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Algebraically independent generators for the invariant field of $\mathrm{SO}_3(\mathbb{R})$ and $\mathrm{O}_3(\mathbb{R})$ representations $\mathbb{R}^3 \oplus \mathbb{H}$

Evelyne Hubert and Martin Jalard*
Inria Côte d'Azur, France

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Abstract

We consider the representations of the group $\mathrm{SO}_3(\mathbb{R})$ that contain the standard representation \mathbb{R}^3 as an irreducible component. This is a large class of representations containing the vector spaces of ternary forms of any odd degree, the space of Piezoelectric tensors and the classical product space $\mathbb{R}^3 \times \dots \times \mathbb{R}^3$ which appears in multiple physical and engineering situations. For any such representation we construct a set of algebraically independent rational invariants that separate orbits outside of an identified hypersurface. As such they generate the field of rational invariants.

Our key ingredient is a constructive use of Seshadri slice lemma. This latter establishes an isomorphism between the $\mathrm{SO}_3(\mathbb{R})$ -invariant field on the full representation $\mathbb{R}^3 \oplus \mathbb{H}$ with the invariant field of $\mathrm{O}_2(\mathbb{R})$ acting on a subspace $\mathbb{R} \oplus \mathbb{H}$. The generating $\mathrm{O}_2(\mathbb{R})$ -invariants are given explicitly as polynomials. We make explicit the denominators of the related $\mathrm{SO}_3(\mathbb{R})$ -invariant and provide formulae for these latter. The results are then extended to $\mathrm{O}_3(\mathbb{R})$ actions and several cases of interest to applications are detailed as examples.

*Contact: martin.jalard@inria.fr

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1 Introduction

Invariants are essential for classifying objects up to a group of transformations. In this article we consider a large class of representations of $\mathrm{SO}_3(\mathbb{R})$ among which some are relevant to domains as continuum mechanics and 3D image analysis. We exhibit a set of algebraically independent rational invariants that is *complete* in the sense that any other rational invariant can be written as a rational function of these. We speak of generating invariants. They furthermore separate orbits outside of a hypersurface that we identify explicitly.

The separation power of rational invariants [PV94; Ros56] is a main draw as compared to polynomial invariants. Though in the case of $\mathrm{SO}_3(\mathbb{R})$, a compact group, a set of generators of the ring of polynomial invariants separate (real) orbits, its cardinality is often much larger than that of a set of generators of the field of rational invariants. For non compact group and nonlinear actions, rational invariants are often the best, if not the only, option [Bür+21; HK07b; Hub12].

A practical and general algorithm to compute a generating set of rational invariants of a group action appeared in [HK07a]; see also [DK15; Hub19; MB99]. A key idea of this construction, the use of a cross-section, was refined to provide specific algorithms for group actions relevant to diverse applications [GHP19; HL13]. In the present article the cross-section we use is actually a Seshadri slice. Its property allows us to lift $\mathrm{O}_2(\mathbb{R})$ -invariants, which have little mystery, to $\mathrm{SO}_3(\mathbb{R})$ -invariant.

We consider the representations of the group $\mathrm{SO}_3(\mathbb{R})$ that contain the standard representation \mathbb{R}^3 as an irreducible component. This is a large class of representations containing the vector spaces of ternary forms of any odd degree, the space of piezoelectric tensors [AG20], 3D moments of a shape [FSZ17], and, last but not least, the classical product space $\mathbb{R}^3 \times \dots \times \mathbb{R}^3$ that appears in multiple physical and engineering situations.

Seshadri slice lemma establishes an isomorphism between the $\mathrm{SO}_3(\mathbb{R})$ -invariant field on the full representation $\mathbb{R}^3 \oplus \mathbb{H}$ with the invariant field of $\mathrm{O}_2(\mathbb{R})$ acting on a subspace $\mathbb{R} \oplus \mathbb{H}$. We exhibit a set of $\dim \mathbb{H} - 1$ algebraically independent invariant polynomials that generate the $\mathrm{O}_2(\mathbb{R})$ invariant field. They correspond unequivocally to algebraically independent rational $\mathrm{SO}_3(\mathbb{R})$ -invariants. Their denominators and the G -invariant hypersurface where they do not separate orbits are made explicit. Their expression can be obtained with simple symbolic manipulations. They are simple enough that we can solve explicitly the inverse problem: recover a representant of any real orbit distinguished by the values of the invariants. The results are furthermore extended to $\mathrm{O}_3(\mathbb{R})$ actions.

As a side product, our construction provides a proof of rationality for the $\mathrm{SO}_3(\mathbb{R})$ and $\mathrm{O}_3(\mathbb{R})$ invariant fields of all the representations considered. We believe this was not known. The most general result known for $\mathrm{SO}_3(\mathbb{R})$ is the *rational stability* of an almost free representation [CS05].

We examine several cases of interest, improving and completing several previous results. First we provide the invariants for action of $\mathrm{O}_3(\mathbb{R})$ on odd degree forms. The invariants for the action of $\mathrm{O}_3(\mathbb{R})$ on even degree forms were given in [GHP19]. Next we give the invariants for the action of $\mathrm{O}_3(\mathbb{R})$ for the space of piezoelectric tensors which stems from continuum mechanics. Other sets of separating invariants were provided in [Oli14; Che+19], but their cardinalities are drastically larger. Finally we provide the G invariants on the space moments which are used in 3D image analysis [FSZ17]. Many such invariants were previously constructed and applied in diverse contexts but there was no firm foundations to their independence and separating power. Some moment invariants up to order 4 were made explicit; We provide a complete and independent set of moment invariants up to any order.

The paper is organized as follows. In Section 2 we make explicit a set of polynomials of small size that form an algebraically independent set of generators for the field of $\mathrm{O}_2(\mathbb{R})$ -invariant for the large class of $\mathrm{O}_2(\mathbb{R})$ -representations relevant to our purpose. In Section 3 we introduce the class of $\mathrm{SO}_3(\mathbb{R})$ -representations of interest to us and show how their rational invariants can be deduced from the $\mathrm{O}_2(\mathbb{R})$ -invariants just constructed. We examine where they separate orbits and solve the inverse problem: given a set of values for the invariants, can we identify a real point where these values are obtained? In Section 4 we extend our

result to the construction of $O_3(\mathbb{R})$ -invariants. In Section 5 we examine closely several cases of interest: odd degree forms, piezoelectric tensors, and moment invariants. Additionally, in Appendix A, we explain how to rewrite any other $SO_3(\mathbb{R})$ -invariant on $\mathbb{R}^3 \oplus \mathbb{H}$ in terms of our generators.

For a real algebraic variety \mathcal{V} , we note $\hat{\mathcal{V}}$ its complexification.

2 Invariants for a class of $O_2(\mathbb{R})$ -representation

The goal of this article is to construct generators of $SO_3(\mathbb{R})$ -invariant fields. For the representations we shall consider these fields are isomorphic to $O_2(\mathbb{R})$ -invariant fields. As a preliminary we thus consider, in this section, the relevant representations of $O_2(\mathbb{R})$ (see Section 2.2). For each of these we make explicit a set of algebraically independent polynomial invariants that generates the field of invariants (Theorem 2.6). We describe precisely the open set on which they separate $O_2(\mathbb{R})$ -orbits.

2.1 Representations of $O_2(\mathbb{C})$

The orthogonal group $O_2(\mathbb{R})$ is often thought of as the one parameter family of linear transformations in the plane given by:

$$O_2(\mathbb{R}) := \left\{ \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}, \alpha \in \mathbb{R} \right\} \cup \left\{ \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ -\sin(\alpha) & -\cos(\alpha) \end{pmatrix}, \alpha \in \mathbb{R} \right\}.$$

If (x_{-1}, x_1) are the coordinates for the plane \mathbb{R}^2 , considering the coordinates

$$\begin{pmatrix} y_{-1} \\ y_1 \end{pmatrix} = P \begin{pmatrix} x_{-1} \\ x_1 \end{pmatrix} \text{ where } P := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

of the \mathbb{R} -vector space $\mathcal{V} = \{(y_{-1}, y_1) \in \mathbb{C}^2 \mid \bar{y}_1 = y_{-1}\}$ we obtain the parametrization

$$O_2(\mathbb{R}) \cong \left\{ \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}, \alpha \in \mathbb{R} \right\} \cup \left\{ \begin{pmatrix} 0 & e^{-i\alpha} \\ e^{i\alpha} & 0 \end{pmatrix}, \alpha \in \mathbb{R} \right\}.$$

Concomitantly [FH13, Exercise 18.2]

$$O_2(\mathbb{C}) \cong \left\{ g_z^+ = \begin{pmatrix} z & 0 \\ 0 & \frac{1}{z} \end{pmatrix}, z \in \mathbb{C}^* \right\} \cup \left\{ g_z^- = \begin{pmatrix} 0 & \frac{1}{z} \\ z & 0 \end{pmatrix}, z \in \mathbb{C}^* \right\}.$$

Proposition 2.1. [GSS88] *The irreducible representations of $O_2(\mathbb{C})$ are indexed by $j \in \mathbb{Z}$ with $j \geq -1$. They consist of representations $\phi_j : O_2(\mathbb{C}) \mapsto \text{GL}(\hat{\mathcal{V}}_j)$ where*

- for $j \in \{0, -1\}$, $\hat{\mathcal{V}}_j \cong \mathbb{C}$ is one dimensional and

$$\phi_0(g_z^+) = (1), \phi_0(g_z^-) = (1); \quad \phi_{-1}(g_z^+) = (1), \phi_{-1}(g_z^-) = (-1);$$

- for $j \geq 1$, $\hat{\mathcal{V}}_j \cong \mathbb{C}^2$ is two dimensional and

$$\phi_j(g_z^+) = \begin{pmatrix} z^j & 0 \\ 0 & z^{-j} \end{pmatrix} \quad \text{and} \quad \phi_j(g_z^-) = \begin{pmatrix} 0 & z^{-j} \\ z^j & 0 \end{pmatrix}$$

The real trace of $\hat{\mathcal{V}}_j$, for $j \geq 1$, is $\mathcal{V}_j = \{(y_{-1}, y_1) \in \mathbb{C}^2 \mid \bar{y}_1 = y_{-1}\}$ and the real irreducible representations of $O_2(\mathbb{R})$ are obtained by restricting the above ones to \mathcal{V}_j and $\{z \in \mathbb{C}^* \mid z\bar{z} = 1\}$.

2.2 Invariants for $\mathrm{SO}_2(\mathbb{C})$

Let p, q, m_1, \dots, m_r be integers with $m_1 = 1, q \geq 1$. We consider the representation

$$\rho : \mathrm{O}_2(\mathbb{C}) \longrightarrow \mathrm{GL}(\hat{\mathcal{V}}) \quad \text{where } \hat{\mathcal{V}} = \bigoplus_{j=1}^p \hat{\mathcal{V}}_0 \oplus \bigoplus_{j=1}^q \hat{\mathcal{V}}_{-1} \oplus \bigoplus_{j=1}^r \hat{\mathcal{V}}_{m_j} \quad (m_1 = 1, q \geq 1).$$

We endow $\hat{\mathcal{V}}$ with coordinates $(t_1, \dots, t_p, d_1, \dots, d_q, y_{-1}, y_1, \dots, y_{-r}, y_r)$. We first proceed to give an algebraically independent generating set for the invariant field of the normal subgroup $\mathrm{SO}_2(\mathbb{C})$ and then deduce the invariant field of the group $\mathrm{O}_2(\mathbb{C})$.

Proposition 2.2. *The following set of polynomials is algebraically independent and generates the invariant field $\mathbb{C}(\hat{\mathcal{V}})^{\mathrm{SO}_2(\mathbb{C})}$.*

$$\mathfrak{e} := \left\{ \begin{array}{ll} \mathfrak{t}_j = t_j & 1 \leq j \leq p \\ \mathfrak{d}_j = d_j & 1 \leq j \leq q \\ \mathfrak{p}_j = y_j y_{-1}^{m_j} & 1 \leq j \leq r \\ \mathfrak{q}_j = y_{-j} y_1^{m_j} & 2 \leq j \leq r \end{array} \right\}$$

Note that the omitted polynomial \mathfrak{q}_1 would be equal to \mathfrak{p}_1 . The proof of Proposition 2.2 rests on the two following lemmas. By Lemmas 2.3 and 2.4, polynomials in \mathfrak{e} are invariant and separate generic orbits. Thanks to Rosenlicht theorem [Ros56], [PV94, Lemma 2.1], they generate $\mathbb{C}(\hat{\mathcal{V}})^{\mathrm{SO}_2(\mathbb{C})}$.

Lemma 2.3. *Each polynomial in \mathfrak{e} is invariant for $\mathrm{SO}_2(\mathbb{C})$*

Proof. Take $z \in \mathbb{C}^*$ to identify some $g_z^+ \in \mathrm{SO}_2(\mathbb{C})$, and $v = (t_1, \dots, t_p, d_1, \dots, d_q, y_{-1}, y_1, \dots, y_{-r}, y_r) \in \hat{\mathcal{V}}$.

- For all $j \leq p$, $\mathfrak{t}_j(\rho(g_z^+)(v)) = t_j = \mathfrak{t}_j(v)$
- For all $j \leq q$, $\mathfrak{d}_j(\rho(g_z^+)(v)) = \det(\rho(g_z^+))d_j = d_j = \mathfrak{d}_j(v)$
- For all $1 \leq j \leq r$, $\mathfrak{p}_j(\rho(g_z^+)(v)) = z^{m_j} y_j z^{-m_j} y_{-1}^{m_j} = \mathfrak{p}_j(v)$
- For all $2 \leq j \leq r$, $\mathfrak{q}_j(\rho(g_z^+)(v)) = z^{m_j} y_1^{m_j} z^{-m_j} y_{-m_j} = \mathfrak{q}_j(v)$ □

Lemma 2.4. *For $u, v \in \{v \in \hat{\mathcal{V}} \mid y_1 y_{-1} \neq 0\}$, we have the separation property:*

$$\forall \mathfrak{f} \in \mathfrak{e}, \mathfrak{f}(u) = \mathfrak{f}(v) \Rightarrow \exists g \in \mathrm{SO}_2(\mathbb{C}), \rho(g)(u) = v.$$

Proof. Take $u = (t_1, \dots, t_p, d_1, \dots, d_q, u_{-1}, u_1, \dots, u_{-r}, u_r)$ and $v = (t'_1, \dots, t'_p, d'_1, \dots, d'_q, v_{-1}, v_1, \dots, v_{-r}, v_r)$ points in $\{v \in \hat{\mathcal{V}} \mid y_1 y_{-1} \neq 0\}$.

- For $1 \leq j \leq p$, $\mathfrak{t}_j(u) = \mathfrak{t}_j(v) \Rightarrow t_j = t'_j$.
Similarly, for $1 \leq j \leq q$, $\mathfrak{d}_j(u) = \mathfrak{d}_j(v) \Rightarrow d_j = d'_j$.
- Since $u_1 \neq 0$ and $v_1 \neq 0$, $z := \frac{v_1}{u_1} \in \mathbb{C}^*$ gives

$$\begin{aligned} \mathfrak{p}_1(u) = \mathfrak{p}_1(v) &\Rightarrow \mathfrak{p}_1(g_z^+(u)) = \mathfrak{p}_1(v) \\ &\Rightarrow z z^{-1} u_1 u_{-1} = v_1 v_{-1} \\ &\Rightarrow z v_1 u_{-1} = v_1 v_{-1} \\ &\Rightarrow z u_{-1} = v_{-1} \end{aligned}$$

That is $\phi_1(g_z^+)(u_{-1}, u_1) = (v_{-1}, v_1)$.

