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Algebraically independent generators for the invariant field of $\mathrm{SO}_3(\mathbb{R})$ and $\mathrm{O}_3(\mathbb{R})$ representations $\mathbb{R}^3 \oplus \mathbb{H}$

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Abstract

We consider the representations of the group $\mathrm{SO}_3(\mathbb{R})$ that contain the standard representation \mathbb{R}^3 as an irreducible component. This is a large class of representations containing the vector spaces of ternary forms of any odd degree, the space of Piezoelectric tensors and the classical product space $\mathbb{R}^3 \times \dots \times \mathbb{R}^3$ which appears in multiple physical and engineering situations. For any such representation we construct a set of algebraically independent rational invariants that separate orbits outside of an identified hypersurface. As such they generate the field of rational invariants.

Our key ingredient is a constructive use of Seshadri slice lemma. This latter establishes an isomorphism between the $\mathrm{SO}_3(\mathbb{R})$ -invariant field on the full representation $\mathbb{R}^3 \oplus \mathbb{H}$ with the invariant field of $\mathrm{O}_2(\mathbb{R})$ acting on a subspace $\mathbb{R} \oplus \mathbb{H}$. The $\mathrm{O}_2(\mathbb{R})$ -invariants are given explicitly as polynomials. We predict the denominators of the related $\mathrm{SO}_3(\mathbb{R})$ -invariants and show how to evaluate these latter thanks to Wigner matrices. The results are then extended to $\mathrm{O}_3(\mathbb{R})$ actions and several cases of interest to applications are detailed as examples.

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1 Introduction

Invariants are useful for classifying objects up to a group of transformations. In this article we consider a large class of representations of $\mathrm{SO}_3(\mathbb{R})$ among which some are relevant to domains as continuum mechanics and machine learning. We construct a generating set of rational invariants that is algebraically independent.

Computational Invariant Theory [DK15; Stu93] has long focused on polynomial invariants. In the case of the group $\mathrm{O}_3(\mathbb{R})$ or $\mathrm{SO}_3(\mathbb{R})$, any two real orbits are separated by a generating set of polynomial invariants. The generating set can nonetheless be very large, of very high degree and difficult to numerically evaluate at a given point robustly. A generating set of rational invariants is typically of smaller cardinality and yet they separate orbits in general position [PV94; Ros56] – this remains true for any group, even for non-reductive groups. Rational invariants can thus prove to be sufficient, and sometimes more relevant, in applications and in connection with other mathematical disciplines [Bür+21; HK07b; Hub12]. A practical and general algorithm to compute a generating set of rational invariants of a group action appeared in [HK07a]; see also [DK15; Hub19; MB99]. The main idea of this construction was refined to provide specific algorithms for group actions relevant to diverse applications [GHP19; HL13]. As corollary of these constructions, the invariant field of these actions is seen to be rational, a problem sometimes referred as Noether’s problem and amply addressed in the literature, as reviewed in [CS05].

In this article we consider the representations of the group $\mathrm{SO}_3(\mathbb{R})$ that contain the standard representation \mathbb{R}^3 as an irreducible component. This is a large class of representations containing the vector spaces of ternary forms of any odd degree, the space of piezoelectric tensors and, last but not least, the classical product space $\mathbb{R}^3 \times \dots \times \mathbb{R}^3$ that appears in multiple physical and engineering situations. The key constructive refinement we introduce, to exhibit a generating set of rational invariants, is based on Seshadri slice lemma.

Seshadri slice lemma establishes an isomorphism between the $\mathrm{SO}_3(\mathbb{R})$ -invariant field on the full representation $\mathbb{R}^3 \oplus \mathbb{H}$ with the invariant field of $\mathrm{O}_2(\mathbb{R})$ acting on a subspace $\mathbb{R} \oplus \mathbb{H}$. We give explicitly a set of $\dim \mathbb{H} - 1$ algebraically independent invariant polynomials that generate the $\mathrm{O}_2(\mathbb{R})$ invariant field. They correspond to algebraically independent rational $\mathrm{SO}_3(\mathbb{R})$ -invariant and we make explicit what are the denominators of these and how to evaluate them where defined. Conversely, we solve the inverse problem: recover a representant of any real orbit distinguished by the values of the invariants.

The results are then extended to $\mathrm{O}_3(\mathbb{R})$ actions and several cases of interest to applications are detailed as examples. As a side product, our construction provides a proof of rationality for the invariant fields of all the representations considered; only *rational stability* is ensured other free $\mathrm{SO}_3(\mathbb{R})$ -representations [CS05].

In continuum mechanics, if a minimal set of generators of the invariant field were determined for the action of $\mathrm{SO}_3(\mathbb{R})$ on the space of elasticity tensors [Des+19], this is the first time a minimal generating set of $\mathrm{SO}_3(\mathbb{R})$ -rational invariants is given for the space of piezoelectric tensors. Other separating sets of invariants are provided in [Oli14; Che+19], but their cardinalities are drastically larger.

The paper is organized as follows. In Section 2 we make explicit a set of polynomials of small size that form an algebraically independent set of generators for the field of $\mathrm{O}_2(\mathbb{R})$ -invariant for the large class of $\mathrm{O}_2(\mathbb{R})$ -representations relevant to our purpose. In Section 3 we introduce the class of $\mathrm{SO}_3(\mathbb{R})$ -representations of interest to us and show how their rational invariants can be deduced from the $\mathrm{O}_2(\mathbb{R})$ -invariants just constructed. We show how to evaluate them numerically on real data and examine where they separate orbits. We furthermore solve the inverse problem: given a set of values for the invariants, can we identify a real point where these values are obtained? In Section 4 we extend our result to the construction of $\mathrm{O}_3(\mathbb{R})$ -invariants. And finally in Section 5 we give explicitly the generating $\mathrm{O}_3(\mathbb{R})$ or $\mathrm{SO}_3(\mathbb{R})$ -invariants for cases of interest in applications. In AppendixA we explain how to rewrite any other $\mathrm{SO}_3(\mathbb{R})$ -invariant on $\mathbb{R}^3 \oplus \mathbb{H}$ in terms of our generators.

For a real algebraic variety \mathcal{V} , we note $\hat{\mathcal{V}}$ its complexification.

2 Invariants for a class of $O_2(\mathbb{R})$ -representation

The goal of this article is to construct generators of $SO_3(\mathbb{R})$ -invariant fields. For the representations we shall consider these fields are isomorphic to $O_2(\mathbb{R})$ -invariant fields. As a preliminary we thus consider, in this section, the relevant representations of $O_2(\mathbb{R})$ (see Section 2.2). For each of these we make explicit a set of algebraically independent polynomial invariants that generates the field of invariants (Theorem 2.8). We describe precisely the open set on which they separate $O_2(\mathbb{R})$ -orbits.

2.1 Representations of $O_2(\mathbb{C})$

The orthogonal group $O_2(\mathbb{R})$ is often thought of as the one parameter family of linear transformations in the plane given by:

$$O_2(\mathbb{R}) := \left\{ \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}, \alpha \in \mathbb{R} \right\} \cup \left\{ \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{pmatrix}, \alpha \in \mathbb{R} \right\}.$$

If (x_1, x_{-1}) are the coordinates for the plane \mathbb{R}^2 , considering the coordinates

$$\begin{pmatrix} y_1 \\ y_{-1} \end{pmatrix} = P \begin{pmatrix} x_1 \\ x_{-1} \end{pmatrix} \text{ where } P := \begin{pmatrix} 1 & \mathbf{i} \\ 1 & -\mathbf{i} \end{pmatrix}$$

of the \mathbb{R} -vector space $\mathcal{V} = \{(y_1, y_{-1}) \in \mathbb{C}^2 \mid \bar{y}_1 = y_{-1}\}$ we obtain the parametrization

$$O_2(\mathbb{R}) \cong \left\{ \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}, \alpha \in \mathbb{R} \right\} \cup \left\{ \begin{pmatrix} 0 & e^{i\alpha} \\ -e^{i\alpha} & 0 \end{pmatrix}, \alpha \in \mathbb{R} \right\}.$$

Concomitantly [FH13, Exercise 18.2]

$$O_2(\mathbb{C}) \cong \left\{ g_z^+ = \begin{pmatrix} z & 0 \\ 0 & \frac{1}{z} \end{pmatrix}, z \in \mathbb{C}^* \right\} \cup \left\{ g_z^- = \begin{pmatrix} 0 & \frac{1}{z} \\ z & 0 \end{pmatrix}, z \in \mathbb{C}^* \right\}.$$

Proposition 2.1. [GSS88] *The irreducible representations of $O_2(\mathbb{C})$ are indexed by $j \in \mathbb{Z}$ with $j \geq -1$. They consist of representations $\phi_j : O_2(\mathbb{C}) \mapsto \text{GL}(\hat{\mathcal{V}}_j)$ where*

- for $j \in \{0, -1\}$, $\hat{\mathcal{V}}_j \cong \mathbb{C}$ is one dimensional and

$$\phi_0(g_z^+) = (1), \phi_0(g_z^-) = (1); \quad \phi_{-1}(g_z^+) = (1), \phi_{-1}(g_z^-) = (-1);$$

- for $j \geq 1$, $\hat{\mathcal{V}}_j \cong \mathbb{C}^2$ is two dimensional and

$$\phi_j(g_z^+) = \begin{pmatrix} z^j & 0 \\ 0 & z^{-j} \end{pmatrix} \quad \text{and} \quad \phi_j(g_z^-) = \begin{pmatrix} 0 & z^{-j} \\ z^j & 0 \end{pmatrix}$$

The real trace of $\hat{\mathcal{V}}_j$, for $j \geq 1$, is $\mathcal{V}_j = \{(y_1, y_{-1}) \in \mathbb{C}^2 \mid \bar{y}_1 = y_{-1}\}$ and the real irreducible representations of $O_2(\mathbb{R})$ are obtained by restricting the above ones to \mathcal{V}_j and $\{z \in \mathbb{C}^* \mid z\bar{z} = 1\}$.

2.2 Invariants for $SO_2(\mathbb{C})$

Let p, q, m_1, \dots, m_r be integers with $m_1 = 1, q \geq 1$. We consider the representation

$$\rho : O_2(\mathbb{C}) \longrightarrow \text{GL}(\hat{\mathcal{V}}) \text{ where } \hat{\mathcal{V}} = \bigoplus_{j=1}^p \hat{\mathcal{V}}_0 \oplus \bigoplus_{j=1}^q \hat{\mathcal{V}}_{-1} \oplus \bigoplus_{j=1}^r \hat{\mathcal{V}}_{m_j} \quad (m_1 = 1, q \geq 1).$$

We endow $\hat{\mathcal{V}}$ with coordinates $(t_1, \dots, t_p, d_1, \dots, d_q, y_1, y_{-1}, \dots, y_r, y_{-r})$. We first proceed to give an algebraically independent generating set for the invariant field of the normal subgroup $\mathrm{SO}_2(\mathbb{C})$ and then deduce the invariant field of the group $\mathrm{O}_2(\mathbb{C})$.

Proposition 2.2. *The following set of polynomials is algebraically independent and generates the invariant field $\mathbb{C}(\hat{\mathcal{V}})^{\mathrm{SO}_2(\mathbb{C})}$.*

$$\mathfrak{e} := \begin{pmatrix} \mathfrak{t}_j = t_j & 1 \leq j \leq p \\ \mathfrak{d}_j = d_j & 1 \leq j \leq q \\ \mathfrak{p}_j = y_j y_{-1}^{m_j} & 1 \leq j \leq r \\ \mathfrak{q}_j = y_{-j} y_1^{m_j} & 2 \leq j \leq r \end{pmatrix}$$

Note that the omitted polynomial \mathfrak{q}_1 would be equal to \mathfrak{p}_1 . The proof of Proposition 2.2 rests on the two following lemmata. By Lemmas 2.3 and 3.5, polynomials in \mathfrak{e} are invariant and separate orbits in the Zariski open set $\hat{\mathcal{U}}_1$. Thanks to Rosenlicht theorem [Ros56], [PV94, Lemma 2.1], they generate $\mathbb{C}(\hat{\mathcal{V}})^{\mathrm{SO}_2(\mathbb{C})}$.

Lemma 2.3. *Each polynomial in \mathfrak{e} is invariant for $\mathrm{SO}_2(\mathbb{C})$*

Proof. Take $z \in \mathbb{C}^*$ to identify some $g_z^+ \in \mathrm{SO}_2(\mathbb{C})$, and $v = (t_1, \dots, t_p, d_1, \dots, d_q, y_1, y_{-1}, \dots, y_r, y_{-r}) \in \hat{\mathcal{V}}$.

- For all $j \leq p$, $\mathfrak{t}_j(\rho(g_z^+)(v)) = t_j = \mathfrak{t}_j(v)$
- For all $j \leq q$, $\mathfrak{d}_j(\rho(g_z^+)(v)) = \det(\rho(g_z^+))d_j = d_j = \mathfrak{d}_j(v)$
- For all $1 \leq j \leq r$, $\mathfrak{p}_j(\rho(g_z^+)(v)) = z^{m_j} y_j z^{-m_j} y_{-1}^{m_j} = \mathfrak{p}_j(v)$
- For all $2 \leq j \leq r$, $\mathfrak{q}_j(\rho(g_z^+)(v)) = z^{m_j} y_1^{m_j} z^{-m_j} y_{-j} = \mathfrak{q}_j(v)$ □

Lemma 2.4. *The set \mathfrak{e} separates orbits in $\hat{\mathcal{U}}_1 := \{v \in \hat{\mathcal{V}} \mid y_1 y_{-1} \neq 0\}$. That is for $u, v \in \hat{\mathcal{U}}_1$ we have*

$$\forall \mathfrak{f} \in \mathfrak{e}, \mathfrak{f}(u) = \mathfrak{f}(v) \Rightarrow \exists g \in \mathrm{SO}_2(\mathbb{C}), \rho(g)(u) = v.$$

Proof. Take $u = (t_1, \dots, t_p, d_1, \dots, d_q, u_1, u_{-1}, \dots, u_r, u_{-r})$ and $v = (t'_1, \dots, t'_p, d'_1, \dots, d'_q, v_1, v_{-1}, \dots, v_r, v_{-r})$ points in $\hat{\mathcal{U}}_1$.

