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► **To cite this version:**

Konstantin Zimenko, Denis Efimov, Andrey Polyakov, Artem Kremlev. Finite-time stability analysis of homogeneous systems with sector nonlinearities \star . Automatica, In press. hal-04613985

HAL Id: hal-04613985

<https://inria.hal.science/hal-04613985>

Submitted on 17 Jun 2024

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Finite-time stability analysis of homogeneous systems with sector nonlinearities [★]

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Abstract

A class of homogeneous systems with sector nonlinearities is considered in the paper. Sufficient conditions of finite-time (input-to-state) stability are established with the use of new constructive modification of the Implicit Lyapunov Function approach. The proposed conditions are given in the form of linear matrix inequalities. The theoretical results are supported with numerical examples.

1 Introduction

Homogeneity is a kind of symmetry of the right-hand sides of dynamical systems implying also a similar property for their solutions, which is widely used in systems and control theory for stability and robustness analysis, nonlinear control and observer design, etc. (see, for example, [1]-[14]). In addition, homogeneous differential equations and inclusions describe a (local or approximate) behaviour of a wide class of dynamical systems, which includes some physical phenomena [2], nonholonomic systems [15], mechanical models with frictions [16], etc. The use of the homogeneity property in control and observer design is supported by a number of useful features, such as behaviour scalability (local (stability) property can be transformed to the whole state space), nonasymptotic convergence rates (finite-time or nearly fixed-time stability) and robustness (e.g., Input-to-State Stability (ISS) and delay robustness) [3], [4], [5], [6].

Stability analysis of nonlinear systems is a basic problem in systems and control theory whose solution is based on the use of Lyapunov function method, then it is hard to solve this problem constructively in a general case since there is no common approach to select a Lyapunov function [17]. In the case of homogeneous systems, in [7],

it was shown that such systems are equivalent to standard homogeneous ones via appropriate change of coordinates. Based on this equivalence it was shown that for a stable homogeneous system there exists an implicitly defined quadratic-like homogeneous Lyapunov function. In some cases, despite significant system nonlinearities, the Implicit Lyapunov Function (ILF) method with quadratic-like function leads to simple stability (stabilizability) criteria in the form of Linear Matrix Inequalities (LMIs) (see, e.g., [8], [9]). Pursuing the aim to enlarge the class of the systems to which the ILF approach can be applied providing efficient LMI-based stability (stabilizability) criteria, the present paper deals with the class of homogeneous systems with sector nonlinearities. Examples of such systems are homogeneous Persidskii systems [1], distributed control applications [18], [19], liquid tanks systems [20], etc.

In [21] stability criteria were presented for generalized Persidskii systems in the form of LMIs. However, in general case, these LMIs are infeasible for *homogeneous* Persidskii systems due to homogeneity imposes some restrictions on system matrices (e.g., linear part of the system is presented or can be reduced to the form A_0x , $x \in \mathbb{R}^n$ with a nilpotent matrix $A_0 \in \mathbb{R}^{n \times n}$), and the approach of [21] does not take into account a possible homogeneity of the system, then the used Lyapunov function cannot be homogeneous. In some cases this problem can be handled with the use of homogeneous approximations as in [10]. However, this approach does not allow to obtain settling-time estimations and does not provide quantitative robustness analysis. Therefore, in the case of homogeneous systems with sector nonlinearities, the analysis becomes more complicated. The main goal of

[★] The material in this paper was partially presented at the 60th IEEE Conference on Decision and Control [1]. This work is supported by RSF under grant 21-71-10032.

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this paper is to propose a new constructive modification of the ILF approach by formulating LMIs to check finite-time stability for homogeneous systems with sector nonlinearities; and provide both quantitative and qualitative robustness analysis. More precisely, in this work a homogeneous ILF is proposed, where a properly dilated sum of quadratic-like part and linear combination of the integrals of the nonlinearities is used. This form of Lyapunov function allows to obtain simple sufficient stability conditions in terms of LMIs. Note, that with the use of the local homogeneity concept the presented result can be developed for nonhomogeneous systems to provide local finite-time stability analysis. With respect to the preliminary results of [1], the key differences are as follows: the results are extended for generalized type of homogeneity and to a class of homogeneous systems with sector nonlinearities (in [1] the weighted homogeneity and homogeneous Persidskii systems are considered); the proofs of all claims are given; new examples are considered.

Notation: $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$; the symbol $\overline{1, m}$ is used to denote a sequence of integers $1, \dots, m$; $\lambda_i(P)$ denotes i^{th} eigenvalue of a matrix $P \in \mathbb{R}^{n \times n}$, $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote its maximum and minimum eigenvalues; $\mathbf{u}_i(A)$ denotes left eigenvector of a matrix A corresponding to i^{th} eigenvalue, i.e., $\mathbf{u}_i(A)A = \lambda_i(A)\mathbf{u}_i(A)$; the set of nonnegative diagonal matrices with dimension $n \times n$ is defined by \mathbb{D}_+^n ; $\text{diag}\{\lambda_i\}_{i=1}^n$ is a diagonal matrix with elements λ_i , $i = \overline{1, n}$; $O_{m \times n}$ denotes zero matrix with dimension $m \times n$; $e_s(i)$ is a vector of the canonical basis of \mathbb{R}^s , where i^{th} component is 1 and the rests are 0; \mathcal{L}_∞ is the set of essentially bounded measurable functions $\mathbb{R} \rightarrow \mathbb{R}^m$ with the norm $\|\cdot\|_{[0, +\infty)} = \text{ess sup}_{t \in [0, +\infty)} \|\cdot(t)\|$; symmetric elements of matrices are denoted by $*$.

2 Preliminaries

Let us consider the system

$$\dot{x} = f(x), \quad x(0) = x_0, \quad (1)$$

where $x \in \mathbb{R}^n$ is the state vector, $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a vector field, $f(0) = 0$. If f is discontinuous with respect to x , then the solutions of (1) are defined through the associated differential inclusions and need to be understood in the sense of Filippov [22]. Assume that (1) has unique solutions defined in forward time (at least locally).

2.1 Finite-time stability

Definition 1 [16], [23] *The origin of (1) is said to be globally finite-time stable if it is globally asymptotically stable and any solution $x(t, x_0)$ of (1) reaches the origin at some finite time moment, i.e. $x(t, x_0) = 0, \forall t \geq T(x_0)$ and $x(t, x_0) \neq 0, \forall t \in [0, T(x_0)), x_0 \neq 0$, where $T : \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{0\}$, $T(0) = 0$ is the settling-time function.*

The following theorem presents the ILF method [24], [25] for finite-time stability analysis.

Theorem 1 [8] *Suppose that there exists a continuous function $Q : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that*

C1) $Q(V, x)$ is continuously differentiable $\forall x \in \mathbb{R}^n \setminus \{0\}$ and $\forall V \in \mathbb{R}_+$;

C2) for any $x \in \mathbb{R}^n \setminus \{0\}$ there exist $V^- \in \mathbb{R}_+$ and $V^+ \in \mathbb{R}_+ : Q(V^-, x) < 0 < Q(V^+, x)$;

C3) for $\Omega = \{(V, x) \in \mathbb{R}^{n+1} : Q(V, x) = 0\}$ $\lim_{\substack{x \rightarrow 0 \\ (V, x) \in \Omega}} V = 0^+, \lim_{\substack{V \rightarrow 0^+ \\ (V, x) \in \Omega}} \|x\| = 0, \lim_{\substack{\|x\| \rightarrow \infty \\ (V, x) \in \Omega}} V = +\infty$;

C4) the inequality $-\infty < \frac{\partial Q(V, x)}{\partial V} < 0$ holds $\forall V \in \mathbb{R}_+$ and $\forall x \in \mathbb{R}^n \setminus \{0\}$;

C5) the inequality $\frac{\partial Q(V, x)}{\partial x} f(x) \leq \alpha V^{1+\mu} \frac{\partial Q(V, x)}{\partial V}$ holds $\forall (V, x) \in \Omega$, where $\mu < 0$ and $\alpha > 0$ are some constants. Then the origin of the system (1) is globally finite-time stable and $T(x_0) \leq -\frac{V_0^{-\mu}}{\alpha \mu}$, where $Q(V_0, x_0) = 0$.

