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# Eigenvalue Methods for Sparse Tropical Polynomial Systems

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**Abstract.** We develop an analogue of eigenvalue methods to construct solutions of systems of tropical polynomial equalities and inequalities. We show that solutions can be obtained by solving parametric mean payoff games, arising to appropriate linearizations of the systems using tropical Macaulay matrices. We implemented specific algorithms adapted to the large scale parametric games that arise in this way, and present numerical experiments.

**Keywords:** Tropical geometry · Polynomial systems · Zero-sum games · Algorithmic complexity

## 1 Introduction

*Motivation* We develop a method to solve systems of equations or of inequations involving multivariate tropical polynomials. Such systems arise in the study of *non-archimedean amoebas*, which are images by a non-archimedean valuation of an algebraic set [17]. The ‘fundamental theorem of tropical geometry’ shows that these amoebas can be described by systems of finitely many tropical polynomial equations [31].

Similarly, the solution sets of systems of tropical polynomial inequalities provide upper approximations of images by a non-archimedean valuation of semi-algebraic sets over real closed non-archimedean fields, and these approximations are exact under genericity conditions [27,8,28]. Owing to their combinatorial nature, tropical polynomial systems are often easier to grasp than their classical analogues. These ideas, which can be traced back to works of Viro [36], Gelfand, Kapranov, Zelevinsky [21], Sturmfels (see [25]), or Mikhalkin [34], are at the heart of tropical geometry, see [26,31] for background.

Systems of tropical polynomial equations and inequations also arise in specific applications, independently of the nonarchimedean interpretation. In particular, they arise in the computation of stationary behaviors of discrete event systems [13], see e.g. [7] for an application to performance evaluation. Other motivations arise from auction theory [9], or chemical reaction networks, see e.g. [14].

Grigoriev and Podolskii established in [23] a *tropical Nullstellensatz*, which states that a system of tropical polynomial equations is solvable if and only if its linearization, obtained by truncating the Macaulay matrix up to an appropriate degree, is solvable. Their results also apply to polynomial inequations. Since systems of tropical linear equations and inequations reduce to mean payoff games [1], this provides both theoretical tools (strong duality theorems) and algorithms. In [2], we provided a refinement of Grigoriev and Podolskii’s tropical nullstellensatz, adapted to sparse polynomial systems, by exploiting the construction of the *Canny-Emiris set* [12,18] and its generalization by Sturmfels [35, §3], used to compute the classical resultants. We provided also an improved truncation degree for dense polynomials (which matches the optimal Macaulay bound). Our approach also applies to hybrid systems, combining tropical equalities with strict and weak inequalities. However, the approach developed in [23,2] only allows one to decide whether the solution set is empty.

*Contribution* Here, we extend this approach to compute solutions. Our method may be thought of as a tropical analogue of the eigenvalue method for polynomial system solving, albeit the notion of ‘eigenvalue’ now has to be understood in a non-linear sense. More precisely, we construct a solution by reduction to parametric families of mean payoff games. In particular, for any coordinate index, we define a spectral function, which provides the value of a game as a function of a parameter, and show that the projection of the solution set on the given coordinate coincides with a super-level set of this spectral function. We present two algorithms building on this idea: a simple dichotomic search, allowing to construct one solution, and a path-following method, computing the graph of the spectral function, allowing one to compute the entire set of solutions when the latter is finite. We present numerical benchmarks.

*Related work* The standard approach to the computation of tropical prevarieties is to exploit the duality between arrangements of tropical hypersurfaces and mixed-polyhedral subdivisions. In that way, decision problems concerning tropical prevarieties can be reduced to the enumeration of mixed cells, see [29,32]. A number of current works deal with the efficient computation of tropical varieties and prevarieties, see [33,22] and the references therein.

In these approaches, a polyhedral complex from which all solutions can be obtained is typically constructed. In contrast, the present method allows one to obtain a more restricted information on the solution set, like a single solution, or a projection on one coordinate. Part of this work relies on ideas of parametric games, going back to [20,19]. A different approach to solve tropical polynomial systems relies on SMT solving [30].

