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Evelyne Hubert, Martin Jalard. Algebraically independent generators for the invariant field of $SO(3, \mathbb{R})$ and $O(3, \mathbb{R})$ representations $\mathbb{R}^3 \oplus \mathbb{H}$. 2026. <hal-04604969v4>

HAL Id: hal-04604969

<https://inria.hal.science/hal-04604969v4>

Preprint submitted on 30 Jan 2026

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Algebraically independent generators for the invariant field of $\mathrm{SO}_3(\mathbb{R})$ and $\mathrm{O}_3(\mathbb{R})$ representations $\mathbb{R}^3 \oplus \mathcal{H}$

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January 30, 2026

Abstract

We consider the representations of the group $\mathrm{SO}_3(\mathbb{R})$ that contain the standard representation \mathbb{R}^3 as an irreducible component. This is a large class of representations containing the vector spaces of ternary forms of any odd degree, the space of piezoelectricity tensors or the classical product space $\mathbb{R}^3 \times \dots \times \mathbb{R}^3$ which appears in multiple physical and engineering situations. For any such representation we construct a set of algebraically independent rational invariants that generates the invariant field.

Our key ingredient is a constructive use of Seshadri slice lemma. This lemma establishes an isomorphism between the $\mathrm{SO}_3(\mathbb{R})$ -invariant field on the full representation $\mathbb{R}^3 \oplus \mathcal{H}$ with the invariant field of a representation of $\mathrm{O}_2(\mathbb{R})$ on a subspace $\mathbb{R} \oplus \mathcal{H}$. We exhibit a set of $\mathrm{O}_2(\mathbb{R})$ -invariant polynomials generating this latter invariant field. The explicit expressions for the generating rational $\mathrm{SO}_3(\mathbb{R})$ -invariants can then be obtained with symbolic manipulations. The results are then extended to actions of $\mathrm{O}_3(\mathbb{R})$ and several cases of interest to applications are detailed as examples.

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Contents

1	Introduction	3
2	Invariants for a class of $O_2(\mathbb{R})$-representation	4
2.1	Representations of $O_2(\mathbb{C})$	4
2.2	Invariants for $SO_2(\mathbb{C})$	5
2.3	Invariants for $O_2(\mathbb{C})$ and $O_2(\mathbb{R})$	6
3	Invariants for $SO_3(\mathbb{R})$	7
3.1	Representations of $SO_3(\mathbb{R})$	8
3.2	Seshadri slice	10
3.3	Restrictions of the invariants to the slice	11
3.4	Evaluating the invariants	12
3.5	Inverse problem	14
4	Generalisation to $O_3(\mathbb{R})$ representations	16
5	Applications	18
5.1	Odd degree ternary forms	18
5.2	Piezoelectricity tensors	19
5.3	3D Moment Invariants	20
A	Rewriting an invariant in terms of the generators	23

1 Introduction

Invariants are essential for classifying objects up to a group of transformations. In this article we consider a large class of representations of $\mathrm{SO}_3(\mathbb{R})$ among which some are relevant to domains as continuum mechanics and 3D image analysis. We exhibit a set of algebraically independent rational invariants that generates the invariant field.

By a result of Rosenlicht [Ros56] [PV94, Theorem 2.3], for a rational action of an algebraic group, a generating set of the invariant field separates generic orbits. This is a clear advantage over the algebra of polynomial invariants for either linear actions of non compact groups [HL12, HL13, BDM⁺21] or rational actions of any algebraic group [HK07b, Hub12]. Even for compact groups, as $\mathrm{SO}_3(\mathbb{R})$, where there exist finite sets of polynomial invariants that separate all orbits, the trade off is that their cardinality is often much larger than a generating set of rational invariants.

A general and practical algorithm to compute a generating set of rational invariants is proposed in [HK07a, Hub19], [DK15, Section 4.10]. The construction is based on a choice of cross-section *a.k.a.* quasisection in [PV94]. The construction presented here exploits the existence of a *Seshadri slice*, *i.e.*, a subvariety \mathcal{S} of the representation space \mathcal{V} , with normalizer $\mathrm{N} < \mathrm{G}$, such that $\mathrm{G} \cdot \mathcal{S}$ is dense in \mathcal{V} . A result attributed to Seshadri [Ses62] indicates that the restriction of \mathcal{V} to \mathcal{S} induces an isomorphism between $\mathbb{R}(\mathcal{V})^{\mathrm{G}}$ and $\mathbb{R}(\mathcal{S})^{\mathrm{N}}$. Variations on this result also appeared in [CTS07, GV78, Nag65, Pop92, PV94]. Seshadri slice lemma is an essential tool for proving the rationality of invariant fields but previous constructive use of it also appeared in [GHP19, LRS20].

We consider the representation the group $\mathrm{SO}_3(\mathbb{R})$ on the space \mathcal{V} that contain the standard representation \mathbb{R}^3 as an irreducible component; we write $\mathcal{V} = \mathbb{R}^3 \oplus \mathcal{H}$. We identify a Seshadri slice $(\mathcal{S}, \mathrm{N})$ with $\mathcal{S} = \mathbb{R} \oplus \mathcal{H}$ and $\mathrm{N} \cong \mathrm{O}_2(\mathbb{R})$. Seshadri slice lemma establishes then the field isomorphism $\mathbb{R}(\mathcal{V})^{\mathrm{SO}_3(\mathbb{R})} \cong \mathbb{R}(\mathcal{S})^{\mathrm{O}_2(\mathbb{R})}$. We exhibit a set of $\dim(\mathcal{H})$ algebraically independent generators for $\mathbb{R}(\mathcal{S})^{\mathrm{O}_2(\mathbb{R})}$. They correspond unequivocally to algebraically independent generators of $\mathbb{R}(\mathcal{V})^{\mathrm{SO}_3(\mathbb{R})}$. Wigner matrices enable us to compute the explicit expressions of the latters, with respect to spherical coordinates. Their expressions in Cartesian coordinates can be obtained with Symbolic manipulations.

As a side product, we get a constructive proof for the rationality of the invariant fields of the representations considered. Nonetheless this could also be deduced from the rationality of the invariant field of any $\mathrm{SL}_2(\mathbb{C})$ -representations, proved in [Kat83, Kat84, BK85], thanks to the two-fold covering $\mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{SO}_3(\mathbb{C})$. Our approach allows to derive directly the rationality over \mathbb{R} , for both representations of $\mathrm{SO}_3(\mathbb{R})$ and $\mathrm{O}_3(\mathbb{R})$. Furthermore, in [HJ25b], the multihomogeneity of the invariants exhibited serve as the base case of an inductive proof of the rationality of the invariant fields for a class of representations of $\mathrm{SO}_n(\mathbb{R})$.

The class of representations of $\mathrm{SO}_3(\mathbb{R})$ containing \mathbb{R}^3 as an irreducible component is quite general, and we apply our method to various examples of interest in mathematics, mechanics and engineering. Firstly, the classical product space $\mathbb{R}^3 \times \dots \times \mathbb{R}^3$ for which generically separating invariants are in demand to compute invariant neural networks [DG24]. We also consider the spaces of homogeneous forms of odd degree¹, as for instance the celebrated ternary cubics. We next give generators for the action of $\mathrm{SO}_3(\mathbb{R})$ for the space of piezoelectricity tensors which stems from continuum mechanics. Other sets of separating invariants were provided in [Oli14, CMQZ25], but their cardinalities are drastically larger. Finally we provide invariants on the space of moments up to any order which are used in 3D image analysis [FSZ17]. Many such invariants were previously constructed and applied in diverse contexts but there was no firm foundations to their independence and generating properties.

The paper is organized as follows. In Section 2 we construct algebraically independent polynomial invariants generating the field of $\mathrm{O}_2(\mathbb{R})$ -invariant for representations relevant to our purpose. In Section 3 we show how to obtain explicit generating rational invariants for the corresponding $\mathrm{SO}_3(\mathbb{R})$ -representations. We next solve the inverse problem: given a set of values for the invariants, can we identify a real point where these

¹The invariants for the action on even degree forms were given in [GHP19].

values are obtained? In Section 4 we extend our result to the construction of $O_3(\mathbb{R})$ -invariants. In Section 5 we examine closely the mentioned cases of interest. Additionally, in Appendix A, we explain how to rewrite any other $SO_3(\mathbb{R})$ -invariant on $\mathbb{R}^3 \oplus \mathcal{H}$ in terms of our generators.

2 Invariants for a class of $O_2(\mathbb{R})$ -representation

The goal of this article is to construct algebraically independent generators for the invariant field of a representation of $SO_3(\mathbb{R})$. For the representations we shall consider these fields are isomorphic to $O_2(\mathbb{R})$ -invariant fields. As a preliminary we thus consider the related representations of $O_2(\mathbb{R})$ for which we make explicit sets of algebraically independent generators for their fields of invariants (Theorem 2.6).

2.1 Representations of $O_2(\mathbb{C})$

The orthogonal group $O_2(\mathbb{R})$ is often thought of as the one parameter family of linear transformations in the plane given by:

$$O_2(\mathbb{R}) := \left\{ \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}, \alpha \in \mathbb{R} \right\} \cup \left\{ \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ -\sin(\alpha) & -\cos(\alpha) \end{pmatrix}, \alpha \in \mathbb{R} \right\}.$$

If (x_{-1}, x_1) are the coordinates for the plane \mathbb{R}^2 , considering the coordinates

$$\begin{pmatrix} y_{-1} \\ y_1 \end{pmatrix} = P \begin{pmatrix} x_{-1} \\ x_1 \end{pmatrix} \text{ where } P := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

of the \mathbb{R} -vector space $\mathcal{V} = \{(y_{-1}, y_1) \in \mathbb{C}^2 \mid \bar{y}_1 = y_{-1}\}$ we obtain the parametrization

$$O_2(\mathbb{R}) \cong \left\{ \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}, \alpha \in \mathbb{R} \right\} \cup \left\{ \begin{pmatrix} 0 & e^{-i\alpha} \\ e^{i\alpha} & 0 \end{pmatrix}, \alpha \in \mathbb{R} \right\}.$$

Concomitantly we have

$$O_2(\mathbb{C}) \cong \left\{ g_z^+ = \begin{pmatrix} z & 0 \\ 0 & \frac{1}{z} \end{pmatrix}, z \in \mathbb{C}^* \right\} \cup \left\{ g_z^- = \begin{pmatrix} 0 & \frac{1}{z} \\ z & 0 \end{pmatrix}, z \in \mathbb{C}^* \right\}.$$

Proposition 2.1. [GSS88] *The irreducible representations of $O_2(\mathbb{C})$ are indexed by $j \in \mathbb{Z}$ with $j \geq -1$. They consist of representations $\phi_j : O_2(\mathbb{C}) \mapsto \text{GL}(\hat{\mathcal{V}}_j)$ where*

- for $j \in \{0, -1\}$, $\hat{\mathcal{V}}_j \cong \mathbb{C}$ is one dimensional and

$$\phi_0(g_z^+) = (1), \phi_0(g_z^-) = (1); \quad \phi_{-1}(g_z^+) = (1), \phi_{-1}(g_z^-) = (-1);$$

- for $j \geq 1$, $\hat{\mathcal{V}}_j \cong \mathbb{C}^2$ is two dimensional and

$$\phi_j(g_z^+) = \begin{pmatrix} z^j & 0 \\ 0 & z^{-j} \end{pmatrix} \quad \text{and} \quad \phi_j(g_z^-) = \begin{pmatrix} 0 & z^{-j} \\ z^j & 0 \end{pmatrix}$$

The real irreducible representations of $O_2(\mathbb{R})$ are obtained by restricting the above ones to the real vector space $\mathcal{V}_j = \{(y_{-1}, y_1) \in \mathbb{C}^2 \mid \bar{y}_1 = y_{-1}\}$ and $\{z \in \mathbb{C}^* \mid z\bar{z} = 1\}$.

2.2 Invariants for $\mathrm{SO}_2(\mathbb{C})$

Let p, n, m_1, \dots, m_r be positive integers with $1 \leq p \leq n$ and $m_1 = 1$. We consider the representation

$$\rho : \mathrm{O}_2(\mathbb{C}) \longrightarrow \mathrm{GL}(\hat{\mathcal{V}}) \quad \text{where } \hat{\mathcal{V}} = \bigoplus_{j=1}^p \hat{\mathcal{V}}_{-1} \oplus \bigoplus_{j=p+1}^n \hat{\mathcal{V}}_0 \oplus \bigoplus_{j=1}^r \hat{\mathcal{V}}_{m_j}.$$

We endow $\hat{\mathcal{V}}$ with coordinates $(v_1, \dots, v_n, y_{-1}, y_1, \dots, y_{-r}, y_r)$. We first proceed to give algebraically independent generators for the invariant field of the normal subgroup $\mathrm{SO}_2(\mathbb{C})$ and then deduce the invariants generating the invariant field of the group $\mathrm{O}_2(\mathbb{C})$.

Proposition 2.2. *The following set of polynomials is algebraically independent and generates the invariant field $\mathbb{C}(\hat{\mathcal{V}})^{\mathrm{SO}_2(\mathbb{C})}$.*

$$\mathfrak{e} := \left\{ \begin{array}{ll} \mathfrak{t}_j & = v_j & 1 \leq j \leq n \\ \mathfrak{p}_j & = y_j y_{-1}^{m_j}, & 1 \leq j \leq r \\ \mathfrak{q}_j & = y_{-j} y_1^{m_j}, & 2 \leq j \leq r \end{array} \right\}.$$

Note that the omitted polynomial \mathfrak{q}_1 would be equal to \mathfrak{p}_1 . The proof of Proposition 2.2 rests on the two following lemmas which show that polynomials in \mathfrak{e} are invariant (Lemma 2.3) and that they separate generic orbits (Lemma 2.4). By a theorem due to Rosenlicht [Ros56, Section 1 Theorem 2], [PV94, Lemma 2.1], they generate $\mathbb{C}(\hat{\mathcal{V}})^{\mathrm{SO}_2(\mathbb{C})}$.

Lemma 2.3. *Each polynomial in \mathfrak{e} is invariant for $\mathrm{SO}_2(\mathbb{C})$.*

Proof. Take $z \in \mathbb{C}^*$ to identify some $g_z^+ \in \mathrm{SO}_2(\mathbb{C})$, and $v = (v_1, \dots, v_n, y_{-1}, y_1, \dots, y_{-r}, y_r) \in \hat{\mathcal{V}}$.

- For all $1 \leq j \leq p$, $\mathfrak{t}_j(\rho(g_z^+)(v)) = \det(g_z^+)v_j = v_j = \mathfrak{t}_j(v)$
- For all $p+1 \leq j \leq n$, $\mathfrak{t}_j(\rho(g_z^+)(v)) = v_j = \mathfrak{t}_j(v)$
- For all $1 \leq j \leq r$, $\mathfrak{p}_j(\rho(g_z^+)(v)) = z^{m_j} y_j z^{-m_j} y_{-1}^{m_j} = \mathfrak{p}_j(v)$
- For all $2 \leq j \leq r$, $\mathfrak{q}_j(\rho(g_z^+)(v)) = z^{m_j} y_1^{m_j} z^{-m_j} y_{-m_j} = \mathfrak{q}_j(v)$. □

Lemma 2.4. *Consider the set $\mathcal{U} = \{v \in \hat{\mathcal{V}} \mid y_1 y_{-1} \neq 0\}$. For any $v, v' \in \mathcal{U}$, we have the following separation property:*

$$\mathfrak{f}(v) = \mathfrak{f}(v'), \forall \mathfrak{f} \in \mathfrak{e} \Rightarrow \exists g \in \mathrm{SO}_2(\mathbb{C}), \rho(g)(v') = v.$$

Proof. Take $v = (v_1, \dots, v_n, y_{-1}, y_1, \dots, y_{-r}, y_r)$ and $v' = (v'_1, \dots, v'_n, y'_{-1}, y'_1, \dots, y'_{-r}, y'_r)$ in \mathcal{U} and define $z := \frac{y_1}{y'_1} \in \mathbb{C}^*$.

