f,T} \times X_{s,T}$ of Problem (3.1)-(3.7) satisfying

$$\|(v, q)\|_{X_{f,T}} + \|(\mathcal{X}, \mathcal{Y})\|_{X_{s,T}} \leq C(R)(\|(v_0, \zeta_0, \zeta_1, \eta_0, \eta_1)\|_{X^0} + \|(f, h, g, 0, 0)\|_{S_T}).$$

It follows that

$$\|\mathcal{L}_T^0(z, r, \bar{\zeta}, \bar{\eta})\|_{X_{f,T} \times X_{s,T}} \leq \|(\zeta_0, \eta_0)\|_{H_{\sharp}^2(0,L) \times H_{\sharp}^3(0,L)} + C(R)(\|(v_0, \zeta_0, \zeta_1, \eta_0, \eta_1)\|_{X^0} + \|(f, h, g, 0, 0)\|_{S_T}). \quad (4.5)$$

Taking $M \geq M_0 = 2(\|(\zeta_0, \eta_0)\|_{H_{\sharp}^2(0,L) \times H_{\sharp}^3(0,L)} + C(R)\|(v_0, \zeta_0, \zeta_1, \eta_0, \eta_1)\|_{X^0})$, we get

$$\|\mathcal{L}_T^0(z, r, \bar{\zeta}, \bar{\eta})\|_{X_{f,T} \times X_{s,T}} \leq \frac{M}{2} + C(R)\|(f, h, g, 0, 0)\|_{S_T}. \quad (4.6)$$

To proceed we need to bound the source terms f, h, g defined through (4.2)-(4.4) respectively in $L^2(0, T; L_{\sharp}^2(\Omega_0))$, $L^2(0, T; H_{\sharp}^1(\Omega_0))$, $L^2(0, T; H_{\sharp}^1(\Omega_0)) \cap H_{00}^1(0, T; (H_{\sharp}^1(\Omega_0))')$ in the right hand side of (4.6). In order to establish such bounds, we gather in the following lemma some properties of the matrix $B_{\zeta, \eta}$ and $A_{\zeta, \eta}$ that are defined by (2.11) and (2.12) respectively.

Lemma 4.1 *Assuming that $(\zeta, \eta) \in L^\infty(0, T; H_{\sharp}^2(0, L)) \cap W^{1, \infty}(0, T; H_{\sharp}^1(0, L))$ and $(\zeta, \eta)|_{t=0} = (\zeta_0, \eta_0) \in H_{\sharp}^2(0, L)$, then, for $0 < s < \frac{1}{2}$*

$$\begin{aligned} \|B_{\zeta, \eta} - B_{\zeta_0, \eta_0}\|_{L^\infty(0, T; H_{\sharp}^{1+s}(\Omega_0))} &\leq CT^{\frac{1}{2}-s} \|(\bar{\zeta}, \bar{\eta})\|_{L^\infty(0, T; H_{\sharp}^2(0, L))}^{\frac{1}{2}+s} \|(\bar{\zeta}, \bar{\eta})\|_{W^{1, \infty}(0, T; H_{\sharp}^1(0, L))}^{\frac{1}{2}-s}, \\ \|\partial_t B_{\zeta, \eta}\|_{L^\infty(0, T; L_{\sharp}^4(\Omega_0))} &\leq C \|(\bar{\zeta}, \bar{\eta})\|_{L^\infty(0, T; H_{\sharp}^2(0, L)) \cap W^{1, \infty}(0, T; H_{\sharp}^1(0, L))}, \end{aligned}$$

where $(\bar{\zeta}, \bar{\eta}) = (\zeta - \zeta_0, \eta - \eta_0)$.

Furthermore, if we assume that

- $\|(\zeta_0, \eta_0)\|_{H_{\sharp}^2(0, L)} + \|(\det \nabla \chi_{\zeta_0, \eta_0})^{-1}\|_{C_{\sharp}(\bar{\Omega}_0)} \leq R$,
- $\|(\zeta, \eta)\|_{L^\infty(0, T; H_{\sharp}^2(0, L)) \cap W^{1, \infty}(0, T; H_{\sharp}^1(0, L))} \leq M$,

we have

$$\|\det \nabla \chi_{\zeta, \eta} - \det \nabla \chi_{\zeta_0, \eta_0}\|_{L^\infty(0, T; H_{\sharp}^{1+s}(\Omega_0))} \leq C(R, M)T^{\frac{1}{2}-s}$$

and there exists a time $T(R, M)$ depending on R and M such that for $T \leq T(R, M)$, $\det(\nabla \chi_{\zeta, \eta})^{-1}$ and $A_{\zeta, \eta}$ are well defined in $C([0, T]; H_{\sharp}^{1+s}(\Omega_0))$, for $0 < s < \frac{1}{2}$, and

$$\begin{aligned} \|(\det \nabla \chi_{\zeta, \eta})^{-1} - (\det \nabla \chi_{\zeta_0, \eta_0})^{-1}\|_{L^\infty(0, T; H_{\sharp}^{1+s}(\Omega_0))} &\leq C(R, M)T^{\frac{1}{2}-s} \\ \|A_{\zeta, \eta} - A_{\zeta_0, \eta_0}\|_{L^\infty(0, T; H_{\sharp}^{1+s}(\Omega_0))} &\leq C(R, M)T^{\frac{1}{2}-s}, \\ \|A_{\zeta, \eta} - A_{\zeta_0, \eta_0}\|_{L^\infty(0, T; H_{\sharp}^{1+s}(\Omega_0))} &\leq C(R, M)T^{\frac{1}{2}-s}, \end{aligned}$$

The proof of the above lemma relies on classical arguments and in particular the fact that $L^\infty(0, T; H_{\sharp}^2(0, L)) \cap W^{1, \infty}(0, T; H_{\sharp}^1(0, L))$ continuously embeds in $C^{0, 1/2-s}([0, T]; H_{\sharp}^{1+s}(\Omega_0)) \subset C([0, T]; C_{\sharp}^1(\bar{\Omega}_0))$, $\det(\nabla \chi_{\zeta, \eta})^{-1}$ and $A_{\zeta, \eta}$ are well defined in $C([0, T]; H_{\sharp}^{1+s}(\Omega_0))$, for $0 < s < \frac{1}{2}$, and that the following inequality is satisfied, for $0 < s < \frac{1}{2}$, for $\bar{\phi} \in L^\infty(0, T; H_{\sharp}^2(0, L)) \cap W^{1, \infty}(0, T; H_{\sharp}^1(0, L))$, such that $\bar{\phi}|_{t=0} = 0$,

$$\|\bar{\phi}\|_{L^\infty(0, T; H_{\sharp}^{\frac{3}{2}+s}(0, L))} \leq CT^{\frac{1}{2}-s} \|\bar{\phi}\|_{L^\infty(0, T; H_{\sharp}^2(0, L))}^{\frac{1}{2}+s} \|\bar{\phi}\|_{W^{1, \infty}(0, T; H_{\sharp}^1(0, L))}^{\frac{1}{2}-s}. \quad (4.7)$$

Moreover since $\|(\det \nabla \chi_{\zeta_0, \eta_0})^{-1}\|_{C_{\sharp}(\bar{\Omega}_0)} \leq R$ and $\|(\zeta, \eta)\|_{C([0, T]; C_{\sharp}^1(\bar{\Omega}_0))} \leq CM$, there exists a time $T(R, M)$ depending on R and M (decreasing with M and R) such that for $T \leq T(R, M)$, $\|(\det \nabla \chi_{\zeta, \eta})^{-1}\|_{C_{\sharp}(\bar{\Omega}_0)} \leq 2R$ ensuring that $\det(\nabla \chi_{\zeta, \eta})^{-1}$ and $A_{\zeta, \eta}$ are well defined in $C([0, T]; H_{\sharp}^{1+s}(\Omega_0))$, for $0 < s < \frac{1}{2}$.

