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# Stability analysis for linear systems with a switched rapidly varying delay

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## Abstract

Linear continuous-time systems with switched pointwise delays are studied. A technique enabling to establish the stability of these systems with a rapidly varying periodic delay is proposed. It relies on two ingredients: a representation of the systems as time-varying systems with constant delays and an averaging approach. Illustrative examples show the effectiveness of the proposed methodology.

*Key words:* Stability, time-varying, switch, delay, averaging

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## 1 Introduction

One of the simplest way to represent time heterogeneity in dynamical systems is by using delays, that can be constant or time-varying, distributed over a bounded or unbounded time-interval. Many continuous-time systems admit *time-varying delays*. They may appear in generic networks when the geometry of the physical channel changes (traffic flows scenarii) or in haptic and teleoperation systems, where the time variation of the communication delay may affect the performances of the overall system. Such arguments motivated the analysis or the control of these systems in many contributions including for instance [3], [15], [17], [18] and [21]. On the other hand, it is well-known that the study of these systems is difficult, except when the delay can be neglected or admits variations that can be neglected. To understand one of the reasons of this difficulty, consider the linear system

$$\dot{x}(t) = Ax(t) + Bx\left(t - h\left(\frac{t}{\epsilon}\right)\right), \quad (1)$$

where  $\epsilon > 0$  and  $h$  is a nonnegative periodic function with  $h_{av}$  as average value. It turns out that the exponential stability of the system

$$\dot{x}_{av}(t) = Ax_{av}(t) + Bx_{av}(t - h_{av}) \quad (2)$$

does not imply that the system (1) is exponentially stable for sufficiently small values of  $\epsilon > 0$ . This fact is established in the paper [12]. Moreover, another reason can be found in [11] where it is shown that a system

$$\dot{x}(t) = Ax(t) + Bx(t - h(t)) \quad (3)$$

with a delay  $h$  such that  $|h|_{\infty} \leq \bar{h}$  may admit unbounded solutions even when

$$\dot{x}(t) = Ax(t) + Bx(t - h_c) \quad (4)$$

is exponentially stable for all constant  $h_c \in [0, \bar{h}]$ .

These remarks prompt us to ask a first question: is it impossible to take advantage of the smallness of  $\epsilon$  to establish in some cases the asymptotic stability of the system (1) ? This question can be more precisely formulated as follows: can we find a method enabling to prove, under appropriate assumptions, that the system (1) is exponentially stable when the constant  $\epsilon$  is small ?

A first answer to this question is given in the contribution [15]. In it, the exponential stability of the system (1) with small constant  $\epsilon$  is studied via a comparison system which includes a term with a distributed delay. This pioneering result presents two limitations: determining the comparison system is not always simple and the delay of the studied system cannot be discontinuous.

These remarks motivate the present paper. We will give an answer to the above question that is significantly different from the one of [15]. We will prove a stability result for a particular family of systems with discontinuous delays: the systems with switches. These systems are fundamental because they are encountered in many applications [4], [20] and because periodic time-varying systems can be approximated by systems with switches, as explained in [12]. This topic will be the subject a future work: its purpose will be to study time-varying systems via an approximation by switched systems.

The main result of the present paper is the following. For a dynamical system with switches, we adopt a representation of the type of the one in [12, Section IV]. The new representation is a time-varying system with several constant pointwise delays. Then an averaging approach is applied. It is based on a time-invariant comparison system with constant pointwise delays whose stability is assumed.

A striking message of the result we establish is that in general the behavior of the solutions of a linear system with a single rapidly varying switched pointwise delay is close to the behavior of the solutions of a linear time-invariant system with several constant pointwise delays.

The averaging technique we use is reminiscent of the one of the recent contributions [9] and [10], which are devoted to the stability analysis of time-varying systems with a single pointwise constant delay. However, the proposed technique is different in the sense that we take advantage of the fact that we handle piecewise constant functions to obtain simpler calculations and stability conditions. Moreover, our technique is completely different from the time delay technique presented in [7] and in other papers by the second author, which applies only when a condition on the size of the delay of the studied system is satisfied. Besides, it is worth observing that we obtain an explicit smallness condition on the parameter which determines how fast the system is varying. We point out also that the main result of the paper [12] is a control design for systems with a delay in the input and no averaging was used in [12] to prove stability or stabilization of the studied of systems.

Let us observe that the proof of exponential stability we will give relies on the vector version of the so-called trajectory-based approach developed for instance in [1] and [13]. Moreover, it is worth mentioning that, contrary to many works devoted to switched systems, ours does not use any dwell-time condition. Finally, we observe that the result we obtain is significantly different from the one of [15] which uses a comparison system with a distributed delay, imposes a constraint on the size of the variations of  $h$  over  $h_{av}$  and does not seem to extend to time-varying systems.

The paper is organized as follows. The main result is stated and proved in Section 2. Illustrative examples are given in Section 3. Concluding remarks are drawn in Section 4 and end the paper. An appendix completes the presentation.