- For $1 \leq j \leq p$, $\mathfrak{t}_j(u) = \mathfrak{t}_j(v) \Rightarrow t_j = t'_j$.
Similarly, for $1 \leq j \leq q$, $\mathfrak{d}_j(u) = \mathfrak{d}_j(v) \Rightarrow d_j = d'_j$.
- Since $u_1 \neq 0$ and $v_1 \neq 0$, $z := \frac{v_1}{u_1} \in \mathbb{C}^*$ gives

$$\begin{aligned} \mathfrak{p}_1(u) = \mathfrak{p}_1(v) &\Rightarrow \mathfrak{p}_1(g_z^+(u)) = \mathfrak{p}_1(v) \\ &\Rightarrow \frac{z u_1 u_{-1}}{z} = v_1 v_{-1} \\ &\Rightarrow \frac{v_1 u_{-1}}{z} = v_1 v_{-1} \\ &\Rightarrow \frac{u_{-1}}{z} = v_{-1} \end{aligned}$$

That is $\phi_1(g_z^+)(u_1, u_{-1}) = (v_1, v_{-1})$.

- Suppose furthermore that for some $2 \leq j \leq r$, $\mathfrak{p}_j(u) = \mathfrak{p}_j(v)$ and $\mathfrak{q}_j(u) = \mathfrak{q}_j(v)$. One then has:

$$\begin{aligned} \mathfrak{p}_j(u) = \mathfrak{p}_j(v) &\Rightarrow \mathfrak{p}_j(g_z^+(u)) = \mathfrak{p}_j(v) \\ &\Rightarrow z^{m_j} u_j \left(\frac{u_{-1}}{z}\right)^{m_j} = v_j v_{-1}^{m_j} \\ &\Rightarrow z^{m_j} u_j v_{-1}^{m_j} = v_j v_{-1}^{m_j} \\ &\Rightarrow z^{m_j} u_j = v_j \end{aligned}$$

Similarly, $\mathfrak{q}_j(u) = \mathfrak{q}_j(v) \Rightarrow z^{-m_j} u_{-j} = v_{-j}$. Hence, $\phi_{m_j}(g_z^+)(u_j, u_{-j}) = (v_j, v_{-j})$.

We showed that if all polynomials in \mathfrak{e} coincide on u and v , $\rho(g_z^+)(u) = v$ and u and v lie on the same orbit. □

2.3 Invariants for $O_2(\mathbb{C})$

$SO_2(\mathbb{C})$ is a normal subgroup of $O_2(\mathbb{C})$ with $O_2(\mathbb{C})/SO_2(\mathbb{C}) = \{g_1^+, g_1^-\} \cong C_2$. There is thus a natural action of C_2 on $\mathbb{C}(\hat{\mathcal{V}})^{SO_2(\mathbb{C})} = \mathbb{C}(\mathfrak{e})$ and

$$\mathbb{C}(\hat{\mathcal{V}})^{O_2(\mathbb{C})} = \left(\mathbb{C}(\hat{\mathcal{V}})^{SO_2(\mathbb{C})} \right)^{C_2} = \mathbb{C}(\mathfrak{e})^{C_2}.$$

The action of C_2 on \mathfrak{e} consists of permutations and sign changes. It thus amounts to a linear action of C_2 on $\mathcal{W} := \text{Vect}(f \mid f \in \mathfrak{e})$. We can thus deduce from Proposition 2.2 an algebraically independent generating invariants for $\mathbb{C}(\hat{\mathcal{V}})^{O_2(\mathbb{C})}$ thanks to the following easy lemma.

Lemma 2.5. *Consider $(\mathfrak{r}, \mathcal{W})$ the representation of $C_2 = \{1, -1\}$ on \mathbb{C}^n where*

$$\mathfrak{r}(-1) = \begin{pmatrix} -I_k & 0 \\ 0 & I_{n-k} \end{pmatrix}.$$

If \mathbb{C}^n is endowed with coordinates $(w_1, \dots, w_k, w_{k+1}, \dots, w_n)$ then $\{w_1 w_i, 1 \leq i \leq k\} \cup \{w_j, k+1 \leq j \leq n\}$ is an algebraically independent set of polynomials generating $\mathbb{C}(\mathcal{W})^{C_2}$.

Indeed, one checks that this set of invariants separates orbits on $\mathcal{U}_2 = \{w \in \mathcal{W} \mid w_1 \neq 0\}$.

Proposition 2.6. *The following set of algebraically independent polynomials generates the field $\mathbb{C}(\hat{\mathcal{V}})^{O_2(\mathbb{C})}$:*

$$\mathcal{E}_{\mathbb{C}} = \left\{ \begin{array}{l} T_j = t_j, \quad 1 \leq j \leq p \\ D_j = d_j d_1, \quad 1 \leq j \leq q \\ P_j = y_j y_{-1}^{m_j} + y_j y_1^{m_j}, \quad 1 \leq j \leq r \\ Q_j = (y_j y_{-1}^{m_j} - y_{-j} y_1^{m_j}) d_1, \quad 2 \leq j \leq r \end{array} \right\}.$$

Proof. The natural action of C_2 on $\mathbb{C}(\hat{\mathcal{V}})^{SO_2(\mathbb{C})}$ is given by

$$\underbrace{[g]}_{\in C_2} \cdot \underbrace{P}_{\in \mathbb{C}(\hat{\mathcal{V}})^{SO_2(\mathbb{C})}} := P \circ g.$$

Take $v = (t_1, \dots, t_p, d_1, \dots, d_q, y_1, y_{-1}, \dots, y_r, y_{-r})$ a point in $\hat{\mathcal{V}}$.

- For $1 \leq j \leq p$, $(g_1^- \cdot t_j)(v) = t_j(\rho(g_1^-) v) = \phi_0(g_1^-) t_j = t_j(v)$;
- For $1 \leq j \leq q$, $(g_1^- \cdot d_j)(v) = d_j(\rho(g_1^-) v) = \phi_{-1}(g_1^-) d_j = -d_j(v)$;
- For $1 \leq j \leq r$, $(g_1^- \cdot p_j)(v) = p_j(\rho(g_1^-) v) = y_{-j} y_1^{m_j} = q_j(v)$
and $(g_1^- \cdot q_j)(v) = q_j(\rho(g_1^-) v) = y_j y_{-1}^{m_j} = p_j(v)$.

Since polynomials in \mathfrak{e} are linearly (and actually algebraically) independent, they form a basis of the vector space $\mathcal{W} := \text{Vect}(f \mid f \in \mathfrak{e})$ and this vector space is invariant by the action of C_2 on \mathfrak{e} ; it is a linear action.

The matrix representation of g_1^- in the basis

$$\mathfrak{e}' := \left\{ \begin{array}{l} t'_j = t_j, \quad 1 \leq j \leq p \\ d'_j = d_j, \quad 1 \leq j \leq q \\ p'_j = p_j + q_j, \quad 1 \leq j \leq r \\ q'_j = p_j - q_j, \quad 2 \leq j \leq r \end{array} \right\} \text{ of } \mathcal{W} \text{ is } M(g_1^-) = \begin{pmatrix} I_p & 0 & 0 & 0 \\ 0 & -I_q & 0 & 0 \\ 0 & 0 & I_r & 0 \\ 0 & 0 & 0 & -I_{r-1} \end{pmatrix}.$$

By Lemma 2.5, the following is a generating set for $\mathbb{C}(\mathcal{W})_2^{\mathbb{C}}$:

$$\mathcal{E}_{\mathbb{C}} := \left\{ \begin{array}{l} T_j = \mathfrak{t}'_j, \quad 1 \leq j \leq p \\ D_j = \mathfrak{d}'_j \mathfrak{d}'_1, \quad 1 \leq j \leq q \\ P_j = \mathfrak{p}'_j, \quad 1 \leq j \leq r \\ Q_j = \mathfrak{q}'_j \mathfrak{d}'_1, \quad 2 \leq j \leq r \end{array} \right\}$$

To check the algebraic independence of $\mathcal{E}_{\mathbb{C}}$, note that $\#\mathcal{E}_{\mathbb{C}} = p+q+2r-1 = \dim(\hat{\mathcal{V}}) - \dim(\mathrm{O}_2(\mathbb{C}))$. However, the transcendence degree of the field $\mathbb{C}(\hat{\mathcal{V}})^{\mathrm{O}_2(\mathbb{C})}$ is equal to $\dim(\hat{\mathcal{V}}) - \max_{v \in \mathcal{V}} (\dim(\mathrm{O}_2(\mathbb{C}) \cdot v))$ [PV94, Section 2.3]. Hence $\#\mathcal{E}_{\mathbb{C}} \leq \deg_{tr}(\mathbb{C}(\hat{\mathcal{V}})^{\mathrm{O}_2(\mathbb{C})})$ while generating this field. Thus $\mathcal{E}_{\mathbb{C}}$ is algebraically independent. \square

The explicit expressions of the elements of $\mathcal{E}_{\mathbb{C}}$ allow us to determine where $\mathcal{E}_{\mathbb{C}}$ fails to separate orbits.

Proposition 2.7. *Polynomials in $\mathcal{E}_{\mathbb{C}}$ separate the orbits of $\mathrm{O}_2(\mathbb{C})$ in $\hat{\mathcal{U}} = \{v \in \hat{\mathcal{V}} \mid d_1 y_1 y_{-1} \neq 0\}$.*

2.4 Invariants for $\mathrm{O}_2(\mathbb{R})$

One observes that the restrictions of elements in $\mathcal{E}_{\mathbb{C}}$ on the real trace

$$\mathcal{V} = \left\{ (t_1, \dots, t_p, d_1, \dots, d_q, y_1, y_{-1}, \dots, y_r, y_{-r}) \in \hat{\mathcal{V}} \mid t_1, \dots, t_p, d_1, \dots, d_q \in \mathbb{R}; \bar{y}_i = y_{-i}, 1 \leq i \leq n \right\}$$

of $\hat{\mathcal{V}}$ are either real or pure imaginary. One can thus determine generating invariants for $\mathbb{R}(\mathcal{V})^{\mathrm{O}_2(\mathbb{R})}$.

Theorem 2.8. *The following set of algebraically independent polynomials generates both $\mathbb{R}(\mathcal{V})^{\mathrm{O}_2(\mathbb{R})}$ or $\mathbb{C}(\hat{\mathcal{V}})^{\mathrm{O}_2(\mathbb{C})}$.*

$$\mathcal{E} = \left\{ \begin{array}{l} T_j = t_j, \quad 1 \leq j \leq p \\ D_j = d_j d_1, \quad 1 \leq j \leq q \\ R_j = P_j = 2\Re(y_{-1}^{m_j} y_j), \quad 1 \leq j \leq r \\ I_j = iQ_j = -2\Im(y_{-1}^{m_j} y_j) d_1, \quad 2 \leq j \leq r \end{array} \right\}$$

Proof. On \mathcal{V} , $\mathfrak{p}_j = y_j y_{-1}^{m_j} = \bar{y}_{-j} \bar{y}_1^{m_j} = \bar{\mathfrak{q}}_j$. Hence, $\mathfrak{p}_j + \mathfrak{q}_j = 2\Re(\mathfrak{p}_j)$ and $\mathfrak{p}_j - \mathfrak{q}_j = 2i\Im(\mathfrak{p}_j)$. Thus R_j and I_j are real and they generate $\mathbb{R}(\mathcal{V})^{\mathrm{O}_2(\mathbb{R})}$ over \mathbb{R} and $\mathbb{C}(\hat{\mathcal{V}})^{\mathrm{O}_2(\mathbb{C})}$ over \mathbb{C} . \square

Since we just multiplied polynomials by a constant, polynomials in \mathcal{E} still separate orbits in

$$\mathcal{U} = \{v \in \mathcal{V} \mid d_1 y_1 y_{-1} \neq 0\}.$$

The polynomials of Theorem 2.8 can be translated into real coordinates through the base change P^{-1} . It involves an expansion by the Newton binomial which makes the complex i disappear. Nevertheless the expressions become rather involved as shown in the following example. This explains why we mostly work with complex coordinates throughout the article.

Example 2.9. *For the 8-dimensional linear representation $S_{\mathcal{T}} = 2\mathcal{V}_{-1} \oplus \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3$, an algebraically independent generating set of $\mathbb{R}(S_{\mathcal{T}})^{\mathrm{O}_2(\mathbb{R})}$ is*

$$\mathcal{E}_{\mathcal{T}} = \left\{ \begin{array}{l} D_1 = d_1^2 \\ D_2 = d_1 d_2 \\ R_1 = 2y_1 y_{-1} \\ R_2 = y_1^2 y_{-2} + y_{-1}^2 y_2 \\ R_3 = y_1^3 y_{-3} + y_{-1}^3 y_3 \\ I_2 = i d_1 (y_1^2 y_{-2} - y_{-1}^2 y_2) \\ I_3 = i d_1 (y_1^3 y_{-3} - y_{-1}^3 y_3) \end{array} \right\} = \left\{ \begin{array}{l} D_1 = d_1^2 \\ D_2 = d_1 d_2 \\ R_1 = 2(x_1^2 + x_{-1}^2) \\ R_2 = 4x_{-2} x_{-1} x_1 - 2x_{-1}^2 x_2 + 2x_1^2 x_2 \\ R_3 = -2x_{-3} x_{-1}^3 + 6x_{-3} x_{-1} x_1^2 - 6x_{-1}^2 x_1 x_3 + 2x_1^3 x_3 \\ I_2 = 2d_1 (x_{-2} x_{-1}^2 - x_{-2} x_1^2 + 2x_{-1} x_1 x_2) \\ I_3 = 2d_1 (3x_{-3} x_{-1}^2 x_1 - x_{-3} x_1^3 - x_{-1}^3 x_3 + 3x_{-1} x_1^2 x_3) \end{array} \right\}$$

3 Invariants for $\mathrm{SO}_3(\mathbb{R})$

In this section we consider the representation of $\mathrm{SO}_3(\mathbb{R})$ on

$$\mathcal{H} = \mathcal{H}_1 \oplus \bigoplus_{j=1}^n \mathcal{H}_{d_j}$$

where \mathcal{H}_d , for $d \in \mathbb{N}$, are the irreducible representations of $\mathrm{SO}_3(\mathbb{R})$, where $\mathcal{H}_1 \cong \mathbb{R}^3$. We shall characterize an algebraically independent set of rational invariants that generates the invariant field $\mathbb{R}(\mathcal{H})^{\mathrm{SO}_3(\mathbb{R})}$. We make clear where these invariants have singularities, on which open set they separate orbits and how to evaluate them. We shall furthermore solve the inverse problem: given values of for these invariants provide firstly conditions for the existence of a (real) orbit where the invariants have these values and secondly an element in this orbit if it exists. All these results rest on a constructive use of Seshadri slice lemma. In the present case this latter asserts that the restriction to $\mathcal{S} = \mathbb{R} \oplus \bigoplus_{j=1}^n \mathcal{H}_{d_j}$ provides an isomorphism between $\mathbb{R}(\mathcal{H})^{\mathrm{SO}_3(\mathbb{R})}$ and $\mathbb{R}(\mathcal{S})^{\mathrm{O}_2(\mathbb{R})}$. We can therefore apply the results of previous section to characterize the invariants of $\mathrm{SO}_3(\mathbb{R})$ on \mathcal{H} .

