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A survey on the enumeration of classes of logical connectives and aggregation functions defined on a finite chain, with new results

Marc Munar^{a,b,*}, Miguel Couceiro^{c,d}, Sebastia Massanet^{a,b} and Daniel Ruiz-Aguilera^{a,b}

^aSoft Computing, Image Processing and Aggregation (SCOPIA) research group, Dept. of Mathematics and Computer Science, University of the Balearic Islands, Palma, 07122, Spain

^bHealth Research Institute of the Balearic Islands (IdISBa), Palma, 07010, Spain

^cUniversité de Lorraine, CNRS, LORIA, Nancy, 54000, France

^dINESC-ID, Instituto Superior Técnico, Universidade de Lisboa, Lisbon, 1000-029, Portugal

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ABSTRACT

The enumeration of logical connectives and aggregation functions defined on a finite chain has been a hot topic in the literature for the last decades. Multiple advantages can be derived from knowing a general formula about their cardinality, for instance, the ability to anticipate the computational cost required for generating operators with different properties. This is of paramount importance in image processing and decision making scenarios, where the identification of the most optimal operator is essential. Furthermore, it facilitates the examination of how constraining a certain property is in relation to its parent class. As a consequence, this paper aims to compile the main existing formulas and the methodologies with which they have been derived. Additionally, we introduce some novel formulas for the number of smooth discrete aggregation functions with neutral element or absorbing element, idempotent conjunctions, and commutative and idempotent conjunctions.

1. Introduction

The exploration of logical connectives and aggregation functions defined on the finite chain $L_n = \{0, 1, \dots, n\}$ has attracted the attention of the research community from its earliest stages. The finite nature of L_n renders it a well-suited framework for modeling ordinal and qualitative information in different applied scenarios. Noteworthy instances include image processing employing discrete mathematical morphology [1, 2] and decision-making processes that hinge upon a predefined set of linguistic labels [3, 4]. In these contexts, the operators under consideration possess finite domain and image sets, so a finite number of operators can be constructed for each n . Consequently, when confronted with the task of determining the optimal configuration of these operators, one possible approach involves generating the complete set of operators, evaluating them, and subsequently selecting the one that best aligns with some criteria. However, this approach has inherent drawbacks. First, the enumeration of the desired class in this way may not be possible, as the cardinality of this class can grow rapidly, even for lower values of n , fact already realized by De Baets and Mesiar in [5] trying to generate all possible discrete t-norms for $n \leq 13$, or Nachtegaele and Kerre in [6] by computationally counting discrete implications and some subclasses. Second, in cases where comprehensive generation is feasible, forecasting the computational resources required for both operator generation and subsequent processing is essential. Third, building upon the previous points, it may not be feasible to generate the entire operator class due to its extensive cardinality, but it might be possible to generate specific subclasses with additional properties. Then, it becomes evident that having a prior formula for the number of operators on L_n addresses several of these challenges. Moreover, this knowledge extends beyond mere enumeration, since it provides a deeper insight into the behavior of specific properties and how constraining the additional properties are in relation to its parent class.

*Corresponding author

✉ marc.munar@uib.es (M. Munar); miguel.couceiro@loria.fr (M. Couceiro); s.massanet@uib.es (S. Massanet); daniel.ruiz@uib.es (D. Ruiz-Aguilera)

Numerous papers in the literature have devoted their efforts to determine a closed formula for the enumeration of different classes of operators on L_n . In general terms, these enumerations have been approached from three distinct perspectives. The first perspective involves characterizing the class of operators, which serves to uncover its inherent structure, as exemplified in [7, 8, 9, 10, 11, 12]. This characterization often enables the establishment of relationships with other operators or structures for which enumeration formulas are previously known. The second perspective centers on the geometric attributes of the operator with some examples in [13, 14, 15, 16, 17]. This approach proves highly advantageous as it allows to relate theoretical properties with geometric characteristics and, in some instances, this relation simplifies the problem significantly. The third perspective revolves around establishing bijections between operators of L_n and other mathematical objects, even if those objects are not intrinsically related to L_n , showing some cases of this strategy in [18, 19, 20, 21]. It is important to note that these three approaches are not mutually exclusive, and they are frequently employed in combination with each other.

This paper thus serves a dual purpose. Firstly, a comprehensive review of the existing literature on the enumeration of operators defined on L_n and the results that lead to such enumerations is given, which can be considered as a continuation of [22], where a survey of operators on L_n and relations between them is offered. In doing so, we consolidate all known formulas into a unified notation. Furthermore, since some of these formulas are originated from other frameworks, such as the study of semigroups, we translate these operators and their enumeration to L_n . In cases where the enumeration of a particular class of operators remains unknown, we take a computational approach to count them, providing some initial values of the enumerations for future reference. Additionally, we verify that these enumerations are appropriately cataloged in the Online Encyclopedia of Integer Sequences (OEIS) [23], the main database for integer sequences. Regarding the second objective, we introduce some novel formulas not previously documented. Namely, (i) the enumeration of smooth discrete aggregation functions that have neutral element or absorbing element, (ii) idempotent conjunctions, and (iii) commutative and idempotent conjunctions.

The paper is organized as follows. It is structured into distinct sections, each dedicated to the examination of a specific class of operators, presenting the corresponding operator definitions and enumerations. We categorize these operators into three main groups. The first group focuses on unary operators, specifically discrete negations, and they are explored in Section 2. The second group delves into discrete aggregation functions that are surveyed in Section 3. We take an in-depth look at various subclasses, including discrete conjunctions and disjunctions in Section 3.2, discrete quasi-copulas and copulas in Section 3.3, discrete t-norms and t-conorms in Section 3.4, discrete uninorms in Section 3.5, and discrete t-operators and nullnorms in Section 3.6. The third category encompasses discrete implications and some additional properties of them, which are discussed in Section 4. Subsequently, in Section 5, we introduce some new enumerations. Namely, formulas for the cardinality of smooth discrete aggregation functions, having neutral element or absorbing element, idempotent discrete conjunctions, and commutative and idempotent discrete conjunctions. The paper then concludes in Section 6, where we offer some final thoughts and provide a summary of the different enumerations along with their cataloging in the OEIS.

2. Enumeration of discrete negations

Discrete negations are unary operators that extend the negation \neg in Boolean algebra. They possess undeniable utility, both on $[0, 1]$ and on L_n . Their significance lies not only in their capacity to embody this broader generalization but also in their potential to generate new classes of operators from other already established.

Definition 2.1. ([8, 22]) *A discrete negation is a unary operator $N : L_n \rightarrow L_n$ that satisfies the following properties:*

- (N1) N is decreasing; that is, $N(x) \geq N(y)$ for all $x, y \in L_n$ with $x \leq y$.
- (N2) $N(0) = n$ and $N(n) = 0$.

Apart from these two properties, additional properties of a discrete negation can be considered. One of them is that of being *strong* (or *involution*), i.e., when $N(N(x)) = x$ holds for all $x \in L_n$. In discrete settings,

it is straightforward to prove that a discrete negation is strong if, and only if, $N(x) = N_C(x) = n - x^1$ for all $x \in L_n$ [8], and therefore N_C (the classical discrete negation) is the only decreasing automorphism on L_n .

When we consider operators defined on $[0, 1]$, depending on the scenario, an interesting property to be considered is continuity, which is not available on L_n . However, several papers have coined the concept of smoothness on L_n (see [8, 24, 25]) and they have considered it as the counterpart of continuity on $[0, 1]$. For unary operators, its definition is the following.

Definition 2.2. ([8]) *A unary operator $f : L_n \rightarrow L_n$ is said to be k -smooth (or simply smooth, when $k = 1$) if*

$$|f(x + 1) - f(x)| \leq k, \quad (1)$$

for all $x \in L_n \setminus \{n\}$.

Because of condition (N2), it is also trivial to show that a discrete negation N is smooth if, and only if, $N(x) = N_C(x)$ for all $x \in L_n$.

The initial study that establishes the enumeration of discrete negations in general is attributed to Munar et al. in [19]. In this study, the authors proved the following formula grounded in elemental combinatorial arguments. To achieve it, they studied the number of monotone unary operators defined between two finite chains L_{m_1} and L_{m_2} .

Lemma 2.1. ([19, Lemma 3.2]) *Let $m_1, m_2 \in \mathbb{N}$ with $m_1, m_2 \geq 1$. Then, the number of monotone unary mappings $f : \{1, \dots, m_1\} \rightarrow \{1, \dots, m_2\}$ is*

$$\binom{m_1 + m_2 - 1}{m_2 - 1}. \quad (2)$$

Remark 2.1. *In the initial version of Lemma 2.1 presented in [19, Lemma 3.2], the condition $m_2 \geq m_1$ is established. Nevertheless, it can be proved in a straightforward manner that this condition is redundant, and its removal does not alter the validity of the lemma.*

Let \mathcal{N}_n be the set of all discrete negations defined on L_n . Using Lemma 2.1, the authors of [19] proposed a formula for $|\mathcal{N}_n|$.

Proposition 2.1. ([19, Proposition 3.1]) *Let $n \geq 1$ and let us denote by \mathcal{N}_n the set of discrete negations defined on L_n . Then,*

$$|\mathcal{N}_n| = \binom{2n - 1}{n}. \quad (3)$$

Mayor and Torrens [8] studied those unary operators $\delta : L_n \rightarrow L_n$, which are (i) monotone increasing, (ii) $\delta(x) \leq x$ for all $x \in L_n$, and (iii) satisfy the boundary conditions $\delta(0) = 0$ and $\delta(n) = n$. They referred to this class of operators as *diagonal sections*, an appellation inspired by their exploration of diagonal sections of binary operators, and they identified in [8, Remark 7.3.5] a formula for the cardinality of the set of diagonal sections, for all $n \geq 1$.

Although not intrinsically linked to discrete negations, it is evident that the operator $N_\delta(x) = n - \delta(x)$, for all $x \in L_n$, is a discrete negation satisfying $N(x) \geq n - x = N_C(x)$, which up to our knowledge have not yet been studied in the literature, and we will refer to them as *over anti-diagonal discrete negations*. Therefore, a bijection between all diagonal sections and over anti-diagonal discrete negations can be established, preserving its cardinality.

Proposition 2.2. ([8, Remark 7.3.5]) *Let $n \geq 1$ and let us denote by $\mathcal{N}_n^{\text{OAD}}$ the set of over anti-diagonal discrete negations defined on L_n . Then,*

$$|\mathcal{N}_n^{\text{OAD}}| = \frac{1}{n + 1} \binom{2n}{n}, \quad (4)$$

which is the n -th Catalan number.

In Table 1 we illustrate for $n \leq 13$ the values of $|\mathcal{N}_n|$ and $|\mathcal{N}_n^{\text{OAD}}|$. To conclude this section, recalling the properties of being smooth or involutive, if $\mathcal{N}_n^{\text{SMT}}$ denotes the set of discrete negations that are smooth, and $\mathcal{N}_n^{\text{INV}}$ denotes the set of discrete negations that are involutive, then $\mathcal{N}_n^{\text{SMT}} = \mathcal{N}_n^{\text{INV}} = \{N_C\}$ for all $n \geq 1$, so $|\mathcal{N}_n^{\text{SMT}}| = |\mathcal{N}_n^{\text{INV}}| = 1$, for all $n \geq 1$.

¹Without loss of generality, we may assume that L_n is equipped with the usual arithmetic of the integer numbers.

n	$ \mathcal{N}_n^*$	$ \mathcal{N}_n^{\text{OAD}}$
1	$1 = 1 \cdot 10^0$	$1 = 2 \cdot 10^0$
2	$3 = 3 \cdot 10^0$	$2 = 2 \cdot 10^0$
3	$10 = 1 \cdot 10^1$	$5 = 5 \cdot 10^0$
4	$35 = 3.5 \cdot 10^1$	$14 = 1.4 \cdot 10^1$
5	$126 = 1.26 \cdot 10^2$	$42 = 4.2 \cdot 10^1$
6	$462 = 4.62 \cdot 10^2$	$132 = 1.32 \cdot 10^2$
7	$1,716 = 1.716 \cdot 10^3$	$429 = 4.29 \cdot 10^2$
8	$6,435 = 6.435 \cdot 10^3$	$1,430 = 1.43 \cdot 10^3$
9	$24,310 = 2.431 \cdot 10^4$	$4,862 = 4.862 \cdot 10^3$
10	$92,378 = 9.2378 \cdot 10^4$	$16,796 = 1.6796 \cdot 10^4$
11	$\approx 3.5272 \cdot 10^5$	$58,786 = 5.8786 \cdot 10^4$
12	$\approx 1.3521 \cdot 10^6$	$\approx 2.0801 \cdot 10^5$
13	$\approx 5.2003 \cdot 10^6$	$\approx 7.429 \cdot 10^5$

Table 1

Enumeration of discrete negations, given by Proposition 2.1, and over anti-diagonal discrete negations, given by Proposition 2.2, for $n \leq 13$.

3. Enumeration of discrete aggregation functions and some subclasses

In this section, we focus on the known formulas for the cardinality of some classes of discrete aggregation functions. In Section 3.1 we recall the concept of discrete aggregation function, its main properties and some existing enumerations. Then, in Sections 3.2, 3.3, 3.4, 3.5 and 3.6 we discuss enumerations of the main classes of discrete aggregation functions.

3.1. Discrete aggregation functions

We start by revisiting discrete aggregation functions. They are defined as monotone binary operations with two boundary conditions.

Definition 3.1. ([22, 26, 27]) *A discrete aggregation function is a binary operator $F : L_n^2 \rightarrow L_n$ which satisfies the following properties:*

- (A1) F is increasing in each argument, that is, $F(x, y) \leq F(x', y')$, for all $x, y, x', y' \in L_n$ with $x \leq x'$ and $y \leq y'$.
- (A2) $F(0, 0) = 0$ and $F(n, n) = n$.

Since this definition is sufficiently generic, some additional properties have been proposed in the literature, which are recalled in the next definition.

Definition 3.2. ([22, 28]) *Let $F : L_n^2 \rightarrow L_n$ be a discrete aggregation function. Then, F is said to be*

- (i) commutative if $F(x, y) = F(y, x)$, for all $x, y \in L_n$;
- (ii) associative if $F(F(x, y), z) = F(x, F(y, z))$, for all $x, y, z \in L_n$;
- (iii) bisymmetric if $F(F(x, y), F(u, v)) = F(F(x, u), F(y, v))$, for all $x, y, u, v \in L_n$;
- (iv) idempotent if $F(x, x) = x$, for all $x \in L_n$, and idempotent-free if $F(x, x) \neq x$ for all $x \in L_n \setminus \{0, n\}$;
- (v) quasitrivial if $F(x, y) \in \{x, y\}$, for all $x, y \in L_n$.

Moreover, F satisfies

- (vi) the 1-Lipschitz condition if $F(z, y) - F(x, y) \leq z - x$ for all $x, y, z \in L_n$ with $x \leq z$;
- (vii) the intermediate value theorem; that is, if $z = F(x, y)$ and $z' = F(x, y')$ with $z < z'$, then for all $z'' \in [z, z']$ there exists some $y'' \in [y, y']$ such that $z'' = F(x, y'')$.

Remark 3.1. *Of the different properties mentioned in Definition 3.2, there are different terminologies to refer to the quasitrivial property. While in this paper we will refer to it as quasitrivial, in the literature there are papers where it is referred to as internal, as in [29, 30], locally internal in [31] or conservative in [16].*

Apart from these properties, the presence of a neutral element and an absorbing element (also known as annihilator) have also been considered. Then, given a binary operator $F : L_n^2 \rightarrow L_n$, an element $e \in L_n$ is said to be a *neutral element of F* if $F(x, e) = F(e, x) = x$, for all $x \in L_n$, and an element $a \in L_n$ is said to be an *absorbing element of F* if $F(x, a) = F(a, x) = a$, for all $x \in L_n$. With these characteristics, it is evident that a neutral and an absorbing element cannot be equal, but they can coexist in the same operator.

Before proceeding with the compilation of known enumerations, it is worth noting that another property of substantial significance for any binary operator, is the concept of smoothness. Much like the smoothness observed in unary operators, smoothness for binary operators is commonly regarded as the property analogous to the continuity of binary operators defined on $[0, 1]$.

Definition 3.3. ([8, 22]) *A binary operator $F : L_n^2 \rightarrow L_n$ is said to be k -smooth (or simply smooth when $k = 1$) if it is k -smooth in each argument; that is, if*

$$|F(x + 1, y) - F(x, y)| \leq k,$$

for all $x \in L_n \setminus \{n\}$ and $y \in L_n$, and

$$|F(x, y + 1) - F(x, y)| \leq k,$$

for all $x \in L_n$ and $y \in L_n \setminus \{n\}$.

A first milestone in obtaining a general formula for the number of discrete aggregation functions was proposed by Munar et al. in [19], where the formula they proposed is based on plane partitions. These objects, initially introduced by MacMahon in [32, 33] and later studied in depth by Stanley in [34], can be understood, intuitively, as a two-dimensional array of integers satisfying certain growth constraints. Their formal definition is the following.

Definition 3.4. ([32, 33, 34]) *A plane partition π with shape $\Lambda = (\lambda_1, \dots, \lambda_r)$ is a two-dimensional array of positive integers such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r$, and represented by*

$$\begin{array}{ccccccc} \pi_{r,1} & \cdots & \pi_{r,\lambda_r} & & & & \\ \vdots & \ddots & & \ddots & & & \\ \pi_{2,1} & \pi_{2,2} & \cdots & \cdots & \pi_{2,\lambda_2} & & \\ \pi_{1,1} & \pi_{1,2} & \cdots & \cdots & \cdots & \cdots & \pi_{1,\lambda_1} \end{array}$$

where each row and column are decreasing; that is, $\pi_{i,j} \geq \pi_{i,j+1}$ and $\pi_{i,j} \geq \pi_{i+1,j}$, whenever both sides are defined.

The enumeration of this object, as it appears in Definition 3.4, is not possible. Although the set of points (i, j) that index the plane partition is finite and the components i and j are bounded, it should be noted that the values that each $\pi_{i,j}$ can take need not be bounded. For this reason, the concept of (a, b, c) -cubic plane partition arises, as the one that satisfies that $r = a$, $\lambda_1 \leq b$ and $1 \leq \pi_{i,j} \leq c$, for all $1 \leq i \leq a$ and $1 \leq j \leq \lambda_i$. MacMahon found a formula for the number of this class of plane partitions [32, 33], which is given by the triple product

$$\prod_{i=1}^a \prod_{j=1}^b \prod_{k=1}^c \frac{i+j+k-1}{i+j+k-2}. \quad (5)$$

Let \mathcal{A}_n denote the set of discrete aggregation functions defined on L_n . In [19], the authors performed a decomposition of \mathcal{A}_n into several mutually disjoint subsets of operators, and established a bijection with different types of (a, b, c) -cubic plane partitions, allowing to establish the cardinality, as shown in the following result.

Proposition 3.1. ([19, Proposition 4.4]) Let $n \geq 1$, and let us denote by \mathcal{A}_n the set of discrete aggregation functions defined on L_n . Then,

$$|\mathcal{A}_n| = \prod_{i=1}^{n+1} \prod_{j=1}^{n+1} \prod_{k=1}^n \frac{i+j+k-1}{i+j+k-2} - 2 \prod_{i=1}^{n+1} \prod_{j=1}^{n+1} \prod_{k=1}^{n-1} \frac{i+j+k-1}{i+j+k-2} + \prod_{i=1}^{n+1} \prod_{j=1}^{n+1} \prod_{k=1}^{n-2} \frac{i+j+k-1}{i+j+k-2}, \quad (6)$$

for all $n \geq 2$, and $|\mathcal{A}_1| = 4$.

Remark 3.2. Equation (6) comprises three nested products which are defined for $n \geq 2$. When $n = 2$, the final term in the sum entails a product in which the upper bound is less than the lower bound. In such instances, this product is conventionally treated as equivalent to 1.

After obtaining a general formula of $|\mathcal{A}_n|$, a particularly interesting line of research arises: the examination of the cardinality of various properties outlined in Definition 3.2. Regarding the commutativity, and building upon plane partitions, Munar et al. in [19] presented a formulation for determining the number of commutative discrete aggregation functions. This formulation employs the concept of *symmetric plane partitions*; that is, a plane partition π satisfying $\pi_{i,j} = \pi_{j,i}$, whenever both sides are defined. MacMahon in his seminal works [32, 33] conjectured a formula for the enumeration of symmetric (a, a, c) -cubic plane partitions, but it was Andrews in [35, 36] who gave a complete proof of the formula. This formula is given by

$$\prod_{i=1}^a \left(\frac{2i+c-1}{2i-1} \cdot \prod_{j=i+1}^a \frac{i+j+c-1}{i+j-1} \right). \quad (7)$$

With a similar strategy, the authors decomposed $|\mathcal{A}_n^{\text{COM}}|$ (the set of commutative discrete aggregation functions) into different mutually disjoint sets, and established bijections with some symmetric (a, a, c) -cubic plane partitions, leading to a general formula.

