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# Local Halanay's Inequality with Application to Feedback Stabilization\*

Michael Malisoff<sup>†</sup> and Frédéric Mazenc<sup>‡</sup>

**Abstract.** We provide a local version of an approach to proving asymptotic stability that is based on Halanay's inequality. Our results are applicable to a family of nonlinear systems that contain state and input delays. We determine input-to-state stability inequalities when the systems contain additive uncertainty. We combine the results with an observer and a Gramian approach, to solve an output feedback stabilization problem. Our numerical examples illustrate how our theorems lead to new basin of attraction estimates.

**Key words.** Stabilization, delayed systems, feedback control

**MSC codes.** 93D25, 93C23

**1. Introduction.** Since its introduction in [6], Halanay's inequality and its generalizations have been very useful in many cases to establish asymptotic stability properties for families of nonlinear systems. These results have been developed in several contributions, including [5], [7], [12], [13], [15], [16], and [17] for both continuous-time and discrete-time systems. In its basic form (e.g., [4, Lemma 4.2, p. 138]), Halanay's inequality calls for finding nonnegative valued differentiable functions  $v$  and constants  $a > 0$ ,  $b \in (0, a)$ , and  $T > 0$  such that

$$(1.1) \quad \dot{v}(t) \leq -av(t) + b \sup_{\ell \in [t-T, t]} v(\ell)$$

holds for all  $t \geq T$ , in order to prove that  $v(t)$  exponentially converges to 0 as  $t \rightarrow +\infty$ . See also works such as [20] for generalizations where, instead of being constants, the  $a$  and  $b$  in (1.1) can depend on  $t$ , which include cases where there are  $t$  values such that  $b(t) > a(t)$ , and which explain the advantages of using Halanay's inequality approaches for stability analysis instead of standard Lyapunov function methods. However, to the best of our knowledge, they only apply to globally exponentially stable systems. On the other hand, in many cases, nonlinear systems are only locally exponentially stable, and in those cases, the global Halanay's results cannot be applied to establish global stability results. [Moreover, surprisingly perhaps, to the best of our knowledge, there is no general result ensuring that a nonlinear system with input or state delays whose linear approximation at the origin is exponentially stabilizable is an open loop locally exponentially stable system \(although substituting a feedback control may make the closed loop system locally exponentially stable\).](#) More generally, we think that the

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31 local stability or stabilization of delayed systems is an under-studied topic even if [1], [3], and  
 32 [9] present results on this subject. This motivates the present work.

33 We establish a local version of the Halanay's inequality based stability result for functions  
 34 that satisfy a nonlinear differential inequality in a suitable local sense. We apply it to systems  
 35 that contain small bounded additive uncertainties, and we determine input-to-state-stability  
 36 (or ISS) inequalities; see below for the definition of ISS. The results can be applied to nonlinear  
 37 systems with delays, and enable us to estimate basins of attraction. In Section 4, we apply  
 38 this result to a local output feedback exponential stabilization problem using an observer and  
 39 an invertible Gramian approach. We revisit the main result of the paper [11], by considering  
 40 systems with poorly known nonlinear terms that violate the usual linear growth conditions.

41 We state and prove our local Halanay's inequality result in Section 2, which we use to prove  
 42 our local exponential stabilization result for state feedback in Section 3, and our generalization  
 43 for systems with outputs in Section 4. Our work covers nonlinear systems with distributed  
 44 state delays and time-varying input delays. We illustrate our results in Section 5 using a  
 45 controlled version of van der Pol's equation, and a system with an output and a saturation,  
 46 whose structures preclude using previous methods to prove global stabilization, but which are  
 47 amenable to our results. This leads to our estimates for basins of attraction and sufficient  
 48 conditions on the bounds for the uncertainties for our local stabilization estimates to hold.  
 49 Then in Section 6, we provide our suggestions for future research.

We use standard notation which we simplify when no confusion would arise. The dimen-  
 sions of our Euclidean spaces are arbitrary, unless indicated otherwise, and  $|\cdot|$  is the usual  
 Euclidean vector norm and corresponding matrix operator norm. For matrices  $A \in \mathbb{R}^{n \times n}$  and  
 $B \in \mathbb{R}^{n \times n}$ , we write  $A \leq B$  provided  $B - A$  is nonnegative definite, and  $I$  denotes the identity  
 matrix. When  $r$  is a time variable, we use the standard notation  $g_r(\ell) = g(r + \ell)$  for functions  
 $g$  and all  $\ell \leq 0$  and  $r \geq 0$  such that  $r + \ell$  is in the domain of  $g$ . We also use the standard  
 family of functions  $\mathcal{K}_\infty$  and the standard definitions of ISS [8, 19] and controllability [18]. Let  
 us recall the definition of ISS systems. A system of the form

$$\dot{X}(t) = f(t, X_t, \delta(t))$$

with initial conditions in the set  $C^0([-\bar{\tau}, 0])$  of all  $\mathbb{R}^n$ -valued continuous functions that are  
 defined in the interval  $[-\bar{\tau}, 0]$  and a delay  $\tau(t)$  valued in  $[0, \bar{\tau}]$  for all  $t \geq 0$  for a given constant  
 $\bar{\tau} \geq 0$  is ISS with respect to  $\delta$ , where  $\delta$  is a locally bounded piecewise continuous function,  
 provided there are a function  $\beta$  of class  $\mathcal{KL}$  and a function  $\alpha$  of class  $\mathcal{K}$  such that all the  
 solutions of the system are such that

$$|X(t)| \leq \beta(|X_s|, t - s) + \alpha \left( \sup_{m \in [s, t]} |\delta(m)| \right)$$

50 for all  $t \geq s$  and all  $s \geq 0$ .

51 **2. Local ISS Halanay's results.** Let  $\alpha : [0, +\infty) \rightarrow [0, +\infty)$  be a continuous, nonnegative  
 52 valued and nondecreasing function such that there are two constants  $v_\star > 0$  and  $a > 0$  such  
 53 that

$$54 \quad (2.1) \quad \alpha(v_\star) = a.$$

55 Let  $t_\star \geq 0$  and  $b > 0$  be two constants. Let  $\zeta : [t_\star, +\infty) \rightarrow [0, +\infty)$  be a nondecreasing  
56 function such that

$$57 \quad (2.2) \quad \zeta(t) < bv_\star$$

58 holds for all  $t \geq t_\star$ . Let  $\mathcal{L}_0 \geq 0$  be a constant, and  $\tau \in (t_\star, +\infty)$  or  $\tau = +\infty$ . Throughout this  
59 section, we let  $V : [t_\star - \mathcal{L}_0, \tau) \rightarrow [0, +\infty)$  be a  $C^1$  function such that

$$60 \quad (2.3) \quad \sup_{m \in [t_\star - \mathcal{L}_0, t_\star]} V(m) < v_\star$$

61 and such that its time derivative  $\dot{V}$  satisfies the inequality

$$62 \quad (2.4) \quad \dot{V}(t) \leq -(a+b)V(t) + \alpha \left( \sup_{m \in [t-\mathcal{L}_0, t]} V(m) \right) \sup_{m \in [t-\mathcal{L}_0, t]} V(m) + \zeta(t)$$

63 for all  $t \in [t_\star, \tau)$ . Notice that if  $\alpha$  is a positive constant, then the inequality (2.4) is a standard  
64 Halanay's inequality. With the preceding notation, we then state and prove a result which is  
65 instrumental in establishing our main result:

66 **Lemma 2.1.** *Consider the function  $V$  introduced above. Then the inequality  $V(t) < v_\star$  is*  
67 *satisfied for all  $t \in [t_\star - \mathcal{L}_0, \tau)$ .*

68 *Proof.* We prove this result by contradiction. Suppose that there were a  $t_c \in [t_\star - \mathcal{L}_0, \tau)$   
69 such that

$$70 \quad (2.5) \quad V(t_c) = v_\star$$

71 and  $V(t) < v_\star$  for all  $t \in [t_\star - \mathcal{L}_0, t_c)$ . Then (2.3) gives  $t_c > t_\star$ , and (2.1), (2.2), and (2.4) give

$$72 \quad (2.6) \quad \dot{V}(t_c) < -(a+b)v_\star + \alpha(v_\star)v_\star + bv_\star = -av_\star + av_\star = 0.$$

73 From the inequality  $\dot{V}(t_c) < 0$  and the continuity of  $V$ , we deduce that there is  $t_d \in (t_\star, t_c)$   
74 such that  $V(t_d) > v_\star$ . This contradicts the definition of  $t_c$ , so the lemma holds.  $\blacksquare$

75 Since  $a > 0$  and  $b > 0$ , there is a unique value  $\lambda > 0$  such that

$$76 \quad (2.7) \quad \lambda = a + b - ae^{\lambda \mathcal{L}_0}.$$

77 Using this  $\lambda$ , we next state and prove the main result of this section.

78 **Theorem 2.2.** *Consider the function  $V$  introduced above, in the case where  $\tau = +\infty$ . Then*  
79

$$80 \quad (2.8) \quad V(t) \leq \sup_{m \in [s-\mathcal{L}_0, s]} V(m)e^{-\lambda(t-s)} + \int_s^t e^{b(m-t)} \zeta(m) dm$$

81 holds for all  $t \geq s$  and for all  $s \geq t_\star$ .

