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The Non-Cancelling Intersections Conjecture

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In this note, we present a conjecture on intersections of set families, and a rephrasing of the conjecture in terms of principal downsets of Boolean lattices. The conjecture informally states that, whenever we can express the measure of a union of sets in terms of the measure of some of their intersections using the inclusion-exclusion formula, then we can express the union as a set from these same intersections via the set operations of disjoint union and subset complement. We also present a partial result towards establishing the conjecture.

Acknowledgements. We thank Louis Jachiet for helpful discussions on the problem, and for having found the counterexample to a stronger version of the conjecture that is mentioned at the end of Section 7. This work was done in part while the authors were visiting the Simons Institute for the Theory of Computing.

1. Introduction

We present in this note a conjecture on intersection lattices of set families, which can be equivalently stated on the Boolean lattice. The original motivation for the conjecture

comes from a problem in database theory about the existence of certain circuit representations for probabilistic query evaluation (see [Monet, 2020a]), but in this note we present the conjecture as a purely abstract claim without any database prerequisites. A positive answer to this abstract conjecture implies a positive answer to the database theory conjecture.

The conjecture can be understood in terms of the inclusion-exclusion formula. Consider a family of sets S_1, \dots, S_n . The inclusion-exclusion formula can be used to express a quantity on the union $S_1 \cup \dots \cup S_n$ (e.g., the cardinality, or the value of some measure) as a function of the intersections of the S_i . In general, some of these intersections may in fact be identical: we can in fact define the *intersection lattice* of the set family to represent the possible intersections that can be obtained. Further, we can have *cancellations*, i.e., some of the possible intersections may end up having a coefficient of zero in the inclusion-exclusion formula, due to cancellations. Thus, in the general case, inclusion-exclusion allows us to express the measure of $S_1 \cup \dots \cup S_n$ in terms of the measure of the intersections I_1, \dots, I_m that have a non-zero coefficient.

Our conjecture asks whether, in this case, one can obtain the set $S_1 \cup \dots \cup S_n$ from the intersections I_1, \dots, I_m , using the set operations of disjoint union and subset-complement. If this is true, then it implies in particular the inclusion-exclusion formulation, provided that our measure is additive in the sense that the measure of $S \dot{\cup} S'$ (where $\dot{\cup}$ denotes disjoint union and $\dot{\setminus}$ subset complement, as per Definition 3.1) is the sum of the measure of S and of S' . Obtaining such a set expression can sometimes be done simply by re-ordering the inclusion-expression formula: for instance, if we write $|X \cup Y| = |X| + |Y| - |X \cap Y|$, we can reorder to $|X \cup Y| = |X| - |X \cap Y| + |Y|$, and then express $X \cup Y = (X \dot{\setminus} (X \cap Y)) \dot{\cup} Y$. However, in general, it is unclear whether such an expression can be obtained. Before continuing, let us give a first toy example.

Example 1.1. Consider the set family $\mathcal{F} = \{S_1, S_2, S_3\}$ with $S_1 = \{a, b, d\}$, $S_2 = \{a, b, c, e\}$, $S_3 = \{a, c, f\}$. Then:

$$\begin{aligned} |S_1 \cup S_2 \cup S_3| &= |S_1| + |S_2| + |S_3| \\ &\quad - (|S_1 \cap S_2| + |S_1 \cap S_3| + |S_2 \cap S_3|) \\ &\quad + |S_1 \cap S_2 \cap S_3|. \end{aligned}$$

Since we have $S_1 \cap S_3 = S_1 \cap S_2 \cap S_3$, we obtain

$$|S_1 \cup S_2 \cup S_3| = |\{a, b, d\}| + |\{a, b, c, e\}| + |\{a, c, f\}| - |\{a, b\}| - |\{a, c\}|.$$

The non-cancelling intersections are the ones that remain, i.e., $\{a, b, d\}$, $\{a, b, c, e\}$, $\{a, c, f\}$, $\{a, b\}$, and $\{a, c\}$, while $\{a\}$ ($= S_1 \cap S_3 = S_1 \cap S_2 \cap S_3$) is a cancelling term.

Then, we can express $S_1 \cup S_2 \cup S_3 = \{a, b, c, d, e, f\}$ with $[((\{a, b, d\} \dot{\setminus} \{a, b\}) \dot{\cup} \{a, b, c, e\}) \dot{\setminus} \{a, c\}] \dot{\cup} \{a, c, f\}$: the reader can easily check that each $\dot{\cup}$ (resp., each $\dot{\setminus}$) is a valid disjoint union (resp., subset complement), and that we have only used the non-cancelling intersections. Note that this is not the only valid expression, for

instance we can also obtain the union with the expression $[\{a, b, d\} \setminus \{a, b\}] \cup [\{a, b, c, e\} \setminus \{a, c\}] \cup \{a, c, f\}$.

Our results imply that it is always possible when there are no cancellations (Fact 4.8), or exactly one cancellation (in the equivalent formulation on Boolean lattices, Theorem 6.2), but in general it seem challenging to do so while avoiding those intersections which cancel out. The goal of this note is to present the current status of our efforts in attacking the conjecture, in particular showcasing some equivalent formulations, presenting examples, and establishing a very partial result. After presenting preliminaries in Section 2, we formally state the conjecture and give examples in Section 3. We show in Section 4 that it suffices to study the conjecture on specific intersection lattices, which we call *tight*, where informally the set family ensures that every possible intersection contains exactly one element that is “specific” to this intersection (in the sense that it is present precisely in this intersection and in larger intersections). We use this in Section 5 to give an alternative formulation: instead of working with intersection lattices, we can work with downsets on the Boolean lattice. We show that this is equivalent to the original conjecture. The hope is that the setting of the Boolean lattice can be more convenient to work with, as its structure is more restricted, and we can define quantities such as the Euler characteristic that seem useful in understanding the structure of the problem.

We next present in Section 6 a partial result that we can show in the context of the Boolean lattice. In the rephrased problem, we must express a downset of that lattice using disjoint union and subset complement on principal downsets spanned by elements which do not “cancel out” in the sense of having non-zero Möbius value. Our result establishes that the rephrased conjecture is true when there is one single node of the downset that has such value; this non-trivially extends the fact that the result is true when no such nodes exist (Lemma 5.7), but falls short of the goal as in general many such zeroes can occur.

We conclude in Section 7 with further questions and directions on the conjecture. We point out that the conjecture can be strengthened, in two different ways, to restrict the shape of the disjoint-union and subset-complement expressions: we also do not know the status of these stronger conjectures. We briefly report on an unsuccessful experimental search for counterexamples. We also hint at an incomplete proof of another partial result where we can avoid more than one zero, provided the targeted downset satisfies a certain decomposability condition.

2. Preliminaries

For a set S we write 2^S its powerset. In this work, by *family of sets*, or simply *set family*, we always mean a finite set of (not necessarily finite) sets. We generally use cursive letters to denote set families, uppercase letters to denote sets, and lowercase letters for elements of sets. For a set family \mathcal{F} (resp., non-empty set family \mathcal{F}) we write $\bigcup \mathcal{F}$ (resp., $\bigcap \mathcal{F}$) for $\bigcup_{X \in \mathcal{F}} X$ (resp., $\bigcap_{X \in \mathcal{F}} X$). For a set X and two functions $f, g: X \rightarrow \mathbb{Z}$,

we write $f + g$ (resp., $f - g$) the function defined by $(f + g)(x) \stackrel{\text{def}}{=} f(x) + g(x)$ (resp., $(f - g)(x) \stackrel{\text{def}}{=} f(x) - g(x)$) for all $x \in X$.

Posets. Recall that a *poset* $P = (A, \leq)$ is a pair consisting of a set A and a binary *partial order* relation \leq over A that is reflexive, antisymmetric and transitive. By slight abuse of notation, we may write $x \in P$ to mean $x \in A$ (and $U \subseteq P$ to mean $U \subseteq A$).

A *greatest* (resp., *least*) element of P is an element $x \in P$ such that for all $x' \in P$ we have $x' \leq x$ (resp., $x \leq x'$). When such an element exists it is unique, and we then denote it by $\hat{1}$ (resp., $\hat{0}$). We may write $\hat{0}_P, \hat{1}_P$ to avoid confusion when multiple posets are involved.

For $G \subseteq A$, we define the *upset of P generated by G* (also called an *order filter*), denoted $\uparrow_P(G)$, by $\uparrow_P(G) \stackrel{\text{def}}{=} \{x \in P \mid \exists y \in G \text{ s.t. } y \leq x\}$. We also define the *downset generated by G* (also called an *order ideal*), denoted $\downarrow_P(G)$, by $\downarrow_P(G) \stackrel{\text{def}}{=} \{x \in P \mid \exists y \in G \text{ s.t. } x \leq y\}$. Note that $\uparrow_P(\emptyset) = \downarrow_P(\emptyset) = \emptyset$. An *upset* (resp., a *downset*) of P is a subset of P of the form $\uparrow_P(G)$ (resp., $\downarrow_P(G)$) for $G \subseteq A$. When $|G| = 1$ we call $\uparrow_P(G)$ (resp., $\downarrow_P(G)$) a *principal upset* (resp., *principal downset*), and by slight abuse of notation we sometimes write $\uparrow_P(x)$ for $x \in P$ to mean $\uparrow_P(\{x\})$ (resp., $\downarrow_P(x)$ to mean $\downarrow_P(\{x\})$).

Two posets $P_1 = (A_1, \leq_1)$ and $P_2 = (A_2, \leq_2)$ are *isomorphic* if there exists a bijection $h: A_1 \rightarrow A_2$ such that for all $x, y \in P_1$ we have $x \leq_1 y$ iff $h(x) \leq_2 h(y)$. When this is the case we write $P_1 \simeq P_2$.

Lattices. A poset $P = (A, \leq)$ is a *meet semilattice* if, for every $x, y \in P$, there exists $z \in P$ such that: (1) $z \leq x$ and $z \leq y$; and (2) for every $z' \in P$ such that $z' \leq x$ and $z' \leq y$, we have $z' \leq z$. This element z is then unique and is called the *meet of x and y* , and written $x \wedge y$. We similarly define *join semilattices* in the expected way, with the join denoted $x \vee y$, and then *lattices* as posets that are both a meet semilattice and a join semilattice.

We recall that a finite join (resp., meet) semilattice that has a least (resp., greatest) element is a lattice (see, e.g., [Stanley, 2011, Proposition 3.3.1]). We also recall the property that in a meet semilattice L , the intersection of two principal downsets $\downarrow_L(x)$ and $\downarrow_L(y)$ generated by $x, y \in L$ is the principal downset generated by $x \wedge y$; the dual property holds about principal upsets in join semilattices with the join operation.

For a set S we write \mathbb{B}_S the Boolean lattice over S , by which we mean the lattice $(2^S, \subseteq)$, where join corresponds to set union and meet to set intersection.

Definition 2.1. A set family \mathcal{F} is called *trivial* if it is empty or if there is $X \in \mathcal{F}$ such that $\bigcup \mathcal{F} = X$, and *non-trivial* otherwise.

We define the standard notion of *intersection lattice* of a non-trivial set family:

Definition 2.2 ([Stanley, 2011], Section 3.7.2). Let \mathcal{F} be a non-trivial set family. For $\mathcal{T} \subseteq \mathcal{F}$, $\mathcal{T} \neq \emptyset$, we define: $S_{\mathcal{T}} \stackrel{\text{def}}{=} \bigcap \mathcal{T}$. We also define $S_{\emptyset} \stackrel{\text{def}}{=} \bigcup \mathcal{F}$ with a slight abuse of notation as S_{\emptyset} depends on the underlying set family \mathcal{F} .

Define the poset $\mathbb{L}_{\mathcal{F}} = (\mathcal{A}, \subseteq)$, where $\mathcal{A} \stackrel{\text{def}}{=} \{S_{\mathcal{T}} \mid \mathcal{T} \subseteq \mathcal{F}\}$. Then one can check that $\mathbb{L}_{\mathcal{F}}$ is a meet semilattice whose meet operation is set intersection. We call $\mathbb{L}_{\mathcal{F}}$ the intersection lattice of \mathcal{F} . We say that a lattice L is an intersection lattice if it is of the form $\mathbb{L}_{\mathcal{F}}$ for some non-trivial set family \mathcal{F} . (Note that we might have $\mathbb{L}_{\mathcal{F}_1} = \mathbb{L}_{\mathcal{F}_2}$ for $\mathcal{F}_1 \neq \mathcal{F}_2$.)

Remark 2.3. Every finite lattice $L = (A, \leq)$ is isomorphic to some intersection lattice. Indeed, define $\mathcal{F} \stackrel{\text{def}}{=} \{\downarrow_L(\{U\}) \mid U \in A, U \neq \hat{1}_L\}$, i.e., the set of principal downsets of L except $\downarrow_L(\{\hat{1}_L\})$ (as otherwise \mathcal{F} is trivial). Then one can easily check that $L \simeq \mathbb{L}_{\mathcal{F}}$, since $\downarrow_L(\{U\}) \cap \downarrow_L(\{V\}) = \downarrow_L(U \wedge_L V)$ for $U, V \in A$.

Such a lattice $L = \mathbb{L}_{\mathcal{F}}$ has a $\hat{0}$, which is $S_{\mathcal{F}} = \bigcap \mathcal{F}$, and a $\hat{1}$, which is $S_{\emptyset} = \bigcup \mathcal{F}$. Notice that $\hat{1} \neq \hat{0}$ because \mathcal{F} is non-trivial. Since L is a finite meet-semilattice that has a $\hat{1}$, we know that L is also a join-semilattice by the property mentioned above, but for the purpose of this note we will not need to know what the join operation corresponds to. Note that by definition $S_{\mathcal{T}_1 \cup \mathcal{T}_2} = S_{\mathcal{T}_1} \cap S_{\mathcal{T}_2}$ for any $\mathcal{T}_1, \mathcal{T}_2 \subseteq \mathcal{F}$. Observe that we may have $S_{\mathcal{T}_1} = S_{\mathcal{T}_2}$ with $\mathcal{T}_1 \neq \mathcal{T}_2$, in which case $S_{\mathcal{T}_1} = S_{\mathcal{T}_2} = S_{\mathcal{T}_1 \cup \mathcal{T}_2}$.

Example 2.4. Figure 1 shows the Hasse diagrams of the various intersection lattices defined next. (Ignore for now the integer annotations next to the nodes and the fact that some nodes are colored.)

L_1 . The intersection lattice L_1 of (the non-trivial set family containing) $S_1 = \{n \in \mathbb{N} \mid n \equiv 0 \pmod{2}\}$, $S_2 = \{n \in \mathbb{N} \mid n \equiv 0 \pmod{3}\}$, and $S_3 = \{n \in \mathbb{N} \mid n \equiv 0 \pmod{12}\}$ is shown in Figure 1a.

L_2 . The intersection lattice L_2 of $S_1 = \{a, d\}$, $S_2 = \{b, d\}$, and $S_3 = \{c, d\}$ is shown in Figure 1b. This example illustrates that an intersection lattice is not necessarily distributive.

L_3 . The intersection lattice L_3 of $S_1 = \{a, b\}$, $S_2 = \{a, c\}$, $S_3 = \{b, c\}$, and $S_4 = \{d\}$ is shown in Figure 1c.

L_4 . The intersection lattice L_4 of $S_1 = \{a, c, d, g\}$, $S_2 = \{a, b, d, f\}$, $S_3 = \{a, b, c, e\}$, and $S_4 = \{a, h\}$ is shown in Figure 1d. Observe that L_3 and L_4 are isomorphic.

Finally, another intersection lattice L_5 is depicted in Figure 2 (ignore for now the tree).