In Section 3.1 \mathcal{H}_d is made explicit as the space of harmonic ternary forms of degree d . The spherical harmonics form bases that decompose the \mathcal{H}_d into irreducible representations of $\mathrm{O}_2(\mathbb{R})$. In Section 3.2 we recall Seshadri's lemma and exhibit a slice \mathcal{S} for \mathcal{H} . In Section 3.3 we characterize an algebraically independent set of invariants that generates $\mathbb{R}(\mathcal{H})^{\mathrm{SO}_3(\mathbb{R})}$. They are determined by their restriction to \mathcal{S} . In Section 3.4 and 3.6 we show how this is sufficient to evaluate these $\mathrm{SO}_3(\mathbb{R})$ -invariants on $\mathcal{H} \setminus \mathcal{Q}$, where \mathcal{Q} is the variety of an invariant quadratic polynomial, solve the inverse problem.

3.1 Representations of $\mathrm{SO}_3(\mathbb{R})$

The irreducible representations \mathcal{H}_d , $d \in \mathbb{N}$, of $\mathrm{SO}_3(\mathbb{R})$ are given explicitly in terms of spaces of homogeneous harmonic polynomials of degree d . The *spherical harmonics* form bases for these that decompose them into irreducible $\mathrm{O}_2(\mathbb{R})$ representations. In this section we recall these notions and introduce the notations we shall use.

\mathcal{H}_d is the vector space of degree d real harmonic homogeneous polynomials on \mathbb{R}^3 :

$$\mathcal{H}_d := \left\{ P \in \mathbb{R}[U, V, W]_d \mid \frac{\partial^2 P}{\partial U^2} + \frac{\partial^2 P}{\partial V^2} + \frac{\partial^2 P}{\partial W^2} = 0 \right\}.$$

The standard action of $\mathrm{SO}_3(\mathbb{R})$ on \mathbb{R}^3 induces, by composition, a linear action on the polynomial ring: $\forall P \in \mathbb{R}[U, V, W]$, $g \in \mathrm{SO}_3(\mathbb{R})$, $g \cdot P = P \circ g^{-1}$. This action stabilises the vector space \mathcal{H}_d for any $d \in \mathbb{N}$. It thus induces the representations $\rho_d : \mathrm{SO}_3(\mathbb{R}) \rightarrow \mathrm{GL}(\mathcal{H}_d)$ for $d \in \mathbb{N}$. They are the only irreducible representations of $\mathrm{SO}_3(\mathbb{R})$ [GSS88]. Among them, (ρ_0, \mathcal{H}_0) is the trivial representation and (ρ_1, \mathcal{H}_1) is the equivalent to the standard representation on \mathbb{R}^3 . The class of representations we consider are thus all the representations where the multiplicity of (\mathcal{H}_1, ρ_1) is strictly positive. This is specific and yet covers a number of relevant cases.

The *spherical harmonics* form bases of the spaces \mathcal{H}_d . They are homogeneous polynomials in $\mathbb{R}[U, V, W]$ that decompose the spaces \mathcal{H}_d into irreducible representations of $\mathrm{O}_2(\mathbb{R})$. Their expressions are nonetheless better known in spherical coordinates than in Cartesian coordinates. We thus introduce spherical coordinates (r, θ, φ) in \mathbb{R}^3 so that

$$u = r \cos(\theta), \quad v = r \sin(\theta) \cos(\phi), \quad w = r \sin(\theta) \sin(\phi).$$

Theorem 3.1. [Dai13] For $1 \leq m \leq d$ note P_d^m the $(d, m)^{\text{th}}$ associated Legendre polynomial. The following

set of functions of (r, θ, φ) is a polynomial basis for $\hat{\mathcal{H}}_d$:

$$\hat{\mathcal{B}}_d := \left\{ \begin{array}{l} Y_d^m = i^{m+d} r^d e^{im\varphi} P_d^m(\cos(\theta)), \quad 1 \leq m \leq d \\ Y_d^{-m} = i^{-m-d} r^d e^{-im\varphi} P_d^m(\cos(\theta)), \quad 1 \leq m \leq d \\ Y_d^0 = r^d P_d^0(\cos(\theta)) \end{array} \right\}$$

With respect to this basis, we put $h_j = (y_{-d_j, j}, y_{-d_j+1, j}, \dots, y_{d_j-1, j}, y_{d_j, j})$ coordinates in $\hat{\mathcal{H}}_{d_j}$, $h = (h_1, \dots, h_n)$ coordinates in $\bigoplus_{j=1}^n \hat{\mathcal{H}}_{d_j}$. When θ, ϕ and r are in \mathbb{R} we have $Y_d^{-m} = \overline{Y_d^m}$, for any $0 \leq m \leq d$. With the change P^{-1} of basis introduced in Section 2.1 we obtain the real harmonic basis:

$$\begin{aligned} X_d^m &= \frac{1}{2}(Y_d^m + Y_d^{-m}) = r^d \cos(m\varphi + (m+d)\frac{\pi}{2}) P_d^m(\cos(\theta)) \\ X_d^{-m} &= \frac{1}{2i}(Y_d^m - Y_d^{-m}) = r^d \sin(m\varphi + (m+d)\frac{\pi}{2}) P_d^m(\cos(\theta)) \end{aligned}$$

Hence, natural real coordinates can be defined on \mathcal{H}_d with $h_j = (x_{-d_j, j}, \dots, x_{d_j, j})$.

Example 3.2. The vector space $\mathcal{H}_1 = \mathbb{R}[U, V, W]_1$ is endowed with the real basis $\mathcal{B}_1 = \{X_1^1 = -W, X_1^0 = U, X_1^{-1} = V\}$.

The real vector space \mathcal{H}_3 is endowed with the basis $\mathcal{B}_3 = \{X_3^m, -3 \leq m \leq 3\}$. We can compute their expressions in Cartesian coordinates up to a normalisation constant:

$$\mathcal{B}_3 = \left\{ \begin{array}{lll} X_3^3 = 3W^2V - V^3 & X_3^2 = UVW & X_3^1 = W(4U^2 - V^2 - W^2) \\ X_3^0 = U(2U^2 - 3V^2 - 3W^2) & & \\ X_3^{-3} = 3WV^2 - W^3 & X_3^{-2} = U(W^2 - V^2) & X_3^{-1} = -V(4U^2 - V^2 - W^2) \end{array} \right\}$$

Proposition 3.3. Write $n := p + q$ such that d_j is even for $1 \leq j \leq p$, and odd for $p + 1 \leq j \leq p + q$. For the restriction to $O_2(\mathbb{C})$, subgroup of $SO_3(\mathbb{R})$, the representation on $\bigoplus_{j=1}^n \hat{\mathcal{H}}_{d_j}$ is isomorphic to

$$\bigoplus_{j=1}^p \hat{\mathcal{V}}_0 \oplus \bigoplus_{j=1}^q \hat{\mathcal{V}}_{-1} \oplus \bigoplus_{j=1}^n \bigoplus_{m=1}^{d_j} \hat{\mathcal{V}}_m$$

where the $\hat{\mathcal{V}}_m$ are the irreducible representations of $O_2(\mathbb{R})$ introduced in Section 2.1.

Proof. Consider the subgroup of rotations of axis U . Its standard representation writes in spherical coordinates:

$$\forall \alpha \in \mathbb{R}, \forall r, \theta, \varphi, \begin{cases} g_{e^{i\alpha}}^+(r, \theta, \varphi) = (r, \theta, \varphi + \alpha) \\ g_{e^{i\alpha}}^-(r, \theta, \varphi) = (r, \pi - \theta, -\varphi - \alpha) \end{cases}$$

Consider the induced action on harmonic polynomials. Let d be a positive integer and $1 \leq m \leq d$. For all $(r, \theta, \varphi) \in \mathbb{R}^3$, and α real,

$$\begin{aligned} \rho_d(g_{e^{i\alpha}}^+) (Y_d^m(r, \theta, \varphi)) &= Y_d^m(g_{e^{-i\alpha}}^+(r, \theta, \varphi)) \\ &= Y_d^m(r, \theta, \varphi - \alpha) \\ &= i^{m+d} r^d e^{im(\varphi+\alpha)} P_d^m(\cos(\theta)) \\ &= e^{-im\alpha} Y_d^m(r, \theta, \varphi) \end{aligned}$$

$$\begin{aligned} \rho_d(g_{e^{i\alpha}}^-) (Y_d^m(r, \theta, \varphi)) &= Y_d^m(g_{e^{-i\alpha}}^-(r, \theta, \varphi)) \\ &= Y_d^m(r, \pi - \theta, -\varphi + \alpha) \\ &= i^{m+d} r^d e^{-im(\varphi-\alpha)} P_d^m(-\cos(\theta)) \end{aligned}$$

Using Legendre associated polynomials parity [Ami11] $P_d^m(-t) = (-1)^{d+m} P_d^m(t)$ we obtain

$$\begin{aligned} \rho_d(g_{e^{i\alpha}}^-)(Y_d^m(r, \theta, \varphi)) &= (-1)^{d+m} i^{m+d} r^d e^{-im(\varphi-\alpha)} P_d^m(\cos(\theta)) \\ &= i^{-d-m} r^d e^{-im(\varphi-\alpha)} P_d^m(\cos(\theta)) \\ \rho_d(g_{e^{i\alpha}}^-)(Y_d^m) &= e^{im\alpha} Y_d^{-m} \end{aligned}$$

Thus, the $O_2(\mathbb{C})$ action on $\text{Vect}(Y_d^m, Y_d^{-m})$ is isomorphic to $\hat{\mathcal{V}}_m$. Meanwhile for $m = 0$, the corresponding

action on $\text{Vect}(Y_d^0)$ is $\begin{cases} \phi_{-1} & \text{when } d \text{ is odd.} \\ \phi_0 & \text{when } d \text{ is even.} \end{cases}$ We have $\rho_j \sim \begin{cases} \phi_{-1} \oplus \bigoplus_{m=1}^{d_j} \phi_m & \text{when } d_j \text{ is odd.} \\ \phi_0 \oplus \bigoplus_{m=1}^{d_j} \phi_m & \text{when } d_j \text{ is even.} \end{cases}$ \square

3.2 Seshadri slice

In this section we identify a Seshadri slice $(\mathcal{S}, \mathcal{N})$, where $\mathcal{N} \cong O_2(\mathbb{R})$, for $\mathcal{H} = \mathcal{H}_1 \oplus \bigoplus_{j=1}^n \mathcal{H}_{d_j}$ that establishes an isomorphism between $\mathbb{R}(\mathcal{H})^{\text{SO}_3(\mathbb{R})}$ and $\mathbb{R}(\mathcal{S})^{\mathcal{N}}$.

Seshadri slice lemma is instrumental in proving the rationality of the invariant field of some algebraic group actions [CS05]. And indeed, in Section 3.3 we make explicit a set of algebraically independent polynomial invariants generating $\mathbb{R}(\mathcal{S})^{\mathcal{N}}$; $\mathbb{R}(\mathcal{S})^{\mathcal{N}}$ is thus rational and so is therefore $\mathbb{R}(\mathcal{H})^{\text{SO}_3(\mathbb{R})}$. We shall nonetheless go further. In Section 3.4 we show how to evaluate the corresponding invariants of $\mathbb{R}(\mathcal{H})^{\text{SO}_3(\mathbb{R})}$ at any point in \mathcal{H} . The evaluation process also provides a way to give the explicit expressions of these invariants after some symbolic manipulations.

Theorem 3.4. *Seshadri slice lemma [CS05, Theorem 3.1]. Let G be a real algebraic group acting on a vector space \mathcal{X} . Let $\mathcal{S} \subset \mathcal{X}$ be a subvector space and $\mathcal{N} < G$ the normalizer of \mathcal{S} in G . Assume that:*

- The closure of $G \times \mathcal{S}$ is the whole of \mathcal{X} .
- There is a non empty \mathcal{N} -stable Zariski open subset $\mathcal{Z} \subset \mathcal{S}$ such that for all $g \in \hat{G}$ and $s \in \hat{\mathcal{Z}}$ satisfying $g(s) \in \hat{\mathcal{Z}}$, there exists $h \in \hat{\mathcal{N}}$ such that $h(s) = g(s)$.

Then, the restriction to \mathcal{Z} of rational functions on \mathcal{X} induces a field isomorphism $\mathbb{R}(\mathcal{X})^G \cong \mathbb{R}(\mathcal{S})^{\mathcal{N}}$.

Definition 3.5. Such a pair $(\mathcal{S}, \mathcal{N})$ is called a Seshadri slice for $G \curvearrowright \mathcal{X}$.