82 *Remark 2.3.* Two key differences between Theorem 2.2 and Theorem 3.2 in [15] are that  
 83 (i) a disturbance is taken into account in Theorem 2.2 and (ii) Theorem 2.2 gives exponential  
 84 stability when the disturbance is not present, whereas the [15, Theorem 3.2] establishes as-  
 85 ymptotic stability. The specifics of the nonlinear functions involved in the nonlinear Halanay's  
 86 inequality considered in [15] makes it possible to prove asymptotic stability of systems which  
 87 are not exponentially stable. Proving a local ISS result for classes of systems that we will study  
 88 below that are locally asymptotically stable but not exponentially stable when no disturbance  
 89 is present is an interesting open problem. On the other hand, see [14] for results for systems  
 90 of the form  $\dot{x}(t) = Ax(t) + f(x(t), u(t))$  having Banach spaces as the state spaces and having  
 91 controls  $u$  that are valued in a normed space, where  $A$  is the generator of a  $C_0$ -semigroup.  
 92 However, this class of systems from [14] do not include our main class of systems, where  $x_t$   
 93 can enter in a nonlinear way and which therefore include nonlinear delay systems; see (3.3)  
 94 below. Moreover, we believe that the methods of [14] do not lend themselves to being adapted  
 95 or extended to cover our general systems, because of the nonlinearity of our systems in  $x_t$ .  
 96 See also [2, Theorem 21] for time invariant cases, for which asymptotic stability is shown to  
 97 imply local ISS.

98 **Proof.** Since the function  $\alpha$  is nondecreasing, Lemma 2.1 and (2.4) ensure that

$$99 \quad (2.9) \quad \dot{V}(t) \leq -(a+b)V(t) + \alpha(v_*) \sup_{m \in [t-\mathcal{L}_0, t]} V(m) + \zeta(t)$$

100 holds for all  $t \geq t_*$ . From (2.1), we deduce that

$$101 \quad (2.10) \quad \dot{V}(t) \leq -(a+b)V(t) + a \sup_{m \in [t-\mathcal{L}_0, t]} V(m) + \zeta(t)$$

102 holds for all  $t \geq t_*$ . Since  $a > 0$  and  $b > 0$ , we can then apply Lemma A.1.1 in Appendix A.1  
 103 below, to conclude.  $\square$

104 *Remark 2.4.* The inequality (2.8) gives the ISS inequality

$$105 \quad (2.11) \quad V(t) \leq \sup_{m \in [s-\mathcal{L}_0, s]} V(m) e^{-\lambda(t-s)} + \frac{1}{b} \sup_{m \in [s, t]} \zeta(m)$$

106 for all  $t \geq s$  and all  $s \geq t_*$ .

107 **3. Local exponential stabilization result.** We use Theorem 2.2 to solve a local stabiliza-  
 108 tion problem for a class of nonlinear systems containing a small time-varying delay in the  
 109 control law.

110 **3.1. Studied system and preliminary result.** Let  $h : [0, +\infty) \rightarrow [0, +\infty)$  be a continuous  
 111 function for which there is a constant  $\bar{h} > 0$  such that

$$112 \quad (3.1) \quad 0 \leq h(t) \leq \bar{h}$$

113 for all  $t \geq 0$ . Let  $\delta : [0, +\infty) \rightarrow \mathbb{R}^n$  be a continuous function that admits a constant  $\bar{\Delta}$  such  
 114 that

$$115 \quad (3.2) \quad |\delta(t)| \leq \bar{\Delta}$$

116 for all  $t \geq 0$ . Consider the system

$$117 \quad (3.3) \quad \dot{x}(t) = Ax(t) + Bu(t - h(t)) + \mathcal{F}(t, x_t) + \delta(t)$$

118 where  $x$  is valued in  $\mathbb{R}^n$ , the input  $u$  is valued in  $\mathbb{R}^p$ , and  $\mathcal{F}$  is locally Lipschitz with respect  
119 to its second argument and piecewise continuous with respect to the first.

120 Throughout this paper, we assume that the dynamics satisfy the usual forward complete-  
121 ness properties, with standard existence and uniqueness properties of solutions. Let  $t_0 \geq 0$ .  
122 We consider initial functions  $x_0 : [t_0 - \bar{h}, t_0] \rightarrow \mathbb{R}^n$ , and we introduce three assumptions:

123 **Assumption 1.** *The pair  $(A, B)$  is controllable.*

124 **Assumption 2.** *There is a continuous nondecreasing function  $\rho : [0, +\infty) \rightarrow [0, +\infty)$  that  
125 is not identically equal to zero such that*

$$126 \quad (3.4) \quad |\mathcal{F}(t, \phi)| \leq \sup_{m \in [-\bar{h}, 0]} |\phi(m)|^2 \rho(|\phi(m)|)$$

127 holds for all functions  $\phi : [-\bar{h}, 0] \rightarrow \mathbb{R}^n$  and all  $t \geq 0$ .

128 It is well known that Assumption 1 provides a matrix  $K \in \mathbb{R}^{p \times n}$  such that the matrix

$$129 \quad (3.5) \quad H = A + BK$$

130 is Hurwitz, and so also a symmetric positive definite matrix  $P \in \mathbb{R}^{n \times n}$  and constants  $c > 0$   
131 and  $\bar{p} > 0$  such that

$$132 \quad (3.6) \quad PH + H^\top P \leq -cP$$

133 and

$$134 \quad (3.7) \quad I \leq P \text{ and } |P| \leq \bar{p},$$

135 e.g., by using the Pole-Shifting Theorem (as stated in [18, p.186]) to find  $K$ , and by then  
136 solving the Riccati equation  $PH + H^\top P = -I$  for  $P$ , and then choosing  $c > 0$  small enough  
137 such that  $cP \leq I$  [18], and then scaling  $P$  by a big enough positive constant if needed to satisfy  
138 our additional requirement that  $P \geq I$ . In what follows, we fix  $\rho$ ,  $K$ ,  $P$ , and  $\bar{p}$  satisfying the  
139 preceding requirements, and we assume that  $BK \neq 0$ . Our last assumption is the following,  
140 which can be viewed as a smallness condition on  $\bar{h}$  or on  $\bar{\Delta}$ :

141 **Assumption 3.** *There is a real value  $s_\star > 0$  such that with the choice*

$$142 \quad (3.8) \quad \omega_0 = (2|A| + 2|BK| + 1 + 2\sqrt{s_\star} \rho(\sqrt{s_\star}))\bar{p},$$

143 the inequality

$$144 \quad (3.9) \quad \left( e^{2.1\omega_0\bar{h}} - 1 \right) \frac{\bar{p}\bar{\Delta}^2}{\omega_0} < s_\star$$

145 is satisfied.

146 In terms of the preceding notation and the positive definite quadratic function

$$147 \quad (3.10) \quad W(x) = x^\top P x,$$

148 where  $x$  is valued in  $\mathbb{R}^n$ , we start with a technical lemma, where Assumption 3 ensures that  
 149 (3.12) is satisfied when the initial function is valued in a small enough neighborhood of the  
 150 origin:

151 **Lemma 3.1.** *Let the system (3.3) satisfy Assumptions 1-3. With the preceding notation,*  
 152 *consider (3.3) in closed-loop with the feedback*

$$153 \quad (3.11) \quad u(t - h(t)) = Kx(t - h(t))$$

154 for all  $t \geq t_0$ . Let  $x$  be a solution of this system such that

$$155 \quad (3.12) \quad \sup_{m \in [t_0 - \bar{h}, t_0]} W(x(m)) e^{2.1\omega_0 \bar{h}} + \left( e^{2.1\omega_0 \bar{h}} - 1 \right) \frac{\bar{p}\bar{\Delta}^2}{\omega_0} < s_\star.$$

156 Then  $x$  is defined over  $[t_0 - \bar{h}, t_0 + 2\bar{h}]$  and

$$157 \quad (3.13) \quad \sup_{m \in [t_0 - \bar{h}, t_0 + 2\bar{h}]} W(x(m)) \leq \sup_{m \in [t_0 - \bar{h}, t_0]} W(x(m)) e^{2.1\omega_0 \bar{h}} + \left( e^{2.1\omega_0 \bar{h}} - 1 \right) \frac{\bar{p}\bar{\Delta}^2}{\omega_0}$$

158 is satisfied.

159 *Proof.* Consider a solution  $x(t)$  of this closed-loop system from the lemma such that (3.12)  
 160 holds. Let  $[t_0 - \bar{h}, t_0 + t_\infty)$  be the domain of definition of  $x(t)$ . Then  $0 < t_\infty < +\infty$  or  $t_\infty = +\infty$ .  
 161 The time derivative of  $W$  defined in (3.10) along this solution satisfies

$$162 \quad (3.14) \quad \begin{aligned} \dot{W}(t) &= 2x(t)^\top P [Ax(t) + BKx(t - h(t)) + \mathcal{F}(t, x_t) + \delta(t)] \\ &\leq 2\bar{p}|x(t)| \left[ |A||x(t)| + |BK||x(t - h(t))| + \sup_{m \in [t - \bar{h}, t]} |x(m)|^2 \rho(|x(m)|) \right] \\ &\quad + 2|x(t)|\bar{p}\bar{\Delta} \end{aligned}$$

163 for all  $t \in (t_0, t_0 + t_\infty)$ , by (3.2), (3.4), (3.7), and the continuity of  $\delta$ . Since  $\rho$  from Assumption  
 164 2 is nondecreasing, it follows from (3.7) that

$$165 \quad (3.15) \quad \begin{aligned} \dot{W}(t) &\leq \bar{p}(2|A| + 2|BK| + 1) \sup_{m \in [t - \bar{h}, t]} W(x(m)) \\ &\quad + 2\bar{p} \sup_{m \in [t - \bar{h}, t]} W(x(m))^{3/2} \rho \left( \sqrt{\sup_{m \in [t - \bar{h}, t]} W(x(m))} \right) + \bar{p}\bar{\Delta}^2 \end{aligned}$$

