



Bidomain Model Coupled To A Realistic Multi-Electrode Device In Cardiac Electrophysiology

Valentin Pannetier, Yves Coudière, Michael Leguèbe

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Bidomain Model Coupled To A Realistic Multi-Electrode Device In Cardiac Electrophysiology

SimCardioTest European Project

Valentin PANNETIER, Yves COUDIÈRE and Michael LEGUÈBE

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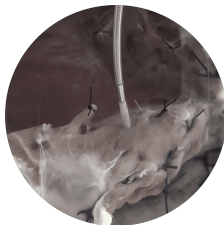

Institut de
Mathématiques de
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Context & Motivation

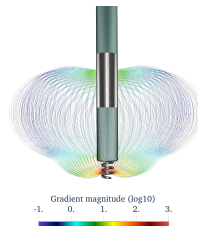
WP2 of SimCardioTest focuses on Pacing Leads and Catheters, especially on **energy required to trigger a cardiac beat** with a pacemaker.



(a) BOREA™ pacemaker



(b) Lead implanted in the septum

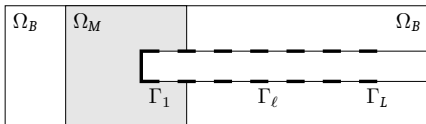


(c) Simulation of electrical field

Problematic

Electrophysiological propagation models already exist, **but we need to describe the stimulation**

- Three discontinuous parts by stimulation \Rightarrow **Device circuit must be modeled**
Structural analysis of electric circuits and consequences for MNA,
D. Estévez Schwarz, C. Tischendorf, 2000
- Presence of a reactive layer at electrode boundaries \Rightarrow **Layer must be modeled too**
Existence and Uniqueness for Electrode Models for Electric Current Computed Tomography,
E. Somersalo, M. Cheney and D. Isaacson, 1992



(a) Geometric setup

+ Contacts +

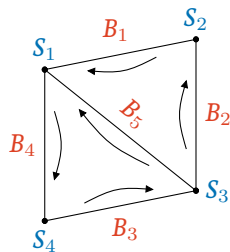


(b) BOREA™

Principle & Notations - I

- An **oriented graph** $\mathcal{G} = (S, B)$ with
 - N_n nodes,
 - N_b branches (dipoles)
- **Potentials** U in Volt and **currents** J in Ampere
- **Incidence matrix** $A = (a_{s,b})_{s \in S, b \in B}$ of size $N_n \times N_b$

$$a_{s,b} = \begin{cases} -1 & \text{if } s \text{ is an end point of } b \\ +1 & \text{if } s \text{ is a start point of } b \\ 0 & \text{elsewhere} \end{cases}$$
- Kirchhoff's **current law** $AJ = 0$ and **voltages** $V := A^T U$.



$$U = (U_{S_1}, U_{S_2}, U_{S_3}, U_{S_4})$$

$$J = (J_{B_1}, J_{B_2}, J_{B_3}, J_{B_4}, J_{B_5})$$

$$\begin{matrix} & B_1 & B_2 & B_3 & B_4 & B_5 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} \end{matrix}$$

Principle & Notations - II

- Only contains **capacitors, resistors and voltage sources**

- Split** graph and matrix accordingly:

$$B = B_c \cup B_g \cup B_s \text{ and } A = [A_c, A_g, A_s]$$

$$AJ = 0 \Rightarrow \boxed{A_c J_c + A_g J_g + A_s J_s = 0}$$

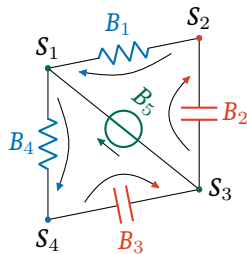
$$V = A^T U \Rightarrow V_c = A_c^T U, V_g = A_g^T U \text{ and } V_s(t) = A_s^T U$$

$$\bullet \text{ Dipole laws: } J_c = C \frac{dV_c}{dt} = C \frac{dA_c^T U}{dt} \text{ and } J_g = GV_g = GA_g^T U$$

- Differential algebraic equations (DAE)**

(U, J_s) solution of

$$\boxed{\begin{cases} A_c C \frac{dA_c^T U}{dt} + A_g G A_g^T U + A_s J_s = 0 \\ A_s^T U = V_s \end{cases}}$$



$$U = (U_{s_1}, U_{s_2}, U_{s_3}, U_{s_4})$$

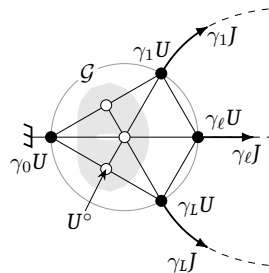
$$J = (J_{B_1}, J_{B_2}, J_{B_3}, J_{B_4}, J_{B_5})$$

$$\begin{matrix} & B_1 & B_2 & B_3 & B_4 & B_5 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} \end{matrix}$$