We use standard notation, which is simplified when no confusion would arise from the context, and where the dimensions of our Euclidean spaces are arbitrary unless otherwise noted. The standard Euclidean 2-norm, and the induced matrix norm, are denoted by  $|\cdot|$ . For vectors  $D = [d_i]$  and  $E = [e_i]$  of the same size, we write  $D \leq E$  provided  $d_i \leq e_i$  for all  $i$ . For any constant  $\mathcal{T} > 0$ ,  $C_{\text{in}}$  is the set of all continuous functions  $\phi : [-\mathcal{T}, 0] \rightarrow \mathbb{R}^a$ . We define  $\Xi_t \in C_{\text{in}}$  by

$$\Xi_t(m) = \Xi(t + m) \quad (5)$$

for all choices of  $\Xi$ ,  $m \leq 0$ , and  $t \geq 0$  such that  $t + m$  is in the domain of  $\Xi$ .

## 2 Main result

This part is devoted to the stability analysis of linear switched systems with a pointwise delay.

### 2.1 Studied system

Let  $r > 0$  be a real number and  $\ell$  be a positive integer.

Let  $s_i$  be  $\ell + 1$  real numbers such that

$$0 = s_0 < \dots < s_\ell = r. \quad (6)$$

Let  $\sigma : [0, +\infty) \rightarrow \{0, \dots, \ell - 1\}$  be the periodic function of period  $r$  defined by:  $\sigma(m) = j$  when  $m \in [s_j, s_{j+1})$  for all  $j \in \{0, \dots, \ell - 1\}$ .

Let  $h_0 > 0, \dots, h_{\ell-1} > 0$  be constants and  $A_0, \dots, A_{\ell-1}, B_0, \dots, B_{\ell-1}$  be matrices in  $\mathbb{R}^{n \times n}$ . Let

$$\bar{h} = \max\{h_0, \dots, h_{\ell-1}\}. \quad (7)$$

Now, consider the system

$$\dot{x}(t) = A_{\sigma(\frac{t}{\epsilon})}x(t) + B_{\sigma(\frac{t}{\epsilon})}x\left(t - h_{\sigma(\frac{t}{\epsilon})}\right) + \delta(t) \quad (8)$$

with  $x$  valued in  $\mathbb{R}^n$  and where  $\delta : [0, +\infty) \rightarrow \mathbb{R}^n$  is a piecewise-continuous disturbance.

**Remark.** Since the matrices  $A_{\sigma(\frac{t}{\epsilon})}$  and  $B_{\sigma(\frac{t}{\epsilon})}$  are time-varying and since the delay  $h$  is discontinuous, the main result of the paper [15] cannot be used to analyze the stability of the system (8).

To ease the stability analysis of the system (8), we propose a representation for it which is reminiscent of the one used in [12].

Let us introduce the following functions:

$$\varphi_j(m) = \begin{cases} 1 & \text{when } m \in [j, j + 1) \\ 0 & \text{when } m \notin [j, j + 1) \end{cases} \quad (9)$$

for  $j = 0$  to  $\ell - 1$ .

From the definition of the functions  $\varphi_i$  and  $\sigma$ , we deduce that

$$\sum_{i=0}^{\ell-1} \varphi_i\left(\sigma\left(\frac{t}{\epsilon}\right)\right) = 1 \quad (10)$$

for all  $t \in \mathbb{R}$ . As an immediate consequence, the system (8) can be rewritten as

$$\begin{aligned} \dot{x}(t) = & \sum_{i=0}^{\ell-1} \varphi_i\left(\sigma\left(\frac{t}{\epsilon}\right)\right) A_{\sigma(\frac{t}{\epsilon})}x(t) \\ & + \sum_{i=0}^{\ell-1} \varphi_i\left(\sigma\left(\frac{t}{\epsilon}\right)\right) B_{\sigma(\frac{t}{\epsilon})}x\left(t - h_{\sigma(\frac{t}{\epsilon})}\right) \\ & + \delta(t). \end{aligned} \quad (11)$$

Now, one can easily prove that

$$\varphi_i\left(\sigma\left(\frac{t}{\epsilon}\right)\right) A_{\sigma(\frac{t}{\epsilon})} = \varphi_i\left(\sigma\left(\frac{t}{\epsilon}\right)\right) A_i \quad (12)$$

and

$$\varphi_i \left( \sigma \left( \frac{t}{\epsilon} \right) \right) B_{\sigma \left( \frac{t}{\epsilon} \right)} x \left( t - h_{\sigma \left( \frac{t}{\epsilon} \right)} \right) = \varphi_i \left( \sigma \left( \frac{t}{\epsilon} \right) \right) B_i x \left( t - h_i \right) \quad (13)$$

for all  $i \in \{0, \dots, \ell - 1\}$  and  $t \in \mathbb{R}$ .

We deduce that the system (8) can be rewritten as

$$\begin{aligned} \dot{x}(t) &= \sum_{i=0}^{\ell-1} \varphi_i \left( \sigma \left( \frac{t}{\epsilon} \right) \right) A_i x(t) \\ &\quad + \sum_{i=0}^{\ell-1} \varphi_i \left( \sigma \left( \frac{t}{\epsilon} \right) \right) B_i x(t - h_i) + \delta(t). \end{aligned} \quad (14)$$

A striking feature of this system is that it is a time-varying system with constant pointwise delays. As a consequence, when  $\epsilon$  is a small constant, we can study its stability via an averaging approach.