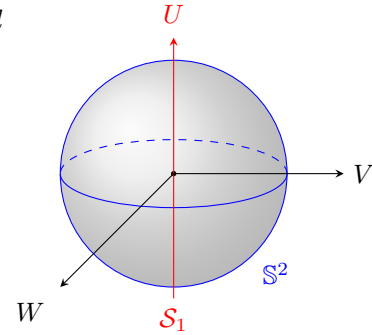
For $P \in \mathbb{R}(\mathcal{S})^{\mathcal{N}}$ there thus exists a unique $\tilde{P} \in \mathbb{R}(\mathcal{X})^G$ such that the restriction of \tilde{P} to \mathcal{Z} equals P ; We shall write $\tilde{P}|_{\mathcal{Z}} = P$.

Proposition 3.6. Consider the subspace $\mathcal{S}_1 = \{\lambda U, \lambda \in \mathbb{R}\} \subset \mathcal{H}_1$ and

$$\mathcal{S} := \mathcal{S}_1 \oplus \bigoplus_{j=1}^n \mathcal{H}_{d_j} \subset \mathcal{H}$$

Then, $(\mathcal{S}, \mathcal{N})$ is a Seshadri slice with normalizer $\mathcal{N} \cong O_2(\mathbb{R})$

$$\mathcal{N} = \left\{ \left(\begin{array}{c|cc} \det(g) & 0 & 0 \\ \hline 0 & & \\ 0 & & g \end{array} \right), g \in O_2(\mathbb{R}) \right\} \subset \text{SO}_3(\mathbb{R})$$



The normalizer \mathcal{N} of \mathcal{S} thus consists of the rotations in \mathbb{R}^3 that leave the U axis invariant. That includes the rotations around the U axis and the rotations by π around any axis in the (V, W) plane.

The open subset \mathcal{Z} in \mathcal{S} that is relevant in the proof and later on in the article is

$$\mathcal{Z} = \{p_1 U + \sum_{j=1}^n P_{d_j} \in \mathcal{S} \text{ with } p_1 \neq 0\}.$$

Proof. We prove the two conditions to be a Seshadri slice:

- Let $P := P_1 + \sum_{j=1}^n P_{d_j}$ be a polynomial in \mathcal{H} . Since (ρ_1, \mathcal{H}_1) is isomorphic to the standard representation, the orbits in \mathcal{H}_1 are the concentric spheres $r\mathbb{S}^2$ of radius $r \in \mathbb{R}^+$. Hence there exists a rotation $g \in \text{SO}_3(\mathbb{R})$ such that $\rho_1(g)(P_1) = \|P_1\|U \in \mathcal{S}_1$. That is, $\rho(g)(P) \in \mathcal{S}_1 \oplus \bigoplus_{j=1}^n \mathcal{H}_{d_j} = \mathcal{S}$.

- We take the non empty subset $\mathcal{Z} = \{p_1 U + \sum_{j=1}^n P_{d_j} \in \mathcal{S} \text{ with } p_1 \neq 0\} \subset \mathcal{S}$. \mathcal{Z} is N -stable and its complexification is $\hat{\mathcal{Z}} = \{p_1 U + \sum_{j=1}^n P_{d_j} \in \hat{\mathcal{S}} \text{ with } p_1 \neq 0\} \subset \hat{\mathcal{S}}$. Then for any $P \in \hat{\mathcal{Z}}$ an element $g \in \text{SO}_3(\mathbb{C})$

such that $\rho(g)(P) \in \hat{\mathcal{Z}}$ satisfies $\rho_1(g) = \begin{pmatrix} \lambda & a & b \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$ with $\lambda, a, b \in \mathbb{C}$. However we have $\rho_1(g)^t = g^t \in \text{SO}_3(\mathbb{C})$

and $\rho_1(g)^t \rho_1(g) = I_3$.

$$\Rightarrow \rho_1(g)^t \rho_1(g) = \begin{pmatrix} \lambda & 0 & 0 \\ a & * & * \\ b & * & * \end{pmatrix} \begin{pmatrix} \lambda & a & b \\ 0 & * & * \\ 0 & * & * \end{pmatrix} = \begin{pmatrix} \lambda^2 & \lambda a & \lambda b \\ \lambda a & * & * \\ \lambda b & * & * \end{pmatrix} = I_3$$

This is $\lambda = \pm 1$, $a = 0$ and $b = 0$. Then, $g \in \hat{\text{N}}$. □

3.3 Restrictions of the invariants to the slice

We saw that

$$\mathcal{S} := \mathcal{S}_1 \oplus \bigoplus_{j=1}^n \mathcal{H}_{d_j} \subset \mathcal{H}_1 \oplus \bigoplus_{j=1}^n \mathcal{H}_{d_j} =: \mathcal{H}$$

with normalizer $\text{N} \cong \text{O}_2(\mathbb{R})$ is a Seshadri slice for $(\mathcal{H}, \text{SO}_3(\mathbb{R}))$. By Theorem 3.3 we know that

$$\bigoplus_{j=1}^n \mathcal{H}_{d_j} \cong \bigoplus_{j=1}^p \mathcal{V}_0 \oplus \bigoplus_{j=1}^q \mathcal{V}_{-1} \oplus \bigoplus_{j=1}^n \bigoplus_{m=1}^{d_j} \mathcal{V}_m$$

where the \mathcal{V}_m are the irreducible representations of $\text{O}_2(\mathbb{R})$ of Section 2.1. This decomposition is furthermore explicitly given by a basis of spherical harmonics as shown in introduced in Proposition 3.3. The action of N on \mathcal{S}_1 is furthermore the determinantal representation ϕ_{-1} . We thus can write

$$\mathcal{S} \cong \mathcal{V}_{-1} \oplus \bigoplus_{j=1}^p \hat{\mathcal{V}}_0 \oplus \bigoplus_{j=1}^q \mathcal{V}_{-1} \oplus \bigoplus_{j=1}^n \bigoplus_{m=1}^{d_j} \mathcal{V}_m.$$

with coordinates $(y_0, z_{m,j}, 1 \leq j \leq r, -d_j \leq m \leq d_j)$. The group action $\text{O}_2(\mathbb{R}) \cong \text{N} \curvearrowright \mathcal{S}$ is a representation satisfying the hypotheses of Theorem 2.8. We apply it to obtain generators:

Theorem 3.7. *The following set of algebraically independent polynomials generates both $\mathbb{R}(\mathcal{S})^N$ and $\mathbb{C}(\hat{\mathcal{S}})^{\hat{N}}$.*

$$\mathcal{E} = \left\{ \begin{array}{lll} T_j(y_0, h) = & y_{0,j}, & 1 \leq j \leq p \\ D_0(y_0, h) = & y_0^2, & \\ D_j(y_0, h) = & y_0 y_{0,j}, & p+1 \leq j \leq n \\ R_{mj}(y_0, h) = & y_{m,j} y_{-1,1}^m + y_{-m,j} y_{1,1}^m, & 1 \leq j \leq n, \quad 1 \leq m \leq d_j \\ I_{mj}(y_0, h) = & i y_0 (y_{m,j} y_{-1,1}^m - y_{-m,j} y_{1,1}^m), & 1 \leq j \leq n, \quad 1 \leq m \leq d_j, \quad (j, m) \neq (1, 1) \end{array} \right\}$$

Corollary 3.8. *Let $\rho : \mathrm{SO}_3(\mathbb{R}) \rightarrow \mathrm{GL}(\mathcal{H})$ be a representation including at least once the irreducible representation (ρ_1, \mathcal{H}_1) . Then, $\mathbb{R}(\mathcal{H})^{\mathrm{SO}_3(\mathbb{R})}$ is rational.*

Proof. The invariant field $\mathbb{R}(\mathcal{S})^N$ is rational (Theorem 2.8). According to Seshadri slice lemma, $\mathbb{R}(\mathcal{H})^{\mathrm{SO}_3(\mathbb{R})}$ is rational. \square

It was already known that an almost free $\mathrm{SO}_n(\mathbb{C})$ action (as $\mathrm{SO}_3(\mathbb{C})$ representation on $\hat{\mathcal{H}}$ whenever $n > 0$) is stably rational [CS05, Proposition 4.12], this being a weaker property than rational. Our result nonetheless also applies to actions which are not almost free.

Example 3.9. *The representation $\mathcal{S}_{\mathcal{T}} = 2\mathcal{V}_{-1} \oplus \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3$ introduced in Example 2.9 is the slice corresponding to the representation $\mathcal{T} = \mathcal{H}_1 \oplus \mathcal{H}_3$. For this $\mathrm{SO}_3(\mathbb{C})$ action, the restriction to \mathcal{S} of an algebraically independent generating invariants on the slice is*

$$\mathcal{E}_{\mathcal{T}} = \left\{ \begin{array}{l} D_1 = y_0^2 \\ D_2 = y_0 y_{0,1} \\ R_1 = 2y_{1,1} y_{-1,1} \\ R_2 = y_{1,1}^2 y_{-2,1} + y_{-1,1}^2 y_{2,1} \\ R_3 = y_{1,1}^3 y_{-3,1} + y_{-1,1}^3 y_{3,1} \\ I_2 = i y_0 (y_{1,1}^2 y_{-2,1} - y_{-1,1}^2 y_{2,1}) \\ I_3 = i y_0 (y_{1,1}^3 y_{-3,1} - y_{-1,1}^3 y_{3,1}) \end{array} \right\}.$$

3.4 Evaluating the invariants

In previous section we characterized the invariants of the action of $\mathrm{SO}_3(\mathbb{R})$ on \mathcal{H} by their restriction to the slice \mathcal{S} . In this section we address how to evaluate these invariants at any other point of \mathcal{H} . We propose to first find a rotation that takes this point to the slice. Such an intersection of the orbit of the point and the slice is not unique, and the rotation to take it there even less so. Yet the evaluation of the slice invariants at any such point of intersection gives the same values. As we are interested by evaluation on the real points of \mathcal{H} we can use of spherical coordinates and describe the rotations by Euler angles.

Definition 3.10. *With respect to the Euclidean basis (u, v, w) , a rotation $g \in \mathrm{SO}_3(\mathbb{R})$ is determined by Euler angles (α, β, γ) where $\alpha \in [0, 2\pi[$, $\beta \in [0, \pi]$ and $\gamma \in [0, 2\pi[$. It is the composition of rotations of axes u, w, u and angles (α, β, γ) :*

$$g(\alpha, \beta, \gamma) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\gamma) & \sin(\gamma) \\ 0 & -\sin(\gamma) & \cos(\gamma) \end{pmatrix} \begin{pmatrix} \cos(\beta) & \sin(\beta) & 0 \\ -\sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

Euler angles are particularly effective to define a rotation $g \in \mathrm{SO}_3(\mathbb{R})$ mapping a point in \mathcal{H} to the slice:

Lemma 3.11. *Let $y+h \in \mathcal{H}_1 \oplus \bigoplus_{j=1}^n \mathcal{H}_{d_j}$ a point in \mathcal{H} , with $y \neq 0$. Note $y = (r, \theta, \varphi)$ in spherical coordinates. Then, for all $\gamma \in \mathbb{R}$, one has $\rho(g(\varphi, \theta, \gamma))(y+h) \in \mathcal{Z} \subset \mathcal{S}$.*

Proof. The representation $\rho_1 : \text{SO}_3(\mathbb{R}) \rightarrow \text{GL}(\mathcal{H}_1)$ is isomorphic to the canonical $\text{SO}_3(\mathbb{R})$ representation on \mathbb{R}^3 . Hence for $g \in \text{SO}_3(\mathbb{R})$ we have $\rho_1(g)(y) \in \mathcal{S}_1 \Leftrightarrow g(y) = (r, 0, 0) \Leftrightarrow g^t(r, 0, 0) = y$. Meanwhile, each rotation in $\{g(\varphi, \theta, \gamma), \gamma \in \mathbb{R}\}$ maps $(r, 0, 0)$ to y . \square

We thus need to address how to compute $\rho(g(\varphi, \theta, \gamma))$. Wigner matrices provide a solution as they make explicit the action of $\text{SO}_3(\mathbb{R})$ on \mathcal{H}_d :

Definition 3.12. [Wig59, p. 15.27]. Let $d \in \mathbb{N}$ and (α, β, γ) some Euler angles. The d^{th} Wigner matrix is the square matrix of size $(2d+1) \times (2d+1)$ defined by

$$W_{(\alpha, \beta, \gamma)}^d = \left(e^{il\alpha} w_{m,l}^d(\beta) e^{im\gamma} \right)_{-d \leq m, l \leq d}$$

where

$$w_{m,l}^d(\beta) := c_{m,l} \sum_{s \in \Gamma_{mm'}^d} \frac{\delta(m,l)(-1)^s i^{l-m}}{(d+m-s)!s!(l-m+s)!(d-l-s)!} \cos\left(\frac{\beta}{2}\right)^{2d+m-l-2s} \sin\left(\frac{\beta}{2}\right)^{l-m+2s}.$$

$\Gamma_{mm'}^d$ is the set of integers s such that $d+m-s$, s , $l-m+s$ and $d-l-s$ are non negative, and

$$\delta(m,l) = \begin{cases} -1 & \text{if } m > 0 \text{ and } l > 0. \\ -1 & \text{if } m \leq 0 \text{ and } l \leq 0. \\ 1 & \text{otherwise.} \end{cases}$$

Note that we slightly modified the coefficients of the functions $w_{m,l}^d$ compared to [Wig59] that uses a different normalisation of the spherical harmonics.

Proposition 3.13. Let $d \in \mathbb{N}$ and (α, β, γ) some Euler angles defining the rotation $g(\alpha, \beta, \gamma) \in \text{SO}_3(\mathbb{R})$. Then, for all vectors $Y = (y_{-d}, \dots, y_d) \in \mathcal{H}_d$, $\rho_d(g(\alpha, \beta, \gamma))(Y) = W_{(\alpha, \beta, \gamma)}^d Y$.

As the third angle γ is basically irrelevant for the rotations taking a point in \mathcal{H} to the slice \mathcal{S} we can give the expressions of the $\text{SO}_3(\mathbb{R})$ invariants as follows.