2.2 Generalized linear homogeneity

In this paper we deal with the one-parameter group $\mathbf{d}(s) : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ of linear dilations given by $\mathbf{d}(s) = e^{G_{\mathbf{d}} s}$, where $s \in \mathbb{R}$, and $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$ is an anti-Hurwitz matrix (i.e., $-G_{\mathbf{d}}$ is Hurwitz) called the generator of the dilation.

Theorem 2 [7] *If \mathbf{d} is a dilation in \mathbb{R}^n , then there exists a symmetric matrix $0 < \bar{P} \in \mathbb{R}^{n \times n}$ such that $\bar{P}G_{\mathbf{d}} + G_{\mathbf{d}}^T \bar{P} > 0$ and*

$$\begin{aligned} e^{\varrho_1 s} &\leq \|\mathbf{d}(s)\|_{\bar{P}} \leq e^{\varrho_2 s} \quad \text{if } s \leq 0, \\ e^{\varrho_2 s} &\leq \|\mathbf{d}(s)\|_{\bar{P}} \leq e^{\varrho_1 s} \quad \text{if } s \geq 0, \end{aligned} \quad (2)$$

where $\|x\|_{\bar{P}} = \sqrt{x^T \bar{P} x}$, $\|\mathbf{d}(s)\|_{\bar{P}} = \sup_{x \neq 0} \frac{\|\mathbf{d}(s)x\|_{\bar{P}}}{\|x\|_{\bar{P}}}$ and $\varrho_1 = \frac{1}{2} \lambda_{\max} \left(\bar{P}^{\frac{1}{2}} G_{\mathbf{d}} \bar{P}^{-\frac{1}{2}} + \bar{P}^{-\frac{1}{2}} G_{\mathbf{d}}^T \bar{P}^{\frac{1}{2}} \right)$, $\varrho_2 = \frac{1}{2} \lambda_{\min} \left(\bar{P}^{\frac{1}{2}} G_{\mathbf{d}} \bar{P}^{-\frac{1}{2}} + \bar{P}^{-\frac{1}{2}} G_{\mathbf{d}}^T \bar{P}^{\frac{1}{2}} \right)$.

Definition 2 [11] *A vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ (a function $h : \mathbb{R}^n \rightarrow \mathbb{R}$) is said to be \mathbf{d} -homogeneous of degree $\mu \in \mathbb{R}$ if $f(\mathbf{d}(s)x) = e^{\mu s} \mathbf{d}(s) f(x)$, $\forall x \in \mathbb{R}^n, \forall s \in \mathbb{R}$, (resp. $h(\mathbf{d}(s)x) = e^{\mu s} h(x)$, $\forall x \in \mathbb{R}^n, \forall s \in \mathbb{R}$).*

If the matrix $G_{\mathbf{d}}$ is diagonal, then \mathbf{d} -homogeneity is reduced to the conventional weighted one [12]. The following theorem provides the relation of homogeneity property and finite-time stability:

Theorem 3 [7], [12] *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a \mathbf{d} -homogeneous vector field with degree $\mu < 0$. If the origin of (1) is locally asymptotically stable, then it is globally finite-time stable and there exists a continuously differentiable Lyapunov function V which is \mathbf{d} -homogeneous of degree $v > -\mu$.*

Under homogeneity conditions, asymptotic stability of a nonlinear system with zero disturbance implies a certain robustness in ISS sense (see, [3], [4, Theorem 4]).

3 Problem statement

Consider the following class of systems:

$$\dot{x}(t) = A_0 x(t) + \sum_{j=1}^M A_j f^j(Ux(t)) + d(t, x(t)), \quad t \geq 0, \quad (3)$$

where $x(t) = [x_1(t) \dots x_n(t)]^T \in \mathbb{R}^n$ is the state vector, $x(0) \in \mathbb{R}^n$, $f^j(Ux) = [f_1^j(e_n^T(1)Ux) \dots f_n^j(e_n^T(n)Ux)]^T$, $j = \overline{1, M}$ are continuous functions ensuring existence of solutions of the system (3) in forward time, $f^j(0) = 0$, the matrices $U, A_k \in \mathbb{R}^{n \times n}$ for $k = \overline{0, M}$, and the function $d: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ describes exogenous disturbances and uncertainties. It is assumed that the system (3) satisfies the following assumptions:

Assumption 1 For any $i = \overline{1, n}$, $j = \overline{1, M}$:

$$s f_i^j(s) > 0 \quad \forall s \in \mathbb{R} \setminus \{0\}. \quad (4)$$

Assumption 2 The system (3) with $d = 0$ is \mathbf{d} -homogeneous of degree $\mu < 0$ with the generator $G_{\mathbf{d}}$, i.e.,

$$\begin{aligned} \mathbf{d}(s) & \left(A_0 x + \sum_{j=1}^M A_j f^j(Ux) \right) \\ & = e^{-s\mu} \left(A_0 \mathbf{d}(s)x + \sum_{j=1}^M A_j f^j(U\mathbf{d}(s)x) \right), \quad \forall s \in \mathbb{R}, \quad \forall x \in \mathbb{R}^n; \end{aligned} \quad (5)$$

and there exists $\rho \in \overline{1, M}$ such that $f^j(Ux)$ are homogeneous of degree μ for $j = \overline{1, \rho}$.

Assumption 3 Each row of the matrix U is a left eigenvector of $G_{\mathbf{d}}$, i.e., $\forall i \in \overline{1, n}$, $\exists j \in \overline{1, k} : e_n^T(i)U = \mathbf{u}_j(G_{\mathbf{d}})$, where k is a number of disjoint real eigenvalues of $G_{\mathbf{d}}$ (it is assumed that $G_{\mathbf{d}}$ has at least one real eigenvalue).

In Assumption 1 and Assumption 3 it is stated that all nonlinearities f_i^j belong to a sector with respect to $\mathbf{u}_i(G_{\mathbf{d}})x$, $i \in \overline{1, k}$ and may take zero values only if $\mathbf{u}_i(G_{\mathbf{d}})x = 0$. Assumption 2 restricts (3) to the class of disturbed \mathbf{d} -homogeneous systems. The presence of homogeneous functions $f^j(Ux)$, $j = \overline{1, \rho}$ can be obtained after a proper re-indexing and decomposition of a nonlinear system dynamics, and the principal restriction of Assumption 2 is (5).

Remark 1 For $U \in \mathbb{D}_+^n$ the diagonal structure of the nonlinearities places (3) to a class of Persidskii systems. Persidskii systems were studied in the context of neural networks [26], [27], electric circuits [28], robust stability analysis [21], gradient systems convergence analysis [29], etc. In [1] LMI based sufficient stability conditions were proposed for a class of homogeneous Persidskii systems with negative degree and $G_{\mathbf{d}} \in \mathbb{D}_+^n$. Homogeneous Persidskii systems with diagonal U describe finite-time controllers (see, e.g., [30]). For $U \notin \mathbb{D}_+^n$ the system (3) has a

structure appearing, for example, in a number of homogeneity based distributed control applications (see, for example, [18], [19], [36]) and observers (e.g., [31], [32]).

Remark 2 According to Theorem 3, to establish the finite-time stability property of (3) with $d = 0$ it is enough to show asymptotic stability of the system. In the case $U \in \mathbb{D}_+^n$, at a first glance, to solve the problem we can use sufficient LMI-based conditions as in [21] to check asymptotic stability, and then the system is finite-time stable due to homogeneity. According to [21] one of the main requirements for stability checking is that there should exist matrices $0 \leq P = P^T \in \mathbb{R}^{n \times n}$ and $\Lambda^j \in \mathbb{D}_+^n$ for $j = \overline{1, M}$ to serve as solutions of the Lyapunov equation for the matrices A_0, A_j for $j = \overline{1, M}$, respectively. However, this approach has the following sticking point: for nonzero linear part of (3) the homogeneity property imposes restrictions on the system matrices (A_0 is nilpotent or A_j is not diagonally stable for some $j \in \{1, \dots, M\}$), that is difficult to fit into the requirements of [21]. Thus, such an approach is more adapted for $A_0 = 0$, that significantly reduces the class of systems under consideration. For the case $U \notin \mathbb{D}_+^n$ (particularly, when U is not invertible) the problem becomes more complicated.