## 2.2 Stability result

We will show that the average value of the function  $\varphi_i \left( \sigma \left( \frac{t}{\epsilon} \right) \right)$  is  $\frac{s_{i+1} - s_i}{r}$ . This prompts us to assume that the system

$$\dot{z}(t) = \sum_{i=0}^{\ell-1} \frac{s_{i+1} - s_i}{r} A_i z(t) + \sum_{i=0}^{\ell-1} \frac{s_{i+1} - s_i}{r} B_i z(t - h_i) \quad (15)$$

is exponentially stable. As a matter of fact, we introduce the following more convenient assumption:

**Assumption 1.** *The system*

$$\begin{aligned} \dot{z}(t) &= \sum_{i=0}^{\ell-1} \frac{s_{i+1} - s_i}{r} A_i z(t) \\ &\quad + \sum_{i=0}^{\ell-1} \frac{s_{i+1} - s_i}{r} B_i z(t - h_i) + \gamma(t), \end{aligned} \quad (16)$$

where  $z$  is valued in  $\mathbb{R}^n$  and  $\gamma : [0, +\infty) \rightarrow \mathbb{R}^n$  is a piecewise continuous function is such that there are constants  $c_i > 0$  such that

$$\begin{aligned} |z(t)| &\leq c_1 e^{-c_2(t-s)} \sup_{m \in [s-2\bar{h}, s]} |z(m)| \\ &\quad + c_3 \sup_{m \in [s-\bar{h}, t]} |\gamma(m)| \end{aligned} \quad (17)$$

for all  $t \geq s \geq \bar{h}$ .

**Remark.**

1. One can prove that Assumption 1 is satisfied by proving that the system (15) is exponentially stable. Indeed, in this case, there is a Lyapunov function as described in [8, Proposition 7.4] from which one can determined through lengthy, but simple, calculations the constants  $c_i$ . They can also be determined through a frequency domain approach. For instance, one can use the  $H_\infty$ -norm evaluation for  $c_3$  for an appropriate transfer matrix (see, e.g., the first illustrative example), a rough evaluation of the spectral abscissa for

$c_2$ , and the asymptotic expansion of the solution for  $c_1$ . For a deeper discussion on such evaluations, we refer to [14].

2. In the ISS (Input-to-State-Stability) inequality (17), we could have replaced  $\sup_{m \in [s-2\bar{h}, s]} |z(m)|$  and  $\sup_{m \in [s-\bar{h}, t]} |\gamma(m)|$  by  $\sup_{m \in [s-\bar{h}, s]} |z(m)|$  and  $\sup_{m \in [s, t]} |\gamma(m)|$ . However, we have selected  $\sup_{m \in [s-2\bar{h}, s]} |z(m)|$  and  $\sup_{m \in [s-\bar{h}, t]} |\gamma(m)|$  because this is more convenient in some cases. Recall that the celebrated notion of ISS has been studied and used in many contributions since its introduction by E. Sontag in [19].

We are ready to state and prove the following result:

**Theorem 1** *Let the system (8) satisfy Assumption 1. Then this system is Input-to-State-Stable with respect to  $\delta$  when*

$$\epsilon \in (0, \epsilon_\#) \quad (18)$$

with

$$\epsilon_\# = \frac{1}{\frac{r}{2} \sum_{i=0}^{\ell-1} (|A_i| + |B_i|) + c_3 r \ell \bar{s}^2} \quad (19)$$

and

$$\bar{s} = \max\{|A_0| + |B_0|, \dots, |A_{\ell-1}| + |B_{\ell-1}|\}. \quad (20)$$

**Remark.** We can easily extend Theorem 1 to systems with several switched delays.

**Proof.** Let us introduce the functions:  $\omega_i : \mathbb{R} \rightarrow (-\infty, 0]$ :

$$\omega_i(t) = -\frac{1}{\epsilon r} \int_t^{t+\epsilon r} (t + \epsilon r - s) \varphi_i \left( \sigma \left( \frac{s}{\epsilon} \right) \right) ds \quad (21)$$

for all  $i \in \{0, \dots, \ell - 1\}$ . One can check readily that

$$\begin{aligned} \dot{\omega}_i(t) &= -\frac{1}{\epsilon r} \int_t^{t+\epsilon r} \varphi_i \left( \sigma \left( \frac{s}{\epsilon} \right) \right) ds + \varphi_i \left( \sigma \left( \frac{t}{\epsilon} \right) \right) \\ &= -\varphi_{i,av} + \varphi_i \left( \sigma \left( \frac{t}{\epsilon} \right) \right) \end{aligned} \quad (22)$$

where

$$\varphi_{i,av} = \frac{1}{r} \int_0^r \varphi_i(\sigma(m)) dm. \quad (23)$$

Now, let us recall that  $\sigma(m) = j$  when  $m \in [s_j, s_{j+1})$  and  $\sigma(m) \notin [j, j + 1)$  when  $m \in [0, s_j) \cup [s_{j+1}, r)$ . It follows that  $\varphi_i(\sigma(m)) = 1$  when  $m \in [s_j, s_{j+1})$  and  $\varphi_i(\sigma(m)) = 0$  when  $m \in [0, s_j) \cup (s_{j+1}, r)$ . Consequently

$$\varphi_{i,av} = \frac{1}{r} \int_{s_i}^{s_{j+1}} \varphi_i(j) dm \quad (24)$$

which implies that

$$\varphi_{i,av} = \frac{s_{j+1} - s_i}{r}. \quad (25)$$

Moreover,

$$|\omega_i(t)| \leq \frac{\epsilon r}{2} \quad (26)$$

for all  $t \in \mathbb{R}$ .