Lemma 4.2 *Given $(z, r, \bar{\zeta}, \bar{\eta}) \in \mathcal{K}_T(M)$ and $(\zeta_0, \eta_0) \in H_{\sharp}^2(0, L)$ such that $\|(\zeta_0, \eta_0)\|_{H_{\sharp}^2(0, L)} + \|(\det \nabla \chi_{\zeta_0, \eta_0})^{-1}\|_{C_{\sharp}(\bar{\Omega}_0)} \leq R$. Let $f[z, \zeta, \eta], h[z, r, \zeta, \eta], g[z, \zeta, \eta]$, be the source terms defined respectively by (4.2)-(4.4) where $(\zeta, \eta) = (\bar{\zeta} + \zeta_0, \bar{\eta} + \eta_0)$, then there exists a time $T(M, R)$ depending on M and R such that for any $T \leq T(M, R)$*

$$\begin{aligned} \|f[z, \zeta, \eta]\|_{L^2(0, T; L_{\sharp}^2(\Omega_0))} &\leq C(R, M)T^{\frac{1}{4}}, \\ \|g[z, \zeta, \eta]\|_{L^2(0, T; H_{\sharp}^1(\Omega_0))} &\leq CT^{\frac{1}{4}}M^2, \\ \|g[z, \zeta, \eta]\|_{H_{00}^1(0, T; (H_{\sharp}^1(\Omega_0))')} &\leq C(R, M)T^{\frac{1}{4}}M^2, \\ \|h[z, r, \zeta, \eta]\|_{L^2(0, T; H_{\sharp}^1(\Omega_0))} &\leq C(R, M)T^{\frac{1}{4}}M^2. \end{aligned}$$

Proof. From the definition of f in (4.2), we have

$$\begin{aligned} \|f[z, \zeta, \eta]\|_{L^2(0, T; L_{\sharp}^2(\Omega_0))} &\leq \rho_f \left(\|(\det \nabla \chi_{\zeta_0, \eta_0} - \det \nabla \chi_{\zeta, \eta})\|_{L^\infty(0, T; L_{\sharp}^\infty(\Omega_0))} \|\partial_t z\|_{L^2(0, T; L_{\sharp}^2(\Omega_0))} \right. \\ &\left. + \|B_{\zeta, \eta}\|_{L^\infty(0, T; L_{\sharp}^\infty(\Omega_0))} \|z \nabla z\|_{L^2(0, T; L_{\sharp}^2(\Omega_0))} + T^{\frac{1}{2}} \|\partial_t \chi_{\zeta, \eta}\|_{L^\infty(0, T; L_{\sharp}^\infty(\Omega_0))} \|B_{\zeta, \eta}\|_{L^\infty(0, T; L_{\sharp}^\infty(\Omega_0))} \|z\|_{L^\infty(0, T; H_{\sharp}^1(\Omega_0))} \right). \end{aligned}$$

By using $\|z\|_{L_{\sharp}^4(\Omega_0)} \leq C \|z\|_{L_{\sharp}^2(\Omega_0)}^{\frac{1}{2}} \|z\|_{H_{\sharp}^1(\Omega_0)}^{\frac{1}{2}}$, we get

$$\|z \nabla z\|_{L^2(0, T; L_{\sharp}^2(\Omega_0))} \leq CT^{\frac{1}{4}} \|z\|_{L^\infty(0, T; L_{\sharp}^2(\Omega_0))}^{\frac{3}{2}} \|z\|_{L^2(0, T; H_{\sharp}^2(\Omega_0))}^{\frac{1}{2}}.$$

It follows that

$$\begin{aligned} \|f[z, \zeta, \eta]\|_{L^2(0, T; L_{\sharp}^2(\Omega_0))} &\leq \rho_f \left(\|(\det \nabla \chi_{\zeta_0, \eta_0} - \det \nabla \chi_{\zeta, \eta})\|_{L^\infty(0, T; L_{\sharp}^\infty(\Omega_0))} \|\partial_t z\|_{L^2(0, T; L_{\sharp}^2(\Omega_0))} \right. \\ &\left. + CT^{\frac{1}{4}} \|B_{\zeta, \eta}\|_{L^\infty(0, T; L_{\sharp}^\infty(\Omega_0))} \|z\|_{L^\infty(0, T; L_{\sharp}^2(\Omega_0))}^{\frac{3}{2}} \|z\|_{L^2(0, T; H_{\sharp}^2(\Omega_0))}^{\frac{1}{2}} \right. \\ &\left. + T^{\frac{1}{2}} \|\partial_t \chi_{\zeta, \eta}\|_{L^\infty(0, T; L_{\sharp}^\infty(\Omega_0))} \|B_{\zeta, \eta}\|_{L^\infty(0, T; L_{\sharp}^\infty(\Omega_0))} \|z\|_{L^\infty(0, T; H_{\sharp}^1(\Omega_0))} \right). \end{aligned}$$

Thus by using (4.7) for $s = \frac{1}{4}$ and the fact that $(z, r, \bar{\zeta}, \bar{\eta}) \in \mathcal{K}_T(M)$, we obtain for $T < 1$

$$\|f[z, \zeta, \eta]\|_{L^2(0, T; L^2_{\sharp}(\Omega_0))} \leq C(R, M)T^{\frac{1}{4}}.$$

By Piola's identity, we have $g[z, \zeta, \eta] = (B_{\zeta_0, \eta_0}^T - B_{\zeta, \eta}^T) : \nabla z$. Hence,

$$\begin{aligned} \|g[z, \zeta, \eta]\|_{L^2(0, T; H^1_{\sharp}(\Omega_0))} &\leq \|B_{\zeta_0, \eta_0} - B_{\zeta, \eta}\|_{L^\infty(0, T; L^\infty_{\sharp}(\Omega_0))} \left(\|\nabla z\|_{L^2(0, T; L^2_{\sharp}(\Omega_0))} + \|\nabla^2 z\|_{L^2(0, T; L^2_{\sharp}(\Omega_0))} \right) \\ &\quad + \|\nabla(B_{\zeta_0, \eta_0}^T - B_{\zeta, \eta}^T) : \nabla z\|_{L^2(0, T; L^2_{\sharp}(\Omega_0))}. \end{aligned}$$

Recalling Lemma 2.1, we have

$$\|\nabla(B_{\zeta_0, \eta_0}^T - B_{\zeta, \eta}^T) : \nabla z\|_{L^2(0, T; L^2_{\sharp}(\Omega_0))} \leq \|B_{\zeta_0, \eta_0} - B_{\zeta, \eta}\|_{L^\infty(0, T; H^{1+s}_{\sharp}(\Omega_0))} \|z\|_{L^2(0, T; H^2_{\sharp}(\Omega_0))},$$

we deduce, since $H^{1+s}_{\sharp}(\Omega_0)$ is continuously embedded in $L^\infty_{\sharp}(\Omega_0)$

$$\|g[z, \zeta, \eta]\|_{L^2(0, T; H^1_{\sharp}(\Omega_0))} \leq C \|B_{\zeta_0, \eta_0} - B_{\zeta, \eta}\|_{L^\infty(0, T; H^{1+s}_{\sharp}(\Omega_0))} \|z\|_{L^2(0, T; H^2_{\sharp}(\Omega_0))}.$$

By Lemma 4.1 and since $\|(\bar{\zeta}, \bar{\eta})\|_{X_{s, T}} \leq M$, we obtain

$$\|B_{\zeta, \eta} - B_{\zeta_0, \eta_0}\|_{L^\infty(0, T; H^{1+s}_{\sharp}(\Omega_0))} \leq CT^{\frac{1}{2}-s}M, \quad (4.8)$$