166 by applying the triangle inequality to upper bound the last right side term in (3.14), in order  
 167 to get  $2|x(t)|\bar{p}\bar{\Delta} \leq \bar{p}|x(t)|^2 + \bar{p}\bar{\Delta}^2$ . In terms of the function

$$168 \quad (3.16) \quad \bar{\omega}(s) = \bar{p}(2|A| + 2|BK| + 1) s + 2s^{3/2}\bar{p}\rho(\sqrt{s}),$$

169 we therefore have

$$170 \quad (3.17) \quad \dot{W}(t) \leq \bar{\omega} \left( \sup_{m \in [t-\bar{h}, t]} W(x(m)) \right) + \bar{p}\bar{\Delta}^2$$

171 for all  $t \in (t_0, t_0 + t_\infty)$  and, according to the definition of  $\omega_0$  in (3.8), we have

$$172 \quad (3.18) \quad \bar{\omega}(s) \leq \omega_0 s$$

173 for all  $s \in [0, s_\star]$ . We apply now Lemma A.2.1 in Appendix A.2 below, with the choices

$$174 \quad (3.19) \quad W(x(t)), \bar{\omega}, \omega_0, \bar{p}^2 \bar{\Delta}^2, t_0, t_\infty, 2.1\bar{h}, \sup_{m \in [t_0-\bar{h}, t_0]} W(x(m)), s_\star, \text{ and } \bar{h}$$

175 playing the roles of  $Z(t)$ ,  $\Psi$ ,  $\Psi_0$ ,  $\bar{\Delta}$ ,  $t_a$ ,  $\tau$ ,  $q$ ,  $\bar{Z}$ ,  $\omega$  and  $T$ , respectively. Assumption 3 ensures  
176 that Assumption A.1 from Appendix A.2 (which is needed to apply Lemma A.2.1) is satisfied.  
177 Also, (A.2.4) holds, since

$$178 \quad (3.20) \quad W(x(t)) \leq \sup_{m \in [t_0-\bar{h}, t_0]} W(x(m))$$

179 for all  $t \in [t_0 - \bar{h}, t_0]$ . Then inequality (3.12) ensures that the inequality (A.2.5) is satisfied.  
180 Therefore, according to Lemma A.2.1, it follows that for all  $t \in [t_0 - \bar{h}, t_0 + \min\{t_\infty, 2.1\bar{h}\}]$ ,  
181 we have

$$182 \quad (3.21) \quad W(x(t)) \leq \sup_{m \in [t_0-\bar{h}, t_0]} W(x(m)) e^{2.1\omega_0 \bar{h}} + \left( e^{2.1\omega_0 \bar{h}} - 1 \right) \frac{\bar{p}\bar{\Delta}^2}{\omega_0}.$$

183 Therefore, the finite escape time phenomenon does not occur over  $[t_0 - \bar{h}, t_0 + 2\bar{h}]$ , which  
184 implies that  $t_\infty > 2\bar{h}$ . ■

185 **3.2. ISS result.** Using the notation from Section 3.1, let us introduce the function

$$186 \quad (3.22) \quad \beta(m) = 2\bar{h}|PBK|(|A| + |BK|) + 2\sqrt{m}(\bar{h}|PBK| + |P|)\rho(\sqrt{m}).$$

187 We now add the following assumption, which can again be regarded as a smallness condition  
188 on  $\bar{h}$ , where  $c > 0$  is the positive constant from (3.6):

189 **Assumption 4.** *The bound  $2\bar{h}|PBK|(|A| + |BK|) < c/4$  is satisfied.*

190 It follows that there is a constant  $w_\star > 0$  such that

$$191 \quad (3.23) \quad \beta(w_\star) = \frac{c}{4}$$

192 and we fix a  $w_\star$  satisfying the preceding requirement in the rest of this subsection. We also  
193 assume:

194 **Assumption 5.** *The inequality*

$$195 \quad (3.24) \quad \frac{4}{c}(|PBK|^2 \bar{h}^2 + |P|^2) \bar{\Delta}^2 \leq \frac{c w_\star}{4}$$

196 *holds.*



197 Assumption 5 can be viewed as a smallness condition on  $\bar{\Delta}$ . Let  $\gamma > 0$  be the constant such  
198 that

$$199 \quad (3.25) \quad \gamma = \frac{c}{2} - \frac{c}{4}e^{2\gamma\bar{h}}.$$

200 We are ready to state and prove the following result:

201 **Theorem 3.2.** *Let the system (3.3) satisfy Assumptions 1-5. Then, with the notation from*  
202 *the preceding subsection, consider (3.3) in closed-loop with the control*

$$203 \quad (3.26) \quad u(t - h(t)) = Kx(t - h(t)).$$

204 Consider any maximal solution  $x(t)$  of the closed-loop system such that

$$205 \quad (3.27) \quad \sup_{m \in [t_0 - \bar{h}, t_0]} W(x(m))e^{2.1\omega_0\bar{h}} + \left( e^{2.1\omega_0\bar{h}} - 1 \right) \frac{\bar{p}\bar{\Delta}^2}{\omega_0} < \min\{s_\star, w_\star\}$$

206 holds. Then, for each  $s \geq t_0 + \bar{h}$ , and with the choice

$$207 \quad (3.28) \quad \mathcal{H}(m) = \frac{4}{c}|PBK|^2\bar{h} \int_0^m |\delta(r)|^2 dr + \frac{4}{c}|P|^2 \sup_{\ell \in [\bar{h}, m]} |\delta(\ell)|^2,$$

208 the inequality

$$209 \quad (3.29) \quad |x(t)| \leq \sqrt{\bar{p} \sup_{m \in [s - 2\bar{h}, s]} |x(m)|^2 e^{-\gamma(t-s)} + \int_s^t e^{\frac{c}{4}(m-t)} \mathcal{H}(m) dm}$$

210 holds for all  $t \geq s$ .

211 *Proof.* We consider a trajectory  $x$  of the closed-loop system satisfying the conditions of  
212 Theorem 3.2. Let  $[t_0 - \bar{h}, t_\infty)$  be the largest domain of definition of  $x$ . Notice for later use  
213 that it follows from Lemma 3.1 that the solution is defined over  $[t_0 - \bar{h}, t_0 + 2\bar{h}]$ , and that the  
214 inequality (3.13) holds. Then necessarily,  $t_\infty > t_0 + 2\bar{h}$ . From the definition of  $H$  in (3.5), we  
215 deduce that

$$216 \quad (3.30) \quad \dot{x}(t) = Hx(t) - BK \int_{t-h(t)}^t \dot{x}(m) dm + \mathcal{F}(t, x_t) + \delta(t)$$

217 for all  $t \in [t_0 + \bar{h}, t_\infty)$ , since the integral in (3.30) is  $x(t) - x(t - h(t))$ . According to (3.6), the  
218 time derivative of  $W$  along (3.30) satisfies

$$219 \quad (3.31) \quad \begin{aligned} \dot{W}(t) \leq & -cW(x(t)) - 2x(t)^\top PBK \int_{t-h(t)}^t \dot{x}(m) dm \\ & + 2x(t)^\top P\mathcal{F}(t, x_t) + 2x(t)^\top P\delta(t) \end{aligned}$$

220 for all  $t \in [t_0 + \bar{h}, t_\infty)$ .

221 Since  $t_\infty > 2\bar{h}$ , it follows that

$$\begin{aligned}
222 \quad (3.32) \quad \dot{W}(t) &\leq -cW(x(t)) \\
&+ 2|x(t)||PBK| \int_{t-h(t)}^t |Ax(m) + BKx(m-h(m)) + \mathcal{F}(m, x_m) + \delta(m)| dm \\
&+ 2x(t)^\top P\mathcal{F}(t, x_t) + 2x(t)^\top P\delta(t)
\end{aligned}$$

223 for all  $t \in [t_0 + \bar{h}, t_\infty)$ . Consequently,

$$\begin{aligned}
224 \quad (3.33) \quad \dot{W}(t) &\leq -cW(x(t)) + 2|PBK||x(t)| \int_{t-h(t)}^t |Ax(m)| dm \\
&+ 2|PBK||x(t)| \int_{t-h(t)}^t |BKx(m-h(m))| dm \\
&+ 2|PBK||x(t)| \int_{t-h(t)}^t |\mathcal{F}(m, x_m)| dm + 2x(t)^\top P\mathcal{F}(t, x_t) \\
&+ 2|PBK||x(t)| \int_{t-h(t)}^t |\delta(m)| dm + 2x(t)^\top P\delta(t)
\end{aligned}$$

225 for all  $t \in [t_0 + \bar{h}, t_\infty)$ . From our bound  $\bar{h}$  on  $h$  from (3.1), Assumption 2, and (3.7), we deduce  
226 that

$$\begin{aligned}
227 \quad (3.34) \quad \dot{W}(t) &\leq -cW(x(t)) + 2\bar{h}|PBK||A| \sup_{m \in [t-\bar{h}, t]} W(x(m)) \\
&+ 2\bar{h}|PBK||BK| \sup_{m \in [t-2\bar{h}, t]} W(x(m)) \\
&+ 2|PBK||x(t)| \int_{t-h(t)}^t \sup_{r \in [m-\bar{h}, m]} |x(r)|^2 \rho(|x(r)|) dm \\
&+ 2|x(t)||P| \sup_{m \in [t-\bar{h}, t]} |x(m)|^2 \rho(|x(m)|) \\
&+ 2|PBK||x(t)| \int_{t-h(t)}^t |\delta(m)| dm + 2|P||x(t)||\delta(t)|
\end{aligned}$$