Now, to apply an averaging technique, we observe that (22) implies that the system (14) can be rewritten as:

$$\begin{aligned} \dot{x}(t) = & \sum_{i=0}^{\ell-1} (\dot{\omega}_i(t) + \varphi_{i,av}) A_i x(t) \\ & + \sum_{i=0}^{\ell-1} (\dot{\omega}_i(t) + \varphi_{i,av}) B_i x(t - h_i) + \delta(t). \end{aligned} \quad (27)$$

The next step of the analysis consists in introducing the operator:

$$a(t) = \left( I - \sum_{i=0}^{\ell-1} \omega_i(t) A_i \right) x(t) - \sum_{i=0}^{\ell-1} \omega_i(t) B_i x(t - h_i), \quad (28)$$

where  $I$  is the identity matrix. This operator is defined for  $t \geq 0$  and through simple calculations, we obtain

$$\begin{aligned} \dot{a}(t) = & \left( I - \sum_{i=0}^{\ell-1} \omega_i(t) A_i \right) \dot{x}(t) \\ & - \sum_{i=0}^{\ell-1} \omega_i(t) B_i \dot{x}(t - h_i) \\ & - \sum_{i=0}^{\ell-1} \dot{\omega}_i(t) A_i x(t) - \sum_{i=0}^{\ell-1} \dot{\omega}_i(t) B_i x(t - h_i). \end{aligned} \quad (29)$$

Using (27), we obtain

$$\begin{aligned} \dot{a}(t) = & \sum_{i=0}^{\ell-1} \varphi_{i,av} A_i x(t) + \sum_{i=0}^{\ell-1} \varphi_{i,av} B_i x(t - h_i) \\ & - \sum_{i=0}^{\ell-1} \omega_i(t) A_i \dot{x}(t) - \sum_{i=0}^{\ell-1} \omega_i(t) B_i \dot{x}(t - h_i) \\ & + \delta(t). \end{aligned} \quad (30)$$

Using the representation (8), we obtain

$$\begin{aligned} \dot{a}(t) = & \sum_{i=0}^{\ell-1} \varphi_{i,av} A_i x(t) + \sum_{i=0}^{\ell-1} \varphi_{i,av} B_i x(t - h_i) \\ & - \sum_{i=0}^{\ell-1} \omega_i(t) A_i \\ & \times \left[ A_{\sigma(\frac{t}{\epsilon})} x(t) + B_{\sigma(\frac{t}{\epsilon})} x(t - h_{\sigma(\frac{t}{\epsilon})}) + \delta(t) \right] \\ & - \sum_{i=0}^{\ell-1} \omega_i(t) B_i \xi_{i,1}(t, x_t) + \delta(t) \end{aligned} \quad (31)$$

with

$$\begin{aligned} \xi_{i,1}(t, x_t) = & A_{\sigma(\frac{t-h_i}{\epsilon})} x(t - h_i) \\ & + B_{\sigma(\frac{t-h_i}{\epsilon})} x\left(t - h_i - h_{\sigma(\frac{t-h_i}{\epsilon})}\right) \\ & + \delta(t - h_i). \end{aligned} \quad (32)$$

Using the equality

$$x(t) = a(t) + \sum_{k=0}^{\ell-1} \omega_k(t) A_k x(t) + \sum_{k=0}^{\ell-1} \omega_k(t) B_k x(t - h_k), \quad (33)$$

which is a consequence of the definition of  $a$  in (28), we obtain

$$\begin{aligned} \dot{a}(t) = & \sum_{i=0}^{\ell-1} \varphi_{i,av} A_i \\ & \times \left[ a(t) + \sum_{k=0}^{\ell-1} \omega_k(t) A_k x(t) + \sum_{k=0}^{\ell-1} \omega_k(t) B_k x(t - h_k) \right] \\ & + \sum_{i=0}^{\ell-1} \varphi_{i,av} B_i a(t - h_i) + \sum_{i=0}^{\ell-1} \varphi_{i,av} B_i \xi_{i,2}(t, x_t) \\ & - \sum_{i=0}^{\ell-1} \omega_i(t) A_i \left[ A_{\sigma(\frac{t}{\epsilon})} x(t) + B_{\sigma(\frac{t}{\epsilon})} x(t - h_{\sigma(\frac{t}{\epsilon})}) \right] \\ & - \sum_{i=0}^{\ell-1} \omega_i(t) B_i \xi_{i,1}(t, x_t) - \sum_{i=0}^{\ell-1} \omega_i(t) A_i \delta(t) + \delta(t) \end{aligned} \quad (34)$$

with

$$\begin{aligned} \xi_{i,2}(t, x_t) = & \sum_{k=0}^{\ell-1} \omega_k(t - h_i) A_k x(t - h_i) \\ & + \sum_{k=0}^{\ell-1} \omega_k(t - h_i) B_k x(t - h_i - h_k). \end{aligned} \quad (35)$$