Proposition 3.2. ([19, Proposition 4.5]) Let $n \geq 1$, and let us denote by $\mathcal{A}_n^{\text{COM}}$ the set of commutative discrete aggregation functions defined on L_n . Then,

$$\begin{aligned} |\mathcal{A}_n^{\text{COM}}| = & \prod_{i=1}^{n+1} \left(\frac{2i+n-1}{2i-1} \cdot \prod_{j=i+1}^{n+1} \frac{i+j+n-1}{i+j-1} \right) - 2 \prod_{i=1}^{n+1} \left(\frac{2i+n-2}{2i-1} \cdot \prod_{j=i+1}^{n+1} \frac{i+j+n-2}{i+j-1} \right) \\ & + \prod_{i=1}^{n+1} \left(\frac{2i+n-3}{2i-1} \cdot \prod_{j=i+1}^{n+1} \frac{i+j+n-3}{i+j-1} \right). \end{aligned} \quad (8)$$

Now, let us direct our attention to the second property shown in Definition 3.2, namely, associativity. Before we proceed further, we want to note an observation. Typically, when formulating formulas for establishing the enumeration of a specific mathematical object, it is customary to translate its properties into geometric characteristics which tend to be more intuitive and, in the vast majority of cases, this enable an easier enumeration process. A paradigmatic illustration of this approach is the enumeration of plane partitions. Interestingly, certain properties lack a known geometric interpretation, associativity being one such example. In fact, a complete characterization of associativity in terms of properties that do not entail a functional equation remains an open challenge in the literature. Several studies have initiated investigations into the general characterization of associativity, as evidenced by some examples in [13, 17, 37, 38]. In [14, 16], it has been recognized that enumerating binary operators fulfilling associativity on its own presents challenges. Nonetheless, they highlight that in combination with other properties, it becomes feasible to characterize the class of operators and establish its enumeration.

We thus turn our attention to [16], where Couceiro et al. scrutinized jointly associativity and quasitriviality on discrete aggregation functions. It is important to emphasize that their approach diverged from the perspective of operators on L_n , instead focusing on binary operators defined on an arbitrary set X (that can be a finite chain), which subsequently yield a semigroup (see [39] for more information about this structure). They establish equivalences between associativity and quasitriviality properties of

a binary operator $F : X_n^2 \rightarrow X_n$ (which is not necessarily monotone) defined on $X_n = \{1, 2, \dots, n\}$, and the representation of F concerning weak orderings induced by F , as proved in [16, Theorem 2.1]. When they impose conditions related to monotonicity, these operators become associative and quasitrivial discrete aggregation functions on L_n (which is isomorphic to X_{n+1}) since quasitriviality ensures that $F(0, 0) = 0$ and $F(n, n) = n$, and they are able to characterize these operators using the framework of weak orderings, as outlined in [16, Theorem 4.5]. These orderings establish connections with the contour lines of the operators and, hence, with their geometry. A similar process was carried out by Kiss in [17] which, chronologically, is simultaneous to [16]. He also characterized these operators, but from a point of view that only involves contour lines. Subsequently, through the enumeration of weak orderings and contour lines, they are able to determine the cardinality of associative and quasitrivial discrete aggregation functions, and some subclasses.

Proposition 3.3. *Let $n \geq 1$, and let us denote by*

- $\mathcal{A}_n^{\text{ASV}}$, *the set of associative discrete aggregation functions defined on L_n ;*
- $\mathcal{A}_n^{\text{QST}}$, *the set of quasitrivial discrete aggregation functions defined on L_n ;*
- $\mathcal{A}_n^{\text{NE}}$, *the set of discrete aggregation functions defined on L_n having some neutral element;*
- $\mathcal{A}_n^{\text{ABS}}$, *the set of discrete aggregation functions defined on L_n having some absorbing element.*

Then, the following statements hold:

- (i) ([16, Proposition 4.11], [17, Theorem 6.1]) *The number of associative and quasitrivial discrete aggregation functions, $|\mathcal{A}_n^{\text{ASV}} \cap \mathcal{A}_n^{\text{QST}}|$, satisfies the recurrence equation*

$$|\mathcal{A}_n^{\text{ASV}} \cap \mathcal{A}_n^{\text{QST}}| - 2 \cdot |\mathcal{A}_{n-1}^{\text{ASV}} \cap \mathcal{A}_{n-1}^{\text{QST}}| - 2 \cdot |\mathcal{A}_{n-2}^{\text{ASV}} \cap \mathcal{A}_{n-2}^{\text{QST}}| = 2, \quad (9)$$

for all $n \geq 3$, with $|\mathcal{A}_1^{\text{ASV}} \cap \mathcal{A}_1^{\text{QST}}| = 4$ and $|\mathcal{A}_2^{\text{ASV}} \cap \mathcal{A}_2^{\text{QST}}| = 12$. Therefore, its general formula is given by

$$\begin{aligned} |\mathcal{A}_n^{\text{ASV}} \cap \mathcal{A}_n^{\text{QST}}| &= \sum_{k=0}^{\lfloor n+1/2 \rfloor} 2 \cdot 3^{k-1} \cdot \binom{n+1}{2k} + \sum_{k=0}^{\lfloor n/2 \rfloor} 3^k \cdot \binom{n+1}{2k+1} - \frac{2}{3} \\ &= \frac{2 + \sqrt{3}}{6} (1 + \sqrt{3})^{n+1} + \frac{2 - \sqrt{3}}{6} (1 - \sqrt{3})^{n+1} - \frac{2}{3}. \end{aligned} \quad (10)$$

- (ii) ([16, Proposition 4.12], [17, Theorem 6.1]) *The number of associative and quasitrivial discrete aggregation functions with neutral element, $|\mathcal{A}_n^{\text{ASV}} \cap \mathcal{A}_n^{\text{QST}} \cap \mathcal{A}_n^{\text{NE}}|$, satisfies the recurrence equation*

$$|\mathcal{A}_n^{\text{ASV}} \cap \mathcal{A}_n^{\text{QST}} \cap \mathcal{A}_n^{\text{NE}}| - 2 \cdot |\mathcal{A}_{n-1}^{\text{ASV}} \cap \mathcal{A}_{n-1}^{\text{QST}} \cap \mathcal{A}_{n-1}^{\text{NE}}| - 2 \cdot |\mathcal{A}_{n-2}^{\text{ASV}} \cap \mathcal{A}_{n-2}^{\text{QST}} \cap \mathcal{A}_{n-2}^{\text{NE}}| = 0, \quad (11)$$

for all $n \geq 3$, with $|\mathcal{A}_1^{\text{ASV}} \cap \mathcal{A}_1^{\text{QST}} \cap \mathcal{A}_1^{\text{NE}}| = 2$ and $|\mathcal{A}_2^{\text{ASV}} \cap \mathcal{A}_2^{\text{QST}} \cap \mathcal{A}_2^{\text{NE}}| = 6$. Therefore, its general formula is given by

$$\begin{aligned} |\mathcal{A}_n^{\text{ASV}} \cap \mathcal{A}_n^{\text{QST}} \cap \mathcal{A}_n^{\text{NE}}| &= \sum_{k=0}^{\lfloor n/2 \rfloor} 3^k \cdot \binom{n+1}{2k+1} \\ &= \frac{\sqrt{3}}{6} (1 + \sqrt{3})^{n+1} - \frac{\sqrt{3}}{6} (1 - \sqrt{3})^{n+1}. \end{aligned} \quad (12)$$

- (iii) ([16, Proposition 4.12]) *The number of associative and quasitrivial discrete aggregation functions with absorbing element, $|\mathcal{A}_n^{\text{ASV}} \cap \mathcal{A}_n^{\text{QST}} \cap \mathcal{A}_n^{\text{ABS}}|$, satisfies the recurrence relation*

$$|\mathcal{A}_n^{\text{ASV}} \cap \mathcal{A}_n^{\text{QST}} \cap \mathcal{A}_n^{\text{ABS}}| = 2 \cdot |\mathcal{A}_{n-1}^{\text{ASV}} \cap \mathcal{A}_{n-1}^{\text{QST}}|, \quad (13)$$

for all $n \geq 2$, with $|\mathcal{A}_1^{\text{ASV}} \cap \mathcal{A}_1^{\text{QST}} \cap \mathcal{A}_1^{\text{ABS}}| = 2$. Therefore, its general formula is given by

$$|\mathcal{A}_n^{\text{ASV}} \cap \mathcal{A}_n^{\text{QST}} \cap \mathcal{A}_n^{\text{ABS}}| = \frac{2 + \sqrt{3}}{3} (1 + \sqrt{3})^n + \frac{2 - \sqrt{3}}{3} (1 - \sqrt{3})^n - \frac{4}{3}. \quad (14)$$

(iv) ([16, Proposition 4.12]) The number of associative and quasitrivial discrete aggregation functions with neutral and absorbing elements, $|\mathcal{A}_n^{\text{ASV}} \cap \mathcal{A}_n^{\text{QST}} \cap \mathcal{A}_n^{\text{NE}} \cap \mathcal{A}_n^{\text{ABS}}|$, satisfies the recurrence relation

$$|\mathcal{A}_n^{\text{ASV}} \cap \mathcal{A}_n^{\text{QST}} \cap \mathcal{A}_n^{\text{NE}} \cap \mathcal{A}_n^{\text{ABS}}| = 2 \cdot |\mathcal{A}_{n-1}^{\text{ASV}} \cap \mathcal{A}_{n-1}^{\text{QST}} \cap \mathcal{A}_{n-1}^{\text{NE}}|, \quad (15)$$

for all $n \geq 2$, with $|\mathcal{A}_1^{\text{ASV}} \cap \mathcal{A}_1^{\text{QST}} \cap \mathcal{A}_1^{\text{NE}} \cap \mathcal{A}_1^{\text{ABS}}| = 2$. Therefore, its general formula is given by

$$|\mathcal{A}_n^{\text{ASV}} \cap \mathcal{A}_n^{\text{QST}} \cap \mathcal{A}_n^{\text{NE}} \cap \mathcal{A}_n^{\text{ABS}}| = \frac{\sqrt{3}}{3} (1 + \sqrt{3})^n - \frac{\sqrt{3}}{3} (1 - \sqrt{3})^n. \quad (16)$$

Another property that presents an added layer of complexity is bisymmetry, which is also defined in terms of a functional equation. Much like associativity, bisymmetry lacks a known geometric interpretation. Interestingly, associativity and bisymmetry exhibit a close relationship.

Given any binary operator $F : L_n^2 \rightarrow L_n$, not necessarily monotone, we have: if it is both commutative and quasitrivial, associativity and bisymmetry become equivalent properties [28, Corollary 2.9]. If the operator is associative and commutative, it is bisymmetric [14, Lemma 22]. Conversely, if it is bisymmetric and possesses a neutral element, then it is associative and commutative [14, Lemma 22]. The bisymmetry is thoroughly examined by Devillet in [28], exploring it from the perspective of orders and contour lines. Additionally, Kiss delves into this subject in [17], keeping the focus on contour lines. Both authors characterize these operators and introduce methods for their enumeration.

Proposition 3.4. Let $n \geq 1$, and let us denote by

- $\mathcal{A}_n^{\text{BSM}}$, the set of bisymmetric discrete aggregation functions defined on L_n ;
- $\mathcal{A}_n^{\text{QST}}$, the set of quasitrivial discrete aggregation functions defined on L_n ;
- $\mathcal{A}_n^{\text{NE}}$, the set of discrete aggregation functions defined on L_n having some neutral element;
- $\mathcal{A}_n^{\text{ABS}}$, the set of discrete aggregation functions defined on L_n having some absorbing element.

Then, the following statements are true:

(i) ([28, Proposition 4.6], [17, Proposition 6.5]) The number of bisymmetric and quasitrivial discrete aggregation functions, $|\mathcal{A}_n^{\text{BSM}} \cap \mathcal{A}_n^{\text{QST}}|$, satisfies the recurrence equation

$$|\mathcal{A}_n^{\text{BSM}} \cap \mathcal{A}_n^{\text{QST}}| = 2 \cdot |\mathcal{A}_{n-1}^{\text{BSM}} \cap \mathcal{A}_{n-1}^{\text{QST}}| + 2, \quad (17)$$

for all $n \geq 2$, with $|\mathcal{A}_1^{\text{BSM}} \cap \mathcal{A}_1^{\text{QST}}| = 4$. Therefore, its general formula is given by

$$|\mathcal{A}_n^{\text{BSM}} \cap \mathcal{A}_n^{\text{QST}}| = 3 \cdot 2^n - 2. \quad (18)$$

(ii) ([28, Proposition 4.6], [17, Proposition 6.5]) The number of bisymmetric and quasitrivial discrete aggregation functions with neutral element, $|\mathcal{A}_n^{\text{BSM}} \cap \mathcal{A}_n^{\text{QST}} \cap \mathcal{A}_n^{\text{NE}}|$, is given by

$$|\mathcal{A}_n^{\text{BSM}} \cap \mathcal{A}_n^{\text{QST}} \cap \mathcal{A}_n^{\text{NE}}| = 2^n. \quad (19)$$

n	$ \mathcal{A}_n $	$ \mathcal{A}_n^{\text{COM}} $	$ \mathcal{A}_n^{\text{BSM}} \cap \mathcal{A}_n^{\text{QST}} $	$ \mathcal{A}_n^{\text{BSM}} \cap \mathcal{A}_n^{\text{QST}} \cap \mathcal{A}_n^{\text{NE}} $	$ \mathcal{A}_n^{\text{BSM}} \cap \mathcal{A}_n^{\text{QST}} \cap \mathcal{A}_n^{\text{ABS}} $
1	$4 = 4 \cdot 10^0$	$2 = 2 \cdot 10^0$	$4 = 4 \cdot 10^0$	$2 = 2 \cdot 10^0$	$2 = 1 \cdot 10^0$
2	$136 = 1.36 \cdot 10^2$	$20 = 2 \cdot 10^1$	$10 = 1 \cdot 10^1$	$4 = 4 \cdot 10^0$	$8 = 8 \cdot 10^0$
3	$21,238 = 2.1238 \cdot 10^4$	$436 = 4.36 \cdot 10^2$	$22 = 2.2 \cdot 10^1$	$8 = 8 \cdot 10^0$	$20 = 2.05 \cdot 10^1$
4	$\approx 1.5374 \cdot 10^7$	$20,328 = 2.0328 \cdot 10^4$	$46 = 4.6 \cdot 10^1$	$16 = 1.6 \cdot 10^1$	$44 = 4.4 \cdot 10^1$
5	$\approx 5.2327 \cdot 10^{10}$	$\approx 2.0315 \cdot 10^6$	$94 = 9.4 \cdot 10^1$	$32 = 3.2 \cdot 10^1$	$92 = 9.2 \cdot 10^1$
6	$\approx 8.4502 \cdot 10^{14}$	$\approx 4.372 \cdot 10^8$	$190 = 1.9 \cdot 10^2$	$64 = 6.4 \cdot 10^1$	$188 = 1.88 \cdot 10^2$
7	$\approx 6.5103 \cdot 10^{19}$	$\approx 2.0348 \cdot 10^{11}$	$382 = 3.82 \cdot 10^2$	$128 = 1.28 \cdot 10^2$	$380 = 3.8 \cdot 10^2$
8	$\approx 2.4006 \cdot 10^{25}$	$\approx 2.0548 \cdot 10^{14}$	$766 = 7.66 \cdot 10^2$	$256 = 2.56 \cdot 10^2$	$764 = 7.64 \cdot 10^2$
9	$\approx 4.2446 \cdot 10^{31}$	$\approx 4.5137 \cdot 10^{17}$	$1,534 = 1.534 \cdot 10^3$	$512 = 5.12 \cdot 10^2$	$1,532 = 1.532 \cdot 10^3$
10	$\approx 3.6026 \cdot 10^{38}$	$\approx 2.1607 \cdot 10^{21}$	$3,070 = 3.07 \cdot 10^3$	$1,024 = 1.024 \cdot 10^3$	$3,068 = 3.068 \cdot 10^3$
11	$\approx 1.4687 \cdot 10^{46}$	$\approx 2.2573 \cdot 10^{25}$	$6,142 = 6.142 \cdot 10^3$	$2,048 = 2.048 \cdot 10^3$	$6,140 = 6.14 \cdot 10^3$
12	$\approx 2.8769 \cdot 10^{54}$	$\approx 5.1522 \cdot 10^{29}$	$12,286 = 1.2286 \cdot 10^4$	$4,096 = 4.096 \cdot 10^3$	$12,284 = 1.2284 \cdot 10^4$
13	$\approx 2.7083 \cdot 10^{63}$	$\approx 2.5713 \cdot 10^{34}$	$24,574 = 2.4574 \cdot 10^4$	$8,192 = 8.192 \cdot 10^3$	$24,572 = 2.4572 \cdot 10^4$

n	$ \mathcal{A}_n^{\text{ASV}} \cap \mathcal{A}_n^{\text{QST}} $	$ \mathcal{A}_n^{\text{ASV}} \cap \mathcal{A}_n^{\text{QST}} \cap \mathcal{A}_n^{\text{NE}} $	$ \mathcal{A}_n^{\text{ASV}} \cap \mathcal{A}_n^{\text{QST}} \cap \mathcal{A}_n^{\text{ABS}} $	$ \mathcal{A}_n^{\text{ASV}} \cap \mathcal{A}_n^{\text{QST}} \cap \mathcal{A}_n^{\text{NE}} \cap \mathcal{A}_n^{\text{ABS}} $
1	$4 = 4 \cdot 10^0$	$2 = 2 \cdot 10^0$	$2 = 2 \cdot 10^0$	$2 = 2 \cdot 10^0$
2	$12 = 1.2 \cdot 10^1$	$6 = 6 \cdot 10^0$	$8 = 8 \cdot 10^0$	$4 = 4 \cdot 10^0$
3	$34 = 3.4 \cdot 10^1$	$16 = 1.6 \cdot 10^1$	$24 = 2.4 \cdot 10^1$	$12 = 1.2 \cdot 10^1$
4	$94 = 9.4 \cdot 10^1$	$44 = 4.4 \cdot 10^1$	$68 = 6.8 \cdot 10^1$	$32 = 3.2 \cdot 10^1$
5	$258 = 2.58 \cdot 10^2$	$120 = 1.2 \cdot 10^2$	$188 = 1.88 \cdot 10^2$	$88 = 8.8 \cdot 10^1$
6	$706 = 7.06 \cdot 10^2$	$328 = 3.28 \cdot 10^2$	$516 = 5.16 \cdot 10^2$	$240 = 2.4 \cdot 10^2$
7	$1,930 = 1.93 \cdot 10^3$	$896 = 8.96 \cdot 10^2$	$1,412 = 1.412 \cdot 10^3$	$656 = 6.56 \cdot 10^2$
8	$5,274 = 5.274 \cdot 10^3$	$2,448 = 2.448 \cdot 10^3$	$3,860 = 3.86 \cdot 10^3$	$1,792 = 1.792 \cdot 10^3$
9	$14,410 = 1.441 \cdot 10^4$	$6,688 = 6.688 \cdot 10^3$	$10,548 = 1.0548 \cdot 10^4$	$4,896 = 4.896 \cdot 10^3$
10	$39,370 = 3.937 \cdot 10^4$	$18,272 = 1.8272 \cdot 10^4$	$28,820 = 2.882 \cdot 10^4$	$13,376 = 1.3376 \cdot 10^4$
11	$\approx 1.0756 \cdot 10^5$	$49,920 = 4.992 \cdot 10^4$	$78,740 = 7.874 \cdot 10^4$	$36,544 = 3.6544 \cdot 10^4$
12	$\approx 2.9387 \cdot 10^5$	$\approx 1.3638 \cdot 10^5$	$\approx 2.1512 \cdot 10^5$	$99,840 = 9.984 \cdot 10^4$
13	$\approx 8.0286 \cdot 10^5$	$\approx 3.7261 \cdot 10^5$	$\approx 5.8773 \cdot 10^5$	$\approx 2.7277 \cdot 10^5$

Table 2

Enumeration of some classes of discrete aggregation functions for $n \leq 13$. From top to bottom, and left to right: discrete aggregation functions in general, given by Proposition 3.1; commutative discrete aggregation functions, given by Proposition 3.2; bisymmetric and quasitrivial discrete aggregation functions, bisymmetric and quasitrivial discrete aggregation functions with neutral element and bisymmetric and quasitrivial discrete aggregation functions with absorbing element, given by Proposition 3.4; and finally associative and quasitrivial discrete aggregation functions, associative and quasitrivial discrete aggregation functions with neutral element, associative and quasitrivial discrete aggregation functions with absorbing element, and associative and quasitrivial discrete aggregation functions with neutral and absorbing elements, given by Proposition 3.3.

(iii) ([28, Proposition 4.6]) *The number of bisymmetric and quasitrivial discrete aggregation functions with absorbing element, $|\mathcal{A}_n^{\text{BSM}} \cap \mathcal{A}_n^{\text{QST}} \cap \mathcal{A}_n^{\text{ABS}}|$, satisfies the recurrence equation*

$$|\mathcal{A}_n^{\text{BSM}} \cap \mathcal{A}_n^{\text{QST}} \cap \mathcal{A}_n^{\text{ABS}}| = |\mathcal{A}_n^{\text{BSM}} \cap \mathcal{A}_n^{\text{QST}}| - 2, \quad (20)$$

for all $n \geq 1$, with $|\mathcal{A}_1^{\text{BSM}} \cap \mathcal{A}_1^{\text{QST}} \cap \mathcal{A}_1^{\text{ABS}}| = 2$. Therefore, its general formula is given by

$$|\mathcal{A}_n^{\text{BSM}} \cap \mathcal{A}_n^{\text{QST}} \cap \mathcal{A}_n^{\text{ABS}}| = 3 \cdot 2^n - 4. \quad (21)$$

In Table 2 we list the first 13 values of all the formulas that have appeared in this section.

3.2. Discrete conjunctions and disjunctions

As noteworthy examples of discrete aggregation functions, we have the so-called discrete conjunctions and discrete disjunctions.

Definition 3.5. ([22]) A discrete conjunction is a discrete aggregation function $C : L_n^2 \rightarrow L_n$ with boundary conditions $C(0, n) = C(n, 0) = 0$.

From this definition, it is straightforward to verify that every discrete conjunction has 0 as absorbing element. On the other hand, we have the concept of discrete disjunction, which has n as absorbing element.

Definition 3.6. ([22]) A discrete disjunction is a discrete aggregation function $D : L_n^2 \rightarrow L_n$ with boundary conditions $D(0, n) = D(n, 0) = n$.