## 2 The sparse tropical Positivstellensatz

The *tropical* (or *max-plus*) *semifield* is the semifield  $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot, \mathbf{0}, \mathbf{1})$  with addition  $\oplus = \max$ , multiplication  $\odot = +$ , zero element  $\mathbf{0} = -\infty$  and unit element  $\mathbf{1} = 0$ . The operations  $\oplus$  and  $\odot$  are respectively referred to as the *tropical addition* and the *tropical multiplication*, and the tropical semifield is denoted by  $\mathbb{R}_\infty$ . The notions of formal polynomials and polynomial functions carry over  $\mathbb{R}_\infty$ . In particular, a polynomial function is a map  $f : \mathbb{R}_\infty^n \rightarrow \mathbb{R}_\infty$  defined by an equation of the form

$$f(x) := \max_{\alpha \in \mathbb{N}^n} (f_\alpha + \langle x, \alpha \rangle) \quad (1)$$

for all  $x \in \mathbb{R}_\infty^n$ , where we assume that  $f_\alpha \neq -\infty$  for finitely many values of  $\alpha \in \mathbb{N}^n$ , and adopt the convention  $(-\infty) \times 0 = 0$  when evaluating  $\langle x, \alpha \rangle$ . The set  $\{\alpha \in \mathbb{N}^n : f_\alpha \neq -\infty\}$  is called the *support* of  $f$  and is denoted  $\text{supp}(f)$ . Although the value  $f(x)$  makes sense for any  $x \in \mathbb{R}_\infty^n$ , we shall restrict our attention here to vectors  $x$  with *finite* values, i.e.  $x \in \mathbb{R}^n$ . In particular, a *root* of the tropical polynomial  $f$  is defined as a point  $x \in \mathbb{R}^n$  such that the maximum in (1) is achieved at least twice. We write  $f(x) \nabla \mathbf{0}$  when  $x$  is a root of  $f$ , to make clear the analogy with the equation ‘ $f(x) = 0$ ’ defining classical roots.

A collection  $f = (f_1, \dots, f_k)$  of  $n$ -variate tropical polynomials defines a *tropical (toric) prevariety*, which is the set of common tropical roots of  $f_1, \dots, f_k$ , i.e., the set of solutions  $x \in \mathbb{R}^n$  of  $f_i(x) \nabla \mathbf{0}$  for all  $1 \leq i \leq k$ .

We also consider tropical equations of the form  $f_i^+(x) = f_i^-(x)$ ,  $1 \leq i \leq k$ , where  $f_i^\pm$  are tropical polynomials, as well as systems of tropical inequalities of the form  $f_i^+(x) \geq f_i^-(x)$  or  $f_i^+(x) > f_i^-(x)$ , for  $1 \leq i \leq k$ . Following [2,3], we call *basic tropical semialgebraic set* the set of solutions of a system involving any combination of weak and strict tropical polynomial inequalities. Tropical prevarieties are easily seen to be particular cases of basic tropical semialgebraic sets.

Given a collection of tropical polynomials  $f = (f_1, \dots, f_k)$ , the *Macaulay matrix*  $\mathcal{M}$  associated to  $f$  is defined as follows: its rows are indexed by pairs  $(i, \alpha)$  where  $1 \leq i \leq k$  and  $\alpha \in \mathbb{N}^n$  and its columns are indexed by integer vectors  $\beta \in \mathbb{N}^n$ , and for given  $(i, \alpha)$  and  $\beta$ , we set the entry  $\mathcal{M}_{(i,\alpha),\beta}$  of  $\mathcal{M}$  equal to the coefficient of the monomial  $X^\beta$  in the polynomial  $X^\alpha f_i(X)$ , or  $-\infty$  if no such monomial exists.

Given a *nonempty* finite subset  $\mathcal{E}$  of  $\mathbb{N}^n$ , and a collection  $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_k)$  of subsets of  $\mathbb{N}^n$ , we denote by  $\mathcal{M}_{\mathcal{E}}^{\mathcal{A}}$  the submatrix of  $\mathcal{M}$  consisting only of the columns with indices  $\beta \in \mathcal{E}$ , and the rows indexed by pairs  $(i, \alpha)$  where  $1 \leq i \leq k$  and  $\alpha \in \mathbb{N}^n$  such that  $\alpha + \mathcal{A}_i \subset \mathcal{E}$ . When the polynomials  $f_i$  have their support equal to  $\mathcal{A}_i$ , we simply write  $\mathcal{M}_{\mathcal{E}}$  instead of  $\mathcal{M}_{\mathcal{E}}^{\mathcal{A}}$ .