Grouping the terms and replacing  $\varphi_{i,av}$  by  $\frac{s_{i+1} - s_i}{r}$  (see (25)) yield

$$\begin{aligned} \dot{a}(t) = & \sum_{i=0}^{\ell-1} \frac{s_{i+1} - s_i}{r} A_i a(t) \\ & + \sum_{i=0}^{\ell-1} \frac{s_{i+1} - s_i}{r} B_i a(t - h_i) + \zeta(t, x_t) + \Delta(t) \end{aligned} \quad (36)$$

with

$$\begin{aligned} \zeta(t, x_t) = & \sum_{i=0}^{\ell-1} \frac{s_{i+1} - s_i}{r} A_i \\ & \times \left[ \sum_{k=0}^{\ell-1} \omega_k(t) A_k x(t) + \sum_{k=0}^{\ell-1} \omega_k(t) B_k x(t - h_k) \right] \\ & + \sum_{i=0}^{\ell-1} \frac{s_{i+1} - s_i}{r} B_i \xi_{i,2}(t, x_t) \\ & - \sum_{i=0}^{\ell-1} \omega_i(t) A_i \left[ A_{\sigma(\frac{t}{\epsilon})} x(t) + B_{\sigma(\frac{t}{\epsilon})} x(t - h_{\sigma(\frac{t}{\epsilon})}) \right] \\ & - \sum_{i=0}^{\ell-1} \omega_i(t) B_i \xi_{i,3}(t, x_t), \end{aligned} \quad (37)$$

$$\begin{aligned} \xi_{i,3}(t, x_t) = & A_{\sigma(\frac{t-h_i}{\epsilon})} x(t - h_i) \\ & + B_{\sigma(\frac{t-h_i}{\epsilon})} x\left(t - h_i - h_{\sigma(\frac{t-h_i}{\epsilon})}\right) \end{aligned} \quad (38)$$

and

$$\Delta(t) = \delta(t) - \sum_{i=0}^{\ell-1} \omega_i(t) A_i \delta(t) - \sum_{i=0}^{\ell-1} \omega_i(t) B_i \delta(t - h_i). \quad (39)$$

From Assumption 1, we deduce that

$$|a(t)| \leq c_1 e^{-c_2(t-s)} \sup_{m \in [s-2\bar{h}, s]} |a(m)| + c_3 \sup_{m \in [s-\bar{h}, t]} |\zeta(m, x_m) + \Delta(m)| \quad (40)$$

for all  $t \geq s \geq \bar{h}$ . Let  $T > 0$  be a real number to be chosen later. Then the inequality (40) gives

$$|a(t)| \leq c_1 e^{-c_2 T} \sup_{m \in [t-T-2\bar{h}, t]} |a(m)| + c_3 \sup_{m \in [t-T-\bar{h}, t]} |\zeta(m, x_m)| + c_3 \sup_{m \in [t-T-\bar{h}, t]} |\Delta(m)| \quad (41)$$

for all  $t \geq T + \bar{h}$ . Through calculations given in Appendix A, one can prove that

$$|\zeta(t, x_t)| \leq K\epsilon \sup_{m \in [t-2\bar{h}, t]} |x(m)| \quad (42)$$

for all  $t \geq 2\bar{h}$  with  $K$  defined in (A.7) and  $\bar{s}$  defined in (20). We deduce that

$$|a(t)| \leq c_1 e^{-c_2 T} \sup_{m \in [t-T-2\bar{h}, t]} |a(m)| + c_3 \sup_{m \in [t-T-\bar{h}, t]} \left( K\epsilon \sup_{\ell \in [m-2\bar{h}, m]} |x(\ell)| \right) + c_3 \sup_{m \in [t-T-\bar{h}, t]} |\Delta(m)| \quad (43)$$

$t \geq T + 3\bar{h}$ . It follows that

$$|a(t)| \leq c_1 e^{-c_2 T} \sup_{m \in [t-T-2\bar{h}, t]} |a(m)| + c_3 K\epsilon \sup_{\ell \in [t-T-3\bar{h}, t]} |x(\ell)| + c_3 \sup_{m \in [t-T-\bar{h}, t]} |\Delta(m)|. \quad (44)$$

On the other hand, the equality (33) and the inequality (26) imply that

$$|x(t)| \leq |a(t)| + \sum_{i=0}^{\ell-1} \frac{\epsilon r}{2} |A_i| |x(t)| + \sum_{k=0}^{\ell-1} \frac{\epsilon r}{2} |B_i| |x(t-h_k)| \leq |a(t)| + \frac{\epsilon r}{2} \left( \sum_{i=0}^{\ell-1} |A_i| + \sum_{k=0}^{\ell-1} |B_i| \right) \sup_{m \in [t-\bar{h}, t]} |x(m)|. \quad (45)$$

By grouping the inequalities (44) and (45), we obtain the

vector inequality

$$\begin{pmatrix} |x(t)| \\ |a(t)| \end{pmatrix} \leq M(T, \epsilon) \begin{pmatrix} \sup_{m \in [t-T-3\bar{h}, t]} |x(m)| \\ \sup_{m \in [t-T-3\bar{h}, t]} |a(m)| \end{pmatrix} + \begin{pmatrix} c_3 \sup_{m \in [t-T-\bar{h}, t]} |\Delta(m)| \\ 0 \end{pmatrix} \quad (46)$$

with

$$M(T, \epsilon) = \begin{bmatrix} \epsilon g & 1 \\ \epsilon c_3 K & c_1 e^{-c_2 T} \end{bmatrix} \quad (47)$$

with  $g = \frac{r}{2} \sum_{i=0}^{\ell-1} (|A_i| + |B_i|)$ .