- For $1 \leq j \leq n$, $\mathfrak{t}_j(v) = \mathfrak{t}_j(v') \Rightarrow v_j = v'_j$.
- We have $\phi_1(g_z^+)(y_{-1}, y_1) = (y'_{-1}, y'_1)$ since

$$\begin{aligned} \mathfrak{p}_1(v) = \mathfrak{p}_1(v') &\Rightarrow \mathfrak{p}_1(g_z^+(y)) = \mathfrak{p}_1(v') \\ &\Rightarrow z z^{-1} y_1 y_{-1} = y'_1 y'_{-1} \\ &\Rightarrow z y'_1 y_{-1} = y'_1 y'_{-1} \\ &\Rightarrow z y_{-1} = y'_{-1}. \end{aligned}$$

- Furthermore for $2 \leq j \leq r$ we have:

$$\begin{aligned} \mathfrak{p}_j(v) = \mathfrak{p}_j(v') &\Rightarrow \mathfrak{p}_j(g_z^+(v)) = \mathfrak{p}_j(v') \\ &\Rightarrow z^{-m_j} y_j (z y_{-1})^{m_j} = y'_j (y'_{-1})^{m_j} \\ &\Rightarrow z^{-m_j} y_j (y'_{-1})^{m_j} = y'_j (y'_{-1})^{m_j} \\ &\Rightarrow z^{-m_j} y_j = y'_j \end{aligned}$$

Similarly, $\mathfrak{q}_j(v) = \mathfrak{q}_j(v') \Rightarrow z^{m_j} y_{-j} = y'_{-j}$. Hence, $\phi_{m_j}(g_z^+)(y_{-j}, y_j) = (y'_{-j}, y'_j)$.

We showed that if all polynomials in \mathfrak{e} coincide on v and v' , $\rho(g_z^+)(v) = v'$, i.e., v and v' lie on the same orbit. \square

2.3 Invariants for $O_2(\mathbb{C})$ and $O_2(\mathbb{R})$

Since $O_2(\mathbb{C}) = SO_2(\mathbb{C}) \rtimes C_2$, the normal subgroup C_2 acts naturally on $\mathbb{C}(\hat{\mathcal{V}})^{SO_2(\mathbb{C})} = \mathbb{C}(\mathfrak{e})$ and

$$\mathbb{C}(\hat{\mathcal{V}})^{O_2(\mathbb{C})} = \left(\mathbb{C}(\hat{\mathcal{V}})^{SO_2(\mathbb{C})} \right)^{C_2} = \mathbb{C}(\mathfrak{e})^{C_2}.$$

The action of C_2 on \mathfrak{e} consists of permutations and sign changes. It thus amounts to a linear action of C_2 on $\mathcal{W} := \text{Vect}(f \mid f \in \mathfrak{e})$. We can thus deduce from Proposition 2.2 a set of algebraically independent invariants generating $\mathbb{C}(\hat{\mathcal{V}})^{O_2(\mathbb{C})}$ thanks to the following lemma.

Lemma 2.5. *For a fixed $k \in \{1, \dots, n\}$, consider $(\mathfrak{r}, \mathcal{W})$ the representation of $C_2 = \{1, -1\}$ on $\mathcal{W} = \mathbb{C}^n$ where*

$$\mathfrak{r}(-1) = \begin{pmatrix} -I_k & 0 \\ 0 & I_{n-k} \end{pmatrix}.$$

If \mathcal{W} is endowed with coordinates $(w_1, \dots, w_k, w_{k+1}, \dots, w_n)$ then $\{w_1 w_i, 1 \leq i \leq k\} \cup \{w_j, k+1 \leq j \leq n\}$ is an algebraically independent set of polynomials generating $\mathbb{C}(\mathcal{W})^{C_2}$.

One can indeed check that this set of invariants separates orbits on $\{w \in \mathcal{W} \mid w_1 \neq 0\}$.

Theorem 2.6. *The following set of algebraically independent polynomials generates the fields $\mathbb{C}(\hat{\mathcal{V}})^{O_2(\mathbb{C})}$ and $\mathbb{R}(\mathcal{V})^{O_2(\mathbb{R})}$:*

$$\mathcal{E} = \left\{ \begin{array}{lll} D_j = & v_j v_1, & 1 \leq j \leq p \\ T_j = & v_j, & p+1 \leq j \leq n \\ R_j = & y_j y_{-1}^{m_j} + y_{-j} y_1^{m_j}, & 1 \leq j \leq r \\ I_j = & i v_1 (y_j y_{-1}^{m_j} - y_{-j} y_1^{m_j}), & 2 \leq j \leq r \end{array} \right\}.$$

Proof. The natural action of C_2 on $\mathbb{C}(\hat{\mathcal{V}})^{SO_2(\mathbb{C})}$ is given by

$$\underbrace{[g]}_{\in C_2} \cdot \underbrace{P}_{\in \mathbb{C}(\hat{\mathcal{V}})^{SO_2(\mathbb{C})}} := P \circ g.$$

Take $v = (v_1, \dots, v_n, y_{-1}, y_1, \dots, y_{-r}, y_r)$ a point in $\hat{\mathcal{V}}$.

- For $1 \leq j \leq p$, $(g_1^- \cdot \mathfrak{t}_j)(v) = \mathfrak{t}_j(\rho(g_1^-) v) = \det(g_1^-) v_j = -\mathfrak{t}_j(v)$;
- For $p+1 \leq j \leq n$, $(g_1^- \cdot \mathfrak{t}_j)(v) = \mathfrak{t}_j(\rho(g_1^-) v) = v_j = \mathfrak{t}_j(v)$;
- For $1 \leq j \leq r$, $(g_1^- \cdot \mathfrak{p}_j)(v) = \mathfrak{p}_j(\rho(g_1^-) v) = y_{-j} y_1^{m_j} = \mathfrak{q}_j(v)$
and $(g_1^- \cdot \mathfrak{q}_j)(v) = \mathfrak{q}_j(\rho(g_1^-) v) = y_j y_{-1}^{m_j} = \mathfrak{p}_j(v)$.

We notice that the linear space $\mathcal{W} := \text{Vect}(\mathfrak{e})$ is stable by the action of C_2 . Furthermore, if we consider the basis of \mathcal{W} given by

$$\left\{ \begin{array}{ll} \mathfrak{t}_j, & 1 \leq j \leq n \\ \mathfrak{p}_j + \mathfrak{q}_j, & 1 \leq j \leq r \\ \mathfrak{p}_j - \mathfrak{q}_j, & 2 \leq j \leq r \end{array} \right\},$$

then the matrix corresponding to the action of g_1^- is

$$\begin{pmatrix} -I_p & 0 & 0 & 0 \\ 0 & I_{n-p} & 0 & 0 \\ 0 & 0 & I_r & 0 \\ 0 & 0 & 0 & -I_{r-1} \end{pmatrix}.$$

By Lemma 2.5, the following is a generating set for $\mathbb{C}(\mathcal{W})_2^{\mathbb{C}}$:

$$\mathcal{E} = \left\{ \begin{array}{l} D_j = \mathfrak{t}_1 \mathfrak{t}_j, \quad 1 \leq j \leq p \\ T_j = \mathfrak{t}_j, \quad p+1 \leq j \leq n \\ R_j = \mathfrak{p}_j + \mathfrak{q}_j, \quad 1 \leq j \leq r \\ I_j = i \mathfrak{t}_1 (\mathfrak{p}_j - \mathfrak{q}_j), \quad 2 \leq j \leq r \end{array} \right\}.$$

Note that $\#\mathcal{E} = n + 2r - 1 = \dim(\hat{\mathcal{V}}) - \dim(\mathrm{O}_2(\mathbb{C}))$. However, the transcendence degree of the field $\mathbb{C}(\hat{\mathcal{V}})^{\mathrm{O}_2(\mathbb{C})}$ is equal to $\dim(\hat{\mathcal{V}}) - \max_{v \in \hat{\mathcal{V}}}(\dim(\mathrm{O}_2(\mathbb{C}) \cdot v))$ [PV94, Section 2.3]. Hence $\#\mathcal{E} \leq \mathrm{tr. deg}(\mathbb{C}(\hat{\mathcal{V}})^{\mathrm{O}_2(\mathbb{C})})$ while generating this field. Thus \mathcal{E} is algebraically independent.

The coefficient i in I_j is added to obtain real polynomials. Indeed, recall that on the real trace

$$\mathcal{V} = \left\{ v \in \hat{\mathcal{V}} \mid v_1, \dots, v_n \in \mathbb{R}; \bar{y}_i = y_{-i}, 1 \leq i \leq n \right\}$$

we have $\mathfrak{p}_j = y_j y_{-1}^{m_j} = \bar{y}_{-j} \bar{y}_1^{m_j} = \bar{\mathfrak{q}}_j$. Hence, $\mathfrak{p}_j + \mathfrak{q}_j = 2\Re(\mathfrak{p}_j)$ and $\mathfrak{p}_j - \mathfrak{q}_j = 2i\Im(\mathfrak{p}_j)$. Thus the restrictions of the polynomials of \mathcal{E} to \mathcal{V} are real. Since $\mathrm{O}_2(\mathbb{R})$ is Zariski dense in $\mathrm{O}_2(\mathbb{C})$, they also generate $\mathbb{R}(\mathcal{V})^{\mathrm{O}_2(\mathbb{R})}$ over \mathbb{R} . \square

The polynomials of \mathcal{E} can be translated into real coordinates through the change of basis P^{-1} defined in Section 2.1. It involves an expansion according to Newton binomial formula which makes the complex i disappear. The expressions become rather involved though, as shown in the following example. This explains why we mostly work with complex coordinates throughout the article.

Example 2.7. *For the 8-dimensional linear representation $S_{\mathcal{T}} = 2\mathcal{V}_{-1} \oplus \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3$, an algebraically independent generating set of $\mathbb{R}(S_{\mathcal{T}})^{\mathrm{O}_2(\mathbb{R})}$ is*

$$\mathcal{E}_{\mathcal{T}} = \left\{ \begin{array}{l} D_1 = v_1^2 \\ D_2 = v_1 v_2 \\ R_1 = 2y_1 y_{-1} \\ R_2 = y_1^2 y_{-2} + y_{-1}^2 y_2 \\ R_3 = y_1^3 y_{-3} + y_{-1}^3 y_3 \\ I_2 = i d_1 (y_1^2 y_{-2} - y_{-1}^2 y_2) \\ I_3 = i d_1 (y_1^3 y_{-3} - y_{-1}^3 y_3) \end{array} \right\} = \left\{ \begin{array}{l} D_1 = v_1^2 \\ D_2 = v_1 v_2 \\ R_1 = x_1^2 + x_{-1}^2 \\ R_2 = \frac{\sqrt{2}}{2} (2x_{-2} x_{-1} x_1 - x_{-1}^2 x_2 + x_1^2 x_2) \\ R_3 = -\frac{1}{2} x_{-3} x_{-1}^3 + \frac{3}{2} x_{-3} x_{-1} x_1^2 - \frac{3}{2} x_{-1}^2 x_1 x_3 + \frac{1}{2} x_1^3 x_3 \\ I_2 = \frac{\sqrt{2}}{2} d_1 (x_{-2} x_{-1}^2 - x_{-2} x_1^2 + 2x_{-1} x_1 x_2) \\ I_3 = -\frac{1}{2} d_1 (3x_{-3} x_{-1}^2 x_1 - x_{-3} x_1^3 - x_{-1}^3 x_3 + 3x_{-1} x_1^2 x_3) \end{array} \right\}$$

3 Invariants for $\mathrm{SO}_3(\mathbb{R})$

We now consider representations of $\mathrm{SO}_3(\mathbb{R})$ on

$$\mathcal{V} = \mathbb{R}^3 \oplus \mathcal{H},$$

where \mathcal{H} is an arbitrary representation of $\mathrm{SO}_3(\mathbb{R})$. We introduce Seshadri slice Lemma and identify a slice $(\mathcal{S}, \mathcal{N})$ where $\mathcal{S} = \mathbb{R} \oplus \mathcal{H}$ and $\mathcal{N} = \mathrm{O}_2(\mathbb{R})$. From the generating set of $\mathbb{R}(\mathcal{S})^{\mathcal{N}}$ computed in the previous section, we derive an algebraically independent generating set for $\mathbb{R}(\mathcal{V})^{\mathrm{SO}_3(\mathbb{R})}$. We show how to evaluate these invariants thanks to Wigner matrices. We shall furthermore solve the inverse problem: given values for these invariants provide, firstly, conditions for the existence of a real orbit where the invariants have these values and, secondly, an element in this orbit if it exists.

3.1 Representations of $\text{SO}_3(\mathbb{R})$

We first classify the irreducible representations of $\text{SO}_3(\mathbb{R})$. They are given in terms of spaces of harmonic polynomials of degree d . We next introduce the bases of *spherical harmonics*, which allow to decompose the spaces of harmonic polynomials into irreducible representations of the subgroup $\text{O}_2(\mathbb{R})$.

Definition 3.1. *The vector space of degree d harmonic homogeneous polynomials on \mathbb{R}^3 is*

$$\mathcal{H}_d := \left\{ P \in \mathbb{R}[U, V, W]_d \mid \frac{\partial^2 P}{\partial U^2} + \frac{\partial^2 P}{\partial V^2} + \frac{\partial^2 P}{\partial W^2} = 0 \right\}.$$

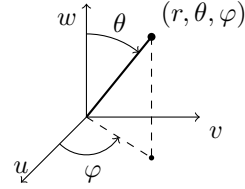
The standard action of $\text{SO}_3(\mathbb{R})$ on \mathbb{R}^3 induces, by composition, a linear action on the polynomial ring:

$$\forall P \in \mathbb{R}[U, V, W], g \in \text{SO}_3(\mathbb{R}), g \cdot P = P \circ g^{-1}.$$

This action stabilises the vector spaces \mathcal{H}_d . It thus induces the representations $\rho_d : \text{SO}_3(\mathbb{R}) \rightarrow \text{GL}(\mathcal{H}_d)$ which are the only irreducible representations of $\text{SO}_3(\mathbb{R})$ [GSS88]. Among them, (ρ_0, \mathcal{H}_0) is the trivial representation and \mathcal{H}_1 is equivalent to the standard representation on \mathbb{R}^3 . The class of representations we consider are thus all the representations containing (\mathcal{H}_1, ρ_1) .