This paper is aimed to derive constructive (in the LMI form) conditions of robust finite-time stability of the homogeneous system (3) with a settling-time estimate.

4 Main result

After proper re-indexing and decomposition of a nonlinear system dynamics the sector condition (4) and homogeneity property imply that:

- for any $i = \overline{1, n}$, $j = \overline{1, \zeta}$ the nonlinearities f_i^j have unbounded integrals, i.e., $\lim_{s \rightarrow \pm\infty} \int_0^s f_i^j(\sigma) d\sigma = +\infty$, where $\zeta \geq \rho$;
- due to $\mu < 0$ for any $i = \overline{1, n}$, $j = \overline{1, \rho}$:

$$s f_i^j(s) > \psi_i^j s^2 \quad \forall s \in [\underline{s}_i, \bar{s}_i] \setminus \{0\}, \quad (6)$$

where $\psi_i^j \in \mathbb{R}_+$ and $-\infty < \underline{s}_i < 0 < \bar{s}_i < +\infty$.

Define $\Psi^j = \text{diag}\{\psi_i^j\}_{i=1}^n \in \mathbb{D}_+^n$, $j = \overline{1, \rho}$ and denote $\mathbf{u}_i = e_n^T(i)U$ for compactness. Define $\mathcal{I} \in \mathbb{R}^{n \times n} : \mathcal{I}_{i,j} = \begin{cases} \text{sign}(\mathbf{u}_i \mathbf{u}_j^T), & \text{if } \mathbf{u}_j, \mathbf{u}_i \text{ are colinear} \\ 0, & \text{otherwise} \end{cases}$ for $i, j = \overline{1, n}$.

Introduce an ILF candidate in the form

$$\begin{aligned} Q(V, x) & = x^T \mathbf{d}^T(-\ln V) P \mathbf{d}(-\ln V) x \\ & + 2 \sum_{j=1}^M \sum_{i=1}^n \Lambda_i^j \int_0^{\mathbf{u}_i \mathbf{d}(-\ln V)x} f_i^j(s) ds - \chi, \end{aligned} \quad (7)$$

where $\Lambda^j = \text{diag}\{\Lambda_i^j\}_{i=1}^n \in \mathbb{D}_+^n$, $0 \leq P = P^T \in \mathbb{R}^{n \times n}$, and $\chi \in \mathbb{R}_+$. All these quantities will be selected below as a solution of LMI. Imposing $\Lambda^j = 0$, $j = \overline{1, M}$ the

equation (7) can be reduced to the ILF considered in [9] for finite-time stabilization of linear MIMO systems.

Theorem 4 *Let Assumptions 1, 2, 3 be satisfied. Let $\alpha, \iota_1, \iota_2 \in \mathbb{R}_+$ be chosen such that the LMI system*

$$P \geq 0; PG_{\mathbf{d}} + G_{\mathbf{d}}^T P \geq 0; \iota_1 P + U^T \sum_{j=1}^{\zeta} \Lambda^j U > 0; \quad (8a)$$

$$\Gamma > 0; U^T \sum_{j=1}^M \Lambda^j U + \iota_2 (PG_{\mathbf{d}} + G_{\mathbf{d}}^T P) > 0; \quad (8b)$$

$$Q_1 + \begin{bmatrix} Q_3 & \mathbf{0}_{(M+1)n \times n} \\ * & \mathbf{0}_{n \times n} \end{bmatrix} + \alpha \begin{bmatrix} Q_2 & \mathbf{0}_{(M+1)n \times n} \\ * & \mathbf{0}_{n \times n} \end{bmatrix} \leq 0, \quad (8c)$$

is feasible for some $P = P^T, \Gamma = \Gamma^T \in \mathbb{R}^{n \times n}; \Lambda^j \in \mathbb{D}_+^n, j = \overline{1, M}, 0 \leq \Xi = \Xi^T \in \mathbb{R}^{(M+1)n \times (M+1)n}$, where for $X = P + \sum_{j=1}^{\rho} U^T \Lambda^j \Psi^j U$ and $\tilde{A}_0 = A_0 + \sum_{j=1}^{\rho} A_j \Psi^j U$

$$Q_1 = \begin{bmatrix} X \tilde{A}_0 + \tilde{A}_0^T X & X A_1 + \tilde{A}_0^T U^T \Lambda^1 & \dots & X A_M + \tilde{A}_0^T U^T \Lambda^M & X \\ * & \Lambda^1 U A_1 + \Lambda_1^T U^T \Lambda^1 & \dots & \Lambda^1 U A_M + \Lambda_1^T U^T \Lambda^M & \Lambda^1 U \\ & & \ddots & & \vdots \\ * & * & & \Lambda^M U A_M + \Lambda_M^T U^T \Lambda^M & \Lambda^M U \\ * & * & & U^T \Lambda^M & -\Gamma \end{bmatrix};$$

$$Q_2 = \begin{bmatrix} X G_{\mathbf{d}} + G_{\mathbf{d}}^T X & U^T \Lambda^1 H & \dots & U^T \Lambda^M H \\ * & \mathbf{0}_{Mn \times Mn} \\ * & * \\ * & * \end{bmatrix};$$

$$Q_3 = \Xi + \begin{bmatrix} \Upsilon_{0,0} & U^T \Upsilon_{0,1} & \dots & U^T \Upsilon_{0,M} \\ * & \Upsilon_{1,1} & \dots & \Upsilon_{1,M} \\ & & \ddots & \vdots \\ * & * & & \Upsilon_{M,M} \end{bmatrix},$$

$H = \text{diag}\{\lambda_i(G_{\mathbf{d}})\}_{i=1}^n, \Upsilon_{0,0} = \mathbf{0}_{n \times n}, \Upsilon_{0,i} \in \mathbb{D}_+^n, \Upsilon_{i,i} = \Upsilon_{i,i}^T \in \mathbb{R}^{n \times n}$,

$$\Upsilon_{i,j} \in \mathbb{R}^{n \times n} : e_n^T(\eta) \Upsilon_{i,j} e_n(l) \begin{cases} \geq 0, & \text{if } e_n^T(\eta) \mathcal{I} e_n(l) = 1 \\ \leq 0, & \text{if } e_n^T(\eta) \mathcal{I} e_n(l) = -1 \\ = 0, & \text{otherwise} \end{cases}$$

for $i = \overline{1, M}, j = \overline{i, M}, \eta, l = \overline{1, n}$, and let the following restriction to the system disturbances be satisfied:

$$d^T \mathbf{d}^T(-\ln V) \Gamma \mathbf{d}(-\ln V) d \leq V^{2\mu} y^T Q_3 y \quad (9)$$

$$\text{for } y = \begin{bmatrix} \mathbf{d}(-\ln V)x \\ f^1(U\mathbf{d}(-\ln V)x) - \Psi^1 U \mathbf{d}(-\ln V)x \\ \vdots \\ f^{\rho}(U\mathbf{d}(-\ln V)x) - \Psi^{\rho} U \mathbf{d}(-\ln V)x \\ f^{\rho+1}(U\mathbf{d}(-\ln V)x) \\ \vdots \\ f^M(U\mathbf{d}(-\ln V)x) \end{bmatrix} \text{ and } V \in \mathbb{R}_+ :$$

$$Q(V, x) = 0 \text{ with } \mathbb{R}_+ \ni \chi < n \min_{i \in \overline{1, n}} \{\underline{s}_i^2, \bar{s}_i^2\} \lambda_{\min}(X) \|U\|^{-2}.$$

Then the origin is globally finite-time stable and $T(x_0) \leq -\frac{V_0^{-\mu}}{\alpha\mu}$, where $Q(V_0, x_0) = 0$.