In an intersection lattice $L = \mathbb{L}_{\mathcal{F}}$, for every $x \in \hat{1}$, the set of elements of L in which x occurs is the principal upset of L generated by $\min_L(x) \stackrel{\text{def}}{=} S_{\{X \in \mathcal{F} \mid x \in X\}}$. Note that $\min_L(x) \neq \hat{1}$ because \mathcal{F} is non-trivial. In particular we have the following:

Fact 2.5. Let L be an intersection lattice and $U \in L$. Then we have $U = \bigcup_{U' \subseteq U} \{x \in \hat{1} \mid U' = \min_L(x)\}$, and this union is disjoint.

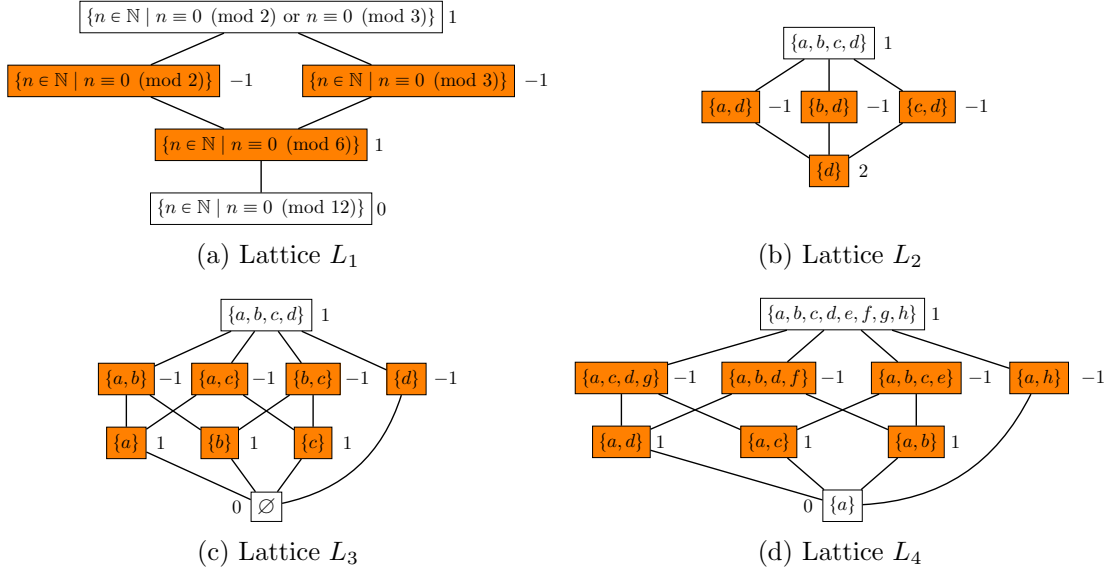


Figure 1: Hasse diagrams of the intersection lattices from Example 2.4. The integer value besides each node n is $\mu_L(n, \hat{1})$ and is computed top-down following Definition 2.6. The orange nodes are the non-cancelling non-trivial intersections.

Möbius inversion formula and inclusion-exclusion. We now introduce the *Möbius function* on finite posets.

Definition 2.6. Let $P = (A, \leq)$ be a finite poset. The Möbius function of P is the function μ_P with value in \mathbb{Z} and domain the set of pairs $(x, y) \in A \times A$ such that $x \leq y$, defined, for each $y \in A$, by induction over the elements x such that $x \leq y$:

- $\mu_P(y, y) \stackrel{\text{def}}{=} 1$;
- $\mu_P(x, y) \stackrel{\text{def}}{=} -\sum_{x < z \leq y} \mu_P(z, y)$.¹

Proposition 2.7 (Möbius inversion formula; see, e.g., Prop. 3.7.1 of [Stanley, 2011]). Let $P = (A, \leq)$ be a finite poset, and let $f, g: A \rightarrow \mathbb{R}$. Then we have:

$$g(x) = \sum_{y \leq x} f(y) \text{ for all } x \in P \quad \text{if and only if} \quad f(x) = \sum_{y \leq x} \mu_P(y, x)g(y) \text{ for all } x \in P.$$

One important application of the Möbius inversion formula is in determining the *measure* of a union of sets from the measure of some of their intersections. For our purposes, given a set S , what we call a *measure on 2^S* is a function $\xi: 2^S \rightarrow \mathbb{R}$ which is nonnegative (i.e., $\xi(U) \geq 0$ for all $U \in 2^S$), which satisfies $\xi(\emptyset) = 0$, and which is additive under countable disjoint unions, i.e., for any countable collection $\{E_k\}_{k=1}^\infty$ of

¹The Möbius function is traditionally defined with $\mu_P(x, y) \stackrel{\text{def}}{=} -\sum_{x \leq z < y} \mu_P(x, z)$, but the two definitions are equivalent [Rota, 1964, Proposition 3]. We use this version as it simplifies our presentation.

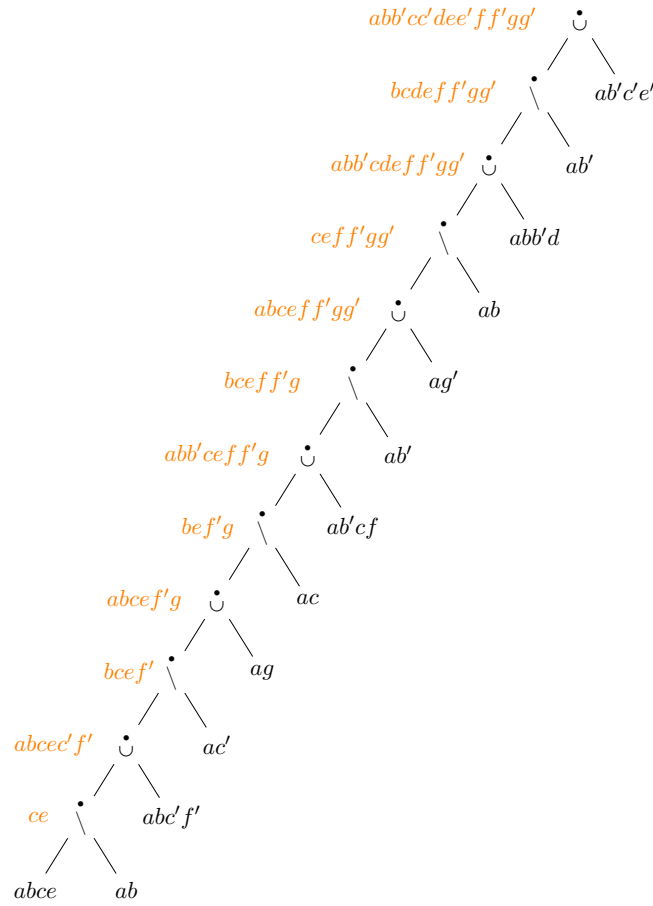
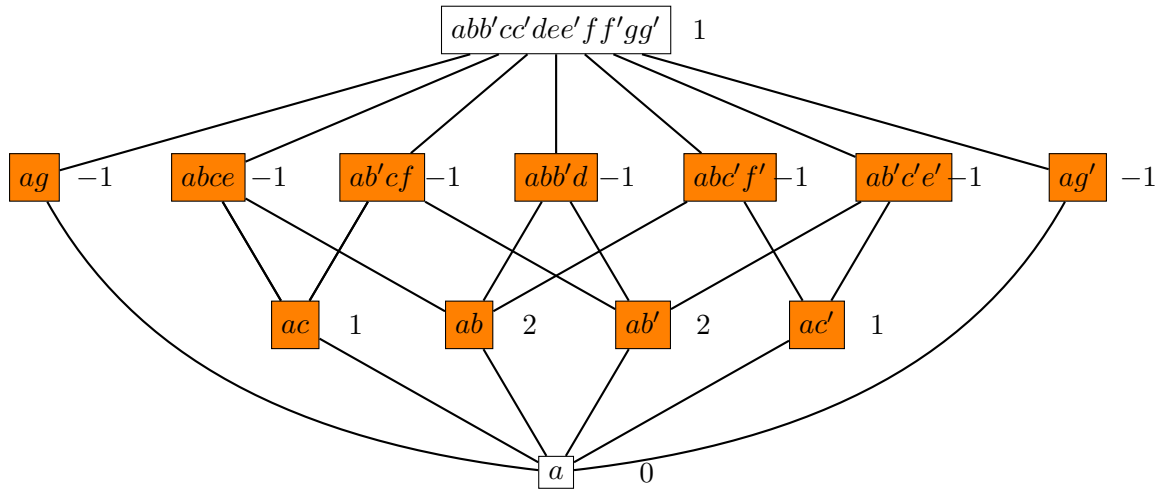


Figure 2: A tight intersection lattice of seven sets, and a witnessing tree showing that it does not violate the conjecture. For brevity we omit curly braces and commas when writing sets, i.e., ab stands for $\{a, b\}$.

pairwise disjoint subsets of S , we have $\xi(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \xi(E_k)$. In particular, ξ is completely determined by the images of the singleton sets $\{x\}$ for $x \in S$.

Proposition 2.8. *Let L be an intersection lattice, and let ξ be a measure on $2^{\hat{1}}$. Then we have*

$$\xi(\hat{1}) = - \sum_{\substack{U \in L \\ U \neq \hat{1}}} \mu_L(U, \hat{1}) \xi(U). \quad (1)$$

Proof. This is a known result but, to be self-contained, we reproduce the proof from [Stanley, 2011] (end of Section 3.7, page 304). Let us define two functions $f, g: L \rightarrow \mathbb{R}$. For $U \in L$, let $g(U) \stackrel{\text{def}}{=} \xi(U)$, and define $f(U)$ to be the measure of all elements of U which belong to no $U' \subsetneq U$, i.e., $f(U) \stackrel{\text{def}}{=} \xi(\{x \in \hat{1} \mid \min_L(x) = U\})$. Observe that $g(U) = \sum_{\substack{U' \in L \\ U' \leq U}} f(U')$ for all $U \in L$, by additivity of ξ and using Fact 2.5. Therefore by Proposition 2.7 we have that $f(\hat{1}) = \sum_{U \in L} \mu_L(U, \hat{1}) g(U)$. But notice that $g(\hat{1}) = \xi(\hat{1})$ and that $f(\hat{1}) = \xi(\emptyset)$ (because $L = \mathbb{L}_{\mathcal{F}}$ for some \mathcal{F} non-trivial), so that $f(\hat{1}) = 0$. Therefore indeed $\xi(\hat{1}) = - \sum_{\substack{U \in L \\ U \neq \hat{1}}} \mu_L(U, \hat{1}) \xi(U)$. \square

We point out that the above proposition can fail if the underlying set family \mathcal{F} is trivial²: this is the reason why we always work with non-trivial families of sets.

Example 2.9. *For the lattices L_1, L_2, L_3, L_4, L_5 from Example 2.4, the values $\mu_{L_i}(U, \hat{1})$ for all elements $U \in L_i$ are shown next to the corresponding nodes in Figures 1 and 2.*

This motivates the following definitions.

Definition 2.10. *Let L be an intersection lattice. We call $\hat{1}$ the trivial intersection. We define the non-trivial intersections of L , denoted $\text{NTI}(L)$, by*

$$\text{NTI}(L) \stackrel{\text{def}}{=} \{U \in L \mid U \neq \hat{1}\},$$

and we define the non-cancelling, non-trivial intersections (or simply non-cancelling intersections) of L , denoted $\text{NCI}(L)$, by

$$\text{NCI}(L) \stackrel{\text{def}}{=} \{U \in L \mid U \neq \hat{1} \text{ and } \mu_L(U, \hat{1}) \neq 0\}.$$

In other words, the non-cancelling intersections are those intersections that do not cancel in the inclusion-exclusion formula, i.e., those U such that $\xi(U)$ occur in Equation (1) with non-zero coefficient.

Last, we will also need the following application of the Möbius inversion formula, which is another formulation of the inclusion-exclusion principle.

Proposition 2.11. *Let S be a finite set, and let $f, g: \mathbb{B}_S \rightarrow \mathbb{R}$. Then we have $g(X) = \sum_{\substack{X' \in \mathbb{B}_S \\ X \subseteq X'}} f(X')$ for all $X \in \mathbb{B}_S$ if and only if $f(X) = \sum_{\substack{X' \in \mathbb{B}_S \\ X \subseteq X'}} (-1)^{|X'| - |X|} g(X')$ for all $X \in \mathbb{B}_S$.*

²Take for instance $\mathcal{F} = \{\{a, b\}, \{a\}\}$.

This result can be obtained by observing that $\mu_{\mathbb{B}_S}(X, Y) = (-1)^{|Y|-|X|}$ for $X \subseteq Y$ as shown in [Stanley, 2011, Example 3.8.1] and using the dual form of the Möbius inversion formula [Stanley, 2011, Proposition 3.7.2].

3. The Non-Cancelling Intersections Conjecture

In this section we state the conjecture and illustrate it with a few toy examples. We adapt to our context notation and terminology from [Hirsch and McLean, 2017] for disjoint unions and subset complements.

Definition 3.1. For two sets S, T , the disjoint union $S \dot{\cup} T$ equals $S \cup T$ if $S \cap T = \emptyset$, else it is undefined. We generalize the disjoint union operator to more than two operands in the expected way. The subset complement $S \dot{\setminus} T$ equals $S \setminus T$ if $T \subseteq S$, else it is undefined.

Notice that

$$S \dot{\cup} T = U \text{ iff } U \dot{\setminus} S = T \text{ iff } U \dot{\setminus} T = S. \quad (2)$$

Definition 3.2. Let \mathcal{F} be a set family. The partial dot-algebra generated by \mathcal{F} , denoted $\bullet(\mathcal{F})$, is the smallest (for inclusion) set family of subsets of $2^{\cup \mathcal{F}}$ which contains \emptyset as well as all the sets of \mathcal{F} and is closed under disjoint union and subset complement. Formally, it is defined inductively by:

1. First base case: $\emptyset \in \bullet(\mathcal{F})$;
2. Second base case: for each $X \in \mathcal{F}$, we have $X \in \bullet(\mathcal{F})$;
3. Induction: for each $X_1, X_2 \in \bullet(\mathcal{F})$, if $X_1 \dot{\cup} X_2$ (resp., $X_1 \dot{\setminus} X_2$) is well-defined, then we have $X_1 \dot{\cup} X_2 \in \bullet(\mathcal{F})$ (resp., $X_1 \dot{\setminus} X_2 \in \bullet(\mathcal{F})$).

Note that when \mathcal{F} is not empty, then item (1) is redundant as it follows from (2) and (3): taking any $X \in \mathcal{F}$, we have $X \in \bullet(\mathcal{F})$ by item (2), and $X \dot{\setminus} X = \emptyset \in \bullet(\mathcal{F})$ by item (3).

Furthermore, observe that $\bullet(\mathcal{F})$ is always finite because \mathcal{F} is. What we call a *witnessing tree* of $X \in \bullet(\mathcal{F})$ is a rooted ordered tree whose leaves are annotated with \emptyset or with elements of \mathcal{F} and whose internal nodes are annotated with either $\dot{\cup}$ or $\dot{\setminus}$ (with the node being binary if it is annotated with $\dot{\setminus}$), with the expected semantics. Obviously, we have $X \in \bullet(\mathcal{F})$ if and only if such a witnessing tree exists.

Example 3.3. Let $\mathcal{F}_1 = \{\{a, d\}, \{b, e\}, \{a, b, c\}\}$. Then $\bullet(\mathcal{F}_1) = \mathcal{F}_1 \cup \{\emptyset, \{a, b, d, e\}\}$. Let $\mathcal{F}_2 = \{\{a, c\}, \{b, c\}, \{c\}\}$. Then we have $\bullet(\mathcal{F}_2) = 2^{\{a, b, c\}}$. Two trees T_0, T'_0 which witness that $\{a, b, c\} \in \bullet(\mathcal{F}_2)$ are shown in Figure 3.