Remark 3.3. Some sources in the literature, such as [27, 40, 41], consider conjunctions as those aggregation functions that are upper bounded by the minimum. However, in this paper (as in [22]) we have chosen to give a more generic definition, which encompasses the definition given in [27, 40, 41]. In the same way, in these sources disjunctions are considered as those aggregation functions which are lower bounded by the maximum.

The discovery of a general formula for the number of discrete conjunctions is attributed to Munar et al. in [19]. Similar to the formulas involved in Propositions 3.1 and 3.2 for discrete aggregation functions, they again used the concept of plane partition to derive the formula for their enumeration. For conjunctions, it is the following one.

Proposition 3.5. ([19, Proposition 4.3]) Let $n \geq 1$, and let us denote by C_n the set of discrete conjunctions defined on L_n . Then,

$$|C_n| = \prod_{i=1}^n \prod_{j=1}^n \prod_{k=1}^n \frac{i+j+k-1}{i+j+k-2} - \prod_{i=1}^n \prod_{j=1}^n \prod_{k=1}^{n-1} \frac{i+j+k-1}{i+j+k-2}. \quad (22)$$

They derived this formula by employing the same procedure used to obtain the formula of Proposition 3.1. The set C_n is decomposed into mutually disjoint sets, each of which corresponds to a specific set of (a, b, c) -cubic plane partitions. Similarly, the formula for the enumeration of commutative discrete conjunctions is obtained by a similar process, but based on Equation (7), which provides the number of symmetric (a, a, c) -cubic plane partitions.

Proposition 3.6. ([19, Proposition 4.5]) Let $n \geq 1$, and let us denote by C_n^{COM} the set of commutative discrete conjunctions defined on L_n . Then,

$$|C_n^{\text{COM}}| = \prod_{i=1}^n \left(\frac{2i+n-1}{2i-1} \cdot \prod_{j=i+1}^n \frac{i+j+n-1}{i+j-1} \right) - \prod_{i=1}^n \left(\frac{2i+n-2}{2i-1} \cdot \prod_{j=i+1}^n \frac{i+j+n-2}{i+j-1} \right). \quad (23)$$

Still building upon the framework of plane partitions, Bejines and Ojeda-Hernández presented in [42] a formula for the number of discrete conjunctions having neutral element n , which they termed *semicopulas* from the perspective of *totally symmetric plane partitions*, originally introduced by Mills et al. in [43]. It is worth highlighting that the literature encompasses a significant variety of plane partition types and related structures (refer to [44] for more information). The authors devoted a whole paper to establishing a bijection between discrete conjunctions with neutral element n and these plane partitions, so we omit the details of this bijection. The formula under discussion is as follows.

Proposition 3.7. ([42, Corollary 17]) Let $n \geq 1$, and let us denote by $C_n^{\text{NE}(n)}$ the set of discrete conjunctions with neutral element n defined on L_n . Then,

$$|C_n^{\text{NE}(n)}| = \prod_{i=0}^{n-1} \frac{(3i+1)(6i)!(2i)!}{(4i+1)!(4i)!}. \quad (24)$$

Another noteworthy property is smoothness, which was fully characterized for discrete conjunctions operators by Munar et al. in [20] using the concept of alternating sign matrix. This type of matrices were introduced by Robbins and Rumsey in [45].

Definition 3.7. ([45, Definition 3.1]) An alternating sign matrix is an $n \times n$ matrix $A = (a_{i,j})$, which satisfies the following properties:

(ASM1) $a_{i,j} \in \{-1, 0, 1\}$, for all $i, j \in \{1, \dots, n\}$.

(ASM2) $\sum_{i=1}^n a_{i,k} = \sum_{j=1}^n a_{l,j} = 1$, for all $k, l \in \{1, \dots, n\}$.

(ASM3) In every row and column the non-zero entries alternate in sign.

Theorem 3.1. ([20, Theorem 3.2]) Let $n \geq 1$, and let $C : L_n^2 \rightarrow L_n$ be a binary operator. Then, C is a smooth discrete conjunction if, and only if, there exists an $n \times n$ alternating sign matrix $A = (a_{i,j})$ such that

$$C(x, y) = \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0, \\ \sum_{i=1}^x \sum_{j=1}^y a_{i,j}, & \text{otherwise,} \end{cases} \quad (25)$$

for all $x, y \in L_n$.

The characterization in question establishes a bijection between C_n^{SMT} , the set of smooth discrete conjunctions on L_n , and \mathcal{M}_n , the set of $n \times n$ alternating sign matrices. Consequently, possessing knowledge of the cardinality of \mathcal{M}_n would suffice for determining the cardinality of C_n^{SMT} . Robbins and Rumsey conjectured a formula for the number of alternating sign matrices. Later, in [46], Zeilberger proved the conjecture through intricate arguments, a fact underscored by the substantial length of the paper, spanning 84 pages, and the meticulous review conducted by 88 reviewers. In between, there were other attempts to prove this conjecture, we refer to [47] for an explanation of the evolution of the problem up to its proof. This formula is given in Proposition 3.9.

Additionally, there is a deeper dimension to this formula. In [46], Zeilberger extended his study by establishing a correspondence between the enumeration of alternating sign matrices and that of n -Gogs and n -Magogs, which are triangular configurations of integers. Subsequently, Munar et al. in [20] introduced another category of triangular groupings, which they termed *rectified n -Magogs*, and can be defined as the set:

$$\mathcal{RT}_n = \left\{ \hat{t} = (\hat{t}_{i,j}), 1 \leq i \leq n, 1 \leq j \leq n - i + 1 \mid 0 \leq \hat{t}_{i,j} \leq j, \hat{t}_{i,j} \geq \hat{t}_{i+1,j}, \hat{t}_{i,j} \leq \hat{t}_{i,j+1} \right\}. \quad (26)$$

They proved two important properties of rectified n -Magogs: their cardinality, and a bijection with the set of commutative discrete conjunctions with neutral element n .

Proposition 3.8. Let $n \geq 1$, and let us denote by

- \mathcal{RT}_n , the set of rectified n -Magogs;
- \mathcal{M}_n , the set of $n \times n$ alternating sign matrices;
- C_n^{COM} , the set of commutative discrete conjunctions defined on L_n ;
- $C_n^{\text{NE}(n)}$, the set of discrete conjunctions with neutral element n defined on L_n .

Then,

(i) ([20, Lemma 4.1]) $|\mathcal{RT}_n| = |\mathcal{M}_{n+1}|$.

(ii) ([20, Lemma 4.2]) The transformation $\omega : \mathcal{RT}_n \rightarrow C_{n+1}^{\text{COM}} \cap C_{n+1}^{\text{NE}(n)}$, given by

$$\omega(\hat{t})(x, y) = \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0, \\ x, & \text{if } y = n + 1, \\ y, & \text{if } x = n + 1, \\ \hat{t}_{n-y,x}, & \text{if } 1 \leq x \leq n \text{ and } x \leq y \leq n, \\ \hat{t}_{n-x,y}, & \text{if } 2 \leq x \leq n \text{ and } 1 \leq y < x, \end{cases} \quad (27)$$

for all $x, y \in L_{n+1}$, is a bijection.

The establishment of $\left|C_{n+1}^{\text{COM}} \cap C_{n+1}^{\text{NE}(n)}\right|$ is straightforward from Proposition 3.8. It is important to highlight, however, that despite $C_n^{\text{COM}} \cap C_n^{\text{NE}(n)}$ and \mathcal{M}_n share the same enumeration, a bijection between these two sets remains unknown. This is partly attributed to the absence of a known bijection between the sets of n -Gogs and n -Magogs as well.

Proposition 3.9. ([20, 46]) *Let $n \geq 2$, and let us denote by*

- \mathcal{M}_n , the set of $n \times n$ alternating sign matrices;
- C_n^{SMT} , the set of smooth discrete conjunctions defined on L_n ;
- C_n^{COM} , the set of commutative discrete conjunctions defined on L_n ;
- $C_n^{\text{NE}(n)}$, the set of discrete conjunctions with neutral element n defined on L_n .

Then,

$$\left|C_n^{\text{COM}} \cap C_n^{\text{NE}(n)}\right| = \left|C_n^{\text{SMT}}\right| = |\mathcal{M}_n| = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} = \prod_{i=1}^n \prod_{j=i}^n \frac{n+i+j-1}{2i+j-1}. \quad (28)$$

Remark 3.4. *Although $\left|C_n^{\text{COM}} \cap C_n^{\text{NE}(n)}\right| = \left|C_n^{\text{SMT}}\right|$, it should be noted that in general $C_n^{\text{COM}} \cap C_n^{\text{NE}(n)} \neq C_n^{\text{SMT}}$, as indicated in [20, Observation 3].*

To conclude our discussion on discrete conjunctions, we revisit the result that provides an enumeration for smooth and associative discrete conjunctions. In certain instances, the smoothness requirement serves to alleviate the complexity inherent to associativity.

Proposition 3.10. ([20, Corollary 3.7.2]) *Let $n \geq 1$ and let us denote by C_n^{SMT} and C_n^{ASV} the sets of smooth discrete conjunctions and associative discrete conjunctions, respectively, defined on L_n . Then,*

$$\left|C_n^{\text{SMT}} \cap C_n^{\text{ASV}}\right| = 2^{n-1}. \quad (29)$$

Let us now consider the class of discrete disjunctions. An approach for determining their enumeration is to again consider plane partitions. However, it is worth noting that in [19, Proposition 4.3] it was established that discrete conjunctions can be bijectively transformed into discrete disjunctions. Therefore, the formulas mentioned above for discrete conjunctions and some of their subclasses can be applied to enumerate discrete disjunctions and some of their subclasses. This bijection is constructed based on the unique strong discrete negation, N_C , which is an automorphism on L_n .

Proposition 3.11. ([19, Proposition 4.3]) *Let $n \geq 1$, and let us denote by C_n and D_n the sets of discrete conjunctions and discrete disjunctions, respectively, defined on L_n . Let $\varphi : C_n \rightarrow D_n$ be defined point-wise by*

$$\varphi(C)(x, y) = n - C(n - x, n - y),$$

for all $x, y \in L_n$, where $C \in C_n$. Then, $\varphi(C)$ is a discrete disjunction, and the transformation φ is a bijection.

This bijection makes it possible to seamlessly use the cardinality of one class for the other, as follows:

- Discrete disjunctions, D_n . Its cardinality coincides with $|C_n|$, given in Proposition 3.5.
- Commutative discrete disjunctions, D_n^{COM} . Its cardinality coincides with $|C_n^{\text{COM}}|$, given in Proposition 3.6.
- Discrete disjunctions with neutral element 0, $D_n^{\text{NE}(0)}$. Its cardinality coincides with the cardinality of discrete conjunctions with neutral element n , $C_n^{\text{NE}(n)}$, given in Proposition 3.7.
- Smooth discrete disjunctions, D_n^{SMT} . Its cardinality coincides with $|C_n^{\text{SMT}}|$, given in Proposition 3.9.

n	$ C_n , D_n $	$ C_n^{\text{COM}} , D_n^{\text{COM}} $	$ C_n^{\text{NE}(n)} , D_n^{\text{NE}(0)} $	$ C_n^{\text{SMT}} , D_n^{\text{SMT}} $	$ C_n^{\text{SMT}} \cap C_n^{\text{ASV}} , D_n^{\text{SMT}} \cap D_n^{\text{ASV}} $
1	$1 = 1 \cdot 10^0$	$1 = 1 \cdot 10^0$	$1 = 1 \cdot 10^0$	$1 = 1 \cdot 10^0$	$1 = 1 \cdot 10^0$
2	$14 = 1.4 \cdot 10^1$	$6 = 6 \cdot 10^0$	$2 = 2 \cdot 10^0$	$2 = 2 \cdot 10^0$	$2 = 2 \cdot 10^0$
3	$805 = 8.05 \cdot 10^2$	$77 = 7.7 \cdot 10^1$	$11 = 1.1 \cdot 10^1$	$7 = 7 \cdot 10^0$	$4 = 4 \cdot 10^0$
4	$\approx 2.0815 \cdot 10^5$	$2,100 = 2.1 \cdot 10^3$	$170 = 1.7 \cdot 10^2$	$42 = 4.2 \cdot 10^1$	$8 = 8 \cdot 10^0$
5	$\approx 2.5041 \cdot 10^8$	$\approx 1.2269 \cdot 10^5$	$7,429 = 7.429 \cdot 10^3$	$429 = 4.29 \cdot 10^2$	$16 = 1.6 \cdot 10^1$
6	$\approx 1.4234 \cdot 10^{12}$	$\approx 1.5459 \cdot 10^7$	$\approx 9.2046 \cdot 10^5$	$7,436 = 7.436 \cdot 10^3$	$32 = 3.2 \cdot 10^1$
7	$\approx 3.8534 \cdot 10^{16}$	$\approx 4.2204 \cdot 10^9$	$\approx 3.238 \cdot 10^8$	$\approx 2.1835 \cdot 10^5$	$64 = 6.4 \cdot 10^1$
8	$\approx 4.9888 \cdot 10^{21}$	$\approx 2.5045 \cdot 10^{12}$	$\approx 3.2367 \cdot 10^{11}$	$\approx 1.085 \cdot 10^7$	$128 = 1.28 \cdot 10^2$
9	$\approx 3.0961 \cdot 10^{27}$	$\approx 3.238 \cdot 10^{15}$	$\approx 9.1986 \cdot 10^{14}$	$\approx 9.1184 \cdot 10^8$	$256 = 2.56 \cdot 10^2$
10	$\approx 9.2223 \cdot 10^{33}$	$\approx 9.1352 \cdot 10^{18}$	$\approx 7.4347 \cdot 10^{18}$	$\approx 1.2953 \cdot 10^{11}$	$512 = 5.12 \cdot 10^2$
11	$\approx 1.3195 \cdot 10^{41}$	$\approx 5.6307 \cdot 10^{22}$	$\approx 1.7094 \cdot 10^{23}$	$\approx 3.1096 \cdot 10^{13}$	$1,024 = 1.024 \cdot 10^3$
12	$\approx 9.0708 \cdot 10^{48}$	$\approx 7.5891 \cdot 10^{26}$	$\approx 1.1183 \cdot 10^{28}$	$\approx 1.2611 \cdot 10^{16}$	$2,048 = 2.048 \cdot 10^3$
13	$\approx 2.997 \cdot 10^{57}$	$\approx 2.2381 \cdot 10^{31}$	$\approx 2.0819 \cdot 10^{33}$	$\approx 8.6394 \cdot 10^{18}$	$4,096 = 4.096 \cdot 10^3$

Table 3

Enumeration of some classes of discrete conjunctions and disjunctions, for $n \leq 13$. From left to right: discrete conjunctions and disjunctions in general, given by Proposition 3.5; commutative discrete conjunctions and disjunctions, given by Proposition 3.6; discrete conjunctions having neutral element n and discrete disjunctions having neutral element 0, given by Proposition 3.7; smooth discrete conjunctions and disjunctions, given by Proposition 3.9; and finally smooth discrete conjunctions and disjunctions which are associative, given by Proposition 3.10.

- Commutative discrete disjunctions with neutral element 0, $D_n^{\text{COM}} \cap D_n^{\text{NE}(0)}$. Its cardinality coincides with the cardinality of commutative discrete conjunctions with neutral element n , $C_n^{\text{COM}} \cap C_n^{\text{NE}(n)}$, given in Proposition 3.9.
- Smooth and associative discrete disjunctions, $D_n^{\text{SMT}} \cap D_n^{\text{ASV}}$. Its cardinality coincides with $|C_n^{\text{SMT}} \cap C_n^{\text{ASV}}|$, given in Proposition 3.10.

In Table 3 we indicate the first 13 terms of the different enumerations for discrete conjunctions and disjunctions.

3.3. Discrete quasi-copulas and copulas

Discrete quasi-copulas and copulas are particular instances of discrete conjunctions. Quasi-copulas on $[0, 1]$ were studied in the framework of theoretical statistics in [48], and on discrete settings, the concept of discrete quasi-copula was introduced in [49], and further studied in [21, 50].

Definition 3.8. ([21, 49, 50]) *A discrete quasi-copula is a discrete conjunction $Q : L_n^2 \rightarrow L_n$ which is 1-Lipschitz and has n as neutral element.*

A comprehensive characterization of discrete quasi-copulas was originally presented by Aguiló et al. in [21], also from the point of view of alternating sign matrices. Intriguingly, they arrived at the same result as Theorem 3.1, yielding to the same enumeration as the one outlined in Proposition 3.9. It is worth noting that the work of Aguiló et al. in [21] chronologically predates the independent research conducted by Munar et al. in [20], and both aimed to characterize different operators whose definitions are also different. A noteworthy conclusion highlighted in [20] is that discrete quasi-copulas do not constitute a proper subset of the set of smooth discrete conjunctions (since 1-Lipschitz property implies smoothness), but rather that the classes of quasi-copulas and smooth discrete conjunctions coincide.

As a subset of discrete quasi-copulas, there is the concept of discrete copula, a notion that also finds extensive application in theoretical statistics. This significance is primarily attributable to Sklar's theorem, originally proposed on $[0, 1]$ in [51], and its equivalent on L_n , stated in [52].

Definition 3.9. ([18]) *A discrete copula is a discrete quasi-copula $K : L_n^2 \rightarrow L_n$ which is 2-increasing, i.e., $K(x_1, y_1) + K(x_2, y_2) \geq K(x_1, y_2) + K(x_2, y_1)$, for all $x_1, x_2, y_1, y_2 \in L_n$ whenever $x_1 \leq x_2$ and $y_1 \leq y_2$.*

Mayor et al. in [18] proposed a complete characterization of discrete copulas using permutation matrices, which are in fact a particular case of alternating sign matrices. The construction of a permutation matrix is derived from the permutation group of n elements, denoted as \mathfrak{S}_n .

Definition 3.10. ([53, Section 5.6]) A permutation matrix $P = (p_{i,j})$ is an $n \times n$ matrix such that there exists a permutation $\sigma \in \mathfrak{S}_n$ of $L_n \setminus \{0\}$ whereby P is expressed as

$$p_{ij} = \begin{cases} 1, & \text{if } i = \sigma(j), \\ 0, & \text{otherwise,} \end{cases} \quad (30)$$

for all $i, j \in \{1, \dots, n\}$.

They characterized discrete copulas using these matrices, yielding a result akin to Theorem 3.1. Subsequently, they were able to determine the cardinality of this class of discrete operators.

Proposition 3.12. ([18, Proposition 6]) Let $n \geq 1$, and let $K : L_n^2 \rightarrow L_n$ be a binary operator. Then, K is a discrete copula if, and only if, there exists an $n \times n$ permutation matrix $P = (p_{ij})$ such that

$$K(x, y) = \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0, \\ \sum_{i=1}^x \sum_{j=1}^y p_{ij}, & \text{otherwise,} \end{cases} \quad (31)$$

for all $x, y \in L_n$.

Proposition 3.13. ([18, Corollary 1]) Let $n \geq 1$, and let us denote by \mathcal{K}_n the set of discrete copulas defined on L_n . Then,

$$|\mathcal{K}_n| = n!. \quad (32)$$

The same characterization of Proposition 3.12 also applies to commutative discrete copulas, but taking symmetric permutation matrices as the equivalent object.

Proposition 3.14. ([18, Corollary 1]) Let $n \geq 1$, and let us denote by $\mathcal{K}_n^{\text{COM}}$ the set of commutative discrete copulas defined on L_n . Then,

$$|\mathcal{K}_n^{\text{COM}}| = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^i (n-2i)! i!}. \quad (33)$$

In addition to these two results, Mayor et al. [18] also studied other properties of discrete copulas, such as the associative and idempotent-free properties.

Proposition 3.15. Let $n \geq 1$, and let us denote by $\mathcal{K}_n^{\text{ASV}}$ and $\mathcal{K}_n^{\text{IDF}}$ the sets of associative discrete copulas and idempotent-free discrete copulas, respectively, defined on L_n . Then, the following statements hold:

- (i) ([18, Corollary 4]) $|\mathcal{K}_n^{\text{ASV}}| = 2^{n-1}$,
- (ii) ([18, Proposition 8]) The only associative and idempotent-free discrete copula is the Łukasiewicz copula, given by $K_L(x, y) = \max\{0, x + y - n\}$, for all $x, y \in L_n$. Therefore, $|\mathcal{K}_n^{\text{ASV}} \cap \mathcal{K}_n^{\text{IDF}}| = 1$.

For discrete quasi-copulas and discrete copulas, we illustrate the first 13 terms of the different enumerations in Table 4. Although the enumeration of \mathcal{Q}_n coincides with that of $\mathcal{C}_n^{\text{SMT}}$, since they are the same class of operators and has been illustrated previously in Table 3, we show it again so that it can be compared with the enumerations of discrete copulas and some subclasses.