Hence,

$$\|g[z, \zeta, \eta]\|_{L^2(0, T; H^1_{\sharp}(\Omega_0))} \leq CT^{\frac{1}{2}-s}M^2,$$

with $0 < s < \frac{1}{2}$. By taking $s = \frac{1}{4}$ we obtain the desired inequality. Next we have to bound $\partial_t g$. Let $w \in H^1_{\sharp}(\Omega_0)$ and $\varphi \in C_c^\infty(0, T)$. We are going to estimate $\int_0^T \int_{\Omega_0} \partial_t g w \varphi$. By integration by parts in space and recalling that $z|_{y=0} = 0$ and $z|_{y=1} = \partial_t \zeta e_1 + \partial_t \eta e_2$, we obtain

$$\begin{aligned} \int_0^T \int_{\Omega_0} g w \partial_t \varphi &= \int_0^T \int_{\Omega_0} \operatorname{div}((B_{\zeta_0, \eta_0}^T - B_{\zeta, \eta}^T)z) w \partial_t \varphi \\ &= - \int_0^T \int_{\Omega_0} ((B_{\zeta_0, \eta_0}^T - B_{\zeta, \eta}^T)z) \cdot \nabla w \partial_t \varphi + \int_0^T \int_0^L ((B_{\zeta_0, \eta_0}^T - B_{\zeta, \eta}^T)z)|_{y=1} \cdot e_2 w \partial_t \varphi \\ &= - \int_0^T \int_{\Omega_0} ((B_{\zeta_0, \eta_0}^T - B_{\zeta, \eta}^T)z) \cdot \nabla w \partial_t \varphi + \int_0^T \int_0^L (\partial_x \bar{\eta} \partial_t \zeta - \partial_x \bar{\zeta} \partial_t \eta) w|_{y=1} \partial_t \varphi. \end{aligned}$$

Note that the second term of the right hand side is equal to zero in the only transversal case. Hence, integrating by parts with respect to the time, we get for all $\varphi \in C_c^\infty(0, T)$

$$\int_0^T \int_{\Omega_0} g w \partial_t \varphi = \int_0^T \int_{\Omega_0} (-\partial_t B_{\zeta, \eta}^T z + (B_{\zeta_0, \eta_0}^T - B_{\zeta, \eta}^T) \partial_t z) \cdot \nabla w \varphi - \int_0^T \int_0^L \partial_t (\partial_x \bar{\eta} \partial_t \zeta - \partial_x \bar{\zeta} \partial_t \eta) w|_{y=1} \varphi.$$

Consequently, we have

$$\begin{aligned} \langle \partial_t g, w \rangle_{(H^1_{\sharp}(\Omega_0))', H^1_{\sharp}(\Omega_0)} &= \int_{\Omega_0} (-\partial_t B_{\zeta, \eta}^T z + (B_{\zeta_0, \eta_0}^T - B_{\zeta, \eta}^T) \partial_t z) \cdot \nabla w \\ &\quad - \int_0^L (\partial_{xt} \bar{\eta} \partial_t \zeta + \partial_x \bar{\eta} \partial_{tt} \zeta - \partial_{xt} \bar{\zeta} \partial_t \eta - \partial_x \bar{\zeta} \partial_{tt} \eta) w|_{y=1}, \quad \forall w \in H^1_{\sharp}(\Omega_0), \quad \text{a.e. } t \in (0, T). \quad (4.9) \end{aligned}$$

By using Holder's inequality and Sobolev injection of $H_{\sharp}^1(\Omega_0)$ in $L_{\sharp}^4(\Omega_0)$, the volumic term can be bounded as follows

$$\begin{aligned} & \left| \int_{\Omega_0} \left(-\partial_t B_{\zeta,\eta}^T z + (B_{\zeta_0,\eta_0}^T - B_{\zeta,\eta}^T) \partial_t z \right) \cdot \nabla w \right| \\ & \leq \left(C \|\partial_t B_{\zeta,\eta}\|_{L_{\sharp}^4(\Omega_0)} \|z\|_{H_{\sharp}^1(\Omega_0)} + \|B_{\zeta,\eta} - B_{\zeta_0,\eta_0}\|_{L_{\sharp}^{\infty}(\Omega_0)} \|\partial_t z\|_{L_{\sharp}^2(\Omega_0)} \right) \|\nabla w\|_{L_{\sharp}^2(\Omega_0)} \end{aligned} \quad (4.10)$$

By using again Holder's inequality and trace inequality, there exists $C > 0$ depending on Ω_0 such that the boundary term can be estimated as

$$\begin{aligned} & \left| \int_0^L (\partial_{xt} \bar{\eta} \partial_t \zeta + \partial_x \bar{\eta} \partial_{tt} \zeta - \partial_{xt} \bar{\zeta} \partial_t \eta - \partial_x \bar{\zeta} \partial_{tt} \eta) w|_{y=1} \right| \leq C (\|\partial_{tx} \bar{\eta}\|_{L_{\sharp}^2(0,L)} \|\partial_t \zeta\|_{L_{\sharp}^{\infty}(0,L)} + \|\partial_x \bar{\eta}\|_{L_{\sharp}^{\infty}(0,L)} \|\partial_{tt} \zeta\|_{L_{\sharp}^2(0,L)} \\ & \quad + \|\partial_{xt} \bar{\zeta}\|_{L_{\sharp}^2(0,L)} \|\partial_t \eta\|_{L_{\sharp}^{\infty}(0,L)} + \|\partial_x \bar{\zeta}\|_{L_{\sharp}^{\infty}(0,L)} \|\partial_{tt} \eta\|_{L_{\sharp}^2(0,L)}) \|w\|_{H_{\sharp}^1(\Omega_0)} \end{aligned}$$

Sobolev injection of $H^{\frac{1}{2}+s}(0,L)$ in $C([0,L])$ for any $s > 0$ leads to

$$\begin{aligned} & \left| \int_0^L (\partial_{xt} \bar{\eta} \partial_t \zeta + \partial_x \bar{\eta} \partial_{tt} \zeta - \partial_{xt} \bar{\zeta} \partial_t \eta - \partial_x \bar{\zeta} \partial_{tt} \eta) w \right| \leq C (\|\partial_t \bar{\eta}\|_{H_{\sharp}^1(0,L)} \|\partial_t \zeta\|_{H_{\sharp}^1(0,L)} + \|\partial_x \bar{\eta}\|_{H_{\sharp}^{\frac{1}{2}+s}(0,L)} \|\partial_{tt} \zeta\|_{L_{\sharp}^2(0,L)} \\ & \quad + \|\partial_{xt} \bar{\zeta}\|_{H_{\sharp}^1(0,L)} \|\partial_t \eta\|_{H_{\sharp}^1(0,L)} + \|\partial_x \bar{\zeta}\|_{H_{\sharp}^{\frac{1}{2}+s}(0,L)} \|\partial_{tt} \eta\|_{L_{\sharp}^2(0,L)}) \|w\|_{H_{\sharp}^1(\Omega_0)}, \end{aligned}$$

and thus

$$\begin{aligned} & \left| \int_0^L (\partial_{xt} \bar{\eta} \partial_t \zeta + \partial_x \bar{\eta} \partial_{tt} \zeta - \partial_{xt} \bar{\zeta} \partial_t \eta - \partial_x \bar{\zeta} \partial_{tt} \eta) w \right| \leq C (\|\partial_t \bar{\eta}\|_{H_{\sharp}^1(0,L)} \|\partial_t \bar{\zeta}\|_{H_{\sharp}^1(0,L)} + \|\bar{\eta}\|_{H_{\sharp}^{\frac{3}{2}+s}(0,L)} \|\partial_{tt} \bar{\zeta}\|_{L_{\sharp}^2(0,L)} \\ & \quad + \|\bar{\zeta}\|_{H_{\sharp}^{\frac{3}{2}+s}(0,L)} \|\partial_{tt} \bar{\eta}\|_{L_{\sharp}^2(0,L)}) \|w\|_{H_{\sharp}^1(\Omega_0)}. \end{aligned} \quad (4.11)$$