We are now ready to state the conjecture.

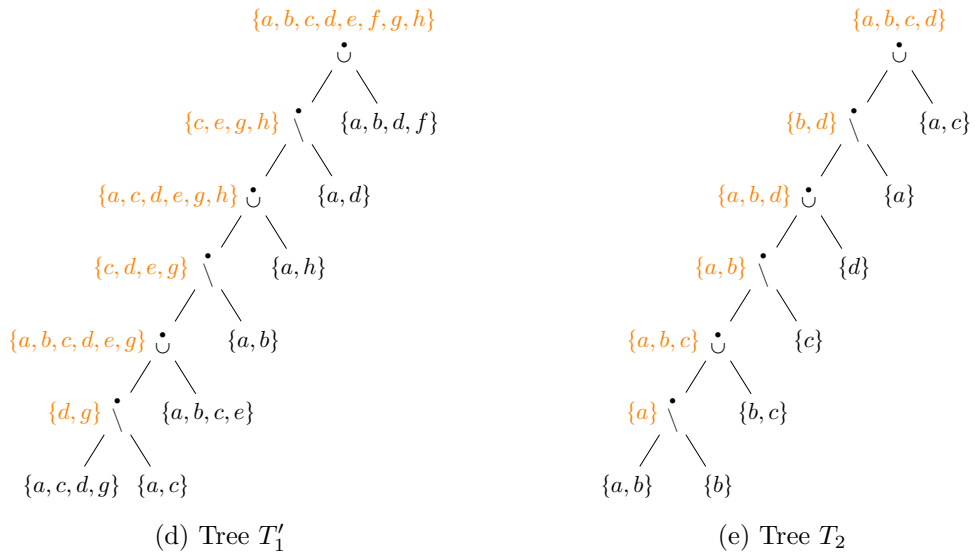
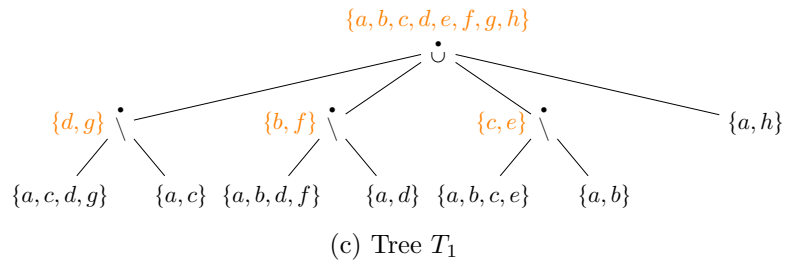
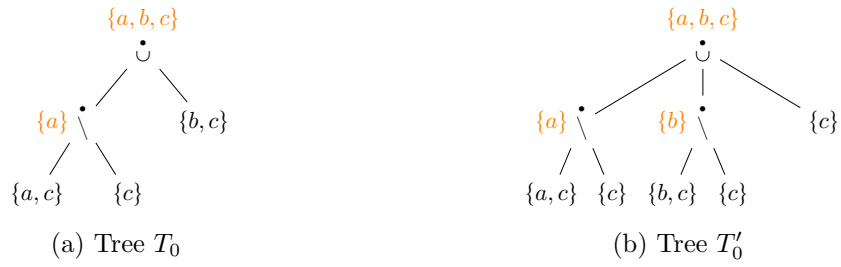


Figure 3: Various witnessing trees. The sets that internal nodes correspond to are shown in orange.

Conjecture 3.4 (NCI conjecture, formulation I). *For every intersection lattice L , we have $\hat{1} \in \bullet(\text{NCI}(L))$.*

In other words, the conjecture states that, given any non-trivial set family, we can always express the union of the sets by starting with the set intersections that do not cancel in the inclusion-exclusion formula and applying the disjoint union and subset complement operators.

Example 3.5. *Continuing Example 2.4, the nodes corresponding to sets of $\text{NCI}(L_i)$ are colored in orange in Figure 1: they are the nodes different from $\hat{1}$ with a non-zero value of the Möbius function. We show next that $\hat{1} \in \bullet(\text{NCI}(L_i))$ for all of them.*

L_1 . We have $\hat{1} = \{n \in \mathbb{N} \mid n \equiv 0 \pmod{2} \text{ or } n \equiv 0 \pmod{3}\} = [\{n \in \mathbb{N} \mid n \equiv 0 \pmod{2}\} \dot{\setminus} \{n \in \mathbb{N} \mid n \equiv 0 \pmod{6}\}] \dot{\cup} \{n \in \mathbb{N} \mid n \equiv 0 \pmod{3}\}$.

L_2 . We have $\hat{1} = \{a, b, c, d\} = [\{a, d\} \dot{\setminus} \{d\}] \dot{\cup} [\{b, d\} \dot{\setminus} \{d\}] \dot{\cup} [\{c, d\}]$.

L_3 . We have $\hat{1} = \{a, b, c, d\} = \{a\} \dot{\cup} \{b\} \dot{\cup} \{c\} \dot{\cup} \{d\}$.

L_4 . Two trees T_1, T'_1 which witness that $\hat{1} \in \bullet(\text{NCI}(L_4))$ can be found in Figure 3.

L_5 . A witnessing tree for L_5 can be found in Figure 2.

The motivation for the conjecture is to understand whether the inclusion-exclusion formula can be understood using the Boolean operations of disjoint union and subset complement. Specifically, let \mathcal{F} be a non-trivial set family, consider its intersection lattice $\mathbb{L}_{\mathcal{F}}$, recall that $\hat{1} = \bigcup \mathcal{F}$, and let ξ be a measure on $2^{\hat{1}}$. We know that the measure of $\hat{1}$ can be obtained from the measure of the non-cancelling intersections, namely, Proposition 2.8 expresses the measure $\xi(\hat{1})$ of the set $\hat{1}$ as an arithmetic combination of the measures $\xi(S)$ with $S \in \text{NCI}(L)$. The conjecture says that we can also express the set $\hat{1}$ from the sets S with $S \in \text{NCI}(L)$ using the Boolean operations of disjoint union and subset complement.

Dual version: non-cancelling unions conjecture. We end this section by presenting a dual version of the NCI conjecture that focuses on unions instead of intersections, and that we call the *non-cancelling unions* (NCU) conjecture. Call a set family \mathcal{F} *co-trivial* if it is empty or if there is $X \in \mathcal{F}$ such that $X = \bigcap \mathcal{F}$, and *non-co-trivial otherwise*. Define the *union lattice* $\mathbb{U}_{\mathcal{F}}$ of a non-co-trivial set family \mathcal{F} as follows: let $R_{\emptyset} \stackrel{\text{def}}{=} \bigcap \mathcal{F}$ and $R_{\mathcal{T}} \stackrel{\text{def}}{=} \bigcup \mathcal{T}$ for $\mathcal{T} \subseteq \mathcal{F}$, $\mathcal{T} \neq \emptyset$, and $\mathbb{U}_{\mathcal{F}} \stackrel{\text{def}}{=} (\{R_{\mathcal{T}} \mid \mathcal{T} \subseteq \mathcal{F}\}, \subseteq)$. Then $\mathbb{U}_{\mathcal{F}}$ is a lattice whose join operation corresponds to set union, has a greatest element $\hat{1} = \bigcup \mathcal{F}$ and a least element $\hat{0} = \bigcap \mathcal{F}$ that are distinct. A lattice is a union lattice if it is of the form $\mathbb{U}_{\mathcal{F}}$ for some non-co-trivial set family. Analogously to Proposition 2.8, it is easy to show that for a union lattice L and measure ξ on $2^{\hat{1}}$, we have

$$\xi(\hat{0}) = - \sum_{\substack{U \in L \\ U \neq \hat{0}}} \mu_L(\hat{0}, U) \xi(U). \quad (3)$$

Define then the *non-cancelling (non-trivial) unions* of L as

$$\text{NCU}(L) \stackrel{\text{def}}{=} \{U \in L \mid U \neq \hat{0}, \mu_L(\hat{0}, U) \neq 0\}.$$

The non-cancelling unions conjecture is then:

Conjecture 3.6 (NCU conjecture). *For every union lattice L , we have $\hat{0} \in \bullet(\text{NCU}(L))$.*

Notice here that the definition of the dot-algebra is unchanged: we still use disjoint complement and disjoint union. Informally again, this is stating that, given a set family, we can always express their intersection by applying the disjoint union and subset complement operations to the set unions that do not cancel in the (“union form” of the) inclusion-exclusion formula.

We show in Appendix A that, unsurprisingly, the NCI and NCU conjectures are in fact equivalent. We decided to focus on the intersections version of the conjecture because inclusion-exclusion is most often presented in “intersection form”.

4. Simplifying to Tight Intersection Lattices

In this section we show that the NCI conjecture can be studied without loss of generality on intersection lattices satisfying a certain *tightness* condition, that we define next. Intuitively, an intersection lattice L is *tight* if for every $U \in L$, $U \neq \hat{1}$, there is exactly one element $x \in \hat{1}$ such that $\min_L(x) = U$. We also define the notion of an intersection lattice being *full*, which will be useful in the next section.

Definition 4.1. *An intersection lattice L is full if for every $U \in L$, $U \neq \hat{1}$, we have $\{x \in \hat{1} \mid \min_L(x) = U\} \neq \emptyset$.³ The lattice L is in addition called tight if for every $U \in L$, $U \neq \hat{1}$, we have $|\{x \in \hat{1} \mid \min_L(x) = U\}| = 1$.*

For instance, the intersection lattices L_2 and L_4 from Figure 1 and L_5 from Figure 2 are tight (hence full), L_1 is full but not tight, and L_3 is not full (hence not tight). Note that if an intersection lattice $L = \mathbb{L}_{\mathcal{F}}$ is tight, then $\hat{1}$ is finite, i.e., the sets in \mathcal{F} are themselves finite: this can be seen by applying Fact 2.5 with $U = \hat{1}$. We then claim that we can reformulate the NCI conjecture as follows:

Conjecture 4.2 (NCI conjecture, formulation II). *For every tight intersection lattice L , we have $\hat{1} \in \bullet(\text{NCI}(L))$.*

We prove this in two steps. First, we show that for every intersection lattice L , there exists a *tight* intersection lattice L' that is isomorphic to L .

Lemma 4.3. *For every intersection lattice L , there exists a tight intersection lattice L' such that $L \simeq L'$.*

³Note that $\{x \in \hat{1} \mid \min_L(x) = \hat{1}\} = \emptyset$ because \mathcal{F} is non-trivial.

Proof. It suffices to observe that the intersection lattice constructed in Remark 2.3 (call it L') is tight. Indeed, letting $U \in L \setminus \{\hat{1}_L\}$, corresponding to $\downarrow_L(\{U\})$ in L' , we have that $\{x \in \hat{1}_{L'} \mid \min_{L'}(x) = \downarrow_L(\{U\})\} = \{U\}$. \square

Second, we show that, for two isomorphic intersection lattices L, L' such that L' is full, then $\hat{1}_{L'} \in \bullet(\text{NCI}(L'))$ implies $\hat{1}_L \in \bullet(\text{NCI}(L))$.

Lemma 4.4. *Let L, L' be isomorphic intersection lattices such that L' is full and such that $\hat{1}_{L'} \in \bullet(\text{NCI}(L'))$. Then we have $\hat{1}_L \in \bullet(\text{NCI}(L))$.*

Proof. Let $h : L \rightarrow L'$ be an isomorphism. Since L' is full, for $U' \in L'$, $U' \neq \hat{1}_{L'}$, let $\alpha_{U'}$ be an element of $\{x' \in \hat{1}_{L'} \mid \min_{L'}(x') = U'\}$. Define $g : \hat{1}_L \rightarrow \hat{1}_{L'}$ by $g(x) \stackrel{\text{def}}{=} \alpha_{h(\min_L(x))}$ for $x \in \hat{1}_L$. Note that g is not necessarily injective. For $x' \in \hat{1}_{L'}$ let $g^{-1}(x') \stackrel{\text{def}}{=} \{x \in \hat{1}_L \mid g(x) = x'\}$, and for $X' \subseteq \hat{1}_{L'}$ let $g^{-1}(X') \stackrel{\text{def}}{=} \bigcup_{x' \in X'} g^{-1}(x')$. From Fact 2.5 and because L' is full it is easy to see that (\dagger) for every $U' \in L'$ we have $g^{-1}(U') = h^{-1}(U')$. In particular we have $g^{-1}(\hat{1}_{L'}) = \hat{1}_L$. Hence, it is enough to prove that for all $X' \in \bullet(\text{NCI}(L'))$ we have $g^{-1}(X') \in \bullet(\text{NCI}(L))$. We show this by induction on $\text{NCI}(L')$. The first base case is when $X' = \emptyset$. But then observe that $g^{-1}(X') = \emptyset$, and that $\emptyset \in \bullet(\text{NCI}(L))$ by definition. The second base case is when $X' \in \text{NCI}(L')$. But then $g^{-1}(X') \in \text{NCI}(L)$ as well by (\dagger) and because h is an isomorphism so clearly $\mu_{L'}(X', \hat{1}_{L'}) = \mu_L(h^{-1}(X'), \hat{1}_L)$. The first inductive case is when $X' = X'_1 \dot{\cup} X'_2$ with $X'_1, X'_2 \in \bullet(\text{NCI}(L'))$. By induction hypothesis we have $g^{-1}(X'_1), g^{-1}(X'_2) \in \bullet(\text{NCI}(L))$. Moreover clearly $g^{-1}(X'_1) \cap g^{-1}(X'_2) = \emptyset$ since X'_1 and X'_2 are disjoint. So indeed $g^{-1}(X') = g^{-1}(X'_1) \dot{\cup} g^{-1}(X'_2) \in \bullet(\text{NCI}(L))$. The second inductive case is when $X' = X'_1 \setminus X'_2$ with $X'_1, X'_2 \in \bullet(\text{NCI}(L'))$. By induction hypothesis again we have $g^{-1}(X'_1), g^{-1}(X'_2) \in \bullet(\text{NCI}(L))$. Moreover clearly $g^{-1}(X'_2) \subseteq g^{-1}(X'_1)$ since $X'_2 \subseteq X'_1$. So indeed $g^{-1}(X') = g^{-1}(X'_1) \setminus g^{-1}(X'_2) \in \bullet(\text{NCI}(L))$. This concludes the proof. \square

This indeed proves that the two formulations are equivalent. We illustrate this last construction in the following example.

Example 4.5. *Consider the intersection lattices L_3 and L_4 from Figure 1, the isomorphism $h : L_3 \rightarrow L_4$ defined following their drawings, and $\alpha_{U'}$ for $U' \in L_4$ $U' \neq \hat{1}_{L_4}$ being the only element in $\{x' \in \hat{1}_{L_4} \mid \min_{L_4}(x') = U'\}$ (since L_4 is tight). Then we have, for instance, $g^{-1}(d) = \{a\}$, $g^{-1}(a) = g^{-1}(f) = \emptyset$. Moreover, consider the tree T'_1 from Figure 3d witnessing that $\hat{1}_{L_4} \in \bullet(\text{NCI}(L_4))$. Applying g^{-1} to every node of this tree yields the tree T_2 from Figure 3e that witnesses $\hat{1}_{L_4} \in \bullet(\text{NCI}(L_4))$. By contrast, consider the expression $\{a\} \dot{\cup} \{b\} \dot{\cup} \{c\} \dot{\cup} \{d\}$ that also witnesses $\hat{1}_{L_3} \in \bullet(\text{NCI}(L_3))$. This expression cannot be translated through h into a tree for L_4 as, e.g., the operation $\{b\} \dot{\cup} \{c\}$ would translate to $\{a, c\} \dot{\cup} \{a, b\}$ in L_4 , which is not valid.*

In fact we claim that, when L is a full intersection lattice, any tree T witnessing that $\hat{1} \in \bullet(\text{NCI}(L))$ must use all of the sets in $\text{NCI}(L)$. (This is not the case of the expression $\{a\} \dot{\cup} \{b\} \dot{\cup} \{c\} \dot{\cup} \{d\}$ for L_3 for instance, but L_3 is not full.) In a sense, this

means that the full intersection lattices are the hardest cases for the NCI conjecture. We formalize and state a stronger claim (on the non-trivial intersections instead of only the non-cancelling ones) in the remaining of this section.