- Suppose furthermore that for some $2 \leq j \leq r$, $\mathfrak{p}_j(u) = \mathfrak{p}_j(v)$ and $\mathfrak{q}_j(u) = \mathfrak{q}_j(v)$. One then has:

$$\begin{aligned} \mathfrak{p}_j(u) = \mathfrak{p}_j(v) &\Rightarrow \mathfrak{p}_j(g_z^+(u)) = \mathfrak{p}_j(v) \\ &\Rightarrow z^{-m_j} u_j (zu_{-1})^{m_j} = v_j v_{-1}^{m_j} \\ &\Rightarrow z^{-m_j} u_j v_{-1}^{m_j} = v_j v_{-1}^{m_j} \\ &\Rightarrow z^{-m_j} u_j = v_j \end{aligned}$$

Similarly, $\mathfrak{q}(u) = \mathfrak{q}(v) \Rightarrow z^{m_j} u_{-j} = v_{-j}$. Hence, $\phi_{m_j}(g_z^+)(u_{-j}, u_j) = (v_{-j}, v_j)$.

We showed that if all polynomials in \mathfrak{e} coincide on u and v , $\rho(g_z^+)(u) = v$ and u and v lie on the same orbit. \square

2.3 Invariants for $O_2(\mathbb{C})$ and $O_2(\mathbb{R})$

There is a semidirect product $O_2(\mathbb{C}) = SO_2(\mathbb{C}) \rtimes C_2$. C_2 acts naturally on $\mathbb{C}(\hat{\mathcal{V}})^{SO_2(\mathbb{C})} = \mathbb{C}(\mathfrak{e})$ and

$$\mathbb{C}(\hat{\mathcal{V}})^{O_2(\mathbb{C})} = \left(\mathbb{C}(\hat{\mathcal{V}})^{SO_2(\mathbb{C})} \right)^{C_2} = \mathbb{C}(\mathfrak{e})^{C_2}.$$

The action of C_2 on \mathfrak{e} consists of permutations and sign changes. It thus amounts to a linear action of C_2 on $\mathcal{W} := \text{Vect}(f \mid f \in \mathfrak{e})$. We can thus deduce from Proposition 2.2 an algebraically independent generating set of invariants for $\mathbb{C}(\hat{\mathcal{V}})^{O_2(\mathbb{C})}$ thanks to the following easy lemma.

Lemma 2.5. *Consider $(\mathfrak{r}, \mathcal{W})$ the representation of $C_2 = \{1, -1\}$ on \mathbb{C}^n where*

$$\mathfrak{r}(-1) = \begin{pmatrix} -I_k & 0 \\ 0 & I_{n-k} \end{pmatrix}.$$

If \mathbb{C}^n is endowed with coordinates $(w_1, \dots, w_k, w_{k+1}, \dots, w_n)$ then $\{w_1 w_i, 1 \leq i \leq k\} \cup \{w_j, k+1 \leq j \leq n\}$ is an algebraically independent set of polynomials generating $\mathbb{C}(\mathcal{W})^{C_2}$.

Indeed, one checks that this set of invariants separates orbits on $\{w \in \mathcal{W} \mid w_1 \neq 0\}$.

Proposition 2.6. *The following set of algebraically independent polynomials generates the fields $\mathbb{C}(\hat{\mathcal{V}})^{O_2(\mathbb{C})}$ and $\mathbb{R}(\mathcal{V})^{O_2(\mathbb{R})}$:*

$$\mathcal{E} = \left\{ \begin{array}{lll} T_j & = & t_j, & 1 \leq j \leq p \\ D_j & = & d_j d_1, & 1 \leq j \leq q \\ R_j & = & y_j y_{-1}^{m_j} + y_j y_1^{m_j}, & 1 \leq j \leq r \\ I_j & = & \text{id}_1 (y_j y_{-1}^{m_j} - y_{-j} y_1^{m_j}), & 2 \leq j \leq r \end{array} \right\}.$$

Proof. The natural action of C_2 on $\mathbb{C}(\hat{\mathcal{V}})^{SO_2(\mathbb{C})}$ is given by

$$\underbrace{[g]}_{\in C_2} \cdot \underbrace{P}_{\in \mathbb{C}(\hat{\mathcal{V}})^{SO_2(\mathbb{C})}} := P \circ g.$$

Take $v = (t_1, \dots, t_p, d_1, \dots, d_q, y_{-1}, y_1, \dots, y_{-r}, y_r)$ a point in $\hat{\mathcal{V}}$.

- For $1 \leq j \leq p$, $(g_1^- \cdot t_j)(v) = t_j(\rho(g_1^-)v) = \phi_0(g_1^-)t_j = t_j(v)$;
- For $1 \leq j \leq q$, $(g_1^- \cdot d_j)(v) = d_j(\rho(g_1^-)v) = \phi_{-1}(g_1^-)d_j = -d_j(v)$;
- For $1 \leq j \leq r$, $(g_1^- \cdot \mathfrak{p}_j)(v) = \mathfrak{p}_j(\rho(g_1^-)v) = y_{-j} y_1^{m_j} = \mathfrak{q}_j(v)$
and $(g_1^- \cdot \mathfrak{q}_j)(v) = \mathfrak{q}_j(\rho(g_1^-)v) = y_j y_{-1}^{m_j} = \mathfrak{p}_j(v)$.

We notice that the linear space $\mathcal{W} := \text{Vect}(\mathfrak{e})$ is stable by the action of \mathbb{C}_2 . Furthermore, if we consider the

basis of \mathcal{W} given by $\left\{ \begin{array}{l} \mathfrak{t}_j, \quad 1 \leq j \leq p \\ \mathfrak{d}_j, \quad 1 \leq j \leq q \\ \mathfrak{p}_j + \mathfrak{q}_j, \quad 1 \leq j \leq r \\ \mathfrak{p}_j - \mathfrak{q}_j, \quad 2 \leq j \leq r \end{array} \right\}$, then the matrix corresponding to the action of g_1^- is

$$\begin{pmatrix} I_p & 0 & 0 & 0 \\ 0 & -I_q & 0 & 0 \\ 0 & 0 & I_r & 0 \\ 0 & 0 & 0 & -I_{r-1} \end{pmatrix}.$$

By Lemma 2.5, the following is a generating set for $\mathbb{C}(\mathcal{W})_2^{\mathbb{C}} : \mathcal{E} = \left\{ \begin{array}{l} T_j = \mathfrak{t}_j, \quad 1 \leq j \leq p \\ D_j = \mathfrak{d}_1 \mathfrak{d}_j, \quad 1 \leq j \leq q \\ R_j = \mathfrak{p}_j + \mathfrak{q}_j, \quad 1 \leq j \leq r \\ I_j = \mathfrak{i} \mathfrak{d}_1 (\mathfrak{p}_j - \mathfrak{q}_j), \quad 2 \leq j \leq r \end{array} \right\}$.

To check the algebraic independence of \mathcal{E} , note that $\#\mathcal{E} = p + q + 2r - 1 = \dim(\hat{\mathcal{V}}) - \dim(\mathbb{O}_2(\mathbb{C}))$. However, the transcendence degree of the field $\mathbb{C}(\hat{\mathcal{V}})^{\mathbb{O}_2(\mathbb{C})}$ is equal to $\dim(\hat{\mathcal{V}}) - \max_{v \in \mathcal{V}} (\dim(\mathbb{O}_2(\mathbb{C}) \cdot v))$ [PV94, Section 2.3]. Hence $\#\mathcal{E} \leq \text{deg}_{tr}(\mathbb{C}(\hat{\mathcal{V}})^{\mathbb{O}_2(\mathbb{C})})$ while generating this field. Thus \mathcal{E} is algebraically independent.

The coefficient \mathfrak{i} in I_j is added to obtain real polynomials. Indeed, recall that on the real trace

$$\mathcal{V} = \left\{ v \in \hat{\mathcal{V}} \mid t_1, \dots, t_p, d_1, \dots, d_q \in \mathbb{R}; \bar{y}_i = y_{-i}, 1 \leq i \leq n \right\}$$

we have $\mathfrak{p}_j = y_j y_{-1}^{m_j} = \bar{y}_{-j} \bar{y}_1^{m_j} = \bar{\mathfrak{q}}_j$. Hence, $\mathfrak{p}_j + \mathfrak{q}_j = 2\Re(\mathfrak{p}_j)$ and $\mathfrak{p}_j - \mathfrak{q}_j = 2\mathfrak{i}\Im(\mathfrak{p}_j)$. Thus the restriction of polynomials of \mathcal{E} to the real trace are real. it implies that they also generate $\mathbb{R}(\mathcal{V})^{\mathbb{O}_2(\mathbb{R})}$ over \mathbb{R} . \square

The polynomials of \mathcal{E} can be translated into real coordinates through the base change P^{-1} . It involves an expansion by the Newton binomial which makes the complex \mathfrak{i} disappear. Nevertheless the expressions become rather involved as shown in the following example. This explains why we mostly work with complex coordinates throughout the article.

Example 2.7. For the 8-dimensional linear representation $S_{\mathcal{T}} = 2\mathcal{V}_{-1} \oplus \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3$, an algebraically independent generating set of $\mathbb{R}(S_{\mathcal{T}})^{\mathbb{O}_2(\mathbb{R})}$ is

$$\mathcal{E}_T = \left\{ \begin{array}{l} D_1 = d_1^2 \\ D_2 = d_1 d_2 \\ R_1 = 2y_1 y_{-1} \\ R_2 = y_1^2 y_{-2} + y_{-1}^2 y_2 \\ R_3 = y_1^3 y_{-3} + y_{-1}^3 y_3 \\ I_2 = \mathfrak{id}_1 (y_1^2 y_{-2} - y_{-1}^2 y_2) \\ I_3 = \mathfrak{id}_1 (y_1^3 y_{-3} - y_{-1}^3 y_3) \end{array} \right\} = \left\{ \begin{array}{l} D_1 = d_1^2 \\ D_2 = d_1 d_2 \\ R_1 = x_1^2 + x_{-1}^2 \\ R_2 = \frac{\sqrt{2}}{2} (2x_{-2} x_{-1} x_1 - x_{-1}^2 x_2 + x_1^2 x_2) \\ R_3 = -\frac{1}{2} x_{-3} x_{-1}^3 + \frac{3}{2} x_{-3} x_{-1} x_1^2 - \frac{3}{2} x_{-1}^2 x_1 x_3 + \frac{1}{2} x_1^3 x_3 \\ I_2 = \frac{\sqrt{2}}{2} d_1 (x_{-2} x_{-1}^2 - x_{-2} x_1^2 + 2x_{-1} x_1 x_2) \\ I_3 = -\frac{1}{2} d_1 (3x_{-3} x_{-1}^2 x_1 - x_{-3} x_1^3 - x_{-1}^3 x_3 + 3x_{-1} x_1^2 x_3) \end{array} \right\}$$

The explicit expressions of the elements of \mathcal{E} allow us to determine where \mathcal{E} fails to separate orbits.

Proposition 2.8. Polynomials in \mathcal{E} separate the orbits of $\mathbb{O}_2(\mathbb{C})$ in $\hat{\mathcal{U}} = \left\{ v \in \hat{\mathcal{V}} \mid d_1 y_1 y_{-1} \neq 0 \right\}$.

3 Invariants for $\text{SO}_3(\mathbb{R})$

In this section we consider the representation of $\text{SO}_3(\mathbb{R})$ on

$$\mathcal{H} = \mathcal{H}_1 \oplus \bigoplus_{j=1}^n \mathcal{H}_{d_j}$$

where \mathcal{H}_d , for $d \in \mathbb{N}$, are the irreducible representations of $\text{SO}_3(\mathbb{R})$, where $\mathcal{H}_1 \cong \mathbb{R}^3$. We shall characterize an algebraically independent set of rational invariants that generates the invariant field $\mathbb{R}(\mathcal{H})^{\text{SO}_3(\mathbb{R})}$. We make clear where these invariants have singularities, on which open set they separate orbits and how to evaluate them. We shall furthermore solve the inverse problem: given values for these invariants provide firstly conditions for the existence of a (real) orbit where the invariants have these values and secondly an element in this orbit if it exists. All these results rest on a constructive use of Seshadri slice lemma. In the present case this latter asserts that the restriction to $\mathcal{S} = \mathbb{R} \oplus \bigoplus_{j=1}^n \mathcal{H}_{d_j}$ provides an isomorphism between $\mathbb{R}(\mathcal{H})^{\text{SO}_3(\mathbb{R})}$ and $\mathbb{R}(\mathcal{S})^{\text{O}_2(\mathbb{R})}$. We can therefore apply the results of previous section to characterize the invariants of $\text{SO}_3(\mathbb{R})$ on \mathcal{H} .

In Section 3.1 \mathcal{H}_d is made explicit as the space of harmonic ternary forms of degree d . The spherical harmonics form bases that decompose the \mathcal{H}_d into irreducible representations of $\text{O}_2(\mathbb{R})$. In Section 3.2 we recall Seshadri's lemma and exhibit a slice \mathcal{S} for \mathcal{H} . In Section 3.3 we characterize an algebraically independent set of invariants that generates $\mathbb{R}(\mathcal{H})^{\text{SO}_3(\mathbb{R})}$. They are determined by their restriction to \mathcal{S} . In Section 3.4 we show how this is sufficient to evaluate these $\text{SO}_3(\mathbb{R})$ -invariants on $\mathcal{H} \setminus \mathcal{Q}$, where \mathcal{Q} is the variety of an invariant quadratic polynomial. We also show how, with some additional symbolic computation, we can obtain the expressions for the $\text{SO}_3(\mathbb{R})$ -invariants. In Section 3.5 we examine the the inverse problem.

3.1 Representations of $\text{SO}_3(\mathbb{R})$

The irreducible representations \mathcal{H}_d , $d \in \mathbb{N}$, of $\text{SO}_3(\mathbb{R})$ are given explicitly in terms of spaces of homogeneous harmonic polynomials of degree d . The *spherical harmonics* form bases for these that decompose them into irreducible $\text{O}_2(\mathbb{R})$ representations. In this section we recall these notions and introduce the notations we shall use.

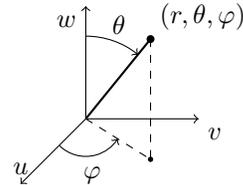
\mathcal{H}_d is the vector space of degree d real harmonic homogeneous polynomials on \mathbb{R}^3 :

$$\mathcal{H}_d := \left\{ P \in \mathbb{R}[U, V, W]_d \mid \frac{\partial^2 P}{\partial U^2} + \frac{\partial^2 P}{\partial V^2} + \frac{\partial^2 P}{\partial W^2} = 0 \right\}.$$

The standard action of $\text{SO}_3(\mathbb{R})$ on \mathbb{R}^3 induces, by composition, a linear action on the polynomial ring: $\forall P \in \mathbb{R}[U, V, W]$, $g \in \text{SO}_3(\mathbb{R})$, $g \cdot P = P \circ g^{-1}$. This action stabilises the vector space \mathcal{H}_d for any $d \in \mathbb{N}$. It thus induces the representations $\rho_d : \text{SO}_3(\mathbb{R}) \rightarrow \text{GL}(\mathcal{H}_d)$ for $d \in \mathbb{N}$. They are the only irreducible representations of $\text{SO}_3(\mathbb{R})$ [GSS88]. Among them, (ρ_0, \mathcal{H}_0) is the trivial representation and \mathcal{H}_1 is equivalent to the standard representation on \mathbb{R}^3 . We shall use the map ∇ that takes $aU + bV + cW \in \mathcal{H}_1$ $\left(\begin{smallmatrix} a & b & c \end{smallmatrix} \right)^T \in \mathbb{R}^3$ to express this equivalence. The class of representations we consider are thus all the representations where the multiplicity of (\mathcal{H}_1, ρ_1) is strictly positive. This is specific and yet covers a number of relevant cases.