3.4. Discrete t-norms and t-conorms

Among all subclasses of conjunctions, it is likely that the classes of discrete t-norms and discrete t-conorms have garnered the most significant interest within the research community. This is partially due to their remarkable versatility in tackling a multitude of practical problems, including applications in image processing [1, 2], and decision-making [3, 4] scenarios. We start with the definition of discrete t-norm.

n	$ \mathcal{Q}_n $	$ \mathcal{K}_n $	$ \mathcal{K}_n^{\text{COM}} $	$ \mathcal{K}_n^{\text{ASV}} $
1	$1 = 1 \cdot 10^0$	$1 = 1 \cdot 10^0$	$1 = 1 \cdot 10^0$	$1 = 1 \cdot 10^0$
2	$2 = 2 \cdot 10^0$	$2 = 2 \cdot 10^0$	$2 = 2 \cdot 10^0$	$2 = 2 \cdot 10^0$
3	$7 = 7 \cdot 10^0$	$6 = 6 \cdot 10^0$	$4 = 4 \cdot 10^0$	$4 = 4 \cdot 10^0$
4	$42 = 4.2 \cdot 10^1$	$24 = 2.4 \cdot 10^1$	$10 = 1 \cdot 10^1$	$8 = 8 \cdot 10^0$
5	$429 = 4.29 \cdot 10^2$	$120 = 1.2 \cdot 10^2$	$26 = 2.6 \cdot 10^1$	$16 = 1.6 \cdot 10^1$
6	$7,436 = 7.436 \cdot 10^3$	$720 = 7.2 \cdot 10^2$	$76 = 7.6 \cdot 10^1$	$32 = 3.2 \cdot 10^1$
7	$\approx 2.1835 \cdot 10^5$	$5,040 = 5.04 \cdot 10^3$	$232 = 2.32 \cdot 10^2$	$64 = 6.4 \cdot 10^1$
8	$\approx 1.085 \cdot 10^7$	$40,320 = 4.032 \cdot 10^4$	$764 = 7.64 \cdot 10^2$	$128 = 1.28 \cdot 10^2$
9	$\approx 9.1184 \cdot 10^8$	$3,628,800 = 3.6288 \cdot 10^5$	$2,620 = 2.62 \cdot 10^3$	$256 = 2.56 \cdot 10^2$
10	$\approx 1.2953 \cdot 10^{11}$	$\approx 3.6288 \cdot 10^6$	$9,496 = 9.496 \cdot 10^3$	$512 = 5.12 \cdot 10^2$
11	$\approx 3.1096 \cdot 10^{13}$	$\approx 3.9917 \cdot 10^7$	$35,696 = 3.5696 \cdot 10^4$	$1,024 = 1.024 \cdot 10^3$
12	$\approx 1.2611 \cdot 10^{16}$	$\approx 4.79 \cdot 10^8$	$\approx 1.4015 \cdot 10^5$	$2,048 = 2.048 \cdot 10^3$
13	$\approx 8.6394 \cdot 10^{18}$	$\approx 6.227 \cdot 10^9$	$\approx 5.685 \cdot 10^5$	$4,096 = 4.096 \cdot 10^3$

Table 4

Enumeration of discrete quasi-copulas and some classes of discrete copulas, for $n \leq 13$. From left to right: discrete quasi-copulas, given by Proposition 3.9; discrete copulas in general, given by Proposition 3.13; commutative discrete copulas, given by Proposition 3.14; and finally associative discrete copulas, given by Proposition 3.15.

Definition 3.11. ([8, 22]) A discrete t-norm is a discrete conjunction $T : L_n^2 \rightarrow L_n$ which is commutative, associative and has n as neutral element.

Let \mathcal{T}_n denote the set of discrete t-norms on L_n . The problem of determining its in general goes back to De Baets and Mesiar [5]. They explored t-norms defined across diverse finite structures, which encompass finite chains among others. They used computational methods to generate all possible discrete t-norms defined on L_n for $n \leq 13$, and therefore they were able to determine $|\mathcal{T}_n|$ for these values of n . These values of $|\mathcal{T}_n|$ are listed on Table 5. Subsequent to this paper, no significant progress has been achieved for the cardinality of discrete t-norms in general; only some progress has been made for the enumeration of t-norms for some specific cases of bounded lattices (see [54, 55]). Nevertheless, certain subclasses of discrete t-norms have garnered attention, which we explain below.

Different properties have been studied for discrete t-norms. One of them is the fact of being Archimedean, whose original definition we do not include for the sake of clarity (refer to [8, Definition 7.2.10] and [22, Definition 2.5] for more information) but which is equivalent for discrete t-norms as being idempotent-free. The number of discrete t-norms that are Archimedean, $|\mathcal{T}_n^{\text{ARC}}|$, remains an open problem. Conversely, if a discrete t-norm T is idempotent, it is necessarily the minimum discrete t-norm, i.e., $T(x, y) = T_M(x, y) = \min\{x, y\}$ for all $x, y \in L_n$, so if $\mathcal{T}_n^{\text{IDE}}$ represents the set of idempotent discrete t-norms, then $|\mathcal{T}_n^{\text{IDE}}| = 1$, for all $n \geq 1$. Particular attention is directed toward the category of smooth discrete t-norms. These were originally characterized by Mayor and Torrens in [56, Theorem 6] through the use of the ordinal sum concept, albeit with notation and terminology that diverged from present conventions. Subsequently, in [8, Theorem 7.2.16], the same authors revisited this result, presenting it using a more present-day notation.

Theorem 3.2. ([8, Proposition 7.3.8]) Let $T : L_n^2 \rightarrow L_n$ be a t-norm on L_n . Then, T is smooth if, and only if, there exists a natural number r with $0 \leq r \leq n - 1$ and a subset

$$I = \{0 = a_0 < a_1 < \dots < a_r < a_{r+1} = n\} \subseteq L_n,$$

such that T is given by

$$T(x, y) = \begin{cases} \max\{a_i, x + y - a_{i+1}\}, & \text{if } (x, y) \in [a_i, a_{i+1}]^2 \text{ for some } 0 \leq i \leq r, \\ \min\{x, y\}, & \text{otherwise,} \end{cases} \quad (34)$$

for all $x, y \in L_n$.

This result implies that the expression of any smooth discrete t-norm T is uniquely determined by the set I . Since as many sets can be constructed as there are elements in the power set of the interval

$[1, n - 1]$, Proposition 3.16 establishes its cardinality. But there is a further consequence of Theorem 3.2. When we simultaneously impose the conditions of being smooth and Archimedean, and bearing in mind that each component $\max\{a_i, x + y - a_{i+1}\}$ of Equation (34) is already idempotent-free on each interval $[a_i, a_{i+1}]$, the sole viable choice is $I = \{0, n\}$. This leads to the specific expression of T as $T(x, y) = T_L(x, y) = \max\{0, x + y - n\}$, for all $x, y \in L_n$, which is renowned as Łukasiewicz discrete t-norm. Therefore, it can be inferred that for a given $n \geq 1$ there exists only one smooth and Archimedean discrete t-norm.

Proposition 3.16. *Let $n \geq 1$, and let us denote by $\mathcal{T}_n^{\text{SMT}}$ and $\mathcal{T}_n^{\text{ARC}}$ the sets of smooth discrete t-norms and Archimedean discrete t-norms, respectively, defined on L_n . Then,*

$$(i) \text{ ([8, Proposition 7.3.7]) } |\mathcal{T}_n^{\text{SMT}} \cap \mathcal{T}_n^{\text{ARC}}| = 1.$$

$$(ii) \text{ ([8, Corollary 7.3.12]) } |\mathcal{T}_n^{\text{SMT}}| = 2^{n-1}.$$

Another class of operators recently proposed by Bejines and Navara [12] is that of maximal Archimedean discrete t-norms.

Definition 3.12. *([12, Definition 2.4]) A discrete t-norm T is maximal Archimedean if it is Archimedean and there is no other Archimedean discrete t-norm T' on L_n satisfying $T(x, y) \leq T'(x, y)$, for all $x, y \in L_n$.*

If we denote by $\mathcal{T}_n^{\text{MACH}}$ the set of maximal Archimedean discrete t-norms on L_n , in [12] it is shown that its cardinality is given by the Fibonacci sequence. In fact, they expressed every discrete t-norm of $\mathcal{T}_n^{\text{MACH}}$ as the extension of a discrete t-norm of $\mathcal{T}_{n-1}^{\text{MACH}}$ or $\mathcal{T}_{n-2}^{\text{MACH}}$, so they ended up with this recursive formula. We omit the details of this decomposition due to its length and because it involves a large number of previous technical results.

Proposition 3.17. *([12, Section 4.4]) Let $n \geq 1$, and let us denote by $\mathcal{T}_n^{\text{MACH}}$ the set of maximal Archimedean discrete t-norms defined on L_n . Then, the number of maximal Archimedean discrete t-norms satisfies the recurrence equation*

$$|\mathcal{T}_n^{\text{MACH}}| = |\mathcal{T}_{n-1}^{\text{MACH}}| + |\mathcal{T}_{n-2}^{\text{MACH}}|, \quad (35)$$

for all $n \geq 3$, with $|\mathcal{T}_1^{\text{MACH}}| = |\mathcal{T}_2^{\text{MACH}}| = 1$. Therefore, $|\mathcal{T}_n^{\text{MACH}}|$ is the n -th element of the Fibonacci sequence, and its general formula is given by

$$|\mathcal{T}_n^{\text{MACH}}| = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}}. \quad (36)$$

As a particular case of discrete disjunction, we have the concept of discrete t-conorm, which is the other class of operators we recall in this section.

Definition 3.13. *([8, 22]) A discrete t-conorm is a discrete disjunction $S : L_n^2 \rightarrow L_n$ which is commutative, associative and has 0 as neutral element.*

Discrete t-norms and t-conorms have very similar definitions, with the position of the neutral element being the only difference. Since the origin of these operators, the duality between discrete t-norms and discrete t-conorms is already known and, in fact, served as the inspiration for the duality between discrete conjunctions and disjunctions presented in Proposition 3.11. As a consequence, cardinalities between some classes of discrete t-norms and t-conorms are also preserved.

Proposition 3.18. *([8, 22]) Let $n \geq 1$, and let us denote by \mathcal{T}_n and \mathcal{S}_n the sets of discrete t-norms and discrete t-conorms, respectively, defined on L_n . Let $\varphi : \mathcal{T}_n \rightarrow \mathcal{S}_n$ be defined point-wise by*

$$\varphi(T)(x, y) = n - T(n - x, n - y),$$

for all $x, y \in L_n$, where $T \in \mathcal{T}_n$. Then, $\varphi(T)$ is a discrete t-conorm, and the transformation φ is a bijection.

n	$ \mathcal{T}_n , \mathcal{S}_n $	$ \mathcal{T}_n^{\text{SMT}}, \mathcal{S}_n^{\text{SMT}} $	$ \mathcal{T}_n^{\text{MACH}}, \mathcal{S}_n^{\text{MACH}} $
1	$1 = 1 \cdot 10^0$	$1 = 1 \cdot 10^0$	$1 = 1 \cdot 10^0$
2	$2 = 2 \cdot 10^0$	$2 = 2 \cdot 10^0$	$1 = 1 \cdot 10^0$
3	$6 = 6 \cdot 10^0$	$4 = 4 \cdot 10^0$	$2 = 2 \cdot 10^0$
4	$22 = 2.2 \cdot 10^1$	$8 = 8 \cdot 10^0$	$3 = 3 \cdot 10^0$
5	$94 = 9.4 \cdot 10^1$	$16 = 1.6 \cdot 10^1$	$5 = 5 \cdot 10^0$
6	$451 = 4.51 \cdot 10^2$	$32 = 3.2 \cdot 10^1$	$8 = 8 \cdot 10^0$
7	$2,386 = 2.386 \cdot 10^3$	$64 = 6.4 \cdot 10^1$	$13 = 1.3 \cdot 10^1$
8	$13,775 = 1.3775 \cdot 10^4$	$128 = 1.28 \cdot 10^2$	$21 = 2.1 \cdot 10^1$
9	$86,417 = 8.6417 \cdot 10^4$	$256 = 2.56 \cdot 10^2$	$34 = 3.4 \cdot 10^1$
10	$590,489 \approx 5.9049 \cdot 10^5$	$512 = 5.12 \cdot 10^2$	$55 = 5.5 \cdot 10^1$
11	$4,446,029 \approx 4.446 \cdot 10^6$	$1,024 = 1.024 \cdot 10^3$	$89 = 8.9 \cdot 10^1$
12	$37,869,449 \approx 3.7869 \cdot 10^7$	$2,048 = 2.048 \cdot 10^3$	$144 = 1.44 \cdot 10^2$
13	$382,549,464 \approx 3.8255 \cdot 10^8$	$4,096 = 4.096 \cdot 10^3$	$233 = 2.33 \cdot 10^2$

Table 5

Enumeration of some classes of discrete t-norms and t-conorms, for $n \leq 13$. From left to right: discrete t-norms and t-conorms in general, obtained computationally; smooth discrete t-norms and t-conorms, given by Proposition 3.16; and finally maximal Archimedean discrete t-norms and minimal Archimedean discrete t-conorms, given by Proposition 3.17.

So, with this result, the enumeration of discrete t-conorms coincides with that of discrete t-norms. However, since $|\mathcal{T}_n|$ lacks a general term, the same applies to $|\mathcal{S}_n|$. Also for the Archimedean property for discrete t-conorms, which is equivalent to $\mathcal{S}(x, x) > x$ for all $x \in L_n \setminus \{0, n\}$: the enumeration of $|\mathcal{S}_n^{\text{ARC}}|$ coincides with the enumeration of $|\mathcal{T}_n^{\text{ARC}}|$, but a general formula for the enumeration of these operators is still unknown.

Regarding the other formulas, they can also be adapted for discrete t-conorms. With the duality between discrete t-norms and t-conorms of Proposition 3.18, a discrete t-norm is smooth if and only if its dual discrete t-conorm is smooth. Also, there exists a unique smooth and Archimedean discrete t-conorm, which is the Łukasiewicz discrete t-conorm $\mathcal{S}_\perp(x, y) = \min\{n, x + y\}$, for all $x, y \in L_n$. Finally, for maximal Archimedean discrete t-norms, there is no precedent in the existing literature regarding this concept for discrete t-conorms. However, it is clear that Definition 3.12 can be adjusted by replacing a discrete t-norm T by a discrete t-conorm \mathcal{S} , keeping the Archimedean property and replacing the property of being maximal with that of being minimal, i.e., when there is not other Archimedean discrete t-conorm \mathcal{S}' such that $\mathcal{S}'(x, y) \leq \mathcal{S}(x, y)$, giving rise to the minimal Archimedean discrete t-conorm. With this discussion, we condense all these formulas into the following three items.

- Smooth discrete t-conorms, $\mathcal{S}_n^{\text{SMT}}$. Its cardinality coincides with $|\mathcal{T}_n^{\text{SMT}}|$, given in Proposition 3.16.
- Smooth and Archimedean discrete t-conorms, $\mathcal{S}_n^{\text{SMT}} \cap \mathcal{S}_n^{\text{ARC}}$. Its cardinality coincides with $|\mathcal{T}_n^{\text{SMT}} \cap \mathcal{T}_n^{\text{ARC}}|$, given in Proposition 3.16.
- Minimal Archimedean discrete t-conorms, $\mathcal{S}_n^{\text{MACH}}$. Its cardinality coincides with $|\mathcal{T}_n^{\text{MACH}}|$, given in Proposition 3.17.

All these classes are listed in Table 5 for $n \leq 13$.

3.5. Discrete uninorms

To generalize the concepts of discrete t-norms and discrete t-conorms, the concept of discrete uninorm appears, which was introduced on discrete settings for the first time in [7]. Discrete uninorms were introduced based on the observation that discrete t-norms and discrete t-conorms primarily differ in the value of their neutral element, and therefore are particular cases of commutative and associative discrete aggregation functions, whose neutral element is any $e \in L_n$. In [29] there is a comprehensive survey about uninorms on $[0, 1]$ and on L_n and in [22] there is a section dedicated solely addressing discrete uninorms.

Definition 3.14. ([7, 29]) A discrete uninorm is a discrete aggregation function $U : L_n^2 \rightarrow L_n$, which is associative, commutative and has neutral element $e \in L_n$.

Note that a discrete uninorm becomes a discrete t-conorm whenever $e = 0$, while it is a discrete t-norm whenever $e = n$. When $0 < e < n$, it is termed a *proper discrete uninorm*. In the latter case, it can be described as follows: its restriction on the region $[0, e]^2$ is a discrete t-norm T_U on $[0, e]$, whereas its restriction to $[e, n]^2$ is a discrete t-conorm S_U on $[e, n]$. We use the notation $U = \langle T_U, e, S_U \rangle$ to represent this behavior. In the remaining domain, denoted by

$$C_e = [0, e - 1] \times [e + 1, n] \cup [e + 1, n] \times [0, e - 1], \quad (37)$$

and referred to as the *compensation space*, it holds that $\min\{x, y\} \leq U(x, y) \leq \max\{x, y\}$, for all $(x, y) \in C_e$. This inequality allows us to obtain that $U(0, n) \in \{0, n\}$, and discrete uninorms such that $U(0, n) = 0$ are called *conjunctive*, whereas those with $U(0, n) = n$ are called *disjunctive*. Since the general behavior of a uninorm depends on the nature of its underlying t-norm and t-conorm, and on the value it takes in the region C_e , several subclasses of uninorms have appeared in the literature according to the behavior of each of these components. As an observation, we note that on L_n there are no smooth discrete uninorms (see [7, Corollary 4.4]), just as there are no continuous uninorms on $[0, 1]$ (see [29]).

As mentioned in Section 3.4, since a general formula for the number of discrete t-norms and discrete t-conorms is not known, it is foreseeable that the number of discrete uninorms is also not known given that its structure is a combination of a discrete t-norm and discrete t-conorm. Indeed, this issue remains an open problem, and to the best of our knowledge there are no references in the literature that have tackled it. Still, a formula can be conjectured for the cardinality of $\mathcal{U}_n^{\text{MIN}}$, the set of discrete uninorms satisfying $U(x, y) = \min\{x, y\}$ for all $(x, y) \in C_e$, and $\mathcal{U}_n^{\text{MAX}}$, the set of discrete uninorms satisfying $U(x, y) = \max\{x, y\}$ for all $(x, y) \in C_e$.

Remark 3.5. *Although the number of discrete t-norms and t-conorms is unknown in general, a formula for $|\mathcal{U}_n^{\text{MIN}}|$ and $|\mathcal{U}_n^{\text{MAX}}|$ can be proposed, since these subclasses of discrete uninorms cannot take any expression in C_e , only the minimum or maximum. A first formula can be*

$$|\mathcal{U}_n^{\text{MIN}}| = |\mathcal{U}_n^{\text{MAX}}| = \sum_{e=1}^{n-1} |\mathcal{T}_e| \cdot |\mathcal{S}_{n-e}|,$$

for all $n \geq 2$. However, bearing in mind that $|\mathcal{S}_n| = |\mathcal{T}_n|$, it can be slightly simplified as

$$|\mathcal{U}_n^{\text{MIN}}| = |\mathcal{U}_n^{\text{MAX}}| = \sum_{e=1}^{n-1} |\mathcal{T}_e| \cdot |\mathcal{T}_{n-e}|,$$

for all $n \geq 2$. Let us note that in these expressions we have that $e \in \{1, \dots, n-1\}$, since in order to define the discrete uninorms belonging to $\mathcal{U}_n^{\text{MIN}}$ and $\mathcal{U}_n^{\text{MAX}}$ the set C_e must be considered, and to be it well defined the extreme values $e = 0$ and $e = n$ must be excluded.

Of the classes of discrete uninorms for which a formula for their cardinality is known, let us start with idempotent discrete uninorms, that have been characterized from three distinct perspectives, but only two focus on their enumeration. The first technique can be found in [10], where De Baets et al. propose a characterization grounded in decreasing unary functions.

Theorem 3.3. ([10, Theorem 3]) *A binary operation U on L_n with neutral element $0 < e < n$ is an idempotent uninorm on L_n if, and only if, there exists a decreasing function $g : [0, e] \rightarrow [e, n]$ with $g(e) = e$ such that*

$$U(x, y) = \begin{cases} \min\{x, y\}, & \text{if } y \leq \bar{g}(x) \text{ and } x \leq \bar{g}(0), \\ \max\{x, y\}, & \text{otherwise,} \end{cases} \quad (38)$$

for all $x, y \in L_n$, where $\bar{g} : L_n \rightarrow L_n$ is given by

$$\bar{g}(x) = \begin{cases} g(x), & \text{if } x \leq e, \\ \max\{z \in [0, e] \mid g(z) \geq x\}, & \text{if } e < x \leq g(0), \\ 0, & \text{if } x > g(0), \end{cases} \quad (39)$$

for all $x \in L_n$.

This result enabled the authors of [10] to determine the number of idempotent discrete uninorms with neutral element $e \in L_n \setminus \{0, n\}$ in particular, and also the number of idempotent discrete uninorms in general. Observe that in this result they convert the enumeration of these binary operators into a count of decreasing unary operators with fixed point e , whose enumeration is simpler. This enumeration can be found in Theorem 3.5.

The second approach is presented by Couceiro et al. in [14, 16]. In these papers, the authors reapply the methodology used in the study for the enumeration of several families of discrete aggregation functions described in Propositions 3.3 and 3.4, again from the perspective of the level curves of binary operators and their geometry. For that, they introduced a powerful tool, namely, the concept of the F -degree sequence of a binary operator $F : L_n^2 \rightarrow L_n$. To the best of our knowledge, this concept has not been previously employed in any other reference dedicated to the examination of binary operators on L_n .

Definition 3.15. ([16]) Let $F : L_n^2 \rightarrow L_n$ be a binary operator. Given $z \in L$, the F -degree of z , denoted with $\deg_F(z)$, is defined as

$$\deg_F(z) = \left| \{(x, y) \in L_n^2 \setminus \{(z, z)\} \mid F(x, y) = F(z, z)\} \right|. \quad (40)$$

The degree sequence of F , denoted with \deg_F , is the increasing sequence of $\{\deg_F(z)\}_{z \in L_n}$.

They established an equivalence between specific properties of a binary operator that are simultaneously satisfied and idempotent discrete uninorms.

Theorem 3.4. ([14, Theorem 12], [16, Theorem 3.7]) For a binary operation $U : L_n^2 \rightarrow L_n$, the following statements are equivalent:

- (i) U is conservative, symmetric and increasing in each argument.
- (ii) U is conservative, increasing in each argument and $\deg_F = (0, 2, 4, \dots, 2n - 2)$.
- (iii) U is an idempotent discrete uninorm.

With this result, they prove by induction the enumeration of the number of idempotent discrete uninorms. For both techniques, the enumeration is in the following result.