228 for all  $t \in [t_0 + \bar{h}, t_\infty)$ . Since  $\rho$  is nondecreasing, we get

$$\begin{aligned}
229 \quad (3.35) \quad \dot{W}(t) &\leq -cW(x(t)) + 2\bar{h}|PBK|(|A| + |BK|) \sup_{m \in [t-2\bar{h}, t]} W(x(m)) \\
&+ 2\bar{h}|PBK|\rho \left( \sup_{m \in [t-2\bar{h}, t]} \sqrt{W(x(m))} \right) dm \sup_{m \in [t-2\bar{h}, t]} W(x(m))^{\frac{3}{2}} \\
&+ 2|P||x(t)||\delta(t)| \\
&+ 2|P| \sup_{m \in [t-2\bar{h}, t]} W(x(m))^{\frac{3}{2}} \rho \left( \sup_{m \in [t-2\bar{h}, t]} \sqrt{W(x(m))} \right) \\
&+ 2|PBK||x(t)| \int_{t-h(t)}^t |\delta(m)| dm
\end{aligned}$$

230 for all  $t \in [t_0 + \bar{h}, t_\infty)$ . We next use the triangle inequality, Jensen's inequality, and (3.7), to  
 231 get

$$232 \quad (3.36) \quad \begin{aligned} 2|PBK||x(t)| \int_{t-h(t)}^t |\delta(m)| dm &\leq \frac{c}{4}W(x(t)) + \frac{4}{c}|PBK|^2\bar{h} \int_{t-h(t)}^t |\delta(m)|^2 dm \text{ and} \\ 2|P||x(t)||\delta(t)| &\leq \frac{c}{4}W(x(t)) + \frac{4}{c}|P|^2|\delta(t)|^2. \end{aligned}$$

233 It follows from (3.35) that

$$234 \quad (3.37) \quad \begin{aligned} \dot{W}(t) &\leq -\frac{c}{2}W(x(t)) + 2\bar{h}|PBK|(|A| + |BK|) \sup_{m \in [t-2\bar{h}, t]} W(x(m)) \\ &\quad + \frac{4}{c}|PBK|^2\bar{h} \int_{t-h(t)}^t |\delta(m)|^2 dm \\ &\quad + 2(\bar{h}|PBK| + |P|) \sup_{m \in [t-2\bar{h}, t]} W(x(m))^{\frac{3}{2}} \rho \left( \sup_{m \in [t-2\bar{h}, t]} \sqrt{W(x(m))} \right) \\ &\quad + \frac{4}{c}|P|^2|\delta(t)|^2 \end{aligned}$$

235 holds for all  $t \in [t_0 + \bar{h}, t_\infty)$ . Therefore, we have

$$236 \quad (3.38) \quad \dot{W}(t) \leq -\frac{c}{2}W(x(t)) + \beta \left( \sup_{m \in [t-2\bar{h}, t]} W(x(m)) \right) \sup_{m \in [t-2\bar{h}, t]} W(x(m)) + \delta_{\#}(t)$$

237 for all  $t \in [t_0 + \bar{h}, t_\infty)$ , where  $\beta$  was defined in (3.22) and

$$238 \quad (3.39) \quad \delta_{\#}(t) = \frac{4}{c}|PBK|^2\bar{h} \int_{t-h(t)}^t |\delta(m)|^2 dm + \frac{4}{c}|P|^2|\delta(t)|^2.$$

239 Note that (3.2) implies that  $|\delta_{\#}(t)| \leq \frac{4}{c}(|PBK|^2\bar{h}^2 + |P|^2)\bar{\Delta}^2$  for all  $t \geq \bar{h}$ . Hence, (3.24)  
 240 gives

$$241 \quad (3.40) \quad |\delta_{\#}(t)| \leq \frac{cw_{\star}}{4}$$

242 for all  $t \geq \bar{h}$ . Then let us recall that (3.13) holds, by (3.27). Consequently (3.27) ensures that  
 243

$$244 \quad (3.41) \quad \sup_{m \in [t_0 - \bar{h}, t_0 + 2\bar{h}]} W(x(m)) < w_{\star}$$

245 We can now apply Theorem 2.2 with  $\lambda = \gamma$ , and with

$$246 \quad (3.42) \quad \begin{aligned} V(t) &= W(x(t)), \quad a = b = c/4, \quad \alpha = \beta, \quad \zeta(t) = \sup_{\ell \in [\bar{h}, t]} |\delta_{\#}(\ell)|, \quad \mathcal{L}_0 = 2\bar{h}, \quad v_{\star} = w_{\star}, \\ \tau &= t_{\infty}, \quad \text{and } t_{\star} = t_0 + \bar{h}. \end{aligned}$$

247 Note that (3.23) ensures that (2.1) is satisfied. Then (3.40)-(3.41) ensure that (2.2)-(2.3) are  
 248 satisfied. Using Lemma 2.1, we can prove that the finite escape time phenomenon does not  
 249 occur, so  $t_{\infty} = +\infty$ , and Theorem 2.2 gives

$$250 \quad (3.43) \quad W(x(t)) \leq \sup_{m \in [s-2\bar{h}, s]} W(x(m))e^{-\gamma(t-s)} + \int_s^t e^{\frac{c}{4}(m-t)} \sup_{\ell \in [\bar{h}, m]} |\delta_{\#}(\ell)| dm$$

251 when  $t \geq s \geq t_0 + \bar{h}$  where  $\gamma$  is the constant defined in (3.25), and where the sup was needed  
 252 in (3.42) and in the integrand in (3.43) because Theorem 2.2 requires its function  $\zeta$  to be  
 253 nondecreasing. Hence, (3.7) gives

$$254 \quad (3.44) \quad |x(t)|^2 \leq \bar{p} \sup_{m \in [s-2\bar{h}, s]} |x(m)|^2 e^{-\gamma(t-s)} + \int_s^t e^{\frac{c}{4}(m-t)} \sup_{\ell \in [\bar{h}, m]} |\delta_{\sharp}(\ell)| dm$$

255 for all  $t \geq s \geq t_0 + \bar{h}$ . This allows us to conclude. ■

## 256 4. Output feedback local stabilization.

257 **4.1. Statement of result.** Consider the system

$$258 \quad (4.1) \quad \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) + \mathcal{G}(x(t)) + \delta(t) \\ y(t) &= Cx(t) \end{cases}$$

259 with  $x$  valued in  $\mathbb{R}^n$ , the input  $u$  is valued in  $\mathbb{R}^p$ , the output  $y$  valued in  $\mathbb{R}^q$ ,  $\mathcal{G}$  being a locally  
 260 Lipschitz nonlinear function and  $\delta$  being continuous. We introduce two assumptions:

261 **Assumption 6.** *The pair  $(A, C)$  is observable, the pair  $(A, B)$  is controllable, and the system*  
 262 *(4.1) is forward complete for each continuous choice of  $u : [0, +\infty) \rightarrow \mathbb{R}^p$ .*

263 **Assumption 7.** *There is a continuous, nondecreasing function  $\kappa : [0, +\infty) \rightarrow [0, +\infty)$  that*  
 264 *is not identically equal to the zero function such that*

$$265 \quad (4.2) \quad |\mathcal{G}(x)| \leq |x|^2 \kappa(|x|)$$

266 holds for all  $x \in \mathbb{R}^n$ .

267 Since  $(A, B)$  is controllable, we can argue as in Section 3.1 to find a matrix  $K \in \mathbb{R}^{p \times n}$ , a  
 268 symmetric and positive definite matrix  $P \in \mathbb{R}^{n \times n}$ , and constants  $c > 0$  and  $\bar{p} > 0$  such that

$$269 \quad (4.3) \quad PH + H^\top P \leq -cP$$

270 where

$$271 \quad (4.4) \quad H = A + BK$$

272 and

$$273 \quad (4.5) \quad I \leq P \quad \text{and} \quad |P| \leq \bar{p}.$$

274 Let  $T > 0$  be a constant and set

$$275 \quad (4.6) \quad E = \int_{-T}^0 e^{A^\top s} C^\top C e^{As} ds.$$

276 Assumption 6 implies that the matrix  $E$  is invertible, e.g., by [18]. We also use the functions  
 277  $\hat{x}$  and  $\mu$  that are defined by

$$278 \quad (4.7) \quad \hat{x}(t) = E^{-1} \int_{t-T}^t e^{A^\top(s-t)} C^\top y(s) ds + E^{-1} \int_{t-T}^t e^{A^\top(s-t)} C^\top C \int_s^t e^{A(s-m)} Bu(m) dm ds$$

279 for all  $t \geq T$  and  $\hat{x}(t) = 0$  for all  $t \in [0, T)$ , and

$$280 \quad (4.8) \quad \mu(x_t) = E^{-1} \int_{t-T}^t e^{A^\top(s-t)} C^\top C \int_s^t e^{A(s-m)} \mathcal{G}(x(m)) dm ds,$$

281 which is defined for all  $t \geq T$ . We also fix any constant  $g$  such that

$$282 \quad (4.9) \quad g \geq \left( \int_{-T}^0 \int_s^0 \left| BKE^{-1} e^{A^\top s} C^\top C e^{A(s-m)} \right| dm ds \right)^2,$$