The *spherical harmonics* form bases of the spaces \mathcal{H}_d . They are homogeneous polynomials in $\mathbb{R}[U, V, W]_d$ that decompose the spaces \mathcal{H}_d into irreducible representations of $\text{O}_2(\mathbb{R})$. Their expressions are nonetheless better known in spherical coordinates than in Cartesian coordinates. We thus introduce spherical coordinates (r, θ, φ) in \mathcal{H}_1 :

$$\begin{aligned} u &= r \sin(\theta) \cos(\varphi) \\ v &= r \sin(\theta) \sin(\varphi) \\ w &= r \cos(\theta) \end{aligned}$$



Theorem 3.2. [Wig59, 15.3] For $0 \leq m \leq d$, note $c_{m,d} = \sqrt{\frac{(2d+1)(d-m)!}{4\pi(d+m)!}}$ a normalisation constant and $P_d^m = (1-t^2)^{\frac{m}{2}} \frac{1}{2^d d!} \frac{\partial^{d+m}}{\partial t^{d+m}} (t^2-1)^d$ the $(d, m)^{\text{th}}$ associated Legendre polynomial. A basis for \mathcal{H}_d is given by $\{Y_d^{-d}, \dots, Y_d^d\} \subset \mathbb{C}[U, V, W]_d$ where

$$\begin{aligned} Y_d^{-m}(r \sin(\theta) \cos(\varphi), r \sin(\theta) \sin(\varphi), r \cos(\theta)) &= i^{-d} c_{m,d} r^d e^{-im\varphi} P_{m,d}(\cos(\theta)), \text{ for } 1 \leq m \leq d, \\ Y_d^0(r \sin(\theta) \cos(\varphi), r \sin(\theta) \sin(\varphi), r \cos(\theta)) &= c_d^0 r^d P_d^0(\cos(\theta)), \\ Y_d^m(r \sin(\theta) \cos(\varphi), r \sin(\theta) \sin(\varphi), r \cos(\theta)) &= i^d c_{m,d} r^d e^{im\varphi} P_{m,d}(\cos(\theta)), \text{ for } 1 \leq m \leq d. \end{aligned}$$

We slightly modified the basis of [Wig59] to fit better the irreducible representations of $\text{O}_2(\mathbb{R})$ defined in Section 2.1. More precisely, we multiplied each Y_d^m by $i^d(-1)^m$ and each Y_d^{-m} by i^{-d} . With the change of basis P introduced in Section 2.1 we obtain the real spherical harmonics:

$$X_d^{-m} = \frac{1}{\sqrt{2}}(Y_d^{-m} + Y_d^m), \quad X_d^0 = Y_d^0, \quad X_d^m = \frac{i}{\sqrt{2}}(Y_d^{-m} - Y_d^m)$$

so that

$$\begin{aligned} X_d^{-m}(r \sin(\theta) \cos(\varphi), r \sin(\theta) \sin(\varphi), r \cos(\theta)) &= \sqrt{2} c_{m,d} r^d \cos\left(m\varphi + \frac{d\pi}{2}\right) P_{m,d}(\cos(\theta)), \\ X_d^0(r \sin(\theta) \cos(\varphi), r \sin(\theta) \sin(\varphi), r \cos(\theta)) &= c_d^0 r^d P_d^0(\cos(\theta)), \\ X_d^m(r \sin(\theta) \cos(\varphi), r \sin(\theta) \sin(\varphi), r \cos(\theta)) &= \sqrt{2} c_{m,d} r^d \sin\left(m\varphi + \frac{d\pi}{2}\right) P_{m,d}(\cos(\theta)). \end{aligned}$$

Example 3.3. The bases of real spherical harmonics for \mathcal{H}_1 , \mathcal{H}_2 and \mathcal{H}_3 are:

$$\mathcal{B}_1 = \left\{ X_1^{-1} = -\sqrt{\frac{3}{4\pi}} V, X_1^0 = \sqrt{\frac{3}{4\pi}} W, X_1^1 = \sqrt{\frac{3}{4\pi}} U \right\},$$

$$\mathcal{B}_2 = \left\{ \begin{array}{l} X_2^{-2} = \frac{\sqrt{15}}{4\sqrt{\pi}} (V^2 - U^2), \quad X_2^{-1} = -\frac{\sqrt{15}}{2\sqrt{\pi}} UW, \quad X_2^0 = \frac{\sqrt{5}}{4\sqrt{\pi}} (2W^2 - U^2 - V^2), \\ X_2^1 = -\frac{\sqrt{15}}{2\sqrt{\pi}} VW, \quad X_2^2 = -\frac{\sqrt{15}}{2\sqrt{\pi}} UV \end{array} \right\},$$

$$\mathcal{B}_3 = \left\{ \begin{array}{l} X_3^{-3} = \frac{\sqrt{70}}{8\sqrt{\pi}} (3U^2 - V^2) V, \quad X_3^{-2} = \frac{\sqrt{105}}{2\sqrt{\pi}} UVW, \quad X_3^{-1} = \frac{\sqrt{42}}{8\sqrt{\pi}} (4W^2 - U^2 - V^2) V, \\ X_3^0 = \frac{\sqrt{7}}{4\sqrt{\pi}} (2W^2 - 3U^2 - 3V^2) W, \\ X_3^1 = \frac{\sqrt{42}}{8\sqrt{\pi}} (U^2 + V^2 - 4W^2) U, \quad X_3^2 = \frac{\sqrt{105}}{4\sqrt{\pi}} (V^2 - U^2) W, \quad X_3^3 = \frac{\sqrt{70}}{8\sqrt{\pi}} U (3V^2 - U^2) \end{array} \right\}.$$

With respect to these bases, we put $(y_{-d_j,j}, y_{-d_j+1,j}, \dots, y_{d_j-1,j}, y_{d_j,j})$ coordinates in $\hat{\mathcal{H}}_{d_j}$, and $(x_{-d_j,j}, \dots, x_{d_j,j})$ on \mathcal{H}_{d_j} . Spherical harmonics allow to decompose the spaces of harmonic polynomials into irreducible representations of the subgroup $O_2(\mathbb{R})$, or $O_2(\mathbb{C})$. To show it, we introduce Euler angles:

Definition 3.4. A rotation $g \in SO_3(\mathbb{R})$ is determined by Euler angles (α, β, γ) where $\alpha \in [0, 2\pi[$, $\beta \in [0, \pi]$ and $\gamma \in [0, 2\pi[$. It is the composition of rotations of axes w, v, w and angles (α, β, γ) :

$$g(\alpha, \beta, \gamma) := \begin{pmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{pmatrix} \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Proposition 3.5. Let d_1, \dots, d_n be $n \geq 1$ non negative integers, sorted so that the first p are odd, and the others are even. The representation of $O_2(\mathbb{R})$ as a subgroup of $SO_3(\mathbb{R})$ on $\bigoplus_{j=1}^n \mathcal{H}_{d_j}$ is isomorphic to

$$\bigoplus_{j=1}^p \mathcal{V}_{-1} \oplus \bigoplus_{j=p+1}^n \mathcal{V}_0 \oplus \bigoplus_{j=1}^n \bigoplus_{m=1}^{d_j} \mathcal{V}_m.$$

Proof. Consider the embedding $O_2(\mathbb{R}) \hookrightarrow SO_3(\mathbb{R})$ given by

$$\forall \alpha \in \mathbb{R}, \begin{cases} g_{e^{i\alpha}}^+ & \rightarrow g(\alpha, 0, 0) \\ g_{e^{i\alpha}}^- & \rightarrow g(\alpha, \pi, 0) \end{cases}$$

It normalises the W -axis and its action on spherical coordinates (r, θ, φ) is

$$\begin{cases} g_{e^{i\alpha}}^+(r, \theta, \varphi) & = (r, \theta, \varphi + \alpha). \\ g_{e^{i\alpha}}^-(r, \theta, \varphi) & = (r, \pi - \theta, \pi - \varphi - \alpha). \end{cases}$$

We compute the induced action on Y_d^m for $d \geq 1$ and $1 \leq m \leq d$. For all $(r, \theta, \varphi) \in \mathbb{R}^3$, and α real,

$$\begin{aligned} \rho_d(g_{e^{i\alpha}}^+) (Y_d^m(r, \theta, \varphi)) &= Y_d^m(g_{e^{-i\alpha}}^+(r, \theta, \varphi)) \\ &= Y_d^m(r, \theta, \varphi - \alpha) \\ &= i^d c_{m,d} r^d e^{im(\varphi - \alpha)} P_{m,d}(\cos(\theta)) \\ &= e^{-im\alpha} Y_d^m(r, \theta, \varphi) \end{aligned}$$

$$\begin{aligned} \rho_d(g_{e^{i\alpha}}^-) (Y_d^m(r, \theta, \varphi)) &= Y_d^m(g_{e^{i\alpha}}^-(r, \theta, \varphi)) \\ &= Y_d^m(r, \pi - \theta, \pi - \varphi - \alpha) \\ &= i^d c_{m,d} r^d e^{im(\pi - \varphi - \alpha)} P_{m,d}(-\cos(\theta)). \end{aligned}$$

But for any $1 \leq m \leq d$ we have $P_{m,d}(-t) = (-1)^{d+m} P_{m,d}(t)$. We thus obtain

$$\begin{aligned} \rho_d(g_{e^{-i\alpha}})(Y_d^m(r, \theta, \varphi)) &= i^d c_{m,d} r^d (-1)^m e^{im(-\varphi-\alpha)} (-1)^{d+m} P_{m,d}(\cos(\theta)) \\ &= i^{-d} c_{m,d} r^d e^{im(-\varphi-\alpha)} P_{m,d}(\cos(\theta)) \\ &= e^{-im\alpha} Y_d^{-m}(r, \theta, \varphi). \end{aligned}$$

Similarly we check that

$$\begin{cases} \rho_d(g_{e^{i\alpha}})(Y_d^{-m}) = e^{im\alpha} Y_d^{-m}, \\ \rho_d(g_{e^{-i\alpha}})(Y_d^{-m}) = e^{im\alpha} Y_d^m. \end{cases}$$

Thus, the representation of $O_2(\mathbb{C})$ on $\text{Vect}(Y_d^{-m}, Y_d^m)$ is isomorphic to $\hat{\mathcal{V}}_m$. Meanwhile for $m = 0$, the corresponding action on $\text{Vect}(Y_d^0)$ is ϕ_{-1} when d is odd and ϕ_0 when d is even. \square

3.2 Seshadri slice

In this section we introduce Seshadri slice lemma and identify a Seshadri slice $(\mathcal{S}, \mathcal{N})$ for the representation of $SO_3(\mathbb{R})$ on $\mathcal{V} = \mathbb{R}^3 \oplus \mathcal{H}$. This establishes an isomorphism between $\mathbb{R}(\mathcal{V})^{SO_3(\mathbb{R})}$ and $\mathbb{R}(\mathcal{S})^{\mathcal{N}}$ that will allow us to compute generating invariants for $\mathbb{R}(\mathcal{V})^{SO_3(\mathbb{R})}$. For \mathcal{A} a real algebraic variety, we note $\hat{\mathcal{A}}$ its complexification, *i.e.*, its Zariski closure over \mathbb{C} .

Theorem 3.6. *Seshadri slice lemma [CTS07, Theorem 3.1]. Let G be a real algebraic group acting on a vector space \mathcal{X} . Let $\mathcal{S} \subset \mathcal{X}$ be a subvector space and $\mathcal{N} < G$ the normalizer of \mathcal{S} in G . Assume that:*

- *The closure of $G \cdot \mathcal{S}$ is the whole of \mathcal{X} .*
- *There is an non empty \mathcal{N} -stable Zariski open subset $\mathcal{Z} \subset \mathcal{S}$ such that for all $g \in \hat{G}$ and $s \in \hat{\mathcal{Z}}$ satisfying $g \cdot s \in \hat{\mathcal{Z}}$, there exists $h \in \hat{\mathcal{N}}$ such that $h \cdot s = g \cdot s$.*

Then, the restriction to \mathcal{Z} of rational functions on \mathcal{X} induces a field isomorphism $\mathbb{R}(\mathcal{X})^G \cong \mathbb{R}(\mathcal{S})^{\mathcal{N}}$.

Definition 3.7. *Such a pair $(\mathcal{S}, \mathcal{N})$ is called a Seshadri slice for $G \curvearrowright \mathcal{X}$.*

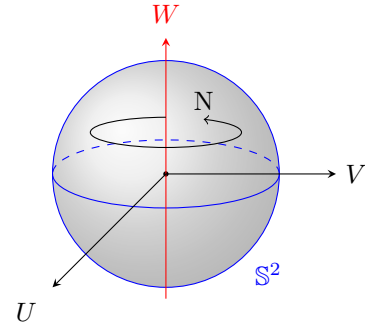
For $P \in \mathbb{R}(\mathcal{S})^{\mathcal{N}}$, we shall write \tilde{P} the unique rational invariant in $\mathbb{R}(\mathcal{X})^G$ such that $P(s) = \tilde{P}(s)$ for generic $s \in \mathcal{S}$. For some set of invariants $\mathcal{E} \subset \mathbb{R}(\mathcal{S})^{\mathcal{N}}$, we note $\tilde{\mathcal{E}} = \{\tilde{P}, P \in \mathcal{E}\} \subset \mathbb{R}(\mathcal{V})^G$.

Proposition 3.8. *Consider the subspace*

$$\mathcal{S} := \mathbb{R}W \oplus \mathcal{H} \subset \mathcal{V}$$

Then, $(\mathcal{S}, \mathcal{N})$ is a Seshadri slice with normalizer

$$\mathcal{N} = \left\{ \left(\begin{array}{c|c} g & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline 0 & \det(g) \end{array} \right), g \in O_2(\mathbb{R}) \right\} \subset SO_3(\mathbb{R}).$$



The normalizer $\mathcal{N} \cong O_2(\mathbb{R})$ of \mathcal{S} consists of the rotations in \mathbb{R}^3 that leave the W axis invariant. That includes the rotations around the W axis and the rotations by π around any axis in the (U, V) plane. The open subset \mathcal{Z} in \mathcal{S} that is relevant in the proof and later on in the article is $\mathcal{Z} = \mathbb{R}^*W \oplus \mathcal{H}$.

Proof. We prove the two conditions to be a Seshadri slice:

• Take $V = H_1 + H$ in $\mathcal{V} = \mathcal{H}_1 \oplus \mathcal{H}$. Since (ρ_1, \mathcal{H}_1) is isomorphic to the standard representation, the orbits in \mathcal{H}_1 are the concentric spheres. Hence there exists a rotation $g \in \text{SO}_3(\mathbb{R})$ such that $\rho_1(g)(H_1) = \|H_1\|W$. That is, $\rho(g)(V) \in \mathcal{S}$.

• \mathcal{Z} is N -stable and its complexification is $\mathcal{Z} = \mathbb{C}^*W \oplus \hat{\mathcal{H}}$. Then for any $V \in \hat{\mathcal{Z}}$, an element $g \in \text{SO}_3(\mathbb{C})$ such that $\rho(g)(V) \in \hat{\mathcal{Z}}$ satisfies

$$\rho_1(g) = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ a & b & c \end{pmatrix}$$

with $a, b, c \in \mathbb{C}$. However we have $\rho_1(g)^t = g^t \in \text{SO}_3(\mathbb{C})$ and $\rho_1(g)^t \rho_1(g) = I_3$. This gives

$$\rho_1(g)^t \rho_1(g) = \begin{pmatrix} * & * & a \\ * & * & b \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ a & b & c \end{pmatrix} \begin{pmatrix} * & * & ac \\ * & * & bc \\ ac & bc & c^2 \end{pmatrix} = I_3.$$

Therefore $a = b = 0$ and $c = \pm 1$. We proved that $g \in \hat{\text{N}}$. □

3.3 Restrictions of the invariants to the slice

From now on, we note $\mathcal{V} = \mathcal{H}_1 \oplus \bigoplus_{j=1}^n \mathcal{H}_{d_j}$. We saw that the subvariety

$$\mathcal{S} := \mathbb{R}W \oplus \bigoplus_{j=1}^n \mathcal{H}_{d_j}$$

with normalizer $\text{N} \cong \text{O}_2(\mathbb{R})$ forms a Seshadri slice for the representation of $\text{SO}_3(\mathbb{R})$ on \mathcal{V} . By Proposition 3.5 we know that

$$\bigoplus_{j=1}^n \mathcal{H}_{d_j} \cong \bigoplus_{j=1}^p \mathcal{V}_{-1} \oplus \bigoplus_{j=p+1}^n \mathcal{V}_0 \oplus \bigoplus_{j=1}^n \bigoplus_{m=1}^{d_j} \mathcal{V}_m$$

where the \mathcal{V}_m are the irreducible representations of $\text{O}_2(\mathbb{R})$ of Section 2.1. The action of $\text{N} \cong \text{O}_2(\mathbb{R})$ on $\mathbb{R}W$ is furthermore the determinantal representation ϕ_{-1} . The representation of $\text{N} \cong \text{O}_2(\mathbb{R})$ on \mathcal{S} decomposes then into

$$\mathcal{S} \cong \mathcal{V}_{-1} \oplus \bigoplus_{j=1}^p \mathcal{V}_{-1} \oplus \bigoplus_{j=p+1}^n \mathcal{V}_0 \oplus \bigoplus_{j=1}^n \bigoplus_{m=1}^{d_j} \mathcal{V}_m.$$