By gathering together (4.10) and (4.11) with (4.9), we obtain

$$\begin{aligned} \|\partial_t g\|_{(H_{\sharp}^1(\Omega_0))'} & \leq C (\|\partial_t B_{\zeta,\eta}\|_{L_{\sharp}^4(\Omega_0)} \|z\|_{H_{\sharp}^1(\Omega_0)} + \|B_{\zeta,\eta} - B_{\zeta_0,\eta_0}\|_{L_{\sharp}^{\infty}(\Omega_0)} \|\partial_t z\|_{L_{\sharp}^2(\Omega_0)} + \|\partial_t \bar{\eta}\|_{H_{\sharp}^1(0,L)} \|\partial_t \bar{\zeta}\|_{H_{\sharp}^1(0,L)} \\ & \quad + \|\bar{\eta}\|_{H_{\sharp}^{\frac{3}{2}+s}(0,L)} \|\partial_{tt} \bar{\zeta}\|_{L_{\sharp}^2(0,L)} + \|\bar{\zeta}\|_{H_{\sharp}^{\frac{3}{2}+s}(0,L)} \|\partial_{tt} \bar{\eta}\|_{L_{\sharp}^2(0,L)}) \end{aligned}$$

It follows that

$$\begin{aligned} \|\partial_t g\|_{L^2(0,T;(H_{\sharp}^1(\Omega_0))')} & \leq C (T^{\frac{1}{2}} \|\partial_t B_{\zeta,\eta}\|_{L^{\infty}(0,T;L_{\sharp}^4(\Omega_0))} \|z\|_{L^{\infty}(0,T;H_{\sharp}^1(\Omega_0))} + \|B_{\zeta,\eta} - B_{\zeta_0,\eta_0}\|_{L^{\infty}(0,T;L_{\sharp}^{\infty}(\Omega_0))} \|\partial_t z\|_{L^2(0,T;L_{\sharp}^2(\Omega_0))} \\ & \quad + T^{\frac{1}{2}} \|\partial_t \bar{\eta}\|_{L^{\infty}(0,T;H_{\sharp}^1(0,L))} \|\partial_t \bar{\zeta}\|_{L^{\infty}(0,T;H_{\sharp}^1(0,L))} + \|\bar{\eta}\|_{L^{\infty}(0,T;H_{\sharp}^{\frac{3}{2}+s}(0,L))} \|\partial_{tt} \bar{\zeta}\|_{L^2(0,T;L_{\sharp}^2(0,L))} \\ & \quad + \|\bar{\zeta}\|_{L^{\infty}(0,T;H_{\sharp}^{\frac{3}{2}+s}(0,L))} \|\partial_{tt} \bar{\eta}\|_{L^2(0,T;L_{\sharp}^2(0,L))}) \end{aligned} \quad (4.12)$$

Since $(z, r, \bar{\zeta}, \bar{\eta}) \in \mathcal{K}_T(M)$, it follows by remembering (4.7), (4.8) and using Lemma 4.1 that for $s < \frac{1}{2}$

$$\|\partial_t g\|_{L^2(0,T;(H_{\sharp}^1(\Omega_0))')} \leq C(R) M^2 T^{\frac{1}{2}-s}.$$

By taking $s = \frac{1}{4}$ we obtain the desired inequality. The source term $h[z, r, \zeta, \eta]$ in (4.3) can be easily estimated using Lemma 4.1. Let us for instance consider $h^a = \mu((A_{\zeta,\eta} - A_{\zeta_0,\eta_0}) \nabla) z$. There exists a time $T(R, M)$ such that this

term is well defined and belong to $L^2(0, T; H_{\sharp}^1(\Omega_0))$ for $T \leq T(R, M)$. By using Holder's inequality, we get that there exists $C > 0$ such that

$$\|h^a\|_{L^2(0, T; H_{\sharp}^1(\Omega_0))} \leq C \left(\|A_{\zeta, \eta} - A_{\zeta_0, \eta_0}\|_{L^\infty(0, T; L_{\sharp}^\infty(\Omega_0))} \|z\|_{L^2(0, T; H_{\sharp}^2(\Omega_0))} + \|(\nabla(A_{\zeta, \eta} - A_{\zeta_0, \eta_0})\nabla)z\|_{L^2(0, T; L_{\sharp}^2(\Omega_0))} \right),$$

Since $\|(z, r)\|_{X_{f, T}} \leq M$ and by using Lemma 2.1, we obtain, for $s > 0$,

$$\|h^a\|_{L^2(0, T; H_{\sharp}^1(\Omega_0))} \leq CM \left(\|A_{\zeta, \eta} - A_{\zeta_0, \eta_0}\|_{L^\infty(0, T; L_{\sharp}^\infty(\Omega_0))} + \|A_{\zeta, \eta} - A_{\zeta_0, \eta_0}\|_{L^\infty(0, T; H_{\sharp}^{1+s}(\Omega_0))} \right),$$

Thus by using the Sobolev injection of $H_{\sharp}^{1+s}(\Omega_0) \subset L_{\sharp}^\infty(\Omega_0)$ and Lemma 4.1, we get, for $0 < s < 1/2$

$$\|h^a\|_{L^2(0, T; H_{\sharp}^1(\Omega_0))} \leq CM^2 T^{\frac{1}{2}-s}. \quad (4.13)$$

The two other terms can be estimated in the same way. By combining (4.6) with the above lemma, we deduce the following corollary.

Corollary 4.1 *For T small enough, \mathcal{L}_T^0 maps $\mathcal{K}_T(M)$ into itself for $M \geq M_0$.*

We turn now to show that for T small enough and $M \geq M_0$, the mapping $\mathcal{L}_T^0 : \mathcal{K}_T(M) \rightarrow \mathcal{K}_T(M)$ is a contraction. Let $(z^{(1)}, r^{(1)}, \bar{\zeta}^{(1)}, \bar{\eta}^{(1)})$ and $(z^{(2)}, r^{(2)}, \bar{\zeta}^{(2)}, \bar{\eta}^{(2)})$ in $\mathcal{K}_T(M)$. For $i = 1, 2$, denote by

$$(v^{(i)}, q^{(i)}, \bar{\mathcal{X}}^{(i)}, \bar{\mathcal{Y}}^{(i)}) = \mathcal{L}_T^0(z^{(i)}, r^{(i)}, \bar{\zeta}^{(i)}, \bar{\eta}^{(i)}).$$

We recall that $(\bar{\zeta}^{(i)}, \bar{\eta}^{(i)}, \bar{\mathcal{X}}^{(i)}, \bar{\mathcal{Y}}^{(i)}) = (\zeta^{(i)} - \zeta_0, \eta^{(i)} - \eta_0, \mathcal{X}^{(i)} - \zeta_0, \mathcal{Y}^{(i)} - \eta_0)$ with $(v^{(i)}, q^{(i)}, \mathcal{X}^{(i)}, \mathcal{Y}^{(i)})$ is a solution of Problem (3.1)-(3.6) with $(a, b) = (\zeta_0, \eta_0)$ and source term $(f^{(i)}, h^{(i)}, g^{(i)}, 0, 0)$ defined as follows:

$$f^{(i)} = f[z^{(i)}, \zeta^{(i)}, \eta^{(i)}], h^{(i)} = h[z^{(i)}, r^{(i)}, \zeta^{(i)}, \eta^{(i)}], g^{(i)} = g[z^{(i)}, \zeta^{(i)}, \eta^{(i)}]. \quad (4.14)$$