Set  $f^+ = (f_1^+, \dots, f_k^+)$  and  $f^- = (f_1^-, \dots, f_k^-)$  two collections of  $k$  tropical polynomials, and for  $1 \leq i \leq k$ ,  $\mathcal{A}_i^\pm = \text{supp}(f_i^\pm)$ ,  $\mathcal{A}_i = \mathcal{A}_i^+ \cup \mathcal{A}_i^-$  and  $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_k)$ . We denote by  $f^+(x) \geq f^-(x)$  the system

$$\max_{\alpha \in \mathcal{A}_i^+} (f_{i,\alpha}^+ + \langle \alpha, x \rangle) \geq \max_{\alpha \in \mathcal{A}_i^-} (f_{i,\alpha}^- + \langle \alpha, x \rangle) \text{ for all } 1 \leq i \leq k, \quad (2)$$

of unknown  $x \in \mathbb{R}^n$ .

Moreover, we denote by  $\mathcal{M}^\pm$  the pair of Macaulay matrices associated to  $f^\pm$ , so with entries  $\mathcal{M}_{(i,\alpha),\beta}^\pm = f_{i,\beta-\alpha}^\pm$ . Then, for any subset  $\mathcal{E}$  of  $\mathbb{N}^n$ , we denote by  $\mathcal{M}_{\mathcal{E}}^\pm$  the submatrices associated to  $\mathcal{E}$  and the collection  $\mathcal{A}$  defined above, *i.e.*  $\mathcal{M}_{\mathcal{E}}^\pm = (\mathcal{M}^\pm)_{\mathcal{E}}^{\mathcal{A}}$ . Finally, we set for  $1 \leq i \leq k$ ,  $r_i = \dim(\text{aff}(\mathcal{A}_i^-)) + 1$ .

We now call *Canny-Emiris subset* of  $\mathbb{N}^n$  associated to the system  $f^+(x) \geq f^-(x)$  any set  $\mathcal{E}$  of the form  $\mathcal{E} := (\tilde{Q} + \delta) \cap \mathbb{N}^n$  with  $\tilde{Q} = r_1 Q_1 + \dots + r_k Q_k$ , where  $Q_i = \text{conv}(\mathcal{A}_i)$  for  $1 \leq i \leq k$ , and  $\delta$  is a generic vector in  $V + \mathbb{N}^n$ , with  $V$  the direction of the affine hull of  $\tilde{Q}$ . Finally, for any subset  $\mathcal{E}'$  of  $\mathbb{N}^n$  containing a Canny-Emiris subset  $\mathcal{E}$  associated to  $f^+(x) \geq f^-(x)$  and  $y \in \mathbb{R}^{\mathcal{E}'}$ , we denote by  $\mathcal{M}_{\mathcal{E}'}^+ \odot y \geq \mathcal{M}_{\mathcal{E}'}^- \odot y$  the following system of tropical linear inequalities:

$$\max_{\beta \in \mathcal{E}'} (m_{(i,\alpha),\beta}^+ + y_\beta) \geq \max_{\beta \in \mathcal{E}'} (m_{(i,\alpha),\beta}^- + y_\beta) \text{ for all } 1 \leq i \leq k \text{ and } \alpha \in \mathcal{A}_i. \quad (3)$$

The following result shows that the solvability of a system of tropical polynomial inequalities is equivalent to the solvability of a ‘linearized’ version obtained from the Macaulay matrices.

**Theorem 1 (Sparse tropical Positivstellensatz, see [3, Theorem 4.1]).** *There exists a solution  $x \in \mathbb{R}^n$  to the system  $f^+(x) \geq f^-(x)$  if and only if there exists a vector  $y \in \mathbb{R}^{\mathcal{E}'}$  satisfying  $\mathcal{M}_{\mathcal{E}'}^+ \odot y \geq \mathcal{M}_{\mathcal{E}'}^- \odot y$ , where  $\mathcal{E}'$  is any subset of  $\mathbb{N}^n$  containing a nonempty Canny-Emiris subset  $\mathcal{E}$  of  $\mathbb{N}^n$  associated to the system  $f^+(x) \geq f^-(x)$ .*

Theorem 4.14 of [3] provides a similar result for general *hybrid* tropical systems, allowing both strict and weak inequalities, as well as equations of the form  $f_i(x) \nabla 0$ . For simplicity of exposition, we consider here solutions of systems of weak inequalities, as in Theorem 1. The present method carries over to hybrid systems.