The *spherical harmonics* form bases of the spaces \mathcal{H}_d . They are homogeneous polynomials in $\mathbb{R}[U, V, W]_d$ that decompose the spaces \mathcal{H}_d into irreducible representations of $\text{O}_2(\mathbb{R})$. Their expressions are nonetheless better known in spherical coordinates than in Cartesian coordinates. We thus introduce spherical coordinates (r, θ, φ) in \mathcal{H}_1 so that

$$\begin{aligned} u &= r \sin(\theta) \cos(\varphi) \\ v &= r \sin(\theta) \sin(\varphi) \\ w &= r \cos(\theta) \end{aligned}$$



Theorem 3.1. [Wig59, p. 15.3] For $0 \leq m \leq d$, note $c_d^m = \sqrt{\frac{(2d+1)(d-m)!}{4\pi(d+m)!}}$ a normalisation constant and $P_d^m = (1-t^2)^{\frac{m}{2}} \frac{1}{2^d d!} \frac{\partial^{d+m}}{\partial t^{d+m}} (t^2-1)^d$ is the $(d, m)^{\text{th}}$ associated Legendre polynomial. A basis for $\hat{\mathcal{H}}_d$ is given

by $\{Y_d^{-d}, \dots, Y_d^d\} \subset \mathbb{C}[U, V, W]_d$ where

$$\begin{aligned} Y_d^{-m}(r \sin(\theta) \cos(\varphi) r \sin(\theta) \sin(\varphi), r \cos(\theta)) &= i^{-d} c_d^m r^d e^{-im\varphi} P_d^m(\cos(\theta)), \text{ for } 1 \leq m \leq d, \\ Y_d^0(r \sin(\theta) \cos(\varphi) r \sin(\theta) \sin(\varphi), r \cos(\theta)) &= c_d^0 r^d P_d^0(\cos(\theta)), \\ Y_d^m(r \sin(\theta) \cos(\varphi) r \sin(\theta) \sin(\varphi), r \cos(\theta)) &= i^d c_d^m r^d e^{im\varphi} P_d^m(\cos(\theta)), \text{ for } 1 \leq m \leq d. \end{aligned}$$

We slightly modified the basis of [Wig59] to fit better the irreducible representations of $O_2(\mathbb{R})$ defined in Section 2.1. Precisely, we multiplied each Y_d^m by $i^d(-1)^m$ and each Y_d^{-m} by i^{-d} . With the change of basis P introduced in Section 2.1 we obtain the real spherical harmonics:

$$X_d^{-m} = \frac{1}{\sqrt{2}}(Y_d^{-m} + Y_d^m), \quad X_d^0 = Y_d^0, \quad X_d^m = \frac{i}{\sqrt{2}}(Y_d^{-m} - Y_d^m)$$

so that

$$\begin{aligned} X_d^{-m}(r \sin(\theta) \cos(\varphi) r \sin(\theta) \sin(\varphi), r \cos(\theta)) &= \sqrt{2} c_d^m r^d \cos(m\varphi + \frac{d\pi}{2}) P_d^m(\cos(\theta)), \\ X_d^0(r \sin(\theta) \cos(\varphi) r \sin(\theta) \sin(\varphi), r \cos(\theta)) &= c_d^0 r^d P_d^0(\cos(\theta)), \\ X_d^m(r \sin(\theta) \cos(\varphi) r \sin(\theta) \sin(\varphi), r \cos(\theta)) &= \sqrt{2} c_d^m r^d \sin(m\varphi + \frac{d\pi}{2}) P_d^m(\cos(\theta)). \end{aligned}$$

Example 3.2. The vector space \mathcal{H}_1 , \mathcal{H}_2 and \mathcal{H}_3 are endowed with the real basis

$$\mathcal{B}_1 = \left\{ X_1^{-1} = -\sqrt{\frac{3}{4\pi}} V, \quad X_1^0 = \sqrt{\frac{3}{4\pi}} W, \quad X_1^1 = \sqrt{\frac{3}{4\pi}} U \right\},$$

$$\mathcal{B}_2 = \left\{ X_2^{-2} = \frac{\sqrt{15}}{4\sqrt{\pi}} (V^2 - U^2), \quad X_2^{-1} = -\frac{\sqrt{15}}{2\sqrt{\pi}} UW, \quad X_2^0 = \frac{\sqrt{5}}{4\sqrt{\pi}} (2W^2 - U^2 - V^2), \quad X_2^1 = -\frac{\sqrt{15}}{2\sqrt{\pi}} VW, \quad X_2^2 = -\frac{\sqrt{15}}{2\sqrt{\pi}} UV \right\},$$

$$\mathcal{B}_3 = \left\{ \begin{array}{lll} X_3^{-3} = \frac{\sqrt{70}}{8\sqrt{\pi}} (3U^2 - V^2) V, & X_3^{-2} = \frac{\sqrt{105}}{2\sqrt{\pi}} UVW, & X_3^{-1} = \frac{\sqrt{42}}{8\sqrt{\pi}} (4W^2 - U^2 - V^2) V, \\ X_3^0 = \frac{\sqrt{7}}{4\sqrt{\pi}} (2W^2 - 3U^2 - 3V^2) W, & & \\ X_{1,3} = \frac{\sqrt{42}}{8\sqrt{\pi}} (U^2 + V^2 - 4W^2) U, & X_{2,3} = \frac{\sqrt{105}}{4\sqrt{\pi}} (V^2 - U^2) W, & X_{3,3} = \frac{\sqrt{70}}{8\sqrt{\pi}} U (3V^2 - U^2) \end{array} \right\}.$$

With respect to these bases, we put $(y_{-d_j, j}, y_{-d_j+1, j}, \dots, y_{d_j-1, j}, y_{d_j, j})$ coordinates in $\hat{\mathcal{H}}_{d_j}$, and $(x_{-d_j, j}, \dots, x_{d_j, j})$ on \mathcal{H}_{d_j} . Spherical harmonics fit very well to the restricted representation of $O_2(\mathbb{C})$. To show it, we introduce Euler angles:

Definition 3.3. A rotation $g \in SO_3(\mathbb{R})$ is determined by Euler angles (α, β, γ) where $\alpha \in [0, 2\pi[$, $\beta \in [0, \pi]$ and $\gamma \in [0, 2\pi[$. It is the composition of rotations of axes w, v, w and angles (α, β, γ) :

$$g(\alpha, \beta, \gamma) := \begin{pmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{pmatrix} \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Proposition 3.4. Write $n := p + q$ such that d_j is even for $1 \leq j \leq p$, and odd for $p + 1 \leq j \leq p + q$. The restricted representation of $O_2(\mathbb{C})$ as a subgroup of $SO_3(\mathbb{C})$ on $\bigoplus_{j=1}^n \hat{\mathcal{H}}_{d_j}$ is isomorphic to

$$\bigoplus_{j=1}^p \hat{\mathcal{V}}_0 \oplus \bigoplus_{j=1}^q \hat{\mathcal{V}}_{-1} \oplus \bigoplus_{j=1}^n \bigoplus_{m=1}^{d_j} \hat{\mathcal{V}}_m.$$

Proof. Consider the embedding $O_2(\mathbb{C}) \hookrightarrow SO_3(\mathbb{C})$ given by

$$\forall \alpha \in \mathbb{R}, \quad \begin{cases} g_{e^{i\alpha}}^+ & \rightarrow g(\alpha, 0, 0) \\ g_{e^{i\alpha}}^- & \rightarrow g(\alpha, \pi, 0) \end{cases}$$

It normalises the axis of W and its action on spherical coordinates (r, θ, φ) is $\begin{cases} g_{e^{i\alpha}}^+(r, \theta, \varphi) = (r, \theta, \varphi + \alpha). \\ g_{e^{i\alpha}}^-(r, \theta, \varphi) = (r, \pi - \theta, \pi - \varphi - \alpha). \end{cases}$
We compute the induced action on Y_d^m for $d \geq 1$ and $1 \leq m \leq d$. For all $(r, \theta, \varphi) \in \mathbb{R}^3$, and α real,

$$\begin{aligned} \rho_d(g_{e^{i\alpha}}^+) (Y_d^m(r, \theta, \varphi)) &= Y_d^m(g_{e^{-i\alpha}}^+(r, \theta, \varphi)) \\ &= Y_d^m(r, \theta, \varphi - \alpha) \\ &= i^d c_d^m r^d e^{im(\varphi - \alpha)} P_d^m(\cos(\theta)) \\ &= e^{-im\alpha} Y_d^m(r, \theta, \varphi) \end{aligned}$$

$$\begin{aligned} \rho_d(g_{e^{i\alpha}}^-) (Y_d^m(r, \theta, \varphi)) &= Y_d^m(g_{e^{i\alpha}}^-(r, \theta, \varphi)) \\ &= Y_d^m(r, \pi - \theta, \pi - \varphi - \alpha) \\ &= i^d c_d^m r^d e^{im(\pi - \varphi - \alpha)} P_d^m(-\cos(\theta)) \end{aligned}$$

But for any $1 \leq m \leq d$ we have $P_d^m(-t) = (-1)^{d+m} P_d^m(t)$ [Ami11]. We thus obtain

$$\begin{aligned} \rho_d(g_{e^{i\alpha}}^-) (Y_d^m(r, \theta, \varphi)) &= i^d c_d^m r^d (-1)^m e^{im(-\varphi - \alpha)} (-1)^{d+m} P_d^m(\cos(\theta)) \\ &= i^{-d} c_d^m r^d e^{im(-\varphi - \alpha)} P_d^m(\cos(\theta)) \\ &= e^{-im\alpha} Y_d^{-m}(r, \theta, \varphi) \end{aligned}$$

Similarly, $\rho_d(g_{e^{i\alpha}}^+) (Y_d^{-m}) = e^{im\alpha} Y_d^{-m}$ and $\rho_d(g_{e^{i\alpha}}^-) (Y_d^{-m}) = e^{im\alpha} Y_d^{-m}$. Thus, the $O_2(\mathbb{C})$ action on $\text{Vect}(Y_d^{-m}, Y_d^m)$ is isomorphic to $\hat{\mathcal{V}}_m$. Meanwhile for $m = 0$, the corresponding action on $\text{Vect}(Y_d^0)$ is ϕ_{-1} when d is odd and

$$\phi_0 \text{ when } d \text{ is even. We have } \rho_j \sim \begin{cases} \phi_{-1} \oplus \bigoplus_{m=1}^{d_j} \phi_m & \text{when } d_j \text{ is odd.} \\ \phi_0 \oplus \bigoplus_{m=1}^{d_j} \phi_m & \text{when } d_j \text{ is even.} \end{cases} \quad \square$$

3.2 Seshadri slice

In this section we identify a Seshadri slice $(\mathcal{S}, \mathbb{N})$, where $\mathbb{N} \cong O_2(\mathbb{R})$, for $\mathcal{H} = \mathcal{H}_1 \oplus \bigoplus_{j=1}^n \mathcal{H}_{d_j}$. It establishes an isomorphism between $\mathbb{R}(\mathcal{H})^{\text{SO}_3(\mathbb{R})}$ and $\mathbb{R}(\mathcal{S})^{\mathbb{N}}$. Seshadri slice lemma is instrumental in proving the rationality of the invariant field of some algebraic group actions [CS05]. And indeed, in Section 3.3 we make explicit a set of algebraically independent polynomial invariants generating $\mathbb{R}(\mathcal{S})^{\mathbb{N}}$; $\mathbb{R}(\mathcal{S})^{\mathbb{N}}$ is thus rational and so is therefore $\mathbb{R}(\mathcal{H})^{\text{SO}_3(\mathbb{R})}$. We shall nonetheless go further. In Section 3.4 we show how to evaluate the corresponding invariants of $\mathbb{R}(\mathcal{H})^{\text{SO}_3(\mathbb{R})}$ at any point in \mathcal{H} . The evaluation process also provides a way to give the explicit expressions of these invariants after some symbolic manipulations.

Theorem 3.5. *Seshadri slice lemma [CS05, Theorem 5.1]. Let G be a real algebraic group acting on a vector space \mathcal{X} . Let $\mathcal{S} \subset \mathcal{X}$ be a subvector space and $\mathbb{N} < G$ the normalizer of \mathcal{S} in G . Assume that:*

- *The closure of $G \times \mathcal{S}$ is the whole of \mathcal{X} .*
- *There is a non empty \mathbb{N} -stable Zariski open subset $\mathcal{Z} \subset \mathcal{S}$ such that for all $g \in \hat{G}$ and $s \in \hat{\mathcal{Z}}$ satisfying $g(s) \in \hat{\mathcal{Z}}$, there exists $h \in \hat{\mathbb{N}}$ such that $h(s) = g(s)$.*

Then, the restriction to \mathcal{Z} of rational functions on \mathcal{X} induces a field isomorphism $\mathbb{R}(\mathcal{X})^G \cong \mathbb{R}(\mathcal{S})^{\mathbb{N}}$.

Definition 3.6. *Such a pair $(\mathcal{S}, \mathbb{N})$ is called a Seshadri slice for $G \curvearrowright \mathcal{X}$.*

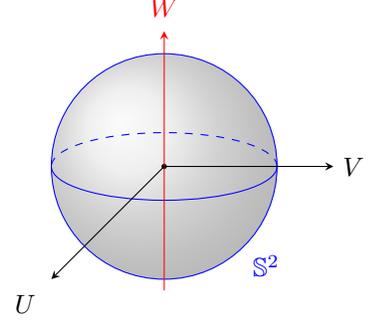
For $P \in \mathbb{R}(\mathcal{S})^{\mathbb{N}}$ there thus exists a unique $\tilde{P} \in \mathbb{R}(\mathcal{X})^G$ such that the restriction of \tilde{P} to \mathcal{Z} equals P ; We shall write $\tilde{P}|_{\mathcal{Z}} = P$.

Proposition 3.7. Consider the subspace

$$\mathcal{S} := \mathbb{R}W \oplus \bigoplus_{j=1}^n \mathcal{H}_{d_j} \subset \mathcal{H}$$

Then, $(\mathcal{S}, \mathbb{N})$ is a Seshadri slice with normalizer $\mathbb{N} \cong \mathrm{O}_2(\mathbb{R})$

$$\mathbb{N} = \left\{ \left(\begin{array}{c|c} g & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline 0 & \det(g) \end{array} \right), g \in \mathrm{O}_2(\mathbb{R}) \right\}$$



The normalizer \mathbb{N} of \mathcal{S} thus consists of the rotations in \mathbb{R}^3 that leave the W axis invariant. That includes the rotations around the W axis and the rotations by π around any axis in the (U, V) plane. The open subset \mathcal{Z} in \mathcal{S} that is relevant in the proof and later on in the article is $\mathcal{Z} = \mathbb{R}^*W \oplus \bigoplus_{j=1}^n \mathcal{H}_{d_j}$

Proof. We prove the two conditions to be a Seshadri slice:

- Take $H = H_1 + H^*$ in $\mathcal{H} = \mathcal{H}_1 \oplus \bigoplus_{j=1}^n \mathcal{H}_{d_j}$. Since (ρ_1, \mathcal{H}_1) is isomorphic to the standard representation, the orbits in \mathcal{H}_1 are the concentric spheres. Hence there exists a rotation $g \in \mathrm{SO}_3(\mathbb{R})$ such that $\rho_1(g)(H_1) = \|H_1\|W \in \mathcal{S}_1$. That is, $\rho(g)(H) \in \mathcal{S}$.