Then

$$\mathcal{L}_T^0(z^{(1)}, r^{(1)}, \bar{\zeta}^{(1)}, \bar{\eta}^{(1)}) - \mathcal{L}_T^0(z^{(2)}, r^{(2)}, \bar{\zeta}^{(2)}, \bar{\eta}^{(2)}) = (v^{(1)}, q^{(1)}, \zeta^{(1)}, \eta^{(1)}) - (v^{(2)}, q^{(2)}, \zeta^{(2)}, \eta^{(2)}) \quad (4.15)$$

Denote by $(v, q, \zeta, \eta) = (v^{(1)}, q^{(1)}, \zeta^{(1)}, \eta^{(1)}) - (v^{(2)}, q^{(2)}, \zeta^{(2)}, \eta^{(2)})$. Then (v, q, ζ, η) is a solution of Problem (3.1) with $(a, b) = (\zeta_0, \eta_0)$, zero initial data and source term $(f, h, g, 0, 0) = (f^{(1)} - f^{(2)}, h^{(1)} - h^{(2)}, g^{(1)} - g^{(2)}, 0, 0)$. Hence if the source term $(f, h, g, 0, 0) \in S_T$, then by using Proposition 3.2 for $(a, b) = (\zeta_0, \eta_0)$ we get

$$\|(v, q)\|_{X_{f, T}} + \|(\zeta, \eta)\|_{X_{s, T}} \leq C(R) \|(f, h, g, 0, 0)\|_{S_T},$$

and thus in view of (4.15) we get

$$\|\mathcal{L}_T^0(z^{(1)}, r^{(1)}, \bar{\zeta}^{(1)}, \bar{\eta}^{(1)}) - \mathcal{L}_T^0(z^{(2)}, r^{(2)}, \bar{\zeta}^{(2)}, \bar{\eta}^{(2)})\|_{X_{f, T} \times X_{s, T}} \leq C(R) \|(f^{(1)} - f^{(2)}, h^{(1)} - h^{(2)}, g^{(1)} - g^{(2)}, 0, 0)\|_{S_T}. \quad (4.16)$$

To proceed we have to bound the source term $(f^{(1)} - f^{(2)}, h^{(1)} - h^{(2)}, g^{(1)} - g^{(2)}, 0, 0)$ defined by (4.14) respectively in S_T in the right hand side of (4.16). In order to establish such bounds, we gather in the following lemma some properties of the matrix $B_{\zeta, \eta}$ and $A_{\zeta, \eta}$ defined respectively by (2.11), (2.12), that can be derived in the very same way as in Lemma 4.1.

Lemma 4.3 Consider two couples $(\zeta^{(1)}, \eta^{(1)})$ and $(\zeta^{(2)}, \eta^{(2)})$ in $L^\infty(0, T; H_{\sharp}^2(0, L)) \cap W^{1, \infty}(0, T; H_{\sharp}^1(0, L))$ such that $(\zeta^{(1)}(0), \eta^{(1)}(0)) = (\zeta^{(2)}(0), \eta^{(2)}(0)) = (\zeta_0, \eta_0)$. Then, for $0 < s < \frac{1}{2}$,

$$\begin{aligned} \|B_{\zeta^{(1)}, \eta^{(1)}} - B_{\zeta^{(2)}, \eta^{(2)}}\|_{L^\infty(0, T; H_{\sharp}^{1+s}(\Omega_0))} &\leq CT^{\frac{1}{2}-s} \|(\bar{\zeta}, \bar{\eta})\|_{L^\infty(0, T; H_{\sharp}^2(0, L))}^{\frac{1}{2}+s} \|(\bar{\zeta}, \bar{\eta})\|_{W^{1, \infty}(0, T; H_{\sharp}^1(0, L))}^{\frac{1}{2}-s}, \\ \|\partial_t B_{\zeta^{(1)}, \eta^{(1)}} - \partial_t B_{\zeta^{(2)}, \eta^{(2)}}\|_{L^\infty(0, T; L_{\sharp}^4(\Omega_0))} &\leq C \|(\bar{\zeta}, \bar{\eta})\|_{L^\infty(0, T; H_{\sharp}^2(0, L)) \cap W^{1, \infty}(0, T; H_{\sharp}^1(0, L))}, \end{aligned}$$

Furthermore, if we assume that

- $\|(\zeta_0, \eta_0)\|_{H_{\sharp}^2(0, L)} + \|(\det \nabla \chi_{\zeta_0, \eta_0})^{-1}\|_{C_{\sharp}(\bar{\Omega}_0)} \leq R$,
- $\|(\zeta^{(i)}, \eta^{(i)})\|_{L^\infty(0, T; H_{\sharp}^2(0, L)) \cap W^{1, \infty}(0, T; H_{\sharp}^1(0, L))} \leq M$,

there exists a time $T(R, M)$ depending on R and M such that for $T \leq T(R, M)$, $\det(\nabla \chi_{\zeta^{(i)}, \eta^{(i)}})^{-1}$ and $A_{\zeta^{(i)}, \eta^{(i)}}$ are well defined in $C([0, T]; H_{\sharp}^{1+s}(\Omega_0))$, for $0 < s < \frac{1}{2}$, and

$$\begin{aligned} \|(\det \nabla \chi_{\zeta^{(1)}, \eta^{(1)}})^{-1} - (\det \nabla \chi_{\zeta^{(2)}, \eta^{(2)}})^{-1}\|_{L^\infty(0, T; H_{\sharp}^{1+s}(\Omega_0))} &\leq C(R, M)T^{\frac{1}{2}-s}, \\ \|A_{\zeta^{(1)}, \eta^{(1)}} - A_{\zeta^{(2)}, \eta^{(2)}}\|_{L^\infty(0, T; H_{\sharp}^{1+s}(\Omega_0))} &\leq C(R, M)T^{\frac{1}{2}-s}, \end{aligned}$$

We move now to bound the source terms.

Lemma 4.4 Given $(z^{(1)}, r^{(1)}, \bar{\zeta}^{(1)}, \bar{\eta}^{(1)})$ and $(z^{(2)}, r^{(2)}, \bar{\zeta}^{(2)}, \bar{\eta}^{(2)})$ in $\mathcal{K}_T(M)$ and $(\zeta_0, \eta_0) \in H_{\sharp}^2(0, L) \times H_{\sharp}^3(0, L)$. Let $f^{(i)}, h^{(i)}, g^{(i)}$ be defined as (4.14) with $(\zeta^{(i)}, \eta^{(i)}) = (\bar{\zeta}^{(i)} + \zeta_0, \bar{\eta}^{(i)} + \eta_0)$ for $i = 1, 2$, then there exists a positive constant C and a time $T(M, R)$ such that for $T \leq T(M, R)$

$$\|f^{(1)} - f^{(2)}\|_{L^2(0, T; L_{\sharp}^2(\Omega_0))} \leq C(R)MT^{\frac{1}{4}} \|(z^{(1)} - z^{(2)}, 0, \bar{\zeta}^{(1)} - \bar{\zeta}^{(2)}, \bar{\eta}^{(1)} - \bar{\eta}^{(2)})\|_{X_{f, T} \times X_{s, T}}, \quad (4.17)$$

$$\|h^{(1)} - h^{(2)}\|_{L^2(0, T; H_{\sharp}^1(\Omega_0))} \leq CT^{\frac{1}{4}}M \|(z^{(1)} - z^{(2)}, r^{(1)} - r^{(2)}, \bar{\zeta}^{(1)} - \bar{\zeta}^{(2)}, \bar{\eta}^{(1)} - \bar{\eta}^{(2)})\|_{X_{f, T} \times X_{s, T}}, \quad (4.18)$$

$$\|g^{(1)} - g^{(2)}\|_{L^2(0, T; H_{\sharp}^1(\Omega_0))} \leq CMT^{\frac{1}{4}} \|(z^{(1)} - z^{(2)}, 0, \bar{\zeta}^{(1)} - \bar{\zeta}^{(2)}, \bar{\eta}^{(1)} - \bar{\eta}^{(2)})\|_{X_{f, T} \times X_{s, T}}, \quad (4.19)$$

$$\|g^{(1)} - g^{(2)}\|_{H_{00}^1(0, T; (H_{\sharp}^1(\Omega_0))')} \leq CMT^{\frac{1}{4}} \|(z^{(1)} - z^{(2)}, 0, \bar{\zeta}^{(1)} - \bar{\zeta}^{(2)}, \bar{\eta}^{(1)} - \bar{\eta}^{(2)})\|_{X_{f, T} \times X_{s, T}}, \quad (4.20)$$