Theorem 3.14. The set of functions of $(y+h) \in \text{SO}_3(\mathbb{R}) \times \mathcal{Z}$ with $y = (r, \theta, \varphi)$ in spherical coordinates

$$\tilde{\mathcal{E}} = \left\{ \begin{array}{l} \tilde{T}_j(y+h) = \sum_{l=-d_j}^{d_j} e^{il\varphi} w_{l,0}^{d_j}(\theta) y_{l,j} \\ \tilde{D}_0(y+h) = r^2 \\ \tilde{D}_j(y+h) = r \sum_{l=-d}^d e^{il\varphi} w_{l,0}^{d_j}(\theta) y_{l,j} \\ \tilde{R}_{mi}(y+h) = \left(\sum_{l=-d_j}^{d_j} e^{il\varphi} w_{l,m}^{d_j}(\theta) y_{l,j} \right) \left(\sum_{l=-d_1}^{d_1} e^{il\varphi} w_{l,-1}^{d_1}(\theta) y_{l,1} \right)^m \\ \quad + \left(\sum_{l=-d_j}^{d_j} e^{il\varphi} w_{l,-m}^{d_j}(\theta) y_{l,j} \right) \left(\sum_{l=-d_1}^{d_1} e^{il\varphi} w_{l,1}^{d_1}(\theta) y_{l,1} \right)^m \\ \tilde{I}_{mi}(y+h) = \text{ir} \left(\sum_{l=-d_j}^{d_j} e^{il\varphi} w_{l,m}^{d_j}(\theta) y_{l,j} \right) \left(\sum_{l=-d_1}^{d_1} e^{il\varphi} w_{l,-1}^{d_1}(\theta) y_{l,1} \right)^m \\ \quad - \text{ir} \left(\sum_{l=-d_j}^{d_j} e^{il\varphi} w_{l,-m}^{d_j}(\theta) y_{l,j} \right) \left(\sum_{l=-d_1}^{d_1} e^{il\varphi} w_{l,1}^{d_1}(\theta) y_{l,1} \right)^m \end{array} \right\}$$

is rational and algebraically independent. It furthermore generates $\mathbb{R}(\mathcal{H})^{\text{SO}_3(\mathbb{R})}$.

Proof. We build a large matrix $W_{(\varphi,\theta,0)} = \text{Diag}(W_{(\varphi,\theta,0)}^{d_j}, 1 \leq j \leq n)$. Take \tilde{P} the invariant function on \mathcal{H} whose restriction to \mathcal{S} is P . Proposition 3.13 claims that $\tilde{P}(y+h) = P(W_{(\varphi,\theta,0)}(y+h))$. We thus evaluated polynomials of $\tilde{\mathcal{E}}$ at $W_{(\varphi,\theta,0)}(y+h)$.

Polynomials in \mathcal{E} are rational, algebraically independent, and they furthermore generate $\mathbb{R}(\mathcal{S})^N$. Meanwhile, Seshadri slice Lemma (Theorem 3.4) claims that the restriction to \mathcal{S} is a field isomorphism between $\mathbb{R}(\mathcal{S})^N$ and $\mathbb{R}(\mathcal{H})^{\text{SO}_3(\mathbb{R})}$, or $\mathbb{C}(\hat{\mathcal{S}})^{\hat{N}}$ and $\mathbb{C}(\hat{\mathcal{H}})^{\text{SO}_3(\mathbb{C})}$. Hence, extensions of polynomials in $\tilde{\mathcal{E}}$ are rational, algebraically independent, and they furthermore generate $\mathbb{R}(\mathcal{H})^{\text{SO}_3(\mathbb{R})}$. \square

Example 3.15. Consider the representation $\mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3 \cong \mathcal{H}_1^4$. The associated slice is then $\mathcal{S}_{\mathcal{H}_1^4} = \mathcal{S}_1 \oplus \mathcal{H}_1^3$ and Theorem 3.7 gives the generating set

$$\mathcal{E}_{\mathcal{H}_1^4} = \left\{ \begin{array}{llll} D_0 & = & y_0^2 & \\ D_1 & = & y_0 y_{0,1} & R_1 = y_{1,1} y_{-1,1} \\ D_2 & = & y_0 y_{0,2} & R_2 = y_{1,1} y_{-1,2} + y_{-1,1} y_{1,2} & I_2 = iy_0 (y_{1,1} y_{-1,2} - y_{-1,1} y_{1,2}) \\ D_3 & = & y_0 y_{0,3} & R_3 = y_{1,1} y_{-1,3} + y_{-1,1} y_{1,3} & I_3 = iy_0 (y_{1,1} y_{-1,3} - y_{-1,1} y_{1,3}) \end{array} \right\}$$

In this simple example translating with Maple the invariants given by Theorem 3.14 in spherical coordinates into Cartesian coordinates leads to expressions that can still each fit on one or two lines:

$$\begin{aligned} \tilde{D}_0 &= 2y_{-1}y_1 + y_0^2 \\ \tilde{D}_1 &= y_{-1}y_{1,1} + y_0y_{0,1} + y_1y_{-1,1} \\ \tilde{D}_2 &= y_{-1}y_{1,2} + y_0y_{0,2} + y_1y_{-1,2} \\ \tilde{D}_3 &= y_{-1}y_{1,3} + y_0y_{0,3} + y_1y_{-1,3} \\ \tilde{R}_1 &= \frac{-y_{-1}^2y_{1,1}^2 - 2y_{-1}y_0y_{0,1}y_{1,1} + 2y_{-1}y_1y_{-1,1}y_{1,1} + 2y_{-1}y_1y_0^2 + 2y_0^2y_{-1,1}y_{1,1} - 2y_0y_1y_{-1,1}y_{0,1} - y_{1,1}^2y_{-1,1}^2}{2y_{-1}y_1 + y_0^2} \\ \tilde{R}_2 &= \frac{-y_{-1}^2y_{1,1}y_{1,2} + ((y_{-1,1}y_{1,2} + y_{-1,2}y_{1,1} + 2y_{0,1}y_{0,2})y_1 - y_0(y_{0,1}y_{1,2} + y_{0,2}y_{1,1}))y_{-1}}{2y_{-1}y_1 + y_0^2} \\ &\quad + \frac{-y_{-1}^2y_{-1,1}y_{-1,2} - y_0(y_{-1,1}y_{0,2} + y_{-1,2}y_{0,1})y_1 + y_0^2(y_{-1,1}y_{1,2} + y_{-1,2}y_{1,1})}{2y_{-1}y_1 + y_0^2} \\ \tilde{R}_3 &= \frac{-y_{-1}^2y_{1,1}y_{1,3} + ((y_{-1,1}y_{1,3} + y_{-1,3}y_{1,1} + 2y_{0,1}y_{0,3})y_1 - y_0(y_{0,1}y_{1,3} + y_{0,3}y_{1,1}))y_{-1}}{2y_{-1}y_1 + y_0^2} \\ &\quad + \frac{-y_{-1}^2y_{-1,1}y_{-1,3} - y_0(y_{-1,1}y_{0,3} + y_{-1,3}y_{0,1})y_1 + y_0^2(y_{-1,1}y_{1,3} + y_{-1,3}y_{1,1})}{2y_{-1}y_1 + y_0^2} \\ \tilde{I}_2 &= i(-y_{-1}y_{0,1}y_{1,2} + y_{-1}y_{0,2}y_{1,1} + y_0y_{-1,1}y_{1,2} - y_0y_{-1,2}y_{1,1} - y_1y_{-1,1}y_{0,2} + y_1y_{-1,2}y_{0,1}) \\ \tilde{I}_3 &= i(-y_{-1}y_{0,1}y_{1,3} + y_{-1}y_{0,3}y_{1,1} + y_0y_{-1,1}y_{1,3} - y_0y_{-1,3}y_{1,1} - y_1y_{-1,1}y_{0,3} + y_1y_{-1,3}y_{0,1}) \end{aligned}$$

3.5 Separation properties

We computed a set of invariants \mathcal{E} generating the invariant field $\mathbb{C}(\hat{\mathcal{S}})^{\hat{N}}$. By Proposition 2.7, it separates orbits in

$$\hat{\mathcal{U}} = \left\{ v \in \hat{\mathcal{S}} \mid y_0 y_{1,1} y_{-1,1} \neq 0 \right\},$$

which is included in $\hat{\mathcal{Z}} = \{v \in \hat{\mathcal{S}} \mid y_0 \neq 0\}$. Here we deduce separation property for the set of rational invariants $\tilde{\mathcal{E}}$ in \mathcal{H} .

Proposition 3.16. Rational functions in $\tilde{\mathcal{E}}$ separate orbits in the open set $\text{SO}_3(\mathbb{C}) \times \hat{\mathcal{U}}$.

Proof. Take u_1 and u_2 two points in $\text{SO}_3(\mathbb{C}) \times \hat{\mathcal{U}}$ which are not in the same orbit. They have s_1 and s_2 representants of their respective orbits in $\hat{\mathcal{U}}$. Since \mathcal{E} separates orbits in \mathcal{U} , there is $f \in \mathcal{E}$ such that $f(s_1) \neq f(s_2)$. Since $\mathcal{U} \subset \mathcal{Z}$, $\tilde{f}(u_1) \neq \tilde{f}(u_2)$. \square

Recall that $y_0^2 = D_0$ and $y_{1,1}y_{-1,1} = \frac{1}{2}R_{11}$. The non separated orbits are thus pointed out by $\tilde{D}_0 = 0$ or $\tilde{R}_{11} = 0$. We also deduce a set of polynomials satisfying the same separation property. For any $P \in \mathcal{E}$, the rational function \tilde{P} is well defined on $\text{SO}_3(\mathbb{C}) \times \hat{\mathcal{Z}}$. We can determine $\hat{\mathcal{H}} \setminus \text{SO}_3(\mathbb{C}) \times \hat{\mathcal{Z}}$ as an irreducible hypersurface.

Proposition 3.17. *Put (u, v, w) the natural coordinates on $\hat{\mathcal{H}}_1 \subset \hat{\mathcal{H}}$, and note $Q = u^2 + v^2 + w^2 \in \mathbb{C}[\hat{\mathcal{H}}]$. Then, $\hat{\mathcal{H}} \setminus (\text{SO}_3(\mathbb{C}) \times \hat{\mathcal{Z}}) = \mathcal{V}(Q)$.*

Proof. Let $z = (u, v, w)$ be a non zero point in \mathcal{H}_1 . Take $g = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ a matrix in $\text{SO}_3(\mathbb{C})$. The form

$$v \mapsto Q(v, z) \text{ is linear and its kernel } K_z \text{ is an hyperplane. Note that } g(z) = (r, 0, 0) \Leftrightarrow \begin{cases} Q(z, v_2) = 0 \\ Q(z, v_3) = 0 \end{cases}$$

By dimension then, v_2, v_3 generate K_z and $\forall w \in K_z, Q(v_1, w) = 0$.

- Suppose that $Q(z, z) = 0$ and $g(z) = (*, 0, 0)$. Then, $z \in K_z$. Thus $Q(z, v_1) = 0$ and $v_2 \in K_z$. Then, $Q(v_1, v_1) = 0$ and that is a contradiction.

- Suppose that $Q(z, z) \neq 0$. Take then r a complex number such that $r^2 = Q(z, z)$. With v_2, v_3 two orthonormal vectors in K_z and $v_1 = \frac{z}{r}$, one has $Q(v_1, v_1) = 1$ and $Q(v_1, v_2) = Q(v_1, v_3) = 0$. This is $g(z) = (r, 0, 0)$. \square

Corollary 3.18. *Let $P \in \mathbb{C}[\hat{\mathcal{S}}]^{\mathbb{N}}$. Then, the denominator of $\tilde{P} \in \mathbb{C}(\hat{\mathcal{H}})^{\text{SO}_3(\mathbb{C})}$ is equal to Q^k , for some $k \in \mathbb{N}$.*

Proof. On an irreducible affine variety, any rational function can be written as a quotient of two relatively prime polynomials $\frac{P}{Q}$ [CLO15]. Then, it is well defined exactly outside $\mathcal{V}(Q)$. \square

Example 3.19. *In Example 3.15 we can observe that the polynomial \tilde{D}_0 is equal to the quadratic form Q introduced in Proposition 3.17. It is the Euclidean norm on the real trace. Corollary 3.18 claims that the denominator of each rational function in $\tilde{\mathcal{E}}$ is thus a power of \tilde{D}_0 .*

As denominators are only powers of $Q = \tilde{D}_0$, one can select only the numerators of the elements of $\tilde{\mathcal{E}}$. It provides a set generating the invariant field composed with polynomials.

Corollary 3.20. *Write $\tilde{\mathcal{E}} = \{P_1 = Q, \frac{P_i}{Q^{k_i}}, 1 \geq i \geq \#\mathcal{E}\}$ with exponents k_i . Then $\mathcal{P} := \{Q, P_i, 2 \leq i \leq \#\mathcal{E}\}$ is a set of algebraically independent polynomials generating $\mathbb{R}(\mathcal{H})^{\text{SO}_3(\mathbb{R})}$. Moreover, it still separates orbits in $\tilde{\mathcal{U}}$.*

Proof. Polynomials in \mathcal{P} generate each generator in $\tilde{\mathcal{E}}$, and thus $\mathbb{R}(\mathcal{H})^{\text{SO}_3(\mathbb{R})}$. By cardinality they are algebraically independent.