Theorem 3.5. The following statements hold:

- (i) ([10, Theorem 4]) The number of idempotent uninorms on L_n , with $n \geq 2$, with neutral element $0 < e < n$, is given by

$$\left| \mathcal{U}_n^{\text{IDE}(e)} \right| = \binom{n}{e}.$$

- (ii) ([14, Theorem 14], [10, Theorem 4]) The total number of idempotent uninorms on L_n , with $n \geq 2$, is given by

$$\left| \mathcal{U}_n^{\text{IDE}} \right| = \sum_{e=0}^n \left| \mathcal{U}_n^{\text{IDE}(e)} \right| = 2^n.$$

Remark 3.6. The formula for $\left| \mathcal{U}_n^{\text{IDE}(e)} \right|$ excludes the extreme values $e = 0$ and $e = n$, since it has been obtained with the characterization of Theorem 3.3, which also excludes them. Now, let us note that in the formula for $\left| \mathcal{U}_n^{\text{IDE}} \right|$ these extreme values are considered, since in order to define idempotent discrete uninorms they are not required to be proper discrete uninorms, and therefore it is possible to include the cases $e = 0$ and $e = n$.

Another class of discrete uninorms that are fully characterized are those whose underlying operators (the discrete t-norm T_U and the discrete t-conorm S_U) are smooth. If $\mathcal{U}_n^{\text{STS}}$ denotes the set of these operators, Ruiz-Aguilera and Torrens determined in [11] its structure using the concept of Id-symmetric extension of a unary operator (see [10, Definition 4]). The general result is as follows.

Theorem 3.6. ([11, Theorem 5]) Let $n \geq 2$. Let U be a binary operator on L_n with neutral element $0 < e < n$ and idempotent elements

$$\text{Ide}(U) = \{0 = a_0 < a_1 < \dots < a_r = e = b_0 < b_1 < \dots < b_s = n\}.$$

Then, the following statements are equivalent:

- (i) U is a uninorm on L_n with smooth underlying operators T_U and S_U .
- (ii) There exists a decreasing function $g : [0, e] \rightarrow [e, n]$ with fixed point e that satisfies simultaneously the following conditions:
 - (a) For all $a_i \in \text{Ide}(U)$, there exists $b_j \in \text{Ide}(U)$ such that $g(a_i) = b_j$.
 - (b) If $x \in [a_i, a_{i+1} - 1]$, then $g(x) = g(a_i)$, for all $i \in \{0, 1, \dots, r - 1\}$.
 - (c) U is given by

$$U(x, y) = \begin{cases} T_U(x, y), & \text{if } (x, y) \in [0, e]^2, \\ S_U(x, y), & \text{if } (x, y) \in [e, n]^2, \\ \min\{x, y\}, & \text{if } (x, y) \in C_e, y \leq \bar{g}(x) \text{ and } x \leq \bar{g}(0), \\ \max\{x, y\}, & \text{otherwise,} \end{cases} \quad (41)$$

where \bar{g} is the only Id-symmetric extension of g .

Once more, they characterized these operators starting from a decreasing unary operator, akin to their approach in Theorem 3.3 for idempotent discrete uninorms. Consequently, it suffices to carry out the counting of these decreasing unary operators, satisfying the conditions outlined in Theorem 3.6.

Theorem 3.7. ([11, Theorem 6]) The following statements hold.

- (i) The number of discrete uninorms $U = \langle T_U, e, S_U \rangle$ on L_n with $n \geq 2$ having smooth underlying operators T_U and S_U with neutral element $e \in [1, n - 1]$, and $r + 1$ idempotent elements on $[0, e]$ and $s + 1$ idempotent elements on $[e, n]$ is given by

$$\binom{r+s}{r} \cdot \binom{e-1}{r-1} \cdot \binom{n-e-1}{s-1}. \quad (42)$$

- (ii) The total number of discrete uninorms $U = \langle T_U, e, S_U \rangle$ on L_n with $n \geq 2$ having smooth underlying operators T_U and S_U with neutral element $e \in [1, n - 1]$, is given by

$$\sum_{r=1}^e \sum_{s=1}^{n-1} \binom{r+s}{r} \cdot \binom{e-1}{r-1} \cdot \binom{n-e-1}{s-1}. \quad (43)$$

- (iii) The total number of discrete uninorms $U = \langle T_U, e, S_U \rangle$ on L_n with $n \geq 2$ having smooth underlying operators T_U and S_U is given by

$$|\mathcal{U}_n^{\text{STS}}| = \sum_{e=1}^{n-1} \sum_{r=1}^e \sum_{s=1}^{n-1} \binom{r+s}{r} \cdot \binom{e-1}{r-1} \cdot \binom{n-e-1}{s-1}. \quad (44)$$

Remark 3.7. In contrast to what has been discussed in Remark 3.6, to define discrete uninorms whose underlying operators are smooth, they are required to be proper discrete uninorms, since the extreme cases $e = 0$ and $e = n$ cause one of the two underlying operators to be ill-defined. For this reason, in the enumeration of statement (iii) of Theorem 3.7 the possibility of directly considering smooth discrete t-norms and smooth discrete t-conorms has been excluded.

Also with Theorem 3.6, they achieved an enumeration of discrete uninorms when the underlying operators are some specific examples of smooth discrete t-norms and discrete t-conorms.

Proposition 3.19. *The following statements hold:*

- (i) ([11, Corollary 2]) *The total number of discrete uninorms on L_n such that their underlying operators are smooth and Archimedean is*

$$\left| \mathcal{U}_n^{\text{STS}} \cap \mathcal{U}_n^{\text{ARC}} \right| = 2n - 2, \quad (45)$$

for all $n \geq 2$.

- (ii) ([11, Corollary 3]) *The total number of discrete uninorms on L_n such that their underlying operators are $T_U = T_M$ and $S_U = S_L$, is*

$$\sum_{e=1}^{n-1} (e + 1) = \frac{n \cdot (n + 1)}{2} - 1, \quad (46)$$

for all $n \geq 2$.

- (iii) ([11, Corollary 4]) *The total number of discrete uninorms on L_n , with $n \geq 1$, such that their underlying operators are $T_U = T_L$ and $S_U = S_M$, is*

$$\sum_{e=1}^{n-1} (n - e + 1) = \frac{n \cdot (n + 1)}{2} - 1, \quad (47)$$

for all $n \geq 2$.

The cases offered by this result allow to determine the number of discrete uninorms whose underlying operators are smooth and Archimedean, or smooth and idempotent, since T_M (the minimum discrete t-norm) and S_M (the maximum discrete t-conorm) are the only discrete t-norm and discrete t-conorm, respectively, that are smooth and idempotent, and T_L and S_L are the only discrete t-norm and discrete t-conorm, respectively, that are smooth and Archimedean. Ruiz-Aguilera and Torrens proposed in [11, Proposition 7] an erroneous formula for the enumeration of these operators, which we reformulate and prove.

Proposition 3.20. ([11, Proposition 7]) *Let $n \geq 2$, and let us denote by $\mathcal{U}_n^{\text{STS}}$ and $\mathcal{U}_n^{\text{ARC,IDE}}$ the sets of discrete uninorms whose underlying operators are smooth and discrete uninorms whose underlying operators are Archimedean or idempotent, respectively, defined on L_n . Then,*

$$\left| \mathcal{U}_n^{\text{STS}} \cap \mathcal{U}_n^{\text{ARC,IDE}} \right| = 2^n + n \cdot (n + 1) - 10, \quad (48)$$

for all $n \geq 3$, with $\left| \mathcal{U}_2^{\text{STS}} \cap \mathcal{U}_2^{\text{ARC,IDE}} \right| = 2$ and $\left| \mathcal{U}_3^{\text{STS}} \cap \mathcal{U}_3^{\text{ARC,IDE}} \right| = 10$.

Proof. Let us focus when $n \geq 4$. Let U be a discrete uninorm whose underlying operators are smooth and idempotent or smooth and Archimedean. In the following, we distinguish some cases according to the position of the neutral element e of U .

- If $e = 1$, then the underlying discrete t-norm is defined on L_1 , and on this domain we have the particularity that $T_L(x, y) = T_M(x, y)$ for all $x, y \in L_1$, so these two operators represent the same situation, and have $\{0, 1\}$ as idempotent elements. Then, we only need to focus on the behavior of S_U . We distinguish two cases:
 - If S_U is smooth and Archimedean, then S_U has only 1 and n as idempotent elements. Applying statement (i) of Theorem 3.7, this case is set up by taking $r = 1$, $s = 1$ and $e = 1$, which yields

$$\binom{1+1}{1} \cdot \binom{0}{0} \cdot \binom{n-1-1}{0} = 2$$

operators.

- If S_U is smooth and idempotent, then it has $n - e + 1$ idempotent elements since it is defined on $[1, n]$. Therefore, applying statement (i) of Theorem 3.7, we have $r = 1$, $s = n - e$ and $e = 1$, giving

$$\binom{1+n-e}{1} \cdot \binom{0}{0} \cdot \binom{n-1-1}{n-1-1} = n$$

operators.

- If $2 \leq e \leq n - 2$, then we distinguish four possible cases, bearing in mind that the above scenario, where different operators collapse and are the same, cannot occur.

- T_U is smooth and Archimedean, and S_U is smooth and Archimedean. Applying statement (i) of Theorem 3.7 with $r = 1$ and $s = 1$, we have

$$\sum_{e=2}^{n-2} \binom{1+1}{1} \cdot \binom{e-1}{0} \cdot \binom{n-e-1}{0} = 2 \cdot (n-3)$$

operators on this case.

- T_U is smooth and Archimedean, and S_U is smooth and idempotent. Applying statement (i) of Theorem 3.7 with $r = 1$ and $s = n - e$, we have

$$\sum_{e=2}^{n-2} \binom{n-e+1}{1} \cdot \binom{e-1}{0} \cdot \binom{n-e-1}{n-e-1} = \frac{n \cdot (n-1) - 6}{2}$$

operators on this case.

- T_U is smooth and idempotent, and S_U is smooth and Archimedean. Applying statement (i) of Theorem 3.7 with $r = e$ and $s = 1$, we have

$$\sum_{e=2}^{n-2} \binom{e+1}{e} \cdot \binom{e-1}{e-1} \cdot \binom{n-e-1}{0} = \frac{n \cdot (n-1) - 6}{2}$$

operators on this case.

- T_U is smooth and idempotent, and S_U is smooth and idempotent. Applying statement (i) of Theorem 3.7 with $r = e$ and $s = n - e$, we have

$$\sum_{e=2}^{n-2} \binom{e+n-e}{e} \cdot \binom{e-1}{e-1} \cdot \binom{n-e-1}{n-e-1} = 2^n - 2n - 2$$

operators on this case.

- If $e = n - 1$. The case $e = n - 1$ is equivalent to the case $e = 1$, so $2 + n$ operators can be counted.

Gathering all the previous enumerations, which are mutually disjoint, the result holds. Finally, the cases $n \in \{2, 3\}$ can be obtained straightforwardly. \square

Remark 3.8. The formula proposed by Ruiz-Aguilera and Torrens in [11, Proposition 7] for the number of discrete uninorms whose underlying operators are smooth and Archimedean, or smooth and idempotent, is

$$2^n + n \cdot (n + 3) - 2.$$

In their proof, they did not take into account that on L_1 it happens that $T_M(x, y) = T_L(x, y)$ and $S_M(x, y) = S_L(x, y)$ for all $x, y \in L_1$. Hence, when they made the final count, these cases are considered as different although they are in fact the same.

To summarize this section, we present in Table 6 the initial values of some of the sequences discussed throughout this section. Specifically, we have computationally determined the total count of discrete uninorms, $|\mathcal{U}_n|$, for $n \leq 9$. Using the formula from Remark 3.5, which relies on \mathcal{T}_n and \mathcal{S}_n (whose values are known for $n \leq 13$), we have established $|\mathcal{U}_n^{\text{MIN}}|$ and $|\mathcal{U}_n^{\text{MAX}}|$ for $n \leq 13$. Employing Theorem 3.5 and Theorem 3.7, we provide the values of $|\mathcal{U}_n^{\text{IDE}}|$ and $|\mathcal{U}_n^{\text{STS}}|$, respectively, for $n \leq 13$. Lastly, we present some values of $|\mathcal{U}_n^{\text{STS}} \cap \mathcal{U}_n^{\text{ARC,IDE}}|$ with the corrected formula presented in Proposition 3.20.

n	$ \mathcal{U}_n $	$ \mathcal{U}_n^{\text{MIN}} , \mathcal{U}_n^{\text{MAX}} $	$ \mathcal{U}_n^{\text{IDE}} $	$ \mathcal{U}_n^{\text{STS}} $	$ \mathcal{U}_n^{\text{STS}} \cap \mathcal{U}_n^{\text{ARC,IDE}} $
1	$2 = 2 \cdot 10^0$	—	—	—	—
2	$6 = 6 \cdot 10^0$	$1 = 1 \cdot 10^0$	$4 = 4 \cdot 10^0$	$2 = 2 \cdot 10^0$	$2 = 6 \cdot 10^0$
3	$22 = 2.2 \cdot 10^1$	$4 = 4 \cdot 10^0$	$8 = 8 \cdot 10^0$	$10 = 1 \cdot 10^0$	$10 = 1 \cdot 10^1$
4	$92 = 9.2 \cdot 10^1$	$16 = 1.6 \cdot 10^1$	$16 = 1.6 \cdot 10^1$	$38 = 3.8 \cdot 10^1$	$26 = 2.6 \cdot 10^1$
5	$426 = 4.26 \cdot 10^2$	$68 = 6.8 \cdot 10^1$	$32 = 3.2 \cdot 10^1$	$130 = 1.3 \cdot 10^2$	$52 = 5.2 \cdot 10^1$
6	$2,146 = 2.146 \cdot 10^3$	$312 = 3.12 \cdot 10^2$	$64 = 6.4 \cdot 10^1$	$422 = 4.22 \cdot 10^2$	$96 = 9.6 \cdot 10^1$
7	$11,634 = 1.1634 \cdot 10^4$	$1,542 = 1.542 \cdot 10^3$	$128 = 1.28 \cdot 10^2$	$1,330 = 1.33 \cdot 10^3$	$174 = 1.74 \cdot 10^2$
8	$67,472 = 6.7472 \cdot 10^4$	$8,188 = 8.188 \cdot 10^3$	$256 = 2.56 \cdot 10^2$	$4,118 = 4.118 \cdot 10^3$	$318 = 3.18 \cdot 10^2$
9	$417,650 = 4.1765 \cdot 10^5$	$46,642 = 4.6642 \cdot 10^5$	$512 = 5.12 \cdot 10^2$	$12,610 = 1.261 \cdot 10^4$	$592 = 5.92 \cdot 10^2$
10	—	$285,246 \approx 2.8525 \cdot 10^5$	$1,024 = 1.024 \cdot 10^3$	$38,342 = 3.8342 \cdot 10^4$	$1,124 = 1.124 \cdot 10^3$
11	—	$1,881,718 \approx 1.8817 \cdot 10^6$	$2,048 = 2.048 \cdot 10^3$	$116,050 = 1.1605 \cdot 10^5$	$2,170 = 2.17 \cdot 10^3$
12	—	$13,549,087 \approx 1.3549 \cdot 10^7$	$4,096 = 4.096 \cdot 10^3$	$\approx 3.502 \cdot 10^5$	$4,242 = 4.242 \cdot 10^3$
13	—	$109,153,102 \approx 1.0915 \cdot 10^8$	$8,192 = 8.192 \cdot 10^3$	$\approx 1.0547 \cdot 10^6$	$8,364 = 8.364 \cdot 10^3$

Table 6

Enumeration of some classes of discrete uninorms for $n \leq 13$. From left to right: discrete uninorms in general, obtained computationally; discrete uninorms such that $U(x, y) = \min\{x, y\}$ or $U(x, y) = \max\{x, y\}$, for all $(x, y) \in C_e$, given by Remark 3.5 which depends on the number of discrete t-norms in general, whose general formula is unknown; idempotent discrete uninorms, given by Theorem 3.5; discrete uninorms having smooth underlying operators, given by Theorem 3.7; and finally discrete uninorms having smooth and idempotent or smooth and Archimedean underlying operators, given by Proposition 3.20.

3.6. Discrete t-operators and nullnorms

A common practice when studying discrete operators is to generalize specific classes with the overarching objective of locating various operators within a more general class, and identifying shared properties among them. This practice is illustrated in the study of discrete t-operators and nullnorms, that were originally thought of as generalizations of discrete t-norms and discrete t-conorms, alternative to the generalization offered by discrete uninorms. We start with the definition of discrete t-operator.

Definition 3.16. ([7, Definition 3.2], [57, Definition 3.9]) A discrete t-operator is an associative and commutative discrete aggregation function $O : L_n^2 \rightarrow L_n$ such that the mappings $O_0(x) = O(0, x)$ and $O_n(x) = O(n, x)$, for all $x \in L_n$, are smooth.

Discrete t-operators were originally introduced in [7] by Mas et al., who proposed these operators including the smoothness in the definition. However, it is noteworthy that t-operators on $[0, 1]$ do not inherently include any hypothesis related to continuity in the whole unit square [58], since continuity is only imposed on the segments $\{0\} \times [0, 1]$ and $\{1\} \times [0, 1]$. Su and Liu in [57] observed the need to eliminate this assumption of smoothness from the definition and, consequently, referred to discrete t-operators as discrete pseudo-t-operators. Even so, for consistency with the existing nomenclature in the literature, we will refer by discrete t-operators in general to the class of discrete aggregation functions that satisfy the properties of Definition 3.16, and by smooth discrete t-operators to those discrete t-operators that are smooth on L_n^2 .

The enumeration of discrete t-operators in the general case remains an open question. Nevertheless, it is worth highlighting that the characterization offered by Su and Liu as a combination of a discrete t-norm and a discrete t-conorm gives a path to derive a formula for their enumeration.

Theorem 3.8. ([57, Corollary 3.10]) A binary operator $O : L_n^2 \rightarrow L_n$ is a discrete t-operator if, and only if, there exist a discrete t-conorm S on $[0, k]$ and a discrete t-norm T on $[k, n]$, where $k = O(0, n)$, such that O is given by

$$O(x, y) = \begin{cases} S(x, y), & \text{if } (x, y) \in [0, k]^2, \\ T(x, y), & \text{if } (x, y) \in [k, n]^2, \\ k, & \text{otherwise,} \end{cases} \quad (49)$$

for all $x, y \in L_n$.

Let \mathcal{O}_n be the set of all discrete t-operators on L_n . A formula for $|\mathcal{O}_n|$ can be obtained in a manner similar to the formula derived for some classes of discrete uninorms, explained in Remark 3.5, also depending on the number of discrete t-norms and discrete t-conorms. This formula is given by

$$|\mathcal{O}_n| = |\mathcal{S}_n| + |\mathcal{T}_n| + \sum_{k=1}^{n-1} |\mathcal{S}_k| \cdot |\mathcal{T}_{n-k}| = 2 \cdot |\mathcal{T}_n| + \sum_{e=1}^{n-1} |\mathcal{T}_e| \cdot |\mathcal{T}_{n-e}|, \quad (50)$$

which interestingly coincides with the formula for $|\mathcal{U}_n^{\text{MIN}}|$ and $|\mathcal{U}_n^{\text{MAX}}|$. But further consequences can be drawn from Theorem 3.8. Mainly, that a discrete t-operator is smooth if, and only if, the underlying discrete t-norm and discrete t-conorm are smooth, a result also reached by Mas et al. in [7] with discrete t-operators. If we substitute $|\mathcal{T}_n|$ in Equation (50) with $|\mathcal{T}_n^{\text{SMT}}|$, we obtain a formula for the number of smooth discrete t-operators, which is equivalent to the number of discrete t-operators. And finally, if we look for the number of discrete t-operators that are idempotent, given that there is only one idempotent discrete t-norm and t-conorm, we can obtain a formula for this class of discrete t-operators.

Proposition 3.21. ([7, Corollary 3.8]) *For $n \geq 2$, the following statements hold:*

- (i) *If $\mathcal{O}_n^{\text{SMT}}$ denotes the set of smooth discrete t-operators, then $|\mathcal{O}_n^{\text{SMT}}| = 2^n + (n-1) \cdot 2^{n-2}$.*
- (ii) *If $\mathcal{O}_n^{\text{IDE}}$ denotes the set of idempotent discrete t-operators, then $|\mathcal{O}_n^{\text{IDE}}| = n + 1$.*

We finish this section with discrete nullnorms, introduced and characterized by Riera and Torrens in [59].

Definition 3.17. ([59, Definition 2.4]) *A discrete nullnorm is an associative and commutative discrete aggregation function $H : L_n^2 \rightarrow L_n$ which has an absorbing element $k \in L_n$ satisfying $H(0, x) = x$, for all $x \leq k$, and $H(n, x) = x$, for all $x \geq k$.*

In a similar way to what happens on $[0, 1]$, discrete nullnorms and discrete t-operators have different definitions (albeit with shared properties), Riera and Torrens characterized in [59, Proposition 2.7] discrete nullnorms with the same result as the one stated in Theorem 3.8; that is, an operator H is a discrete nullnorm if, and only if, it satisfies Equation (49). Given that these two classes of operators are essentially the same, it is evident that the enumerations discussed previously are also applicable to discrete nullnorms. Specifically, the number of discrete nullnorms, $|\mathcal{H}_n|$, is governed by the formula of Equation (50). Moreover, $|\mathcal{H}_n^{\text{SMT}}|$ and $|\mathcal{H}_n^{\text{IDE}}|$, representing the number of smooth and idempotent discrete nullnorms, respectively, are determined in Proposition 3.21. Table 7 provides examples of the enumerations discussed in this section for $n \leq 13$.

4. Enumeration of discrete implications and some subclasses

Discrete implications extend the concept of implication from Boolean logic, mirroring how discrete negations, discrete t-norms and discrete t-conorms broaden the notions of negation, conjunction and disjunction in Boolean logic, respectively. The two surveys [60, 61] compile a large number of properties of fuzzy implication functions², while in the two surveys [62, 63] we find sections dedicated to discrete implications. Initially, the concept of discrete implication was proposed by Nachtegaele and Kerre [6] under the name of *implicator*, and was studied in detail by Mas et al. in [64] establishing the usual nomenclature.