Definition 4.6. Let \mathcal{F} be a set family, $X \in \bullet(\mathcal{F})$ and T a tree witnessing this. For a leaf ℓ of T , the polarity of ℓ in T , denoted $\text{pol}_T(\ell)$, is 1 if the number of times we take the right edge of a \setminus node in the path from the root of T of ℓ is even; and -1 otherwise. The multiplicity of $U \in \mathcal{F} \cup \{\emptyset\}$ in T is then

$$\text{mult}_T(U) \stackrel{\text{def}}{=} \sum_{\substack{\ell \text{ leaf of } T \\ \text{of the form } U}} \text{pol}_T(\ell).$$

In other words, if we see each $U \in \mathcal{F} \cup \{\emptyset\}$ as a variable and see T as an arithmetic expression over these variables (by replacing $\dot{\cup}$ nodes by $+$ and $\dot{\setminus}$ nodes by $-$), then the multiplicity of U in T is the coefficient of the corresponding variable once we develop this expression and factorize. We then have:

Proposition 4.7. Let L be a full intersection lattice such that $\hat{1} \in \bullet(\text{NTI}(L))$, and T a tree witnessing this. Then we have $\text{mult}_T(U) = -\mu_L(U, \hat{1})$ for every $U \in \text{NTI}(L)$.

Since Proposition 4.7 will not be strictly needed to establish our main results, we defer its proof to Appendix B. (We include this proposition as we think that it is relevant to motivate the conjecture.) Notice that Proposition 4.7 can be false if we do not impose intersection lattices to be full, see e.g., the expression for L_3 given in Example 3.5.

Furthermore, one can show that for any tight intersection lattice L we have $\bullet(\text{NTI}(L)) = 2^{\hat{1}_L}$. In other words, intuitively, if we use the non-trivial intersections instead of the non-canceling intersections in the conjecture, then we can achieve all possible sets of elements:

Fact 4.8. For every tight intersection lattice L we have $\bullet(\text{NTI}(L)) = 2^{\hat{1}_L}$.

Proof. Generalizing from tight intersection lattices, we will show that the result holds more generally when we assume that for every $U \in L$, $U \neq \hat{1}$, there is *at most* one element $x \in \hat{1}$ such that $\min_L(x) = U$. Let L be an intersection lattice satisfying this weaker condition. We proceed by induction on the downsets of L , showing the following by induction on $0 \leq n < |L|$: for any downset D of L containing n nodes, letting $X_D \subseteq \hat{1}$ be the set of elements introduced at nodes of D , formally, $\{x \in \hat{1} \mid \min_L(x) \in D\}$, then we have $\bullet(D) = 2^{X_D}$. Note that, when taking this claim with $n = |L| - 1$, then the only downset D of L containing $|L| - 1$ nodes is the downset D containing all intersections except S_\emptyset , i.e., precisely $\text{NTI}(L)$, and then $X_D = \hat{1}$, so this claim with $n = |L| - 1$ is the result we want to show.

The base case is $n = 0$, i.e., the downset $D = \emptyset$. Then $X_D = \emptyset$ and $\bullet(\emptyset) = \{\emptyset\} = 2^{X_D}$ and we are done.

For the induction case, let D be a downset containing at least one node, let U be some maximal node of D , and let $D' = D \setminus \{U\}$. The first case is then there is no element

introduced at U , i.e., $U = \bigcup_{U' \in L, U' \subsetneq U} U'$, then we have $X_D = X_{D'}$, and the induction hypothesis gives us that $\bullet(D') = 2^{X_{D'}} = 2^{X_D}$, which concludes because $\bullet(D) \supseteq \bullet(D')$ (as $D' \subseteq D$) and clearly $\bullet(D) \subseteq 2^{X_D}$. Let us thus focus on the second case, where there is an element x introduced at U .

We first show that $\{x\} \in \bullet(D)$. Indeed, by definition we have $U \in \bullet(D)$ because $U \in D$. Further, all elements of U except x were introduced below U and in D' , and therefore we have that $U \setminus \{x\} \subseteq X_{D'}$. Hence, by induction hypothesis, we have $U \setminus \{x\} \in \bullet(D')$, so $U \setminus \{x\} \in \bullet(D)$ because $D' \subseteq D$. From $U \in \bullet(D)$ and $U \setminus \{x\} \in \bullet(D)$ we deduce $\{x\} \in \bullet(D)$.

Now let us show that $\bullet(D) = 2^{X_D}$. Pick $Y \in 2^{X_D}$. If $x \notin Y$, then $Y \in 2^{X_{D'}}$, so by induction hypothesis $Y \in \bullet(D')$, hence $Y \in \bullet(D)$, so we conclude immediately. If $x \in Y$, let $Y' = Y \setminus \{x\}$. We have $Y' \in 2^{X_{D'}}$, so by induction hypothesis we have $Y' \in \bullet(D')$ hence $Y' \in \bullet(D)$. Further, we have established in the previous paragraph that $\{x\} \in \bullet(D)$. We conclude that $Y \in \bullet(D)$. Thus, we have shown the inductive claim $\bullet(D) = 2^{X_D}$.

We have concluded the induction, which as we explained establishes the claim. \square

By Proposition 4.7, in a tree T which witnesses that $\hat{1} \in \bullet(\text{NTI}(L))$ for a full intersection lattice L , the multiplicity of a cancelling intersection is zero. Intuitively, we can see this as suggesting that we do not really need such intersections, and so that the NCI conjecture should morally be true. In addition notice that, for a measure ξ on $2^{\hat{1}}$, applying ξ to T (and replacing \circ nodes by $+$ and \setminus nodes by $-$) and then factoring gives back exactly Equation (1) according to Proposition 4.7.

5. Alternative Formulation: Dot-Algebra of Non-Cancelling Principal Downsets in the Boolean Lattice

In this section we give an alternative formulation of the conjecture which is about the Boolean lattice. It will in particular use results from the previous section. As the structure of the Boolean lattice is “simpler”, the hope is that the alternative formulation of the conjecture may be easier to attack. We present the alternative formulation and show that it is equivalent in Section 5.1, and show in Section 5.2 some useful properties that we can deduce from this alternative formulation.

5.1. The NCPD conjecture

For S finite, we call a *configuration of \mathbb{B}_S* , or simply *configuration* when clear from context, a subset of \mathbb{B}_S . Since configurations are in particular set families, we will generally denote them with cursive letters.

We start by defining what we call *generalized Möbius functions*, that are parameterized by a configuration, and that take only one argument.

Definition 5.1. *Let S finite and $\mathcal{C} \subseteq \mathbb{B}_S$ a configuration. We define the (generalized)*

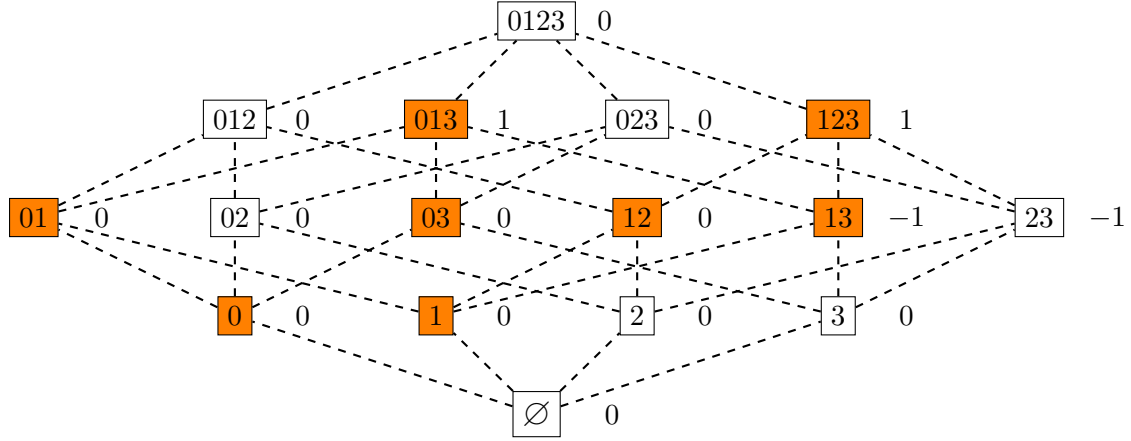


Figure 4: Visual representation of the configuration \mathcal{C} from Example 5.2 and of its associated Möbius function $\hat{\mu}_{\mathbb{B}_S, \mathcal{C}}$ beside each node. Here, \mathcal{C} consists of all the colored nodes.

Möbius function $\hat{\mu}_{\mathbb{B}_S, \mathcal{C}} : \mathbb{B}_S \rightarrow \mathbb{Z}$ associated to \mathcal{C} by top-down induction by:

$$\hat{\mu}_{\mathbb{B}_S, \mathcal{C}}(X) \stackrel{\text{def}}{=} \begin{cases} 1 - \sum_{X' \subsetneq X} \hat{\mu}_{\mathbb{B}_S, \mathcal{C}}(X') & \text{if } X \in \mathcal{C} \\ - \sum_{X' \subsetneq X} \hat{\mu}_{\mathbb{B}_S, \mathcal{C}}(X') & \text{if } X \notin \mathcal{C} \end{cases}.$$

In particular, the value $\hat{\mu}_{\mathbb{B}_S, \mathcal{C}}(S)$ is 0 if $S \notin \mathcal{C}$, and 1 if $S \in \mathcal{C}$.

Example 5.2. Let \mathcal{C} be the configuration

$$\{\emptyset, \{1\}, \{0, 1\}, \{0, 3\}, \{1, 2\}, \{1, 3\}, \{0, 1, 3\}, \{1, 2, 3\}\}$$

of $\mathbb{B}_{\{0,1,2,3\}}$. We have depicted in Figure 4 this configuration and its associated Möbius function.

Next, we define the *non-cancelling principal downsets* of such a configuration.

Definition 5.3. Let S finite and \mathcal{C} a configuration of \mathbb{B}_S . The set of non-cancelling principal downsets of \mathbb{B}_S with respect to \mathcal{C} , denoted $\text{NCPD}_{\mathbb{B}_S}(\mathcal{C})$, is defined by

$$\text{NCPD}_{\mathbb{B}_S}(\mathcal{C}) \stackrel{\text{def}}{=} \{\downarrow_{\mathbb{B}_S}(X) \mid X \in \mathbb{B}_S, \hat{\mu}_{\mathbb{B}_S, \mathcal{C}}(X) \neq 0\}.$$

(Recall that we write $\downarrow_{\mathbb{B}_S}(X)$ for $\downarrow_{\mathbb{B}_S}(\{X\})$.)

Note that the elements of $\text{NCPD}_{\mathbb{B}_S}(\mathcal{C})$ are subsets of \mathbb{B}_S . The notation might appear heavy at first sight, e.g., using subscripts in $\hat{\mu}_{\mathbb{B}_S, \mathcal{C}}$ and $\text{NCPD}_{\mathbb{B}_S}$ to always recall which Boolean lattice we are considering. This will however help to avoid confusion in later proofs, when multiple Boolean lattices and configurations will be involved.

We are now ready to state the alternative formulation, which we call for convenience the *NCPD conjecture*.

Conjecture 5.4 (NCPD conjecture). *For every finite S , for every configuration \mathcal{I} of \mathbb{B}_S that is a downset of \mathbb{B}_S , we have that $\mathcal{I} \in \bullet(\text{NCPD}_{\mathbb{B}_S}(\mathcal{I}))$.*

We point out that, if we do not ask the configuration to be a downset of \mathbb{B}_S then the statement is false, i.e., there exists a finite S and configuration \mathcal{C} of \mathbb{B}_S such that $\mathcal{C} \notin \bullet(\text{NCPD}_{\mathbb{B}_S}(\mathcal{C}))$: see Section 7.

We prove in the remaining of this section that this conjecture is equivalent to the NCI conjecture. We start with a general-purpose lemma that gives a correspondence between both formulations:

Lemma 5.5. *Let S be a finite set and let \mathcal{I} be a downset of \mathbb{B}_S of the form $\mathcal{I} = \downarrow_{\mathbb{B}_S}(\mathcal{F})$ for \mathcal{F} a non-trivial set family. For $X \in \mathcal{F}$ let $X' \stackrel{\text{def}}{=} \downarrow_{\mathbb{B}_S}(X)$ (which is 2^X), and let $\mathcal{F}' \stackrel{\text{def}}{=} \{X' \mid X \in \mathcal{F}\}$. Note that \mathcal{F}' is non-trivial as well, and let then L be its intersection lattice. Then we have $\text{NCPD}_{\mathbb{B}_S}(\mathcal{I}) = \text{NCI}(L)$.*

The proof of this lemma is tedious and we defer it to Appendix C. We then show each direction of the equivalence in turn.

NCI conjecture (formulation I) \implies NCPD conjecture. Assume Conjecture 3.4 to be true, let \mathcal{I} be a configuration of \mathbb{B}_S for some finite set S , assume that \mathcal{I} is a downset, and let us show that $\mathcal{I} \in \bullet(\text{NCPD}_{\mathbb{B}_S}(\mathcal{I}))$. If \mathcal{I} is empty this is clear because then we have $\bullet(\text{NCPD}_{\mathbb{B}_S}(\mathcal{I})) = \bullet(\emptyset) = \{\emptyset\}$. If \mathcal{I} is a principal downset of \mathbb{B}_S this is also clear, because in that case we have that $\hat{\mu}_{\mathbb{B}_S, \mathcal{I}}(X)$ for $X \in \mathbb{B}_S$ equals 1 if $\mathcal{I} = \downarrow_{\mathbb{B}_S}(X)$ and 0 otherwise, so assume that \mathcal{I} is not empty and is not a principal downset. Then \mathcal{I} is of the form $\downarrow_{\mathbb{B}_S}(\mathcal{F})$ for some non-trivial set family \mathcal{F} . Let then L be the intersection lattice constructed from S and \mathcal{F} as in Lemma 5.5. By this Lemma we have that $\text{NCPD}_{\mathbb{B}_S}(\mathcal{I}) = \text{NCI}(L)$. By the hypothesis of Conjecture 3.4 being true we have $\hat{1}_L \in \bullet(\text{NCI}(L))$, but $\hat{1}_L$ is \mathcal{I} so indeed $\mathcal{I} \in \bullet(\text{NCPD}_{\mathbb{B}_S}(\mathcal{I}))$.