This decomposition satisfies the hypotheses of Section 2. We can thus apply Theorem 2.6 to get the following set of algebraically independent invariants generating $\mathbb{R}(\mathcal{S})^{\text{N}}$:

$$\mathcal{E} = \left\{ \begin{array}{lll} D_j = & y_{0,0}y_{0,j}, & 0 \leq j \leq p \\ T_j = & y_{0,j}, & p+1 \leq j \leq n \\ R_{mj} = & y_{m,j}y_{-1,1}^m + y_{-m,j}y_{1,1}^m, & 1 \leq j \leq n, \quad 1 \leq m \leq d_j \\ I_{mj} = & iy_{0,0}(y_{m,j}y_{-1,1}^m - y_{-m,j}y_{1,1}^m), & 1 \leq j \leq n, \quad 1 \leq m \leq d_j, \quad (j, m) \neq (1, 1) \end{array} \right\}$$

Theorem 3.9. *The set $\tilde{\mathcal{E}}$ is algebraically independent and generates both $\mathbb{R}(\mathcal{V})^{\text{SO}_3(\mathbb{R})}$ and $\mathbb{C}(\hat{\mathcal{V}})^{\text{SO}_3(\mathbb{C})}$.*

Since invariants in $\tilde{\mathcal{E}}$ are algebraically independent, they provide a constructive proof of the rationality of the invariant field as a corollary:

Corollary 3.10. *Let $\rho : \text{SO}_3(\mathbb{R}) \rightarrow \text{GL}(\mathcal{V})$ be a representation including at least once the irreducible representation (ρ_1, \mathcal{H}_1) . Then, $\mathbb{R}(\mathcal{V})^{\text{SO}_3(\mathbb{R})}$ is rational.*

The rationality of the invariant field is a problem well addressed in invariant theory [Noe83, CTS07, Dol85, PV94]. The rationality of the invariant field of any representation of $\mathrm{SL}_2(\mathbb{C})$ was proved in [Kat83, Kat84, BK85]. For irreducible representations of even degree, Katsylo uses Seshadri slices (that he calls (G, N) -sections) similar to the one we identified in Proposition 3.8. One could thus deduce the rationality for representations of $\mathrm{SO}_3(\mathbb{C})$ thanks to the two fold covering $\mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{SO}_3(\mathbb{C})$. Nevertheless, the explicit generating set provided by our constructive proof is valuable. It allows us to derive directly the rationality of the *real* invariant fields (for representations of $\mathrm{SO}_3(\mathbb{R})$ and $\mathrm{O}_3(\mathbb{R})$). Furthermore [HJ25b] uses the multi-homogeneity of the obtained set to show the rationality of the invariant fields of some representations of the larger orthogonal groups $\mathrm{SO}_n(\mathbb{R})$.

Example 3.11. *The $\mathrm{O}_2(\mathbb{R})$ -representation $\mathcal{S}_{\mathcal{T}} = 2\mathcal{V}_{-1} \oplus \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3$ introduced in Example 2.7 is the slice corresponding to the $\mathrm{SO}_3(\mathbb{R})$ -representation $\mathcal{T} = \mathcal{H}_1 \oplus \mathcal{H}_3$. The restrictions to \mathcal{S} of a generating set of $\mathrm{SO}_3(\mathbb{R})$ -invariants are:*

$$\mathcal{E}_{\mathcal{T}} = \left\{ \begin{array}{l} D_0 = y_{0,0}^2 \\ D_1 = y_{0,0}y_{0,1} \\ R_{11} = 2y_{1,1}y_{-1,1} \\ R_{21} = y_{1,1}^2y_{-2,1} + y_{-1,1}^2y_{2,1} \\ R_{31} = y_{1,1}^3y_{-3,1} + y_{-1,1}^3y_{3,1} \\ I_{21} = iy_{0,0}(y_{1,1}^2y_{-2,1} - y_{-1,1}^2y_{2,1}) \\ I_{31} = iy_{0,0}(y_{1,1}^3y_{-3,1} - y_{-1,1}^3y_{3,1}) \end{array} \right\}.$$

Example 3.12. *Consider the representation $\mathbb{R}^3 \oplus \dots \oplus \mathbb{R}^3 \cong \mathcal{H}_1^{n+1}$. The associated slice is then $\mathcal{S}_{\mathcal{H}_1^{n+1}} = \mathbb{R}W \oplus \mathcal{H}_1^n$ and Theorem 3.9 gives the restrictions to the slice of a generating set of $\mathrm{SO}_3(\mathbb{R})$ -invariants:*

$$\mathcal{E}_{\mathcal{H}_1^{n+1}} = \left\{ \begin{array}{ll} D_j = y_{0,0}y_{0,i}, & 0 \leq j \leq n \\ R_j = y_{1,1}y_{-1,j} + y_{-1,1}y_{1,j}, & 1 \leq j \leq n \\ I_j = iy_{0,0}(y_{1,1}y_{-1,j} - y_{-1,1}y_{1,j}), & 2 \leq j \leq n \end{array} \right\}.$$

We note (r_0, \dots, r_n) a $(n+1)$ -tuple of vectors in $\mathcal{H}_1^{n+1} \cong (\mathbb{R}^3)^{n+1}$. The scalar products (r_i, r_j) , $0 \leq i \leq j \leq n$ and the determinants $\det(r_i | r_j | r_k)$, $0 \leq i < j < k \leq n$, generate the invariant algebra $\mathbb{R}[\mathcal{H}_1^n]^{\mathrm{SO}_3(\mathbb{R})}$ [Wey46, Theorem 2.9.A]. Examining their restrictions to the slice, we can find their relationships with the D_j, R_j , and I_j above. By Theorem 3.6 we deduce that the corresponding $\mathrm{SO}_3(\mathbb{R})$ -invariants are:

$$\begin{aligned} \tilde{D}_j &= (r_0, r_i), & 0 \leq j \leq n, \\ \tilde{R}_j &= 2 \left((r_1, r_j) - \frac{(r_0, r_j)^2}{(r_0, r_0)} \right), & 1 \leq j \leq n, \\ \tilde{I}_j &= \det(r_0 | r_1 | r_j), & 2 \leq j \leq n. \end{aligned}$$

3.4 Evaluating the invariants

In the previous section we characterized the invariants of the action of $\mathrm{SO}_3(\mathbb{R})$ on $\mathcal{V} = \mathbb{R}^3 \oplus \mathcal{H}$ by their restrictions to the slice $\mathcal{S} = \mathbb{R} \oplus \mathcal{H}$. In this section we address how to evaluate these invariants at any other point $v \in \mathcal{V}$. We first find a rotation g that takes v to \mathcal{S} and next evaluate our invariants at $g \cdot v$. Neither g nor the point of intersection of the orbit with the slice is unique, yet the values of the invariants is the same for any choice among those. As we are interested by evaluation on the real points, we introduce spherical coordinates and Euler angles, and then use Wigner matrices.

Lemma 3.13. *Let $H_1 + H \in \mathcal{H}_1 \oplus \mathcal{H}$ be a point in \mathcal{V} , with $H_1 \neq 0$. There is $(r, \theta, \varphi) \in \mathbb{R}_{>0} \times [0, \pi[\times [0, 2\pi[$ such that $H_1 = r \sin(\theta) \cos(\varphi) U + r \sin(\theta) \sin(\varphi) V + r \cos(\theta) W$. Then one has*

$$\rho(g(-\varphi, -\theta, 0))((H_1 + H)) \in \mathcal{Z} \subset \mathcal{S}.$$

Proof. The gradient map $\nabla : \mathcal{H}_1 \rightarrow \mathbb{R}^3$ is an isomorphism of $\text{SO}_3(\mathbb{R})$ -representations. Thus

$$\rho_1(g(-\varphi, -\theta, 0))(H_1) = rW.$$

This implies $\rho(g(-\varphi, -\theta, 0))(H_1 + H) \in \mathcal{Z}$. \square

We thus have determined a rotation $g(-\varphi, -\theta, 0) \in \text{SO}_3(\mathbb{R})$ that takes a point in \mathcal{V} to the slice. We now need to make explicit the action of such a rotation on \mathcal{V} . For each integer d , the matricial representation of $\rho_d(g(-\varphi, -\theta, 0))$ with respect to the basis of spherical harmonics is provided by Wigner matrices.

Definition 3.14. [Wig59, 15.27]. Let $d \in \mathbb{N}$ and (α, β, γ) Euler angles. The d^{th} Wigner matrix is the square matrix of size $(2d+1) \times (2d+1)$ defined by

$$W_{(\alpha, \beta, \gamma)}^d = \left(e^{-im\gamma} w_{m, m'}^d(-\beta) e^{-im'\alpha} \right)_{-d \leq m, m' \leq d}$$

where

$$w_{m, m'}^d(\beta) := \delta(m, m') \sum_{s=\max(0, m'-m)}^{\min(d-m, d+m')} \frac{(-1)^s \sqrt{(d+m')!(d-m')!(d+m)!(d-m)!}}{(d+m'-s)!s!(m-m'+s)!(d-m-s)!} \cos\left(\frac{\beta}{2}\right)^{2d+m'-m-2s} \sin\left(\frac{\beta}{2}\right)^{m-m'+2s},$$

$$\text{and } \delta(m, m') = i^{d(\text{sgn}(m') - \text{sgn}(m))} (-1)^{\frac{m'+|m'|-m-|m|}{2}}.$$

We used the convention $\text{sgn}(0) = 0$. The coefficient $\delta(m, m')$ comes from the slight change of basis between Theorem 3.2 and [Wig59]. This requires to conjugate by a diagonal matrix.

Proposition 3.15. [Wig59] Let $d \in \mathbb{N}$ and (α, β, γ) be the Euler angles defining the rotation $g(\alpha, \beta, \gamma) \in \text{SO}_3(\mathbb{R})$. For $H = y_{-d} Y_d^{-d} + \dots + y_d Y_d^d \in \mathcal{H}_d$, $\rho_d(g(\alpha, \beta, \gamma))(H) = \tilde{y}_{-d} Y_d^{-d} + \dots + \tilde{y}_d Y_d^d$ where

$$\begin{pmatrix} \tilde{y}_{-d} \\ \vdots \\ \tilde{y}_d \end{pmatrix} = W_{(\alpha, \beta, \gamma)}^d \begin{pmatrix} y_{-d} \\ \vdots \\ y_d \end{pmatrix}.$$

As the third angle γ is irrelevant for the rotations taking a point in \mathcal{V} to the slice \mathcal{S} we can give the expressions of the $\text{SO}_3(\mathbb{R})$ invariants as follows.

Theorem 3.16. Consider the representation of $\text{SO}_3(\mathbb{R})$ on $\mathcal{V} = \mathcal{H}_1 \oplus \bigoplus_{j=1}^n \mathcal{H}_{d_j}$. For $H_1 \in \mathcal{H}_1 \setminus \{0\}$ there is $(r, \theta, \varphi) \in \mathbb{R}_{>0} \times [0, \pi[\times [0, 2\pi[$ such that $H_1 = r \sin(\theta) \cos(\varphi) U + r \sin(\theta) \sin(\varphi) V + r \cos(\theta) W$ so that its coordinates in the basis of spherical harmonics are

$$y_{-1,0} = i \sqrt{\frac{2\pi}{3}} r \sin(\theta) e^{i\varphi}, \quad y_{0,0} = \sqrt{\frac{4\pi}{3}} r \cos(\theta), \quad y_{1,0} = -i \sqrt{\frac{2\pi}{3}} r \sin(\theta) e^{-i\varphi}.$$

Consider the large matrix $W(-\varphi, -\theta, 0) = \text{Diag}(W_{(-\varphi, -\theta, 0)}^{d_j}, 1 \leq j \leq n)$. The set of rational functions $\tilde{\mathcal{E}} := \{P \circ W(-\varphi, -\theta, 0), P \in \mathcal{E}\}$ is algebraically independent and generates $\mathbb{R}(\mathcal{V})^{\text{SO}_3(\mathbb{R})}$.

Proof. Take \tilde{P} the invariant function on \mathcal{V} whose restriction to \mathcal{S} is P , and $(H_1 + H)$ a point in $\mathcal{V} = \mathcal{H}_1 \oplus \mathcal{H}$. Choose $(-\varphi, -\theta, 0)$ according to Lemma 3.13 so that $\rho(g(-\varphi, -\theta, 0))(H_1 + H) \in \mathcal{S}$. Then $\tilde{P}(H_1 + H) = P(g(-\varphi, -\theta, 0)(H_1 + H))$. By Proposition 3.15 this latter is $P \circ W(-\varphi, -\theta, 0)(H_1 + H)$.

Polynomials in \mathcal{E} are algebraically independent and generate $\mathbb{R}(\mathcal{S})^N$. Meanwhile, Seshadri slice Lemma (Theorem 3.6) asserts that the restriction to \mathcal{S} provides a field isomorphism between $\mathbb{R}(\mathcal{V})^{\text{SO}_3(\mathbb{R})}$ and $\mathbb{R}(\mathcal{S})^N$, or $\mathbb{C}(\hat{\mathcal{V}})^{\text{SO}_3(\mathbb{C})}$ and $\mathbb{C}(\hat{\mathcal{S}})^{\hat{N}}$. Hence, extensions of polynomials in $\tilde{\mathcal{E}}$ are rational, algebraically independent, and they furthermore generate $\mathbb{R}(\mathcal{V})^{\text{SO}_3(\mathbb{R})}$. \square

Polynomials in \mathcal{E} are multihomogeneous with respect to the decomposition of \mathcal{S} into irreducible representations of $O_2(\mathbb{R})$. Rational invariants in $\tilde{\mathcal{E}}$ inherit this multihomogeneity:

Proposition 3.17. *Rational functions in $\tilde{\mathcal{E}}$ are multihomogeneous with respect to the decomposition of \mathcal{V} into irreducible representations of $SO_3(\mathbb{R})$.*

The multihomogeneity of a generating set allows to derive the rationality of the invariant fields of representations of a direct product $SO_3(\mathbb{R}) \times C_2^{n-3}$, $n \in \mathbb{N}$. Seshadri slices give then the rationality for some representations of $SO_n(\mathbb{R})$ [HJ25b].

Theorem 3.16 provides explicit expressions of the invariants in spherical coordinates. One can obtain the expression in Cartesian coordinates with symbolic manipulation and rewriting with Groebner bases. Examples of these computations are given in Example 5.3 for ternary cubics. Note in particular that $\tilde{D}_0 = y_{0,0}^2 + 2y_{-1,0}y_{1,0}$, or in real coordinates $x_{-1,0}^2 + x_{0,0}^2 + x_{1,0}^2$, is an irreducible polynomial. Its variety over \mathbb{C} contains the points whose orbit does not intersect $\hat{\mathcal{S}}$. As proved in [HJ25a], powers of \tilde{D}_0 are the only possible denominators that can appear in $\tilde{\mathcal{E}}$. This appears in the explicit computations in Example 5.3.

Proposition 3.18. *The set $\hat{\mathcal{V}} \setminus (SO_3(\mathbb{C}) \cdot \hat{\mathcal{Z}})$ is $\mathcal{V}(\tilde{D}_0)$, the variety where \tilde{D}_0 vanishes.*

Proof. Consider $(H_1 + H) \in \mathbb{C}^3 \oplus \hat{\mathcal{H}}$. Assume $g \in SO_3(\mathbb{C})$ is such that $\rho_1(g)(H_1) \in \mathbb{C}W$ so that $\rho(g)(H_1 + H) \in \mathcal{S}$. Then $\rho(g)(H_1 + H) \notin \hat{\mathcal{Z}}$ iff $D_0(\rho(g)(H_1 + H)) \neq 0$, and equivalently $\tilde{D}_0(H_1 + H) \neq 0$.