Definition 4.1. ([64, Definition 3]) *A discrete implication is a binary operation $I : L_n^2 \rightarrow L_n$ such that it is decreasing in the first argument, increasing in the second argument, $I(0, 0) = I(n, n) = n$ and $I(n, 0) = 0$.*

The cardinality of these operators already attracted the attention of Nachtegaele and Kerre in their seminal paper [6]. Computationally, they determined the number of discrete implications on L_n for $n \leq 5$, and it was not until years later that Munar et al. in [19] determined a general formula for the enumeration of discrete implications. Their procedure was based on establishing a bijection between discrete implications and discrete conjunctions.

²When the domain is $[0, 1]$, the implications defined on it are generally referred to as *fuzzy implications*.

n	$ \mathcal{O}_n , \mathcal{H}_n $	$ \mathcal{O}_n^{\text{SMT}}, \mathcal{H}_n^{\text{SMT}} $	$ \mathcal{O}_n^{\text{IDE}}, \mathcal{H}_n^{\text{IDE}} $
1	$2 = 2 \cdot 10^0$	$2 = 2 \cdot 10^0$	$2 = 2 \cdot 10^0$
2	$5 = 5 \cdot 10^0$	$5 = 5 \cdot 10^0$	$3 = 3 \cdot 10^0$
3	$16 = 1.6 \cdot 10^1$	$12 = 1.2 \cdot 10^1$	$4 = 4 \cdot 10^0$
4	$60 = 6 \cdot 10^1$	$28 = 2.8 \cdot 10^1$	$5 = 5 \cdot 10^0$
5	$256 = 2.56 \cdot 10^2$	$64 = 6.4 \cdot 10^1$	$6 = 6 \cdot 10^0$
6	$1,214 = 1.214 \cdot 10^3$	$144 = 1.44 \cdot 10^2$	$7 = 7 \cdot 10^0$
7	$6,314 = 6.314 \cdot 10^3$	$320 = 3.2 \cdot 10^2$	$8 = 8 \cdot 10^0$
8	$35,738 = 3.5738 \cdot 10^4$	$704 = 7.04 \cdot 10^2$	$9 = 9 \cdot 10^0$
9	$219,476 \approx 2.1948 \cdot 10^5$	$1,536 = 1.536 \cdot 10^3$	$10 = 1 \cdot 10^1$
10	$1,466,224 \approx 1.4662 \cdot 10^6$	$3,328 = 3.328 \cdot 10^3$	$11 = 1.1 \cdot 10^1$
11	$10,773,776 \approx 1.0774 \cdot 10^7$	$7,168 = 7.168 \cdot 10^3$	$12 = 1.2 \cdot 10^1$
12	$89,287,985 \approx 8.9288 \cdot 10^7$	$15,360 = 1.536 \cdot 10^4$	$13 = 1.3 \cdot 10^1$
13	$874,252,030 \approx 8.7425 \cdot 10^8$	$32,768 = 3.2768 \cdot 10^4$	$14 = 1.4 \cdot 10^1$

Table 7

Enumeration of some classes of discrete t-operators and nullnorms, for $n \leq 13$. From left to right: discrete t-operators and nullnorms in general, obtained with Equation (50), which depends on the number of discrete t-norms in general, which is unknown; smooth discrete t-operators and nullnorms, given by Proposition 3.21; and finally idempotent discrete t-operators and nullnorms, given by Proposition 3.21.

Proposition 4.1. ([19, Proposition 4.1]) *Let $n \geq 1$, and let us denote by C_n and I_n the sets of discrete conjunctions and discrete implications, respectively, defined on L_n . Let $\varphi : C_n \rightarrow I_n$ be defined point-wise by*

$$\varphi(C)(x, y) = n - C(x, n - y),$$

for all $x, y \in L_n$, where $C \in C_n$. Then, $\varphi(C)$ is a discrete implication, and the transformation φ is a bijection.

From this result, we have the following consequences. Firstly, that the number of discrete implications defined on L_n , $|I_n|$, coincides with $|C_n|$, which is given in Proposition 3.5 (see [19, Corollary 4.1]). Secondly, if we denote by C_n^{SMT} and I_n^{SMT} the sets of smooth discrete conjunctions and smooth discrete implications, respectively, the same formula of φ given in Proposition 4.1 but restricted to these sets of smooth operators also remains a bijection, so that I_n^{SMT} is given by Proposition 3.9 (see [20, Corollary 3.2.2]).

The definition of discrete implication is quite generic, and as such, it does not allow to derive certain properties which are useful in different situations. Consequently, the literature has proposed several additional properties for discrete implications, and also their cardinality has been investigated. The main ones are listed below.

- The *contrapositive symmetry* with respect to a discrete negation N ,

$$I(x, y) = I(N(y), N(x)), \quad x, y \in L_n. \quad (\text{CP}(N))$$

- The *left neutrality principle*,

$$I(n, y) = y, \quad y \in L_n. \quad (\text{NP})$$

- The *consequent boundary*,

$$I(x, y) \geq y, \quad x, y \in L_n. \quad (\text{CB})$$

- The *ordering principle*,

$$I(x, y) = n \Leftrightarrow x \leq y, \quad x, y \in L_n. \quad (\text{OP})$$

- The *identity principle*,

$$I(x, x) = n, \quad x \in L_n. \quad (\text{IP})$$

- The *exchange principle*,

$$I(x, I(y, z)) = I(y, I(x, z)), \quad x, y, z \in L_n. \quad (\text{EP})$$

- The *generalized modus ponens*, with respect to a discrete t-norm T ,

$$T(x, I(x, y)) \leq y, \text{ for all } x, y \in L_n. \quad (\text{MP}(T))$$

- The *generalized modus tollens*, with respect to a discrete t-norm T and a discrete negation N ,

$$T(N(y), I(x, y)) \leq N(x), \text{ for all } x, y \in L_n. \quad (\text{MT}(T, N))$$

- The *law of importation*, with respect to a discrete t-norm T ,

$$I(T(x, y), z) = I(x, I(y, z)), \text{ for all } x, y, z \in L_n. \quad (\text{LI}(T))$$

In the following sections we compile enumerations of discrete implications satisfying some of the above properties. In particular, formulas are known for $(\text{CP}(N_C))$ being N_C the classical discrete negation, (NP) , (CB) , (OP) and (IP) . In Table 8 we list the first values of these enumerations. For the remaining properties, a general enumeration remains an open problem.

4.1. Discrete implications satisfying $(\text{CP}(N))$

We start with the contrapositive symmetry with respect to a discrete negation N . In [19, Proposition 4.6] it was observed that the transformation φ of Proposition 4.1 establishes a bijection between C_n^{COM} and $I_n^{\text{CP}(N_C)}$, which denotes the set of discrete implications satisfying $(\text{CP}(N_C))$, i.e., satisfying the contrapositive symmetry with respect to the only strong discrete negation, $N_C(x) = n - x$ for all $x \in L_n$. For this reason, the enumeration of $I_n^{\text{CP}(N_C)}$ is given by Proposition 3.6. In addition to N_C , it is also possible to determine the number of discrete implications satisfying $(\text{CP}(N_{D_1}))$ and $(\text{CP}(N_{D_2}))$, where N_{D_1} and N_{D_2} are the least and the greatest discrete negations, respectively, and are given by

$$N_{D_1}(x) = \begin{cases} n, & \text{if } x = 0, \\ 0, & \text{otherwise,} \end{cases} \quad N_{D_2}(x) = \begin{cases} n, & \text{if } x \neq n, \\ 0, & \text{otherwise,} \end{cases}$$

for all $x \in L_n$. Imposing the functional equation of $(\text{CP}(N))$ for each discrete negation and distinguishing different cases according to the position of $x, y \in L_n$, we have that there is only one discrete implication that satisfies $(\text{CP}(N_{D_1}))$, and it is given by

$$I_1(x, y) = \begin{cases} 0, & \text{if } y = 0 \text{ and } 1 \leq x \leq n, \\ n, & \text{otherwise,} \end{cases}$$

for all $x, y \in L_n$. Similarly, we find that there is only one discrete implication that satisfies $(\text{CP}(N_{D_2}))$, and it is given by

$$I_2(x, y) = \begin{cases} 0, & \text{if } x = n \text{ and } 0 \leq y \leq n - 1, \\ n, & \text{otherwise,} \end{cases}$$

for all $x, y \in L_n$. Beyond these cases, to date, there is no formula for $|I_n^{\text{CP}}|$ in general, i.e., the number of discrete implications satisfying $(\text{CP}(N))$ with respect to some discrete negation N , nor is there a computationally obtained enumeration of it.

4.2. Discrete implications satisfying (NP)

Nachtegaal and Kerre [6] conducted a computational analysis to determine the number of discrete implications satisfying (NP) , denoted as I_n^{NP} , for $n \leq 6$. Subsequently, Munar et al. in [19] introduced a recursive formula to calculate $|I_n^{\text{NP}}|$ for any value of n . This approach also relied on plane partitions, specifically on those called (α, β) -monotonic plane partitions, which were initially proposed by Krattenthaler in [65, 66]. These plane partitions can be described as those satisfying $\pi_{i,j} \geq \pi_{i,j+1} + \alpha$ and $\pi_{i,j} \geq \pi_{i+1,j} + \beta$ (following the same nomenclature as in Definition 3.4) whenever both sides are defined. Krattenthaler [65]

also provided a closed formula for enumerating (α, β) -plane partitions whose components of each column are upper-bounded by the corresponding entry of a decreasing c -tuple \mathbf{v} , where c is equal to the number of columns. Although this formula seems adequate at first sight to count these plane partitions, even setting $\alpha = 1$ and $\mathbf{v} = (n, n - 1, \dots, 2, 1)$ does not give the desired count, since this combination of parameters determines the number of plane partitions whose first row is bounded component-wise by \mathbf{v} , and not the plane partitions whose first row is equal to \mathbf{v} , which is the one we are looking for. Consequently, Munar et al. extended this formula and devised a formula for $(1, 1)$ -monotonic plane partitions that the first row is equal to the identity. This formula centers around two operators, denoted as Θ and Υ , which are defined as follows. Given a decreasing n -tuple $\mathbf{v} = (v_1, \dots, v_n) \in L_n^n$:

- $\Theta(\mathbf{v})$ is the number of binary discrete operators $F : (L_n \setminus \{0\})^2 \rightarrow L_n$ which are increasing in each argument and $F(n, y) \leq v_{n-y+1}$ for all $y \in L_n \setminus \{0\}$.
- $\Upsilon(\mathbf{v})$ is the number of binary discrete operators $F : (L_n \setminus \{0\})^2 \rightarrow L_n$ which are increasing in each argument and $F(n, y) = v_{n-y+1}$ for all $y \in L_n \setminus \{0\}$.

That said, it only remains to calculate $\Theta(\mathbf{v})$ and $\Upsilon(\mathbf{v})$ for any decreasing n -tuple \mathbf{v} .

Proposition 4.2. *Let $\mathbf{v} \in L_n^n$ be a decreasing n -tuple. Then, the following statements hold:*

- ([19, Proposition 5.2]) $\Theta(\mathbf{v}) = \det A(\mathbf{v})$, where $A(\mathbf{v})$ is the $n \times n$ matrix whose entries are

$$A(\mathbf{v})_{i,j} = \binom{n + v_j}{n - i + j},$$

for all $i, j \in \{1, \dots, n\}$.

- ([19, Proposition 5.3]) If \mathbf{v} is reduced, i.e., $\mathbf{v} = (k, 0, \dots, 0) \in L_n^n$ for some $k \in L_n$, then

$$\Upsilon(\mathbf{v}) = \binom{n + k - 1}{k}.$$

- [19, Section 5.1] The general formula of $\Upsilon(\mathbf{v})$ is

$$\Upsilon(\mathbf{v}) = \begin{cases} \binom{n+v_1-1}{v_1}, & \text{if } \mathbf{v} \text{ is reduced,} \\ \Theta(\mathbf{v}) - \sum_{\mathbf{x} \in V_n(\mathbf{v}) \setminus \mathbf{v}} \Upsilon(\mathbf{x}), & \text{otherwise,} \end{cases}$$

where $V_n(\mathbf{v}) = \{\mathbf{x} \in L_n^n \mid x_i \leq v_i \text{ and } x_i \geq x_{i+1} \text{ for all } i \in \{1, \dots, n\}\}$.

Clearly, the formula being sought for $|\mathcal{I}_n^{\text{NP}}|$ is the operator Υ evaluated on the n -tuple equal to the identity. Table 8 summarizes some of its values, and we note that since it is a recursive formula it has a high computational cost and therefore it has only been listed for $n \leq 10$.

Proposition 4.3. ([19, Proposition 5.5]) *Let $n \geq 1$ and let us denote by $\mathcal{I}_n^{\text{NP}}$ the set of discrete implications defined on L_n satisfying (NP). Then,*

$$|\mathcal{I}_n^{\text{NP}}| = \Upsilon((n, n - 1, \dots, 2, 1)). \quad (51)$$

Remark 4.1. *In the definition of Θ and Υ , it is stated that they correspond to the enumeration of a certain type of increasing operators in both arguments. This is because in [19] the authors actually exploited the bijection between discrete implications and discrete conjunctions to determine this formula. Indeed, the enumeration of Proposition 4.3 is also true for discrete conjunctions that have neutral element n only in the first argument [19, Proposition 5.1].*

Remark 4.2. *It is indeed remarkable that the computation of $\Theta(\mathbf{v})$, where $\mathbf{v} \in L_n^n$, is achieved through the use of a determinant. This approach is a direct consequence of the Lindström-Gessel-Viennot theorem, which was independently proven by Lindström in [67] and by Gessel and Viennot in [68]. This theorem states that the number of paths within a directed graph (a technique commonly employed for counting plane partitions) is equal to the determinant of a matrix. The entries of this matrix correspond to enumerations of simpler cases within this counting process.*

4.3. Discrete implications satisfying (CB)

For discrete implications satisfying the consequent boundary, $\mathcal{I}_n^{\text{CB}}$, Nachtegaele and Kerre [6] determined their cardinality for $n \leq 5$. Similarly, the authors of [19] identified that from the operators Θ and Υ , introduced in Section 4.2, it is possible to determine a general formula for $|\mathcal{I}_n^{\text{CB}}|$: instead of evaluating Υ only on $(n, n-1, \dots, 2, 1)$, it is evaluated on all decreasing n -tuples \mathbf{v} satisfying $v_i \leq i$, and all partial counts are summed. The following proposition establishes this count. As in the enumeration of (NP), for (CB) the proposed formula is also recursive with a high computational cost, so that in Table 8 its values have been illustrated for $n \leq 9$.

Proposition 4.4. ([19, Proposition 5.6]) *Let $n \geq 1$ and let us denote by $\mathcal{I}_n^{\text{CB}}$ the set of discrete implications defined on L_n satisfying (CB). Then,*

$$|\mathcal{I}_n^{\text{CB}}| = \sum_{\mathbf{v} \in \tilde{V}_n} \Upsilon(\mathbf{v}), \quad (52)$$

where $\tilde{V}_n = \{\mathbf{v} \in L_n^n \mid v_1 = n, v_n \leq 1, v_i \leq n - i + 1 \text{ and } v_i \geq v_{i+1} \text{ for all } 1 \leq i \leq n - 1\}$.

Remark 4.3. *Similar to Remark 4.1, the formula of Proposition 4.4 also applies for the enumeration of those discrete conjunctions satisfying $C(x, y) \leq y$ for all $x, y \in L_n$, as stated in [19, Proposition 5.1].*

Remark 4.4. *It is trivial to show that if a discrete implication satisfies (NP), then it also satisfies (CB), so that $\mathcal{I}_n^{\text{NP}} \subseteq \mathcal{I}_n^{\text{CB}}$. This is confirmed in the enumeration shown in Table 8, since $|\mathcal{I}_n^{\text{NP}}| \leq |\mathcal{I}_n^{\text{CB}}|$ for all $n \geq 1$.*

4.4. Discrete implications satisfying (OP)

Setting aside the formulations outlined in Sections 4.2 and 4.3, which are centered on the operators Θ and Υ , and rooted in the framework of (α, β) -monotonic plane partitions, the authors of [19] also determined the enumeration of discrete implications satisfying (OP), using the so called *staircased plane partitions*, whose origins and their enumeration are also attributed to Krattenthaler in [65]. Nachtegaele and Kerre [6] also determined the number of discrete implications that satisfy (OP), for $n \leq 6$.

Following the notation in Definition 3.4, a plane partition is staircased when there exist positive and non-zero integers k and l such that its shape $\Lambda = (\lambda_1, \dots, \lambda_r)$ satisfies $\lambda_i = l - k \cdot i$ for all $1 \leq i \leq r$. The enumeration proposed by Krattenthaler in [65, Corollary 10] does not directly yield the formula of this class of discrete implications. Therefore, similarly to their approach with conjunctions, the authors of [19] decomposed $\mathcal{I}_n^{\text{OP}}$ into distinct subsets, each of which is in bijection with certain sets of staircased plane partitions. The resulting formula they derived is given in the following proposition.

Proposition 4.5. ([19, Proposition 5.7]) *Let $n \geq 1$, and let us denote by $\mathcal{I}_n^{\text{OP}}$ the set of discrete implications defined on L_n satisfying (OP). Then,*

$$\begin{aligned} |\mathcal{I}_n^{\text{OP}}| = & \left(\left(\prod_{i=1}^n \frac{(2n-i)!}{(n-2+i)!} \right) \left(\prod_{i=2}^n \prod_{j=0}^{i-2} \frac{3n+1-i+2j}{2} \right) - \left(\prod_{i=1}^n \frac{(2n-i-1)!}{(n-3+i)!} \right) \left(\prod_{i=2}^n \prod_{j=0}^{i-2} \frac{3n-1-i+2j}{2} \right) \right) \\ & \cdot \left(2^{\frac{n(n-1)}{2}} \right) \left(\prod_{i=1}^{n-1} i^{n-i} \right) \left(\prod_{i=1}^n \frac{1}{(2n+1-2i)!} \right) \end{aligned} \quad (53)$$

for all $n \geq 2$, with $|\mathcal{I}_1^{\text{OP}}| = 1$.

Remark 4.5. *As stated in [19, Proposition 5.1], the formula of Proposition 4.5 for $|\mathcal{I}_n^{\text{OP}}|$ is applicable to enumerate those conjunctions that satisfy $C(x, y) = 0$ if, and only if, $y \leq n - x$, with $x, y \in L_n$.*

4.5. Discrete implications satisfying (IP)

In this section, we present the formula proposed in [19] for the enumeration of discrete implications satisfying (IP), which is also founded on the concept of staircased plane partitions. Similar to the preceding sections, Nachtegaele and Kerre computationally determined its cardinality for $n \leq 5$.

n	$ \mathcal{I}_n $	$ \mathcal{I}_n^{\text{SMT}} $	$ \mathcal{I}_n^{\text{CP}(N_C)} $	$ \mathcal{I}_n^{\text{NP}} $	$ \mathcal{I}_n^{\text{CB}} $	$ \mathcal{I}_n^{\text{OP}} $	$ \mathcal{I}_n^{\text{IP}} $
1	$= 1 \cdot 10^0$	$= 1 \cdot 10^0$	$= 1 \cdot 10^0$	$= 1 \cdot 10^0$	$= 1 \cdot 10^0$	$= 1 \cdot 10^0$	$= 1 \cdot 10^0$
2	$= 1.4 \cdot 10^1$	$= 2 \cdot 10^0$	$= 6 \cdot 10^0$	$= 5 \cdot 10^0$	$= 8 \cdot 10^0$	$= 4 \cdot 10^0$	$= 9 \cdot 10^0$
3	$= 8.05 \cdot 10^2$	$= 7 \cdot 10^0$	$= 7.7 \cdot 10^1$	$= 8.4 \cdot 10^1$	$= 2.05 \cdot 10^2$	$= 7 \cdot 10^1$	$= 2.46 \cdot 10^2$
4	$\approx 2.0815 \cdot 10^5$	$= 4.2 \cdot 10^1$	$= 2.1 \cdot 10^3$	$= 4.719 \cdot 10^3$	$= 1.7108 \cdot 10^4$	$= 4.125 \cdot 10^3$	$= 2.1307 \cdot 10^4$
5	$\approx 2.5041 \cdot 10^8$	$= 4.29 \cdot 10^2$	$\approx 1.2269 \cdot 10^5$	$\approx 8.8488 \cdot 10^5$	$\approx 4.6936 \cdot 10^6$	$\approx 8.0309 \cdot 10^5$	$\approx 5.9679 \cdot 10^6$
6	$\approx 1.4234 \cdot 10^{12}$	$= 7.436 \cdot 10^3$	$\approx 1.5459 \cdot 10^7$	$\approx 5.5336 \cdot 10^8$	$\approx 4.2538 \cdot 10^9$	$\approx 5.1624 \cdot 10^8$	$\approx 5.4648 \cdot 10^9$
7	$\approx 3.8534 \cdot 10^{16}$	$\approx 2.1835 \cdot 10^5$	$\approx 4.2204 \cdot 10^9$	$\approx 1.1535 \cdot 10^{12}$	$\approx 1.2769 \cdot 10^{13}$	$\approx 1.0977 \cdot 10^{12}$	$\approx 1.6465 \cdot 10^{13}$
8	$\approx 4.9888 \cdot 10^{21}$	$\approx 1.085 \cdot 10^7$	$\approx 2.5045 \cdot 10^{12}$	$\approx 8.0122 \cdot 10^{15}$	$\approx 1.2715 \cdot 10^{17}$	$\approx 7.7337 \cdot 10^{15}$	$\approx 1.6387 \cdot 10^{17}$
9	$\approx 3.0961 \cdot 10^{27}$	$\approx 9.1184 \cdot 10^8$	$\approx 3.238 \cdot 10^{15}$	$\approx 1.8542 \cdot 10^{20}$	$\approx 4.2044 \cdot 10^{21}$	$\approx 1.808 \cdot 10^{20}$	$\approx 5.4014 \cdot 10^{21}$
10	$\approx 9.2223 \cdot 10^{33}$	$\approx 1.2953 \cdot 10^{11}$	$\approx 9.1352 \cdot 10^{18}$	$\approx 4.6196 \cdot 10^{26}$	—	$\approx 1.404 \cdot 10^{25}$	$\approx 5.9067 \cdot 10^{26}$
11	$\approx 1.3195 \cdot 10^{41}$	$\approx 3.1096 \cdot 10^{13}$	$\approx 5.6307 \cdot 10^{22}$	—	—	$\approx 3.6241 \cdot 10^{30}$	$\approx 2.1453 \cdot 10^{32}$
12	$\approx 9.0708 \cdot 10^{48}$	$\approx 1.2611 \cdot 10^{16}$	$\approx 7.5891 \cdot 10^{26}$	—	—	$\approx 3.1111 \cdot 10^{36}$	$\approx 2.59 \cdot 10^{38}$
13	$\approx 2.997 \cdot 10^{57}$	$\approx 8.6394 \cdot 10^{18}$	$\approx 2.2381 \cdot 10^{31}$	—	—	$\approx 8.8857 \cdot 10^{42}$	$\approx 1.0399 \cdot 10^{45}$

Table 8

Enumeration of some classes of discrete implications, for $n \leq 13$. From left to right: discrete implications in general, given by Proposition 3.5; smooth discrete implications, given by Proposition 3.9; discrete implications satisfying $(\text{CP}(N_C))$, given by Proposition 3.6; discrete implications satisfying (NP) , given by Proposition 4.3; discrete implications satisfying (CB) , given by Proposition 4.4; discrete implications satisfying (OP) , given by Proposition 4.5; and finally discrete implications satisfying (IP) , given by Proposition 4.6.