Proof. We skip the proof of the first three estimates as they can be derived thanks to Lemma 2.1, Lemma 4.3 and by following similar arguments as in [28]. We focus on the last estimate as it makes appear terms that vanish in the only transversal case and require a special attention. The estimate will once again be done by duality. Let $w \in H_{\sharp}^1(\Omega_0)$ and $\varphi \in C_c^\infty(0, T)$. We have

$$\begin{aligned} \int_0^T \int_{\Omega_0} (g^{(1)} - g^{(2)}) w \partial_t \varphi &= \int_0^T \int_{\Omega_0} \operatorname{div}((B_{\zeta_0, \eta_0}^T - B_{\zeta^{(2)}, \eta^{(2)}}^T)(z^{(1)} - z^{(2)})) w \partial_t \varphi \\ &\quad - \int_0^T \int_{\Omega_0} \operatorname{div}((B_{\zeta^{(1)}, \eta^{(1)}}^T - B_{\zeta^{(2)}, \eta^{(2)}}^T)z^{(1)}) w \partial_t \varphi \quad (4.21) \end{aligned}$$

By performing integration by parts with respect to space we obtain

$$\begin{aligned} \int_0^T \int_{\Omega_0} (g^{(1)} - g^{(2)}) w \partial_t \varphi &= - \int_0^T \int_{\Omega_0} ((B_{\zeta_0, \eta_0}^T - B_{\zeta^{(2)}, \eta^{(2)}}^T)(z^{(1)} - z^{(2)})) \cdot \nabla w \partial_t \varphi \\ + \int_0^T \int_{\Omega_0} ((B_{\zeta^{(1)}, \eta^{(1)}}^T - B_{\zeta^{(2)}, \eta^{(2)}}^T)z^{(1)}) \cdot \nabla w \partial_t \varphi &+ \int_0^T \int_0^L [\partial_x \bar{\eta}^{(2)}(\partial_t \zeta^{(1)} - \partial_t \zeta^{(2)}) - \partial_x \bar{\zeta}^{(2)}(\partial_t \eta^{(1)} - \partial_t \eta^{(2)})] w|_{y=1} \partial_t \varphi \\ &- \int_0^T \int_0^L ((\partial_x \eta^{(1)} - \partial_x \eta^{(2)}) \partial_t \zeta^{(1)} - (\partial_x \zeta^{(1)} - \partial_x \zeta^{(2)}) \partial_t \eta^{(1)}) w|_{y=1} \partial_t \varphi. \end{aligned}$$

Now, by performing integration by parts with respect to time we get for any $\varphi \in C_c^\infty(0, T)$

$$\begin{aligned} \int_0^T \int_{\Omega_0} (g^{(1)} - g^{(2)}) w \partial_t \varphi &= \int_0^T \int_{\Omega_0} \left(-\partial_t B_{\zeta^{(2)}, \eta^{(2)}}^T(z^{(1)} - z^{(2)}) + (B_{\zeta_0, \eta_0}^T - B_{\zeta^{(2)}, \eta^{(2)}}^T)(\partial_t z^{(1)} - \partial_t z^{(2)}) \right) \cdot \nabla w \varphi \\ &\quad - \int_0^T \int_{\Omega_0} \left((\partial_t B_{\zeta^{(1)}, \eta^{(1)}}^T - \partial_t B_{\zeta^{(2)}, \eta^{(2)}}^T) z^{(1)} + (B_{\zeta^{(1)}, \eta^{(1)}}^T - B_{\zeta^{(2)}, \eta^{(2)}}^T) \partial_t z^{(1)} \right) \cdot \nabla w \varphi \\ &\quad - \int_0^T \int_0^L \partial_t (\partial_x \bar{\eta}^{(2)} (\partial_t \zeta^{(1)} - \partial_t \zeta^{(2)}) - \partial_x \bar{\zeta}^{(2)} (\partial_t \eta^{(1)} - \partial_t \eta^{(2)})) w|_{y=1} \varphi \\ &\quad + \int_0^T \int_0^L \partial_t ((\partial_x \eta^{(1)} - \partial_x \eta^{(2)}) \partial_t \zeta^{(1)} - (\partial_x \zeta^{(1)} - \partial_x \zeta^{(2)}) \partial_t \eta^{(1)}) w|_{y=1} \varphi. \end{aligned}$$

Consequently, we have

$$\begin{aligned} \langle \partial_t (g^{(1)} - g^{(2)}), w \rangle_{(H_{\sharp}^1(\Omega_0))', H_{\sharp}^1(\Omega_0)} &= \int_{\Omega_0} \left(-\partial_t B_{\zeta^{(2)}, \eta^{(2)}}^T(z^{(1)} - z^{(2)}) + (B_{\zeta_0, \eta_0}^T - B_{\zeta^{(2)}, \eta^{(2)}}^T)(\partial_t z^{(1)} - \partial_t z^{(2)}) \right) \cdot \nabla w \\ &\quad - \int_{\Omega_0} \left((\partial_t B_{\zeta^{(1)}, \eta^{(1)}}^T - \partial_t B_{\zeta^{(2)}, \eta^{(2)}}^T) z^{(1)} + (B_{\zeta^{(1)}, \eta^{(1)}}^T - B_{\zeta^{(2)}, \eta^{(2)}}^T) \partial_t z^{(1)} \right) \cdot \nabla w \\ &\quad - \int_0^L (\partial_{xt} \bar{\eta}^{(2)} (\partial_t \zeta^{(1)} - \partial_t \zeta^{(2)}) - \partial_{xt} \bar{\zeta}^{(2)} (\partial_t \eta^{(1)} - \partial_t \eta^{(2)}) - \partial_x \bar{\eta}^{(2)} (\partial_{tt} \zeta^{(1)} - \partial_{tt} \zeta^{(2)}) + \partial_x \bar{\zeta}^{(2)} (\partial_{tt} \eta^{(1)} - \partial_{tt} \eta^{(2)})) w|_{y=1} \\ &\quad + \int_0^L ((\partial_{xt} \eta^{(1)} - \partial_{xt} \eta^{(2)}) \partial_t \zeta^{(1)} - (\partial_{xt} \zeta^{(1)} - \partial_{xt} \zeta^{(2)}) \partial_t \eta^{(1)} + (\partial_x \eta^{(1)} - \partial_x \eta^{(2)}) \partial_{tt} \zeta^{(1)} - (\partial_x \zeta^{(1)} - \partial_x \zeta^{(2)}) \partial_{tt} \eta^{(1)}) w|_{y=1}. \end{aligned}$$