Proof. Consider a candidate Lyapunov function V defined implicitly by $Q(V, x) = 0$. The function $Q(V, x)$ satisfies the conditions C1)-C3) of Theorem 1. Indeed, it is continuously differentiable for

all $V \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$. Assumption 1 and the inequality $\iota_1 P + U^T \sum_{j=1}^{\zeta} \Lambda^j U > 0$ from (8a) imply C2) and C3). Indeed, for $x \in \mathcal{B} := \{x \in \mathbb{R}^n \setminus \{0\} : 2 \sum_{j=1}^M \sum_{i=1}^n \Lambda_i^j \int_0^{\mathbf{u}_i \mathbf{d}(-\ln V)x} f_i^j(s) ds = 0\}$ due to Assumption 1 $x^T \mathbf{d}^T(-\ln V) U^T \sum_{j=1}^{\zeta} \Lambda^j U \mathbf{d}(-\ln V)x = 0$ and by Finsler's lemma $x^T \mathbf{d}^T(-\ln V) P \mathbf{d}(-\ln V)x > 0$. Then for $x \in \mathcal{B}$ the following inequalities are satisfied

$$\frac{\lambda_{\min}(\bar{P}^{-1}) \lambda_{\min}(P + \frac{1}{\iota_1} U^T \sum_{j=1}^{\zeta} \Lambda^j U) \|x\|_{\bar{P}}^2}{\min\{V^{e_1}, V^{e_2}\}} \leq x^T \mathbf{d}^T(-\ln V) (P + \frac{1}{\iota_1} U^T \sum_{j=1}^{\zeta} \Lambda^j U) \mathbf{d}(-\ln V)x = x^T \mathbf{d}^T(-\ln V) P \mathbf{d}(-\ln V)x = Q(V, x) + \chi \leq \frac{\lambda_{\max}(\bar{P}^{-1}) \lambda_{\max}(P) \|x\|_{\bar{P}}^2}{\min\{V^{e_1}, V^{e_2}\}}, \quad (10)$$

where $\varrho_1, \varrho_2, \bar{P}$ are as in Theorem 2. For $x \notin \mathcal{B} \cup \{0\}$ we have $2 \sum_{j=1}^M \sum_{i=1}^n \Lambda_i^j \int_0^{\mathbf{u}_i \mathbf{d}(-\ln V)x} f_i^j(s) ds > 0$ and, taking into account $\mathbf{u}_i \mathbf{d}(-\ln V)x = V^{-\lambda_i(G_{\mathbf{d}})} \mathbf{u}_i x$, by mean value theorem

$$0 < 2 \sum_{j=1}^M \sum_{i=1}^n \Lambda_i^j V^{-\lambda_i(G_{\mathbf{d}})} \mathbf{u}_i x f_i^j(\kappa_{i,j}) \leq Q(V, x) + \chi \leq \frac{\lambda_{\max}(\bar{P}^{-1}) \lambda_{\max}(P) \|x\|_{\bar{P}}^2}{\min\{V^{e_1}, V^{e_2}\}} + 2 \sum_{j=1}^M \sum_{i=1}^n \Lambda_i^j V^{-\lambda_i(G_{\mathbf{d}})} \mathbf{u}_i x f_i^j(\kappa_{i,j}), \quad (11)$$

where $\kappa_{i,j} \in (0, V^{-\lambda_i(G_{\mathbf{d}})} \mathbf{u}_i x)$. Then (10), (11) guarantee that the lower and upper estimates are positive definite functions of x , and there exist $V^- \in \mathbb{R}_+$ and $V^+ \in \mathbb{R}_+ : Q(V^-, x) < 0 < Q(V^+, x)$. Thus, the condition C2) holds for any $x \in \mathbb{R}^n \setminus \{0\}$. Moreover, if $Q(V, x) = 0$ then by the same arguments it follows that the condition C3) is verified. Since

$$\begin{aligned} \frac{\partial Q}{\partial V} &= -V^{-1} x^T \mathbf{d}^T(-\ln V) (P G_{\mathbf{d}} + G_{\mathbf{d}}^T P) \mathbf{d}(-\ln V)x \\ &\quad - 2V^{-1} \sum_{j=1}^M \sum_{i=1}^n \Lambda_i^j \mathbf{u}_i G_{\mathbf{d}} \mathbf{d}(-\ln V) x f_i^j(\mathbf{u}_i \mathbf{d}(-\ln V)x) \\ &= -V^{-1} x^T \mathbf{d}^T(-\ln V) (P G_{\mathbf{d}} + G_{\mathbf{d}}^T P) \mathbf{d}(-\ln V)x \\ &\quad - 2V^{-1} \sum_{j=1}^M \sum_{i=1}^n \Lambda_i^j \lambda_i(G_{\mathbf{d}}) \mathbf{u}_i \mathbf{d}(-\ln V) x f_i^j(\mathbf{u}_i \mathbf{d}(-\ln V)x), \end{aligned}$$

and by Assumption 1

$$\begin{aligned} \{x \in \mathbb{R}^n \setminus \{0\} : \\ \sum_{j=1}^M \sum_{i=1}^n \Lambda_i^j \lambda_i(G_{\mathbf{d}}) \mathbf{u}_i \mathbf{d}(-\ln V) x f_i^j(\mathbf{u}_i \mathbf{d}(-\ln V)x) = 0\} \\ \equiv \{x \in \mathbb{R}^n \setminus \{0\} : \sum_{j=1}^M \sum_{i=1}^n x^T \mathbf{d}^T(-\ln V) \mathbf{u}_i^T \Lambda_i^j \mathbf{u}_i \mathbf{d}(-\ln V)x = 0\} \\ \equiv \{x \in \mathbb{R}^n \setminus \{0\} : \sum_{j=1}^M x^T \mathbf{d}^T(-\ln V) U^T \Lambda^j U \mathbf{d}(-\ln V)x = 0\}. \end{aligned}$$

Then by Finsler's lemma the inequalities $U^T \sum_{j=1}^M \Lambda^j U + \iota_2 (P G_{\mathbf{d}} + G_{\mathbf{d}}^T P) > 0$ for $\iota_2 \in \mathbb{R}$ and $P G_{\mathbf{d}} + G_{\mathbf{d}}^T P \geq 0$ imply that $\frac{\partial Q}{\partial V} < 0$ for all $V \in \mathbb{R}_+$ and $x \in \mathbb{R}^n \setminus \{0\}$, and the condition C4) of Theorem 1 holds.

Now, in order to obtain more efficient LMIs (see Remark 5 below), let us rewrite (3) and (7) in the following

equivalent forms:

$$\begin{aligned} \dot{x}(t) &= (A_0 + \sum_{j=1}^{\rho} A_j \Psi^j U)x(t) \\ &+ \sum_{j=1}^{\rho} A_j (f^j(Ux(t)) - \Psi^j Ux(t)) \\ &+ \sum_{m=\rho+1}^M A_m f^m(Ux(t)) + d(t, x(t)), \quad t \geq 0, \end{aligned} \quad (12)$$

$$\begin{aligned} Q(V, x) &= x^T \mathbf{d}^T(-\ln V) X \mathbf{d}(-\ln V) x \\ &+ 2 \sum_{j=1}^M \sum_{i=1}^n \Lambda_i^j \int_0^{\mathbf{u}_i \mathbf{d}(-\ln V) x} g_i^j(s) ds - \chi, \end{aligned} \quad (13)$$

where $g_i^j(s) = \begin{cases} f_i^j(s) - \psi_i^j s & \text{for } j = \overline{1, \rho}, i = \overline{1, n}, \\ f_i^j(s) & \text{for } j = \overline{\rho+1, M}, i = \overline{1, n} \end{cases}$,