Let  $\epsilon > 0$  be such that

$$\epsilon < \epsilon_{\#} \quad (48)$$

where  $\epsilon_{\#}$  is the constant defined in (19). Then

$$\epsilon g + \sqrt{\epsilon^2 g^2 + 4\epsilon c_3 K} < 2. \quad (49)$$

We can prove that this inequality imply that the matrix

$$\begin{bmatrix} \epsilon g & 1 \\ \epsilon c_3 K & 0 \end{bmatrix} \quad (50)$$

is Schur stable by studying its eigenvalues.

Then, the continuity of the eigenvalues of a matrix with respect to the entries implies that there is a constant  $T_p > 0$  such that  $M(T_p, \epsilon)$  is Schur stable. Finally, by noticing that (26) implies that

$$|\Delta(t)| \leq \frac{\epsilon r}{2} \sum_{i=0}^{\ell-1} |A_i| |\delta(t)| + \frac{\epsilon r}{2} \sum_{i=0}^{\ell-1} |B_i| |\delta(t-h_i)| + |\delta(t)| \quad (51)$$

and by applying the trajectory based approach [1], [13], the vector inequality (46) allows us to conclude.

### 3 Illustrations

In this section, we illustrate Theorem 1 through two examples which show that Theorem 1 applies easily.

#### 3.1 First illustration

Consider the system (8) with  $n = 1$ ,  $r = 2$ :

$$A_0 = -2 \quad , \quad A_1 = 0 \quad , \quad (52)$$

$$B_0 = 0 \quad , \quad B_1 = \frac{3}{2} \quad , \quad (53)$$

$$h_0 > 0 \quad , \quad h_1 > 0 \quad , \quad (54)$$

$s_1 = 1$  and  $s_2 = 2$ .

In this particular case, the system (16) is represented by a scalar delay-differential equation of the form:

$$\dot{z}(t) = -z(t) + \frac{3}{4} z(t-h_1) + \gamma(t) \quad (55)$$

Since  $\frac{3}{4} < 1$ , this system is ISS. Indeed, it is easy to observe that in the case  $\gamma = 0$ , (55) is exponentially stable independently of the delay value  $h_1$ . Many proofs of this fact can be proposed. For instance, using a similar argument to the one proposed in [16], the system free of delay is exponentially stable and the corresponding characteristic function  $\Delta(s) = s + 1 - 3/4e^{-sh_1}$  has no roots on the imaginary axis independently of the delay  $h_1$  since by  $|1 + (j\omega)^2| > |3/4e^{-j\omega h_1}|^2$ , for all  $\omega \in \mathbb{R}$  and all  $h_1 \in \mathbb{R}_+$ . Next, by explicitly computing  $H_{z\gamma}$ , the transfer function from  $\gamma$  to  $z$ , one can estimate an upper bound for  $c_3$  and it corresponds to the  $H_\infty$ -norm of the transfer  $H_{z\gamma}$ . For instance, in the case  $h_1 = 0.1$ , one gets  $c_{3,m} \leq 4$  (see, e.g., [14,2] for further details).

Next, by choosing  $c_3 = 4$ , Theorem 1 allows concluding that the studied system is ISS with respect to  $\delta$  when

$$\epsilon < \frac{2}{153}. \quad (56)$$

### 3.2 Second illustration

Let us consider the system (8) with  $n = 2$ ,

$$A_0 = \begin{bmatrix} 0 & 1 \\ -1 & -\kappa \end{bmatrix}, \quad (57)$$

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B_0 = 0, \quad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & -\kappa \end{bmatrix}, \quad (58)$$

with  $\kappa > 0$ ,  $h_0 > 0$ ,  $h_1 = 2\pi$ ,  $s_1 = 1$  and  $s_2 = r = 2$ .

Now, notice that, in this particular case, the system (16) is given by the following delay-differential equation:

$$\dot{z}(t) = \frac{1}{2}(A_0 + A_1)z(t) + \frac{1}{2}B_1z(t - 2\pi) + \gamma(t). \quad (59)$$

It can be rewritten as

$$\dot{z}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -\frac{\kappa}{2} \end{bmatrix} z(t) + \begin{bmatrix} 0 & 0 \\ 0 & -\frac{\kappa}{2} \end{bmatrix} z(t - 2\pi) + \gamma(t) \quad (60)$$

with  $\gamma(t) = (\gamma_1(t), \gamma_2(t))^T$ . This system is globally exponentially stable when the constant  $\kappa$  is sufficiently small. Indeed, consider  $\gamma = 0$ , and consider the system

$$\dot{z}(t) = \frac{1}{2}(A_0 + A_1)z(t) + \frac{1}{2}B_1z(t - h_1), \quad (61)$$

where  $h_1$  and  $\kappa$  are considered as free parameters. Under these assumptions, the characteristic function  $\Delta : \mathbb{C} \mapsto \mathbb{C}$  of (61) is:

$$\Delta(s) = s^2 + \frac{\kappa}{2}s + 1 + \frac{\kappa}{2}se^{-sh_1}. \quad (62)$$