Conversely for any $(H_1 + H) \in \mathbb{C}^3 \oplus \hat{\mathcal{H}}$ we can find $g \in SO_3(\mathbb{C})$ such that $\rho(g)(H_1 + H) \in \mathcal{S}$. Write $H_1 = uU + vV + wW$. If both u and v are zero we can take g to be the identity. Otherwise we can choose $z_1, z_2 \in \mathbb{C}^3$ that form a basis of the kernel of the linear form over \mathbb{C}^3 $z \rightarrow (u, v, w)z$ and furthermore satisfy $z_1^T z_1 = 1$, $z_2^T z_2 = 1$, and $z_1^T z_2 = 0$. One can then determine z_3 such that $g = (z_1, z_2, z_3)^T \in SO_3(\mathbb{R})$. This g is such that $\rho(g)(H_1 + H) \in \mathcal{S}$. \square

3.5 Inverse problem

With the functions in $\tilde{\mathcal{E}}$ we construct the rational map $\Phi : \mathcal{V} \rightarrow \mathbb{R}^{\#\tilde{\mathcal{E}}}$ that separates orbits almost everywhere. The converse problem occurs: for a point $z \in \mathbb{R}^{\#\tilde{\mathcal{E}}}$, can we determine if z is in the image $\Phi(\mathcal{V})$? If so, can we find a preimage $v \in \mathcal{V}$ with $\Phi(v) = z$?

Any point in \mathcal{V} has a representant of its orbit in \mathcal{S} . Thus $\Phi(\mathcal{V}) := \Phi(\mathcal{S})$. Since the restrictions of the functions in $\tilde{\mathcal{E}}$ to the slice are polynomial functions, $\Phi(\mathcal{V})$ is a semialgebraic set, *i.e.*, a finite union of set in $\mathbb{R}^{\#\tilde{\mathcal{E}}}$ defined by polynomial equalities and inequalities.

Proposition 3.19. *Consider the real vector $z \in \mathbb{R}^{\#\tilde{\mathcal{E}}}$ defined by*

$$z = \begin{pmatrix} d_i, & 0 \leq i \leq p \\ t_j, & p+1 \leq j \leq n \\ r_{mj}, & 1 \leq j \leq n, \quad 1 \leq m \leq d_j \\ i_{mj}, & 1 \leq j \leq n, \quad 1 \leq m \leq d_j \text{ and } (m, j) \neq (1, 1) \end{pmatrix}.$$

Then, $z \in \Phi(\mathcal{V})$ iff one of the following conditions holds:

- $d_0 > 0$ and $r_{11} > 0$
- $d_0 > 0$ and $\forall j, m, r_{mj} = i_{mj} = 0$
- $d_0 = 0$ and $r_{11} > 0$ and $\forall j, m, i_{mj} = 0$ and $\forall j, d_j = 0$
- $d_0 = 0$ and $\forall j, m, r_{mj} = i_{mj} = 0$ and $\forall j, d_j = 0$.

The following proof is constructive; Its gives a primage in \mathcal{S} when it exists.

Proof. Take $z \in \mathbb{R}^{\#\tilde{\mathcal{E}}}$ a vector as in Proposition 3.19. The slice \mathcal{S} intersects all real orbits. Hence, there exists some real preimage $v \in \mathcal{V}$ iff there exists also a preimage in \mathcal{S} . Take then v a real point in the slice.

Firstly, $D_0(H_1) = y_{0,0}^2 \geq 0$ and $R_{11}(H_1) = 2y_{1,1}y_{-1,1} = 2y_{1,1}\overline{y_{1,1}} \geq 0$. Then $\Phi_{\tilde{\mathcal{E}}}(v) = z \Rightarrow d_0 \geq 0$ and $r_{11} \geq 0$. We choose $y_{0,0} = \sqrt{d_0}$ and $y_{1,1}, y_{-1,1}$ conjugate complex numbers of module $\frac{1}{2}\sqrt{r_{11}}$.

Suppose that nor d_0 or r_{11} vanish. Since $y_{1,1} \neq 0$ and $y_{0,0} \neq 0$ one can choose for $m \leq j$, $y_{-m,j} = \frac{y_{0,0}r_{mj} - iy_{mj}}{2y_{0,0}y_{1,1}^m}$ and $y_{m,j} = \overline{y_{-m,j}} = \frac{y_{0,0}r_{mj} + iy_{mj}}{2y_{0,0}y_{1,1}^m}$. That way,

$$\begin{aligned} R_{mj}(H_1) &= y_{1,1}^m y_{-mj} + y_{-1,1}^m y_{m,j} \\ &= y_{1,1}^m \frac{y_{0,0}r_{mj} - iy_{mj}}{2y_{0,0}y_{1,1}^m} + y_{-1,1}^m \frac{y_{0,0}r_{mj} + iy_{mj}}{2y_{0,0}y_{1,1}^m} \\ &= r_{mj} \\ I_{mj}(H_1) &= iy_{0,0} (y_{1,1}^m y_{-m,j} - y_{-1,1}^m y_{m,j}) \\ &= iy_{0,0} \left(y_{1,1}^m \frac{y_{0,0}r_{mj} - iy_{mj}}{2y_{0,0}y_{1,1}^m} - y_{-1,1}^m \frac{y_{0,0}r_{mj} + iy_{mj}}{2y_{0,0}y_{1,1}^m} \right) \\ &= iy_{mj}. \end{aligned}$$

Take finally $y_{0,j} = \frac{d_j}{y_{0,0}}$ for $0 \leq j \leq p$ and $y_{0,j} = t_j$ for $p+1 \leq j \leq n$. Then $\Phi_{\tilde{\mathcal{E}}}(v) = z$.

Suppose that $R_{11}(H_1) = r_{11} = 0$. Then $y_{1,1} = y_{-1,1} = 0$ and $\Phi_{\tilde{\mathcal{E}}}(v) = z \Rightarrow \forall j, m, r_{mj} = iy_{mj} = 0$. Take just $y_{0,j} = \frac{d_j}{y_{0,0}}$ for $0 \leq j \leq p$ and $y_{0,j} = t_j$ for $p+1 \leq j \leq n$ to have $\Phi_{\tilde{\mathcal{E}}}(v) = z$.

Suppose that $D_0(H_1) = d_0 = 0$. Then $y_{0,0} = 0$ and $\Phi_{\tilde{\mathcal{E}}}(v) = z \Rightarrow \begin{cases} \forall j, & iy_{mj} = y_{0,0}\Im(y_{1,1}y_{-m,j}) = 0. \\ \forall j, & d_j = y_{0,0}y_{0i} = 0. \end{cases}$ One can then choose, for all $j \leq n$ and $1 \leq m \leq d_j$, $y_{-m,j} = \frac{r_{mj}}{y_{1,1}^m}$ and $y_{m,j} = \overline{y_{-m,j}} = \frac{r_{mj}}{y_{-1,1}^m}$ to have $R_{mj}(v) = r_{mj}$. Take just $y_{0,j} = t_j$ for $j \leq p$ to have $\Phi_{\tilde{\mathcal{E}}}(v) = z$.

Suppose that $R_{11}(H_1) = r_{11} = 0$ and $D_0(H_1) = d_0 = 0$. Then $y_{1,1} = y_{-1,1} = 0$ and $y_{0,0} = 0$. One has then $\Phi_{\tilde{\mathcal{E}}}(v) = z \Rightarrow \begin{cases} \forall j, & iy_{mj} = r_{mj} = 0. \\ \forall j, & d_j = y_{0,0}y_{0i} = 0. \end{cases}$ Take just $y_{0,j} = t_j$ for $j \leq p$ to have $\Phi_{\tilde{\mathcal{E}}}(v) = z$. \square

The above inequalities define a semialgebraic set $\Phi(\mathcal{V})$ birationally equivalent to the orbit space $\mathcal{V}/\text{SO}_3(\mathbb{R})$. We furthermore determine a subset birationally equivalent to the stratum of principal orbit type:

Proposition 3.20. *If $\tilde{D}_0(v) > 0$ and $\tilde{R}_{11}(v) > 0$, then the stabilizer $\text{Stab}(v)$ is trivial.*

Proof. Take v a point in \mathcal{V} and s a point of its orbit lying on the slice. Write $(y_{0,0}, 0, 0, y_{1,1}, \dots, y_{-1,1})$ the coordinates of s . Suppose that $\tilde{D}_0(v) = D_0(s) > 0$ and $\tilde{R}_{11}(v) = R_{11}(s) > 0$. That is, $y_{0,0}, y_{1,1}, y_{-1,1}$ are non zero. Meanwhile, $\text{SO}_3(\mathbb{R})$ action on \mathcal{H}_1 is dual to the canonical action of $\text{SO}_3(\mathbb{R})$ on \mathbb{R}^3 . A rotation $g \in \text{SO}_3(\mathbb{R})$ stabilizes $(0, y_{0,0}, 0)$ in \mathcal{H}_1 iff

$$g \in \left\{ \left(\begin{array}{ccc} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{array} \right), \alpha \in \mathbb{R} \right\} \cong \text{SO}_2(\mathbb{R}).$$

This subgroup is normal in N and its action on $\text{Vect}(y_{1,-1}, y_{1,1})$ is isomorphic to \mathcal{V}_1 . Its only point with a non trivial stabilizer is $(0, 0)$. Then, $\text{Stab}(s) = \{I_3\}$. But, the stabilizers $\text{Stab}(v)$ and $\text{Stab}(s)$ are conjugate in $\text{SO}_3(\mathbb{R})$. Thus, $\text{Stab}(v) = \{I_3\}$. \square

However, the converse of Proposition 3.20 is not true: consider the point with $y_{0,1} = y_{-1,2} = y_{1,2} = 1$, all the other coordinates being zero. Then $\text{Stab}(v)$ is trivial while $\tilde{D}_0(v) = \tilde{R}_{11}(v) = 0$. On the other hand all the orbits with non trivial isotropy groups lie in $\mathcal{V}(\tilde{D}_0)$ and are mapped to zero. Our generating set has thus important limitations to stratify the orbit space. In [HJ25a] we design a strategy that goes beyond the present result by finding other Seshadri slices in the remaining variety $\mathcal{V}(\tilde{D}_0)$. This allows to compute step by step a complete separating system, and to stratify \mathcal{V} according to the orbit types.

4 Generalisation to $O_3(\mathbb{R})$ representations

We consider the representation of the group $O_3(\mathbb{R}) \triangleright SO_3(\mathbb{R})$ on

$$\mathcal{H} := \mathcal{H}_1 \oplus \bigoplus_{j=1}^n \mathcal{H}_{d_j} \oplus \bigoplus_{j=1}^{n^\sharp} \mathcal{H}_{e_j^\sharp}.$$

where $d_1, \dots, d_n, e_1, \dots, e_{n^\sharp}$ are integers. Both \mathcal{H}_d^\sharp and \mathcal{H}_d are the space of harmonic homogeneous polynomials of degree d . The representation $\rho_d : O_3(\mathbb{R}) \rightarrow GL(\mathcal{H}_d)$ is defined as for $SO_3(\mathbb{R})$ in Section 3.1. The representation $\rho_d^\sharp : O_3(\mathbb{R}) \rightarrow GL(\mathcal{H}_d^\sharp)$ is obtained as

$$\rho_d^\sharp : \begin{cases} O_3(\mathbb{R}) & \longrightarrow & GL(\mathcal{H}_d^\sharp) \\ g & \longmapsto & \det(g) \rho_d(g) \end{cases}$$

The representations (ρ_d, \mathcal{H}_d) and $(\rho_d^\sharp, \mathcal{H}_d^\sharp)$, $d \in \mathbb{N}$, are the only irreducible representations of $O_3(\mathbb{R})$ [GSS88]. For simplicity we assume that d_1, \dots, d_n are sorted decreasingly. A new Seshadri slice $(\mathcal{S}, \mathbb{N}^\sharp)$ can be identified with the same subspace \mathcal{S} as in Proposition 3.8, and with the normalizer $\mathbb{N}^\sharp \triangleright \mathbb{N}$, generated by \mathbb{N} and $-I_3$. This forms the direct product $\mathbb{N}^\sharp = \mathbb{N} \times C_2$. We consider the polynomials:

$$\begin{aligned} D_j^\sharp &= y_{0,0} y_{0,j}^\sharp & 1 \leq j \leq p^\sharp \\ T_j^\sharp &= y_{0,j}^\sharp & p^\sharp + 1 \leq j \leq n^\sharp \\ R_{mj}^\sharp &= y_{m,j}^\sharp y_{-1,1}^m + y_{-m,j}^\sharp y_{1,1}^m & j \leq n^\sharp & -e_j \leq m \leq e_j \\ I_{mj}^\sharp &= iy_{0,0} \left(y_{m,j}^\sharp y_{-1,1}^m - y_{-m,j}^\sharp y_{1,1}^m \right) & j \leq n^\sharp & -e_j \leq m \leq e_j \end{aligned}$$

Theorem 4.1. *Take P a polynomial in $\mathbb{R}(\mathcal{S})^\mathbb{N}$ as follows:*

- If $n^\sharp \geq 1$, choose $P := T_j^\sharp$ (for some e_j even) or $P := D_j^\sharp$ (for some e_j odd).
- Else, if $d_1 \geq 2$, choose $P := I_{d_1 1}$.
- Else, if $n \geq 2$ and $d_2 = 1$, choose $P := I_{12}$.
- Else, choose $P := 1$.

Then, the following set of algebraically independent polynomials generates both $\mathbb{R}(\mathcal{S})^{\mathbb{N}^\sharp}$ or $\mathbb{C}(\hat{\mathcal{S}})^{\hat{\mathbb{N}}^\sharp}$:

$$\mathcal{E}^\dagger = \left\{ \begin{array}{l} D_j^\dagger = D_j, \quad d_j \text{ odd} \\ T_j^\dagger = T_j, \quad d_j \text{ even} \\ R_{mj}^\dagger = \begin{cases} R_{mj}, & d_1 m + d_j \text{ even} \\ PR_{mj}, & d_1 m + d_j \text{ odd} \end{cases} \\ I_{mj}^\dagger = \begin{cases} PI_{mj}, & d_1 m + d_j \text{ even} \\ I_{mj}, & d_1 m + d_j \text{ odd} \end{cases} \end{array} \right\} \cup \left\{ \begin{array}{l} D_j^\dagger = PD_j^\sharp, \quad e_j \text{ odd} \\ T_j^\dagger = PT_j^\sharp, \quad e_j \text{ even} \\ R_{mj}^\dagger = \begin{cases} PR_{mj}^\sharp, & d_1 m + e_j \text{ even} \\ R_{mj}^\sharp, & d_1 m + e_j \text{ odd} \end{cases} \\ I_{mj}^\dagger = \begin{cases} I_{mj}^\sharp, & d_1 m + e_j \text{ even} \\ PI_{mj}^\sharp, & d_1 m + e_j \text{ odd} \end{cases} \end{array} \right\}$$

Proof. The direct product $\hat{\mathbb{N}}^\sharp = \hat{\mathbb{N}} \times C_2$ gives $\mathbb{C}(\hat{\mathcal{S}})^{\hat{\mathbb{N}}^\sharp} = (\mathbb{C}(\hat{\mathcal{S}})^{\hat{\mathbb{N}}})^{C_2}$. Theorem 3.9 gives the same minimal set of generators for $\mathbb{C}(\hat{\mathcal{S}})^{\hat{\mathbb{N}}}$:

$$\mathcal{E} = \left\{ \begin{array}{l} D_j, \quad d_j \text{ odd} \\ T_j, \quad d_j \text{ even} \\ R_{mj}, \quad 1 \leq j \leq n, m \leq d_j \\ I_{mj}, \quad 1 \leq j \leq n, m \leq d_j, (j, m) \neq (1, 1) \end{array} \right\} \cup \left\{ \begin{array}{l} T_j^\sharp, \quad e_j \text{ even} \\ D_j^\sharp, \quad e_j \text{ odd} \\ R_{mj}^\sharp, \quad 1 \leq j \leq n^\sharp, m \leq e_j \\ I_{mj}^\sharp, \quad 1 \leq j \leq n^\sharp, m \leq e_j \end{array} \right\}.$$