283 and, in terms of the function  $\kappa$  from Assumption 7, we use the function

$$284 \quad (4.10) \quad \gamma_0(u) = \frac{\underline{c}}{\bar{c}} \bar{p} (1+g) \kappa^2(\sqrt{u}) u.$$

285 From our assumption on  $\kappa$ , we then fix a constant  $u_\star > 0$  such that

$$286 \quad (4.11) \quad \gamma_0(u_\star) = \frac{\underline{c}}{4}.$$

287 Using the preceding notation, our final assumption is the following smallness condition on the  
288  $\delta_i$ 's:

289 **Assumption 8.** *There are a vector  $M = [m_1, \dots, m_n]^\top \in [0, +\infty)^n$ , a constant  $\bar{\Delta} > 0$ , and  
290 a constant  $d > 0$  such that*

$$291 \quad (4.12) \quad d \geq \left( 1 + \int_{-T}^0 \int_s^0 \left| BKE^{-1} e^{A^\top s} C^\top C e^{A(s-m)} M \right| dm ds \right) |M|$$

292 *for which the conditions*

$$293 \quad (4.13) \quad \sup_{t \geq 0} |\delta_i(t)| \leq m_i \bar{\Delta} \text{ for } i = 1, \dots, n$$

294 *and*

$$295 \quad (4.14) \quad \bar{\Delta} < \frac{\underline{c}}{4d} \sqrt{\frac{u_\star}{\bar{p}}}$$

296 *are satisfied.*

297 We also use the function  $\delta^\sharp : [T, +\infty) \rightarrow \mathbb{R}^n$  that is defined by

$$298 \quad (4.15) \quad \delta^\sharp(t) = \delta(t) - BKE^{-1} \int_{t-T}^t e^{A^\top(s-t)} C^\top C \int_s^t e^{A(s-m)} \delta(m) dm ds.$$

299 Let  $\nu > 0$  be the constant such that

$$300 \quad (4.16) \quad \nu = \frac{\underline{c}}{2} - \frac{\underline{c}}{4} e^{\nu T}.$$

301 In terms of the preceding notation, our main result of this subsection is:

302 **Theorem 4.1.** Consider the system (4.1) in closed-loop with the output feedback

$$303 \quad (4.17) \quad u(t) = K\hat{x}(t).$$

304 Let Assumptions 6-8 hold. Consider any maximal solution  $x(t)$  of this system such that

$$305 \quad (4.18) \quad \sup_{\ell \in [0, T]} x(\ell)^\top P x(\ell) < u_\star.$$

306 Then, for each  $s \geq T$ , the inequality

$$307 \quad (4.19) \quad |x(t)| \leq \sqrt{\bar{p}} \sqrt{\sup_{m \in [s-T, s]} |x(m)|^2 e^{-\nu(t-s)} + \frac{4}{c} \int_s^t e^{\frac{c}{4}(m-t)} \sup_{\ell \in [T, m]} |\delta^\sharp(\ell)|^2 dm}$$

308 holds for all  $t \geq s$ .

309 **Remark 4.2.** From the inequality (4.19) and the subadditivity of the square root, we can  
 310 deduce a standard ISS inequality. A crucial difference between our result and the one of [11]  
 311 is that we do not assume that the function  $\mathcal{G}$  is known. Instead, we assume that the function  
 312  $\kappa$  in Assumption 7 is known. Note that the observer  $\hat{x}$  is not present in the inequality (4.19)  
 313 and the rate of convergence of  $x(t)$  depends only on the choices of  $K$  and  $T$ . Therefore we can  
 314 view  $\hat{x}$  as an ‘‘almost finite time observer’’. We have assumed that (4.1) is forward complete.  
 315 This is a technical assumption, but we conjecture that it can be relaxed. Since  $u_\star > 0$ , one  
 316 can often use continuous dependence arguments to check (4.18), e.g., when  $\delta = 0$  because  
 317 then the closed loop system admits a 0 equilibrium, which allows us to satisfy (4.18) in cases  
 318 where  $|\delta(t)|$  is known to be small enough for all values  $t \in [0, T]$ .

319 **4.2. Proof of Theorem 4.1.** The system (4.1) in closed loop with (4.17) is

$$320 \quad (4.20) \quad \dot{x}(t) = Ax(t) + BK\hat{x}(t) + \mathcal{G}(x(t)) + \delta(t).$$

321 By applying the method of variation of parameters to (4.20) on the interval  $[s, t]$  for any  
 322  $s \in [t - T, t]$ , then left multiplying the result by  $e^{A^\top(s-t)} C^\top C e^{A(s-t)}$  and then integrating the  
 323 result over all  $s \in [t - T, t]$ , it follows that

$$324 \quad (4.21) \quad x(t) = \hat{x}(t) + \mu(x_t) + E^{-1} \int_{t-T}^t e^{A^\top(s-t)} C^\top C \int_s^t e^{A(s-m)} \delta(m) dm ds$$

325 for all  $t \geq T$ . By solving for  $\hat{x}$  in (4.21) and substituting the result into (4.20), it follows that

$$326 \quad (4.22) \quad \dot{x}(t) = Ax(t) + BK \left( x(t) - \mu(x_t) - E^{-1} \int_{t-T}^t e^{A^\top(s-t)} C^\top C \int_s^t e^{A(s-m)} \delta(m) dm ds \right) \\ + \mathcal{G}(x(t)) + \delta(t)$$

327 for all  $t \geq T$ . From the definitions of  $H = A + BK$  from (4.4) and the  $\delta^\sharp$  formula in (4.15), it  
 328 follows that for all  $t \geq T$ , we obtain

$$329 \quad (4.23) \quad \dot{x}(t) = Hx(t) - BK\mu(x_t) + \mathcal{G}(x(t)) + \delta^\sharp(t).$$

330 Notice for later use that Assumption 8 and the Cauchy-Schwarz inequality imply that

$$331 \quad (4.24) \quad |\delta^\sharp(t)| \leq |M|\bar{\Delta} + |M| \int_{t-T}^t \int_s^t \left| BKE^{-1}e^{A^\top(s-t)}C^\top Ce^{A(s-m)}M \right| dm ds \bar{\Delta} \leq d\bar{\Delta}$$

332 for all  $t \geq T$ , where  $d$  was introduced in (4.12). Now, we introduce the Lyapunov function:

$$333 \quad (4.25) \quad U(x) = x^\top Px.$$

334 Its time derivative along all trajectories of (4.23) satisfies

$$335 \quad (4.26) \quad \dot{U}(t) = 2x(t)^\top P[Hx(t) - BK\mu(x_t) + \mathcal{G}(x(t)) + \delta^\sharp(t)]$$

336 for all  $t \geq T$ . From (4.3), it follows that

$$337 \quad (4.27) \quad \begin{aligned} \dot{U}(t) &\leq -cU(x(t)) + 2x(t)^\top P[-BK\mu(x_t) + \mathcal{G}(x(t)) + \delta^\sharp(t)] \\ &\leq -cU(x(t)) + 2 \left\{ x(t)^\top \sqrt{P} \right\} \left\{ \sqrt{P}[-BK\mu(x_t) + \mathcal{G}(x(t)) + \delta^\sharp(t)] \right\} \\ &\leq -\frac{c}{2}U(x(t)) + \frac{2}{c}[-BK\mu(x_t) + \mathcal{G}(x(t)) + \delta^\sharp(t)]^\top P[-BK\mu(x_t) \\ &\quad + \mathcal{G}(x(t)) + \delta^\sharp(t)] \end{aligned}$$

338 where the last inequality came from applying Young's inequality to get  $2ab \leq \frac{c}{2}|a|^2 + \frac{2}{c}|b|^2$ ,  
339 where  $a$  and  $b$  are the terms in curly braces in (4.27). Consequently, the Cauchy-Schwarz  
340 inequality gives

$$341 \quad (4.28) \quad \begin{aligned} \dot{U}(t) &\leq -\frac{c}{2}U(x(t)) + \frac{4}{c}\bar{p} \left[ |-BK\mu(x_t) + \mathcal{G}(x(t))|^2 + |\delta^\sharp(t)|^2 \right] \\ &\leq -\frac{c}{2}U(x(t)) + \frac{8}{c}\bar{p} \left[ |BK\mu(x_t)|^2 + |\mathcal{G}(x(t))|^2 \right] + \frac{4\bar{p}}{c}|\delta^\sharp(t)|^2 \end{aligned}$$

342 for all  $t \geq T$ . We can then use our choice (4.8) of  $\mu$  to obtain

$$343 \quad (4.29) \quad \begin{aligned} |BK\mu(x_t)| &= \left| \int_{t-T}^t \int_s^t BKE^{-1}e^{A^\top(s-t)}C^\top Ce^{A(s-m)}\mathcal{G}(x(m))dm ds \right| \\ &\leq \int_{t-T}^t \int_s^t \left| BKE^{-1}e^{A^\top(s-t)}C^\top Ce^{A(s-m)} \right| |x(m)|^2 \kappa(|x(m)|) dm ds \end{aligned}$$