NCPD conjecture \implies NCI conjecture (formulation II). Assume now that Conjecture 5.4 is true, let \mathcal{F} be a non-trivial set family such that the intersection lattice $L = \mathbb{L}_{\mathcal{F}}$ is tight, and let us show that $\hat{1}_L \in \bullet(\text{NCI}(L))$. For $X \in \mathcal{F}$ let $X' \stackrel{\text{def}}{=} 2^X$, and let $\mathcal{F}' \stackrel{\text{def}}{=} \{X' \mid X \in \mathcal{F}\}$. Note that \mathcal{F}' is non-trivial as well, so let us consider $L' = \mathbb{L}_{\mathcal{F}'}$. For $\mathcal{T} \subseteq \mathcal{F}$, let \mathcal{T}' be the corresponding subset of \mathcal{F}' , i.e., $\mathcal{T}' = \{X' \mid X \in \mathcal{T}\}$, and vice versa. Then observe that for $\mathcal{T} \subseteq \mathcal{F}, \mathcal{T} \neq \emptyset$ we have $S_{\mathcal{T}'} = 2^{S_{\mathcal{T}}}$, so that $L \simeq L'$. Moreover we have that L' is full: indeed it is clear that for $\mathcal{T} \subseteq \mathcal{F}, \mathcal{T} \neq \emptyset$ we have $S_{\mathcal{T}} \in \{X \in \hat{1}_{L'} \mid \min_{L'}(X) = S_{\mathcal{T}'}\}$.⁴ Define now $S = \hat{1}_L$ (which is finite because L is tight), and let \mathcal{I} be the downset of \mathbb{B}_S that is $\downarrow_{\mathbb{B}_S}(\mathcal{F})$. Observe then that L' is exactly the intersection lattice that we would obtain from the construction described in Lemma 5.5 applied on S and \mathcal{F} , so that $\text{NCPD}_{\mathbb{B}_S}(\mathcal{I}) = \text{NCI}(L')$ by that lemma. Since Conjecture 5.4 is true by hypothesis we have $\mathcal{I} \in \bullet(\text{NCPD}_{\mathbb{B}_S}(\mathcal{I}))$, hence $\mathcal{I} \in \bullet(\text{NCI}(L'))$.

⁴But note that L' is never tight: indeed we have, e.g., $\{X \in \hat{1}_{L'} \mid \min_{L'}(X) = \hat{0}_{L'}\} = \{\hat{0}_{L'}, \emptyset\}$.

But $\mathcal{I} = \{Y \in \mathbb{B}_S \mid \exists X \in \mathcal{F}, Y \subseteq X\}$ by definition, which is equal to $\bigcup \mathcal{F}'$, which is $\hat{1}_L$ by definition. Therefore $\hat{1}_L \in \bullet(\text{NCI}(L'))$. This implies $\hat{1}_L \in \bullet(\text{NCI}(L))$ as well by Lemma 4.4.

5.2. Useful facts

We now present useful facts about the NCPD formulation of the conjecture that motivate its introduction.

First, we observe that the generalized Möbius functions are linear (in their parameters) under disjoint union and subset complement.

Fact 5.6. *Let S finite and $\mathcal{C}_1, \mathcal{C}_2$ be two configurations of \mathbb{B}_S such that $\mathcal{C}_1 \dot{\cup} \mathcal{C}_2$ (resp., $\mathcal{C}_1 \setminus \mathcal{C}_2$) is well defined. Then $\hat{\mu}_{\mathbb{B}_S, \mathcal{C}_1 \dot{\cup} \mathcal{C}_2} = \hat{\mu}_{\mathbb{B}_S, \mathcal{C}_1} + \hat{\mu}_{\mathbb{B}_S, \mathcal{C}_2}$ (resp., $\hat{\mu}_{\mathbb{B}_S, \mathcal{C}_1 \setminus \mathcal{C}_2} = \hat{\mu}_{\mathbb{B}_S, \mathcal{C}_1} - \hat{\mu}_{\mathbb{B}_S, \mathcal{C}_2}$).*

Proof. One can simply check by top-down induction that for all $X \in \mathbb{B}_S$ we have $\hat{\mu}_{\mathbb{B}_S, \mathcal{C}_1 \dot{\cup} \mathcal{C}_2}(X) = \hat{\mu}_{\mathbb{B}_S, \mathcal{C}_1}(X) + \hat{\mu}_{\mathbb{B}_S, \mathcal{C}_2}(X)$ (resp., $\hat{\mu}_{\mathbb{B}_S, \mathcal{C}_1 \setminus \mathcal{C}_2}(X) = \hat{\mu}_{\mathbb{B}_S, \mathcal{C}_1}(X) - \hat{\mu}_{\mathbb{B}_S, \mathcal{C}_2}(X)$). \square

Second, we show that, if we allow any principal downset in the dot algebra, then we can build any configuration: this is the analogue of Fact 4.8. In fact we will use the following stronger result:⁵

Lemma 5.7. *Let \mathcal{C} be a configuration of \mathbb{B}_S (for S finite), and let*

$$A_{\mathcal{C}} \stackrel{\text{def}}{=} \{\downarrow_{\mathbb{B}_S}(\{X\}) \mid X \in \downarrow_{\mathbb{B}_S}(\mathcal{C})\}.$$

Then we have $\mathcal{C} \in \bullet(A_{\mathcal{C}})$.

Proof. By induction on the size of $\downarrow_{\mathbb{B}_S}(\mathcal{C})$. The base case is when $|\downarrow_{\mathbb{B}_S}(\mathcal{C})| = 0$, which can only happen when \mathcal{C} is \emptyset . Then clearly $\mathcal{C} \in \bullet(A_{\mathcal{C}})$ as $A_{\mathcal{C}} = \emptyset$ and $\bullet(\emptyset) = \{\emptyset\}$.

For the inductive case, assume the claim is true for all \mathcal{C}' with $|\downarrow_{\mathbb{B}_S}(\mathcal{C}')| \leq n$, and let us show it is true for \mathcal{C} with $|\downarrow_{\mathbb{B}_S}(\mathcal{C})| = n + 1$. Let $X \in \mathcal{C}$ be maximal, and consider $\mathcal{C}' \stackrel{\text{def}}{=} \mathcal{C} \setminus \{X\}$. Then $|\downarrow_{\mathbb{B}_S}(\mathcal{C}')| \leq n$, hence by induction hypothesis we have $\mathcal{C}' \in \bullet(A_{\mathcal{C}'})$. But it is clear that $A_{\mathcal{C}'} \subseteq A_{\mathcal{C}}$, so that $\mathcal{C}' \in \bullet(A_{\mathcal{C}})$ as well. Moreover, consider $\mathcal{C}_1 \stackrel{\text{def}}{=} \downarrow_{\mathbb{B}_S}(\{X\}) \setminus \{X\}$. By induction hypothesis and for the same reasons we have $\mathcal{C}_1 \in \bullet(A_{\mathcal{C}})$. Then $\mathcal{C}' \dot{\cup} (\downarrow_{\mathbb{B}_S}(\{X\}) \setminus \mathcal{C}_1)$ is a valid expression witnessing that $\mathcal{C} \in \bullet(A_{\mathcal{C}})$. \square

Third, we show that all instances of the NCPD conjecture are already “full”, in the sense that an analogous version of Proposition 4.7 automatically holds for them (whereas we saw that Proposition 4.7 might be false for intersection lattices that are not full). Let $\text{PD}(\mathbb{B}_S)$ be the set of principal downsets of \mathbb{B}_S , i.e., $\text{PD}(\mathbb{B}_S) \stackrel{\text{def}}{=} \{\downarrow_{\mathbb{B}_S}(X) \mid X \in \mathbb{B}_S\}$. We show:

⁵In this lemma and its proof we correctly use the notation $\downarrow_{\mathbb{B}_S}(\{X\})$ for $X \in \mathbb{B}_S$ instead of $\downarrow_{\mathbb{B}_S}(X)$ to avoid confusion.

Proposition 5.8. *Let S finite and \mathcal{I} be a configuration of \mathbb{B}_S that is a downset and T a tree witnessing that $\mathcal{I} \in \bullet(\text{PD}(\mathbb{B}_S))$. Then we have $\text{mult}_T(\downarrow_{\mathbb{B}_S}(X)) = \hat{\mu}_{\mathbb{B}_S, \mathcal{I}}(X)$ for every $X \in \mathbb{B}_S$.*

Proof. It is routine to show by (bottom-up) induction on T that, for every node $n \in T$, letting \mathcal{C}_n be the corresponding configuration represented by the subtree of T rooted at n we have that $\text{mult}_{T_n}(\downarrow_{\mathbb{B}_S}(X)) = \hat{\mu}_{\mathbb{B}_S, \mathcal{C}_n}(X)$ for every $X \in \mathbb{B}_S$; this in particular uses the linearity of generalized Möbius functions under disjoint unions and subset complements. Applying this claim to the root of T gives the claim. \square

Last, we establish a connection between the generalized Möbius function of a configuration \mathcal{C} and the *Euler characteristic* of \mathcal{C} , that will be crucial in the next section.

Definition 5.9. *Let \mathcal{C} be a set family of finite sets. We define the Euler characteristic of \mathcal{C} , denoted $e(\mathcal{C})$, by $e(\mathcal{C}) \stackrel{\text{def}}{=} \sum_{X \in \mathcal{C}} (-1)^{|X|}$.*

Proposition 5.10. *Let S finite and \mathcal{C} a configuration of \mathbb{B}_S . Then we have $\hat{\mu}_{\mathbb{B}_S, \mathcal{C}}(X) = (-1)^{|X|} \times e(\mathcal{C} \cap \uparrow_{\mathbb{B}_S}(X))$ for every $X \in \mathbb{B}_S$.*

Proof. Define $f, g : \mathbb{B}_S \rightarrow \mathbb{Z}$ by $f(X) \stackrel{\text{def}}{=} \hat{\mu}_{\mathbb{B}_S, \mathcal{C}}(X)$ and $g(X) = 1$ if $X \in \mathcal{C}$ and 0 otherwise. Observe then that, by definition of $\hat{\mu}_{\mathbb{B}_S, \mathcal{C}}$ we have $g(X) = \sum_{\substack{X' \in \mathbb{B}_S \\ X \subseteq X'}} f(X')$ for all $X \in \mathbb{B}_S$. Then by Proposition 2.11 we have that $f(X) = \sum_{\substack{X' \in \mathbb{B}_S \\ X \subseteq X'}} (-1)^{|X'| - |X|} g(X')$ for all $X \in \mathbb{B}_S$, i.e.,

$$\begin{aligned} \hat{\mu}_{\mathbb{B}_S, \mathcal{C}}(X) &= \sum_{\substack{X' \in \mathbb{B}_S \\ X \subseteq X'}} (-1)^{|X'| - |X|} g(X') \\ &= \sum_{X' \in \mathcal{C} \cap \uparrow_{\mathbb{B}_S}(X)} (-1)^{|X'| - |X|} \\ &= (-1)^{|X|} \times \sum_{X' \in \mathcal{C} \cap \uparrow_{\mathbb{B}_S}(X)} (-1)^{|X'|} \\ &= (-1)^{|X|} \times e(\mathcal{C} \cap \uparrow_{\mathbb{B}_S}(X)) \end{aligned}$$

for all $X \in \mathbb{B}_S$, which is what we wanted to show. \square

Connections between the Möbius function and the Euler characteristic have already been shown, for instance see Philip Hall's theorem [Rota, 1964, Proposition 6 and Theorem 3]. As far as we can tell, however, the connection from Proposition 5.10 seems new.

6. Partial result on NCPD: avoiding a given zero

In this section we present a partial result on the NCPD conjecture intuitively saying that we can “avoid” any given non-trivial zero. We first define what these are.

Definition 6.1. Let \mathcal{I} be a downset of \mathbb{B}_S . We define the non-trivial zeros of \mathcal{I} , denoted $\text{NTZ}_{\mathbb{B}_S}(\mathcal{I})$, by

$$\text{NTZ}_{\mathbb{B}_S}(\mathcal{I}) \stackrel{\text{def}}{=} \{Z \in \mathcal{I} \mid \hat{\mu}_{\mathbb{B}_S, \mathcal{I}}(Z) = 0\}.$$

By opposition, the trivial zeros of \mathcal{I} are the elements $Z \in \mathbb{B}_S \setminus \mathcal{I}$ such that $\hat{\mu}_{\mathbb{B}_S, \mathcal{I}}(Z) = 0$ (notice that these form an upset of \mathbb{B}_S).

Observe that if \mathcal{I} does not have non-trivial zeros, i.e., $\text{NTZ}_{\mathbb{B}_S}(\mathcal{I}) = \emptyset$, then we have indeed that $\mathcal{I} \in \bullet(\text{NCPD}_{\mathbb{B}_S}(\mathcal{I}))$, that is, the NCPD conjecture holds, as can be seen by taking $\mathcal{C} = \mathcal{I}$ in Lemma 5.7. So we can focus on downsets \mathcal{I} having as least one such non-trivial zero.

The partial result that we show in this section is that, for any given such non-trivial zero Z , if we are allowed to use all the non-trivial principal downsets *except the one generated by Z* , then we can construct \mathcal{I} . Formally we will prove the following:

Theorem 6.2. Let \mathcal{I} be a downset of \mathbb{B}_S such that $\text{NTZ}_{\mathbb{B}_S}(\mathcal{I})$ is not empty, and let $Z \in \text{NTZ}_{\mathbb{B}_S}(\mathcal{I})$. Then we have $\mathcal{I} \in \bullet(\{\downarrow_{\mathbb{B}_S}(X) \mid X \in \mathbb{B}_S \setminus \{Z\}\})$.

The proof reuses and extends some ideas and notions from [Monet, 2020a]. We first define some of these notions in Section 6.1, in particular the notion of *adjacent pairs* and certain equivalence classes. In Section 6.2 we show what we call the *lifting lemma*, and show adjacent pairs can be simulated with principal downsets. We combine everything in Section 6.3 to prove Theorem 6.2.

6.1. Adjacent pairs

Definition 6.3. An adjacent pair of \mathbb{B}_S is a configuration \mathcal{P} of the form $\{X, X \setminus \{x\}\}$ (with $X \in \mathbb{B}_S$ and $x \in X$).

Definition 6.4. Let A be a set of adjacent pairs of \mathbb{B}_S , and let $\mathcal{C}, \mathcal{C}'$ be two configurations of \mathbb{B}_S . We then write $\mathcal{C} \overset{+A}{\rightsquigarrow} \mathcal{C}'$ when there exists an adjacent pair $\mathcal{P} \in A$ such that $\mathcal{C}' = \mathcal{C} \dot{\cup} \mathcal{P}$. Similarly we write $\mathcal{C} \overset{-A}{\rightsquigarrow} \mathcal{C}'$ whenever $\mathcal{C}' \overset{+A}{\rightsquigarrow} \mathcal{C}$ (i.e., when there exists $\mathcal{P} \in A$ such that $\mathcal{C}' = \mathcal{C} \setminus \mathcal{P}$), and write $\mathcal{C} \overset{\pm A}{\rightsquigarrow} \mathcal{C}'$ when $\mathcal{C} \overset{+A}{\rightsquigarrow} \mathcal{C}'$ or $\mathcal{C} \overset{-A}{\rightsquigarrow} \mathcal{C}'$. Observe that $\overset{\pm A}{\rightsquigarrow}$ is symmetric. We write $\overset{\pm A}{\rightsquigarrow}^*$ the reflexive transitive closure of $\overset{\pm A}{\rightsquigarrow}$, and write \simeq_A the induced equivalence relation.

In other words, we have $\mathcal{C} \simeq_A \mathcal{C}'$ when we can go from \mathcal{C} to \mathcal{C}' by iteratively (1) adding an adjacent pair from A to the current configuration if the pair is disjoint from the configuration; or (2) removing an adjacent pair from A to the current configuration if the pair was included in the configuration.