Take then the symmetry $\sigma = -I_3 \in \hat{N}^\sharp$. In spherical coordinates it acts as

$$\forall (r, \theta, \varphi), \sigma((r, \theta, \varphi)) = (r, \pi - \theta, \varphi + \pi).$$

By composition, for a vector $Y_{d_j}^m \in \mathcal{H}_{d_j}$, σ acts as

$$\begin{aligned} \rho_{d_j}(\sigma)(Y_{d_j}^m)((r, \theta, \varphi)) &= Y_{d_j}^m(\sigma((r, \theta, \varphi))) \\ &= Y_{d_j}^m(r, \pi - \theta, \varphi + \pi) \\ &= r^d c_d^m e^{im(\varphi + \pi)} P_{d_j}^m(-\cos(\theta)) \\ &= (-1)^{d_j} Y_{d_j}^m((r, \theta, \varphi)) \end{aligned}$$

and in a sharped representation $\mathcal{H}_{d_j}^\sharp$,

$$\begin{aligned} \rho_{d_j}^\sharp(\sigma)(Y_{d_j}^m)((r, \theta, \varphi)) &= -Y_{d_j}^m(\sigma((r, \theta, \varphi))) \\ &= -Y_{d_j}^m(r, \pi - \theta, \varphi + \pi) \\ &= -r^d c_d^m e^{im(\varphi + \pi)} P_{d_j}^m(-\cos(\theta)) \\ &= (-1)^{d_j+1} Y_{d_j}^m((r, \theta, \varphi)). \end{aligned}$$

It thus has an induced action on each elements in \mathcal{E} :

$$\left\{ \begin{array}{l} \sigma(D_j) = D_j, \quad d_j \text{ odd} \\ \sigma(D_j^\sharp) = -D_j^\sharp, \quad e_j \text{ odd} \\ \sigma(T_j) = T_j, \quad d_j \text{ even} \\ \sigma(T_j^\sharp) = -T_j^\sharp, \quad e_j \text{ even} \end{array} \right\} \text{ and } \left\{ \begin{array}{l} \sigma(R_{mj}) = (-1)^{d_j+d_1 m} R_{mj}, \\ \sigma(R_{mj}^\sharp) = (-1)^{1+e_j+d_1 m} R_{mj}^\sharp \\ \sigma(I_{mj}) = (-1)^{1+d_j+d_1 m} I_{mj} \\ \sigma(I_{mj}^\sharp) = (-1)^{e_j+d_1 m} I_{mj}^\sharp. \end{array} \right.$$

We next apply Lemma 2.5. We need a polynomial $P \in \mathcal{E}$ flipped by σ . Whenever $n^\sharp \geq 1$, one can choose $P := T_j^\sharp$ (for e_j even) or $P := D_j^\sharp$ (for e_j odd). Else it depends on d_1 . If $d_1 \geq 2$ then I_{d_1} suits. If $d_1 = 1, n \geq 2$ and $d_2 = 1$ then I_{12} suits. Otherwise, no polynomial in \mathcal{E} is flipped and $\mathbb{C}(\hat{\mathcal{S}})^{\hat{N}} = \mathbb{C}(\hat{\mathcal{S}})^{\hat{N}^\sharp}$. \square

Almost free $O_3(\mathbb{C})$ actions have a stably rational invariant field [CTS07, Proposition 4.10]. The above theorem provides the following stronger result, yet with a different hypothesis as already discussed after Theorem 3.9.

Corollary 4.2. *Let $\rho : O_3(\mathbb{R}) \rightarrow GL(\mathcal{H})$ a linear representation including at least once the irreducible representation (ρ_1, \mathcal{H}_1) . Then, both $\mathbb{R}(\mathcal{V})^{O_3(\mathbb{R})}$ and $\mathbb{C}(\hat{\mathcal{V}})^{O_3(\mathbb{C})}$ are rational.*

Example 4.3. *We follow up on Example 3.11 where the representation space was $\mathcal{T} = \mathcal{H}_1 \oplus \mathcal{H}_3$. We examined the $SO_3(\mathbb{R})$ -invariants, giving their restrictions to the slice. The restrictions to the slice $\mathcal{S}_{\mathcal{T}}$ of a generating set of rational $O_3(\mathbb{R})$ -invariant is given by the following algebraically independent generating set for $\mathbb{R}(\mathcal{S}_{\mathcal{T}})^{N^\sharp}$:*

$$\mathcal{E}_{\mathcal{T}}^\dagger = \left\{ \begin{array}{l} D_1^\dagger = y_{0,0}^2 \\ D_2^\dagger = y_{0,0} y_{0,1} \\ R_1^\dagger = 2y_{1,1} y_{-1,1} \\ R_2^\dagger = (y_{1,1}^2 y_{-2,1} + y_{-1,1}^2 y_{2,1})^2 \\ R_3^\dagger = y_{1,1}^3 y_{-3,1} + y_{-1,1}^3 y_{3,1} \\ I_2^\dagger = iy_{0,0} (y_{1,1}^2 y_{-2,1} - y_{-1,1}^2 y_{2,1}) \\ I_3^\dagger = iy_{0,0} (y_{1,1}^3 y_{-3,1} - y_{-1,1}^3 y_{3,1}) (y_{1,1}^2 y_{-2,1} + y_{-1,1}^2 y_{2,1}) \end{array} \right\}$$

5 Applications

We have exhibited invariants for representations of $\mathrm{SO}_3(\mathbb{R})$ and $\mathrm{O}_3(\mathbb{R})$ on spaces $\mathbb{R}^3 \oplus \mathcal{H}$. Among those is the well studied case of $\mathbb{R}^3 \times \dots \times \mathbb{R}^3$ treated to some extent in Example 3.12. This representation and its invariants are relevant in several areas, in particular to design invariant neural networks [BSV23, DCCP21, DG24, GD04]. In this section we apply our method for three other cases of relevance.

5.1 Odd degree ternary forms

Among the cases of interest to which our construction applies is the representation of $\mathrm{SO}_3(\mathbb{R})$ or $\mathrm{O}_3(\mathbb{R})$ on the vector space of homogeneous polynomials $\mathbb{R}[U, V, W]_d$ of odd degree d . The case of even degrees was treated in [GHP19]. The starting point for both odd and even degrees is the harmonic decomposition.

Lemma 5.1. [AH12, Theorem 2.18] $\mathbb{R}[U, V, W]_d = \bigoplus_{j=0}^{\lfloor d/2 \rfloor} Q^j \mathcal{H}_{d-2j}$ where $Q = U^2 + V^2 + W^2$.

Whenever d is odd, the last term of this decomposition is \mathcal{H}_1 and thus we can apply the construction provided in previous sections.

Corollary 5.2. For any $d \in \mathbb{N}$, the invariant field of the $\mathrm{SO}_3(\mathbb{R})$ or $\mathrm{O}_3(\mathbb{R})$ action on $\mathbb{R}[U, V, W]_d$ is rational.

Example 5.3. The vector space $\mathcal{T} = \mathcal{H}_1^3 \oplus \mathcal{H}_3^3$ introduced in Example 3.11 is isomorphic to the space of ternary cubics $\mathbb{R}[U, V, W]_3$. The basis of $\mathbb{C}[U, V, W]_3$ we use is:

$$\{QY_{-1,1}, QY_{0,1}, QY_{1,1}; Y_{-3,3}, Y_{-2,3}, Y_{-1,3}, Y_{0,3}, Y_{1,3}, Y_{2,3}, Y_{3,3}\}.$$

In this basis an element of $\mathbb{C}[U, V, W]_3$ is given by its vector of coefficients

$$(y_{-1,0}, y_{0,0}, y_{1,0}; y_{-3,1}, y_{-2,1}, y_{-1,1}, y_{0,1}, y_{1,1}, y_{2,1}, y_{3,1})^T.$$

The Seshadri slice for $\mathbb{R}[U, V, W]_3$ is $\mathcal{S} = \mathbb{R}QW \oplus \mathcal{H}_3^3$ with stabilizer $N \cong \mathrm{O}_2(\mathbb{R})$. The set $\mathcal{E}_{\mathcal{T}}$ in Example 3.11 is an algebraically independent generating set of $\mathbb{R}(\mathcal{S})^N$. By Theorem 3.6, $\mathcal{E}_{\mathcal{T}}$ is an algebraically generating set of $\mathbb{R}(\mathcal{T})^{\mathrm{SO}_3(\mathbb{R})}$. Wigner matrices give the expressions of the pulled back invariants with respect to spherical coordinates. Symbolic computations, that we implemented in Maple, provide their expressions in terms of

Cartesian coordinates w.r.t. bases of spherical harmonics. We give the three first ones:

$$\tilde{D}_0 = y_{0,0}^2 + 2y_{-1,0}y_{1,0}$$

$$\tilde{D}_1 = \frac{\sqrt{3}y_{-1,0}y_{1,0}(y_{-1,0}y_{1,3}+y_{1,0}y_{-1,3})+\sqrt{5}(y_{1,0}^3y_{-3,3}+y_{-1,0}^3y_{3,3})}{\sqrt{2}(y_{0,0}^2+2y_{1,0}y_{-1,0})} - \frac{i\sqrt{3}y_{0,0}(y_{1,0}^2y_{-2,3}-y_{-1,0}^2y_{2,3})}{2(y_{0,0}^2+2y_{1,0}y_{-1,0})} \\ - \frac{2\sqrt{3}y_{0,0}^2(y_{-1,0}y_{1,3}+y_{1,0}y_{-1,3})+y_{0,0}y_{0,3}(y_{0,0}^2-3\sqrt{5}y_{-1,0}y_{1,0})}{(y_{0,0}^2+2y_{1,0}y_{-1,0})}$$

$$\begin{aligned} \tilde{R}_{11} = & \frac{1}{4D_0^3}(-12i\sqrt{2}\sqrt{3}\sqrt{5}y_{-2,1}y_{-1,0}y_{0,0}^2y_{0,1}y_{1,0}^3 + 4i\sqrt{2}\sqrt{3}\sqrt{5}y_{-2,1}y_{-1,0}^2y_{0,0}y_{1,0}^4 + 8i\sqrt{2}\sqrt{3}\sqrt{5}y_{-2,1}y_{0,0}^4y_{0,1}y_{1,0}^2 \\ & + 20i\sqrt{5}y_{-1,0}^2y_{-1,1}y_{0,0}^3y_{1,0}y_{2,1} + 30i\sqrt{3}y_{-3,1}y_{-2,1}y_{0,0}y_{1,0}^5 + 8i\sqrt{5}y_{-2,1}y_{0,0}^5y_{1,0}y_{1,1} + 32i\sqrt{5}y_{-1,0}^3y_{0,0}^3y_{1,1}y_{2,1} \\ & + 10i\sqrt{3}y_{-3,1}y_{-1,0}^2y_{0,0}^3y_{1,0}^3y_{2,1} + 20i\sqrt{3}y_{-3,1}y_{-1,0}y_{0,0}^3y_{1,0}^2y_{2,1} + 18i\sqrt{5}y_{-2,1}y_{-1,0}^2y_{0,0}y_{1,0}^3y_{1,1} \\ & + 26i\sqrt{5}y_{-2,1}y_{-1,0}y_{-1,1}y_{0,0}y_{1,0}^4 - y_{-1,0}^4y_{1,0}^2y_{1,1}^2 - y_{-1,0}^2y_{-1,1}^2y_{1,0}^4 + 30y_{-3,1}y_{-1,0}^3y_{1,0}^3y_{3,1} + 20y_{-2,1}y_{-1,0}^3y_{1,0}^3y_{2,1} \\ & + 64y_{-1,0}^3y_{0,0}^2y_{1,0}y_{1,1}^2 - 48y_{-1,0}^2y_{0,0}^2y_{1,0}^2y_{1,1}^2 + 64y_{-1,0}y_{0,0}^2y_{1,0}^2y_{1,1}^3 + 48y_{-1,0}y_{0,0}^4y_{1,0}^2y_{1,1} + 2y_{-1,0}^3y_{-1,1}y_{1,0}^3y_{1,1} \\ & + 8\sqrt{3}\sqrt{5}y_{-1,0}^3y_{-1,1}y_{0,0}^2y_{1,0}y_{3,1} + 22\sqrt{3}\sqrt{2}y_{-1,0}^2y_{0,0}y_{0,1}y_{1,0}^2y_{1,1} + 22\sqrt{3}\sqrt{2}y_{-1,0}^2y_{-1,1}y_{0,0}y_{1,0}y_{1,1}^3 \\ & - 48\sqrt{3}\sqrt{2}y_{-1,0}^2y_{0,0}^3y_{0,1}y_{1,0}y_{1,1} - 48\sqrt{3}\sqrt{2}y_{-1,0}y_{-1,1}y_{0,0}^3y_{0,1}y_{1,0}^2 - 24y_{-1,0}^2y_{0,0}^4y_{1,1}^2 \\ & + 6\sqrt{5}\sqrt{2}y_{-3,1}y_{-1,0}y_{0,0}y_{0,1}y_{1,0}^4 + 6\sqrt{5}\sqrt{2}y_{-1,0}^4y_{0,0}y_{0,1}y_{1,0}y_{3,1} - 10i\sqrt{3}y_{-2,1}y_{-1,0}^3y_{0,0}y_{1,0}^2y_{3,1} \\ & - 20i\sqrt{3}y_{-2,1}y_{-1,0}^2y_{-1,1}y_{0,0}^3y_{1,0}y_{3,1} - 20i\sqrt{5}y_{-2,1}y_{-1,0}y_{0,0}^3y_{1,0}^2y_{1,1} - 26i\sqrt{5}y_{-1,0}^4y_{0,0}y_{0,1}y_{1,0}y_{2,1} \\ & - 18i\sqrt{5}y_{-1,0}^3y_{-1,1}y_{0,0}y_{1,0}^2y_{2,1} + 8\sqrt{3}\sqrt{5}y_{-3,1}y_{-1,0}y_{0,0}^2y_{1,0}^3y_{1,1} + 12y_{-1,0}^3y_{0,0}^2y_{1,0}^3 - 24y_{-1,1}^2y_{0,0}^4y_{1,0}^2 \\ & - 10y_{-2,1}^2y_{-1,0}y_{1,0}^5 + 40y_{-2,1}^2y_{0,0}^2y_{1,0}^4 - 10y_{-1,0}^5y_{1,0}y_{2,1}^2 + 40y_{-1,0}^4y_{0,0}^2y_{2,1}^2 + 8y_{-1,1}y_{0,0}^6y_{1,1} \\ & - 4i\sqrt{2}\sqrt{3}\sqrt{5}y_{-1,0}^4y_{0,0}y_{1,0}^2y_{2,1} - 8i\sqrt{2}\sqrt{3}\sqrt{5}y_{-1,0}^2y_{0,0}^4y_{0,1}y_{2,1} - 15y_{-3,1}^2y_{1,0}^6 - 15y_{-1,0}^6y_{3,1}^2 \\ & + 2\sqrt{3}\sqrt{5}y_{-3,1}y_{-1,0}^2y_{1,0}^4y_{1,1} - 2\sqrt{3}\sqrt{5}y_{-3,1}y_{-1,0}y_{-1,1}y_{1,0}^5 + 14\sqrt{3}\sqrt{5}y_{-3,1}y_{-1,1}y_{0,0}^4y_{1,0} \\ & - 2\sqrt{3}\sqrt{5}y_{-1,0}^5y_{0,0}y_{1,1}y_{3,1} + 2\sqrt{3}\sqrt{5}y_{-1,0}^4y_{-1,1}y_{1,0}^2y_{3,1} + 14\sqrt{3}\sqrt{5}y_{-1,0}^4y_{0,0}^2y_{1,1}y_{3,1} - 4\sqrt{3}\sqrt{5}y_{-1,0}^2y_{-1,1}y_{0,0}^4y_{3,1} \\ & + 8\sqrt{3}\sqrt{2}y_{-1,1}y_{0,0}^5y_{0,1}y_{1,0} - 12\sqrt{5}\sqrt{2}y_{-3,1}y_{0,0}^3y_{0,1}y_{1,0}^3 - 12\sqrt{5}\sqrt{2}y_{-1,0}^3y_{0,0}^3y_{0,1}y_{3,1} \\ & - 30i\sqrt{3}y_{-1,0}^5y_{0,0}y_{2,1}y_{3,1} - 32i\sqrt{5}y_{-2,1}y_{-1,1}y_{0,0}^3y_{1,0}^3 - 8i\sqrt{5}y_{-1,0}y_{-1,1}y_{0,0}^5y_{2,1} - 4\sqrt{3}\sqrt{5}y_{-3,1}y_{0,0}^4y_{1,0}^2y_{1,1} \\ & + 12i\sqrt{2}\sqrt{3}\sqrt{5}y_{-1,0}^3y_{0,0}^2y_{0,1}y_{1,0}y_{2,1} + 30y_{-3,1}y_{-1,0}^2y_{0,0}^2y_{1,0}^2y_{3,1} + 40y_{-2,1}y_{-1,0}y_{0,0}^4y_{1,0}y_{2,1} \\ & + 114y_{-1,0}^2y_{-1,1}y_{0,0}^2y_{1,0}^2y_{1,1} - 40y_{-1,0}y_{-1,1}y_{0,0}^4y_{1,0}y_{1,1} + 8\sqrt{3}\sqrt{2}y_{-1,0}y_{0,0}^5y_{0,1}y_{1,1}) \end{aligned}$$

The expressions of \tilde{R}_{12} , \tilde{R}_{13} , \tilde{I}_{12} and \tilde{I}_{13} are too large to be exposed here. One can nevertheless find them, and the codes to compute them, at <https://sites.google.com/view/martin-jalard/accueil>.