- \mathcal{Z} is \mathbb{N} -stable and its complexification is $\mathcal{Z} = \mathbb{C}^*W \oplus \bigoplus_{j=1}^n \hat{\mathcal{H}}_{d_j}$. Then for any $H \in \hat{\mathcal{Z}}$, an element $g \in \mathrm{SO}_3(\mathbb{C})$ such that $\rho(g)(H) \in \hat{\mathcal{Z}}$ satisfies $\rho_1(g) = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ a & b & c \end{pmatrix}$ with $a, b, c \in \mathbb{C}$. However we have $\rho_1(g)^t = g^t \in \mathrm{SO}_3(\mathbb{C})$ and $\rho_1(g)^t \rho_1(g) = I_3$.

$$\Rightarrow \rho_1(g)^t \rho_1(g) = \begin{pmatrix} * & * & a \\ * & * & b \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ a & b & c \end{pmatrix} \begin{pmatrix} * & * & ac \\ * & * & bc \\ ac & bc & c^2 \end{pmatrix} = I_3$$

This is $a = b = 0$ and $c = \pm 1$. Then, $g \in \hat{\mathbb{N}}$. □

3.3 Restrictions of the invariants to the slice

We saw that

$$\mathcal{S} := \mathbb{R}W \oplus \bigoplus_{j=1}^n \mathcal{H}_{d_j} \subset \mathcal{H}_1 \oplus \bigoplus_{j=1}^n \mathcal{H}_{d_j} =: \mathcal{H}$$

with normalizer $\mathbb{N} \cong \mathrm{O}_2(\mathbb{R})$ is a Seshadri slice for $(\mathcal{H}, \mathrm{SO}_3(\mathbb{R}))$. By Theorem 3.4 we know that

$$\bigoplus_{j=1}^n \mathcal{H}_{d_j} \cong \bigoplus_{j=1}^p \mathcal{V}_0 \oplus \bigoplus_{j=1}^q \mathcal{V}_{-1} \oplus \bigoplus_{j=1}^n \bigoplus_{m=1}^{d_j} \mathcal{V}_m$$

where the \mathcal{V}_m are the irreducible representations of $\mathrm{O}_2(\mathbb{R})$ of Section 2.1. This decomposition is furthermore explicitly given by a basis of spherical harmonics as introduced in Proposition 3.4. The action of \mathbb{N} on $\mathbb{R}W$ is furthermore ϕ_{-1} , the determinantal representation. We thus have the following representation of $\mathrm{O}_2(\mathbb{R}) \cong \mathbb{N}$ satisfying the hypotheses of Theorem 2.6:

$$\mathcal{S} \cong \mathcal{V}_{-1} \oplus \bigoplus_{j=1}^p \hat{\mathcal{V}}_0 \oplus \bigoplus_{j=1}^q \mathcal{V}_{-1} \oplus \bigoplus_{j=1}^n \bigoplus_{m=1}^{d_j} \mathcal{V}_m.$$

Theorem 3.8. *The following set of algebraically independent polynomials generates both $\mathbb{R}(\mathcal{S})^N$ and $\mathbb{C}(\hat{\mathcal{S}})^{\tilde{N}}$.*

$$\mathcal{E} = \left\{ \begin{array}{lll} D_0 = & y_{0,0}^2, & \\ T_j = & y_{0,j}, & 1 \leq j \leq p \\ D_j = & y_{0,0}y_{0,j}, & p+1 \leq j \leq n \\ R_{mj} = & y_{m,j}y_{-1,1}^m + y_{-m,j}y_{1,1}^m, & 1 \leq j \leq n, \quad 1 \leq m \leq d_j \\ I_{mj} = & iy_{0,0}(y_{m,j}y_{-1,1}^m - y_{-m,j}y_{1,1}^m), & 1 \leq j \leq n, \quad 1 \leq m \leq d_j, (j,m) \neq (1,1) \end{array} \right\}$$

Corollary 3.9. *Let $\rho : \mathrm{SO}_3(\mathbb{R}) \rightarrow \mathrm{GL}(\mathcal{H})$ be a representation including at least once the irreducible representation (ρ_1, \mathcal{H}_1) . Then, $\mathbb{R}(\mathcal{H})^{\mathrm{SO}_3(\mathbb{R})}$ is rational.*

Proof. The invariant field $\mathbb{R}(\mathcal{S})^N$ is rational (Theorem 2.6). According to Seshadri slice lemma, $\mathbb{R}(\mathcal{H})^{\mathrm{SO}_3(\mathbb{R})}$ is rational. \square

It was already known that an almost free $\mathrm{SO}_n(\mathbb{C})$ action (as $\mathrm{SO}_3(\mathbb{C})$ representation on $\hat{\mathcal{H}}$ whenever $n > 0$) is stably rational [CS05, Proposition 4.12], this being a weaker property than rational. Our result nonetheless also applies to actions which are not almost free.

Example 3.10. *The $\mathrm{O}_2(\mathbb{R})$ -representation $\mathcal{S}_{\mathcal{T}} = 2\mathcal{V}_{-1} \oplus \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3$ introduced in Example 2.7 is the slice corresponding to the $\mathrm{SO}_3(\mathbb{R})$ -representation $\mathcal{T} = \mathcal{H}_1 \oplus \mathcal{H}_3$. The restrictions to \mathcal{S} of an algebraically independent generating set of rational $\mathrm{SO}_3(\mathbb{R})$ -invariants are:*

$$\mathcal{E}_{\mathcal{T}} = \left\{ \begin{array}{ll} D_0 = & y_{0,0}^2 \\ D_1 = & y_{0,0}y_{0,1} \\ R_{11} = & 2y_{1,1}y_{-1,1} \\ R_{21} = & y_{1,1}^2y_{-2,1} + y_{-1,1}^2y_{2,1} \\ R_{31} = & y_{1,1}^3y_{-3,1} + y_{-1,1}^3y_{3,1} \\ I_{21} = & iy_{0,0}(y_{1,1}^2y_{-2,1} - y_{-1,1}^2y_{2,1}) \\ I_{31} = & iy_{0,0}(y_{1,1}^3y_{-3,1} - y_{-1,1}^3y_{3,1}) \end{array} \right\}.$$

Example 3.11. *Consider the representation $\mathbb{R}^3 \oplus \dots \oplus \mathbb{R}^3 \cong \mathcal{H}_1^{n+1}$. The associated slice is then $\mathcal{S}_{\mathcal{H}_1^4} = \mathcal{S}_1 \oplus \mathcal{H}_1^n$ and Theorem 3.8 gives the restrictions to the slice of a generating set of $\mathrm{SO}_3(\mathbb{R})$ -invariants.*

$$\mathcal{E}_{\mathcal{H}_1^4} = \left\{ \begin{array}{lll} D_j = & y_{0,0}y_{0,i}, & 0 \leq j \leq n \\ R_j = & y_{1,1}y_{-1,j} + y_{-1,1}y_{1,j}, & 1 \leq j \leq n \\ I_j = & iy_{0,0}(y_{1,1}y_{-1,j} - y_{-1,1}y_{1,j}), & 2 \leq j \leq n \end{array} \right\}.$$

We note (r_0, \dots, r_n) a $n+1$ -tuple of vectors in $\mathcal{H}_1^{n+1} \cong (\mathbb{R}^3)^{n+1}$. The scalar products (r_i, r_j) , $0 \leq i \leq j \leq n$ and the determinants $\det(r_i | r_j | r_k)$, $0 \leq i < j < k \leq n$, generate the invariant algebra $\mathbb{R}[\mathcal{H}_1^n]^{\mathrm{SO}_3(\mathbb{R})}$ [Wey46, Theorem 2.9.A]. Examining their restrictions to the slice, we can find their relationships with the D_j, R_j , and I_j above. By Theorem 3.5 we deduce that the corresponding $\mathrm{SO}_3(\mathbb{R})$ -invariants are:

$$\begin{aligned} \tilde{D}_j &= (r_0, r_i), & 0 \leq j \leq n, \\ \tilde{R}_j &= 2 \left((r_1, r_j) - \frac{(r_0, r_j)^2}{(r_0, r_0)} \right), & 1 \leq j \leq n, \\ \tilde{I}_j &= \det(r_0 | r_1 | r_j), & 2 \leq j \leq n. \end{aligned}$$

3.4 Evaluating the invariants

In previous section we characterized the invariants of the action of $\mathrm{SO}_3(\mathbb{R})$ on \mathcal{H} by their restrictions to the slice \mathcal{S} . In this section we address how to evaluate these invariants at any other point of \mathcal{H} . We propose to first find a rotation that takes this point to the slice. Such an intersection of the orbit of the point and the

slice is not unique, and the rotation to take it there even less so. Yet the evaluation of the slice invariants at any such point of intersection gives the same values. As we are interested by evaluation on the real points of \mathcal{H} we can use of spherical coordinates and describe the rotations by Euler angles. Euler angles are particularly effective to define a rotation $g \in \text{SO}_3(\mathbb{R})$ mapping a point in \mathcal{H} to the slice.

Lemma 3.12. *Let $H_1 + H^* \in \mathcal{H}_1 \oplus \bigoplus_{j=1}^n \mathcal{H}_{d_j}$ be a point in \mathcal{H} , with $H_1 \neq 0$. There is $(r, \theta, \varphi) \in \mathbb{R}_{>0} \times [0, \pi[\times [0, 2\pi[$ such that $H_1 = r \sin(\theta) \cos(\varphi) U + r \sin(\theta) \sin(\varphi) V + r \cos(\theta) W$. Then one has*

$$\rho(g(-\varphi, -\theta, 0))(H_1 + H^*) \in \mathcal{Z} \subset \mathcal{S}.$$

Proof. Since the gradient map $\nabla : \mathcal{H}_1 \rightarrow \mathbb{R}^3$ is an isomorphism of $\text{SO}_3(\mathbb{R})$ -representations we have $\rho_1(g(-\varphi, -\theta, 0))(H_1) = rW$. This implies $\rho(H_1 + H^*) \in \mathcal{Z}$. \square

Note that the coordinates of H_1 as above in the basis $\{X_1^{-1}, X_1^0, X_1^1\}$ are

$$x_{-1,0} = -\sqrt{\frac{4\pi}{3}} r \sin(\theta) \sin(\varphi), \quad x_{0,0} = \sqrt{\frac{4\pi}{3}} r \cos(\theta), \quad x_{1,0} = \sqrt{\frac{4\pi}{3}} r \sin(\theta) \cos(\varphi)$$

while they are

$$y_{-1,0} = i\sqrt{\frac{2\pi}{3}} r \sin(\theta) e^{i\varphi}, \quad y_{0,0} = \sqrt{\frac{4\pi}{3}} r \cos(\theta), \quad y_{1,0} = -i\sqrt{\frac{2\pi}{3}} r \sin(\theta) e^{-i\varphi}$$

in the basis $\{Y_1^{-1}, Y_1^0, Y_1^1\}$.

We thus need to compute $\rho(g(-\varphi, -\theta, 0))$. Wigner matrices provide a solution as they make explicit the action of $\text{SO}_3(\mathbb{R})$ on \mathcal{H}_d :

Definition 3.13. [Wig59, p. 15.27]. *Let $d \in \mathbb{N}$ and (α, β, γ) some Euler angles. The d^{th} Wigner matrix is the square matrix of size $(2d+1) \times (2d+1)$ defined by*

$$W_{(\alpha, \beta, \gamma)}^d = \left(e^{-im\gamma} w_{m, m'}^d(-\beta) e^{-im'\alpha} \right)_{-d \leq m, m' \leq d}$$

where

$$w_{m, m'}^d(\beta) := \delta(m, m') \sum_{s=\max(0, m'-m)}^{\min(d-m, d+m')} \frac{(-1)^s \sqrt{(d+m')!(d-m')!(d+m)!(d-m)!}}{(d+m'-s)!s!(m-m'+s)!(d-m-s)!} \cos\left(\frac{\beta}{2}\right)^{2d+m'-m-2s} \sin\left(\frac{\beta}{2}\right)^{m-m'+2s},$$

$$\text{and } \delta(m, m') = i^{d(\text{sgn}(m') - \text{sgn}(m))} (-1)^{\frac{m'+|m'|-m-|m|}{2}}.$$

The coefficient $\delta(m, m')$ comes from the base change we effected in Theorem 3.1 as compared to [Wig59], which requires to conjugate by a diagonal matrix. We use the convention $\text{sgn}(0) = 0$.

Proposition 3.14. [Wig59] *Let $d \in \mathbb{N}$ and (α, β, γ) some Euler angles defining the rotation $g(\alpha, \beta, \gamma) \in \text{SO}_3(\mathbb{R})$. For $H = y_{-d} Y_d^{-d} + \dots + y_d Y_d^d \in \mathcal{H}_d$, $\rho_d(g(\alpha, \beta, \gamma))(H) = \tilde{y}_{-d} Y_d^{-d} + \dots + \tilde{y}_d Y_d^d$ where*

$$\begin{pmatrix} \tilde{y}_{-d} \\ \vdots \\ \tilde{y}_d \end{pmatrix} = W_{(\alpha, \beta, \gamma)}^d \begin{pmatrix} y_{-d} \\ \vdots \\ y_d \end{pmatrix}.$$

As the third angle γ is basically irrelevant for the rotations taking a point in \mathcal{H} to the slice \mathcal{S} we can give the expressions of the $\mathrm{SO}_3(\mathbb{R})$ invariants as follows.

Theorem 3.15. *Consider the representation of $\mathrm{SO}_3(\mathbb{R})$ on $\mathcal{H} = \mathcal{H}_1 \oplus \bigoplus_{j=1}^n \mathcal{H}_{d_j}$. For $H_1 \in \mathcal{H}_1 \setminus \{0\}$ there is $(r, \theta, \varphi) \in \mathbb{R}_{>0} \times [0, \pi[\times [0, 2\pi[$ such that $H_1 = r \sin(\theta) \cos(\varphi) U + r \sin(\theta) \sin(\varphi) V + r \cos(\theta) W$ so that its coordinates in the spherical harmonic basis can be written*

$$y_{-1,0} = i\sqrt{\frac{2\pi}{3}} r \sin(\theta) e^{i\varphi}, \quad y_{0,0} = \sqrt{\frac{4\pi}{3}} r \cos(\theta), \quad y_{1,0} = -i\sqrt{\frac{2\pi}{3}} r \sin(\theta) e^{-i\varphi}.$$

Consider the large matrix $W(-\varphi, -\theta, 0) = \mathrm{diag}(W_{(-\varphi, -\theta, 0)}^{d_j}, 1 \leq j \leq n)$. The set of rational functions $\tilde{\mathcal{E}} := \{P \circ W(-\varphi, -\theta, 0), P \in \mathcal{E}\}$ is algebraically independent and generates $\mathbb{R}(\mathcal{H})^{\mathrm{SO}_3(\mathbb{R})}$.