Definition 6.5. For a subset \mathcal{G} of \mathbb{B}_S , define the set of allowed adjacent pairs of \mathbb{B}_S relative to \mathcal{G} , denoted $\text{AP}_{\mathbb{B}_S}(\mathcal{G})$, by

$$\text{AP}_{\mathbb{B}_S}(\mathcal{G}) \stackrel{\text{def}}{=} \{\mathcal{P} \mid \mathcal{P} \text{ is an adjacent pair of } \mathbb{B}_S \text{ and } \mathcal{P} \subseteq \mathcal{G}\}.$$

Definition 6.6. For S finite, define the undirected graph G_S with vertex set 2^S whose edges are all adjacent pairs of \mathbb{B}_S . A subset \mathcal{G} of \mathbb{B}_S is connected if it is a connected set of nodes in G_S .

The goal of this section is to establish the following proposition:

Proposition 6.7. Let \mathcal{G} be a connected subset of \mathbb{B}_S , and $\mathcal{C}_1, \mathcal{C}_2$ two configurations that are included in \mathcal{G} . We have $e(\mathcal{C}_1) = e(\mathcal{C}_2)$ if and only if $\mathcal{C}_1 \simeq_{\text{AP}_{\mathbb{B}_S}(\mathcal{G})} \mathcal{C}_2$.

This result already appears in [Monet, 2020a, Proposition 6.1], but only when \mathcal{G} is the whole Boolean lattice, i.e., when⁶ we have $\mathcal{G} = 2^S$. Observe that the “if” direction in Proposition 6.7 is trivial, since adding or removing an adjacent pair (with disjoint union or subset complement) does not modify the Euler characteristic. Hence we need to prove the “only if” direction. To this end, we reproduce the following lemma from [Monet, 2020a]:

Lemma 6.8 (Lemma 5.10 of [Monet, 2020a]). Let \mathcal{C} be a configuration of \mathbb{B}_S , and $X \neq X'$ be two subsets of S such that there is a simple path P of the form $X = X_0 - \dots - X_{n+1} = X'$ from X to X' in G_S with $n \geq 0$ and $X_i \notin \mathcal{C}$ for $1 \leq i \leq n$. Then we have the following:

Erasing. If $(-1)^{|X|} \neq (-1)^{|X'|}$ (i.e., n is even) and $\{X, X'\} \subseteq \mathcal{C}$ then, defining \mathcal{C}' by $\mathcal{C}' \stackrel{\text{def}}{=} \mathcal{C} \setminus \{X, X'\}$, we have $\mathcal{C} \xrightarrow{\pm \text{AP}_{\mathbb{B}_S}(P)}^* \mathcal{C}'$. We say that we go from \mathcal{C} to \mathcal{C}' by erasing X and X' .

Teleporting. If $(-1)^{|X|} = (-1)^{|X'|}$ (i.e., n is odd) and $X \in \mathcal{C}$ and $X' \notin \mathcal{C}$ then, defining \mathcal{C}' by $\mathcal{C}' \stackrel{\text{def}}{=} (\mathcal{C} \setminus \{X\}) \cup \{X'\}$, we have $\mathcal{C} \xrightarrow{\pm \text{AP}_{\mathbb{B}_S}(P)}^* \mathcal{C}'$. We say that we go from \mathcal{C} to \mathcal{C}' by teleporting X to X' .

Proof. We only explain the erasing part, as teleporting works similarly. Let $n = 2i$. For $0 \leq j < i$, do the following: add the adjacent pair $\{X_{2j+1}, X_{2j+2}\}$ and remove the adjacent pair $\{X_{2j}, X_{2j+1}\}$. Finally, remove the adjacent pair $\{X_{2i}, X_{2i+1}\}$. \square

Teleporting is illustrated in Figure 5 (adapted from [Monet, 2020a]).

We will also reuse the *fetching lemma* from [Monet, 2020a], extending it to connected subsets of \mathbb{B}_S (instead of the full Boolean lattice). Given the right circumstances, this lemma fetches for us two sets $X, X' \in \mathcal{C}$ and a suitable path so that we can erase X and X' . Formally:

Lemma 6.9 (Fetching lemma, slightly extending Lemma 5.11 of [Monet, 2020a]). Let \mathcal{G} be a subset of \mathbb{B}_S and $\mathcal{C} \subseteq \mathcal{G}$ a configuration such that $|\mathcal{C}| \neq |e(\mathcal{C})|$. Then there exist $X, X' \in \mathcal{C}$ with $(-1)^{|X|} \neq (-1)^{|X'|}$ and a simple path $X = X_0 - \dots - X_{n+1} = X'$ from X to X' in G_S (hence with n even) with all nodes in this path being in \mathcal{G} such that $X_i \notin \mathcal{C}$ for $1 \leq i \leq n$.

⁶We will actually only use this result when \mathcal{G} is a downset of \mathbb{B}_S , but we still prove this more general version for completeness and because it is not much more complicated.

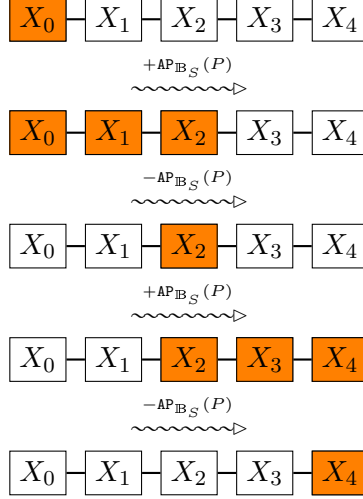


Figure 5: Imagine that the path at the top occurs in G_S for some configuration \mathcal{C} of \mathbb{B}_S (in orange), and let $P = \{X_0, \dots, X_4\}$. The consecutive orange configurations of \mathbb{B}_S are obtained by single steps of the transformation. The total transformation illustrates what is called *teleporting* in Lemma 6.8: we go from \mathcal{C} to \mathcal{C}' in the bottom path by teleporting X_1 to X_4 .

Proof. Since $|\mathcal{C}| \neq |\mathcal{e}(\mathcal{C})|$, there exist $X'', X''' \in \mathcal{C}$ with $(-1)^{|X''|} \neq (-1)^{|X'''|}$. Let $X'' = X''_0 - \dots - X''_{m+1} = X'''$ be an arbitrary simple path from X'' to X''' in G_S with all nodes being in \mathcal{G} (such a path clearly exists because \mathcal{G} is a connected subset of G_S). Now, let $k_1 \stackrel{\text{def}}{=} \max(0 \leq j \leq m \mid (-1)^{|X''_j|} = (-1)^{|X''|} \text{ and } X''_j \in \mathcal{C})$, and then let $k_2 \stackrel{\text{def}}{=} \min(k_1 < j \leq m+1 \mid (-1)^{|X''_j|} = (-1)^{|X''|} \text{ and } X''_j \in \mathcal{C})$, which are well-defined. Then we can take X to be X''_{k_1} and X' to be X''_{k_2} , which satisfy the desired property. \square

With these in place we are ready to prove Proposition 6.7.

Proof of Proposition 6.7. As mentioned above, we only need to prove the “only if” part of the statement. Starting from \mathcal{C}_1 , we repeatedly use the fetching lemma and Erasing until we obtain a configuration $\mathcal{C}'_1 \subseteq \mathcal{G}$ such that $\mathcal{C}'_1 \simeq_{\text{AP}_{\mathbb{B}_S}(\mathcal{G})} \mathcal{C}_1$ and $|\mathcal{C}'_1| = \mathbf{e}(\mathcal{C}'_1)$. We do the same with \mathcal{C}_2 , obtaining $\mathcal{C}'_2 \subseteq \mathcal{G}$ with $\mathcal{C}'_2 \simeq_{\text{AP}_{\mathbb{B}_S}(\mathcal{G})} \mathcal{C}_2$ and $|\mathcal{C}'_2| = \mathbf{e}(\mathcal{C}'_2)$. If $\mathcal{C}'_1 = \mathcal{C}'_2$ we are done. Otherwise, let $n \stackrel{\text{def}}{=} |\mathcal{C}'_1 \setminus \mathcal{C}'_2|$, which is > 0 because $|\mathcal{C}'_1| = |\mathcal{C}'_2|$. We build by induction a sequence of configurations $\mathcal{C}''_1 = \mathcal{C}''_0, \dots, \mathcal{C}''_n = \mathcal{C}'_2$ that are all included in \mathcal{G} and that satisfy (1) $\mathcal{C}''_i \simeq_{\text{AP}_{\mathbb{B}_S}(\mathcal{G})} \mathcal{C}''_{i+1}$ for all $0 \leq i \leq n-1$ and (2) $|\mathcal{C}''_i| = \mathbf{e}(\mathcal{C}''_i)$ and $|\mathcal{C}''_i \setminus \mathcal{C}'_2| = n - i$ for all $0 \leq i \leq n$. It is clear that this indeed implies the claim. As \mathcal{C}''_0 satisfies condition (2), it is enough to explain how to build \mathcal{C}''_{i+1} from \mathcal{C}''_i for $i < n$, and then to argue that the \mathcal{C}''_n from this construction is indeed equal to \mathcal{C}'_2 . We have $|\mathcal{C}''_i \setminus \mathcal{C}'_2| = n - i > 1$ by induction hypothesis, so there exists $X' \in \mathcal{C}''_i \setminus \mathcal{C}'_2$ and $X'' \in \mathcal{C}'_2 \setminus \mathcal{C}''_i$ (because $|\mathcal{C}''_i| = \mathbf{e}(\mathcal{C}''_i) = \mathbf{e}(\mathcal{C}'_1) = |\mathcal{C}'_1| = |\mathcal{C}'_2|$). Let $X' = X'_0 - \dots - X'_{m+1} = X''$ be an arbitrary simple path from X' to X'' in G_S with all nodes being in \mathcal{G} (such a path

exists because \mathcal{G} is a connected subset of G_S , and m is odd and ≥ 1). Observe that all nodes X'_k with k odd are not in $\mathcal{C}''_i \cup \mathcal{C}'_2$, because $|\mathcal{C}''_i| = e(\mathcal{C}''_i)$ and $|\mathcal{C}'_2| = e(\mathcal{C}'_2)$. Now, let $k_1 \stackrel{\text{def}}{=} \max(0 \leq j \leq m \mid X'_j \in \mathcal{C}''_i \setminus \mathcal{C}'_2)$, and then let $k_2 \stackrel{\text{def}}{=} \min(k_1 < j \leq m + 1 \mid X'_j \in \mathcal{C}'_2 \setminus \mathcal{C}''_i)$, which are well-defined. Define $\mathcal{C}''_{i+1} \stackrel{\text{def}}{=} \mathcal{C}''_i \setminus \{X_{k_1}\} \cup \{X_{k_2}\}$. Notice that \mathcal{C}''_{i+1} satisfies condition (2), so all we need to show is that $\mathcal{C}''_i \simeq_{\text{AP}_{\mathbb{B}_S}(\mathcal{G})} \mathcal{C}''_{i+1}$. We know the following: for a node X'_k with $k_1 < k < k_2$, if k is odd then $X'_k \notin \mathcal{C}''_i \cup \mathcal{C}'_2$, and if k is even then either $X'_k \in \mathcal{C}''_i \cap \mathcal{C}'_2$ or $X'_k \notin \mathcal{C}''_i \cup \mathcal{C}'_2$. If there is no node X'_k with $k_1 < k < k_2$, k even such that $X'_k \in \mathcal{C}''_i$, then we can simply use Lemma 6.8 to teleport X_{k_1} to X_{k_2} and we are done. Otherwise, let $X'_{\ell_1}, \dots, X'_{\ell_m}$ with $m \geq 1$ be all the nodes with $k_1 < \ell_p < k_2$ even that are in \mathcal{C}''_i , in order, i.e., $\ell_1 < \dots < \ell_m$. Then we can successively teleport X'_{ℓ_m} to X'_{k_2} , $X'_{\ell_{m-1}}$ to X'_{ℓ_m} , and so on, until we teleport X'_{ℓ_1} to X'_{ℓ_2} and finally we teleport X'_{k_1} to X'_{ℓ_1} , thus obtaining \mathcal{C}''_{i+1} as promised. It is then clear that \mathcal{C}''_n is \mathcal{C}'_2 , because we have $|\mathcal{C}''_n \setminus \mathcal{C}'_2| = 0$ by condition (2), and $|\mathcal{C}''_n| = |\mathcal{C}'_2|$ by construction. This concludes the proof. \square

6.2. Lifting lemma and simulating adjacent pairs with principal downsets

A simple, yet important component of the proof of Theorem 6.2 will be what we call the *lifting lemma*. We first state and prove it before giving intuition.

Lemma 6.10 (Lifting lemma). *Let $Z \in \mathbb{B}_S$ and $\mathcal{A} \subseteq \uparrow_{\mathbb{B}_S}(Z)$. Let $I_{\mathcal{A}} \stackrel{\text{def}}{=} \{\downarrow_{\mathbb{B}_S}(X) \cap \uparrow_{\mathbb{B}_S}(Z) \mid X \in \mathcal{A}\}$ and $I'_{\mathcal{A}} \stackrel{\text{def}}{=} \{\downarrow_{\mathbb{B}_S}(X) \mid X \in \mathcal{A}\}$. Then, for any $\mathcal{C} \in \bullet(I_{\mathcal{A}})$, defining the configuration*

$$\mathbf{lift}_{\mathbb{B}_S, Z}(\mathcal{C}) \stackrel{\text{def}}{=} \{X \in \mathbb{B}_S \mid X \cup Z \in \mathcal{C}\},$$

we have that $\mathbf{lift}_{\mathbb{B}_S, Z}(\mathcal{C}) \in \bullet(I'_{\mathcal{A}})$.

Proof. This is shown by induction on $\mathcal{C} \in \bullet(I_{\mathcal{A}})$. The first base case is when $\mathcal{C} = \emptyset$, but then we have $\mathbf{lift}_{\mathbb{B}_S, Z}(\mathcal{C}) = \emptyset$ as well, which is in $\bullet(I'_{\mathcal{A}})$ by definition. The second base case is when $\mathcal{C} = \downarrow_{\mathbb{B}_S}(X) \cap \uparrow_{\mathbb{B}_S}(Z)$ for some X in \mathcal{A} . In this case, it is easy to show that $\mathbf{lift}_{\mathbb{B}_S, Z}(\mathcal{C}) = \downarrow_{\mathbb{B}_S}(X)$; this uses in particular that set union is the join operation of the lattice \mathbb{B}_S . But then this implies $\mathbf{lift}_{\mathbb{B}_S, Z}(\mathcal{C}) \in \bullet(I'_{\mathcal{A}})$ indeed. For the inductive case, we focus on $\dot{\cup}$ as \setminus works similarly. Let $\mathcal{C}_1, \mathcal{C}_2 \in \bullet(I_{\mathcal{A}})$ such that $\mathcal{C} \stackrel{\text{def}}{=} \mathcal{C}_1 \dot{\cup} \mathcal{C}_2$ is well-defined, and let us show that $\mathbf{lift}_{\mathbb{B}_S, Z}(\mathcal{C}) \in \bullet(I'_{\mathcal{A}})$. By induction hypothesis we have that $\mathbf{lift}_{\mathbb{B}_S, Z}(\mathcal{C}_1) \in \bullet(I'_{\mathcal{A}})$ and $\mathbf{lift}_{\mathbb{B}_S, Z}(\mathcal{C}_2) \in \bullet(I'_{\mathcal{A}})$. Now, observe that $\mathbf{lift}_{\mathbb{B}_S, Z}(\mathcal{C}_1) \dot{\cup} \mathbf{lift}_{\mathbb{B}_S, Z}(\mathcal{C}_2)$ is well defined, and furthermore that we have $\mathbf{lift}_{\mathbb{B}_S, Z}(\mathcal{C}_1 \dot{\cup} \mathcal{C}_2) = \mathbf{lift}_{\mathbb{B}_S, Z}(\mathcal{C}_1) \dot{\cup} \mathbf{lift}_{\mathbb{B}_S, Z}(\mathcal{C}_2)$. This implies $\mathbf{lift}_{\mathbb{B}_S, Z}(\mathcal{C}) \in \bullet(I'_{\mathcal{A}})$ as wanted, and concludes the proof. \square

The intuition behind the lifting lemma is the following. Notice that $\uparrow_{\mathbb{B}_S}(Z)$ is isomorphic to the Boolean lattice $\mathbb{B}_{S'}$ with $S' \stackrel{\text{def}}{=} S \setminus Z$. Consider a witnessing tree T' representing a configuration \mathcal{C} of $\mathbb{B}_{S'}$ that is formed from principal downsets of $\mathbb{B}_{S'}$. Consider the “lifted tree” T that is obtained from T' by replacing every leaf of T of the

form $\downarrow_{\mathbb{B}_{S'}}(X)$ for $X \subseteq S'$ by the principal downset of \mathbb{B}_S that is $\downarrow_{\mathbb{B}_S}(X \cup Z)$. The lifting lemma says that the obtained tree T' is valid (i.e., the internal nodes are well-defined disjoint unions and subset complements) and that it represents the configuration of \mathbb{B}_S that is $\text{lift}_{\mathbb{B}_S, Z}(\mathcal{C})$.