5.2 Piezoelectricity tensors

For some material of industrial interest (single crystals, ceramics, thin films, . . .), an electric displacement is induced in response to an applied mechanical stress. This property is called the piezoelectricity effect and a third-order tensor relates the induced electric displacement vector to the second-order stress tensor [AG20]. The orthogonal group $\text{SO}_3(\mathbb{R})$ acts naturally on the 18-dimensional space of piezoelectricity tensors

$$\text{Piez} := \{e \in \otimes^3 \mathbb{R}^3, e_{ijk} = e_{ikj}\}.$$

As such it decomposes into $\text{SO}_3(\mathbb{R})$ -irreducible subspaces as [ADKP23, Section 7.1]

$$\text{Piez} \cong \mathcal{H}_1 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3.$$

A tensor in Piez is accordingly given by a coordinate vector

$$(y_{-1,0}, y_{0,0}, y_{1,0}; y_{-1,1}, y_{0,1}, y_{1,1}; y_{-2,2}, y_{-1,2}, y_{0,2}, y_{1,2} y_{2,2}; y_{-3,3}, y_{-2,3}, y_{-1,3}, y_{0,3}, y_{1,3}, y_{2,3}, y_{3,3}),$$

where $y_{-m,j} = \bar{y}_{m,j}$, when the components \mathcal{H}_d are endowed the bases of Theorem 3.2. A Seshadri slice for $\text{SO}_3(\mathbb{R})$ acting on Piez is given by the subspace \mathcal{S} defined by the equations $y_{-1,0} = 0$ and $y_{1,0} = 0$ and which is stabilized by $\text{N} \cong \text{O}_2(\mathbb{R})$. Theorem 2.6 gives an algebraically independent set of generators for $\mathbb{R}(\mathcal{S})^{\text{N}}$:

Proposition 5.4. *The following set is an algebraically independent and generates $\mathbb{R}(\mathcal{S})^N$.*

$$\mathcal{E}_{\text{Piez}} = \left\{ \begin{array}{lll} T_2 = y_{0,2} & R_{11} = 2y_{-1,1}y_{1,1} & I_{12} = iy_{0,0}(y_{1,2}y_{-1,1} - y_{-1,2}y_{1,1}) \\ D_0 = y_{0,0}^2 & R_{12} = y_{1,2}y_{-1,1} + y_{-1,2}y_{1,1} & I_{22} = iy_{0,0}(y_{2,2}y_{-1,1}^2 - y_{-2,2}y_{1,1}^2) \\ D_1 = y_{0,0}y_{0,1} & R_{22} = y_{2,2}y_{-1,1}^2 + y_{-2,2}y_{1,1}^2 & I_{13} = iy_{0,0}(y_{1,3}y_{-1,1} - y_{-1,3}y_{1,1}) \\ D_3 = y_{0,0}y_{0,3} & R_{13} = y_{1,3}y_{-1,1} + y_{-1,3}y_{1,1} & I_{23} = iy_{0,0}(y_{2,3}y_{-1,1}^2 - y_{-2,3}y_{1,1}^3) \\ & R_{23} = y_{2,3}y_{-1,1}^2 + y_{-2,3}y_{1,1}^2 & I_{33} = iy_{0,0}(y_{3,3}y_{-1,1}^3 - y_{-3,3}y_{1,1}^3) \\ & R_{33} = y_{3,3}y_{-1,1}^3 + y_{-3,3}y_{1,1}^3 & \end{array} \right\}$$

Here is its expression in real coordinates (see Section 3.3):

$$\mathcal{E}_{\text{Piez}} = \left\{ \begin{array}{ll} T_2 = x_{0,2} & R_{1,1} = x_{-1,1}^2 + x_{1,1}^2 \\ D_0 = x_{0,0}^2 & R_{1,2} = x_{-1,1}x_{-1,2} + x_{1,1}x_{1,2} \\ D_1 = x_{0,0}x_{0,1} & R_{2,2} = \frac{\sqrt{2}}{2}(2x_{-2,2}x_{-1,1}x_{1,1} - x_{-1,1}^2x_{2,2} + x_{1,1}^2x_{2,2}) \\ D_3 = x_{0,0}x_{0,3} & R_{1,3} = x_{-1,1}x_{-1,3} + x_{1,1}x_{1,3} \\ & R_{2,3} = \frac{\sqrt{2}}{2}(2x_{-2,3}x_{-1,1}x_{1,1} - x_{-1,1}^2x_{2,3} + x_{1,1}^2x_{2,3}) \\ & R_{3,3} = -\frac{1}{2}x_{-3,3}x_{-1,1}^3 + \frac{3}{2}x_{-3,3}x_{-1,1}x_{1,1}^2 - \frac{3}{2}x_{-1,1}^2x_{1,1}x_{3,3} + \frac{1}{2}x_{1,1}^3x_{3,3} \\ & I_{1,2} = -x_{0,0}(x_{-1,1}x_{1,2} - x_{-1,2}x_{1,1}) \\ & I_{2,2} = -\frac{\sqrt{2}}{2}x_{0,0}(x_{-2,2}x_{-1,1}^2 - x_{-2,2}x_{1,1}^2 + 2x_{-1,1}x_{1,1}x_{2,2}) \\ & I_{1,3} = -x_{0,0}(x_{-1,1}x_{1,3} - x_{-1,3}x_{1,1}) \\ & I_{2,3} = -\frac{\sqrt{2}}{2}x_{0,0}(x_{-2,3}x_{-1,1}^2 - x_{-2,3}x_{1,1}^2 + 2x_{-1,1}x_{1,1}x_{2,3}) \\ & I_{3,3} = -\frac{1}{2}x_{0,0}(3x_{-3,3}x_{-1,1}^2x_{1,1} - x_{-3,3}x_{1,1}^3 - x_{-1,1}^2x_{3,3} + 3x_{-1,1}x_{1,1}^2x_{3,3}) \end{array} \right\}$$

Theorem 3.16 allows us to deduce an algebraically independent set $\tilde{\mathcal{E}}_{\text{Piez}}$ generating $\mathbb{R}(\text{Piez})^{\text{SO}_3(\mathbb{R})}$. In [ADKP23, Section 7.1], the group acting on the space of piezoelectricity tensors is actually $\text{O}_3(\mathbb{R})$. The $\text{O}_3(\mathbb{R})$ representation is then $\text{Piez} \cong \mathcal{H}_1 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2^{\#} \oplus \mathcal{H}_3$. Here the first term $d_1 = 1$ is odd, and $T_2^{\#} = y_{0,2}$ of Theorem 4.1 can be chosen as the flipped polynomial. Applying Theorem 4.1:

Proposition 5.5. *The following set of cardinal 15 is an algebraically independent generating set for both $\mathbb{C}(\hat{\mathcal{S}})^{N^{\#}}$ or $\mathbb{R}(\mathcal{S})^{N^{\#}}$.*

$$\mathcal{E}_{\text{Piez}}^{\#} = \left\{ \begin{array}{lll} T_2^{\#} = y_{0,2}^2 & R_{11} = y_{1,1}y_{-1,1} + y_{-1,1}y_{1,1} & I_{12}^{\#} = iy_{0,0}(y_{1,2}y_{-1,1} - y_{-1,2}y_{1,1}) \\ D_0 = y_{0,0}^2 & R_{12}^{\#} = y_{0,2}(y_{1,2}y_{-1,1} + y_{-1,2}y_{1,1}) & I_{22}^{\#} = iy_{0,0}y_{0,2}(y_{2,2}y_{-1,1}^2 - y_{-2,2}y_{1,1}^2) \\ D_1 = y_{0,0}y_{0,1} & R_{22}^{\#} = y_{2,2}y_{-1,1}^2 + y_{-2,2}y_{1,1}^2 & I_{13} = iy_{0,0}y_{0,2}(y_{1,3}y_{-1,1} - y_{-1,3}y_{1,1}) \\ D_3 = y_{0,0}y_{0,3} & R_{13} = y_{1,3}y_{-1,1} + y_{-1,3}y_{1,1} & I_{23} = iy_{0,0}(y_{2,3}y_{-1,1}^2 - y_{-2,3}y_{1,1}^3) \\ & R_{23} = y_{0,2}(y_{2,3}y_{-1,1}^2 + y_{-2,3}y_{1,1}^2) & I_{33} = iy_{0,0}y_{0,2}(y_{3,3}y_{-1,1}^3 - y_{-3,3}y_{1,1}^3) \\ & R_{33} = y_{3,3}y_{-1,1}^3 + y_{-3,3}y_{1,1}^3 & \end{array} \right\}$$

Here again Theorem 3.16 allows us to deduce an algebraically independent set generating $\mathbb{R}(\text{Piez})^{\text{O}_3(\mathbb{R})}$.

5.3 3D Moment Invariants

Several field of engineering are confronted with differentiating or classifying 3D shapes up to a group of transformations, and in particular the group $\text{SO}_3(\mathbb{R})$ of rotations. Moments are descriptors of shapes that are obtained by acquisition devices or by computation. Moment invariants are then the descriptors of the shape independently of rotation. For two dimensional images, many approaches to the computation and application of moment invariants were developed. The books [FSZ09, FSZ17] are entry points to the intricate literature for mathematical and methodological advances together with their concrete applications. The seminal papers in the 3D case seem to be [Guo93, Hu62, LD89]. A sample of applications in medicine and biology can be found in [GDV⁺13, MPD⁺04, SMD⁺07].

Finding analytical expressions of moment invariants is the one facet that we wish to expand on here. Several methods have been designed to produce 3D moment invariants under $\text{SO}_3(\mathbb{R})$ with recent contributions [BB23, LHXL17, SFB15, Sun24]. Explicit expressions for some moment invariants up to tenth order were exhibited. The main result of this article, Theorem 3.14, applies straightforwardly and provides explicit expressions of an algebraically independent set of moment invariants, up to any order, that is generating.

Consider a measure μ with compact support in \mathbb{R}^3 . The associated moment functional is the linear form

$$\begin{aligned} \mathbf{m}_\mu : \mathbb{R}[U, V, W] &\rightarrow \mathbb{R} \\ p &\mapsto \int_{\mathbb{R}^3} p \, d\mu \end{aligned}$$

The measure can for instance be given by a density function $\omega : \mathbb{R}^3 \rightarrow \mathbb{R}_{\geq 0}$ with compact support so that the moments are

$$\mathbf{m}_\mu(p) = \int_{\mathbb{R}^3} p(U, V, W) \omega(U, V, W) \, dudvdw.$$

For instance ω can be the characteristic function of a 3D shape. Another important case is when μ is an atomic measure, *i.e.*, a linear combination of a finite number Diracs centered at $\xi_1, \dots, \xi_r \in \mathbb{R}^3$ with positive coefficients ω_i . Then

$$\mathbf{m}_\mu(p) = \sum_{i=1}^r \omega_i p(\xi_i).$$

Assume ω is the characteristic function of a shape and that this shape is transformed by an invertible linear map $a : \mathbb{R}^3 \rightarrow \mathbb{R}^3$. Then the characteristic function of the transformed shape is $\omega \circ a^{-1}$. More generally, for a measure μ and its transform μ^a by a we have $\mathbf{m}_{\mu^a}(p) = \mathbf{m}_\mu(p \circ a)$.

For a group representation $G \rightarrow \text{GL}(\mathbb{R}^3)$, if $A_p^{(d)} : G \rightarrow \text{GL}_{r_d}(\mathbb{R})$ is the matrix of the induced representation on $\mathbb{R}[U, V, W]_d$ in the basis $\{p_1, \dots, p_{r_d}\}$, *i.e.* $[p_1 \circ g^{-1} \ \dots \ p_{r_d} \circ g^{-1}] = [p_1 \ \dots \ p_{r_d}] A_p^{(d)}(g)$, then, in the dual basis, the matrix of the induced representation on the dual space $(\mathbb{R}[U, V, W]_d)^*$ is $M_{p^*}^{(d)} : G \rightarrow \text{GL}_{r_d}(\mathbb{R})$ with $M_{p^*}^{(d)}(g) = \left(A_p^{(d)}(g)\right)^{-T}$. Concomitantly

$$\begin{bmatrix} \mathbf{m}_{\mu^g}(p_1) \\ \dots \\ \mathbf{m}_{\mu^g}(p_{r_d}) \end{bmatrix} = \left(A^{(d)}(g^{-1})\right)^T \begin{bmatrix} \mathbf{m}_\mu(p_1) \\ \dots \\ \mathbf{m}_\mu(p_{r_d}) \end{bmatrix}.$$

In the case of a compact group we can always find bases where the matrices $A_p^{(d)}(g)$ are orthogonal so that the induced representations on $\mathbb{R}[U, V, W]_d$ and its dual $(\mathbb{R}[U, V, W]_d)^*$ are equivalent. For $\text{SO}_3(\mathbb{R})$, the real spherical harmonics defined in Section 3.1 provide such bases.