Generic Models

Multi-Electrode Device Model

- Graph (S, B) oriented with $L + 1$ **specialized nodes**
- **Potentials** split as $U = (U^\circ, \gamma U)$ of size N_n , internal potentials U°
- **Dirichlet boundary condition**, $\gamma_0 U = 0$ (ground)
- **Coupling boundary potentials** $\gamma_1 U, \dots, \gamma_L U$
- **Coupling output currents** $\gamma J = (\gamma_1 J, \dots, \gamma_L J)$



(U, J_s) solution of DAE

$$\begin{cases} A_c C \frac{dA_c^\top U}{dt} + A_g G A_g^\top U + A_s J_s + \gamma A \gamma J = 0 \\ A_s^\top U = V_s \end{cases}$$

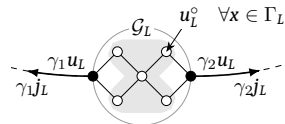
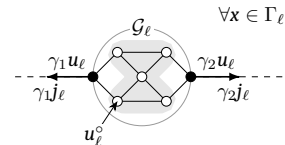
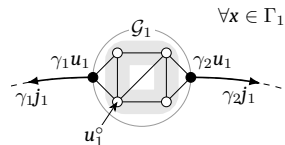
with $\gamma A := [0, \text{Id}_L]^\top$.

Generic Models

Equivalent Models for each electrode

On each electrode $\ell = 1, \dots, L$ (without voltage sources)

- **Defined spatially:** $\forall x \in \Gamma_\ell$
- **Potentials** split as $u_\ell = (u_\ell^\circ, \gamma u_\ell)$ of size N_ℓ
- **Coupling boundary potentials** $\gamma_1 u_\ell, \gamma_2 u_\ell$
- **Coupling output currents** $\gamma j_\ell = (\gamma_1 j_\ell, \gamma_2 j_\ell)$



u_ℓ solution of x-ODE

$$A_{\ell,c} C_\ell \frac{\partial A_{\ell,c}^\top u_\ell}{\partial t} + A_{\ell,g} G_\ell A_{\ell,g}^\top u_\ell + \gamma A_\ell \gamma j_\ell = 0$$

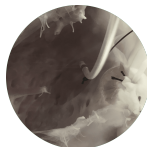
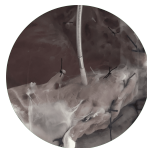
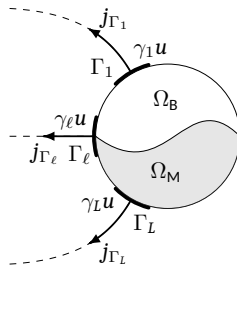
on Γ_ℓ

with $\gamma A_\ell := [0, \text{Id}_2]^\top$.

Generic Models

Bidomain Model (PDE)

- **Tissue** Ω_M – intracellular and extracellular
- **Bath/Blood** Ω_B – passive media
- **Transmembrane voltage** $[u] = u_i - u$
- **Conductivity tensors** σ and σ_i
- **L electrodes** at boundary $\bigcup_{\ell=1}^L \Gamma_\ell \subset \partial(\overline{\Omega_M} \cup \overline{\Omega_B})$
- $\gamma_\ell u = u|_{\Gamma_\ell}$, and $-\sigma \nabla u \cdot \mathbf{n} = j_{\Gamma_\ell}$ on Γ_ℓ

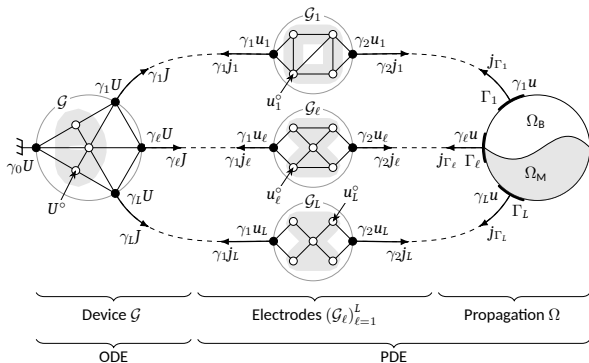


$$(u, u_i) \text{ solution of PDE } \begin{cases} -\operatorname{div}(\sigma_i \nabla u_i) = -j_m & \text{in } \Omega_M \\ -\operatorname{div}(\sigma \nabla u) = 0 & \text{in } \Omega_B \end{cases}$$

with membrane current density $j_m = \chi (c_m \partial_t [u] + I_{\text{ion}}([u], h, t))$, and Ionic Model on h .