Proof. Take \tilde{P} the invariant function on \mathcal{H} whose restriction to \mathcal{S} is P . Proposition 3.14 claims that $\tilde{P}(H) = \underbrace{P(W(-\varphi, -\theta, 0)(H))}_{\in \mathcal{S}}$. We thus evaluated invariants of $\tilde{\mathcal{E}}$ at $W_{(-\varphi, -\theta, 0)}(H_1 + H^*)$.

Polynomials in \mathcal{E} are rational, algebraically independent, and they furthermore generate $\mathbb{R}(\mathcal{S})^N$. Meanwhile, Seshadri slice Lemma (Theorem 3.5) claims that the restriction to \mathcal{S} is a field isomorphism between $\mathbb{R}(\mathcal{S})^N$ and $\mathbb{R}(\mathcal{H})^{\mathrm{SO}_3(\mathbb{R})}$, or $\mathbb{C}(\hat{\mathcal{S}})^{\hat{N}}$ and $\mathbb{C}(\hat{\mathcal{H}})^{\mathrm{SO}_3(\mathbb{C})}$. Hence, extensions of polynomials in $\tilde{\mathcal{E}}$ are rational, algebraically independent, and they furthermore generate $\mathbb{R}(\mathcal{H})^{\mathrm{SO}_3(\mathbb{R})}$. \square

Example 3.16. *We gave in Example 3.10 the restrictions to the slice of a generating set of rational $\mathrm{SO}_3(\mathbb{R})$ -invariants for the $\mathrm{SO}_3(\mathbb{R})$ -representation $\mathcal{T} = \mathcal{H}_1 \oplus \mathcal{H}_3$. With Theorem 3.15 and some symbolic manipulations we can give their explicit expressions as for instance $\tilde{D}_0 = y_{0,0}^2 + 2y_{-1,0}y_{1,0}$ but also*

$$\tilde{D}_1 = \frac{\sqrt{3}y_{-1,0}y_{1,0}(y_{-1,0}y_{1,3} + y_{1,0}y_{-1,3}) + \sqrt{5}(y_{1,0}^3y_{-3,3} + y_{-1,0}^3y_{3,3})}{\sqrt{2}(y_{0,0}^2 + 2y_{1,0}y_{-1,0})} - \frac{i\sqrt{3}y_{0,0}(y_{1,0}^2y_{-2,3} - y_{-1,0}^2y_{2,3})}{2(y_{0,0}^2 + 2y_{1,0}y_{-1,0})} \\ - \frac{2\sqrt{3}y_{0,0}^2(y_{-1,0}y_{1,3} + y_{1,0}y_{-1,3}) + y_{0,0}y_{0,3}(y_{0,0}^2 - 3\sqrt{5}y_{-1,0}y_{1,0})}{(y_{0,0}^2 + 2y_{1,0}y_{-1,0})}$$

The field isomorphism of Seshadri slice lemma enables us to deduce some properties for $\tilde{\mathcal{E}}$:

Proposition 3.17. *Rational functions in $\tilde{\mathcal{E}}$ separate orbits in the open set $\mathrm{SO}_3(\mathbb{R}) \times \mathcal{U}$, where $\mathcal{U} = \{v \in \mathcal{V} \mid y_{0,0}y_{1,1}y_{-1,1} \neq 0\}$.*

Proof. Take u_1 and u_2 two points in $\mathrm{SO}_3(\mathbb{R}) \times \mathcal{U}$ which are not in the same orbit. They have s_1 and s_2 representants of their respective orbits in \mathcal{U} . Since \mathcal{E} separates orbits in $\mathcal{U} \subset \mathcal{S}$, there is $f \in \mathcal{E}$ such that $f(s_1) \neq f(s_2)$. However, note that $\mathcal{U} \subset \mathcal{Z}$ and that by Seshadri slice Lemma, restriction from \mathcal{H} to \mathcal{Z} is a field isomorphism. That is $\tilde{f}(u_1) \neq \tilde{f}(u_2)$. \square

The non separated orbits are thus pointed out by $\tilde{D}_0 = 0$ or $\tilde{R}_{11} = 0$. The function \tilde{D}_0 is a polynomial, and defines furthermore the variety where the restriction of Seshadri slice Lemma does not hold:

Proposition 3.18. *The set $\hat{\mathcal{H}} \setminus (\mathrm{SO}_3(\mathbb{C}) \times \hat{\mathcal{Z}})$ is $\mathcal{V}(\tilde{D}_0)$, the variety where \tilde{D}_0 vanishes.*

Proof. Let z be a non zero point in $\mathbb{C}^3 \cong \mathcal{H}_1$. The form $v \mapsto (v^t z)$ is linear and its kernel K_z is an hyperplane.

Take $g = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ a matrix in $\mathrm{SO}_3(\mathbb{C})$. Note that $gz = (r, 0, 0) \Leftrightarrow \begin{cases} (v_2, z) = 0 \\ (v_3, z) = 0 \end{cases}$. By dimension

then, v_2, v_3 generate K_z and $K_z = v_1^\perp$.

• Suppose that $\tilde{D}_0(z) = (z, z) = 0$ and $g(z) = (*, 0, 0)$. Then, $z \in K_z$, and $(v_1, z) = 0$. Then, $(v_1, v_1) = 0$ and that is a contradiction.

• Suppose that $\tilde{D}_0(z) \neq 0$. Take then r a complex number such that $r^2 = \tilde{D}_0(z)$. With v_2, v_3 two orthonormal vectors in K_z and $v_1 = \frac{z}{r}$, one has $(v_1, v_1) = 1$ and $(v_1, v_2) = (v_1, v_3) = 0$. This is $g(z) = (r, 0, 0)$. \square

The rational functions in $\tilde{\mathcal{E}}$ can only admit singularities where the restriction of Seshadri slice Lemma fails, that is on $\mathcal{V}(\tilde{D}_0)$. Since \tilde{D}_0 is prime, it is the only possible denominator for the functions of $\tilde{\mathcal{E}}$. This appears in the expanded expressions in Example 3.16.

3.5 Inverse problem

By concatenation, we construct with functions of $\tilde{\mathcal{E}}$ a rational map $\Phi : \mathcal{H} \rightarrow \mathbb{R}^{\#\tilde{\mathcal{E}}}$ that separates orbits almost everywhere. Then the converse problem occurs: for a point $z \in \mathbb{R}^{\#\tilde{\mathcal{E}}}$, how can we determine if z is in the image $\Phi(\mathcal{H})$? And if the answer is yes, can we find a preimage $H \in \mathcal{H}$ with $\Phi(H) = z$?

Any point in \mathcal{H} has a representant of its orbit in \mathcal{S} , or it is a singular point for Φ . Thus $\Phi(\mathcal{H}) := \Phi(\mathcal{S})$. Since functions in $\tilde{\mathcal{E}}$ are polynomials on the slice, $\Phi(\mathcal{H})$ is semialgebraic, *i.e.*, a finite union of set in $\mathbb{R}^{\#\tilde{\mathcal{E}}}$ defined by polynomial equalities and inequalities.

Proposition 3.19. Let $z = \left(\begin{array}{l} t_i, \quad 1 \leq i \leq p \\ d_j, \quad 0 \leq j \leq q \\ r_{mj}, \quad 1 \leq j \leq n, \quad 1 \leq m \leq d_j \\ i_{mj}, \quad 1 \leq j \leq n, \quad 1 \leq m \leq d_j \text{ and } (m, j) \neq (1, 1) \end{array} \right) \in \mathbb{R}^{\#\tilde{\mathcal{E}}}$ a real vector.

Then, $z \in \Phi(\mathcal{H})$ iff one of the following conditions holds:

- $d_0 > 0$ and $r_{11} > 0$
- $d_0 > 0$ and $\forall j, m, r_{mj} = i_{mj} = 0$
- $d_0 = 0$ and $r_{11} > 0$ and $\forall j, m, i_{mj} = 0$ and $\forall j, d_j = 0$
- $d_0 = 0$ and $\forall j, m, r_{mj} = i_{mj} = 0$ and $\forall j, d_j = 0$

The following proof is constructive, giving thus a preimage in \mathcal{S} at the same time.

Proof. Take $z \in \mathbb{R}^{\#\tilde{\mathcal{E}}}$ a vector as in Proposition 3.19. The slice \mathcal{S} intersects all real orbits. Hence, there exists some real preimage $H \in \mathcal{H}$ iff there exists also a preimage in \mathcal{S} . Take then H a real point in the slice.

Firstly, $D_0(H_1) = y_{0,0}^2 \geq 0$ and $R_{11}(H_1) = 2y_{1,1}y_{-1,1} = 2y_{1,1}\overline{y_{1,1}} \geq 0$. Then $\Phi_{\tilde{\mathcal{E}}}(H) = z \Rightarrow d_0 \geq 0$ and $r_{11} \geq 0$. We choose $y_{0,0} = \sqrt{d_0}$ and $y_{1,1}, y_{-1,1}$ conjugated complex numbers of module $\frac{1}{2}\sqrt{r_{11}}$.

Suppose that nor d_0 or r_{11} vanish. Since $y_{1,1} \neq 0$ and $y_{0,0} \neq 0$ one can choose for $m \leq j$, $y_{-m,j} = \frac{y_{0,0}r_{mj} - i_{mj}}{2y_{0,0}y_{1,1}^m}$ and $y_{m,j} = \overline{y_{-m,j}} = \frac{y_{0,0}r_{mj} + i_{mj}}{2y_{0,0}y_{1,1}^m}$. That way,

$$\begin{aligned} R_{mj}(H_1) &= y_{1,1}^m y_{-mj} + y_{-1,1}^m y_{m,j} \\ &= y_{1,1}^m \frac{y_{0,0}r_{mj} - i_{mj}}{2y_{0,0}y_{1,1}^m} + y_{-1,1}^m \frac{y_{0,0}r_{mj} + i_{mj}}{2y_{0,0}y_{1,1}^m} \\ &= r_{mj} \\ I_{mj}(H_1) &= iy_{0,0} (y_{1,1}^m y_{-m,j} - y_{-1,1}^m y_{m,j}) \\ &= iy_{0,0} \left(y_{1,1}^m \frac{y_{0,0}r_{mj} - i_{mj}}{2y_{0,0}y_{1,1}^m} - y_{-1,1}^m \frac{y_{0,0}r_{mj} + i_{mj}}{2y_{0,0}y_{1,1}^m} \right) \\ &= i_{mj} \end{aligned}$$

Take finally $y_{0,j} = t_j$ for $j \leq p$ and $y_{0,j} = \frac{d_j}{y_{0,0}}$ for $j \leq q$. Then $\Phi_{\tilde{\mathcal{E}}}(H) = z$.

Suppose that $R_{11}(H_1) = r_{11} = 0$. Then $y_{1,1} = y_{-1,1} = 0$ and $\Phi_{\tilde{\varepsilon}}(H) = z \Rightarrow \forall j, m, r_{mj} = i_{mj} = 0$. Take just $y_{0,j} = t_j$ for $j \leq p$ and $y_{0,j} = \frac{d_j}{y_{0,0}}$ for $j \leq q$ to have $\Phi_{\tilde{\varepsilon}}(H) = z$.

Suppose that $D_0(H_1) = d_0 = 0$. Then $y_{0,0} = 0$ and $\Phi_{\tilde{\varepsilon}}(H) = z \Rightarrow \begin{cases} \forall j, & i_{mj} = y_{0,0} \Im(y_{1,1} y_{-m,j}) = 0. \\ \forall j, & d_j = y_{0,0} y_{0i} = 0. \end{cases}$

One can then choose, for all $j \leq n$ and $1 \leq m \leq d_j$, $y_{-m,j} = \frac{r_{mj}}{y_{1,1}^m}$ and $y_{m,j} = \frac{r_{mj}}{y_{-1,1}^m}$ to have $R_{mj}(H) = r_{mj}$. Take just $y_{0,j} = t_j$ for $j \leq p$ to have $\Phi_{\tilde{\varepsilon}}(H) = z$.

Suppose that $R_{11}(H_1) = r_{11} = 0$ and $D_0(H_1) = d_0 = 0$. Then $y_{1,1} = y_{-1,1} = 0$ and $y_{0,0} = 0$. One has then $\Phi_{\tilde{\varepsilon}}(H) = z \Rightarrow \begin{cases} \forall j, & i_{mj} = r_{mj} = 0. \\ \forall j, & d_j = y_{0,0} y_{0i} = 0. \end{cases}$ Take just $y_{0,j} = t_j$ for $j \leq p$ to have $\Phi_{\tilde{\varepsilon}}(H) = z$. \square

The above inequalities define a semialgebraic set $\Phi(\mathcal{H})$ birationally equivalent to the orbit space $\mathcal{H}/\text{SO}_3(\mathbb{R})$. We furthermore determine a subset birationally equivalent to the stratum of principal orbit type:

Proposition 3.20. *If $\tilde{D}_0(H) > 0$ and $\tilde{R}_{11}(H) > 0$, then the stabilizer $\text{Stab}(H)$ is trivial.*

Proof. Take H a point in \mathcal{H} and S a point of its orbit lying on the slice. Write $(y_{0,0}, 0, 0, y_{1,1}, \dots, y_{-1,1})$ the coordinates of S . Suppose that $\tilde{D}_0(H) = D_0(S) > 0$ and $\tilde{R}_{11}(H) = R_{11}(S) > 0$. That is, $y_{0,0}, y_{1,1}, y_{-1,1}$ are non zero. Meanwhile, $\text{SO}_3(\mathbb{R})$ action on \mathcal{H}_1 is dual to the canonic action of $\text{SO}_3(\mathbb{R})$ on \mathbb{R}^3 . A rotation $g \in \text{SO}_3(\mathbb{R})$ stabilizes $(0, y_{0,0}, 0)$ in \mathcal{H}_1 iff $g \in \left\{ \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \alpha \in \mathbb{R} \right\} \cong \text{SO}_2(\mathbb{R})$. This subgroup is normal in N and its action on $\text{Vect}(y_{1,-1}, y_{1,1})$ is isomorphic to \mathcal{V}_1 . Its only point with a non trivial stabilizer is $(0, 0)$. Then, $\text{Stab}(S) = \{I_3\}$. But, the stabilizers $\text{Stab}(H)$ and $\text{Stab}(S)$ are conjugated in $\text{SO}_3(\mathbb{R})$. Thus, $\text{Stab}(H) = \{I_3\}$. \square

However, the converse of Proposition 3.20 is not true: consider the point with $y_{0,1} = y_{-1,2} = y_{1,2} = 1$, all the other coordinates being zero. Then $\text{Stab}(H)$ is trivial while $\tilde{D}_0(H) = \tilde{R}_{11}(H) = 0$. On the other hand all the orbits with non trivial isotropy groups lie in $\mathcal{V}(\tilde{D}_0)$ and are mapped to zero. Our separating set has thus important limitations to caputre and stratify the orbit space. One can nonetheless design a strategy that goes beyond the present result by finding other Seshadri slices in the remaining variety $\mathcal{V}(\tilde{D}_0)$. It allows to compute step by step a complete separating system, and to stratify \mathcal{H} according to the orbit types. This is examined in for the space of piezoelectric tensors in [HJ24b].