The action of $\text{SO}_3(\mathbb{R})$ on moments of order 0 up to order d is thus equivalent to the action of $\text{SO}_3(\mathbb{R})$ on $\mathfrak{M}_d = \mathbb{R} \oplus \mathbb{R}[U, V, W]_1 \oplus \dots \oplus \mathbb{R}[U, V, W]_d$. Given that

$$\mathbb{R}[U, V, W]_d = \bigoplus_{k=0}^{\lfloor \frac{d}{2} \rfloor} Q^k \mathcal{H}_{d-2k}$$

we can apply Theorem 3.14 to provide the moment invariants up to any order. For instance for $d = 4$ the basis of \mathfrak{M}_4 is, with $Q = U^2 + V^2 + W^2$,

$$\{1\} \cup \{X_1^{-1}, X_1^0, X_1^1\} \cup \{Q; X_2^{-2}, \dots, X_2^2\} \cup \{QX_1^{-1}, QX_1^0, QX_1^1; X_3^{-3}, \dots, X_3^3\} \cup \{Q^2; QX_2^{-2}, \dots, QX_2^2; X_4^{-4}, \dots, X_4^4\}.$$

The moments $\mathbf{m}(1)$, $\mathbf{m}(Q)$, $\mathbf{m}(Q^2)$ are already invariants. To obtain a generating set of moment invariants one has to add the invariants obtained from Theorem 3.14 with the following coordinates that are arranged according to parity of the degrees ($p = 3$, $n = 5$ in the notation of Theorem 3.14).

$$\begin{aligned}
y_{-1,0} &= \frac{1}{\sqrt{2}} (\mathfrak{m}(X_1^{-1}) + i \mathfrak{m}(X_1^1)), \quad y_{0,0} = \mathfrak{m}(X_1^0), \quad y_{1,0} = \frac{1}{\sqrt{2}} (\mathfrak{m}(X_1^{-1}) - i \mathfrak{m}(X_1^1)); \\
y_{-2,1} &= \frac{1}{\sqrt{2}} (\mathfrak{m}(X_2^{-2}) + i \mathfrak{m}(X_2^2)), \quad \dots, \quad y_{2,1} = \frac{1}{\sqrt{2}} (\mathfrak{m}(X_2^{-2}) - i \mathfrak{m}(X_2^2)); \\
y_{-2,2} &= \frac{1}{\sqrt{2}} (\mathfrak{m}(QX_2^{-2}) + i \mathfrak{m}(QX_2^2)), \quad \dots, \quad y_{2,2} = \frac{1}{\sqrt{2}} (\mathfrak{m}(QX_2^{-2}) - i \mathfrak{m}(QX_2^2)); \\
y_{-4,3} &= \frac{1}{\sqrt{2}} (\mathfrak{m}(X_2^{-4}) + i \mathfrak{m}(X_2^4)), \quad \dots, \quad y_{4,3} = \frac{1}{\sqrt{2}} (\mathfrak{m}(X_2^{-4}) - i \mathfrak{m}(X_2^4)); \\
y_{-1,4} &= \frac{1}{\sqrt{2}} (\mathfrak{m}(QX_1^{-1}) + i \mathfrak{m}(QX_1^1)), \quad \dots, \quad y_{1,4} = \frac{1}{\sqrt{2}} (\mathfrak{m}(QX_1^{-1}) - i \mathfrak{m}(QX_1^1)); \\
y_{-3,5} &= \frac{1}{\sqrt{2}} (\mathfrak{m}(X_3^{-3}) + i \mathfrak{m}(X_3^3)), \quad \dots, \quad y_{3,5} = \frac{1}{\sqrt{2}} (\mathfrak{m}(X_3^{-3}) - i \mathfrak{m}(X_3^3))
\end{aligned}$$

It is nonetheless often assumed that $\mathfrak{m}(\mathbb{R}[U, V, W]_1) = \{0\}$. This is obtained by applying an appropriate translation to the measure. One thus has to deal with the action of $\mathrm{SO}_3(\mathbb{R})$ on $\mathfrak{M}_d^c = \mathbb{R} \oplus \mathbb{R}[U, V, W]_2 \oplus \mathbb{R}[U, V, W]_3 \oplus \dots \oplus \mathbb{R}[U, V, W]_d$. One can still apply Theorem 3.14 since $\mathbb{R}[U, V, W]_3 = Q\mathcal{H}_1 \oplus \mathcal{H}_3$. We reorganize the components of \mathfrak{M}_d^c as

$$\mathfrak{M}_d^c = Q\mathcal{H}_1 \oplus \mathbb{R} \oplus (Q\mathbb{R} \oplus \mathcal{H}_2) \oplus (Q^2\mathbb{R} \oplus Q\mathcal{H}_2 \oplus \mathcal{H}_4) \oplus \mathcal{H}_3.$$

Again, $\mathfrak{m}(1)$, $\mathfrak{m}(Q)$, $\mathfrak{m}(Q^2)$ being already invariants, we can apply Theorem 3.14 on the following coordinates ($p = 3$):

$$\begin{aligned}
y_{-1,0} &= \frac{1}{\sqrt{2}} (\mathfrak{m}(QX_1^{-1}) + i \mathfrak{m}(QX_1^1)), \quad y_{0,0} = \mathfrak{m}(QX_1^0), \quad y_{-1,0} = \frac{1}{\sqrt{2}} (\mathfrak{m}(QX_1^{-1}) - i \mathfrak{m}(QX_1^1)); \\
y_{-2,1} &= \frac{1}{\sqrt{2}} (\mathfrak{m}(X_2^{-2}) + i \mathfrak{m}(X_2^2)), \quad \dots, \quad y_{2,1} = \frac{1}{\sqrt{2}} (\mathfrak{m}(X_2^{-2}) - i \mathfrak{m}(X_2^2)); \\
y_{-2,2} &= \frac{1}{\sqrt{2}} (\mathfrak{m}(QX_2^{-2}) + i \mathfrak{m}(QX_2^2)), \quad \dots, \quad y_{2,2} = \frac{1}{\sqrt{2}} (\mathfrak{m}(QX_2^{-2}) - i \mathfrak{m}(QX_2^2)); \\
y_{-4,3} &= \frac{1}{\sqrt{2}} (\mathfrak{m}(X_4^{-4}) + i \mathfrak{m}(X_4^4)), \quad \dots, \quad y_{4,3} = \frac{1}{\sqrt{2}} (\mathfrak{m}(X_4^{-4}) - i \mathfrak{m}(X_4^4)). \\
y_{-3,4} &= \frac{1}{\sqrt{2}} (\mathfrak{m}(X_3^{-3}) + i \mathfrak{m}(X_3^3)), \quad \dots, \quad y_{3,4} = \frac{1}{\sqrt{2}} (\mathfrak{m}(X_3^{-3}) - i \mathfrak{m}(X_3^3));
\end{aligned}$$

The expressions of the invariants can be obtained with symbolic computation. Nevertheless, they become quickly unwieldy. To evaluate these invariants numerically, the better way would probably be to apply a rotation that brings

$$\left(-\frac{1}{\sqrt{2}}(y_{1,0} + y_{-1,0}), y_{0,0}, \frac{i}{\sqrt{2}}(y_{1,0} - y_{-1,0}) \right)^T \quad \text{to} \quad \left(0, 0, \sqrt{y_{0,0}^2 + 2y_{-1,0}y_{1,0}} \right)^T$$

and then evaluate the restriction of the invariants to the slice as given in Theorem 3.7.

An alternative to provide explicit expressions of moment invariants is to extend the method of [GHP19] that is based on a natural Seshadri slice for $\mathbb{R}[U, V, W]_2$. The natural bases for the polynomial rings are then the cubic harmonics. Note that if the measure is centrally symmetric then all moments of odd order are zero and therefore the rational invariants obtained by applying Theorem 3.14 are zero or undefined. The method based on [GHP19] would provide generically separating moment invariants then though.

A Rewriting an invariant in terms of the generators

We exhibited algebraically independent generators of the invariant field $\mathbb{R}(\mathcal{V})^{\text{SO}_3(\mathbb{R})}$ emphasizing their separation properties. One can also use a generating set of invariants as new *variables* of an *invariant problem*, thus *factoring out* the symmetry; see for instance [HL12, HL13, HL16] where the invariants are used to solve polynomial systems with symmetry or reduce the number of parameters in dynamical systems. In this appendix we show how to rewrite any invariants in $\mathbb{R}(\mathcal{V})^{\text{SO}_3(\mathbb{R})}$ in terms of the generators exhibited.

Consider $\tilde{R} \in \mathbb{R}(\mathcal{V})^{\text{SO}_3(\mathbb{R})}$. Then the restriction of \tilde{R} to the slice \mathcal{S} is an $\text{O}_2(\mathbb{R})$ -invariant; in other words $R = \tilde{R}|_{\mathcal{S}} \in \mathbb{R}(\mathcal{S})^{\text{O}_2(\mathbb{R})}$. If we can identify a rational function F such that $R = F(T_j, D_j, R_{mj}, I_{mj})$ then $\tilde{R} = F(\tilde{T}_j, \tilde{D}_j, \tilde{R}_{mj}, \tilde{I}_{mj})$ by Theorem 3.6. One thus can rewrite any element of $\mathbb{R}(\mathcal{V})^{\text{SO}_3(\mathbb{R})}$ as a rational function of $\tilde{\mathcal{E}}$ thanks to the following rewriting rules.

Theorem A.1. *Assume $P, Q \in \mathbb{R}[\mathcal{S}]$ and take p and q in*

$$\mathbb{R}(T_j, D_j, R_{mj}, I_{mj})[y_{0,0}; y_{1,1}; y_{-1,1}; y_{-d_j,j}, \dots, y_{-d_j,j}, 1 \leq j \leq n]$$

the reductions of P and Q by the following confluent system of rewrite rules:

$$\begin{aligned} y_{0,j} &\longrightarrow \frac{D_j}{D_0} y_{0,0} & 1 \leq j \leq p \\ y_{0,j} &\longrightarrow T_j & p+1 \leq j \leq n \\ y_{m,j} &\longrightarrow 2^{m-1} \left(\frac{R_{11}^{mj}}{R_{11}^m} - \frac{iI_{mj}}{y_{0,0}R_{11}^m} \right) y_{1,1}^m & 1 \leq j \leq n, 1 \leq m \leq d_j \\ y_{-m,j} &\longrightarrow 2^{m-1} \left(\frac{R_{11}^{mj}}{R_{11}^m} + \frac{iI_{mj}}{y_{0,0}R_{11}^m} \right) y_{-1,1}^m & 1 \leq j \leq n, 1 \leq m \leq d_j \\ y_{1,1}y_{-1,1} &\longrightarrow R_{11} \\ y_{0,0}^2 &\longrightarrow D_0 \end{aligned}$$

Then $\frac{P}{Q}$ belongs to $\mathbb{R}(\mathcal{S})^N$ iff $p = \check{p}r$ and $q = \check{q}r$ with $\check{p}, \check{q} \in \mathbb{R}(T_j, D_j, R_{mj}, I_{mj})$ and $r \in \mathbb{R}[y]$. In which case

$$\frac{P}{Q} = \frac{\check{p}(T_j(H_1), D_j(H_1), R_{mj}(H_1), I_{mj}(H_1))}{\check{q}(T_j(H_1), D_j(H_1), R_{mj}(H_1), I_{mj}(H_1))}$$

Proof. The proof relies on [MQB99]. A Gröbner basis of the ideal in $\mathbb{R}(y)[Y]$ generated by

$$\begin{aligned} &\{Y_0 Y_{0,j} - y_{0,0} y_{0,j}, 1 \leq j \leq p\} \cup \{Y_{0,j} - y_{0,j}, p+1 \leq j \leq n\} \\ &\cup \{(Y_{-1,1}^m Y_{m,j} + Y_{1,1}^m Y_{-m,j}) - (y_{-1,1}^m y_{m,j} + y_{1,1}^m y_{-m,j}), 1 \leq j \leq n, 1 \leq m \leq d_j, (m,j) \neq (1,1)\} \\ &\cup \{Y_0(Y_{-1,1}^m Y_{m,j} - Y_{1,1}^m Y_{-m,j}), -y_{0,0}(y_{-1,1}^m y_{m,j} - y_{1,1}^m y_{-m,j}), 1 \leq j \leq n, 1 \leq m \leq d_j, (m,j) \neq (1,1)\} \\ &\cup \{Y_{-1,1} Y_{1,1} - y_{-1,1} y_{1,1}, Y_0^2 - y_{0,0}^2\} \end{aligned}$$

w.r.t. to a term order that eliminates $Y_{0,j}$, $Y_{m,j}$ and $Y_{-m,j}$ is given by

$$\begin{aligned} &\{\mathbf{Y}_0^2 - y_0^2, \mathbf{Y}_{1,1} \mathbf{Y}_{-1,1} - y_{1,1} y_{-1,1}\} \\ &\cup \{\mathbf{Y}_{0,j} - \frac{y_{0,j}}{y_0} Y_0 \mid 1 \leq j \leq p\} \cup \{\mathbf{Y}_{0,j} - y_{0,j} \mid p+1 \leq j \leq n\} \\ &\cup \left\{ \mathbf{Y}_{\mathbf{m},\mathbf{j}} - \left(\frac{(y_{m,j} y_{-1,1}^m + y_{-m,j} y_{1,1}^m)}{2(y_{-1,1} y_{1,1})^m} + \frac{(y_{m,j} y_{-1,1}^m - y_{-m,j} y_{1,1}^m)}{2y_0(y_{-1,1} y_{1,1})^m} Y_0 \right) Y_{1,1}^m \mid 1 \leq j \leq n, 1 \leq m \leq d_j, (m,j) \neq (1,1) \right\}, \\ &\cup \left\{ \mathbf{Y}_{-\mathbf{m},\mathbf{j}} - \left(\frac{(y_{m,j} y_{-1,1}^m + y_{-m,j} y_{1,1}^m)}{2y_{-1,1}^m y_{1,1}^m} - \frac{(y_{m,j} y_{-1,1}^m - y_{-m,j} y_{1,1}^m)}{2y_0(y_{-1,1} y_{1,1})^m} Y_0 \right) Y_{-1,1}^m \mid 1 \leq j \leq n, 1 \leq m \leq d_j, (m,j) \neq (1,1) \right\} \end{aligned}$$

where the leading terms are in bold font. This latter can be written as :

$$\begin{aligned} &\{\mathbf{Y}_0^2 - D_0, \mathbf{Y}_{1,1} \mathbf{Y}_{-1,1} - \frac{1}{2} R_{11}\} \\ &\cup \left\{ \mathbf{Y}_{0,j} - \frac{y_{0,j}}{y_0} Y_0 \mid 1 \leq j \leq p \right\} \cup \{\mathbf{Y}_{0,j} - y_{0,j} \mid p+1 \leq j \leq n\} \\ &\cup \left\{ \mathbf{Y}_{\mathbf{m},\mathbf{j}} - 2^{m-1} \left(\frac{R_{11}^{mj}}{R_{11}^m} - \frac{iI_{mj}}{D_0 R_{11}^m} Y_0 \right) Y_{1,1}^m \mid 1 \leq j \leq n, 1 \leq m \leq d_j, (m,j) \neq (1,1) \right\}, \\ &\cup \left\{ \mathbf{Y}_{-\mathbf{m},\mathbf{j}} - 2^{m-1} \left(\frac{R_{11}^{mj}}{R_{11}^m} + \frac{iI_{mj}}{D_0 R_{11}^m} Y_0 \right) Y_{-1,1}^m \mid 1 \leq j \leq n, 1 \leq m \leq d_j, (m,j) \neq (1,1) \right\}. \end{aligned}$$

By [MQB99, Lemma 1.7 and Algorithm 1.10], we obtain the announced result. \square

Example A.2. *In the space of piezoelectricity tensors, the invariant \tilde{P} whose restriction to the slice equals $P = y_{-2,1}^3 y_{3,1}^2 + y_{2,1}^3 y_{-3,1}^2$ can be rewritten as*

$$\tilde{P} = \frac{256}{\tilde{R}_{1,1}^6} \left(\tilde{R}_{21}^3 \tilde{R}_{31}^2 - \frac{\tilde{R}_{21}^3 \tilde{I}_{31}^2}{\tilde{D}_0} + \frac{6\tilde{R}_{21}^2 \tilde{I}_{21} \tilde{R}_{31} \tilde{I}_{31}}{\tilde{D}_0} - \frac{3\tilde{R}_{21} \tilde{I}_{21}^2 \tilde{R}_{31}^2}{\tilde{D}_0} + \frac{3\tilde{R}_{21} \tilde{I}_{21}^2 \tilde{I}_{31}^2}{\tilde{D}_0^2} - \frac{2\tilde{I}_{21}^3 \tilde{R}_{31} \tilde{I}_{31}}{\tilde{D}_0^2} \right).$$

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