$X = P + \sum_{j=1}^{\rho} U^T \Lambda^j \Psi^j U$. Then, denoting $\tilde{A}_0 = A_0 + \sum_{j=1}^{\rho} A_j \Psi^j U$ and using (5), (8) we have

$$\begin{aligned} \dot{V} &= -\left(\frac{\partial Q}{\partial V}\right)^{-1} \frac{\partial Q}{\partial x} \dot{x} \\ &= 2V \left(x^T \mathbf{d}^T(-\ln V) (P G_{\mathbf{d}} + G_{\mathbf{d}}^T P) \mathbf{d}(-\ln V) x \right. \\ &\quad \left. + 2 \sum_{j=1}^M \sum_{i=1}^n \Lambda_i^j \lambda_i(G_{\mathbf{d}}) \mathbf{u}_i \mathbf{d}(-\ln V) x f_i^j(\mathbf{u}_i \mathbf{d}(-\ln V) x) \right)^{-1} \\ &\quad \times \left[x^T \mathbf{d}^T(-\ln V) X \mathbf{d}(-\ln V) \dot{x} \right. \\ &\quad \left. + \sum_{j=1}^M \sum_{i=1}^n \Lambda_i^j g_i^j(\mathbf{u}_i \mathbf{d}(-\ln V) x) \mathbf{u}_i \mathbf{d}(-\ln V) \dot{x} \right] \\ &= 2V^{1+\mu} \left(x^T \mathbf{d}^T(-\ln V) (X G_{\mathbf{d}} + G_{\mathbf{d}}^T X) \mathbf{d}(-\ln V) x \right. \\ &\quad \left. + 2 \sum_{j=1}^M \sum_{i=1}^n x^T \mathbf{d}^T(-\ln V) U^T \Lambda^j H g^j(U \mathbf{d}(-\ln V) x) \right)^{-1} \\ &\quad \times \left[x^T \mathbf{d}^T(-\ln V) X \tilde{A}_0 \mathbf{d}(-\ln V) x \right. \\ &\quad \left. + x^T \mathbf{d}^T(-\ln V) X \sum_{j=1}^M A_j g^j(U \mathbf{d}(-\ln V) x) \right. \\ &\quad \left. + V^{-\mu} \sum_{j=1}^M \dot{x}^T \mathbf{d}^T(-\ln V) U^T \Lambda^j g^j(U \mathbf{d}(-\ln V) x) \right. \\ &\quad \left. + V^{-\mu} x^T \mathbf{d}^T(-\ln V) X \mathbf{d}(-\ln V) \dot{x} \right] \\ &= V^{1+\mu} (y^T Q_2 y)^{-1} \left(\left[V^{-\mu} \mathbf{d}(-\ln V) \right]^T Q_1 \left[V^{-\mu} \mathbf{d}(-\ln V) \right] \right. \\ &\quad \left. + V^{-2\mu} \mathbf{d}^T(-\ln V) \Gamma \mathbf{d}(-\ln V) \right) \\ &\leq \frac{V^{1+\mu} (V^{-2\mu} \mathbf{d}^T(-\ln V) \Gamma \mathbf{d}(-\ln V) - \alpha y^T Q_2 y - y^T Q_3 y)}{y^T Q_2 y} \\ &\leq -\alpha V^{1+\mu}, \end{aligned}$$

where the aggregated vector variable y is defined in the formulation of the theorem. Thus, the condition $C5)$ holds. Note, that due to the condition $C4)$ of Theorem 1, we have $y^T Q_2 y > 0$ for $x \in \mathbb{R}^n \setminus \{0\}$.

Now, let us show that $y^T Q_3 y \geq 0$, i.e., the class of feasible disturbances is not empty. Denote the set $\mathcal{A} := \{x \in \mathbb{R}^n \setminus \{0\} : \underline{s}_i \leq V^{-\lambda_i(G_{\mathbf{d}})} \mathbf{u}_i x \leq \bar{s}_i, i = \overline{1, n}\}$ and $\omega_1 = \min_{1 \leq i \leq n} \bar{s}_i^2$, $\omega_2 = \min_{1 \leq i \leq n} \underline{s}_i^2$. Since $\chi < n \min\{\omega_1, \omega_2\} \lambda_{\min}(X) \|U\|^{-2}$, then accordingly to (6) and (13) for $Q(V, x) = 0$ we have $n \min\{\omega_1, \omega_2\} > \frac{\|U\|^2 \chi}{\lambda_{\min}(X)} \geq \|U\|^2 \|\mathbf{d}(-\ln V) x\|^2 \geq \|U \mathbf{d}(-\ln V) x\|^2$ if $2 \sum_{j=1}^M \sum_{i=1}^n \Lambda_i^j \int_0^{\mathbf{u}_i \mathbf{d}(-\ln V) x} g_i^j(s) ds \geq 0$. Note that

$$\begin{aligned} \{x \in \mathbb{R}^n \setminus \{0\} : \|U \mathbf{d}(-\ln V) x\|^2 \leq n \min\{\omega_1, \omega_2\}\} &\subseteq \mathcal{A} \\ \subseteq \mathcal{C} := \{x \in \mathbb{R}^n \setminus \{0\} : 2 \sum_{j=1}^M \sum_{i=1}^n \Lambda_i^j \int_0^{\mathbf{u}_i \mathbf{d}(-\ln V) x} g_i^j(s) ds \geq 0\}, \end{aligned}$$

i.e., the solution of $Q(V, x) = 0$ implies $\mathbf{u}_i \mathbf{d}(-\ln V) x \in [\underline{s}_i, \bar{s}_i]$ on the set \mathcal{C} . Consider the set

$\mathcal{E} := \{x \in \mathbb{R}^n \setminus \{0\} : -\varsigma < 2 \sum_{j=1}^M \sum_{i=1}^n \Lambda_i^j \int_0^{\mathbf{u}_i \mathbf{d}(-\ln V) x} g_i^j(s) ds < \varsigma\}$, where $\varsigma \in \mathbb{R}_+$ is arbitrary small. Note that $\mathcal{E} \cap \mathcal{A} = \emptyset$, i.e., if $x \in \mathcal{E}$, then $\exists \mathbf{u}_i \mathbf{d}(-\ln V) x \notin [\underline{s}_i, \bar{s}_i]$ for some $i \in \{1, \dots, n\}$. Assume that there exists a solution of $Q(V, x) = 0$ for $x \in \mathcal{E}$. Then taking into account ς is arbitrary small, we have $n \min\{\omega_1, \omega_2\} \geq \|U \mathbf{d}(-\ln V) x\|^2$, that contradicts the assumption $\mathbf{u}_i \mathbf{d}(-\ln V) x \notin [\underline{s}_i, \bar{s}_i]$. Finally, due to continuity of the Lyapunov function we conclude that $\mathbf{u}_i \mathbf{d}(-\ln V) x \in [\underline{s}_i, \bar{s}_i]$ for any $x \in \mathbb{R}^n \setminus \{0\}$, $i = \overline{1, n}$, where $\underline{s}_i, \bar{s}_i$ are defined in (6). In this case $y^T Q_3 y \geq 0$ for any $x \in \mathbb{R}^n \setminus \{0\}$. Thus, all conditions of Theorem 1 are satisfied, and the system (3) is globally finite-time stable if (8) holds. \blacksquare

Remark 3 Let us define $\tilde{\chi} = \max_{\{x \in \mathbb{R}^n : V(x)=1\}} (\chi - \|x\|_P^2) \in \mathbb{R}_+$,

that can be calculated on a numerical grid. Then the implicit restriction (9) to the system disturbances can be replaced by explicit (but more conservative) inequality

$$\|d\|_P^2 \leq q \gamma^{-1} \times \min \left\{ \chi^{-\frac{\mu+\varrho_1-\varrho_2}{\varrho_2}} \|x\|_P^{\frac{2(\mu+\varrho_1)}{\varrho_2}}, \beta^{2(\mu-\varrho_1+\varrho_2)} \|x\|_P^{\frac{2(\mu-\varrho_1+2\varrho_2)}{\varrho_2}} \right\} \quad (14)$$

for $\beta = (\chi - \tilde{\chi})^{\frac{-1}{2\varrho_2}}$ under additional conditions $\Gamma \leq \gamma P$, $\sum_{j=1}^M \Lambda^j \neq 0$, $P G_{\mathbf{d}} + G_{\mathbf{d}}^T P > 0$ and $Q_3 \geq \begin{bmatrix} qP & O_{n \times n} \\ O_{n \times n} & O_{Mn \times Mn} \end{bmatrix}$ for some $q, \gamma \in \mathbb{R}_+$.