Take two point u_1, u_2 in $\tilde{\mathcal{U}}$ which are not in the same orbit. Then $P_1 = Q$ does not vanish at u_1 and u_2 . Suppose that Q does not separate orbits of u_1 and u_2 . Then, there is some $2 \leq i \leq \#\mathcal{E}$ such that $\frac{P_i}{Q^{k_i}}(u_1) \neq \frac{P_i}{Q^{k_i}}(u_2)$. By multiplication by $Q(u_1)^{k_i} = Q(u_2)^{k_i} \neq 0$ one has $P_i(u_1) \neq P_i(u_2)$. \square

3.6 Inverse problem

Functions of \mathcal{E} enable to construct the rational map

$$\Phi_{\tilde{\mathcal{E}}} : \begin{cases} \mathcal{H} & \longrightarrow \mathbb{R}^{\#\tilde{\mathcal{E}}} \\ h & \longmapsto (P(h), P \in \tilde{\mathcal{E}}) \end{cases}$$

that separates orbits almost everywhere. Note \mathcal{X} the image $\Phi_{\tilde{\mathcal{E}}}(\mathcal{H})$. Then the converse problem occurs: for a point $z \in \mathbb{R}^{\#\tilde{\mathcal{E}}}$, how can we determine if $z \in \mathcal{X}$? And if the answer is yes, can we find a preimage $h \in \mathcal{H}$ with $\Phi_{\tilde{\mathcal{E}}}(h) = z$?

Any point in \mathcal{H} has a representant of its orbit in \mathcal{S} , or it is a singular point for $\Phi_{\tilde{\mathcal{E}}}$. Thus $\mathcal{X} := \Phi_{\tilde{\mathcal{E}}}(\mathcal{S})$. Since functions in $\tilde{\mathcal{E}}$ are polynomials on the slice, \mathcal{X} is semialgebraic. That is, it is a finite union of set in $\mathbb{R}^{\#\tilde{\mathcal{E}}}$ defined by polynomial equalities and inequalities:

Proposition 3.21. Let $z = \left(\begin{array}{ll} t_i, & 1 \leq i \leq p \\ d_j, & 0 \leq j \leq q \\ r_{mj}, & 1 \leq j \leq n, \quad 1 \leq m \leq d_j \\ i_{mj}, & 1 \leq j \leq n, \quad 1 \leq m \leq d_j \text{ and } (m, j) \neq (1, 1) \end{array} \right) \in \mathbb{R}^{\#\tilde{\mathcal{E}}}$ a real vector.

Then, $z \in \mathcal{X}$ iff one of the following conditions holds:

- $d_0 > 0$ and $r_{11} > 0$
- $d_0 > 0$ and $\forall j, m, r_{mj} = i_{mj} = 0$
- $d_0 = 0$ and $r_{11} > 0$ and $\forall j, m, i_{mj} = 0$ and $\forall j, d_j = 0$
- $d_0 = 0$ and $\forall j, m, r_{mj} = i_{mj} = 0$ and $\forall j, d_j = 0$

The following proof is constructive, giving thus a primage in \mathcal{S} at the same time.

Proof. Take $z \in \mathbb{R}^{\#\tilde{\mathcal{E}}}$ a vector as in Proposition 3.21. The slice \mathcal{S} intersects all real orbits. Hence, there exists some real preimage $h \in \mathcal{H}$ iff there exists also a preimage in \mathcal{S} . Take then h a real point in the slice.

Firstly, $D_0(y_0, h) = y_0^2 \geq 0$ and $R_{11}(y_0, h) = 2y_{1,1}y_{-1,1} = 2y_{1,1}\overline{y_{1,1}} \geq 0$. Then $\Phi_{\tilde{\mathcal{E}}}(h) = z \Rightarrow d_0 \geq 0$ and $r_{11} \geq 0$. We choose $y_0 = \sqrt{d_0}$ and y_{11}, y_{-11} conjugated complex numbers of module $\frac{1}{2}\sqrt{r_{11}}$.

Suppose that nor d_0 or r_{11} vanish. Since $y_{1,1} \neq 0$ and $y_0 \neq 0$ one can choose for $m \leq j$, $y_{-m,j} = \frac{y_0 r_{mj} - i_{mj}}{2y_0 y_{1,1}^m}$ and $y_{m,j} = \overline{y_{-m,j}} = \frac{y_0 r_{mj} + i_{mj}}{2y_0 y_{1,1}^m}$. That way,

$$\begin{aligned} R_{mj}(y_0, h) &= y_{1,1}^m y_{-mj} + y_{-1,1}^m y_{m,j} \\ &= y_{1,1}^m \frac{y_0 r_{mj} - i_{mj}}{2y_0 y_{1,1}^m} + y_{-1,1}^m \frac{y_0 r_{mj} + i_{mj}}{2y_0 y_{1,1}^m} \\ &= r_{mj} \\ I_{mj}(y_0, h) &= iy_0 (y_{1,1}^m y_{-m,j} - y_{-1,1}^m y_{m,j}) \\ &= iy_0 \left(y_{1,1}^m \frac{y_0 r_{mj} - i_{mj}}{2y_0 y_{1,1}^m} - y_{-1,1}^m \frac{y_0 r_{mj} + i_{mj}}{2y_0 y_{1,1}^m} \right) \\ &= i_{mj} \end{aligned}$$

Take finally $y_{0,j} = t_j$ for $j \leq p$ and $y_{0,j} = \frac{d_j}{y_0}$ for $j \leq q$. Then $\Phi_{\tilde{\mathcal{E}}}(h) = z$.

Suppose that $R_{11}(y_0, h) = r_{11} = 0$. Then $y_{1,1} = y_{-1,1} = 0$ and $\Phi_{\tilde{\mathcal{E}}}(h) = z \Rightarrow \forall j, m, r_{mj} = i_{mj} = 0$. Take just $y_{0,j} = t_j$ for $j \leq p$ and $y_{0,j} = \frac{d_j}{y_0}$ for $j \leq q$ to have $\Phi_{\tilde{\mathcal{E}}}(h) = z$.

Suppose that $D_0(y_0, h) = d_0 = 0$. Then $y_0 = 0$ and $\Phi_{\tilde{\mathcal{E}}}(h) = z \Rightarrow \begin{cases} \forall j, & i_{mj} = y_0 \Im(y_{11} y_{-mj}) = 0. \\ \forall j, & d_j = y_0 y_{0i} = 0. \end{cases}$ One can then choose, for all $j \leq n$ and $1 \leq m \leq d_j$, $y_{-m,j} = \frac{r_{mj}}{y_{1,1}^m}$ and $y_{m,j} = \overline{y_{-m,j}} = \frac{r_{mj}}{y_{-1,1}^m}$ to have $R_{mj}(h) = r_{mj}$. Take just $y_{0,j} = t_j$ for $j \leq p$ to have $\Phi_{\tilde{\mathcal{E}}}(h) = z$.

Suppose that $R_{11}(y_0, h) = r_{11} = 0$ and $D_0(y_0, h) = d_0 = 0$. Then $y_{1,1} = y_{-1,1} = 0$ and $y_0 = 0$. One has then $\Phi_{\tilde{\mathcal{E}}}(h) = z \Rightarrow \begin{cases} \forall j, & i_{mj} = r_{mj} = 0. \\ \forall j, & d_j = y_0 y_{0i} = 0. \end{cases}$ Take just $y_{0,j} = t_j$ for $j \leq p$ to have $\Phi_{\tilde{\mathcal{E}}}(h) = z$. \square

These inequalities are precious to study the orbit set, giving a semialgebraic subset \mathcal{X} related to $\mathcal{H}/\text{SO}_3(\mathbb{R})$. The values of generating polynomials given by $\Phi_{\tilde{\mathcal{E}}}$ allows almost everywhere to determine when a point $h \in \mathcal{H}$ is of principal orbit type, namely points whose stabilizer in $\text{SO}_3(\mathbb{R})$ is $\text{Stab}(h) = \{I_3\}$:

Proposition 3.22. *If $\tilde{D}_0(h) > 0$ and $\tilde{R}_{11}(h) > 0$, then the stabilizer $\text{Stab}(h)$ is trivial.*

Proof. Take h a point in \mathcal{H} and s a point of its orbit lying on the slice. Write $(y_0, 0, 0, y_{1,1}, \dots, y_{-1,1})$ the coordinates of s in \mathcal{S} . Suppose that $\tilde{D}_0(h) = D_0(s) > 0$ and $\tilde{R}_{11}(h) = R_{11}(s) > 0$. That is, $y_0, y_{1,1}, y_{-1,1}$ are non zero. Meanwhile, $\text{SO}_3(\mathbb{R})$ action on \mathcal{H}_1 is dual to the canonic action of $\text{SO}_3(\mathbb{R})$ on \mathbb{R}^3 . A rotation $g \in \text{SO}_3(\mathbb{R})$ stabilizes $(y_0, 0, 0)$ in \mathcal{H}_1 iff $g \in \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}, \theta \in \mathbb{R} \right\} \cong \text{SO}_2(\mathbb{R})$. This subgroup is normal in N and its action on $\text{Vect}(y_{1,1}, y_{-1,1})$ is isomorphic to \mathcal{V}_1 . Its only point with a non trivial stabilizer is $(0, 0)$. Then, $\text{Stab}(s) = \{Id\}$. But, the stabilizers $\text{Stab}(h)$ and $\text{Stab}(s)$ are conjugated in $\text{SO}_3(\mathbb{R})$. Thus, $\text{Stab}(h) = \{I_3\}$. \square

The inequalities in the above proposition do not capture the whole of the principal stratum though. Indeed the converse of Proposition 3.22 is not true: consider the point with $y_{0,1} = y_{-1,2} = y_{1,2} = 1$, all the other coordinates being zero. Then $\text{Stab}(h)$ is trivial while $\tilde{D}_0(h) = \tilde{R}_{11}(h) = 0$.

On the other hand all the orbits with non trivial isotropy groups lie in $\mathcal{V}(Q)$ and are mapped to zero. Our separating set has thus important limitations to capture and stratify the orbit space. One can nonetheless design a strategy that goes beyond the present result by finding other Seshadri slices in the remaining variety $\mathcal{V}(Q)$. It allows to compute step by step a complete separating system, and to stratify the orbit space. This is examined in for the space of piezoelectric tensors in [HJ24b].

4 Generalisation to $\text{O}_3(\mathbb{R})$ representations

We consider the representations of the group $\text{O}_3(\mathbb{R}) \triangleright \text{SO}_3(\mathbb{R})$

$$\mathcal{H} := \mathcal{H}_1 \oplus \bigoplus_{j=1}^n \mathcal{H}_{d_j} \oplus \bigoplus_{j=1}^{n^\sharp} \mathcal{H}_{e_j^\sharp}.$$

where $d_1, \dots, d_n, e_1, \dots, e_{n^\sharp}$ are integers. Both \mathcal{H}_d^\sharp and \mathcal{H}_d can be thought of as the space of harmonic homogeneous polynomials of degree d . The representation on \mathcal{H}_d is the restriction to \mathcal{H}_d of the representation of $\text{O}_3(\mathbb{R})$ on the polynomial ring $\mathbb{R}[U, V, W]$ obtained by composition with the standard representation of $\text{O}_3(\mathbb{R})$ on \mathbb{R}^3 . We call it $\rho_d : \text{O}_3(\mathbb{R}) \rightarrow \text{GL}(\mathcal{H}_d)$ as was done for $\text{SO}_3(\mathbb{R})$ in Section 3.1. The $\text{O}_3(\mathbb{R})$ representation ρ_d^\sharp on \mathcal{H}_d^\sharp is defined by

$$\rho_d^\sharp : \begin{cases} \text{O}_3(\mathbb{R}) & \longrightarrow & \text{GL}(\mathcal{H}_d^\sharp) \\ g & \longmapsto & \det(g) \rho_d(g) \end{cases}$$

The representations (ρ_d, \mathcal{H}_d) and $(\rho_d^\sharp, \mathcal{H}_d^\sharp)$, $d \in \mathbb{N}$, are the only irreducible representations of $\text{O}_3(\mathbb{R})$ [GSS88].

For simplicity we assume that d_1, \dots, d_n are sorted decreasingly. A new Seshadri slice $(\mathcal{S}, \text{N}^\sharp)$ can be identified with the same subspace \mathcal{S} (Proposition 3.2) and with the normalizer $\text{N}^\sharp \triangleright \text{N}$, generated by N and $-I_3$. It forms the direct product $\text{N}^\sharp = \text{N} \times \text{C}_2$. We create the polynomials

$$\begin{aligned} T_j^\sharp &= & t_j^\sharp & & j \leq p^\sharp \\ D_j^\sharp &= & d_j^\sharp & & j \leq q^\sharp \\ R_{mj}^\sharp &= & y_{m,j}^\sharp y_{-1,1}^m + y_{-m,j}^\sharp y_{1,1}^m & & j \leq n^\sharp \quad -e_j \leq m \leq e_j \\ I_{mj}^\sharp &= & iy_0 \left(y_{m,j}^\sharp y_{-1,1}^m - y_{-m,j}^\sharp y_{1,1}^m \right) & & j \leq n^\sharp \quad -e_j \leq m \leq e_j \end{aligned}$$

Theorem 4.1. *Take P a polynomial in $\mathbb{R}(\mathcal{S})^{\text{N}}$ as follows:*

- If $n^\sharp \geq 1$, choose $P := T_j^\sharp$ (for some e_j even) or $P := D_j^\sharp$ (for some e_j odd).
- Else, if $d_1 \geq 2$, choose $P := I_{d_1}$.
- Else, if $n \geq 2$ and $d_2 = 1$, choose $P := I_{12}$.
- Else, choose $P := 1$.