### 3 Deciding the solvability of tropical linear systems with mean payoff games

We next recall how the linearized system of inequalities can be solved by a reduction to mean payoff games, referring to [1, Section 2] for details.

Let  $G$  be a (finite) oriented weighted bipartite graph given with its set of vertices  $I \sqcup J$  and its set of arcs  $E \subseteq (I \times J) \cup (J \times I)$ . The vertices of  $G$  are referred to as the *states* or *positions* of

the game, and the arcs of  $G$  are referred to as the *actions* or *moves*. The payments of the game are described by two matrices  $A = (a_{ij})_{(i,j) \in I \times J}$  and  $B = (b_{ij})_{(i,j) \in I \times J}$ , where we set  $a_{ij} = -\infty$  whenever the arc  $(j, i)$  does not exist in the graph  $G$  and likewise  $b_{ij} = -\infty$  whenever the arc  $(i, j)$  does not exist. We assume that every node of  $G$  has at least one successor. The associated *mean payoff game* is the zero-sum two player game defined as follows. The first player is called the *minimizer* and the second one the *maximizer*. At turn  $N$ , if the current state is  $j_N \in J$ , the minimizer chooses the next state  $i_N \in I$  along an arc  $(j_N, i_N)$  makes a payment of  $-a_{i_N j_N}$  to the maximizer. Then the maximizer, from the current state  $i_k \in I$ , chooses a state  $j_{N+1} \in J$  such that  $(i_N, j_{N+1})$  is an arc of  $G$ , and receives a payment of  $b_{i_N j_{N+1}}$  from the minimizer. These steps repeat indefinitely. The maximizer wishes to maximize the average payment received per time unit whereas the minimizer wishes to minimize it. This game is known to have a value [16,37], denoted by  $\chi_{j_0}$  where  $j_0 \in J$  is the initial state. We say that the game, from this initial state, is winning for the maximizer if this value is nonnegative.

We consider the max-plus linear operator  $y \mapsto By$  defined from  $(\mathbb{R} \cup \{\pm\infty\})^J$  to  $(\mathbb{R} \cup \{\pm\infty\})^I$ , with the convention that  $(-\infty) + (+\infty) = (-\infty)$ , by  $(By)_i := \max_{j \in J} (b_{ij} + y_j)$  for all  $i \in I$ . Likewise, we consider the min-plus linear operator  $z \mapsto A^\sharp z$  defined from  $(\mathbb{R} \cup \{\pm\infty\})^I$  to  $(\mathbb{R} \cup \{\pm\infty\})^J$ , with the convention  $(+\infty) + (-\infty) = (+\infty)$  this time, by  $(A^\sharp z)_j := \min_{i \in I} (-a_{ij} + z_i)$  for all  $j \in J$ . The requirement that every node of  $G$  has at least one successor is equivalent to the following:

- Assumption 2.** (a) for all  $j \in J$ , there exists  $i \in I$  such that  $a_{ij} \neq -\infty$ ;  
(b) for all  $i \in I$ , there exists  $j \in J$  such that  $b_{ij} \neq -\infty$ .

Finally, the *Shapley operator* of this game is the operator  $T : (\mathbb{R} \cup \{\pm\infty\})^J \rightarrow (\mathbb{R} \cup \{\pm\infty\})^J$  defined by  $T(y) := A^\sharp By$  for all  $y \in \mathbb{R} \cup \{\pm\infty\}$ . In other words, one has

$$T(y) = \left( \min_{i \in I} (-a_{ij} + \max_{k \in J} (b_{ik} + y_k)) \right)_{j \in J} \quad (4)$$

for all  $y \in (\mathbb{R} \cup \{\pm\infty\})^J$ . Note in particular that the tropical linear system  $A \odot y \leq B \odot y$  is equivalent to the inequality  $y \leq T(y)$ . The vector of values  $\chi(T) = (\chi_j)_{j \in J} \in \mathbb{R}^J$  only depends on the operator  $T$ , as it coincides with the limit  $\lim_{N \rightarrow +\infty} T^N(0)/N$ . The following non-linear spectral theorem characterizes the minimal value of an initial state.