Generic Models

Transmission conditions



— Potentials

$$\gamma_\ell U = \gamma_1 u_\ell(x), \text{ for } x \in \Gamma_\ell$$

$$\gamma_2 u_\ell(x) = \gamma_\ell u(x), \text{ for } x \in \Gamma_\ell$$

— Currents

$$\gamma_\ell J + \int_{\Gamma_\ell} \gamma_1 j_\ell(x) = 0$$

$$\gamma_2 j_\ell(x) + j_{\Gamma_\ell}(x) = 0, \text{ for } x \in \Gamma_\ell$$

Sketch of the proof – I

Global variational formulation

For all $\phi = (\phi, \phi_i, \phi_1, \dots, \phi_L, \Phi) \in \mathcal{C}_c^\infty$ of $[0, T) \times \{\overline{\Omega}, \overline{\Omega_M}, (\overline{\Gamma_\ell})^{N_\ell}, \mathbb{R}^{N_n}\}$,

find $\mathbf{u} := (u, u_i, u_1, \dots, u_L, U)$ such that $\gamma_0 U = 0$ + transmission conditions on potentials, and J_s , verifying

$$\left\{ \begin{array}{l} \int_0^T \int_{\Omega_M} \sigma_i \nabla u_i \cdot \nabla \phi_i + \int_0^T \int_{\Omega} \sigma \nabla u \cdot \nabla \phi + \int_0^T \int_{\Omega_M} \left(c_m \frac{\partial [u]}{\partial t} [\phi] + f([u], h, t) [\phi] \right) \\ \quad + \int_0^T \left(c \frac{dA_c^\top U}{dt} \cdot A_c^\top \Phi + GA_g^\top U \cdot A_g^\top \Phi + A_s J_s \cdot \Phi \right) \\ \quad + \sum_{\ell=1}^L \int_0^T \int_{\Gamma_\ell} \left(c_\ell \frac{\partial A_{\ell,c}^\top u_\ell}{\partial t} \cdot A_{\ell,c}^\top \phi_\ell + G_\ell A_{\ell,g}^\top u_\ell \cdot A_{\ell,g}^\top \phi_\ell \right) = 0, \\ A_s^\top U = V_s \end{array} \right.$$

+ IPP \Rightarrow weak solution

Sketch of the proof – II

Global variational formulation semi-discretized in time

Find $\mathbf{u}^n := (u^n, u_i^n, u_1^n, \dots, u_L^n, U^n) \in \mathcal{H}_0$, and $J_s^n \in \mathbb{R}^{N_s}$ with

$$\mathcal{H} := H^1(\Omega_M) \times H^1(\Omega) \times \prod_{\ell=1}^L L^2(\Gamma_\ell)^{N_\ell} \times \mathbb{R}^{N_n},$$

$$\mathcal{H}_0 := \{\mathbf{u} \in \mathcal{H} \text{ s.t. } \boxed{\gamma_0 U = 0, \gamma_\ell U = \gamma_1 u_\ell, \gamma \mathbf{u} = \gamma_2 u_\ell}, \ell = 1, \dots, L\},$$

verifying, for all $\phi := (\phi, \phi_i, \phi_1, \dots, \phi_L, \Phi) \in \mathcal{H}_0$,

$$a\left(\frac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\Delta t}, \phi\right) + b(\mathbf{u}^n, \phi) + (f([u^n]), [\phi])_{\Omega_M} + A_s J_s^n \cdot \Phi = 0,$$

$$A_s^\top U^n = V_s^n$$

$$a(\delta \mathbf{u}, \phi) = (c_m [\delta u], [\phi])_{\Omega_M} + \sum_{\ell=1}^L \left(C_\ell A_{\ell,c}^\top \delta u_\ell, A_{\ell,c}^\top \phi_\ell \right)_{\Gamma_\ell} + C A_c^\top \delta U \cdot A_c^\top \Phi,$$

$$b(\mathbf{u}, \phi) = (\sigma_i \nabla u_i, \nabla \phi_i)_{\Omega_M} + (\sigma \nabla \mathbf{u}, \nabla \phi)_\Omega + \sum_{\ell=1}^L \left(G_\ell A_{\ell,g}^\top u_\ell, A_{\ell,g}^\top \phi_\ell \right)_{\Gamma_\ell} + G A_g^\top U \cdot A_g^\top \Phi.$$