This lemma will have two uses in the proof of Theorem 6.2. The first is that said proof will work by first constructing a configuration in the principal upset generated by the non-trivial zero Z that is fixed in the theorem statement, and then this lemma will be used to “lift” the result to the whole Boolean lattice \mathbb{B}_S . The second use of this lemma is that it helps us show that adjacent pairs can be simulated by principal downsets, while avoiding the principal downset that is at the bottom. We explain this second use in the following lemma.

Lemma 6.11. *Let $\mathcal{P} = \{X, X \setminus \{x\}\}$ be an adjacent pair of \mathbb{B}_S . Then we have $\mathcal{P} \in \bullet(\{\downarrow_{\mathbb{B}_S}(Y) \mid \emptyset \subsetneq Y \subseteq X\})$.*

In this statement, pay attention that the principal downset $\downarrow_{\mathbb{B}_S}(\emptyset)$ is not in the base set of the dot-algebra.⁷ In other words, we are allowed to use any principal downset that is generated by an element that is below X and strictly above \emptyset in \mathbb{B}_S . The reason is that in the proof of Theorem 6.2, Z will correspond to \emptyset here and it generates the only downset we are not allowed to use. We now prove Lemma 6.11.

Proof of Lemma 6.11. Clearly, it suffices to show the claim in the case that $X = S$. To this end, it is enough to show that we have $\mathbb{B}_S \setminus \mathcal{P} \in \bullet(\{\downarrow_{\mathbb{B}_S}(Y) \mid \emptyset \subsetneq Y \subseteq X\})$: indeed we can then obtain \mathcal{P} as $\downarrow_{\mathbb{B}_S}(S) \setminus (\mathbb{B}_S \setminus \mathcal{P})$. Let \mathcal{C}' be the configuration $\{X \in \mathbb{B}_S \mid \{x\} \subseteq X \subsetneq S\}$. Using Lemma 5.7 appropriately we have that $\mathcal{C}' \in \bullet(\{\downarrow_{\mathbb{B}_S}(Y) \cap \uparrow_{\mathbb{B}_S}(\{x\}) \mid \{x\} \subseteq Y \subsetneq S\})$. We now use the lifting lemma (invoked with $Z = \{x\}$) to obtain that $\text{lift}_{\mathbb{B}_S, \{x\}}(\mathcal{C}') \in \bullet(\{\downarrow_{\mathbb{B}_S}(Y) \mid \{x\} \subseteq Y \subsetneq S\})$, hence in particular $\text{lift}_{\mathbb{B}_S, \{x\}}(\mathcal{C}') \in \bullet(\{\downarrow_{\mathbb{B}_S}(Y) \mid \emptyset \subsetneq Y \subseteq S\})$. But notice that $\text{lift}_{\mathbb{B}_S, \{x\}}(\mathcal{C}')$ is actually $\mathbb{B}_S \setminus \mathcal{P}$, which concludes the proof. \square

6.3. Proof of Theorem 6.2

We now prove Theorem 6.2. Fix the finite set S and the configuration \mathcal{I} that is a downset of \mathbb{B}_S such that $\text{NTZ}_{\mathbb{B}_S}(\mathcal{I})$ is not empty, and let $Z \in \text{NTZ}_{\mathbb{B}_S}(\mathcal{I})$. By Proposition 5.10 we have $\hat{\mu}_{\mathbb{B}_S, \mathcal{I}}(Z) = (-1)^{|Z|} \times e(\mathcal{I} \cap \uparrow_{\mathbb{B}_S}(Z))$, therefore $e(\mathcal{I} \cap \uparrow_{\mathbb{B}_S}(Z)) = 0$. We now instantiate Proposition 6.7 with $\mathcal{G} = \mathcal{C}_1 = \mathcal{I} \cap \uparrow_{\mathbb{B}_S}(Z)$ and $\mathcal{C}_2 = \emptyset$ to obtain $\mathcal{I} \cap \uparrow_{\mathbb{B}_S}(Z) \simeq_{\text{AP}_{\mathbb{B}_S}(\mathcal{I} \cap \uparrow_{\mathbb{B}_S}(Z))} \emptyset$. This gives us a left-linear tree T for $\mathcal{I} \cap \uparrow_{\mathbb{B}_S}(Z)$ with disjoint unions and subset complements and whose leaves are adjacent pairs from $\text{AP}_{\mathbb{B}_S}(\mathcal{I} \cap \uparrow_{\mathbb{B}_S}(Z))$. Next we use Lemma 6.11 to simulate, in $\uparrow_{\mathbb{B}_S}(Z)$, each such adjacent pair using the principal downsets of $\uparrow_{\mathbb{B}_S}(Z)$, except the one generated by Z (thanks to the fact that $\emptyset \subsetneq Y$ in the statement of the lemma). Therefore, we obtain that $\mathcal{I} \cap \uparrow_{\mathbb{B}_S}(Z) \in \bullet(\{\downarrow_{\mathbb{B}_S}(X) \cap \uparrow_{\mathbb{B}_S}(Z) \mid X \in \mathcal{I}, Z \subsetneq X\})$. We now use Lemma 6.10 to obtain that $\text{lift}_{\mathbb{B}_S, Z}(\mathcal{I} \cap \uparrow_{\mathbb{B}_S}(Z)) \in \bullet(\{\downarrow_{\mathbb{B}_S}(X) \mid X \in \mathcal{I} \setminus \{Z\}\})$. Notice that

⁷Recall that to alleviate the notation we write $\downarrow_{\mathbb{B}_S}(\emptyset)$ to mean $\downarrow_{\mathbb{B}_S}(\{\emptyset\})$, which is then $\{\emptyset\}$.

$\text{lift}_{\mathbb{B}_S, Z}(\mathcal{I} \cap \uparrow_{\mathbb{B}_S}(Z)) \cap \uparrow_{\mathbb{B}_S}(Z) = \mathcal{I} \cap \uparrow_{\mathbb{B}_S}(Z)$, i.e., restricted to $\uparrow_{\mathbb{B}_S}(Z)$ we have the correct configuration, and we now only need to “fix” what is outside.

Now let $\mathcal{C}'_1 = \text{lift}_{\mathbb{B}_S, Z}(\mathcal{I} \cap \uparrow_{\mathbb{B}_S}(Z)) \setminus (\mathcal{I} \cap \uparrow_{\mathbb{B}_S}(Z))$ and $\mathcal{C}'_2 = \mathcal{I} \setminus (\mathcal{I} \cap \uparrow_{\mathbb{B}_S}(Z))$. We use Lemma 5.7 to obtain that \mathcal{C}'_1 is in $\bullet(\{\downarrow_{\mathbb{B}_S}(X) \mid X \in \mathcal{C}'_1\})$, and likewise for \mathcal{C}'_2 . Observe that \mathcal{C}'_1 and \mathcal{C}'_2 only consist of elements outside of $\uparrow_{\mathbb{B}_S}(Z)$, in particular $\mathcal{C}'_1, \mathcal{C}'_2 \subseteq \mathcal{I} \setminus \{Z\}$, so that \mathcal{C}'_1 and \mathcal{C}'_2 are in $\bullet(\{\downarrow_{\mathbb{B}_S}(X) \mid X \in \mathcal{I} \setminus \{Z\}\})$. Finally we can obtain \mathcal{I} with the expression $(\text{lift}_{\mathbb{B}_S, Z}(\mathcal{I} \cap \uparrow_{\mathbb{B}_S}(Z)) \dot{\setminus} \mathcal{C}'_1) \dot{\cup} \mathcal{C}'_2$, which is in $\bullet(\{\downarrow_{\mathbb{B}_S}(X) \mid X \in \mathcal{I} \setminus \{Z\}\})$ by what precedes. This concludes the proof.

7. Extensions, Variants and Counterexample Search

In this section we present a generalization of Theorem 6.2 that we believe to be true, present variants of the conjectures, and talk about our experimental search for counterexamples.

Extension of Theorem 6.2. We sketch here a proposed generalization that allows us to avoid not one single zero, but subsets of zeros while requiring that the lattice has a certain structure. We first define a few notions.

Definition 7.1. Let \mathcal{I} be a downset of \mathbb{B}_S . For $X \in \mathbb{B}_S$, the non-trivial covering zeros of X relative to \mathcal{I} are

$$\text{NTCZ}_{\mathbb{B}_S, \mathcal{I}}(X) \stackrel{\text{def}}{=} \{Z \in \text{NTZ}_{\mathbb{B}_S}(\mathcal{I}) \mid X \subsetneq Z \text{ and there is no } Z' \in \text{NTZ}_{\mathbb{B}_S}(\mathcal{I}) \text{ s.t. } X \subsetneq Z' \subsetneq Z\}.$$

In other words, $\text{NTCZ}_{\mathbb{B}_S, \mathcal{I}}(X)$ simply consists of the minimal non-trivial zeros of \mathcal{I} that are strictly above X . Observe that we have $\text{NTCZ}_{\mathbb{B}_S, \mathcal{I}}(X) = \emptyset$ whenever $X \notin \mathcal{I}$. We now define k -decomposable downsets.

Definition 7.2. A configuration \mathcal{I} of \mathbb{B}_S that is a downset is called k -decomposable if for every $X \in \mathbb{B}_S$ we have $|\text{NTCZ}_{\mathbb{B}_S, \mathcal{I}}(X)| \leq k$.

This intuitively means that, in every upset, there are at most two non-trivial covering zeroes. We believe that the following is true, but have not completely formalized the proof.

Conjecture 7.3. For every finite S , for every configuration \mathcal{I} of \mathbb{B}_S that is a 2-decomposable downset of \mathbb{B}_S , we have that $\mathcal{I} \in \bullet(\text{NCPD}_{\mathbb{B}_S}(\mathcal{I}))$.

Proof sketch. For an element $X \in \mathbb{B}_S$, let $I_X \stackrel{\text{def}}{=} \{\downarrow_{\mathbb{B}_S}(Y) \cap \uparrow_{\mathbb{B}_S}(X) \mid Y \in \uparrow_{\mathbb{B}_S}(X), \hat{\mu}_{\mathbb{B}_S, \mathcal{I}}(Y) \neq 0\}$. The idea of the proof would be to show by top-down induction on \mathbb{B}_S that for every $X \in \mathbb{B}_S$, we have $\mathcal{I} \cap \uparrow_{\mathbb{B}_S}(X) \in \bullet(I_X)$. Applying this claim to $X = \emptyset$ yields the desired result. To show this, we use similar tools to those developed for the proof of Theorem 6.2, extended with an accounting of Euler characteristics of diverse sets of configurations. It does not seem that the proposed proof would directly generalize to the general case, or indeed to 3-decomposable configurations. \square

Strengthenings. We present here two possible orthogonal ways in which we can strengthen the conjecture; we do not know whether the stronger conjectures are true, nor whether they are equivalent to the original conjecture.

First, in view of Proposition 4.7, we could require that the non-trivial intersections are only used positively or only negatively depending on the sign of their Möbius value. Indeed, Proposition 4.7 implies that, on tight intersection lattices, the non-trivial intersections have total multiplicity equal to their Möbius value, but they may be used both positively and negatively. Likewise, in view of Proposition 5.8, we could require that the principal downsets are only used positively and negatively.

Second, we do not know if the witnessing trees can be required to be left-linear, i.e., whether we can obtain $\hat{1}$ (for the NCI conjecture) or the desired configuration (for the NCPD conjecture) by a series of operations where, at each step, we add a (disjoint) non-trivial intersection (or principal downset), or subtract one (which must be a subset). For instance, this formulation does not allow us to express a disjoint union of two configurations themselves obtained with more complex witnessing trees. This stronger conjecture is the topic of the question asked in [Amarilli, 2019] (up to replacing upsets by downsets and to working in general lattices instead of only on Boolean lattices). This question also explains why the conjecture is false if asked about general DAGs (instead of lattices), and shows that the conjecture can be true for so-called *crown-free lattices*.

We note that the construction for the proof of Theorem 6.2 (or the one that we have in mind for Conjecture 7.3) does not satisfy either of these strengthenings. However, the proof of Fact 4.8 can be adjusted to satisfy the left-linear condition.

Counterexample search. We have implemented a search for counterexamples of the NCI conjecture. We consider Sperner families [Monet, 2020b], which give all sets of subsets of a base set of n elements that are non-equivalent (i.e., that are not the same up to permuting the elements). We have checked in a brute-force fashion that the conjecture holds up to $n = 5$, i.e., on all intersection lattices such that $\hat{1}$ has cardinality at most 5. Unfortunately, already for $n = 6$ there are intersection lattices that are too large for the computation to finish sufficiently quickly. We also generated random intersection lattices over larger sets of elements, but could not find a counterexample (assuming that our code is correct). The code checks the strong version of the NCI conjecture in the previous terminology, i.e., it searches for left-linear trees and requires the non-trivial intersections to be used precisely with the right polarity. The code is available as-is [Amarilli, 2023]. We have also improved the implementation to search for solutions on each lattice using a SAT-solver rather than a brute-force search: this speeds up the computation but also did not yield any counterexample.

We had also implemented, earlier, a search for counterexamples of an alternative phrasing of the NCPD conjecture: see [Amarilli et al., 2020]. This also illustrates that the NCPD conjecture is false if the configuration to reach is not a downset; see Figure 4 of [Amarilli et al., 2020] (up to reversing directions, i.e., considering downsets instead of upsets).

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A. Proofs for Section 3 (The Non-Cancelling Intersections Conjecture)

We show in this section that the NCI and NCU conjectures are equivalent. We only sketch the proofs as details are easy to fill.

Definition A.1. Let \mathcal{F} be a set family. For $S \in 2^{\cup \mathcal{F}}$, let $\text{dual}_{\mathcal{F}}(S) \stackrel{\text{def}}{=} (\cup \mathcal{F}) \setminus S$, and for $\mathcal{T} \subseteq \mathcal{F}$ let $\text{dual}_{\mathcal{F}}(\mathcal{T}) \stackrel{\text{def}}{=} \{\text{dual}_{\mathcal{F}}(X) \mid X \in \mathcal{T}\}$.

Observe then that \mathcal{F} is non-trivial if and only if $\text{dual}_{\mathcal{F}}(\mathcal{F})$ is non-co-trivial.

Fact A.2. Let $L = \mathbb{L}_{\mathcal{F}}$ be an intersection lattice, and let L'' be the union lattice $\mathbb{U}_{\text{dual}_{\mathcal{F}}(\mathcal{F})}$ of $\text{dual}_{\mathcal{F}}(\mathcal{F})$, but with the order reversed. Then we have $L \simeq L''$.