When  $h_1 = 0$ , the system (61) is exponentially stable if and only if  $\kappa > 0$ . By using the continuity property of the characteristic roots with respect to the system's parameters (see, e.g. [14]), the exponential stability is preserved for sufficiently small delays as long as there are no characteristic roots crossing the imaginary axis. In other words, to find critical parameters' values (in our case, gain and

delay margins), one needs to explore the existence of characteristic roots located on the imaginary axis. More precisely, in our case, simple computations show that, independently of the gain value  $\kappa > 0$ ,  $\Delta$  has a pair of roots on the imaginary axis  $s = \pm j$  and this happens if and only if  $h_1 = \pi, 3\pi, \dots, (2m+1)\pi$ , with  $m$  a non-negative integer.

At this stage, as suggested in [14] for the analysis of the spectrum location with respect to the delay parameter, one needs to explicitly compute the crossing direction at the "critical" delay values  $h_1$  given above with the observation that if the crossing is towards instability (resp. stability) then  $Re \left\{ \frac{ds}{dh_1} \right\} \Big|_{s=j\omega} > 0$  (resp.  $< 0$ ). Using the procedure proposed in [6] (see also [5]), we observe that:

$$\begin{aligned} Re \left\{ \frac{ds}{dh_1} \right\} \Big|_{s=j\omega, h_1=\pi} &= Re \left\{ -\frac{\frac{\partial \Delta}{\partial h_1}}{\frac{\partial \Delta}{\partial s}} \right\} \Big|_{s=j\omega, h_1=\pi} \\ &= \frac{\kappa s^2}{-skh_1 + (\kappa + 4s)e^{sh_1} + \kappa} \\ &= Re \left\{ -\frac{j\kappa}{\pi\kappa + 4} \right\} = 0. \end{aligned} \quad (63)$$

Since  $\text{sgn}(Re \left\{ \frac{ds}{dh_1} \right\} \Big|_{s=j\omega}) = 0$ , we cannot conclude about the crossing direction and, thus, a second-order derivative should be explicitly computed. Simple computations lead to:

$$\begin{aligned} \frac{d^2s}{dh_1^2} &= \\ &= -\frac{\kappa s^3(\kappa^2(2s\tau - 3) - 2k(\kappa + 2s)e^{sh_1} + (\kappa + 4s)^2e^{2sh_1})}{(-skh_1 + (\kappa + 4s)e^{sh_1} + \kappa)^3}. \end{aligned} \quad (64)$$

The evaluation of this quantity at the "point" of interest (" $s = j$ ") gives us the following relation

$$\begin{aligned} Re \left\{ \frac{d^2s}{dh_1^2} \right\} \Big|_{s=j, h_1=\pi} &= Re \left\{ \frac{2s\kappa(k(\pi\kappa + 6) + 8j)}{(\pi\kappa + 4)^3} \right\} \Big|_{s=j, h_1=\pi} \\ &= -16\kappa. \end{aligned} \quad (65)$$

Since  $\kappa > 0$ , it follows that the crossing direction is towards stability. To summarize the above analysis, the corresponding system is exponentially stable for all positive delays  $h_1 \in \mathbb{R}_+ \setminus \{(2m+1)\pi, m \in \mathbb{Z}_+\}$ , and marginally stable (one simple pair of characteristic roots located on the imaginary axis, all the other roots located in  $\mathbb{C}_-$ , i.e. with negative real part) for all  $h_1 \in \{(2m+1)\pi, m \in \mathbb{Z}_+\}$ . In fact, the (rightmost) characteristic root, seen as a function depending on the delay parameter, arrives (from the left-half complex plane) by "touching" ("tangentially") the imaginary axis and "turns" back to the left. Such a behavior is depicted in Fig 1. For a different example depicting a similar property, we refer to [6]. Thus, in conclusion, *if  $h_1 = 2\pi$ , the dynamical system (60) is exponentially stable and, furthermore, this property holds for all positive gains  $\kappa > 0$ .*

Finally, by using the toolbox proposed in [2], for  $\kappa = \frac{1}{2}$  in (60), we obtain an estimation for  $c_3$ :  $c_3 = 6.3$  (corresponding frequency:  $\omega = 0.78$ ). If  $\kappa = 1$ , one gets  $c_3 = 6.65$  (for

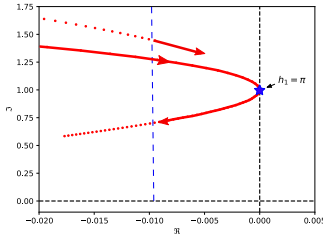


Fig. 1. Evolution of the (rightmost characteristic) roots of  $\Delta$  in (62) as the delay parameter  $h_1$  increases inside the open interval  $(0, 2\pi)$  (under the assumption that  $\kappa > 0$ ).