Then, the following set of algebraically independent polynomials generates both $\mathbb{R}(\mathcal{S})^{N^\sharp}$ or $\mathbb{C}(\hat{\mathcal{S}})^{\hat{N}^\sharp}$:

$$\mathcal{E}^\dagger = \left\{ \begin{array}{l} D_0^\dagger = D_0 \\ T_j^\dagger = T_j, \quad d_j \text{ even} \\ D_j^\dagger = D_j, \quad d_j \text{ odd} \\ R_{mj}^\dagger = \begin{cases} R_{mj}, & d_1 m + d_j \text{ even} \\ PR_{mj}, & d_1 m + d_j \text{ odd} \end{cases} \\ I_{mj}^\dagger = \begin{cases} PI_{mj}, & d_1 m + d_j \text{ even} \\ I_{mj}, & d_1 m + d_j \text{ odd} \end{cases} \end{array} \right\} \cup \left\{ \begin{array}{l} T_j^\ddagger = PT_j^\sharp, \quad e_j \text{ even} \\ D_j^\ddagger = PD_j^\sharp, \quad e_j \text{ odd} \\ R_{mj}^\ddagger = \begin{cases} PR_{mj}^\sharp, & d_1 m + e_j \text{ even} \\ R_{mj}^\sharp, & d_1 m + e_j \text{ odd} \end{cases} \\ I_{mj}^\ddagger = \begin{cases} I_{mj}^\sharp, & d_1 m + e_j \text{ even} \\ PI_{mj}^\sharp, & d_1 m + e_j \text{ odd} \end{cases} \end{array} \right\}$$

Proof. We notice the direct product $\hat{N}^\sharp = \hat{N} \times C_2$. This gives $\mathbb{C}(\hat{\mathcal{S}})^{\hat{N}^\sharp} = \mathbb{C}(\mathbb{C}(\hat{\mathcal{S}})^{\hat{N}})^{C_2}$. Theorem 3.7 gives the same minimal set of generators for $\mathbb{C}(\hat{\mathcal{S}})^{\hat{N}}$, independantly with the sharp:

$$\mathcal{E} = \left\{ \begin{array}{l} T_j(y_0, h) = y_{0,j} \quad d_j \text{ even} \\ T_j^\sharp(y_0, h) = y_{0,j}^\sharp \quad e_j \text{ even} \\ D_0(y_0, h) = y_0^0 \\ D_j(y_0, h) = y_0 \cdot y_{0,j}, \quad d_j \text{ odd} \\ D_j^\sharp(y_0, h) = y_0 y_{0,j}^\sharp, \quad e_j \text{ odd} \\ R_{mj}(y_0, h) = y_{m,j} y_{-1,1}^m + y_{-m,j} y_{1,1}^m, \quad 1 \leq j \leq n, m \leq d_j \\ R_{mj}^\sharp(y_0, h) = y_{m,j}^\sharp y_{-1,1}^m + y_{-m,j}^\sharp y_{1,1}^m, \quad 1 \leq j \leq n^\sharp, m \leq e_j \\ I_{mj}(y_0, h) = iy_0 (y_{m,j} y_{-1,1}^m - y_{-m,j} y_{1,1}^m), \quad 1 \leq j \leq n, m \leq d_j, (j, m) \neq (1, 1) \\ I_{mj}^\sharp(y_0, h) = iy_0 (y_{m,j}^\sharp y_{-1,1}^m - y_{-m,j}^\sharp y_{1,1}^m), \quad 1 \leq j \leq n^\sharp, m \leq e_j \end{array} \right\}$$

Take then the symmetry $\sigma = -I_3 \in \hat{N}^\sharp$. In spherical coordinates it acts as $\forall (r, \theta, \varphi), \sigma((r, \theta, \varphi)) = (r, \pi - \theta, \varphi + \pi)$. By composition, for a vector $Y_{d_j}^m \in \mathcal{H}_{d_j}$, σ acts as

$$\begin{aligned} \rho_{d_j}(\sigma)(Y_{d_j}^m)((r, \theta, \varphi)) &= Y_{d_j}^m(\sigma((r, \theta, \varphi))) \\ &= Y_{d_j}^m(r, \pi - \theta, \varphi + \pi) \\ &= r^d c_d^m e^{im(\varphi + \pi)} P_{d_j}^m(-\cos(\theta)) \\ &= (-1)^{d_j} Y_{d_j}^m((r, \theta, \varphi)) \end{aligned}$$

and in a sharped representation $\mathcal{H}_{d_j}^\sharp$,

$$\begin{aligned} \rho_{d_j}^\sharp(\sigma)(Y_{d_j}^m)((r, \theta, \varphi)) &= -Y_{d_j}^m(\sigma((r, \theta, \varphi))) \\ &= -Y_{d_j}^m(r, \pi - \theta, \varphi + \pi) \\ &= -r^d c_d^m e^{im(\varphi + \pi)} P_{d_j}^m(-\cos(\theta)) \\ &= (-1)^{d_j + 1} Y_{d_j}^m((r, \theta, \varphi)) \end{aligned}$$

It thus has an induced action on each elements in \mathcal{E} :

$$\left\{ \begin{array}{l} \sigma(T_j) = T_j, \quad d_j \text{ even} \\ \sigma(T_j^\sharp) = -T_j^\sharp, \quad e_j \text{ even} \\ \sigma(D_0) = D_0 \\ \sigma(D_j) = D_j, \quad d_j \text{ odd} \\ \sigma(D_j^\sharp) = -D_j^\sharp, \quad e_j \text{ odd} \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \sigma(R_{mj}) = (-1)^{d_j+d_1m} R_{mj}, \\ \sigma(R_{mj}^\sharp) = (-1)^{1+e_j+d_1m} R_{mj}^\sharp \\ \sigma(I_{mj}) = (-1)^{1+d_j+d_1m} I_{mj} \\ \sigma(I_{mj}^\sharp) = (-1)^{e_j+d_1m} I_{mj}^\sharp, \end{array} \right.$$

We next apply lemma 2.5. We need a polynomial $P \in \mathcal{E}$ flipped by σ . Whenever $n^\sharp \geq 1$, one can choose $P := T_j^\sharp$ (for e_j even) or $P := D_j^\sharp$ (for e_j odd). Else it depends on d_1 . If $d_1 \geq 2$ then I_{11} suits. If $d_1 = 1, n \geq 2$ and $d_2 = 1$ then I_{12} suits. Otherwise, no polynomial in \mathcal{E} is flipped and $\mathbb{C}(\hat{\mathcal{S}})^{\hat{N}} = \mathbb{C}(\hat{\mathcal{S}})^{\hat{N}^\sharp}$. \square

Almost free $O_3(\mathbb{C})$ actions have a stably rational invariant field [CS05, Proposition 4.10]. The above theorem provides the following stronger result, yet with a different hypothesis as already discussed after Theorem 3.7.

Corollary 4.2. *Let $\rho : O_3(\mathbb{R}) \rightarrow GL(\mathcal{H})$ a linear representation including at least once the irreducible representation (ρ_1, \mathcal{H}_1) . Then, both $\mathbb{R}(\mathcal{H})^{O_3(\mathbb{R})}$ and $\mathbb{C}(\mathcal{H})^{O_3(\mathbb{C})}$ are rational.*

Example 4.3. *The ternary cubics $O_3(\mathbb{R})$ -representation $\mathbb{R}[U, V, W]_3$ is isomorphic to $\mathcal{T} = \mathcal{H}_1 \oplus \mathcal{H}_3$. With the basis introduced in Example 3.2, an algebraically independent generating set for the slice invariants $\mathbb{R}(\mathcal{S}_{\mathcal{T}})^{N^\sharp}$ is thus*

$$\mathcal{E}_{\mathcal{T}}^\dagger = \left\{ \begin{array}{l} D_1^\dagger = y_0^2 \\ D_3^\dagger = y_0 y_{0,3} \\ R_1^\dagger = 2y_{1,3} y_{-1,3} \\ R_2^\dagger = (y_{1,3}^2 y_{-2,3} + y_{-1,3}^2 y_{2,3})^2 \\ R_3^\dagger = y_{1,3}^3 y_{-3,3} + y_{-1,3}^3 y_{3,3} \\ I_2^\dagger = iy_0 (y_{1,3}^2 y_{-2,3} - y_{-1,3}^2 y_{2,3}) \\ I_3^\dagger = iy_0 (y_{1,3}^3 y_{-3,3} - y_{-1,3}^3 y_{3,3}) (y_{1,3}^2 y_{-2,3} + y_{-1,3}^2 y_{2,3}) \end{array} \right.$$

5 Applications

We have exhibited invariants for representations of $SO_3(\mathbb{R})$ and $O_3(\mathbb{R})$ on spaces $\mathbb{R}^3 \oplus \mathbb{H}$. Among those is the well studied case of $\mathbb{R}^3 \times \dots \times \mathbb{R}^3$ treated to some extent in Example 3.15. This representation and its invariants are relevant in several areas [BV23; DCP21; DG22; GD04; Wey46]. In [HJ24a] we treat the general case, that of $SO_n(\mathbb{R})$ and $O_n(\mathbb{R})$ acting on $\mathbb{R}^n \times \dots \times \mathbb{R}^n$ and provide algebraically independent sets of generators for the invariant field, relating them with the classical Weyl invariants [Wey46]. In this section we expand on two other cases of relevance for which we give explicitly the restrictions of the invariants to the slice. As explained in Section 3, from these expressions we can evaluate the invariants at any point outside of an hypersurface, as well as solve the inverse problem.

5.1 Odd degree ternary forms

Among the cases of interest to which our construction applies is the representation of $SO_3(\mathbb{R})$ or $O_3(\mathbb{R})$ on the vector space of homogeneous polynomials $\mathbb{R}[U, V, W]_d$ of odd degree d . The case of even degrees was treated in [GHP19]. The starting point for both odd and even degree is the harmonic decomposition.

Lemma 5.1. [AH12, Theorem 2.18] *The usual representation of $SO_3(\mathbb{R})$ on $\mathbb{R}[U, V, W]_d$ is isomorphic to*

$$\bigoplus_{j=0}^{\lfloor d/2 \rfloor} \mathcal{H}_{d-2j}.$$

Whenever d is odd, the last term of this decomposition is \mathcal{H}_1 and thus we can apply the construction provided in previous sections.

Corollary 5.2. *For any $d \in \mathbb{N}$, the invariant field of the $\mathrm{SO}_3(\mathbb{R})$ or $\mathrm{O}_3(\mathbb{R})$ action on $\mathbb{R}[U, V, W]_d$ is rational.*

Example 5.3. *The vector space $T = \mathcal{H}_1 \oplus \mathcal{H}_3$ introduced in Example 3.9 is isomorphic to the space of ternary cubics $\mathbb{R}[U, V, W]_3$. Completing the basis obtained in Example 3.2, one endows \mathcal{T} with the real harmonic basis*

$$\mathcal{B}_{\mathcal{T}} = \left\{ \begin{array}{lll} X_{-1} = QV, & X_0 = QU, & X_1 = -QW \\ X_3^3 = 3W^2V - V^3 & X_3^2 = UVW & X_3^1 = W(4U^2 - V^2 - W^2) \\ X_3^{-3} = 3WV^2 - W^3 & X_3^0 = U(2U^2 - 3V^2 - 3W^2) & X_3^{-1} = -V(4U^2 - V^2 - W^2) \\ & X_3^{-2} = U(W^2 - V^2) & \end{array} \right\}$$

The set $\mathcal{E}_{\mathcal{T}}$ in Example 3.9 generates then $\mathbb{R}(\mathcal{S}_{\mathcal{T}})^N$ (with respect to the associated complex basis). Theorem 3.14 gives a set of rational generators for $\mathbb{R}(\mathcal{T})^{\mathrm{SO}_3(\mathbb{R})}$. It separates orbits everywhere but in $\{\tilde{D}_0 = 0\} = \{x_{-1}^2 + x_0^2 + x_1^2 = 0\}$ and $\{\tilde{R}_{11} = 0\}$.

5.2 Piezoelectricity tensors

In Continuum Mechanics, the orthogonal group $\mathrm{SO}_3(\mathbb{R})$ acts on the piezoelectricity tensors space $\mathbb{P}\mathrm{iez}$ as presented in [Che+19, Section 2.1] and in [Azz+22]:

$$\mathbb{P}\mathrm{iez} := \{e \in \otimes^3 \mathbb{R}^3, e_{ijk} = e_{ikj}\}$$

For physical reasons this action is mostly studied over the reals. The 18-dimensional space $\mathbb{P}\mathrm{iez}$ decomposes into $\mathrm{SO}_3(\mathbb{R})$ -irreducible subspaces as [Azz+22, Section 7.1]

$$\mathbb{P}\mathrm{iez} \cong \mathcal{H}_1 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3.$$

One can choose $(\mathcal{S}, \mathcal{N})$ as in Theorem 2.8 and get:

Proposition 5.4. *The following set is an algebraically independent and generates $\mathbb{R}(\mathcal{S})^N$.*

$$\mathcal{E}_{\mathbb{P}\mathrm{iez}} = \left\{ \begin{array}{lll} T_2 = y_{0,2} & R_{11} = 2y_{-1,1}y_{1,1} & I_{12} = iy_0(y_{1,2}y_{-1,1} - y_{-1,2}y_{1,1}) \\ D_0 = y_0^2 & R_{12} = y_{1,2}y_{-1,1} + y_{-1,2}y_{1,1} & I_{22} = iy_0(y_{2,2}y_{-1,1}^2 - y_{-2,2}y_{1,1}^2) \\ D_1 = y_0y_{0,1} & R_{22} = y_{2,2}y_{-1,1}^2 + y_{-2,2}y_{1,1}^2 & I_{13} = iy_0(y_{1,3}y_{-1,1} - y_{-1,3}y_{1,1}) \\ D_3 = y_0y_{0,3} & R_{13} = y_{1,3}y_{-1,1} + y_{-1,3}y_{1,1} & I_{23} = iy_0(y_{2,3}y_{-1,1}^2 - y_{-2,3}y_{1,1}^2) \\ & R_{23} = y_{2,3}y_{-1,1}^2 + y_{-2,3}y_{1,1}^2 & I_{33} = iy_0(y_{3,3}y_{-1,1}^3 - y_{-3,3}y_{1,1}^3) \\ & R_{33} = y_{3,3}y_{-1,1}^3 + y_{-3,3}y_{1,1}^3 & \end{array} \right\}$$