Proposition 4.6. ([19, Proposition 5.8]) *Let $n \geq 1$, and let us denote by $\mathcal{I}_n^{\text{IP}}$ the set of discrete implications defined on L_n satisfying (IP) . Then,*

$$|\mathcal{I}_n^{\text{IP}}| = \left(\left(\prod_{i=1}^n \frac{(2n-i+1)!}{(n+i-1)!} \right) \left(\prod_{i=2}^n \prod_{j=0}^{i-2} \frac{3n+3-i+2j}{2} \right) - \left(\prod_{i=1}^n \frac{(2n-i)!}{(n-2+i)!} \right) \left(\prod_{i=2}^n \prod_{j=0}^{i-2} \frac{3n+1-i+2j}{2} \right) \right) \cdot \left(2^{\frac{n(n-1)}{2}} \right) \left(\prod_{i=1}^{n-1} i^{n-i} \right) \left(\prod_{i=1}^n \frac{1}{(2n+1-2i)!} \right) \quad (54)$$

for all $n \geq 2$, with $|\mathcal{I}_1^{\text{IP}}| = 1$.

Remark 4.6. *As stated in [19, Proposition 5.1], the formula of Proposition 4.6 for $|\mathcal{I}_n^{\text{IP}}|$ is applicable to enumerate those conjunctions that satisfy $C(x, y) = 0$ for all $y \leq n - x$, with $x, y \in L_n$.*

Remark 4.7. *Remark 4.4 shows that the relationship between (NP) and (CB) is observed through the enumeration of both classes of discrete implications. Similarly, this is also the case for (OP) and (IP) . If a discrete implication satisfies (OP) , it also satisfies (IP) , so $\mathcal{I}_n^{\text{OP}} \subseteq \mathcal{I}_n^{\text{IP}}$ and consequently $|\mathcal{I}_n^{\text{OP}}| \leq |\mathcal{I}_n^{\text{IP}}|$ for all $n \geq 1$, as it can be shown in Table 8.*

5. New formulas for the enumeration of some classes of operators

In this section, we further explore the enumerations of operators on L_n , presenting some novel formulas. In Section 5.1, we prove the enumeration of smooth discrete aggregation functions with a neutral element, or with an absorbing element. In Section 5.2, we present the enumeration of idempotent discrete conjunctions and commutative idempotent discrete conjunctions.

5.1. Enumeration of smooth discrete aggregation functions having neutral or absorbing element

In this section, we focus on the enumeration of two classes of smooth discrete aggregation functions: those with a neutral element, and those with an absorbing element. The proposed formulas are constructed from the existing characterizations of these discrete aggregation functions, proposed by Mas et al. in [9].

These results rely on the combination of a discrete quasi-copula (see Section 3.3 for more information), and a discrete dual quasi-copula, which we recall below.

Definition 5.1. ([9]) A discrete dual quasi-copula is a discrete disjunction $Q' : L_n^2 \rightarrow L_n$ which is 1-Lipschitz and has 0 as neutral element.

Discrete dual quasi-copulas were originally introduced by Mas et al. in [9]; however, the primary focus of their work was to study smooth discrete aggregation functions so an enumeration of discrete dual quasi-copulas is not yet known. It is important to highlight that the main difference between discrete quasi-copulas and discrete dual quasi-copulas lies in the location of the neutral element. This distinction serves as a reminder of the duality observed between discrete conjunctions and discrete disjunctions, and therefore we can derive the following result.

Proposition 5.1. Let $n \geq 1$, and let us denote by \mathcal{Q}_n and \mathcal{Q}'_n the sets of discrete quasi-copulas and discrete dual quasi-copulas, respectively, defined on L_n . Let $\varphi : \mathcal{Q}_n \rightarrow \mathcal{Q}'_n$ be defined point-wise by

$$\varphi(Q)(x, y) = n - Q(n - x, n - y),$$

for all $x, y \in L_n$, where $Q \in \mathcal{Q}_n$. Then, $\varphi(Q)$ is a discrete dual quasi-copula, and the transformation φ is a bijection.

Proof. Let $Q' = \varphi(Q)$. It is straightforward to show that Q is monotone if, and only if, Q' is; Q is 1-Lipschitz if, and only if, Q' is; and finally that Q has neutral element n if, and only if, Q' has neutral element 0. Moreover, $\varphi^{-1}(Q') = n - Q'(n - x, n - y)$, for all $x, y \in L_n$, so φ is a bijection. \square

We have discussed in Section 3.3 that, actually, discrete quasi-copulas are nothing more than smooth discrete conjunctions. With this result, we can therefore conclude that discrete dual quasi-copulas proposed in [9] are really smooth discrete disjunctions. Then, their enumeration is given by Proposition 3.9. Said that, we reformulate the characterization of smooth discrete aggregation functions with neutral element or absorbing element $\alpha \in L_n$ on the basis of smooth discrete conjunctions and disjunctions.

Theorem 5.1. ([9, Theorem 2]) Let $\alpha \in L_n$ and $F : L_n^2 \rightarrow L_n$ be a binary operator on L_n . Then, the following statements are equivalent:

- (i) F is a smooth discrete aggregation function with absorbing element α .
- (ii) There exist a smooth discrete conjunction C on $[0, \alpha]$ and a smooth discrete disjunction D on $[\alpha, n]$ such that F is given by

$$F(x, y) = \begin{cases} D(x, y), & \text{if } (x, y) \in [0, \alpha]^2, \\ C(x, y), & \text{if } (x, y) \in [\alpha, n]^2, \\ \alpha, & \text{otherwise,} \end{cases} \quad (55)$$

for all $x, y \in L_n$.

Theorem 5.2. ([9, Theorem 3]) Let $\alpha \in L_n$ and $F : L_n^2 \rightarrow L_n$ be a binary operator on L_n . Then, the following statements are equivalent:

- (i) F is a smooth discrete aggregation function with neutral element α .
- (ii) There exist a smooth discrete conjunction C on $[0, \alpha]$ and a smooth discrete disjunction D on $[\alpha, n]$ such that F is given by

$$F(x, y) = \begin{cases} C(x, y), & \text{if } (x, y) \in [0, \alpha]^2, \\ D(x, y), & \text{if } (x, y) \in [\alpha, n]^2, \\ x + y - \alpha, & \text{otherwise,} \end{cases} \quad (56)$$

for all $x, y \in L_n$.

With these results, we propose the enumeration of discrete aggregation functions that have absorbing element or neutral element $\alpha \in L_n$. Table 9 illustrates the first 13 values of this enumeration.

Proposition 5.2. *For all $n \geq 2$, the following statements hold:*

- (i) *The number $|\mathcal{A}_n^{\text{SMT}} \cap \mathcal{A}_n^{\text{ABS}(\alpha)}|$ of smooth discrete aggregation functions on L_n with absorbing element $\alpha \in L_n \setminus \{0, n\}$, coincides with the number $|\mathcal{A}_n^{\text{SMT}} \cap \mathcal{A}_n^{\text{NE}(\alpha)}|$ of smooth discrete aggregation functions on L_n with neutral element $\alpha \in L_n \setminus \{0, n\}$, and are given by*

$$|\mathcal{A}_n^{\text{SMT}} \cap \mathcal{A}_n^{\text{ABS}(\alpha)}| = |\mathcal{A}_n^{\text{SMT}} \cap \mathcal{A}_n^{\text{NE}(\alpha)}| = \left(\prod_{i=0}^{\alpha-1} \frac{(3i+1)!}{(\alpha+i)!} \right) \left(\prod_{i=0}^{n-\alpha-1} \frac{(3i+1)!}{(n-\alpha+i)!} \right). \quad (57)$$

- (ii) *The number $|\mathcal{A}_n^{\text{SMT}} \cap \mathcal{A}_n^{\text{ABS}}|$ of smooth discrete aggregation functions on L_n with absorbing element, coincides with the number $|\mathcal{A}_n^{\text{SMT}} \cap \mathcal{A}_n^{\text{NE}}|$ of smooth discrete aggregation functions on L_n with neutral element, and are given by*

$$|\mathcal{A}_n^{\text{SMT}} \cap \mathcal{A}_n^{\text{ABS}}| = |\mathcal{A}_n^{\text{SMT}} \cap \mathcal{A}_n^{\text{NE}}| = 2 \cdot \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} + \sum_{\alpha=1}^{n-1} \left(\prod_{i=0}^{\alpha-1} \frac{(3i+1)!}{(\alpha+i)!} \right) \left(\prod_{i=0}^{n-\alpha-1} \frac{(3i+1)!}{(n-\alpha+i)!} \right). \quad (58)$$

Proof. Let us prove each statement. Regarding the first one, we prove the case for the absorbing element α . Using the characterization of Theorem 5.1, given $\alpha \in L_n \setminus \{0, n\}$, the number of smooth discrete aggregation functions with absorbing element α is given by the product of the number of smooth discrete disjunctions that can be constructed on $[0, \alpha]$, and the number of smooth discrete conjunctions that can be constructed on $[\alpha, n]$. The first of these two enumerations is straightforward, and is given by Proposition 3.9. Regarding the second, let us observe that a binary operator C is a discrete conjunction on an interval $[a, b]$, with $a < b$ if, and only if, there exists a discrete conjunction C' in L_{b-a} such that $C'(x, y) = C(x+a, y+a) - a$, for all $x, y \in L_{b-a}$. Therefore, the number of smooth discrete conjunctions that can be constructed on $[\alpha, n]$ is equal to the number of smooth discrete conjunctions that can be constructed on $L_{n-\alpha}$, which is also given by Proposition 3.9. For the case involving the neutral element α , an analogous argument proves the equality.

Let us look at the second statement, and focus on the enumeration of $\mathcal{A}_n^{\text{SMT}} \cap \mathcal{A}_n^{\text{ABS}}$, which trivially is equal to

$$|\mathcal{A}_n^{\text{SMT}} \cap \mathcal{A}_n^{\text{ABS}}| = \sum_{\alpha \in L_n} |\mathcal{A}_n^{\text{SMT}} \cap \mathcal{A}_n^{\text{ABS}(\alpha)}|.$$

When $\alpha = 0$ or $\alpha = n$, the operators of $\mathcal{A}_n^{\text{SMT}} \cap \mathcal{A}_n^{\text{ABS}(\alpha)}$ are smooth discrete conjunctions or smooth discrete disjunctions, respectively, whose enumeration is the same and they are given by Proposition 3.9. When $1 \leq \alpha \leq n-1$, $|\mathcal{A}_n^{\text{SMT}} \cap \mathcal{A}_n^{\text{ABS}(\alpha)}|$ is given in the previous statement, and therefore the formula holds. Finally, for the enumeration of $\mathcal{A}_n^{\text{SMT}} \cap \mathcal{A}_n^{\text{NE}}$, it suffices to reproduce a similar argument. \square

Remark 5.1. *As indicated in the proof of Proposition 5.2, if the absorbing element in statement (i) is $\alpha = 0$ or $\alpha = n$, then the formula of $|\mathcal{A}_n^{\text{SMT}} \cap \mathcal{A}_n^{\text{ABS}(\alpha)}|$ coincides with that of Proposition 3.9. The same fact occurs if α is a neutral element.*

5.2. Enumeration of idempotent discrete conjunctions

In this section, we delve into the realm of idempotent discrete conjunctions. To initiate our exploration, we shall consider an idempotent discrete conjunction C on L_n , which can be visually represented by the following diagram:

n	$ \mathcal{A}_n^{\text{SMT}} \cap \mathcal{A}_n^{\text{ABS}} $	$ \mathcal{A}_n^{\text{SMT}} \cap \mathcal{A}_n^{\text{NE}} $	$ \mathcal{C}_n^{\text{COM}} \cap \mathcal{C}_n^{\text{IDE}} $	$ \mathcal{C}_n^{\text{IDE}} $
1	$2 = 2 \cdot 10^0$	$2 = 2 \cdot 10^0$	$1 = 1 \cdot 10^0$	$1 = 1 \cdot 10^0$
2	$5 = 5 \cdot 10^0$	$5 = 5 \cdot 10^0$	$2 = 2 \cdot 10^0$	$4 = 4 \cdot 10^0$
3	$18 = 1.8 \cdot 10^1$	$18 = 1.8 \cdot 10^1$	$8 = 8 \cdot 10^0$	$64 = 6.4 \cdot 10^1$
4	$102 = 1.02 \cdot 10^2$	$102 = 1.02 \cdot 10^2$	$64 = 6.4 \cdot 10^1$	$4,096 = 4.096 \cdot 10^3$
5	$970 = 9.7 \cdot 10^2$	$970 = 9.7 \cdot 10^2$	$1,024 = 1.024 \cdot 10^3$	$\approx 1.0486 \cdot 10^6$
6	$15,947 = 1.5947 \cdot 10^4$	$15,947 = 1.5947 \cdot 10^4$	$32,768 = 3.2768 \cdot 10^4$	$\approx 1.0737 \cdot 10^9$
7	$\approx 4.5387 \cdot 10^5$	$\approx 4.5387 \cdot 10^5$	$\approx 2.0972 \cdot 10^6$	$\approx 4.398 \cdot 10^{12}$
8	$\approx 2.2175 \cdot 10^7$	$\approx 2.2175 \cdot 10^7$	$\approx 2.6844 \cdot 10^8$	$\approx 7.2058 \cdot 10^{16}$
9	$\approx 1.8464 \cdot 10^9$	$\approx 1.8464 \cdot 10^9$	$\approx 6.8719 \cdot 10^{10}$	$\approx 4.7224 \cdot 10^{21}$
10	$\approx 2.6094 \cdot 10^{11}$	$\approx 2.6094 \cdot 10^{11}$	$\approx 3.5184 \cdot 10^{13}$	$\approx 1.2379 \cdot 10^{27}$
11	$\approx 6.2454 \cdot 10^{13}$	$\approx 6.2454 \cdot 10^{13}$	$\approx 3.6029 \cdot 10^{16}$	$\approx 1.2981 \cdot 10^{33}$
12	$\approx 2.5285 \cdot 10^{16}$	$\approx 2.5285 \cdot 10^{16}$	$\approx 7.3787 \cdot 10^{19}$	$\approx 5.4445 \cdot 10^{39}$
13	$\approx 1.7304 \cdot 10^{19}$	$\approx 1.7304 \cdot 10^{19}$	$\approx 3.0223 \cdot 10^{23}$	$\approx 9.1344 \cdot 10^{46}$

Table 9

Enumeration of the proposed formulas. From left to right: smooth discrete aggregation function with neutral element or with absorbing element, given by Proposition 5.2; commutative and idempotent discrete conjunctions, and idempotent discrete conjunctions, given by Proposition 5.4.

$$C = \begin{matrix} & \begin{matrix} n & 0 & C(1, n) & C(2, n) & C(3, n) & C(4, n) & \cdots & C(n-1, n) & n \end{matrix} \\ \begin{matrix} n-1 \\ \vdots \\ 4 \\ 3 \\ 2 \\ 1 \\ 0 \\ 0 \end{matrix} & \begin{bmatrix} 0 & C(1, n-1) & C(2, n-1) & C(3, n-1) & C(4, n-1) & \cdots & n-1 & C(n, n-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & C(1, 4) & C(2, 4) & C(3, 4) & 4 & \cdots & C(n-1, 4) & C(n, 4) \\ 0 & C(1, 3) & C(2, 3) & 3 & C(4, 3) & \cdots & C(n-1, 3) & C(n, 3) \\ 0 & C(1, 2) & 2 & C(3, 2) & C(4, 2) & \cdots & C(n-1, 2) & C(n, 2) \\ 0 & 1 & C(2, 1) & C(3, 1) & C(4, 1) & \cdots & C(n-1, 1) & C(n, 1) \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & \cdots & n-1 & n \end{bmatrix} \end{matrix}$$

Let us represent the set of idempotent discrete conjunctions as $\mathcal{C}_n^{\text{IDE}}$. Looking at the above diagram, it is possible to decompose any idempotent discrete conjunction C into two commutative idempotent discrete conjunctions. This arises from the fact that values of C located above the diagonal $y = x$, with $x \in L_n$, and those below the diagonal, have no relation and are independent. Consequently, determining the enumeration of $|\mathcal{C}_n^{\text{COM}} \cap \mathcal{C}_n^{\text{IDE}}|$ provides a solution to the initial problem. We formalize this insight in the following proposition.

Proposition 5.3. Let $n \geq 1$, and let us denote by $\mathcal{C}_n^{\text{IDE}}$ and $\mathcal{C}_n^{\text{COM}}$ the sets of idempotent discrete conjunctions and commutative discrete conjunctions, respectively, defined on L_n . Let $\varphi : \mathcal{C}_n^{\text{IDE}} \rightarrow (\mathcal{C}_n^{\text{COM}} \cap \mathcal{C}_n^{\text{IDE}}) \times (\mathcal{C}_n^{\text{COM}} \cap \mathcal{C}_n^{\text{IDE}})$ be defined by $\varphi(C) = (C_1, C_2)$, with $C \in \mathcal{C}_n^{\text{IDE}}$, C_1 and C_2 are given point-wise by $C_1(x, y) = C_1(y, x) = C(x, y)$ and $C_2(x, y) = C_2(y, x) = C(y, x)$ for all $x \in L_n$ and $y \in [x, n]$. Then, the transformation φ is a bijection.

Proof. First, φ is well defined since C_1 and C_2 are commutative and idempotent by construction. Next, the injective and exhaustive nature of φ follows directly from the expressions of C_1 and C_2 , since the common points of C_1 and C_2 are only those on the diagonal of C . Then, it is a bijective transformation, and φ^{-1} is given by $\varphi^{-1}(C_1, C_2) = C$, where $C(x, y) = C_1(x, y)$ and $C(y, x) = C_2(x, y)$ for all $x \in L_n$ and $y \in [x, n]$. \square

With this result, we can observe that $|\mathcal{C}_n^{\text{IDE}}| = |\mathcal{C}_n^{\text{COM}} \cap \mathcal{C}_n^{\text{IDE}}|^2$ for all $n \geq 1$. To find the enumeration of $\mathcal{C}_n^{\text{COM}} \cap \mathcal{C}_n^{\text{IDE}}$, let us consider the values of any commutative and idempotent discrete conjunction C located above the diagonal, excluding the values of the segments $\{0\} \times L_n$ and $L_n \times \{0\}$, since above them C is equal to zero because it is an absorbing element. So, instead of seeing these values as a part of the discrete conjunction, let us see them as a simple triangular array, which satisfies certain monotone constraints between rows and columns. These triangular arrays are known as Gelfand-Tsetlin patterns, which were introduced in [69] by Gelfand and Tsetlin when studying the set of unimodular matrices.

Definition 5.2. ([69, 70]) Let $n \geq 2$. A Gelfand-Tsetlin pattern $G = (g_{i,j})$ of size n is a finite triangular array of natural numbers, with $1 \leq i \leq n$ and $1 \leq j \leq n - i + 1$, represented by

$$\begin{array}{cccccccc} g_{1,1} & g_{1,2} & g_{1,3} & \cdots & \cdots & \cdots & & g_{1,n} \\ g_{2,1} & g_{2,2} & g_{2,3} & \cdots & \cdots & & g_{2,n-1} & \\ g_{3,1} & g_{3,2} & g_{3,3} & \cdots & & g_{3,n-2} & & \\ \vdots & \vdots & \vdots & \ddots & & & & \\ g_{n-2,1} & g_{n-2,2} & g_{n-2,3} & & & & & \\ g_{n-1,1} & g_{n-1,2} & & & & & & \\ g_{n,1} & & & & & & & \end{array}$$

which satisfies $g_{j+1,i} \leq g_{j,i} \leq g_{j,i+1}$, for all $1 \leq i \leq n - 1$ and $1 \leq j \leq n - i$.