4 Generalisation to $\text{O}_3(\mathbb{R})$ representations

We consider the representations of the group $\text{O}_3(\mathbb{R}) \triangleright \text{SO}_3(\mathbb{R})$

$$\mathcal{H} := \mathcal{H}_1 \oplus \bigoplus_{j=1}^n \mathcal{H}_{d_j} \oplus \bigoplus_{j=1}^{n^\sharp} \mathcal{H}_{e_j^\sharp}.$$

where $d_1, \dots, d_n, e_1, \dots, e_{n^\sharp}$ are integers. Both \mathcal{H}_d^\sharp and \mathcal{H}_d can be thought of as the space of harmonic homogeneous polynomials of degree d . The representation on \mathcal{H}_d is the restriction to \mathcal{H}_d of the representation of $\text{O}_3(\mathbb{R})$ on the polynomial ring $\mathbb{R}[U, V, W]$ obtained by composition with the standard representation of $\text{O}_3(\mathbb{R})$ on \mathbb{R}^3 . We call it $\rho_d : \text{O}_3(\mathbb{R}) \rightarrow \text{GL}(\mathcal{H}_d)$ as was done for $\text{SO}_3(\mathbb{R})$ in Section 3.1. The $\text{O}_3(\mathbb{R})$ representation ρ_d^\sharp on \mathcal{H}_d^\sharp is defined by

$$\rho_d^\sharp : \begin{cases} \text{O}_3(\mathbb{R}) & \longrightarrow & \text{GL}(\mathcal{H}_d^\sharp) \\ g & \longmapsto & \det(g) \rho_d(g) \end{cases}$$

The representations (ρ_d, \mathcal{H}_d) and $(\rho_d^\sharp, \mathcal{H}_d^\sharp)$, $d \in \mathbb{N}$, are the only irreducible representations of $O_3(\mathbb{R})$ [GSS88].

For simplicity we assume that d_1, \dots, d_n are sorted decreasingly. A new Seshadri slice $(\mathcal{S}, \mathbb{N}^\sharp)$ can be identified with the same subspace \mathcal{S} (Proposition 3.2) and with the normalizer $\mathbb{N}^\sharp \triangleright \mathbb{N}$, generated by \mathbb{N} and $-I_3$. It forms the direct product $\mathbb{N}^\sharp = \mathbb{N} \times C_2$. We create the polynomials

$$\begin{aligned} T_j^\sharp &= t_j^\sharp & j \leq p^\sharp \\ D_j^\sharp &= d_j^\sharp & j \leq q^\sharp \\ R_{mj}^\sharp &= y_{m,j}^\sharp y_{-1,1}^m + y_{-m,j}^\sharp y_{1,1}^m & j \leq n^\sharp \quad -e_j \leq m \leq e_j \\ I_{mj}^\sharp &= iy_{0,0}^\sharp \left(y_{m,j}^\sharp y_{-1,1}^m - y_{-m,j}^\sharp y_{1,1}^m \right) & j \leq n^\sharp \quad -e_j \leq m \leq e_j \end{aligned}$$

Theorem 4.1. *Take P a polynomial in $\mathbb{R}(\mathcal{S})^\mathbb{N}$ as follows:*

- If $n^\sharp \geq 1$, choose $P := T_j^\sharp$ (for some e_j even) or $P := D_j^\sharp$ (for some e_j odd).
- Else, if $d_1 \geq 2$, choose $P := I_{d_1 1}$.
- Else, if $n \geq 2$ and $d_2 = 1$, choose $P := I_{12}$.
- Else, choose $P := 1$.

Then, the following set of algebraically independent polynomials generates both $\mathbb{R}(\mathcal{S})^{\mathbb{N}^\sharp}$ or $\mathbb{C}(\hat{\mathcal{S}})^{\hat{\mathbb{N}}^\sharp}$:

$$\mathcal{E}^\dagger = \left\{ \begin{array}{l} D_0^\dagger = D_0 \\ T_j^\dagger = T_j, \quad d_j \text{ even} \\ D_j^\dagger = D_j, \quad d_j \text{ odd} \\ R_{mj}^\dagger = \begin{cases} R_{mj}, & d_1 m + d_j \text{ even} \\ PR_{mj}, & d_1 m + d_j \text{ odd} \end{cases} \\ I_{mj}^\dagger = \begin{cases} PI_{mj}, & d_1 m + d_j \text{ even} \\ I_{mj}, & d_1 m + d_j \text{ odd} \end{cases} \end{array} \right\} \cup \left\{ \begin{array}{l} T_j^\sharp = PT_j^\sharp, \quad e_j \text{ even} \\ D_j^\sharp = PD_j^\sharp, \quad e_j \text{ odd} \\ R_{mj}^\sharp = \begin{cases} PR_{mj}^\sharp, & d_1 m + e_j \text{ even} \\ R_{mj}^\sharp, & d_1 m + e_j \text{ odd} \end{cases} \\ I_{mj}^\sharp = \begin{cases} I_{mj}^\sharp, & d_1 m + e_j \text{ even} \\ PI_{mj}^\sharp, & d_1 m + e_j \text{ odd} \end{cases} \end{array} \right\}$$

Proof. We notice the direct product $\hat{\mathbb{N}}^\sharp = \hat{\mathbb{N}} \times C_2$. This gives $\mathbb{C}(\hat{\mathcal{S}})^{\hat{\mathbb{N}}^\sharp} = \mathbb{C}(\mathbb{C}(\hat{\mathcal{S}})^{\hat{\mathbb{N}}})^{C_2}$. Theorem 3.8 gives the same minimal set of generators for $\mathbb{C}(\hat{\mathcal{S}})^{\hat{\mathbb{N}}}$, independantly with the sharp:

$$\mathcal{E} = \left\{ \begin{array}{l} T_j, \quad d_j \text{ even} \\ D_0 \\ D_j, \quad d_j \text{ odd} \\ R_{mj}, \quad 1 \leq j \leq n, m \leq d_j \\ I_{mj}, \quad 1 \leq j \leq n, m \leq d_j, (j, m) \neq (1, 1) \end{array} \right\} \cup \left\{ \begin{array}{l} T_j^\sharp, \quad e_j \text{ even} \\ D_j^\sharp, \quad e_j \text{ odd} \\ R_{mj}^\sharp, \quad 1 \leq j \leq n^\sharp, m \leq e_j \\ I_{mj}^\sharp, \quad 1 \leq j \leq n^\sharp, m \leq e_j \end{array} \right\}$$

Take then the symmetry $\sigma = -I_3 \in \hat{\mathbb{N}}^\sharp$. In spherical coordinates it acts as $\forall (r, \theta, \varphi), \sigma((r, \theta, \varphi)) = (r, \pi - \theta, \varphi + \pi)$. By composition, for a vector $Y_{d_j}^m \in \mathcal{H}_{d_j}$, σ acts as

$$\begin{aligned} \rho_{d_j}(\sigma)(Y_{d_j}^m)((r, \theta, \varphi)) &= Y_{d_j}^m(\sigma((r, \theta, \varphi))) \\ &= Y_{d_j}^m(r, \pi - \theta, \varphi + \pi) \\ &= r^{d_j} c_d^m e^{im(\varphi + \pi)} P_{d_j}^m(-\cos(\theta)) \\ &= (-1)^{d_j} Y_{d_j}^m((r, \theta, \varphi)) \end{aligned}$$

and in a sharped representation $\mathcal{H}_{d_j}^\sharp$,

$$\begin{aligned} \rho_{d_j}^\sharp(\sigma)(Y_{d_j}^m)((r, \theta, \varphi)) &= -Y_{d_j}^m(\sigma((r, \theta, \varphi))) \\ &= -Y_{d_j}^m(r, \pi - \theta, \varphi + \pi) \\ &= -r^d c_d^m e^{im(\varphi + \pi)} P_{d_j}^m(-\cos(\theta)) \\ &= (-1)^{d_j+1} Y_{d_j}^m((r, \theta, \varphi)) \end{aligned}$$

It thus has an induced action on each elements in \mathcal{E} :

$$\left\{ \begin{array}{l} \sigma(T_j) = T_j, \quad d_j \text{ even} \\ \sigma(T_j^\sharp) = -T_j^\sharp, \quad e_j \text{ even} \\ \sigma(D_0) = D_0 \\ \sigma(D_j) = D_j, \quad d_j \text{ odd} \\ \sigma(D_j^\sharp) = -D_j^\sharp, \quad e_j \text{ odd} \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \sigma(R_{mj}) = (-1)^{d_j+d_1m} R_{mj}, \\ \sigma(R_{mj}^\sharp) = (-1)^{1+e_j+d_1m} R_{mj}^\sharp \\ \sigma(I_{mj}) = (-1)^{1+d_j+d_1m} I_{mj} \\ \sigma(I_{mj}^\sharp) = (-1)^{e_j+d_1m} I_{mj}^\sharp, \end{array} \right.$$

We next apply Lemma 2.5. We need a polynomial $P \in \mathcal{E}$ flipped by σ . Whenever $n^\sharp \geq 1$, one can choose $P := T_j^\sharp$ (for e_j even) or $P := D_j^\sharp$ (for e_j odd). Else it depends on d_1 . If $d_1 \geq 2$ then I_{11} suits. If $d_1 = 1, n \geq 2$ and $d_2 = 1$ then I_{12} suits. Otherwise, no polynomial in \mathcal{E} is flipped and $\mathbb{C}(\hat{\mathcal{S}})^{\hat{N}} = \mathbb{C}(\hat{\mathcal{S}})^{\hat{N}^\sharp}$. \square

Almost free $O_3(\mathbb{C})$ actions have a stably rational invariant field [CS05, Proposition 4.10]. The above theorem provides the following stronger result, yet with a different hypothesis as already discussed after Theorem 3.8.

Corollary 4.2. *Let $\rho : O_3(\mathbb{R}) \rightarrow GL(\mathcal{H})$ a linear representation including at least once the irreducible representation (ρ_1, \mathcal{H}_1) . Then, both $\mathbb{R}(\mathcal{H})^{O_3(\mathbb{R})}$ and $\mathbb{C}(\mathcal{H})^{O_3(\mathbb{C})}$ are rational.*

Example 4.3. *We follow up on Example 3.10 and ?? where the representation space was $\mathcal{T} = \mathcal{H}_1 \oplus \mathcal{H}_3$. We examined the $SO_3(\mathbb{R})$ -invariants, giving their restrictions to the slice. In the basis given in Example 3.2, the restrictions to the slice $\mathcal{S}_{\mathcal{T}}$ of a generating set of rational $O_3(\mathbb{R})$ -invariant is given by the following algebraically independent generating set for $\mathbb{R}(\mathcal{S}_{\mathcal{T}})^{N^\sharp}$:*

$$\mathcal{E}_{\mathcal{T}}^\dagger = \left\{ \begin{array}{l} D_1^\dagger = y_{0,0}^2 \\ D_2^\dagger = y_{0,0}y_{0,1} \\ R_1^\dagger = 2y_{1,1}y_{-1,1} \\ R_2^\dagger = (y_{1,1}^2y_{-2,1} + y_{-1,1}^2y_{2,1})^2 \\ R_3^\dagger = y_{1,1}^3y_{-3,1} + y_{-1,1}^3y_{3,1} \\ I_2^\dagger = iy_{0,0}(y_{1,1}^2y_{-2,1} - y_{-1,1}^2y_{2,1}) \\ I_3^\dagger = iy_{0,0}(y_{1,1}^3y_{-3,1} - y_{-1,1}^3y_{3,1})(y_{1,1}^2y_{-2,1} + y_{-1,1}^2y_{2,1}) \end{array} \right\}$$

5 Applications

We have exhibited invariants for representations of $SO_3(\mathbb{R})$ and $O_3(\mathbb{R})$ on spaces $\mathbb{R}^3 \oplus \mathbb{H}$. Among those is the well studied case of $\mathbb{R}^3 \times \dots \times \mathbb{R}^3$ treated to some extent in Example 3.11. This representation and its invariants are relevant in several areas [BV23; DCP21; DG24; GD04; Wey46]. In [HJ24a] we treat the general case, that of $SO_n(\mathbb{R})$ and $O_n(\mathbb{R})$ acting on $\mathbb{R}^n \times \dots \times \mathbb{R}^n$ and provide algebraically independent sets of generators for the invariant field, relating them with the classical Weyl invariants [Wey46]. In this section we expand on two other cases of relevance for which we give explicitly the restrictions of the invariants to the slice. As explained in Section 3, from these expressions we can evaluate the invariants at any point outside of an hypersurface, as well as solve the inverse problem.

5.1 Odd degree ternary forms

Among the cases of interest to which our construction applies is the representation of $\mathrm{SO}_3(\mathbb{R})$ or $\mathrm{O}_3(\mathbb{R})$ on the vector space of homogeneous polynomials $\mathbb{R}[U, V, W]_d$ of odd degree d . The case of even degrees was treated in [GHP19]. The starting point for both odd and even degree is the harmonic decomposition.

Lemma 5.1. [AH12, Theorem 2.18] $\mathbb{R}[U, V, W]_d = \bigoplus_{j=0}^{\lfloor d/2 \rfloor} Q^j \mathcal{H}_{d-2j}$ where $Q = U^2 + V^2 + W^2$.

Whenever d is odd, the last term of this decomposition is \mathcal{H}_1 and thus we can apply the construction provided in previous sections.

Corollary 5.2. For any $d \in \mathbb{N}$, the invariant field of the $\mathrm{SO}_3(\mathbb{R})$ or $\mathrm{O}_3(\mathbb{R})$ action on $\mathbb{R}[U, V, W]_d$ is rational.

Example 5.3. The vector space $\mathcal{T} = \mathcal{H}_1 \oplus \mathcal{H}_3$ introduced in Example 3.10 is isomorphic to the space of ternary cubics $\mathbb{R}[U, V, W]_3$. The basis of $\mathbb{C}[U, V, W]_3$ we use is:

$$\{QY_1^{-1}, QY_1^0, QY_1^1; Y_3^{-3}, Y_3^{-2}, Y_3^{-1}, Y_3^0, Y_3^1, Y_3^2, Y_3^3\}.$$

In this basis an element of $\mathbb{C}[U, V, W]_3$ is given by its vector of coefficients

$$(y_{-1,0}, y_{0,0}, y_{1,0}; y_{-3,1}, y_{-2,1}, y_{-1,1}, y_{0,1}, y_{1,1}, y_{2,1}, y_{3,1})^T.$$

This element belongs to $\mathbb{R}[U, V, W]_3$ when $y_{-m,j} = \bar{y}_{m,j}$.

The Seshadri slice for $\mathbb{R}[U, V, W]_3$ is $S_{\mathcal{T}} = \langle QV \rangle \oplus \mathcal{H}_3$ with stabilizer $N \cong \mathrm{O}_2(\mathbb{R})$. The set $\mathcal{E}_{\mathcal{T}}$ in Example 3.10 is an algebraically independent generating set of $\mathbb{R}(S_{\mathcal{T}})^N$. By Theorem 3.5, each of its element is the restriction of a unique rational $\mathrm{SO}_3(\mathbb{R})$ -invariant for the action $\mathrm{SO}_3(\mathbb{R})$ on $\mathbb{R}[U, V, W]_3$. The expression of this latter can be obtained by applying Theorem 3.15 and symbolic manipulations. By Theorem 3.5 these lifted invariants form an algebraically independent generating set of $\mathrm{SO}_3(\mathbb{R})$ -invariants for $\mathbb{R}[U, V, W]_3$. They separate orbits everywhere but in $\{\tilde{D}_0 = 0\} = \{y_{0,0}^2 + 2y_{-1,1}y_{1,1} = 0\}$ and $\{\tilde{R}_{11} = 0\}$.