344 where the last inequality is a consequence of (4.2). Since  $\kappa$  is nondecreasing, it follows that

$$345 \quad (4.30) \quad \begin{aligned} &|BK\mu(x_t)| \\ &\leq \int_{t-T}^t \int_s^t \left| BKE^{-1}e^{A^\top(s-t)}C^\top Ce^{A(s-m)} \right| dm ds \sup_{r \in [t-T, t]} |x(r)|^2 \kappa \left( \sup_{r \in [t-T, t]} |x(r)| \right) \\ &= \int_{-T}^0 \int_s^0 \left| BKE^{-1}e^{A^\top s}C^\top Ce^{A(s-m)} \right| dm ds \sup_{r \in [t-T, t]} |x(r)|^2 \kappa \left( \sup_{r \in [t-T, t]} |x(r)| \right) \\ &\leq \int_{-T}^0 \int_s^0 \left| BKE^{-1}e^{A^\top s}C^\top Ce^{A(s-m)} \right| dm ds \sup_{r \in [t-T, t]} U(x(r)) \kappa \left( \sup_{r \in [t-T, t]} \sqrt{U(x(r))} \right) \end{aligned}$$

346 where the last inequality is a consequence of (4.5). We deduce that

$$347 \quad (4.31) \quad |BK\mu(x_t)|^2 \leq g \sup_{m \in [t-T, t]} U^2(x(m)) \kappa^2 \left( \sup_{m \in [t-T, t]} \sqrt{U(x(m))} \right)$$

348 with  $g$  defined in (4.9). Similarly, we have

$$349 \quad (4.32) \quad |\mathcal{G}(x)|^2 \leq U^2(x) \kappa^2 \left( \sqrt{U(x)} \right)$$

350 for all  $x \in \mathbb{R}^n$ . It follows that

$$351 \quad (4.33) \quad |BK\mu(x_t)|^2 + |\mathcal{G}(x(t))|^2 \leq (1+g) \sup_{m \in [t-T, t]} U^2(x(m)) \kappa^2 \left( \sup_{m \in [t-T, t]} \sqrt{U(x(m))} \right).$$

352 Consequently, for all  $t \geq T$ , (4.28) gives

$$353 \quad (4.34) \quad \dot{U}(t) \leq -\frac{c}{2}U(x(t)) + \gamma_0 \left( \sup_{m \in [t-T, t]} U(x(m)) \right) \sup_{m \in [t-T, t]} U(x(m)) + \frac{4\bar{p}}{c} |\delta^\#(t)|^2$$

354 with  $\gamma_0$  defined in (4.10).

355 Now, let us apply Theorem 2.2 with  $U(x(t))$  playing the role of  $V(t)$ ,  $\gamma_0$  playing the role  
356 of  $\alpha$ ,  $v_\star = u_\star$ ,  $\mathcal{L}_0 = t_\star = T$ ,  $\tau = +\infty$  and  $a = b = \frac{c}{4}$ . The inequality (4.14) gives

$$357 \quad (4.35) \quad \bar{\Delta}^2 < \frac{c^2}{16d^2\bar{p}} u_\star.$$

358 Then, according to (4.24),

$$359 \quad (4.36) \quad \frac{4\bar{p}}{c} |\delta^\#(t)|^2 < \frac{c}{4} u_\star$$

360 for all  $t \geq T$ . Consider any maximal solution  $x(t)$  of the closed loop system satisfying (4.18).  
361 Since (4.16) and (4.36) also hold, we deduce from Theorem 2.2 that  $U(x(t))$  satisfies the ISS  
362 inequality

$$363 \quad (4.37) \quad U(x(t)) \leq \sup_{m \in [s-T, s]} U(x(m)) e^{-\nu(t-s)} + \frac{4\bar{p}}{c} \int_s^t e^{\frac{c}{4}(m-t)} \sup_{\ell \in [T, m]} |\delta^\#(\ell)|^2 dm$$

364 if  $t \geq s \geq T$ . This inequality and (4.5) allow us to conclude.

## 365 5. Illustrations.

366 **5.1. Illustration of Theorem 3.2.** Consider the controlled van der Pol equation

$$367 \quad (5.1) \quad \begin{cases} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -x_1(t) + \epsilon(1 - x_1^2)x_2 + u(t - h(t)) \end{cases}$$

368 for constants  $\epsilon > 0$  and a continuous choice  $h(t)$  of the delay; see, e.g., [8, Section 13.2] for  
369 simpler cases with no delays. The dynamics are used to represent oscillations in vacuum tube



370 circuits, and provide a fundamental equation in nonlinear oscillation theory. The system has  
 371 the form (3.3) with the choices

$$372 \quad (5.2) \quad A = \begin{bmatrix} 0 & 1 \\ -1 & \epsilon \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{and } \mathcal{F}(t, x_t) = \begin{bmatrix} 0 \\ -\epsilon x_1^2(t)x_2(t) \end{bmatrix}$$

373 with  $\delta = 0$ . Using Mathematica [10], we can check that Assumption 1-2 hold with  $\bar{\Delta} = 0$ ,  
 374  $K = [-1, -2]$ ,  $\rho(s) = \epsilon s$ ,

$$375 \quad (5.3) \quad P = \begin{bmatrix} 4.09112 & 0.722222 \\ 0.722222 & 1.17951 \end{bmatrix},$$

376 and  $\epsilon = 0.01$ , where  $P$  was found by first solving for a positive definite symmetric matrix  
 377  $P_1 \in \mathbb{R}^{2 \times 2}$  such that  $P_1 H + H^\top P_1 = -I$  holds with  $H = A + BK$ , then choosing  $c = 0.75$   
 378 in order to satisfy  $cP_1 \leq I$ , and then multiplying  $P_1$  by 3.25 to satisfy the requirement that  
 379  $P \geq I$  with the choice  $P = 3.25P_1$ . Also, since  $\bar{\Delta} = 0$ , Assumptions 3 and 5 hold for any  
 380  $s_* > 0$ . We can then also use Mathematica to compute the basin of attraction from Theorem  
 381 3.2. For instance, when the delay  $h$  is the zero function, we can check that we can satisfy  
 382 the requirements of Theorem 3.2 with  $w_* = s_* = 2.20049$  and all initial functions whose  
 383 norms are bounded by 0.718677. If we instead use the delay bound  $\bar{h} = 0.008$  and keep all  
 384 other parameter values the same as before, then the basin of attraction consists of all initial  
 385 functions that are bounded by 0.137212. This illustrates the trade-off that increasing the  
 386 bound  $\bar{h}$  on the allowable input delays  $h(t)$  can reduce the basin of attraction.

387 **5.2. Illustration of Theorem 4.1.** Consider the two dimensional system

$$388 \quad (5.4) \quad \begin{cases} \dot{x}_1(t) &= \text{sat}(x_2(t)) + \delta_1(t) \\ \dot{x}_2(t) &= u(t) + \delta_2(t) \\ y(t) &= x_1(t) \end{cases}$$

389 where  $\text{sat}$  is the standard saturation that is defined by  $\text{sat}(s) = s$  when  $|s| \leq 1$ ,  $\text{sat}(s) = 1$  if  
 390  $s > 1$ , and  $\text{sat}(s) = -1$  if  $s < -1$ . We illustrate Theorem 4.1. To satisfy Assumption 7 with

$$391 \quad (5.5) \quad \mathcal{G}(x) = [\text{sat}(x_2) - x_2, 0]^\top,$$

392 we prove that

$$393 \quad (5.6) \quad |\mathcal{G}(x)| \leq |x|^2 \kappa(|x|)$$

394 for all  $x \in \mathbb{R}^2$ , where  $\kappa$  is defined by

$$395 \quad (5.7) \quad \kappa(r) = \max\{r - 1, 0\},$$

396 by considering two cases:

- 397 1) If  $|x| \leq 1$ , then  $|x_2| \leq 1$ . It follows that  $\mathcal{G}(x) = 0$ . Consequently, (5.6) is satisfied.  
 398 2) If  $|x| > 1$  and  $|x_2| \leq 1$ , then  $\mathcal{G}(x) = 0$  which implies that (5.6) is satisfied. Next, consider  
 399 the case where  $x_2 > 1$ . Then  $|\mathcal{G}(x)| = x_2 - 1$ . Thus (5.6) is satisfied if and only if  $x_2 - 1 \leq$

400  $|x|^2\kappa(|x|)$ , which is satisfied because  $x_2 - 1 \leq x_2^2(x_2 - 1) = x_2^2\kappa(x_2)$ . If  $x_2 < -1$ , then we get  
 401  $|\mathcal{G}(x)| = -x_2 - 1$  and we obtain a similar result.

402 It follows that Assumptions 6-7 are satisfied, once we choose

$$403 \quad (5.8) \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ and } C = [1 \quad 0].$$

404 Since the preceding choices give

$$405 \quad (5.9) \quad e^{As} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

406 for all  $s \in \mathbb{R}$ , our formulas (4.6)-(4.7) give

$$407 \quad (5.10) \quad E = \int_{-T}^0 \begin{bmatrix} 1 & s \\ s & s^2 \end{bmatrix} ds = \begin{bmatrix} T & -\frac{T^2}{2} \\ -\frac{T^2}{2} & \frac{T^3}{3} \end{bmatrix}$$

408 and therefore also

$$409 \quad (5.11) \quad \hat{x}(t) = \frac{12}{T^3} \begin{bmatrix} \frac{T^2}{3} & \frac{T}{2} \\ \frac{T}{2} & 1 \end{bmatrix} \int_{t-T}^t \begin{bmatrix} 1 \\ s-t \end{bmatrix} y(s) ds \\ + \frac{12}{T^3} \begin{bmatrix} \frac{T^2}{3} & \frac{T}{2} \\ \frac{T}{2} & 1 \end{bmatrix} \int_{t-T}^t \begin{bmatrix} 1 \\ s-t \end{bmatrix} \int_s^t (s-m)u(m)dm ds$$

410 for all  $t \geq T$ . Let us choose

$$411 \quad (5.12) \quad K = [-1, -2].$$

412 We can then use Mathematica to check that our requirements are met with

$$413 \quad (5.13) \quad P = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

414  $c = 0.845$ ,  $T = 0.1$ ,  $g = 9.3218$ ,  $\gamma_0(s) = 293.164\kappa^2(\sqrt{s})s$ , and  $u_* = 1.053$ . Using the inequality  
 415

$$416 \quad (5.14) \quad 2|x|^2 \leq x^\top Px$$

417 we obtain the value 0.592453 for the bound on the norm of the initial function from Theorem  
 418 4.1. Moreover, the maximum allowable upper bound on  $\bar{\delta}$  is 0.0309026 when  $M = [1, 0]^\top$ , and  
 419 that the maximum allowable bound on  $\bar{\delta}$  is 0.112342 when  $M = [0, 1]^\top$ .