Indeed, since $2 \sum_{j=1}^M \sum_{i=1}^n \Lambda_i^j \int_0^{\mathbf{u}_i \mathbf{d}(-\ln V) x} f_i^j(s) ds$ is homogeneous with zero degree and $P > 0$, then for $Q(V, x) = 0$ we have $0 < \chi - \tilde{\chi} \leq x^T \mathbf{d}^T(-\ln V) P \mathbf{d}(-\ln V) x \leq \chi$.

Then, $V(x) \geq \chi^{-\frac{1}{2\varrho_2}} \|x\|_P^{\frac{\varrho_2}{\varrho_2}}$ for $\{x \in \mathbb{R}^n : V(x) \leq 1\}$

and $V(x) \leq \beta \|x\|_P^{\frac{1}{\varrho_2}}$ for $\{x \in \mathbb{R}^n : V(x) \geq 1\}$ by Theorem 2 and Assumption 1. Finally, under conditions on Γ and Q_3 , (9) follows from $d^T \mathbf{d}^T(-\ln V) P \mathbf{d}(-\ln V) d \leq \frac{q}{\gamma} V^{2\mu} x^T \mathbf{d}^T(-\ln V) P \mathbf{d}(-\ln V) x$, and using (2) and obtained bounds on $V(x)$, we have that (14) implies (9).

Example 1 Consider the three tank hydraulic system considered in [20]:

$$\dot{x} = \begin{bmatrix} -\frac{\alpha_{13}}{S} & 0 & 0 \\ 0 & -\frac{\alpha_{20}}{S} & \frac{\alpha_{32}}{S} \\ \frac{\alpha_{13}}{S} & 0 & -\frac{\alpha_{32}}{S} \end{bmatrix} \begin{bmatrix} |x_1 - x_3|^{1/2} \text{sign}(x_1 - x_3) \\ |x_2|^{1/2} \text{sign}(x_2) \\ |x_3 - x_2|^{1/2} \text{sign}(x_3 - x_2) \end{bmatrix} + \begin{bmatrix} \frac{1}{S} u_1 \\ \frac{1}{S} u_2 \\ 0 \end{bmatrix},$$

where $x = [x_1 \ x_2 \ x_3]^T = [h_1 \ h_2 \ h_3]^T$ with $h_i \in \mathbb{R}_+$ being the liquid levels in each tank for $i = \overline{1, 3}$, the input flows are $u = [u_1 \ u_2]^T = [Q_1 \ Q_2]^T$. The system is in the form (3) with zero matrix A_0 and $d = [\frac{1}{S} u_1 \ \frac{1}{S} u_2 \ 0]^T$, and it is homogeneous of degree -0.5 with $G_{\mathbf{d}} = I_3$ for $u_1 = u_2 = 0$, i.e., Assumption 2 is satisfied. Let $S = a_{13} = a_{32} = 1$, $a_{20} = 2$ and $u_1 = 0.5x_2$, $u_2 = 0.5 \sin(x_2)$. Then for $A_0 = 0$, $A_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 1 \\ 1 & 0 & -1 \end{bmatrix}$, $\Psi = I_3$,

$U = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ the LMI (8) is feasible and the condition (9) is satisfied. Thus, the system is globally finite-time stable with the settling-time function estimate provided by Theorem 3.

Remark 4 If U is invertible then under change of coordinates $z = Ux$ the given result can be reduced to the case of stability analysis of homogeneous Persidskii systems considered in [1].

Stable homogeneous systems have such qualitative stability properties as ISS with respect to additive perturbations and measurement noises (see, for example, [3], [4]). In this case to establish the ISS property it is enough to analyse system stability with $d = 0$.

Corollary 1 *Let all conditions of Theorem 4 be satisfied for $\Gamma = 0$ and*

$$Q_1 = \begin{bmatrix} X\tilde{A}_0 + \tilde{A}_0^T X & XA_1 + \tilde{A}_0^T U^T \Lambda^1 & \dots & XA_M + \tilde{A}_0^T U^T \Lambda^M & \mathbf{O}_{n \times n} \\ * & \Lambda^1 U A_1 + A_1^T U^T \Lambda^1 & \dots & \Lambda^1 U A_M + A_1^T U^T \Lambda^M & \mathbf{O}_{n \times n} \\ & & \ddots & \vdots & \vdots \\ * & * & \dots & \Lambda^M U A_M + A_1^T U^T \Lambda^M & \mathbf{O}_{n \times n} \\ * & * & \dots & \mathbf{O}_{n \times n} & \mathbf{O}_{n \times n} \end{bmatrix}.$$

Then the system (3) is ISS for $d \in \mathcal{L}_\infty$, and for $d = 0$ the origin is globally finite-time stable with $T(x_0) \leq -\frac{V_0^{-\mu}}{\alpha\mu}$, where $Q(V_0, x_0) = 0$.

Proof. The proof follows exactly the same arguments as one of Theorem 4 for $d(t) = 0$ and ISS properties of homogeneous systems given in [3], [4, Theorem 4]. ■

Example 2 Consider the problem of finite-time control for uniform allocation of agents on a segment. The dynamical model of each agent is described by a simple integrator $\dot{x}_i(t) = u_i(x(t)) + d_i(t, x)$, $i = \overline{1, n}$, where $u_i \in \mathbb{R}$ is the control input, $x \in \mathbb{R}^n$, $d_i(t, x)$ is a bounded exogenous disturbance. Let x_{p1}, x_{p2} denote the fixed endpoints of the segment. Then, according to [18] the control protocol $u_i(x) = a|s_i(x)|^{1+\mu} \text{sign}(s_i(x))$ uniformly allocates agents on a segment, where $s_i(x) = \frac{1}{2}(x_{i-1} - x_i) + \frac{1}{2}(x_{i+1} - x_i)$, $a \in \mathbb{R}_+$, $\mu \in (-1, 0)$, $x_{-1} = x_{p1}$ and $x_{n+1} = x_{p2}$. Using the change of variables $\bar{x} = x + U^{-1}b$ we obtain the system

$$\dot{\bar{x}} = f(U\bar{x}) + d, \quad (15)$$

where $d \in \mathbb{R}^n$, $d(t) \in \mathcal{L}_\infty$,

$$f_i(U\bar{x}) = a \left| \frac{1}{2}(\bar{x}_{i-1} - \bar{x}_i) + \frac{1}{2}(\bar{x}_{i+1} - \bar{x}_i) \right|^{1+\mu} \times \text{sign} \left(\frac{1}{2}(\bar{x}_{i-1} - \bar{x}_i) + \frac{1}{2}(\bar{x}_{i+1} - \bar{x}_i) \right),$$

$$U = \begin{bmatrix} -1 & 0.5 & 0 & \dots & 0 \\ 0.5 & -1 & 0.5 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \dots & 0.5 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 0.5x_{p1} \\ 0 \\ \vdots \\ 0.5x_{p2} \end{bmatrix}.$$

Note that U is Hurwitz (see [18]). Obviously the system (15) is in the form (3) with $M = 1$, $A_0 = \mathbf{O}_{n \times n}$, $A_1 = I_n$, and for $d = 0$ it is \mathbf{d} -homogeneous of degree μ with $G_{\mathbf{d}} = I_n$ and $e_n^T(i)U = \mathbf{u}_i$, i.e., Assumption 2 and Assumption 3 are satisfied. Due to UA_1 is diagonally stable the conditions of Corollary 1 are satisfied and the system is finite-time stable for $d = 0$ and ISS with respect to disturbances d . In comparison with [18] the proposed approach allows to obtain new settling time estimations. According to the simulation results for different initial conditions, the settling time estimations based on Corollary 1 are significantly closer to the true values than the

estimations derived from the results of [18]. For example, for $n = 5$, $a = 5$ and $x_i(0) = 0$, $i = \overline{1, 5}$, $x_{p1} = 5$, $x_{p2} = 11$ the settling time is 3.25, while the estimates obtained using the proposed result and [18] are 3.96 and 5.22, respectively. Moreover, this approach is promising for further development of finite-time control protocols for systems with more complex agent models, that is one of the possible directions for further research.