**Theorem 3 (Collatz-Wielandt property [1, Theorem 2.8 and Remark 2.10]).** *Let  $T$  be the Shapley operator defined in (4). Then, the following quantities coincide and they are all equal to  $\underline{\chi}(T) := \min\{\chi_j : j \in J\}$*

$$\sup\{\lambda \in \mathbb{R} : \exists u \in \mathbb{R}^J, T(u) \geq \lambda + u\} \quad (5a)$$

$$\inf\{\lambda \in \mathbb{R} \cup \{+\infty\} : \exists u \in (\mathbb{R} \cup \{+\infty\})^J, u \not\equiv +\infty, T(u) \leq \lambda + u\} \quad (5b)$$

$$\inf\{\lambda \in \mathbb{R} \cup \{+\infty\} : \exists u \in (\mathbb{R} \cup \{+\infty\})^J, u \not\equiv +\infty, T(u) = \lambda + u\} . \quad (5c)$$

Moreover, the infima and supremum in (5) are attained.

**Theorem 4 ([1, Corollary 3.4]).** *The tropical linear system  $A \odot y \leq B \odot y$  has a solution  $y \in \mathbb{R}^J$  if and only if all the initial states of the associated game have a nonnegative value, i.e.  $\chi(T) \geq 0$ .*

## 4 Mean payoff games oracles

To check the solvability of the linearized system (3) using Theorem 4, it suffices to compute the value vector of a mean payoff game. Actually, a weaker information will be enough for some of

our results. We call *weak mean payoff game oracle* a procedure which takes as input two tropical matrices  $A, B$ , and decides whether  $\underline{\chi}(T) \geq 0$  with  $T = A \sharp B$ . We denote by  $\text{w-MPG}(|I|, |J|, W)$  the number of arithmetic operations of a mean payoff oracle taking as input  $|I| \times |J|$  matrices  $A, B$  whose entries are either relative integers of absolute values bounded by  $W$  or  $-\infty$ . We observe that  $\text{w-MPG}(|I|, |J|, W) \geq |I||J|$  since the input size is  $\Omega(|I| \times |J|)$ . We shall also use the notion of *strong mean payoff game oracle*, which not only decides whether  $\underline{\chi}(T) \geq 0$ , as a weak oracle does, but also, whenever  $\chi(T) \equiv \lambda \in \mathbb{R}$ , returns a vector  $u \in \mathbb{R}^n$  such that  $T(u) = \lambda + u$ . The vector  $u$  is called a *bias*. The bias vector serves as an optimality certificate, allowing one to identify optimal policies, see [6] for background. We denote by  $\text{MPG}(|I|, |J|, W)$  the number of arithmetic operations of a strong mean payoff oracle.

A classical algorithm to solve mean payoff games is value iteration, analyzed in [37]. It consists in computing the sequence  $T^N(0)$  and inferring the limit  $\lim_{N \rightarrow +\infty} T^N(0)/N$  by specializing  $N$  to an explicit sufficiently large value.

**Theorem 5 (Corollary of [37, Theorem 2.4]).** *Value iteration provides a weak mean payoff oracle requiring  $\mathcal{O}(|J|^2 W)$  evaluations of the Shapley operator  $T$ , leading to  $\text{w-MPG}(|I|, |J|, W) = \mathcal{O}(|I||J|^3 W)$ .*

The number of iterations of the method of [37] is always in  $\Omega(|J|^2 W)$ , which is unpracticable in our application ( $J$  will be exponentially large in the input size). We presented in [2] a refinement of value iteration, exploiting the ideas of Krasnoselskii-Mann damping with an acceleration or ‘widening’ step. This accelerated version allows in practice for a much quicker check of feasibility. We also have the following result.