Sketch of the proof – III

Existence

— Hypothesis

A_s has full column rank

— Consequence

$\ker(A_s) = \{0\}$ and $\text{Im}(A_s^T) = \ker(A_s)^\perp = \mathbb{R}^{N_s}$, $\Rightarrow \forall V_s^n \in \mathbb{R}^{N_s}$ there exists $U_0^n \in \mathbb{R}^{N_n}$ such as $A_s^T U_0^n = V_s^n$ and $\|\mathbf{u}_0^n\|_{\mathcal{H}} \leq \text{cst} \|V_s^n\|_{\mathbb{R}^{N_n}}$.

— Variational formulation in the constraint space

Find $\mathbf{u}_\star^n \in \mathcal{H}_0^{\ker}$, with $\mathbf{u}^n = \mathbf{u}_\star^n + \mathbf{u}_0^n$ and $\boxed{\mathcal{H}_0^{\ker} := \{\mathbf{u} \in \mathcal{H}_0 \mid U \in \ker(A_s^T)\}}$, verifying for all $\phi_\star \in \mathcal{H}_0^{\ker}$,

$$a\left(\frac{\mathbf{u}_\star^n + \mathbf{u}_0^n - \mathbf{u}^{n-1}}{\Delta t}, \phi_\star\right) - b(\mathbf{u}_\star^n + \mathbf{u}_0^n, \phi_\star) + (f([u^n]), [\phi])_{\Omega_M} = 0$$

Lax-Milgram and Banach fixed-point theorems: existence of \mathbf{u}_\star^n in \mathcal{H}_0^{\ker} .

— Resolution of J_s^n equation in \mathbb{R}^{N_s}

- $\forall \{e_k\}$ basis of \mathbb{R}^{N_s} , $\exists \{\phi_k\} \subset \mathcal{H}_0$ s.t. $e_k = A_s^T \Phi_k$, and J_s^n solves the linear systems $J_s^n \cdot e_k = w_k$,
- w_k are independent of the choice of $\{\phi_k\}$.

Sketch of the proof – IV

Compactness & passing to the limit

We have a sequence (\mathbf{u}_0^n) bounded by data, and (\mathbf{u}_\star^n) in $\mathcal{H}_0^{\text{ker}}$ with *a priori* estimates.

— *A priori* estimates on

$$a(\mathbf{u}^n, \mathbf{u}^n) \quad \sum_n \Delta t b(\mathbf{u}^n, \mathbf{u}^n) \quad \sum_n \Delta t a \left(\frac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\Delta t}, \frac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\Delta t} \right)$$

for compactness (as f is nonlinear)

— **Compactness:** discrete Aubin-Simon (Gallouët-Latché theorem)

\Rightarrow up to subsequence $\mathbf{u}^n \rightarrow \mathbf{u}$ (strong convergence)

— **Passing to the limit**

$\Rightarrow \mathbf{u}$ solution of the continuous problem.



SIM
CARDIO
TEST

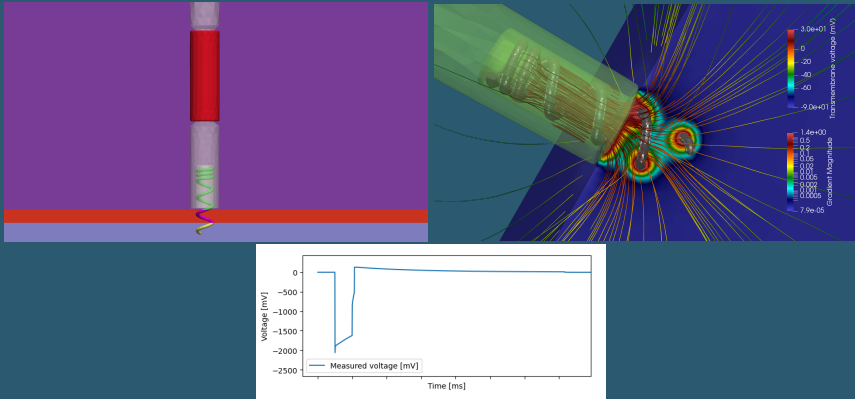


Figure: Pacemaker + 2 R//C contacts + geometry inspired of a standard VEGA lead