5.2 Piezoelectricity tensors

For some material of industrial interest (single crystals, ceramics, and thin films, . . .), an electric displacement is induced in response to an applied mechanical stress. This property is called the piezoelectric effect and a third-order tensor relates the induced electric displacement vector to the second-order stress tensor [AG20].

The orthogonal group $\mathrm{SO}_3(\mathbb{R})$ acts naturally on the 18-dimensional space of piezoelectric tensors

$$\mathbb{P}\mathrm{iez} := \{e \in \otimes^3 \mathbb{R}^3, e_{ijk} = e_{ikj}\}.$$

As such it decomposes into $\mathrm{SO}_3(\mathbb{R})$ -irreducible subspaces as [Azz+22, Section 7.1]

$$\mathbb{P}\mathrm{iez} \cong \mathcal{H}_1 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3.$$

A tensor in $\mathbb{P}\mathrm{iez}$ is accordingly given by a coordinate vector

$$(y_{-1,0}, y_{0,0}, y_{1,0}; y_{-1,1}, y_{0,1}, y_{1,1}; y_{-2,2}, y_{-1,2}, y_{0,2}, y_{1,2}, y_{2,2}; y_{-3,3}, y_{-2,3}, y_{-1,3}, y_{0,3}, y_{1,3}, y_{2,3}, y_{3,3}),$$

where $y_{-m,j} = \bar{y}_{m,j}$, when the components \mathcal{H}_d are endowed the bases of Theorem 3.1.

A Seshadri slice for $\mathrm{SO}_3(\mathbb{R})$ acting on $\mathbb{P}\mathrm{iez}$ is given by the subspace \mathcal{S} defined by the equations $y_{-1,0} = 0$ and $y_{1,0} = 0$ that is stabilized by $N \cong \mathrm{O}_2(\mathbb{R})$. Theorem 2.6 gives us an algebraically independent set of generators for $\mathbb{R}(\mathcal{S})^N$, namely:

Proposition 5.4. *The following set is an algebraically independent and generates $\mathbb{R}(\mathcal{S})^{\mathbb{N}}$.*

$$\mathcal{E}_{\text{Piez}} = \left\{ \begin{array}{lll} T_2 = y_{0,2} & R_{11} = 2y_{-1,1}y_{1,1} & I_{12} = iy_{0,0}(y_{1,2}y_{-1,1} - y_{-1,2}y_{1,1}) \\ D_0 = y_{0,0}^2 & R_{12} = y_{1,2}y_{-1,1} + y_{-1,2}y_{1,1} & I_{22} = iy_{0,0}(y_{2,2}y_{-1,1}^2 - y_{-2,2}y_{1,1}^2) \\ D_1 = y_{0,0}y_{0,1} & R_{22} = y_{2,2}y_{-1,1}^2 + y_{-2,2}y_{1,1}^2 & I_{13} = iy_{0,0}(y_{1,3}y_{-1,1} - y_{-1,3}y_{1,1}) \\ D_3 = y_{0,0}y_{0,3} & R_{13} = y_{1,3}y_{-1,1} + y_{-1,3}y_{1,1} & I_{23} = iy_{0,0}(y_{2,3}y_{-1,1}^2 - y_{-2,3}y_{1,1}^3) \\ & R_{23} = y_{2,3}y_{-1,1}^2 + y_{-2,3}y_{1,1}^2 & I_{33} = iy_{0,0}(y_{3,3}y_{-1,1}^3 - y_{-3,3}y_{1,1}^3) \\ & R_{33} = y_{3,3}y_{-1,1}^3 + y_{-3,3}y_{1,1}^3 & \end{array} \right\}$$

Here is its expression in real coordinates (see Section 3.3):

$$\mathcal{E}_{\text{Piez}} = \left\{ \begin{array}{ll} T_2 = x_{0,2} & R_{1,1} = x_{-1,1}^2 + x_{1,1}^2 \\ D_0 = x_{0,0}^2 & R_{1,2} = x_{-1,1}x_{-1,2} + x_{1,1}x_{1,2} \\ D_1 = x_{0,0}x_{0,1} & R_{2,2} = \frac{\sqrt{2}}{2}(2x_{-2,2}x_{-1,1}x_{1,1} - x_{-1,1}^2x_{2,2} + x_{1,1}^2x_{2,2}) \\ D_3 = x_{0,0}x_{0,3} & R_{1,3} = x_{-1,1}x_{-1,3} + x_{1,1}x_{1,3} \\ & R_{2,3} = \frac{\sqrt{2}}{2}(2x_{-2,3}x_{-1,1}x_{1,1} - x_{-1,1}^2x_{2,3} + x_{1,1}^2x_{2,3}) \\ & R_{3,3} = -\frac{1}{2}x_{-3,3}x_{-1,1}^3 + \frac{3}{2}x_{-3,3}x_{-1,1}x_{1,1}^2 - \frac{3}{2}x_{-1,1}^2x_{1,1}x_{3,3} + \frac{1}{2}x_{1,1}^3x_{3,3} \\ & I_{1,2} = -x_{0,0}(x_{-1,1}x_{1,2} - x_{-1,2}x_{1,1}) \\ & I_{2,2} = -\frac{\sqrt{2}}{2}x_{0,0}(x_{-2,2}x_{-1,1}^2 - x_{-2,2}x_{1,1}^2 + 2x_{-1,1}x_{1,1}x_{2,2}) \\ & I_{1,3} = -x_{0,0}(x_{-1,1}x_{1,3} - x_{-1,3}x_{1,1}) \\ & I_{2,3} = -\frac{\sqrt{2}}{2}x_{0,0}(x_{-2,3}x_{-1,1}^2 - x_{-2,3}x_{1,1}^2 + 2x_{-1,1}x_{1,1}x_{2,3}) \\ & I_{3,3} = -\frac{1}{2}x_{0,0}(3x_{-3,3}x_{-1,1}^2x_{1,1} - x_{-3,3}x_{1,1}^3 - x_{-1,1}^3x_{3,3} + 3x_{-1,1}x_{1,1}^2x_{3,3}) \end{array} \right\}$$

Each of these invariant is the restriction to the slice of a unique rational $\text{SO}_3(\mathbb{R})$ -invariant on Piez . The expression for this $\text{SO}_3(\mathbb{R})$ -invariant can be obtained by Theorem 3.15. The set of these provides an algebraically independent set $\tilde{\mathcal{E}}_{\text{Piez}}$ generating $\mathbb{R}(\text{Piez})^{\text{SO}_3(\mathbb{R})}$. This set separates orbits wherever \tilde{D}_0 and the denominator of \tilde{R}_{11} do not vanish.

In [Azz+22, Section 7.1], the group acting on the Piezoelectricity tensors is actually $\text{O}_3(\mathbb{R})$. The $\text{O}_3(\mathbb{R})$ representation is then $\text{Piez} \cong \mathcal{H}_1 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2^{\#} \oplus \mathcal{H}_3$. Here the first term $d_1 = 1$ is odd, and $T_2^{\#} = y_{0,2}$ of Theorem 4.1 can be chosen as the flipped polynomial P . Applying Theorem 4.1:

Proposition 5.5. *The following set of cardinal 15 is an algebraically independent generating set for both $\mathbb{C}(\hat{\mathcal{S}})^{\mathbb{N}^{\#}}$ or $\mathbb{R}(\mathcal{S})^{\mathbb{N}^{\#}}$.*

$$\mathcal{E}_{\text{Piez}}^{\#} = \left\{ \begin{array}{lll} T_2^{\#} = y_{0,2}^2 & R_{11} = y_{1,1}y_{-1,1} + y_{-1,1}y_{1,1} & I_{12}^{\#} = iy_{0,0}(y_{1,2}y_{-1,1} - y_{-1,2}y_{1,1}) \\ D_0 = y_{0,0}^2 & R_{12}^{\#} = y_{0,2}(y_{1,2}y_{-1,1} + y_{-1,2}y_{1,1}) & I_{22}^{\#} = iy_{0,0}y_{0,2}(y_{2,2}y_{-1,1}^2 - y_{-2,2}y_{1,1}^2) \\ D_1 = y_{0,0}y_{0,1} & R_{22}^{\#} = y_{2,2}y_{-1,1}^2 + y_{-2,2}y_{1,1}^2 & I_{13} = iy_{0,0}y_{0,2}(y_{1,3}y_{-1,1} - y_{-1,3}y_{1,1}) \\ D_3 = y_{0,0}y_{0,3} & R_{13} = y_{1,3}y_{-1,1} + y_{-1,3}y_{1,1} & I_{23} = iy_{0,0}(y_{2,3}y_{-1,1}^2 - y_{-2,3}y_{1,1}^3) \\ & R_{23} = y_{0,2}(y_{2,3}y_{-1,1}^2 + y_{-2,3}y_{1,1}^2) & I_{33} = iy_{0,0}y_{0,2}(y_{3,3}y_{-1,1}^3 - y_{-3,3}y_{1,1}^3) \\ & R_{33} = y_{3,3}y_{-1,1}^3 + y_{-3,3}y_{1,1}^3 & \end{array} \right\}$$

Each of these invariant is the restriction to the slice of a unique rational $\text{O}_3(\mathbb{R})$ -invariant on Piez . The set of these provides an algebraically independent set $\tilde{\mathcal{E}}_{\text{Piez}}$ generating $\mathbb{R}(\text{Piez})^{\text{O}_3(\mathbb{R})}$.

5.3 3D Moment Invariants

Several field of engineering are confronted with differentiating or classifying 3D shapes up to a group of transformations, and in particular the group $\text{SO}_3(\mathbb{R})$ of rotations. Moments are descriptors of a shape that are obtained by acquisition devices or by computation. Moment invariants are then the descriptors

of the shape independently of rotation. For two dimensional images, many approaches to the computation and application of moment invariants were developed. The books [FSZ09; FSZ17] are entry points to the intricate literature for mathematical and methodological advances together with their concrete applications. The seminal papers in the 3D case seem to be [Guo93; Hu62; LD89]. A sample of applications in medicine and biology can be found in [Gon+13; Man+04; Som+07].

Finding analytical expressions of moment invariants is the one facet that we wish to expand on here. Several methods have been designed to produce 3D moment invariants under $\text{SO}_3(\mathbb{R})$ with recent contributions [BB23; Li+17; SFB15; Sun24]. Explicit expressions for some moment invariants up to tenth order were exhibited. Yet their *completeness* and *independence* has not been settled. The main result of this article, Theorem 3.14, applies straightforwardly and provide explicit expressions of an algebraically independent set of moment invariants, up to any order, that is complete and minimal in the sense that any other rational invariants can be written as a rational function of these.

Consider a measure μ with compact support in \mathbb{R}^3 . The associated moment functional is the linear form

$$\begin{aligned} \mathbf{m}_\mu : \mathbb{R}[u, v, w] &\rightarrow \mathbb{R} \\ p &\mapsto \int_{\mathbb{R}^3} p d\mu \end{aligned}$$

The measure can for instance be given by a density function $\omega : \mathbb{R}^3 \rightarrow \mathbb{R}_{\geq 0}$ with compact support so that the moments are

$$\mathbf{m}_\mu(p) = \int_{\mathbb{R}^3} p(u, v, w) \omega(u, v, w) du dv dw.$$

For instance ω can be the characteristic function of a 3D shape. Another important case is when μ is an atomic measure, *i.e.*, a linear combination of a finite number Diracs centered at $\xi_1, \dots, \xi_r \in \mathbb{R}^3$ with positive coefficients ω_i . Then

$$\mathbf{m}_\mu(p) = \sum_{i=1}^r \omega_i p(\xi_i).$$

Assume ω is the characteristic function of a shape and that this shape is transformed by an invertible linear map $a : \mathbb{R}^3 \rightarrow \mathbb{R}^3$. Then the characteristic function of the transformed shape is $\omega \circ a^{-1}$. More generally, for a measure μ and its transform μ^a by a we have $\mathbf{m}_{\mu^a}(p) = \mathbf{m}_\mu(p \circ a)$.

For a group representation $G \rightarrow \text{GL}(\mathbb{R}^3)$, if $A_p^{(d)} : G \rightarrow \text{GL}_{r_d}(\mathbb{R})$ is the matrix of the induced representation on $\mathbb{R}[u, v, w]_d$ in the basis $\{p_1, \dots, p_{r_d}\}$, *i.e.* $[p_1 \circ g^{-1} \ \dots \ p_{r_d} \circ g^{-1}] = [p_1 \ \dots \ p_{r_d}] A_p^{(d)}(g)$, then, in the dual basis, the matrix of the induced representation on the dual space $(\mathbb{R}[u, v, w]_d)^*$ is $M_{p^*}^{(d)} : G \rightarrow \text{GL}_{r_d}(\mathbb{R})$ with $M_{p^*}^{(d)}(g) = \left(A_p^{(d)}(g)\right)^{-T}$. Concomitantly

$$\begin{bmatrix} \mathbf{m}_{\mu^g}(p_1) \\ \dots \\ \mathbf{m}_{\mu^g}(p_{r_d}) \end{bmatrix} = \left(A^{(d)}(g^{-1})\right)^T \begin{bmatrix} \mathbf{m}_\mu(p_1) \\ \dots \\ \mathbf{m}_\mu(p_{r_d}) \end{bmatrix}.$$

In the case of a compact group we can always find bases where the matrices $A_p^{(d)}(g)$ are orthogonal so that the induced representations on $\mathbb{R}[u, v, w]_d$ and its dual $(\mathbb{R}[u, v, w]_d)^*$ are equivalent. For $\text{SO}_3(\mathbb{R})$, the real spherical harmonics defined in Section 3.1 provide such bases.