Thanks to Holder's inequality, Sobolev injections and trace theorem we have

$$\begin{aligned} &|\langle \partial_t (g^{(1)} - g^{(2)}), w \rangle_{(H_{\sharp}^1(\Omega_0))', H_{\sharp}^1(\Omega_0)}| \\ &\leq \left(C \|\partial_t B_{\zeta^{(2)}, \eta^{(2)}}^T\|_{L_{\sharp}^4(\Omega_0)} \|z^{(1)} - z^{(2)}\|_{H_{\sharp}^1(\Omega_0)} + \|B_{\zeta_0, \eta_0} - B_{\zeta^{(2)}, \eta^{(2)}}\|_{L_{\sharp}^\infty(\Omega_0)} \|\partial_t z^{(1)} - \partial_t z^{(2)}\|_{L_{\sharp}^2(\Omega_0)} \right. \\ &\quad \left. + C \|\partial_t B_{\zeta^{(1)}, \eta^{(1)}}^T - \partial_t B_{\zeta^{(2)}, \eta^{(2)}}^T\|_{L_{\sharp}^4(\Omega_0)} \|z^{(1)}\|_{H_{\sharp}^1(\Omega_0)} + \|B_{\zeta^{(1)}, \eta^{(1)}} - B_{\zeta^{(2)}, \eta^{(2)}}\|_{L_{\sharp}^\infty(\Omega_0)} \|\partial_t z^{(1)}\|_{L_{\sharp}^2(\Omega_0)} \right) \|w\|_{H_{\sharp}^1(\Omega_0)} \\ &\quad + C \left(\|\partial_{xt} \bar{\eta}^{(2)}\|_{L_{\sharp}^2(0, L)} \|\partial_t \zeta^{(1)} - \partial_t \zeta^{(2)}\|_{L_{\sharp}^\infty(0, L)} + \|\partial_{xt} \bar{\zeta}^{(2)}\|_{L_{\sharp}^2(0, L)} \|\partial_t \eta^{(1)} - \partial_t \eta^{(2)}\|_{L_{\sharp}^\infty(0, L)} \right. \\ &\quad \left. + \|\partial_x \bar{\eta}^{(2)}\|_{L_{\sharp}^\infty(0, L)} \|\partial_{tt} \zeta^{(1)} - \partial_{tt} \zeta^{(2)}\|_{L_{\sharp}^2(0, L)} + \|\partial_x \bar{\zeta}^{(2)}\|_{L_{\sharp}^\infty(0, L)} \|\partial_{tt} \eta^{(1)} - \partial_{tt} \eta^{(2)}\|_{L_{\sharp}^2(0, L)} \right. \\ &\quad \left. \|\partial_{xt} \eta^{(1)} - \partial_{xt} \eta^{(2)}\|_{L_{\sharp}^2(0, L)} \|\partial_t \zeta^{(1)}\|_{L_{\sharp}^\infty(0, L)} + \|\partial_{xt} \zeta^{(1)} - \partial_{xt} \zeta^{(2)}\|_{L_{\sharp}^2(0, L)} \|\partial_t \eta^{(1)}\|_{L_{\sharp}^\infty(0, L)} \right. \\ &\quad \left. + \|\partial_x \eta^{(1)} - \partial_x \eta^{(2)}\|_{L_{\sharp}^\infty(0, L)} \|\partial_{tt} \zeta^{(1)}\|_{L_{\sharp}^2(0, L)} + \|\partial_x \zeta^{(1)} - \partial_x \zeta^{(2)}\|_{L_{\sharp}^\infty(0, L)} \|\partial_{tt} \eta^{(1)}\|_{L_{\sharp}^2(0, L)} \right) \|w\|_{H_{\sharp}^1(\Omega_0)}. \end{aligned}$$

Using the Sobolev injection of $H^{\frac{1}{2}+s}(0, L) \subset C_{\sharp}^0(0, L)$ for any $s > 0$ it follows that

$$\begin{aligned} \|\partial_t (g^{(1)} - g^{(2)})\|_{(H_{\sharp}^1(\Omega_0))'} &\leq C \|\partial_t B_{\zeta^{(2)}, \eta^{(2)}}^T\|_{L_{\sharp}^4(\Omega_0)} \|z^{(1)} - z^{(2)}\|_{H_{\sharp}^1(\Omega_0)} + \|B_{\zeta_0, \eta_0} - B_{\zeta^{(2)}, \eta^{(2)}}\|_{L_{\sharp}^\infty(\Omega_0)} \|\partial_t z^{(1)} - \partial_t z^{(2)}\|_{L_{\sharp}^2(\Omega_0)} \\ &\quad + C \|\partial_t B_{\zeta^{(1)}, \eta^{(1)}}^T - \partial_t B_{\zeta^{(2)}, \eta^{(2)}}^T\|_{L_{\sharp}^4(\Omega_0)} \|z^{(1)}\|_{H_{\sharp}^1(\Omega_0)} + \|B_{\zeta^{(1)}, \eta^{(1)}} - B_{\zeta^{(2)}, \eta^{(2)}}\|_{L_{\sharp}^\infty(\Omega_0)} \|\partial_t z^{(1)}\|_{L_{\sharp}^2(\Omega_0)} \\ &\quad + C \left(\|\partial_t \bar{\eta}^{(2)}\|_{H_{\sharp}^1(0, L)} \|\partial_t \zeta^{(1)} - \partial_t \zeta^{(2)}\|_{H_{\sharp}^1(0, L)} + \|\partial_t \bar{\zeta}^{(2)}\|_{H_{\sharp}^1(0, L)} \|\partial_t \eta^{(1)} - \partial_t \eta^{(2)}\|_{H_{\sharp}^1(0, L)} \right. \\ &\quad \left. + \|\bar{\eta}^{(2)}\|_{H_{\sharp}^{\frac{3}{2}+s}(0, L)} \|\partial_{tt} \zeta^{(1)} - \partial_{tt} \zeta^{(2)}\|_{L_{\sharp}^2(0, L)} + \|\bar{\zeta}^{(2)}\|_{H_{\sharp}^{\frac{3}{2}+s}(0, L)} \|\partial_{tt} \eta^{(1)} - \partial_{tt} \eta^{(2)}\|_{L_{\sharp}^2(0, L)} \right. \\ &\quad \left. \|\partial_t \eta^{(1)} - \partial_t \eta^{(2)}\|_{H_{\sharp}^1(0, L)} \|\partial_t \zeta^{(1)}\|_{H_{\sharp}^1(0, L)} + \|\partial_t \zeta^{(1)} - \partial_t \zeta^{(2)}\|_{H_{\sharp}^1(0, L)} \|\partial_t \eta^{(1)}\|_{H_{\sharp}^1(0, L)} \right. \\ &\quad \left. + \|\eta^{(1)} - \eta^{(2)}\|_{H_{\sharp}^{\frac{3}{2}+s}(0, L)} \|\partial_{tt} \zeta^{(1)}\|_{L_{\sharp}^2(0, L)} + \|\zeta^{(1)} - \zeta^{(2)}\|_{H_{\sharp}^{\frac{3}{2}+s}(0, L)} \|\partial_{tt} \eta^{(1)}\|_{L_{\sharp}^2(0, L)} \right). \end{aligned}$$