Remark 5 In order to satisfy the condition (8) the matrices X and Λ^j serve as solutions of the Lyapunov equation for the matrices \tilde{A}_0 and UA_j for $j = \overline{1, M}$, respectively. In this case it is important to choose appropriate representation of the system in the form (12) to better take the features of the LMIs (8). Indeed, without the use of the representation (12), the linear part of the system (3) in general case can be represented with nilpotent matrix A_0 for which the inequalities (8) are not feasible (i.e., $\tilde{A}_0 = A_0$, $\Psi^j = 0$ and in this case the matrix $X\tilde{A}_0 + \tilde{A}_0^T X$ in Q_1 cannot be negative definite).

The matrix Q_3 is used in order to relax the stability requirements for each matrix A_k , $k = \overline{0, M}$. The matrices $\Upsilon_{\eta, j}$ for $\eta = \overline{0, M-1}$ and $j = \eta+1, M$ are introduced in order to reduce the conservatism of the condition (8c). Indeed, since for $\chi < n \min\{\underline{s}_i^2, \bar{s}_i^2\} \lambda_{\min}(X) \|U\|^{-2}$ we have that $\mathbf{u}_i \mathbf{d}(-\ln V)x \in [\underline{s}_i, \bar{s}_i] \setminus \{0\}$ for any $x \in \mathbb{R}^n \setminus \{0\}$, $i = \overline{1, n}$, then due to the sector properties (Assumption 1), all terms in $y^T Q_3 y$ are nonnegative for $x \in \mathbb{R}^n \setminus \{0\}$. In other words, all cross-terms, which appear on the left-hand side of (8c), should not be considered as additional "perturbations" in verification of the restriction (8c) provided that they have a non-positive multiplier.

The advantages of using the representation (12) are based on the following observations:

- the linear term $\sum_{j=1}^{\rho} A_j \Psi^j Ux$ has slower dynamics in a neighbourhood of 0 with respect to $\sum_{j=1}^{\rho} A_j f^j(Ux)$;
- the sector restriction (4) is generic, and it does not introduce in the analysis the lower and upper bounds usual in the absolute stability theory [33], then inclusion of the matrices Ψ^j allows the lower one to be taken into account;
- due to homogeneity property of the system and the chosen Lyapunov function, the global stability properties can be analysed in the vicinity of 0, where the sector condition is satisfied for nonlinearities $f_i^j(\mathbf{u}_i x) - \psi_i^j \mathbf{u}_i x$, $i = \overline{1, n}$, $j = \overline{1, \rho}$.

Similarly to the representation (12), the same trick can be used for other terms of (3) with a slow dynamics in a neighbourhood of 0, and the obtained results stay relevant. It is shown in the following corollary, that for stability analysis we can use the representation with the terms $A_j(f^j(Ux) - \Psi^j Ux - \Phi^j h^j(Ux))$, where $\Phi^j \in \mathbb{D}_+^n$, $j = \overline{1, \rho}$ and $h^j(Ux)$ have slow dynamics close to 0. Assume that in (3) the terms $f^i(Ux(t))$, $i = \rho+1, \vartheta$, $\vartheta \in [\rho+1, M]$ have slow dynamics close to 0, and (8) is not feasible. For any $i = \overline{1, n}$, $j = \overline{1, \rho}$, $q = \overline{1, \vartheta} - \rho$ we have $sf_i^j(s) >$

$\phi_i f_i^{\rho+q}(s)s, \forall s \in [\underline{s}_i, \bar{s}_i] \setminus \{0\}$, where $\phi_i \in \mathbb{R}_+$ and $-\infty < \underline{s}_i < 0 < \bar{s}_i < +\infty$ are as before. Define $\Phi = \text{diag}\{\phi_i\}_{i=1}^n \in \mathbb{D}_+^n$ and $\hat{f}^i(Ux(t)) = \Phi f^i(Ux(t))$ for $i = \overline{\rho+1, \vartheta}$, then for $i = \overline{1, n}, j = \overline{1, \rho}, q = \overline{1, \vartheta - \rho}$ we have $s f_i^j(s) > \hat{f}_i^{\rho+q}(s)s, \forall s \in [\underline{s}_i, \bar{s}_i] \setminus \{0\}$ and by (6)

$$s f_i^j(s) > \frac{1}{1+\vartheta-\rho} \left(\psi_i^j s + \sum_{q=1}^{\vartheta-\rho} \hat{f}_i^{\rho+q}(s) \right) s \quad \forall s \in [\underline{s}_i, \bar{s}_i] \setminus \{0\}. \quad (16)$$

Therefore, the following result can be used:

Corollary 2 *Let Assumptions 1, 2, 3 be satisfied. Let $\alpha \in \mathbb{R}_+, \iota_1, \iota_2 \in \mathbb{R}_+$ be chosen such that (8), (9) are satisfied for some $P = P^T \in \mathbb{R}^{n \times n}; \Lambda^j \in \mathbb{D}_+^n, j = \overline{1, M}, 0 \leq \Xi = \Xi^T \in \mathbb{R}^{(M+1)n \times (M+1)n}$ with*

$$y = \begin{bmatrix} \mathbf{d}(-\ln V)x \\ f^1(U\mathbf{d}(-\ln V)x) - \frac{1}{1+\vartheta-\rho} \left[\Psi^1 U\mathbf{d}(-\ln V)x + \sum_{q=1}^{\vartheta-\rho} \hat{f}^{\rho+q}(U\mathbf{d}(-\ln V)x) \right] \\ \vdots \\ f^\rho(U\mathbf{d}(-\ln V)x) - \frac{1}{1+\vartheta-\rho} \left[\Psi^\rho U\mathbf{d}(-\ln V)x + \sum_{q=1}^{\vartheta-\rho} \hat{f}^{\rho+q}(U\mathbf{d}(-\ln V)x) \right] \\ \hat{f}^{\rho+1}(U\mathbf{d}(-\ln V)x) \\ \vdots \\ \hat{f}^\vartheta(U\mathbf{d}(-\ln V)x) \\ f^{\vartheta+1}(U\mathbf{d}(-\ln V)x) \\ \vdots \\ f^M(U\mathbf{d}(-\ln V)x) \end{bmatrix},$$

$$X = P + \frac{1}{1+\vartheta-\rho} \sum_{j=1}^{\rho} U^T \Lambda^j \Psi^j U, \tilde{A}_0 = A_0 + \frac{1}{1+\vartheta-\rho} \sum_{j=1}^{\rho} A_j \Psi^j U,$$

$\tilde{A}_i = A_i \Phi^{-1} + \frac{1}{1+\vartheta-\rho} \sum_{j=1}^{\rho} A_j$ substituted instead of A_i for $i = \overline{\rho+1, \vartheta}$, and $\tilde{\Lambda}^j = \Lambda^j \Phi^{-1} + \frac{1}{1+\vartheta-\rho} \sum_{q=1}^{\rho} \Lambda^q$ substituted instead of Λ^j for $j = \overline{\rho+1, \vartheta}$.

Then the origin is globally finite-time stable and $T(x_0) \leq -\frac{V_0^-}{\alpha\mu}$, where $Q(V_0, x_0) = 0$.