The action of $\text{SO}_3(\mathbb{R})$ on moments of order 0 up to order d is thus equivalent to the action of $\text{SO}_3(\mathbb{R})$ on $\mathfrak{M}_d = \mathbb{R} \oplus \mathbb{R}[u, v, w]_1 \oplus \dots \oplus \mathbb{R}[u, v, w]_d$. Given that

$$\mathbb{R}[u, v, w]_d = \bigoplus_{k=0}^{\lfloor \frac{d}{2} \rfloor} (u^2 + v^2 + w^2)^k \mathcal{H}_{d-2k}$$

we can apply Theorem 3.14 to provide the moment invariants up to any order. For instance for $d = 4$ the basis of \mathfrak{M}_4 is, with $q = u^2 + v^2 + w^2$,

$$\{1\} \cup \{X_1^{-1}, X_1^0, X_1^0\} \cup \{q; X_2^{-2}, \dots, X_2^2\} \cup \{qX_1^{-1}, qX_1^0, qX_1^0; X_3^{-3}, \dots, X_3^3\} \cup \{q^2; qX_2^{-2}, \dots, qX_2^2; X_4^{-4}, \dots, X_4^4\}.$$

The moments $\mathbf{m}(1)$, $\mathbf{m}(q)$, $\mathbf{m}(q^2)$ are already invariants. To obtain a generating set of moment invariants one has to add the invariants obtained from Theorem 3.14 with the following coordinates that are arranged according to parity of the degrees ($p = 3$, $n = 5$ in the notation of Theorem 3.14).

$$\begin{aligned} y_{-1,0} &= \frac{1}{\sqrt{2}} (\mathbf{m}(X_1^{-1}) + i \mathbf{m}(X_1^1)), \quad y_{0,0} = \mathbf{m}(X_1^0), \quad y_{1,0} = \frac{1}{\sqrt{2}} (\mathbf{m}(X_1^{-1}) - i \mathbf{m}(X_1^1)); \\ y_{-2,1} &= \frac{1}{\sqrt{2}} (\mathbf{m}(X_2^{-2}) + i \mathbf{m}(X_2^2)), \quad \dots, \quad y_{2,1} = \frac{1}{\sqrt{2}} (\mathbf{m}(X_2^{-2}) - i \mathbf{m}(X_2^2)); \\ y_{-2,2} &= \frac{1}{\sqrt{2}} (\mathbf{m}(qX_2^{-2}) + i \mathbf{m}(qX_2^2)), \quad \dots, \quad y_{2,2} = \frac{1}{\sqrt{2}} (\mathbf{m}(qX_2^{-2}) - i \mathbf{m}(qX_2^2)); \\ y_{-4,3} &= \frac{1}{\sqrt{2}} (\mathbf{m}(X_2^{-4}) + i \mathbf{m}(X_2^4)), \quad \dots, \quad y_{-4,3} = \frac{1}{\sqrt{2}} (\mathbf{m}(X_2^{-4}) - i \mathbf{m}(X_2^4)); \\ y_{-1,4} &= \frac{1}{\sqrt{2}} (\mathbf{m}(qX_1^{-1}) + i \mathbf{m}(qX_1^1)), \quad \dots, \quad y_{1,4} = \frac{1}{\sqrt{2}} (\mathbf{m}(qX_1^{-1}) - i \mathbf{m}(qX_1^1)); \\ y_{-3,5} &= \frac{1}{\sqrt{2}} (\mathbf{m}(X_3^{-3}) + i \mathbf{m}(X_3^3)), \quad \dots, \quad y_{3,5} = \frac{1}{\sqrt{2}} (\mathbf{m}(X_3^{-3}) - i \mathbf{m}(X_3^3)) \end{aligned}$$

It is nonetheless often assumed that $\mathbf{m}(\mathbb{R}[u, v, w]_1) = \{0\}$. This is obtained by applying an appropriate translation to the measure. One thus has to deal with the action of $\text{SO}_3(\mathbb{R})$ on $\mathfrak{M}_d^o = \mathbb{R} \oplus \mathbb{R}[u, v, w]_2 \oplus \mathbb{R}[u, v, w]_3 \oplus \dots \oplus \mathbb{R}[u, v, w]_d$. One can still apply Theorem 3.14 since $\mathbb{R}[u, v, w]_3 = q\mathcal{H}_1 \oplus \mathcal{H}_3$. We reorganize the components of \mathfrak{M}_d^o as

$$\mathfrak{M}_d^o = q\mathcal{H}_1 \oplus \mathbb{R} \oplus (q\mathbb{R} \oplus \mathcal{H}_2) \oplus (q^2\mathbb{R} \oplus q\mathcal{H}_2 \oplus \mathcal{H}_4) \oplus \mathcal{H}_3.$$

Again, $\mathbf{m}(1)$, $\mathbf{m}(q)$, $\mathbf{m}(q^2)$ being already invariants, we can apply Theorem 3.14 on the following coordinates ($p = 3$):

$$\begin{aligned} y_{-1,0} &= \frac{1}{\sqrt{2}} (\mathbf{m}(qX_1^{-1}) + i \mathbf{m}(qX_1^1)), \quad y_{0,0} = \mathbf{m}(qX_1^0), \quad y_{-1,0} = \frac{1}{\sqrt{2}} (\mathbf{m}(qX_1^{-1}) - i \mathbf{m}(qX_1^1)), ; \\ y_{-2,1} &= \frac{1}{\sqrt{2}} (\mathbf{m}(X_2^{-2}) + i \mathbf{m}(X_2^2)), \quad \dots, \quad y_{2,1} = \frac{1}{\sqrt{2}} (\mathbf{m}(X_2^{-2}) - i \mathbf{m}(X_2^2)); \\ y_{-2,2} &= \frac{1}{\sqrt{2}} (\mathbf{m}(qX_2^{-2}) + i \mathbf{m}(qX_2^2)), \quad \dots, \quad y_{2,2} = \frac{1}{\sqrt{2}} (\mathbf{m}(qX_2^{-2}) - i \mathbf{m}(qX_2^2)); \\ y_{-4,3} &= \frac{1}{\sqrt{2}} (\mathbf{m}(X_4^{-4}) + i \mathbf{m}(X_4^4)), \quad \dots, \quad y_{4,3} = \frac{1}{\sqrt{2}} (\mathbf{m}(X_4^{-4}) - i \mathbf{m}(X_4^4)). \\ y_{-3,4} &= \frac{1}{\sqrt{2}} (\mathbf{m}(X_3^{-3}) + i \mathbf{m}(X_3^3)), \quad \dots, \quad y_{3,4} = \frac{1}{\sqrt{2}} (\mathbf{m}(X_3^{-3}) - i \mathbf{m}(X_3^3)); \end{aligned}$$

Though they can be obtained with symbolic computation the expressions of the invariants become quickly unwieldy. To evaluate these invariants numerically, the better way would probably be to apply a rotation that brings

$$\left(-\frac{1}{\sqrt{2}}(y_{1,0} + y_{-1,0}), y_{0,0}, \frac{i}{\sqrt{2}}(y_{1,0} - y_{-1,0}) \right)^T \quad \text{to} \quad \left(0, 0, \sqrt{y_{0,0}^2 + 2y_{-1,0}y_{1,0}} \right)^T$$

as soon as possible, and then evaluate the restriction of the invariants to the slice as given in Theorem 3.7.

An alternative to provide explicit expressions of moment invariants is to extend the method of [GHP19] that is based on a natural Seshadri slice for $\mathbb{R}[u, v, w]_2$. The natural bases for the polynomial rings are then the cubic harmonics. Note that if the measure is centrally symmetric then all moments of odd order are zero and therefore the rational invariants obtained by applying Theorem 3.14 are zero or undefined. The method based on [GHP19] would provide separating moment invariants then though.

A Rewriting an invariant in terms of the generators

In this article we exhibited algebraically independent generators of the invariant field $\mathbb{R}(\mathcal{H})^{\text{SO}_3(\mathbb{R})}$ emphasizing their separation properties. One can also use a generating set of invariants as new *variables* of an *invariant problem*, thus *factoring out* the symmetry; see for instance [HL12; HL13; HL16] where are examined the use of invariants of Abelian groups to solve polynomial systems or reduce the number of parameters in dynamical systems. In this appendix we show how to rewrite any invariants in $\mathbb{R}(\mathcal{H})^{\text{SO}_3(\mathbb{R})}$ in terms of the generators exhibited.

Consider $\tilde{R} \in \mathbb{R}(\mathcal{H})^{\text{SO}_3(\mathbb{R})}$. Then the restriction of \tilde{R} to the slice \mathcal{S} is an $\text{O}_2(\mathbb{R})$ -invariant; in other words $R = \tilde{R}|_{\mathcal{S}} \in \mathbb{R}(\mathcal{S})^{\text{O}_2(\mathbb{R})}$. If we can identify a rational function F such that $R = F(T_j, D_j, R_{mj}, I_{mj})$ then $\tilde{R} = F(\tilde{T}_j, \tilde{D}_j, \tilde{R}_{mj}, \tilde{I}_{mj})$ by Theorem 3.5. One thus can rewrite any element of $\mathbb{R}(\mathcal{H})^{\text{SO}_3(\mathbb{R})}$ as a rational function of $\tilde{\mathcal{E}}$ thanks to the following rewrite rules.

Theorem A.1. *Assume $P, Q \in \mathbb{R}[\mathcal{S}]$ and take p and q in*

$$\mathbb{R}(T_j, D_j, R_{mj}, I_{mj})[y_{0,0}; y_{1,1}; y_{-1,1}; y_{-d_j,j}, \dots, y_{-d_j,j}, 1 \leq j \leq n]$$

the reductions of P and Q by the following confluent system of rewrite rules:

$$\begin{aligned} y_{0,j} &\longrightarrow T_j & 1 \leq j \leq p \\ y_{0,j} &\longrightarrow \frac{D_j}{D_0} y_{0,0} & p+1 \leq j \leq n \\ y_{m,j} &\longrightarrow 2^{m-1} \left(\frac{R_{mj}}{R_{11}^m} - \frac{i I_{mj}}{y_{0,0} R_{11}^m} \right) y_{1,1}^m & 1 \leq j \leq n, 1 \leq m \leq d_j \\ y_{-m,j} &\longrightarrow 2^{m-1} \left(\frac{R_{mj}}{R_{1,1}^m} + \frac{i I_{mj}}{y_{0,0} R_{1,1}^m} \right) y_{-1,1}^m & 1 \leq j \leq n, 1 \leq m \leq d_j \\ y_{1,1} y_{-1,1} &\longrightarrow R_{11} \\ y_{0,0}^2 &\longrightarrow D_0 \end{aligned}$$

Then $\frac{P}{Q}$ belongs to $\mathbb{R}(\mathcal{S})^N$ iff $p = \check{p}r$ and $q = \check{q}r$ with $\check{p}, \check{q} \in \mathbb{R}(T_j, D_j, R_{mj}, I_{mj})$ and $r \in \mathbb{R}[y]$. In which case

$$\frac{P}{Q} = \frac{\check{p}(T_j(H_1), D_j(H_1), R_{mj}(H_1), I_{mj}(H_1))}{\check{q}(T_j(H_1), D_j(H_1), R_{mj}(H_1), I_{mj}(H_1))}$$

Proof. The proof relies on [MS99]. A Gröbner basis of the ideal in $\mathbb{R}(y)[Y]$ generated by

$$\begin{aligned} &\{Y_{0,j} - y_{0,j}, 1 \leq j \leq p\} \cup \{Y_0 Y_{0,j} - y_{0,0} y_{0,j}, p+1 \leq j \leq n\} \\ &\cup \{(Y_{-1,1}^m Y_{m,j} + Y_{1,1}^m Y_{-m,j}) - (y_{-1,1}^m y_{m,j} + y_{1,1}^m y_{-m,j}), 1 \leq j \leq n, 1 \leq m \leq d_j, (m, j) \neq (1, 1)\} \\ &\cup \{Y_0 (Y_{-1,1}^m Y_{m,j} - Y_{1,1}^m Y_{-m,j}), -y_{0,0} (y_{-1,1}^m y_{m,j} - y_{1,1}^m y_{-m,j}), 1 \leq j \leq n, 1 \leq m \leq d_j, (m, j) \neq (1, 1)\} \\ &\cup \{Y_{-1,1} Y_{1,1} - y_{-1,1} y_{1,1}, Y_0^2 - y_{0,0}^2\} \end{aligned}$$

w.r.t. to a term order that eliminates $Y_{0,j}$, $Y_{m,j}$ and $Y_{-m,j}$ is given by

$$\begin{aligned} & \{ \mathbf{Y}_0^2 - y_0^2, \mathbf{Y}_{1,1} \mathbf{Y}_{-1,1} - y_{1,1} y_{-1,1} \} \\ & \cup \{ \mathbf{Y}_{0,j} - y_{0,j} \mid 1 \leq j \leq p \} \cup \left\{ \mathbf{Y}_{0,j} - \frac{y_{0,j}}{y_0} Y_0 \mid p+1 \leq j \leq n \right\} \\ & \cup \left\{ \mathbf{Y}_{\mathbf{m},\mathbf{j}} - \left(\frac{(y_{m,j} y_{-1,1}^m + y_{-m,j} y_{1,1}^m)}{2(y_{-1,1} y_{1,1})^m} + \frac{(y_{m,j} y_{-1,1}^m - y_{-m,j} y_{1,1}^m)}{2y_0 (y_{-1,1} y_{1,1})^m} Y_0 \right) Y_{1,1}^m \mid 1 \leq j \leq n, 1 \leq m \leq d_j, (m,j) \neq (1,1) \right\}, \\ & \cup \left\{ \mathbf{Y}_{-\mathbf{m},\mathbf{j}} - \left(\frac{(y_{m,j} y_{-1,1}^m + y_{-m,j} y_{1,1}^m)}{2y_{-1,1}^m y_{1,1}^m} - \frac{(y_{m,j} y_{-1,1}^m - y_{-m,j} y_{1,1}^m)}{2y_0 (y_{-1,1} y_{1,1})^m} Y_0 \right) Y_{-1,1}^m \mid 1 \leq j \leq n, 1 \leq m \leq d_j, (m,j) \neq (1,1) \right\} \end{aligned}$$

where the leading terms are in bold font. This latter can be written as :

$$\begin{aligned} & \{ \mathbf{Y}_0^2 - D_0, \mathbf{Y}_{1,1} \mathbf{Y}_{-1,1} - \frac{1}{2} R_{11} \} \\ & \cup \{ \mathbf{Y}_{0,j} - T_j \mid 1 \leq j \leq p \} \cup \left\{ \mathbf{Y}_{0,j} - \frac{D_j}{D_0} Y_0 \mid p+1 \leq j \leq n \right\} \\ & \cup \left\{ \mathbf{Y}_{\mathbf{m},\mathbf{j}} - 2^{m-1} \left(\frac{R_{mj}}{R_{11}^m} - \frac{i I_{mj}}{D_0 R_{11}^m} Y_0 \right) Y_{1,1}^m \mid 1 \leq j \leq n, 1 \leq m \leq d_j, (m,j) \neq (1,1) \right\}, \\ & \cup \left\{ \mathbf{Y}_{-\mathbf{m},\mathbf{j}} - 2^{m-1} \left(\frac{R_{mj}}{R_{11}^m} + \frac{i I_{mj}}{D_0 R_{11}^m} Y_0 \right) Y_{-1,1}^m \mid 1 \leq j \leq n, 1 \leq m \leq d_j, (m,j) \neq (1,1) \right\}. \end{aligned}$$

By [MS99, Lemma 1.7 and Algorithm 1.10], we obtain the announced result. \square

Example A.2. In the space of piezoelectricity tensor, the invariant \tilde{P} whose restriction to the slice equals $P = y_{-2,1}^3 y_{3,1}^2 + y_{2,1}^3 y_{-3,1}^2$ can be rewritten as

$$\tilde{P} = \frac{256}{\tilde{R}_{1,1}^6} \left(\tilde{R}_{21}^3 \tilde{R}_{31}^2 - \frac{\tilde{R}_{21}^3 \tilde{I}_{31}^2}{\tilde{D}_0} + \frac{6 \tilde{R}_{21}^2 \tilde{I}_{21} \tilde{R}_{31} \tilde{I}_{31}}{\tilde{D}_0} - \frac{3 \tilde{R}_{21} \tilde{I}_{21}^2 \tilde{R}_{31}^2}{\tilde{D}_0} + \frac{3 \tilde{R}_{21} \tilde{I}_{21}^2 \tilde{I}_{31}^2}{\tilde{D}_0^2} - \frac{2 \tilde{I}_{21}^3 \tilde{R}_{31} \tilde{I}_{31}}{\tilde{D}_0^2} \right).$$

B Conflicts of interest

The authors have no conflict of interest to declare that are relevant to this article.

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