Thus,

$$\begin{aligned}
& \|\partial_t(g^{(1)} - g^{(2)})\|_{L^2(0,T;(H_{\frac{1}{2}}^1(\Omega_0))')} \leq CT^{\frac{1}{2}} \|\partial_t B_{\zeta^{(2)},\eta^{(2)}}^T\|_{L^\infty(0,T;L_{\frac{1}{2}}^4(\Omega_0))} \|z^{(1)} - z^{(2)}\|_{L^\infty(0,T;H_{\frac{1}{2}}^1(\Omega_0))} \\
& + \|B_{\zeta_0,\eta_0} - B_{\zeta^{(2)},\eta^{(2)}}\|_{L^\infty(0,T;L_{\frac{1}{2}}^\infty(\Omega_0))} \|\partial_t z^{(1)} - \partial_t z^{(2)}\|_{L^2(0,T;L_{\frac{1}{2}}^2(\Omega_0))} + CT^{\frac{1}{2}} \|\partial_t B_{\zeta^{(1)},\eta^{(1)}}^T - \partial_t B_{\zeta^{(2)},\eta^{(2)}}^T\|_{L^\infty(0,T;L_{\frac{1}{2}}^4(\Omega_0))} \|z^{(1)}\|_{L^\infty(0,T;H_{\frac{1}{2}}^1(\Omega_0))} \\
& \quad + \|B_{\zeta^{(1)},\eta^{(1)}} - B_{\zeta^{(2)},\eta^{(2)}}\|_{L^\infty(0,T;L_{\frac{1}{2}}^\infty(\Omega_0))} \|\partial_t z^{(1)}\|_{L^2(0,T;L_{\frac{1}{2}}^2(\Omega_0))} \\
& + C \left(T^{\frac{1}{2}} \|\partial_t \bar{\eta}^{(2)}\|_{L^\infty(0,T;H_{\frac{1}{2}}^1(0,L))} \|\partial_t \zeta^{(1)} - \partial_t \zeta^{(2)}\|_{L^\infty(0,T;H_{\frac{1}{2}}^1(0,L))} + T^{\frac{1}{2}} \|\partial_t \bar{\zeta}^{(2)}\|_{L^\infty(0,T;H_{\frac{1}{2}}^1(0,L))} \|\partial_t \eta^{(1)} - \partial_t \eta^{(2)}\|_{L^\infty(0,T;H_{\frac{1}{2}}^1(0,L))} \right. \\
& \quad + \|\bar{\eta}^{(2)}\|_{L^\infty(0,T;H_{\frac{1}{2}}^{\frac{3}{2}+s}(0,L))} \|\partial_{tt} \zeta^{(1)} - \partial_{tt} \zeta^{(2)}\|_{L^2(0,T;L_{\frac{1}{2}}^2(0,L))} + \|\bar{\zeta}^{(2)}\|_{L^\infty(0,T;H_{\frac{1}{2}}^{\frac{3}{2}+s}(0,L))} \|\partial_{tt} \eta^{(1)} - \partial_{tt} \eta^{(2)}\|_{L^2(0,T;L_{\frac{1}{2}}^2(0,L))} \\
& \quad \left. T^{\frac{1}{2}} \|\partial_t \eta^{(1)} - \partial_t \eta^{(2)}\|_{L^\infty(0,T;H_{\frac{1}{2}}^1(0,L))} \|\partial_t \zeta^{(1)}\|_{L^\infty(0,T;H_{\frac{1}{2}}^1(0,L))} + T^{\frac{1}{2}} \|\partial_t \zeta^{(1)} - \partial_t \zeta^{(2)}\|_{L^\infty(0,T;H_{\frac{1}{2}}^1(0,L))} \|\partial_t \eta^{(1)}\|_{L^\infty(0,T;H_{\frac{1}{2}}^1(0,L))} \right. \\
& \quad \left. + \|\eta^{(1)} - \eta^{(2)}\|_{L^\infty(0,T;H_{\frac{1}{2}}^{\frac{3}{2}+s}(0,L))} \|\partial_{tt} \zeta^{(1)}\|_{L^2(0,T;L_{\frac{1}{2}}^2(0,L))} + \|\zeta^{(1)} - \zeta^{(2)}\|_{L^\infty(0,T;H_{\frac{1}{2}}^{\frac{3}{2}+s}(0,L))} \|\partial_{tt} \eta^{(1)}\|_{L^2(0,T;L_{\frac{1}{2}}^2(0,L))} \right).
\end{aligned}$$

By using inequality (4.7), Lemma 4.1 and noting that $(z^{(i)}, 0, \zeta^{(i)}, \eta^{(i)}) \in \mathcal{K}_{T,M}$ for $i = 1, 2$, we obtain

$$\begin{aligned}
& \|\partial_t(g^{(1)} - g^{(2)})\|_{L^2(0,T;H_{\frac{1}{2}}^1(\Omega_0))'} \leq CM \left(T^{\frac{1}{2}} \|z^{(1)} - z^{(2)}\|_{L^\infty(0,T;H_{\frac{1}{2}}^1(\Omega_0))} + T^s \|\partial_t z^{(1)} - \partial_t z^{(2)}\|_{L^2(0,T;L_{\frac{1}{2}}^2(\Omega_0))} \right. \\
& \quad + T^{\frac{1}{2}} \|\partial_t B_{\zeta^{(1)},\eta^{(1)}}^T - \partial_t B_{\zeta^{(2)},\eta^{(2)}}^T\|_{L^\infty(0,T;L_{\frac{1}{2}}^4(\Omega_0))} + \|B_{\zeta^{(1)},\eta^{(1)}} - B_{\zeta^{(2)},\eta^{(2)}}\|_{L^\infty(0,T;L_{\frac{1}{2}}^\infty(\Omega_0))} \\
& \quad + T^{\frac{1}{2}} \|\partial_t \zeta^{(1)} - \partial_t \zeta^{(2)}\|_{L^\infty(0,T;H_{\frac{1}{2}}^1(0,L))} + T^{\frac{1}{2}} \|\partial_t \eta^{(1)} - \partial_t \eta^{(2)}\|_{L^\infty(0,T;H_{\frac{1}{2}}^1(0,L))} \\
& \quad + T^{\frac{1}{2}-s} \|\partial_{tt} \zeta^{(1)} - \partial_{tt} \zeta^{(2)}\|_{L^2(0,T;L_{\frac{1}{2}}^2(0,L))} + T^{\frac{1}{2}-s} \|\partial_{tt} \eta^{(1)} - \partial_{tt} \eta^{(2)}\|_{L^2(0,T;L_{\frac{1}{2}}^2(0,L))} \\
& \quad + T^{\frac{1}{2}} \|\partial_t \eta^{(1)} - \partial_t \eta^{(2)}\|_{L^\infty(0,T;H_{\frac{1}{2}}^1(0,L))} + T^{\frac{1}{2}} \|\partial_t \zeta^{(1)} - \partial_t \zeta^{(2)}\|_{L^\infty(0,T;H_{\frac{1}{2}}^1(0,L))} \\
& \quad \left. + \|\eta^{(1)} - \eta^{(2)}\|_{L^\infty(0,T;H_{\frac{1}{2}}^{\frac{3}{2}+s}(0,L))} + \|\zeta^{(1)} - \zeta^{(2)}\|_{L^\infty(0,T;H_{\frac{1}{2}}^{\frac{3}{2}+s}(0,L))} \right).
\end{aligned}$$

Finally, by using Lemma 4.3 and (4.7), and taking $s = 1/4$ we obtain we establish the estimate (4.20).

Combining the result of the above Lemma with (4.16), we deduce the following:

Corollary 4.2 *For T small enough and $M \geq M_0$, the mapping $\mathcal{L}_T^0 : \mathcal{K}_T(M) \rightarrow \mathcal{K}_T(M)$ is a contraction.*

Consequently we have proved existence and uniqueness of strong solution of a periodic two dimensional fluid-beam interaction problem for which no additional viscosity has been added to the structure equations and without neglecting the longitudinal displacement. There no gap between the initial regularity and the one obtain at any positive time. Note that additional regularity may surely be derived for the transversal displacement by using properties of the equation with inertia of rotation, yet this is out of the scope of the present study. As for the fluid-wave case the dissipation of the fluid enables to control the velocity of the structure and it is therefore likely to prove that the semi-group associated to the linear problem is analytic paving the way to the study of the long time behavior of this coupled system. Note that considering other boundary conditions at the inlet and outlet induce additional difficulties and in particular on may loose the elliptic result.

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