Proof. The proof follows the same arguments as one of Theorem 4, where:

- instead of (12) the following representation is used:

$$\begin{aligned} \dot{x}(t) &= (A_0 + \frac{1}{1+\vartheta-\rho} \sum_{j=1}^{\rho} A_j \Psi^j U)x(t) \\ &+ \sum_{j=1}^{\rho} A_j \left(f^j(x(t)) - \frac{1}{1+\vartheta-\rho} \left[\Psi^j Ux(t) + \sum_{q=1}^{\vartheta-\rho} \hat{f}^{\rho+q}(Ux(t)) \right] \right) \\ &+ \sum_{q=1}^{\vartheta-\rho} (A_{q+\rho} \Phi^{-1} + \frac{1}{1+\vartheta-\rho} \sum_{j=1}^{\rho} A_j) \hat{f}^{q+\rho}(Ux(t)) \\ &+ \sum_{m=\vartheta+1}^M A_m (f^m(Ux(t))) + d(t, x(t)), \quad t \geq 0, \end{aligned}$$

- For ILF (7) the representation

$$Q(V, x) = x^T \mathbf{d}^T(-\ln V) X \mathbf{d}(-\ln V)x + 2 \sum_{j=1}^M \sum_{i=1}^n \tilde{\Lambda}_i^j \int_0^{\mathbf{u}_i \mathbf{d}(-\ln V)x} g_i^j(s) ds - \chi,$$

is used, where $X = P + \frac{1}{1+\vartheta-\rho} \sum_{j=1}^{\rho} U^T \Lambda^j \Psi^j U$,

$$g_i^j(s) = \begin{cases} f_i^j(s) - \frac{1}{1+\vartheta-\rho} \left(\psi_i^j s + \sum_{q=1}^{\vartheta-\rho} \hat{f}_i^{\rho+q}(s) \right) & \text{for } j = \overline{1, \rho}, i = \overline{1, n}, \\ \hat{f}_i^j(s) & \text{for } j = \overline{\rho+1, \vartheta}, i = \overline{1, n}, \\ f_i^j(s) & \text{for } j = \overline{\vartheta+1, M}, i = \overline{1, n} \end{cases},$$

$$\tilde{\Lambda}^j = \begin{cases} \Lambda^j & \text{for } j = \overline{1, \rho} \text{ and } j = \overline{\vartheta+1, M}, \\ \Lambda^j \Phi^{-1} + \frac{1}{1+\vartheta-\rho} \sum_{q=1}^{\rho} \Lambda^q & \text{for } j = \overline{\rho+1, \vartheta} \end{cases},$$

- for $Q(V, x) = 0$ the inequality (16) implies the sector condition for the terms

$$f^j(U\mathbf{d}(-\ln V)x)$$

$$- \frac{1}{1+\vartheta-\rho} \left[\Psi^j U\mathbf{d}(-\ln V)x + \sum_{q=1}^{\vartheta-\rho} \hat{f}^{\rho+q}(U\mathbf{d}(-\ln V)x) \right]. \quad \blacksquare$$

Remark 6 According to Corollary 2 for the case $U \in \mathbb{D}_+^n$, in order to establish finite-time stability and ISS property, it is sufficient that all terms f^i with fast dynamics in a neighbourhood of 0 are multiplied with diagonally stable matrices A_i .

Remark 7 If in (3) there exists a locally bounded and piecewise continuous $f_j, j \in \{1, \dots, M\}$, then under the assumption that the trajectories do not stay on the discontinuity set (except at the origin), the results above are preserved by interpreting the inequality for the derivative of Lyapunov function in the almost everywhere sense.

Remark 8 In order to obtain less conservative LMIs the following approach can be used. In (8) some elements $e_n^T(\eta) \Upsilon_{i,j} e_n(l), \eta, l \in \overline{1, n}$ of the matrices $\Upsilon_{i,j}, i, j \in \overline{0, M}$ that correspond to the case $e_n^T(\eta) \mathcal{I} e_n(l) = 1$ can be selected to be negative if there are positive elements of $\Upsilon_{l,q}, l, q \in \overline{0, M}$ that correspond to terms with the same multipliers, but having faster or the same dynamics in the vicinity of the origin; and the sum of these elements is greater or equal than the sum of absolute values of negative counterparts that correspond to terms with slower dynamics. The similar remark with inverted signs can be made for the case $e_n^T(\eta) \mathcal{I} e_n(l) = -1$.

Example 3 Consider the problem of observer design for the second order linear system $\dot{x} = Ax, y = Cx$, where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, C = [1 \ 0]$. According to [31], [32] let us choose the observer $\hat{x} = A\hat{x} + \begin{bmatrix} l_1 & 0 \\ 0 & l_2 \end{bmatrix} \begin{bmatrix} |\hat{x}_1 - y|^{1-\mu} \text{sign}(\hat{x}_1 - y) \\ |\hat{x}_1 - y|^{1-2\mu} \text{sign}(\hat{x}_1 - y) \end{bmatrix}$, where $\mu \in (0, 1/2)$, and $l_1, l_2 \in \mathbb{R}$ are such that $A + \begin{bmatrix} l_1 & 0 \\ 0 & l_2 \end{bmatrix} C$ to be Hurwitz. In [34], [35] such an observer was studied for $\mu = 1/2$ (Super-Twisting Algorithm). For $e = \hat{x} - x$ the error system takes the form

$$\dot{e} = Ae + \begin{bmatrix} l_1 & 0 \\ 0 & l_2 \end{bmatrix} \begin{bmatrix} |e_1|^{1-\mu} \text{sign}(e_1) \\ |e_1|^{1-2\mu} \text{sign}(e_1) \end{bmatrix} \quad (17)$$

that corresponds to (3) with $A_0 = A, M = 1, A_1 = \text{diag}\{l_i\}_{i=1}^n, U = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, f^1(Ue) = \begin{bmatrix} |e_1|^{1-\mu} \text{sign}(e_1) \\ |e_1|^{1-2\mu} \text{sign}(e_1) \end{bmatrix}$. Note that the system (17) is homogeneous of degree $-\mu$ with $G_{\mathbf{d}} = \begin{bmatrix} 1 & 0 \\ 0 & 1-\mu \end{bmatrix}$. Note that the paper [32] provides an additional rather conservative bound for the degree of homogeneity: $\mu < \frac{\theta}{n(\sqrt{n}+\theta)}, \theta = \frac{\exp(1)}{2} \frac{\lambda_{\min}(-\tilde{A}_0^T X - X \tilde{A}_0)}{\lambda_{\max}(X) \|X A_1\|}$,

where $\tilde{A}_0 = A + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} C, 0 < X \in \mathbb{R}^{2 \times 2}$ is such that $-\tilde{A}_0^T X - X \tilde{A}_0 > 0$. Let $l_1 = l_2 = 1$. For $\mu = 0.49, \Psi = I_2$ all conditions of Corollary 1 are satisfied with $X = \begin{bmatrix} 6.3356 & -1.4619 \\ -1.4619 & 3.7064 \end{bmatrix}, Q_3 = \begin{bmatrix} -15.1548 & 0 & 5.3517 & 2.2257 \\ 0 & 0 & 0 & 0 \\ 5.3517 & 0 & -7.4128 & 3.7064 \\ 2.2257 & 0 & 3.7064 & 0 \end{bmatrix}$, where Remark 8 was used. Thus, the system (17) is finite-time stable, while for the same matrix X the result of [32] is applicable only for $\mu < 0.0097$ (i.e., for an observer close to the standard Luenberger one).

5 Conclusions

In the paper, sufficient conditions of robust finite-time stability are presented for a class of homogeneous systems with sector nonlinearities. The conditions are given in terms of LMI. The proposed approach extends the results of [1], [21]: with the use of system representation (12), simple LMI-based stability conditions were extended on the class of \mathbf{d} -homogeneous systems with sector nonlinearities; and nonasymptotic (finite-time) stability was studied. The proposed results allow to obtain settling-time estimates. The main advantages of the proposed results are as follows: a non quadratic-like ILF is proposed with related LMIs that can be applied to a wider class of homogeneous systems (quadratic-like ILF [7], [8], [9] is a special case of (7) with $\Lambda_i^j \equiv 0$) as it is shown in Example 3; the results of [21] are extended to the class of *homogeneous* Persidskii systems (LMIs given in [21] are rather restrictive in general case); in comparison with results based on homogeneous approximations the proposed method provides quantitative robustness analysis (Theorem 4, Remark 3); in some cases our approach allows to obtain better settling-time estimations (see, e.g., Example 2). Possible directions for future research include an extension of the results for systems with positive degree of homogeneity, finite-time control and observer design.

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