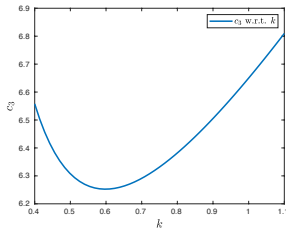


Fig. 2. Evaluation of  $c_3$  as a function of  $\kappa \in [0.4, 1.1]$  for the system (60).

a corresponding frequency  $\omega = 0.69$ ). For other values, we refer to Fig. 2 with the observation that for smaller/larger gains the value of  $c_3$  is increasing. Now, one can prove that

$$\begin{aligned} |A_0| &= \frac{2+\kappa^2+\kappa\sqrt{\kappa^2+4}}{2}, & |A_1| &= 1, \\ |B_0| &= 0, & |B_1| &= \frac{1}{2}. \end{aligned} \quad (66)$$

Let us consider the case where  $\kappa = \frac{1}{2}$ . Then Theorem 1 ensures that the system we study is ISS when

$$\epsilon < \frac{1}{103}. \quad (67)$$

**Remark.** Let us observe that in this example the average value of the delay  $h$  is  $\pi$  and therefore the average system similar to (2) is

$$\dot{x}_{av}(t) = \frac{1}{2}[A_0 + A_1]x_{av}(t) + \frac{1}{2}B_1x_{av}(t - \pi). \quad (68)$$

We have already explained why this system is not exponentially stable. This fact shows that the average value of the delay does not give any (explicit) information on the stability of the studied system.

## 4 Conclusion

We proposed a new technique of stability analysis of linear continuous-time systems with a periodic switched delay and additive disturbances.

The result we obtain can be easily extended to many families of systems. In particular, we expect to extend it to dynamical systems with time-varying delays by approximating these systems by switched systems.

Also, we conjecture that our result can be adapted to systems with time-varying distributed delays. Our technique can also be applied to control problems for systems with a rapidly time-varying delay in the input and thereby an extension of the main result of the paper [12] could be obtained.

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## A The constant $K$

Let us introduce the simplifying notation

$$\mathcal{S}(x_t) = \sup_{l \in [t-2\bar{h}, t]} |x(l)|. \quad (\text{A.1})$$

Let us observe first that (26) gives

$$|\xi_{i,2}(t, x_t)| \leq \frac{\epsilon r}{2} \sum_{k=0}^{\ell-1} (|A_k| + |B_k|) \mathcal{S}(x_t) \quad (\text{A.2})$$

and

$$|\xi_{i,3}(t, x_t)| \leq \left( |A_{\sigma(\frac{t-h_i}{\epsilon})}| + |B_{\sigma(\frac{t-h_i}{\epsilon})}| \right) \mathcal{S}(x_t), \quad (\text{A.3})$$

where  $\xi_{i,2}$  and  $\xi_{i,3}$  are the functionals defined respectively in (35) and (38). Now, consider the functional  $\zeta$  defined in (37). Using again (26), one can prove that the inequality

$$\begin{aligned} |\zeta(t, x_t)| &\leq \sum_{i=0}^{\ell-1} \frac{s_{i+1} - s_i}{r} |A_i| \\ &\times \frac{\epsilon r}{2} \sum_{k=0}^{\ell-1} (|A_k| + |B_k|) \mathcal{S}(x_t) \\ &+ \sum_{i=0}^{\ell-1} \frac{s_{i+1} - s_i}{r} |B_i| \\ &\times \frac{\epsilon r}{2} \sum_{k=0}^{\ell-1} (|A_k| + |B_k|) \mathcal{S}(x_t) \\ &+ \frac{\epsilon r}{2} \sum_{i=0}^{\ell-1} |A_i| \left[ |A_{\sigma(\frac{t}{\epsilon})}| + |B_{\sigma(\frac{t}{\epsilon})}| \right] \mathcal{S}(x_t) \\ &+ \frac{\epsilon r}{2} \sum_{i=0}^{\ell-1} |B_i| \\ &\times \left( |A_{\sigma(\frac{t-h_i}{\epsilon})}| + |B_{\sigma(\frac{t-h_i}{\epsilon})}| \right) \mathcal{S}(x_t) \end{aligned} \quad (\text{A.4})$$

is satisfied. Consequently, the inequality

$$\begin{aligned} |\zeta(t, x_t)| &\leq \frac{\epsilon}{2} \sum_{i=0}^{\ell-1} (s_{i+1} - s_i) \bar{s} \sum_{k=0}^{\ell-1} \bar{s} \mathcal{S}(x_t) \\ &+ \frac{\epsilon r}{2} \sum_{i=0}^{\ell-1} \bar{s}^2 \mathcal{S}(x_t) \end{aligned} \quad (\text{A.5})$$

where  $\bar{s}$  is the constant defined in (20), is satisfied. We deduce that

$$|\zeta(t, x_t)| \leq \epsilon r \bar{s}^2 \ell \sup_{l \in [t-2\bar{h}, t]} |x(l)|. \quad (\text{A.6})$$

We deduce that we can choose

$$K = r \bar{s}^2 \ell. \quad (\text{A.7})$$