**Theorem 6.** *A strong mean payoff oracle can be implemented by making  $\mathcal{O}(|J|^3 W)$  evaluations of the Shapley operator  $T$ , leading to  $\text{MPG}(|I|, |J|, W) = \mathcal{O}(|I||J|^4 W)$ .*

Indeed, we first compute  $\chi(T)$  by means of [37, Theorem 2.3]. Moreover, when  $\chi(T) \equiv \lambda \in \mathbb{R}$  is a constant vector, we first perform the iteration  $u^{k+1} = (-\lambda + T)(u^k) \wedge u^k$ , starting from  $u^0 = 0$ , and show it converges to a vector  $u$  such that  $u \leq (-\lambda + T)(u)$ , in  $\mathcal{O}(|J|^3 W)$  iterations, then we perform the iteration  $v^{k+1} = (-\lambda + T)(v^k)$ , starting from  $v^0 = u$ , and show it converges to a bias vector  $v$ , satisfying  $v = (-\lambda + T)(v)$ , again in  $\mathcal{O}(|J|^3 W)$  iterations, leading to Theorem 6.

Another approach to solve mean payoff games is policy iteration, see [15, Algorithm 2], which is strongly polynomial for discounted problems with a fixed discount factor [24,4], and the mean payoff problem reduces to the discounted problem, with a discount rate in  $1 - 1/\Omega(|J|^3 W)$  [37, p. 353]. This leads to pseudo-polynomial bounds, which are weaker than the one obtained for value iteration. However, policy iteration is experimentally the fastest method for our purpose.

## 5 Existence of short solutions of tropical polynomial systems

We shall need an a priori bound on the solutions of a tropical polynomial system. The set of these solutions is a closed polyhedral complex. The next result shows that if this set is nonempty, there is always an element in this set with a bitsize polynomially bounded in the input size.

**Theorem 7.** *Let  $f^\pm = (f_1^\pm, \dots, f_k^\pm)$  be two collections of tropical polynomials and let*

$$d = \max_{1 \leq i \leq k} \deg(f_i^\pm) \quad \text{and} \quad W = \max_{1 \leq i \leq k} \|f_i^\pm\|_\infty,$$

*and for  $\epsilon \in \{\pm 1\}^n$ , denote by  $\epsilon \mathbb{R}_{\geq 0}^n$  the orthant  $\{x \in \mathbb{R}^n : \epsilon_j x_j \geq 0 \text{ for all } 1 \leq j \leq n\}$ . Then:*

- (i) *the vertices of every polyhedral complex  $\{x \in \mathbb{R}^n : f_i^+(x) \geq f_i^-(x)\} \cap \epsilon \mathbb{R}_{\geq 0}^n$  are included in a  $\|\cdot\|_\infty$ -ball of radius  $2n(2d)^{n-1}W$  centered at point 0;*

(ii) if moreover all the coefficients of the polynomials  $f_i^\pm$  are integer, these vertices have coordinates that are rational numbers with a denominator bounded above by  $(2d)^n$ .

Taking the intersection with the orthant  $\epsilon\mathbb{R}_{\geq 0}^n$  is a technical convenience, which makes sure that a vertex always exists as soon as the system of inequalities has a solution in this orthant. This allows us to tackle situations in which for instance solutions sets contains affine lines. Then, the coordinates and denominator of this vertex are bounded using Hadamard's inequality for determinants, leading to the above estimate.

## 6 The dichotomic search method

We now present a first method allowing us to construct one solution of a system of tropical polynomial inequalities. We saw in Section 3 that checking whether a system of weak tropical polynomial inequalities admits a solution in  $\mathbb{R}^n$ , using the tropical Positivstellensatz, reduces to solving a mean payoff game. We enrich this system, by adding extra inequalities of the form  $a \leq x_1$  or  $x_1 \leq b$ . In this way, we can decide whether there is a solution such that  $x_1 \in [a, b]$ . Moreover, Theorem 7 provides an a priori bound for a solution, allowing us to reduce the search space to a sup-norm box centered at the origin with bounded radius, and to rational numbers with a bounded denominator. In this way, a rational number  $\bar{x}_1$  which belongs to the projection of the solution set on the first variable can be obtained. Then, we substitute  $x_1$  by the fixed value  $\bar{x}_1$  in the polynomial system, and perform again a dichotomic search, now on the variable  $x_2$ , leading to a rational value  $\bar{x}_2$  such that  $(\bar{x}_1, \bar{x}_2)$  belongs to the projection of the solution set on the first two variables. We pursue this procedure by fixing gradually the variables  $x_1, \dots, x_n$ . We call *dichotomic search* this method. Observe that dichotomic search stops at the first step if the solution set is empty. We arrive at the following complexity result.