Proof sketch. One can check that the function that sends $\hat{1}_L$ to $\hat{1}_{L''}$ ($= \hat{0}_{\mathbb{U}_{\text{dual}_{\mathcal{F}}(\mathcal{F})}}$) and $S_{\mathcal{T}} = \bigcap \mathcal{T}$ for $\mathcal{T} \subseteq \mathcal{F}$, $\mathcal{T} \neq \emptyset$ to $R_{\text{dual}_{\mathcal{F}}(\mathcal{T})} = \bigcup \text{dual}_{\mathcal{F}}(\mathcal{T})$ is an isomorphism between L and L'' . This uses the following two simple facts:

1. For any sets A, B, C with $B \cup C \subseteq A$ we have $A \setminus B \subseteq A \setminus C$ if and only if $C \subseteq B$.
2. For a set A and set family \mathcal{T} we have $\bigcap_{X \in \mathcal{T}} A \setminus X = A \setminus \bigcup \mathcal{T}$. □

The dual version of this fact also holds, i.e., for a union lattice $L = \mathbb{U}_{\mathcal{F}}$, letting L'' be the intersection lattice $\mathbb{L}_{\text{dual}_{\mathcal{F}}(\mathcal{F})}$ of $\text{dual}_{\mathcal{F}}(\mathcal{F})$ but with the order reversed, we have $L \simeq L''$. (This uses point (1) above and the dual of point (2), namely, $\bigcup_{X \in \mathcal{T}} A \setminus X = A \setminus \bigcap \mathcal{T}$.)

We can then show that the NCU conjecture implies the NCI conjecture as follows. Let $L = \mathbb{L}_{\mathcal{F}}$ be an intersection lattice, and let $L' \stackrel{\text{def}}{=} \mathbb{U}_{\text{dual}_{\mathcal{F}}(\mathcal{F})}$ be the union lattice of $\text{dual}_{\mathcal{F}}(\mathcal{F})$. Using Fact A.2 and [Rota, 1964, Proposition 3], it is routine to show by induction on $\bullet(\text{NCU}(L'))$ the following claim (\dagger): for every $R \in \bullet(\text{NCU}(L'))$ we have $\text{dual}_{\mathcal{F}}(R) \in \bullet(\text{NCI}(L))$. Since the NCU conjecture holds we have $\hat{0}_{L'} \in \bullet(\text{NCU}(L'))$, hence by (\dagger) we have $\text{dual}_{\mathcal{F}}(\hat{0}_{L'}) \in \bullet(\text{NCI}(L))$. But observe that $\hat{0}_{L'} = \bigcap_{X \in \mathcal{F}} \text{dual}_{\mathcal{F}}(X)$ by definition, which is equal to $\bigcup \mathcal{F} \setminus \bigcup \mathcal{F} = \emptyset$ by Item (2) above. So $\text{dual}_{\mathcal{F}}(\hat{0}_{L'}) = \bigcup \mathcal{F}$, and this is $\hat{1}_L$, so indeed $\hat{1}_L \in \bullet(\text{NCI}(L))$.

We can show that the NCI conjecture implies the NCU conjecture in the same way, using the dual version of Fact A.2. This completes the proof.

B. Proofs for Section 4 (Simplifying to Tight Intersection Lattices)

Proposition 4.7. Let L be a full intersection lattice such that $\hat{1} \in \bullet(\text{NTI}(L))$, and T a tree witnessing this. Then we have $\text{mult}_T(U) = -\mu_L(U, \hat{1})$ for every $U \in \text{NTI}(L)$.

Proof. For $U \in L$, $U \neq \hat{1}$, let α_U be an element of $\{x \in \hat{1} \mid \min_L(x) = U\}$. We first define, for each $X \subseteq \hat{1}$, a function $\check{\mu}_{L,X} : L \rightarrow \mathbb{Z}$, and state three useful properties of these functions. The function $\check{\mu}_{L,X}$ for $X \subseteq \hat{1}$ is defined by $\check{\mu}_{L,X}(\hat{1}) \stackrel{\text{def}}{=} 0$, and, for $U \in \text{NTI}(L)$,

by $\check{\mu}_{L,X}(U) \stackrel{\text{def}}{=} 1 - \sum_{\substack{U' \in L \\ U \subsetneq U'}} \check{\mu}_{L,X}(U')$ if $\alpha_U \in X$, and $\check{\mu}_{L,X}(U) \stackrel{\text{def}}{=} - \sum_{\substack{U' \in L \\ U \subsetneq U'}} \check{\mu}_{L,X}(U')$ if $\alpha_U \notin X$.

The three properties are:

1. We have $\check{\mu}_{L,\emptyset}(U) = 0$ for $U \in L$.
2. For every $U \in \text{NTI}(L)$ we have $\check{\mu}_{L,\hat{1}}(U) = -\mu_L(U, \hat{1})$.
3. For $X_1, X_2 \subseteq \hat{1}$ such that $X_1 \cap X_2 = \emptyset$ (resp., such that $X_2 \subseteq X_1$) we have $\check{\mu}_{L, X_1 \dot{\cup} X_2} = \check{\mu}_{L, X_1} + \check{\mu}_{L, X_2}$ (resp., $\check{\mu}_{L, X_1 \setminus X_2} = \check{\mu}_{L, X_1} - \check{\mu}_{L, X_2}$).
4. For $U \in \text{NTI}(L)$ and $U' \in L$, the value $\check{\mu}_{L,U}(U')$ is 1 if $U' = U$ and 0 otherwise.

These can all easily be established by top-down induction on L . Now for a node $n \in T$, let X_n be the subset of $\hat{1}$ represented by the subtree of T rooted at n . We show by bottom-up induction on T that for every node n of T and $U \in \text{NTI}(L)$ we have $\text{mult}_{T_n}(U) = \check{\mu}_{L, X_n}(U)$. This will indeed imply the result by applying it to the root of T and using item (2), since we then have that $X_n = \hat{1}$. The base case, when n is a leaf of T , follows from item (4) and item (1), while the inductive case, when n is an internal node of T , follows from (3). \square

C. Proofs for Section 5 (Alternative Formulation: Dot-Algebra of Non-Cancelling Principal Downsets in the Boolean Lattice)

Lemma 5.5. *Let S be a finite set and let \mathcal{I} be a downset of \mathbb{B}_S of the form $\mathcal{I} = \downarrow_{\mathbb{B}_S}(\mathcal{F})$ for \mathcal{F} a non-trivial set family. For $X \in \mathcal{F}$ let $X' \stackrel{\text{def}}{=} \downarrow_{\mathbb{B}_S}(X)$ (which is 2^X), and let $\mathcal{F}' \stackrel{\text{def}}{=} \{X' \mid X \in \mathcal{F}\}$. Note that \mathcal{F}' is non-trivial as well, and let then L be its intersection lattice. Then we have $\text{NCPD}_{\mathbb{B}_S}(\mathcal{I}) = \text{NCI}(L)$.*

Proof. Observe that $\hat{1}_L = \mathcal{I}$. For $\mathcal{T} \subseteq \mathcal{F}$, we write \mathcal{T}' for the corresponding subset of \mathcal{F}' , i.e., $\mathcal{T}' = \{X' \mid X \in \mathcal{T}\}$, and vice versa. For $\mathcal{T} \subseteq \mathcal{F}, \mathcal{T} \neq \emptyset$ let $X_{\mathcal{T}} \stackrel{\text{def}}{=} \bigcap \mathcal{T}$. We show that the following hold:

1. For $\mathcal{T}' \subseteq \mathcal{F}', \mathcal{T}' \neq \emptyset$ we have $S_{\mathcal{T}'} = \downarrow_{\mathbb{B}_S}(X_{\mathcal{T}})$.
2. Let L' be the lattice $(\{X_{\mathcal{T}} \mid \mathcal{T} \subseteq \mathcal{F}, \mathcal{T} \neq \emptyset\} \cup \{\hat{1}_{L'}\}, \leq_{L'})$ where $\hat{1}_{L'}$ is a fresh element and $Y \leq_{L'} \hat{1}_{L'}$ for all $Y \in L'$ and $X_{\mathcal{T}_1} \leq_{L'} X_{\mathcal{T}_2}$ for $\mathcal{T}_1, \mathcal{T}_2 \subseteq \mathcal{F}$ if $X_{\mathcal{T}_1} \subseteq X_{\mathcal{T}_2}$. Then $L \simeq L'$ via the isomorphism that sends $\hat{1}_L$ to $\hat{1}_{L'}$ and $S_{\mathcal{T}'}$ for $\mathcal{T}' \subseteq \mathcal{F}', \mathcal{T}' \neq \emptyset$ to $X_{\mathcal{T}}$.
3. For every $Y \in \mathbb{B}_S$, if Y is of the form $X_{\mathcal{T}}$ then $\hat{\mu}_{\mathbb{B}_S, \mathcal{I}}(Y) = -\mu_L(S_{\mathcal{T}'}, \hat{1}_L)$, otherwise $\hat{\mu}_{\mathbb{B}_S, \mathcal{I}}(Y) = 0$.

Notice that Item (3) together with Item (1) implies that $\text{NCPD}_{\mathbb{B}_S}(\mathcal{I}) = \text{NCI}(L)$, which is what we need to prove. Item (1) follows from the chain of trivialities $S_{\mathcal{T}'} \stackrel{\text{def}}{=} \bigcap \mathcal{T}' \stackrel{\text{def}}{=} \bigcap_{X' \in \mathcal{T}'} X' = \bigcap_{X \in \mathcal{T}} 2^X = 2^{\bigcap \mathcal{T}} \stackrel{\text{def}}{=} 2^{X_{\mathcal{T}}} \stackrel{\text{def}}{=} \downarrow_{\mathbb{B}_S}(X_{\mathcal{T}})$, and Item (2) is easy to prove using Item (1). For Item (3), we show that $\hat{\mu}_{\mathbb{B}_S, \mathcal{I}}$ is as claimed. For $Y \in \mathbb{B}_S$, $Y \notin \mathcal{I}$, it is clear that $\hat{\mu}_{\mathbb{B}_S, \mathcal{I}}(Y) = 0$, by definition of $\hat{\mu}_{\mathbb{B}_S, \mathcal{I}}$ and because \mathcal{I} is a downset, so we can focus on $Y \in \mathcal{I}$. We show by top-down induction on \mathcal{I} that $\hat{\mu}_{\mathbb{B}_S, \mathcal{I}}$ is as claimed on \mathcal{I} . The base case is when Y is a maximal element of \mathcal{I} , i.e., $Y = X$ for some maximal $X \in \mathcal{F}$. Then we have $\hat{\mu}_{\mathbb{B}_S, \mathcal{I}}(Y) = 1 = -\mu_L(X')$ indeed. The inductive case is when Y is not a maximal element of \mathcal{I} . Let $\mathcal{T}_Y \stackrel{\text{def}}{=} \{X \in \mathcal{F} \mid Y \subseteq X\}$, which is non-empty. Notice the following fact (\dagger): we have $Y \subseteq X_{\mathcal{T}_Y}$, and more specifically we have $Y \subseteq X_{\mathcal{T}}$ for $\mathcal{T} \subseteq \mathcal{F}$, $\mathcal{T} \neq \emptyset$ if and only if $\mathcal{T} \subseteq \mathcal{T}_Y$. We then distinguish two cases.

- The first case is when $Y = X_{\mathcal{T}_Y}$. We have $\hat{\mu}_{\mathbb{B}_S, \mathcal{I}}(Y) = 1 - \sum_{Y' \subsetneq Y'} \hat{\mu}_{\mathbb{B}_S, \mathcal{I}}(Y')$ by definition. Now by what precedes, if $Y' \notin \mathcal{I}$ then $\hat{\mu}_{\mathbb{B}_S, \mathcal{I}}(Y') = 0$, and by induction hypothesis if Y' is not of the form $X_{\mathcal{T}}$ then $\hat{\mu}_{\mathbb{B}_S, \mathcal{I}}(Y') = 0$. Hence, we have $\hat{\mu}_{\mathbb{B}_S, \mathcal{I}}(Y) = 1 - \sum_{\substack{Y' \subsetneq Y' \\ Y' = X_{\mathcal{T}}}} \hat{\mu}_{\mathbb{B}_S, \mathcal{I}}(X_{\mathcal{T}})$, and the suitable \mathcal{T} are the subsets of \mathcal{T}_Y by (\dagger), i.e., $\hat{\mu}_{\mathbb{B}_S, \mathcal{I}}(Y) = 1 - \sum_{Y' \in \{X_{\mathcal{T}} \mid \mathcal{T} \subseteq \mathcal{T}_Y\} \setminus \{Y\}} \hat{\mu}_{\mathbb{B}_S, \mathcal{I}}(X_{\mathcal{T}})$. Now, by induction hypothesis, we have $\hat{\mu}_{\mathbb{B}_S, \mathcal{I}}(X_{\mathcal{T}}) = -\mu_L(S_{\mathcal{T}'}, \hat{1}_L)$ for all $\mathcal{T} \subseteq \mathcal{T}_Y$ such that $X_{\mathcal{T}} \neq Y$, therefore $\hat{\mu}_{\mathbb{B}_S, \mathcal{I}}(Y) = 1 + \sum_{Y' \in \{X_{\mathcal{T}} \mid \mathcal{T} \subseteq \mathcal{T}_Y\} \setminus \{Y\}} \mu_L(S_{\mathcal{T}'}, \hat{1}_L)$. Now by Item (2) this is equal to $1 + \sum_{\substack{U \in L \\ U \neq \hat{1}_L \\ S_{\mathcal{T}'_Y} \subsetneq U}} \mu_L(U, \hat{1}_L)$, which is equal to $\sum_{\substack{U \in L \\ S_{\mathcal{T}'_Y} \subsetneq U}} \mu_L(U, \hat{1}_L)$, which is $-\mu_L(S_{\mathcal{T}'_Y})$ by definition. So indeed $\hat{\mu}_{\mathbb{B}_S, \mathcal{I}}(Y) = -\mu_L(S_{\mathcal{T}'_Y})$.
- The second case is when $Y \neq X_{\mathcal{T}_Y}$. Then, the nodes of \mathcal{I} in the principal upset of Y minus $\{Y\}$ consists of the nodes of \mathcal{I} in the principal upset of $X_{\mathcal{T}_Y}$, and of nodes that are not of the form $X_{\mathcal{T}}$ by (\dagger) and that are strict supersets of Y . By induction hypothesis, the $\hat{\mu}_{\mathbb{B}_S, \mathcal{I}}$ -value of the latter is zero. So $\hat{\mu}_{\mathbb{B}_S, \mathcal{I}}(Y) = 1 - \sum_{X_{\mathcal{T}_Y} \subsetneq Y'} \hat{\mu}_{\mathbb{B}_S, \mathcal{I}}(Y') = 1 - \hat{\mu}_{\mathbb{B}_S, \mathcal{I}}(X_{\mathcal{T}_Y}) - \sum_{X_{\mathcal{T}_Y} \subsetneq Y'} \hat{\mu}_{\mathbb{B}_S, \mathcal{I}}(Y')$, but by definition of $\hat{\mu}_{\mathbb{B}_S, \mathcal{I}}$ we have $\hat{\mu}_{\mathbb{B}_S, \mathcal{I}}(X_{\mathcal{T}_Y}) = 1 - \sum_{X_{\mathcal{T}_Y} \subsetneq Y'} \hat{\mu}_{\mathbb{B}_S, \mathcal{I}}(Y')$, so indeed $\hat{\mu}_{\mathbb{B}_S, \mathcal{I}}(Y) = 0$. This concludes the proof. \square