420 **6. Conclusion.** We advanced the state of the art for stability analysis of nonlinear sys-  
 421 tems, by providing a local version of Halanay's inequality that is conducive to proving local  
 422 asymptotic stability properties for nonlinear systems that contain state or input delays and  
 423 uncertainties. Our new results are significant, because of the well-known benefits of using  
 424 global versions of Halanay's inequality to prove global asymptotic stability for systems with  
 425 unknown delays, and because many significant systems are only locally asymptotically stable  
 426 and therefore are beyond the scope of earlier global versions of Halanay's inequality. Another  
 427 significant benefit of our work is that we allow the dynamics to contain unknown nonlinearities  
 428 that violate the standard linear growth conditions and that can contain distributed state  
 429 delays. We used our approach to prove a local feedback stabilization result for a large class of  
 430 nonlinear systems with outputs. We illustrated how our methods provide new estimates for  
 431 basins of attraction for a controlled Van der Pol equation and other cases that contain input  
 432 delays and outputs. We hope to find analogs for discrete-time systems, and vector versions  
 433 providing local analogs of the corresponding continuous-time Halanay's inequality results from  
 434 [12].

#### 435 **Appendices: Proofs of key lemmas.**

436 **A.1. ISS inequality.** We prove a key lemma that we used in our proof of Theorem 2.2.  
 437 To this end, first let  $t_\star \geq 0$  and  $\mathcal{L}_0 \geq 0$  be given constants. Consider a  $C^1$  function  $V : [t_\star - \mathcal{L}_0, +\infty) \rightarrow [0, +\infty)$ , a nonnegative valued nondecreasing continuous function  $\zeta$ , and constants  $a > 0$  and  $b > 0$  such that

$$440 \quad (\text{A.1.1}) \quad \dot{V}(t) \leq -(a+b)V(t) + a \sup_{m \in [t-\mathcal{L}_0, t]} V(m) + \zeta(t)$$

441 holds for all  $t \geq t_\star$ . Let  $\lambda > 0$  be the constant defined as is (2.7). We then have the following  
 442 result, where we can use the inequality (A.1.2) to easily deduce an ISS inequality:

443 **Lemma A.1.1.** *Consider the function  $V$  introduced in Section 2 above. The inequality*

$$444 \quad (\text{A.1.2}) \quad V(t) \leq \sup_{m \in [s-\mathcal{L}_0, s]} V(m)e^{-\lambda(t-s)} + \int_s^t e^{b(m-t)} \zeta(m) dm$$

445 *holds for all  $t \geq s$  and all  $s \geq t_\star$ .*

446 **Proof.** Let  $s \geq t_\star$  be a constant. Let us introduce the function

$$447 \quad (\text{A.1.3}) \quad \theta(t) = \int_s^{\max\{t, s\}} e^{b(m-t)} \zeta(m) dm.$$

448 Notice that  $\dot{\theta}(t) = -b\theta(t) + \zeta(t)$  for all  $t > s$ . Using the nondecreasing property of  $\zeta$  to get

$$449 \quad (\text{A.1.4}) \quad \theta(t) \leq \int_s^t e^{b(m-t)} dm \zeta(t) \leq \frac{1}{b} \zeta(t)$$

450 for all  $t \geq s$ , it follows that  $\theta$  is nondecreasing over  $[s, +\infty)$ . Moreover,  $\theta(t) = 0$  for all  
 451  $t \in [s - \mathcal{L}_0, s]$ . Hence,

$$452 \quad (\text{A.1.5}) \quad \dot{\theta}(t) = -(a+b)\theta(t) + a \sup_{m \in [t-\mathcal{L}_0, t]} \theta(m) + \zeta(t)$$

453 holds for all  $t > s$ . We next choose

$$454 \quad (\text{A.1.6}) \quad v_l = \sup_{m \in [s - \mathcal{L}_0, s]} V(m) \text{ and } \chi_\epsilon(t) = \theta(t) + v_l e^{-\lambda(t-s)} + \epsilon,$$

455 where  $\epsilon > 0$  is a constant. Then

$$456 \quad (\text{A.1.7}) \quad \chi_\epsilon(t) > \theta(t) + v_l e^{-\lambda(t-s)} \geq v_l = \sup_{m \in [s - \mathcal{L}_0, s]} V(m)$$

457 holds for all  $t \in [s - \mathcal{L}_0, s]$ . Therefore,

$$458 \quad (\text{A.1.8}) \quad \chi_\epsilon(t) > V(t) \text{ for all } t \in [s - \mathcal{L}_0, s].$$

459 We next prove that

$$460 \quad (\text{A.1.9}) \quad \chi_\epsilon(t) > V(t)$$

461 holds for all  $t \geq s - \mathcal{L}_0$ . Let us start to prove this by observing that

$$462 \quad (\text{A.1.10}) \quad \dot{\chi}_\epsilon(t) = -(a+b)\theta(t) + a \sup_{m \in [t - \mathcal{L}_0, t]} \theta(m) + \zeta(t) - \lambda v_l e^{-\lambda(t-s)}$$

463 for all  $t > s$ . From (2.7), we deduce that

$$\begin{aligned} \dot{\chi}_\epsilon(t) &= -(a+b)\theta(t) + a \sup_{m \in [t - \mathcal{L}_0, t]} \theta(m) + \zeta(t) \\ &\quad - (a+b - a e^{\lambda \mathcal{L}_0}) v_l e^{-\lambda(t-s)} \\ 464 \quad (\text{A.1.11}) \quad &= -(a+b) (\theta(t) + v_l e^{-\lambda(t-s)}) + a \left( \sup_{m \in [t - \mathcal{L}_0, t]} \theta(m) + v_l e^{-\lambda(t-s-\mathcal{L}_0)} \right) \\ &\quad + \zeta(t). \end{aligned}$$

465 Now, observe that

$$466 \quad (\text{A.1.12}) \quad \sup_{m \in [t - \mathcal{L}_0, t]} (\theta(m) + v_l e^{-\lambda(m-s)}) \leq \sup_{m \in [t - \mathcal{L}_0, t]} \theta(m) + v_l e^{-\lambda(t-s-\mathcal{L}_0)}.$$

467 As an immediate consequence,

$$468 \quad (\text{A.1.13}) \quad \dot{\chi}_\epsilon(t) \geq -(a+b) (\theta(t) + v_l e^{-\lambda(t-s)}) + a \sup_{m \in [t - \mathcal{L}_0, t]} (\theta(m) + v_l e^{-\lambda(m-s)}) + \zeta(t)$$

469 for all  $t > s$ . Using the definition of  $\chi_\epsilon$ , we obtain

$$\begin{aligned} \dot{\chi}_\epsilon(t) &\geq -(a+b) (\chi_\epsilon(t) - \epsilon) + a \sup_{m \in [t - \mathcal{L}_0, t]} (\chi_\epsilon(m) - \epsilon) + \zeta(t) \\ 470 \quad (\text{A.1.14}) \quad &= -(a+b) \chi_\epsilon(t) + a \sup_{m \in [t - \mathcal{L}_0, t]} \chi_\epsilon(m) + \zeta(t) + b\epsilon. \end{aligned}$$

471 We next proceed by contradiction. Bearing in mind (A.1.8), suppose that there were a  
472  $t_c > s$  such that  $\chi_\epsilon(t) > V(t)$  for all  $t \in [s - \mathcal{L}_0, t_c)$  and  $\chi_\epsilon(t_c) = V(t_c)$ . Then (A.1.14) gives

$$473 \quad (\text{A.1.15}) \quad \dot{\chi}_\epsilon(t_c) \geq -(a+b)V(t_c) + a \sup_{m \in [t_c - \mathcal{L}_0, t_c]} \chi_\epsilon(m) + \zeta(t_c) + b\epsilon.$$

474 On the other hand (A.1.1) gives

$$475 \quad (\text{A.1.16}) \quad -\dot{V}(t_c) \geq (a+b)V(t_c) - a \sup_{m \in [t_c - \mathcal{L}_0, t_c]} V(m) - \zeta(t_c).$$

476 By adding (A.1.15) and (A.1.16), we obtain

$$477 \quad (\text{A.1.17}) \quad \dot{\chi}_\epsilon(t_c) - \dot{V}(t_c) \geq a \left[ \sup_{m \in [t_c - \mathcal{L}_0, t_c]} \chi_\epsilon(m) - \sup_{m \in [t_c - \mathcal{L}_0, t_c]} V(m) \right] + b\epsilon.$$

The definition of  $t_c$  ensures that  $\sup_{m \in [t_c - \mathcal{L}_0, t_c]} \chi_\epsilon(m) \geq V(t)$  for all  $t \in [t_c - \mathcal{L}_0, t_c]$ . It follows that

$$\sup_{m \in [t_c - \mathcal{L}_0, t_c]} \chi_\epsilon(m) - \sup_{m \in [t_c - \mathcal{L}_0, t_c]} V(m) \geq 0.$$

478 We deduce from (A.1.17) that  $\dot{\chi}_\epsilon(t_c) - \dot{V}(t_c) > 0$ . Since  $\chi_\epsilon(t_c) - V(t_c) = 0$ , we deduce  
 479 that there is  $t_d \in (s, t_c)$  such that  $\chi_\epsilon(t_d) - V(t_d) < 0$ . This contradicts the definition of  
 480  $t_c$ . Hence,  $\chi_\epsilon(t) > V(t)$  holds for all  $t \geq s - \mathcal{L}_0$ . Since  $\epsilon > 0$  is arbitrary, we deduce that  
 481  $\theta(t) + v_l e^{-\lambda(t-s)} \geq V(t)$  for all  $t \geq s - \mathcal{L}_0$ . Therefore, the conclusion of the lemma follows.