Here is its expression in real coordinates (see Section 3.3):

$$\mathcal{E}_{\mathbb{P}\mathrm{iez}} = \left\{ \begin{array}{lll} T_2 = x_{0,2} & R_{11} = 2(x_{1,1}^2 + x_{-1,1}^2) & \\ D_0 = x_0^2 & R_{12} = 4x_{-2}x_{-1}x_1 - 2x_{-1}^2x_2 + 2x_1^2x_2 & \\ D_1 = x_0x_{0,1} & R_{22} = 4x_{-2}x_{-1}x_1 - 2x_{-1}^2x_2 + 2x_1^2x_2 & \\ D_3 = x_0x_{0,3} & R_{13} = -2x_{-3}x_{-1}^3 + 6x_{-3}x_{-1}x_1^2 - 6x_{-1}^2x_1x_3 + 2x_1^3x_3 & \\ & R_{23} = -2x_{-3}x_{-1}^3 + 6x_{-3}x_{-1}x_1^2 - 6x_{-1}^2x_1x_3 + 2x_1^3x_3 & \\ & R_{33} = -2x_{-3}x_{-1}^3 + 6x_{-3}x_{-1}x_1^2 - 6x_{-1}^2x_1x_3 + 2x_1^3x_3 & \\ & I_{12} = 2x_0(x_{-2}x_{-1}^2 - x_{-2}x_1^2 + 2x_{-1}x_1x_2) & \\ & I_{22} = 2x_0(x_{-2}x_{-1}^2 - x_{-2}x_1^2 + 2x_{-1}x_1x_2) & \\ & I_{13} = 2x_0(3x_{-3}x_{-1}^2x_1 - x_{-3}x_1^3 - x_{-1}^3x_3 + 3x_{-1}x_1^2x_3) & \\ & I_{23} = 2x_0(3x_{-3}x_{-1}^2x_1 - x_{-3}x_1^3 - x_{-1}^3x_3 + 3x_{-1}x_1^2x_3) & \\ & I_{33} = 2x_0(3x_{-3}x_{-1}^2x_1 - x_{-3}x_1^3 - x_{-1}^3x_3 + 3x_{-1}x_1^2x_3) & \end{array} \right\}$$

This set provides an algebraically independent set $\mathcal{E}_{\text{Piez}}^{\sim}$ generating $\mathbb{R}(\mathcal{H})^{\text{SO}_3(\mathbb{R})}$, which separates orbits wherever \tilde{D}_0 and the denominator of \tilde{R}_{11} do not vanish. The denominator of \tilde{R}_{11} is the same as in Example 3.15:

$$-y_{-1}^2 y_{1,1}^2 - 2y_{-1} y_0 y_{0,1} y_{1,1} + 2y_{-1} y_1 y_{-1,1} y_{1,1} + 2y_{-1} y_1 y_{0,1}^2 + 2y_0^2 y_{-1,1} y_{1,1} - 2y_0 y_1 y_{-1,1} y_{0,1} - y_1^2 y_{-2,1}^2$$

In [Azz+22, Section 7.1], the group acting on the Piezoelectricity tensors is actually $\text{O}_3(\mathbb{R})$. The $\text{O}_3(\mathbb{R})$ representation is then $\text{Piez} \cong \mathcal{H}_1 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2^{\sharp} \oplus \mathcal{H}_3$. Here the first term $d_1 = 1$ is odd, and $T_2^{\sharp} = y_{0,2}$ of Theorem 4.1 can be chosen as the flipped polynomial P . Applying Theorem 4.1:

Proposition 5.5. *The following set of cardinal 15 is an algebraically independent generating set for both $\mathbb{C}(\hat{\mathcal{S}})^{\hat{N}^{\sharp}}$ or $\mathbb{R}(\mathcal{S})^{N^{\sharp}}$.*

$$\mathcal{E}_{\text{Piez}} = \left\{ \begin{array}{lll} T_2^{\sharp} = y_{0,2}^2 & R_{11} = y_{1,1} y_{-1,1} + y_{-1,1} y_{1,1} & I_{12}^{\sharp} = iy_0 (y_{1,2} y_{-1,1} - y_{-1,2} y_{1,1}) \\ D_0 = y_0^2 & R_{12}^{\sharp} = y_{0,2} (y_{1,2} y_{-1,1} + y_{-1,2} y_{1,1}) & I_{22}^{\sharp} = iy_0 y_{0,2} (y_{2,2} y_{-1,1}^2 - y_{-2,2} y_{1,1}^2) \\ D_1 = y_0 y_{0,1} & R_{22}^{\sharp} = y_{2,2} y_{-1,1}^2 + y_{-2,2} y_{1,1}^2 & I_{13} = iy_0 y_{0,2} (y_{1,3} y_{-1,1} - y_{-1,3} y_{1,1}) \\ D_3 = y_0 y_{0,3} & R_{13} = y_{1,3} y_{-1,1} + y_{-1,3} y_{1,1} & I_{23} = iy_0 (y_{2,3} y_{-1,1}^2 - y_{-2,3} y_{1,1}^2) \\ & R_{23} = y_{0,2} (y_{2,3} y_{-1,1}^2 + y_{-2,3} y_{1,1}^2) & I_{33} = iy_0 y_{0,2} (y_{3,3} y_{-1,1}^3 - y_{-3,3} y_{1,1}^3) \\ & R_{33} = y_{3,3} y_{-1,1}^3 + y_{-3,3} y_{1,1}^3 & \end{array} \right\}$$

A Rewriting an invariant in terms of the generators

In this article we exhibited algebraically independent generators of the invariant field $\mathbb{R}(\mathcal{H})^{\text{SO}_3(\mathbb{R})}$ emphasizing their separation properties. One can also use a generating set of invariants as new *variables* of an *invariant problem*, thus *factoring out* the symmetry; see for instance [HL12; HL13; HL16] where are examined the use of invariants of Abelian groups to solve polynomial systems or reduce the number of parameters in dynamical systems. In this appendix we show how to rewrite any invariants in $\mathbb{R}(\mathcal{H})^{\text{SO}_3(\mathbb{R})}$ in terms of the generators exhibited.

Consider $\tilde{R} \in \mathbb{R}(\mathcal{H})^{\text{SO}_3(\mathbb{R})}$. Then the restriction of \tilde{R} to the slice \mathcal{S} is an $\text{O}_2(\mathbb{R})$ -invariant; in other words $R = \tilde{R}|_{\mathcal{S}} \in \mathbb{R}(\mathcal{S})^{\text{O}_2(\mathbb{R})}$. If we can identify a rational function F in $\mathbb{R}(T_j, D_j, R_{mj}, I_{mj})$ such that $R = F(\mathcal{E})$ then $\tilde{R} = F(\tilde{\mathcal{E}})$ by Theorem 3.4. One thus can rewrite any element of $\mathbb{R}(\mathcal{H})^{\text{SO}_3(\mathbb{R})}$ as a rational function of $\tilde{\mathcal{E}}$ thanks to the following rewrite rules.

Theorem A.1. *Assume $P, Q \in \mathbb{R}[\mathcal{S}]$ and take p and q in*

$$\mathbb{R}(T_j, D_j, R_{mj}, I_{mj})[y_0; y_{1,1}; y_{-1,1}; y_{-d_j,j}, \dots, y_{-d_j,j}, 1 \leq j \leq n]$$

the reductions of P and Q by the following confluent system of rewrite rules:

$$\begin{array}{lll} y_{0,j} & \longrightarrow & T_j & 1 \leq j \leq p \\ y_{0,j} & \longrightarrow & \frac{D_j}{D_0} y_0 & p+1 \leq j \leq n \\ y_{m,j} & \longrightarrow & 2^{m-1} \left(\frac{R_{mj}}{R_{11}^m} - \frac{i I_{mj}}{y_0 R_{11}^m} \right) y_{1,1}^m & 1 \leq j \leq n, 1 \leq m \leq d_j \\ y_{-m,j} & \longrightarrow & 2^{m-1} \left(\frac{R_{mj}}{R_{1,1}^m} + \frac{i I_{mj}}{y_0 R_{1,1}^m} \right) y_{-1,1}^m & 1 \leq j \leq n, 1 \leq m \leq d_j \\ y_{1,1} y_{-1,1} & \longrightarrow & R_{11} \\ y_0^2 & \longrightarrow & D_0 \end{array}$$

Then $\frac{P}{Q}$ belongs to $\mathbb{R}(\mathcal{S})^N$ iff $p = \check{p}r$ and $q = \check{q}r$ with $\check{p}, \check{q} \in \mathbb{R}(T_j, D_j, R_{mj}, I_{mj})$ and $r \in \mathbb{R}[y]$. In which case

$$\frac{P}{Q} = \frac{\check{p}(T_j(y_0, h), D_j(y_0, h), R_{mj}(y_0, h), I_{mj}(y_0, h))}{\check{q}(T_j(y_0, h), D_j(y_0, h), R_{mj}(y_0, h), I_{mj}(y_0, h))}$$

Proof. The proof relies on [MS99]. A Gröbner basis of the ideal in $\mathbb{R}(y)[Y]$ generated by

$$\begin{aligned} & \{Y_{0,j} - y_{0,j}, 1 \leq j \leq p\} \cup \{Y_0 Y_{0,j} - y_0 y_{0,j}, p+1 \leq j \leq n\} \\ & \cup \{(Y_{-1,1}^m Y_{m,j} + Y_{1,1}^m Y_{-m,j}) - (y_{-1,1}^m y_{m,j} + y_{1,1}^m y_{-m,j}), 1 \leq j \leq n, 1 \leq m \leq d_j, (m,j) \neq (1,1)\} \\ & \cup \{Y_0(Y_{-1,1}^m Y_{m,j} - Y_{1,1}^m Y_{-m,j}), -y_0(y_{-1,1}^m y_{m,j} - y_{1,1}^m y_{-m,j}), 1 \leq j \leq n, 1 \leq m \leq d_j, (m,j) \neq (1,1)\} \\ & \cup \{Y_{-1,1} Y_{1,1} - y_{-1,1} y_{1,1}, Y_0^2 - y_0^2\} \end{aligned}$$

w.r.t. to a term order that eliminates $Y_{0,j}$, $Y_{m,j}$ and $Y_{-m,j}$ is given by

$$\begin{aligned} & \{\mathbf{Y}_0^2 - y_0^2, \mathbf{Y}_{1,1} \mathbf{Y}_{-1,1} - y_{1,1} y_{-1,1}\} \\ & \cup \{\mathbf{Y}_{0,j} - y_{0,j} \mid 1 \leq j \leq p\} \cup \left\{ \mathbf{Y}_{0,j} - \frac{y_{0,j}}{y_0} Y_0 \mid p+1 \leq j \leq n \right\} \\ & \cup \left\{ \mathbf{Y}_{\mathbf{m},\mathbf{j}} - \left(\frac{(y_{m,j} y_{-1,1}^m + y_{-m,j} y_{1,1}^m)}{2(y_{-1,1} y_{1,1})^m} + \frac{(y_{m,j} y_{-1,1}^{m-1} - y_{-m,j} y_{1,1}^m)}{2y_0(y_{-1,1} y_{1,1})^m} Y_0 \right) Y_{1,1}^m \mid 1 \leq j \leq n, 1 \leq m \leq d_j, (m,j) \neq (1,1) \right\}, \\ & \cup \left\{ \mathbf{Y}_{-\mathbf{m},\mathbf{j}} - \left(\frac{(y_{m,j} y_{-1,1}^m + y_{-m,j} y_{1,1}^m)}{2y_{-1,1}^m y_{1,1}^m} - \frac{(y_{m,j} y_{-1,1}^{m-1} - y_{-m,j} y_{1,1}^m)}{2y_0(y_{-1,1} y_{1,1})^m} Y_0 \right) Y_{-1,1}^m \mid 1 \leq j \leq n, 1 \leq m \leq d_j, (m,j) \neq (1,1) \right\} \end{aligned}$$

where the leading terms are in bold font. This latter can be written as :

$$\begin{aligned} & \{\mathbf{Y}_0^2 - D_0, \mathbf{Y}_{1,1} \mathbf{Y}_{-1,1} - \frac{1}{2} R_{11}\} \\ & \cup \{\mathbf{Y}_{0,j} - T_j \mid 1 \leq j \leq p\} \cup \left\{ \mathbf{Y}_{0,j} - \frac{D_j}{D_0} Y_0 \mid p+1 \leq j \leq n \right\} \\ & \cup \left\{ \mathbf{Y}_{\mathbf{m},\mathbf{j}} - 2^{m-1} \left(\frac{R_{mj}}{R_{11}^m} - \frac{i I_{mj}}{D_0 R_{11}^m} Y_0 \right) Y_{1,1}^m \mid 1 \leq j \leq n, 1 \leq m \leq d_j, (m,j) \neq (1,1) \right\}, \\ & \cup \left\{ \mathbf{Y}_{-\mathbf{m},\mathbf{j}} - 2^{m-1} \left(\frac{R_{mj}}{R_{11}^m} + \frac{i I_{mj}}{D_0 R_{11}^m} Y_0 \right) Y_{-1,1}^m \mid 1 \leq j \leq n, 1 \leq m \leq d_j, (m,j) \neq (1,1) \right\}. \end{aligned}$$

By [MS99, Lemma 1.7 and Algorithm 1.10], we obtain the announced result. \square

Example A.2. In the space of piezoelectricity tensor, the invariant \tilde{P} whose restriction to the slice equals $P = y_{-2,1}^3 y_{3,1}^2 + y_{2,1}^3 y_{-3,1}^2$ can be rewritten as

$$\tilde{P} = \frac{256}{\tilde{R}_{1,1}^6} \left(\tilde{R}_{21}^3 \tilde{R}_{31}^2 - \frac{\tilde{R}_{21}^3 \tilde{I}_{31}^2}{\tilde{D}_0} + \frac{6\tilde{R}_{21}^2 \tilde{I}_{21} \tilde{R}_{31} \tilde{I}_{31}}{\tilde{D}_0} - \frac{3\tilde{R}_{21} \tilde{I}_{21}^2 \tilde{R}_{31}^2}{\tilde{D}_0} + \frac{3\tilde{R}_{21} \tilde{I}_{21}^2 \tilde{I}_{31}^2}{\tilde{D}_0^2} - \frac{2\tilde{I}_{21}^3 \tilde{R}_{31} \tilde{I}_{31}}{\tilde{D}_0^2} \right).$$

As predicted by Corollary 3.18, the denominator is a power of $\tilde{D}_0 = Q$ though the rewriting system of Theorem A.1 could lead us to believe that \tilde{R}_1 could appear as denominator as well.

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