Remark 5.2. Since Gelfand-Tsetlin patterns appear in various fields of combinatorics, there are several ways to represent and index them. In this case, we have chosen to index them with the usual matrix indexing. See [69] or [70] for two examples where the indexing variables are different.

Our objective is to determine the enumeration of Gelfand-Tsetlin patterns that satisfy the condition $g_{n-i+1,i} = i$ for all $1 \leq i \leq n$. There are studies that have dealt with this problem. In [71], Zeilberger made an historical overview of enumerations for various structures, including plane partitions, alternating sign matrices and Young tableaux, among others. Zeilberger provided a formula for enumerating Gelfand-Tsetlin patterns meeting the condition of $g_{n-i+1,i} = i$ for all $1 \leq i \leq n$, though he did not provide a proof and categorized the process to obtain the formula as *straightforward*. Subsequently, Fischer, in [72, Theorem 1] established a formula for counting monotone triangles, where Gelfand-Tsetlin patterns' row monotonicity is replaced by strict monotonicity, and the condition to be met is $g_{n-i+1,i} = x_i$, where $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{N}^n$ is a strictly increasing n -tuple. However, these structures are not relevant for modelling commutative idempotent discrete conjunctions. Years later, Fischer, in [73, Theorem 1], was the first to offer an enumeration for Gelfand-Tsetlin patterns with $g_{n-i+1,i} = x_i$, where $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{N}^n$ is an increasing n -tuple. The result is as follows.

Theorem 5.3. ([72, Theorem 1]) Let $n \geq 2$, and let $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{N}^n$ such that $x_i \leq x_{i+1}$ for all $1 \leq i \leq n - 1$. Then, the number of Gelfand-Tsetlin patterns $G = (g_{i,j})$ of size n such that $g_{n-i+1,i} = x_i$ is given by

$$\prod_{i=1}^n \prod_{j=i+1}^n \frac{x_j - x_i + j - i}{j - i}. \quad (59)$$

Based on this result, we can establish the enumeration of $C_n^{\text{COM}} \cap C_n^{\text{IDE}}$.

Proposition 5.4. Let $n \geq 2$, and let us denote by C_n^{IDE} and C_n^{COM} the sets of idempotent discrete conjunctions and commutative discrete conjunctions, respectively, defined on L_n . Then, the following statements hold.

(i) The number of commutative and idempotent discrete conjunctions is given by

$$\left| C_n^{\text{COM}} \cap C_n^{\text{IDE}} \right| = 2^{\frac{n(n-1)}{2}}. \quad (60)$$

(ii) The number of idempotent discrete conjunctions is given by

$$\left| C_n^{\text{IDE}} \right| = 2^{n(n-1)}. \quad (61)$$

Proof. We prove each statement.

(i) Let \mathcal{G}_n be the set of Gelfand-Tsetlin patterns of size n such that $g_{n-i+1,i} = i$, for all $1 \leq i \leq n$. First of all, the transformation $F : \mathcal{G}_n \rightarrow C_n^{\text{COM}} \cap C_n^{\text{IDE}}$, defined as $F(G) = C$, where C is given point-wise by $C(x, y) = 0$ if $x = 0$ or $y = 0$, or $C(x, y) = C(y, x) = g_{n-y+1,x}$ for all $1 \leq x \leq n$ and $1 \leq y \leq x - 1$, is well

defined and also is a bijective correspondence. Then, applying Theorem 5.3 with $\mathbf{x} = (1, 2, \dots, n)$, we have

$$\left| \mathcal{C}_n^{\text{COM}} \cap \mathcal{C}_n^{\text{IDE}} \right| = \prod_{i=1}^n \prod_{j=i+1}^n \frac{j-i+j-i}{j-i} = \prod_{i=1}^n \prod_{j=i+1}^n 2 = 2^{\frac{n(n-1)}{2}},$$

as we wanted to prove.

- (ii) It is deduced straightforwardly from statement (i) and the bijective transformation defined in Proposition 5.3. □

These two formulas are tabulated in Table 9. Considering the bijection outlined in Proposition 3.11, which relates discrete conjunctions and discrete disjunctions, and the bijection described in Proposition 4.1, linking discrete conjunctions and discrete implications, we can extend the two formulas proposed in Proposition 5.4 to the following cases:

- Statement (i) of Proposition 5.4 also applies for the enumeration of idempotent and commutative discrete disjunctions, and discrete implications satisfying $(\text{CP}(N_C))$ and $I(x, n-x) = n-x$, for all $x \in L_n$.
- Statement (ii) of Proposition 5.4 also applies for the enumeration of idempotent discrete disjunctions, and discrete implications satisfying $I(x, n-x) = n-x$, for all $x \in L_n$.

6. An overview of the existing enumerations and open issues

Since the advent of operators defined on L_n , there has been a notable surge in the interest surrounding their enumeration. Furthermore, these studies of enumeration have not only been confined to L_n but have also emanated from diverse frameworks, including semigroups, plane partitions, and several classes of special matrices, among others. In this paper we have compiled and unified the established formulas for the main families of operators on L_n . In addition, during the process of reviewing the different formulas, we have corrected the proof of the enumeration of discrete uninorms having smooth and idempotent or smooth and Archimedean underlying operators in Proposition 3.20.

Moreover, we have made further progress in enumerating operators defined on L_n . By leveraging the existing characterizations of smooth discrete aggregation functions with neutral and absorbing elements, we have proposed their enumeration in Proposition 5.2, which is grounded in the cardinality of smooth discrete conjunctions and disjunctions. Then, we have also proposed an enumeration for commutative idempotent discrete conjunctions and idempotent discrete conjunctions in Proposition 5.4, based on Gelfand-Tsetlin patterns and their enumerations.

In Table 12, we provide a comprehensive summary of all known formulas, categorized by the family to which they belong to, the results with the formula, and the corresponding entry in the Online Encyclopedia of Integer Sequences (OEIS) [23]. Nonetheless, several enumerations of significant importance still pose an open problem. These are as follows:

- Smooth discrete aggregation functions. The study of this enumeration on L_n has, to our knowledge, not yet been addressed. But it has been obtained computationally in OEIS/A306372 for matrices $A = (a_{i,j})$ of size $(n+1) \times (n+1)$, which satisfy $a_{1,1} = 1$, $a_{n+1,n+1} = n+1$, the entries in each row and column are increasing, $a_{i+1,j} - a_{i,j} \leq 1$ and $a_{j,i+1} - a_{j,i} \leq 1$ for all $i \in \{1, \dots, n-1\}$ and $j \in \{1, \dots, n\}$, whose enumeration clearly coincides with the number of smooth discrete aggregation functions. This enumeration is shown in Table 10.
- Smooth and commutative discrete aggregation functions. Its enumeration has not been addressed on L_n nor tabulated in OEIS. The authors of this paper, using computational methods, have established their enumeration for $n \leq 6$, which is shown in Table 10, and tabulated in OEIS/A367432.

- Smooth and commutative discrete conjunctions. Theorem 3.1 establishes a bijection between smooth discrete conjunctions and alternating sign matrices. Munar et al. in [20, Proposition 3.6] already observed that the same correspondence holds between smooth and commutative discrete conjunctions and symmetric alternating sign matrices. However, Bousquet-Mélou and Habsieger in [74] indicated that a formula for the enumeration of this particular type of matrices remains an open problem, providing their enumeration for $n \leq 20$. Recently, Behrend et al. in [75] reaffirmed that this remains unsolved. Some values of this sequence are exemplified in Table 10. It is tabulated in OEIS/A005163.
- Discrete t-norms. De Baets and Mesiar in [5] determined its cardinality for $n \leq 13$ and, since then, no further progress has been known. In part, the difficulty in determining a formula is due to the associativity property. It should be noted that the formula for the number of commutative conjunctions with neutral element n given in Proposition 3.9 serves as an upper bound on the number of discrete t-norms. The known values are illustrated in Table 10 and it is tabulated in OEIS/A030453.
- Archimedean discrete t-norms. To our knowledge, no progress has been made on this enumeration. We can only highlight the work of Petrik and Vetterlein in [76], where they propose an algorithm to generate the enumeration of these operators, although they only manage to reach $n \leq 12$. These values are listed in Table 10. It is tabulated in OEIS/A253948.
- Discrete uninorms. Its structure as a combination of a discrete t-norm and a discrete t-conorm suggests that its enumeration is not straightforward. Indeed, no study has been found in the literature that deals with their enumeration. We have established their enumeration for $n \leq 9$ using computational methods, as shown in Table 10.
- Discrete implications satisfying (EP), (CP(N)), (LI(T)) and (MP(T)) in general. Nachtegael and Kerre in [6] computationally enumerated discrete implications satisfying (EP) for $n \leq 5$ and, to the best of our knowledge, no further study has dealt with the enumeration of the other properties. For (CP(N)), Munar et al. in [19] addressed it when $N = N_C$, but the general case is still unexplored. The authors of this paper, using computational methods, have established their enumeration for $n \leq 5$, which are shown in Table 11 and tabulated it in OEIS/A367444, OEIS/A367445, OEIS/A367447 and OEIS/A367446, respectively. It is important to note that the obtained enumeration for (CP(N)) encompasses all those discrete implications satisfying (CP(N)) with some negation N , i.e., the negation N is not fixed and if one discrete implication satisfies the contrapositive symmetry with more than one discrete negation, it only appears once in the enumeration. The enumerations of (LI(T)) and (MP(T)) are computed in an analogous way. Observe, finally, that the number of discrete implications satisfying (MT(T,N)) for some discrete t-conorm T and when the discrete negation is N_C is also shown in Table 11. It coincides with the number of discrete implications satisfying (MP(T)) with respect to some discrete t-norm T , since in this case the bijection of Proposition 4.1 transforms one class into the other.

To conclude, the examination of operators defined on L_n has a pronounced computational dimension. In light of this, it is worth noting that some of the authors of this study developed `DiscreteFuzzyOperators` in [77], a pure-Python package devoted to managing some classes of operators mentioned in this paper, along with the incorporation of most of the formulas for their enumeration compiled herein. Additionally, we have taken the initiative to construct the *Discrete Dataset* [78], wherein, when feasible, we have generated all operators from diverse classes, for some values of n . It is publicly accessible in [78], with the aim of facilitating future research work. The repository also includes the code used to generate the different enumerations that have appeared in the paper.

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n	$ \mathcal{A}_n^{\text{SMT}} $	$ \mathcal{A}_n^{\text{SMT}} \cap \mathcal{A}_n^{\text{COM}} $	$ \mathcal{C}_n^{\text{SMT}} \cap \mathcal{C}_n^{\text{COM}} $	$ \mathcal{T}_n $	$ \mathcal{T}_n^{\text{ARC}} $	$ \mathcal{U}_n $
1	$4 = 4 \cdot 10^0$	$2 = 2 \cdot 10^0$	$1 = 1 \cdot 10^0$	$1 = 1 \cdot 10^0$	$1 = 1 \cdot 10^0$	$2 = 2 \cdot 10^0$
2	$44 = 4.4 \cdot 10^1$	$10 = 1 \cdot 10^1$	$2 = 2 \cdot 10^0$	$2 = 2 \cdot 10^0$	$1 = 1 \cdot 10^0$	$6 = 6 \cdot 10^0$
3	$1,236 = 1.236 \cdot 10^3$	$80 = 8 \cdot 10^1$	$5 = 5 \cdot 10^0$	$6 = 6 \cdot 10^0$	$2 = 2 \cdot 10^0$	$22 = 2.2 \cdot 10^1$
4	$84,910 = 8.491 \cdot 10^3$	$1,008 = 1.008 \cdot 10^3$	$16 = 1.6 \cdot 10^1$	$22 = 2.2 \cdot 10^1$	$6 = 6 \cdot 10^0$	$92 = 9.2 \cdot 10^1$
5	$\approx 1.4024 \cdot 10^7$	$19,764 = 1.9764 \cdot 10^4$	$67 = 6.7 \cdot 10^1$	$94 = 9.4 \cdot 10^1$	$22 = 2.2 \cdot 10^1$	$426 = 4.26 \cdot 10^2$
6	$\approx 5.531 \cdot 10^9$	$600,028 \approx 6.0003 \cdot 10^5$	$368 = 3.68 \cdot 10^2$	$451 = 4.51 \cdot 10^2$	$95 = 9.5 \cdot 10^1$	$2,146 = 2.146 \cdot 10^3$
7	$\approx 5.1922 \cdot 10^{12}$	—	$2,630 = 2.63 \cdot 10^3$	$2,386 = 2.386 \cdot 10^3$	$471 = 4.71 \cdot 10^2$	$11,634 = 1.1634 \cdot 10^4$
8	$\approx 1.1583 \cdot 10^{16}$	—	$24,376 = 2.4376 \cdot 10^4$	$13,775 = 1.3775 \cdot 10^4$	$2,670 = 2.67 \cdot 10^3$	$67,472 = 6.7472 \cdot 10^4$
9	$\approx 6.1353 \cdot 10^{19}$	—	$293,770 = 2.9377 \cdot 10^5$	$86,417 = 8.6417 \cdot 10^4$	$17,387 = 1.7387 \cdot 10^4$	$417,650 = 4.1765 \cdot 10^5$
10	$\approx 7.7117 \cdot 10^{23}$	—	$\approx 4.6106 \cdot 10^6$	$\approx 5.9049 \cdot 10^5$	$\approx 1.3175 \cdot 10^5$	—
11	$\approx 2.2994 \cdot 10^{28}$	—	$\approx 9.4081 \cdot 10^7$	$\approx 4.446 \cdot 10^6$	$\approx 1.1841 \cdot 10^6$	—
12	$\approx 1.626 \cdot 10^{33}$	—	$\approx 2.4927 \cdot 10^9$	$\approx 3.7869 \cdot 10^7$	$\approx 1.2897 \cdot 10^7$	—
13	$\approx 2.7265 \cdot 10^{38}$	—	$\approx 8.5828 \cdot 10^{10}$	$\approx 3.8255 \cdot 10^8$	—	—
	OEIS/A306372	OEIS/A367432	OEIS/A005163	OEIS/A030453	OEIS/A253948	OEIS/A367442

Table 10

Enumeration of some classes of discrete aggregation functions, for $n \leq 13$, obtained using computational methods since a general formula for their enumeration has not been found. From left to right: smooth discrete aggregation functions, smooth and commutative discrete aggregation functions, smooth and commutative discrete conjunctions, discrete t-norms, Archimedean discrete t-norms and discrete uninorms.

n	$ \mathcal{I}_n^{\text{EP}} $	$ \mathcal{I}_n^{\text{CP}} $	$ \mathcal{I}_n^{\text{MP}} $	$ \mathcal{I}_n^{\text{LI}} $	$ \mathcal{I}_n^{\text{MT}(N_C)} $
1	$1 = 1 \cdot 10^0$	$1 = 1 \cdot 10^0$	$1 = 1 \cdot 10^0$	$1 = 1 \cdot 10^0$	$1 = 1 \cdot 10^0$
2	$10 = 1 \cdot 10^1$	$8 = 8 \cdot 10^0$	$9 = 9 \cdot 10^0$	$9 = 9 \cdot 10^0$	$9 = 9 \cdot 10^0$
3	$165 = 1.65 \cdot 10^2$	$99 = 9.9 \cdot 10^1$	$519 = 7.7 \cdot 10^1$	$102 = 1.02 \cdot 10^2$	$519 = 7.7 \cdot 10^1$
4	$3,863 = 3.863 \cdot 10^3$	$2,828 = 2.828 \cdot 10^3$	$150,120 \approx 1.5012 \cdot 10^3$	$1,529 = 1.529 \cdot 10^3$	$150,120 \approx 1.5012 \cdot 10^3$
5	$117,096 \approx 1.171 \cdot 10^5$	$152,474 \approx 1.5247 \cdot 10^5$	$202,728,377 \approx 2.0273 \cdot 10^8$	$28,702 = 2.8702 \cdot 10^4$	$202,728,377 \approx 2.0273 \cdot 10^8$
	OEIS/A367444	OEIS/A367445	OEIS/A367446	OEIS/A367447	OEIS/A367446

Table 11

Enumeration of some classes of discrete implications, for $n \leq 5$, obtained using computational methods since a general formula for their enumeration has not been found. From left to right: discrete implications satisfying (EP), (CP(N)) with respect to some discrete negation N , (MP(T)) with respect to some discrete t-norm T , (LI(T)) with respect to some discrete t-norm T , and (MT(T, N)) with respect to some discrete t-norm T and the classical discrete negation N_C .

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A survey on the enumeration of classes of logical connectives and aggregation functions defined on a finite chain, with new results

Class	Subclass	Result	Original source	OEIS	
Discrete negations	Discrete negations	Prop. 2.1	[19]	A001700	
	Over anti-diagonal discrete negations	Prop. 2.2	[8]	A000108	
Discrete aggregation functions	Discrete aggregation functions	Prop. 3.1	[19]	A366446	
	Commutative discrete aggregation functions	Prop. 3.2	[19]	A366447	
	Disc. agg. func. associative and quasitrivial	Prop. 3.3	[14, 17]	A293005	
	Disc. agg. func. associative, quasitrivial and neutral element	Prop. 3.3	[14, 17]	A002605	
	Disc. agg. func. associative, quasitrivial and absorbing element	Prop. 3.3	[14]	A293006	
	Disc. agg. func. associative, quasitrivial and with neutral and absorbing elements	Prop. 3.3	[14]	A293007	
	Disc. agg. func. bisymmetric and quasitrivial	Prop. 3.4	[28]	A296953	
	Disc. agg. func. bisymmetric, quasitrivial and neutral element	Prop. 3.4	[28]	A131577	
	Disc. agg. func. bisymmetric, quasitrivial and absorbing element	Prop. 3.4	[28]	A296954	
	Smooth disc. agg. func. with absorbing element	Prop. 5.2	Sec. 5.1	A366449	
	Smooth disc. agg. func. with neutral element	Prop. 5.2	Sec. 5.1	A366449	
Discrete conjunctions	Discrete conjunctions	Prop. 3.5	[19]	A360612	
	Commutative discrete conjunctions	Prop. 3.6	[19]	A366540	
	Discrete conjunctions with neutral element n	Prop. 3.7	[42]	A051255	
	Smooth discrete conjunctions	Prop. 3.9	[20]	A005130	
	Smooth and associative discrete conjunctions	Prop. 3.10	[20]	A000079	
	Commutative discrete conjunctions with neutral element n	Prop. 3.9	[20]	A005130	
	Idempotent and commutative discrete conjunctions	Prop. 5.4	Sec. 5.2	A006125	
	Idempotent discrete conjunctions	Prop. 5.4	Sec. 5.2	A053763	
Discrete disjunctions	Discrete disjunctions	Prop. 3.5	[19]	A360612	
	Commutative discrete disjunctions	Prop. 3.6	[19]	A366540	
	Discrete disjunctions with neutral element 0	Prop. 3.7	[42]	A051255	
	Smooth discrete disjunctions	Prop. 3.9	[20]	A005130	
	Smooth and associative discrete disjunctions	Prop. 3.10	[20]	A000079	
	Commutative discrete disjunctions with neutral element 0	Prop. 3.9	[20]	A005130	
	Idempotent and commutative discrete disjunctions	Prop. 5.4	Sec. 5.2	A006125	
	Idempotent discrete disjunctions	Prop. 5.4	Sec. 5.2	A053763	
Discrete quasi-copulas		Prop. 3.9	[20, 21]	A005130	
Discrete copulas	Discrete copulas	Prop. 3.13	[18]	A000142	
	Commutative discrete copulas	Prop. 3.14	[18]	A000085	
	Associative discrete copulas	Prop. 3.15	[18]	A000079	
Discrete t-norms	Smooth discrete t-norms	Prop. 3.16	[8]	A000079	
	Maximal Archimedean discrete t-norms	Prop. 3.17	[12]	A000045	
Discrete t-conorms	Smooth discrete t-conorms	Prop. 3.16	[8]	A000079	
	Minimal Archimedean discrete t-conorms	Prop. 3.17	[12]	A000045	
Discrete uninorms	Idempotent discrete uninorms with neutral element e	Thm. 3.5	[10]	—	
	Idempotent discrete uninorms	Thm. 3.5	[10, 14]	A000079	
	Discrete uninorms having smooth underlying operators	Disc. uni. having smooth underlying operators, neutral element e , $r + 1$ idempotent elements on $[0, e]$ and $s + 1$ idempotent elements on $[e, n]$	Thm. 3.7	[11]	—
		Disc. uni. having smooth underlying operators and neutral element e	Thm. 3.7	[11]	—
		Disc. uni. having smooth underlying operators	Thm. 3.7	[11]	A025192
		Disc. uni. having smooth and idempotent or smooth and Archimedean underlying operators	Prop. 3.20	[11]	A366542
		Disc. uni. having T_L and S_L as underlying operators.	Prop. 3.19	[11]	A005843
		Disc. uni. having T_M and S_L as underlying operators.	Prop. 3.19	[11]	A000124
Disc. uni. having T_L and S_M as underlying operators.	Prop. 3.19	[11]	A000124		
Discrete t-operators/nullnorms	Smooth discrete t-operators/nullnorms	Prop. 3.21	[7]	A000027	
	Smooth and idempotent discrete t-operators/nullnorms	Prop. 3.21	[7]	A045623	
Discrete implications	Discrete implications	Prop. 3.5	[19]	A360612	
	Smooth discrete implications	Prop. 3.9	[20]	A005130	
	Discrete implications satisfying $(CP(N_C))$	Prop. 3.6	[19]	A366540	
	Discrete implications satisfying (NP)	Prop. 4.3	[19]	A367192	
	Discrete implications satisfying (CB)	Prop. 4.4	[19]	A367540	
	Discrete implications satisfying (OP)	Prop. 4.5	[19]	A367434	
Discrete implications satisfying (IP)	Prop. 4.6	[19]	A367541		

Table 12

Summary of the different enumerations considered in this paper, the result containing the formula, the original reference where it was proposed and the code associated to its entry in OEIS.