482 **A.2. Technical result.** We prove the key lemma that we used in the proof of Lemma 3.1.  
 483 We use constants  $T > 0$ ,  $q > 0$ ,  $\Psi_0 > 0$ ,  $\omega > 0$ ,  $\tau > 0$ ,  $\bar{\Delta} \geq 0$  and  $t_a \geq 0$  and a continuous,  
 484 nondecreasing function  $\Psi : [0, +\infty) \rightarrow [0, +\infty)$  such that

$$485 \quad (\text{A.2.1}) \quad \Psi(\ell) \leq \Psi_0 \ell$$

486 for all  $\ell \in [0, \omega]$ . Let  $Z : [t_a - T, t_a + \tau) \rightarrow [0, +\infty)$  be a nonnegative valued function of class  
 487  $C^1$  such that

$$488 \quad (\text{A.2.2}) \quad \dot{Z}(t) \leq \Psi \left( \sup_{\ell \in [t-T, t]} Z(\ell) \right) + \bar{\Delta}$$

489 for all  $t \in [t_a, t_a + \tau)$ . We use the following assumption:

490 **Assumption A.1.** *The inequality*

$$491 \quad (\text{A.2.3}) \quad (e^{\Psi_0 q} - 1) \frac{\bar{\Delta}}{\Psi_0} < \omega$$

492 *is satisfied.*

493 In the following lemma, the existence of values  $\bar{Z} > 0$  such that (A.2.5) is satisfied follows  
 494 from (A.2.3):

495 **Lemma A.2.1.** *Let Assumption A.1 be satisfied. Let  $Z$  be such that*

$$496 \quad (\text{A.2.4}) \quad Z(\ell) \leq \bar{Z} \text{ for all } \ell \in [t_a - T, t_a]$$

497 *where  $\bar{Z} \in \mathbb{R}$  is such that*

$$498 \quad (\text{A.2.5}) \quad \bar{Z} e^{\Psi_0 q} + (e^{\Psi_0 q} - 1) \frac{\bar{\Delta}}{\Psi_0} < \omega.$$

499 *Then*

$$500 \quad (\text{A.2.6}) \quad Z(t) \leq \bar{Z} e^{\Psi_0 q} + (e^{\Psi_0 q} - 1) \frac{\bar{\Delta}}{\Psi_0}$$

501 *holds for all  $t \in [t_a - T, t_a + \min\{\tau, q\})$ .*

502 *Proof.* We first prove that  $Z(t) < \omega$  for all  $t \in [t_a - T, t_a + \min\{\tau, q\})$ . To prove this,  
 503 we proceed by contradiction. Note that (A.2.4)-(A.2.5) imply that  $Z(t) < \omega$  for all  $t \in$   
 504  $[t_a - T, t_a]$ , and let us suppose that there were a  $t_c \in [0, \min\{\tau, q\})$  such that  $Z(t) < \omega$  for all  
 505  $t \in [t_a - T, t_a + t_c)$  and  $Z(t_a + t_c) = \omega$ . Then

$$506 \quad (\text{A.2.7}) \quad \dot{Z}(t) \leq \Psi_0 \sup_{\ell \in [t-T, t]} Z(\ell) + \bar{\Delta}$$

507 for all  $t \in [t_a, t_a + t_c]$ , by the bounds (A.2.1) and (A.2.2). Now, let us introduce the function

$$508 \quad (\text{A.2.8}) \quad \xi_\epsilon(t) = (\bar{Z} + \epsilon)e^{(\Psi_0 + \epsilon)(t-t_a)} + [e^{(\Psi_0 + \epsilon)(t-t_a)} - 1] \frac{\bar{\Delta}}{\Psi_0 + \epsilon}$$

509 with  $\epsilon > 0$ . Let us observe that  $Z(t_a) \leq \bar{Z} < \bar{Z} + \epsilon \leq \xi_\epsilon(t_a)$ . We next show that  $Z(t) < \xi_\epsilon(t)$   
 510 for all  $t \in [t_a, t_a + t_c]$ . We argue by contradiction. Suppose there were a  $t_f \in [t_a, t_a + t_c]$  such  
 511 that

$$512 \quad (\text{A.2.9}) \quad Z(t) < \xi_\epsilon(t) \text{ for all } t \in [t_a, t_f) \text{ and } Z(t_f) = \xi_\epsilon(t_f).$$

513 Simple calculations based on the formula (A.2.8) then give

$$\begin{aligned} \dot{\xi}_\epsilon(t) &= (\Psi_0 + \epsilon)(\bar{Z} + \epsilon)e^{(\Psi_0 + \epsilon)(t-t_a)} + e^{(\Psi_0 + \epsilon)(t-t_a)} \bar{\Delta} \\ 514 \quad (\text{A.2.10}) \quad &= (\Psi_0 + \epsilon) \left[ \xi_\epsilon(t) + (-e^{(\Psi_0 + \epsilon)(t-t_a)} + 1) \frac{\bar{\Delta}}{\Psi_0 + \epsilon} \right] + e^{(\Psi_0 + \epsilon)(t-t_a)} \bar{\Delta} \\ &= (\Psi_0 + \epsilon)\xi_\epsilon(t) + \bar{\Delta} \end{aligned}$$

515 for all  $t > t_a$ . On the other hand, (A.2.7) and (A.2.9) give  $\dot{Z}(t_f) \leq \Psi_0 \xi_\epsilon(t_f) + \bar{\Delta}$ , by our  
 516 choice of  $\xi_\epsilon$ . Consequently, (A.2.10) gives

$$517 \quad (\text{A.2.11}) \quad \dot{Z}(t_f) < \dot{\xi}_\epsilon(t_f),$$

518 which we can combine with (A.2.9) to deduce that there is a  $t_g \in [t_a, t_f)$  such that

$$519 \quad (\text{A.2.12}) \quad \xi_\epsilon(t_g) < Z(t_g).$$

520 This yields a contradiction with the definition of  $t_f$ . We deduce that

$$521 \quad (\text{A.2.13}) \quad Z(t) < \xi_\epsilon(t) \text{ for all } t \in [t_a, t_a + t_c].$$

522 Since  $\epsilon$  is an arbitrary positive number, we deduce that

$$523 \quad (\text{A.2.14}) \quad Z(t) \leq \bar{Z}e^{\Psi_0(t-t_a)} + (e^{\Psi_0(t-t_a)} - 1) \frac{\bar{\Delta}}{\Psi_0} \text{ for all } t \in [t_a, t_a + t_c].$$

524 Since  $t_c < q$ , it follows that

$$525 \quad (\text{A.2.15}) \quad Z(t_a + t_c) \leq \bar{Z}e^{\Psi_0 q} + (e^{\Psi_0 q} - 1) \frac{\bar{\Delta}}{\Psi_0}.$$

526 Since  $Z(t_a + t_c) = \omega$ , we obtain

$$527 \quad (\text{A.2.16}) \quad \omega \leq \bar{Z}e^{\Psi_0 q} + (e^{\Psi_0 q} - 1) \frac{\bar{\Delta}}{\Psi_0}.$$

528 This contradicts (A.2.5). Hence,

$$529 \quad (\text{A.2.17}) \quad Z(t) < \omega \text{ for all } t \in [t_a - T, t_a + \min\{\tau, q\}).$$

530 It follows from (A.2.1) that

$$531 \quad (\text{A.2.18}) \quad \dot{Z}(t) \leq \Psi_0 \sup_{\ell \in [t-T, t]} Z(\ell) + \bar{\Delta}$$

532 for all  $t \in [t_a, t_a + \min\{\tau, q\})$ . Arguing as in (A.2.7)-(A.2.13) except with  $t_c$  replaced by  
533  $\min\{\tau, q\}$ , we obtain  $Z(t) \leq \xi_\epsilon(t)$  for all  $t \in [t_a, t_a + \min\{\tau, q\})$  which implies that

$$534 \quad (\text{A.2.19}) \quad Z(t) \leq \bar{Z}e^{\Psi_0(t-t_a)} + (e^{\Psi_0(t-t_a)} - 1) \frac{\bar{\Delta}}{\Psi_0}$$

535 for all  $t \in [t_a, t_a + \min\{\tau, q\})$ . Since

$$536 \quad (\text{A.2.20}) \quad Z(t) \leq \bar{Z} \leq \bar{Z}e^{\Psi_0 q} + (e^{\Psi_0 q} - 1) \frac{\bar{\Delta}}{\Psi_0}$$

537 for all  $t \in [t_a - T, t_a]$ , we can conclude. ■

538

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