**Theorem 8.** *Consider a system of weak polynomial inequalities, as in Theorem 7. Then, the dichotomic search method returns a rational solution of this system (or decides that there is none) in  $\mathcal{O}(\log(n(2d)^{2n-1}W))$  calls to a weak mean payoff oracle.*

## 7 The path following method

Let  $\zeta$  be a real parameter. Using Theorem 1, the feasibility of the system  $f_i^+(x) \geq f_i^-(x)$  for all  $1 \leq i \leq k$  with the added constraint  $x_1 = \zeta$  can be expressed as a system of homogeneous tropical linear inequalities of the form  $A_\zeta \odot y \leq B_\zeta \odot y$  of unknown  $y \in \mathbb{R}^J$ , in which the entries of the tropical matrices  $A_\zeta = (a_{ij}(\zeta))_{(i,j) \in I \times J}$  and  $B_\zeta = (b_{ij}(\zeta))_{(i,j) \in I \times J}$  are piecewise affine functions of the scalar  $\zeta$  with non  $-\infty$  coefficients bounded by  $W_\zeta := W + |\zeta|d$ . We consider the parametric Shapley operator  $T_\zeta := A_\zeta^\# B_\zeta$  which, up to replacing it by the operator  $u \mapsto u \wedge T_\zeta(u)$ , can be assumed to send  $\mathbb{R}^J$  onto itself. We define the *spectral function* of this operator to be the map  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $\phi(\zeta) = \underline{\chi}(T_\zeta)$ .

**Theorem 9.** *The spectral function  $\phi$  is continuous and piecewise affine. Moreover, the projection on the first coordinate of the solution set  $S = \{x \in \mathbb{R}^n : f_i^+(x) \geq f_i^-(x) \text{ for all } 1 \leq i \leq k\}$  coincides with the super-level set  $\{\zeta \in \mathbb{R} : \phi(\zeta) \geq 0\}$ .*

We next show how to compute the restriction of the spectral function to an interval  $[\underline{\zeta}, \bar{\zeta}]$ . We define the matrix  $\bar{A}_\zeta$ , obtained by replacing every  $-\infty$  entry of  $A_\zeta$  by a number  $-M$ . If  $M$  is chosen larger than  $4|\mathcal{E}| \max(W_\zeta, W_{\bar{\zeta}})$ , then  $\underline{\chi}(\bar{A}_\zeta^\# B_\zeta)$  is a constant vector whose entries coincide with  $\underline{\chi}(A_\zeta^\# B_\zeta)$ . Moreover, for all  $\zeta \in \mathbb{R}$ , the eigenproblem  $\bar{A}_\zeta^\# B_\zeta u = \lambda + u$  with  $\lambda \in \mathbb{R}$  and

$u \in \mathbb{R}^n$ , depending on  $\zeta$ , is solvable, and  $\lambda = \lambda(\zeta) := \underline{\chi}(A_\zeta^\sharp B_\zeta)$ . We will construct piecewise linear functions  $\lambda : [\underline{\zeta}, \bar{\zeta}] \rightarrow \mathbb{R}$  and  $u : [\underline{\zeta}, \bar{\zeta}] \rightarrow \mathbb{R}^J$  and  $v : [\underline{\zeta}, \bar{\zeta}] \rightarrow \mathbb{R}^I$  satisfying

$$\begin{aligned} \min_{i \in I} -\bar{a}_{ij}(\zeta) + v_i(\zeta) &= \lambda(\zeta) + u_j(\zeta) & \text{for all } j \in J \\ \max_{j \in J} b_{ij}(\zeta) + u_j(\zeta) &= v_i(\zeta) & \text{for all } i \in I . \end{aligned} \quad (6)$$

Suppose that  $\lambda$ ,  $u$  and  $v$  have been evaluated at a point  $\zeta_0$ . Then, we look for a solution of (6) of the form

$$\lambda(\zeta) = \lambda(\zeta_0) + (\zeta - \zeta_0)\lambda'(\zeta_0) , \quad u(\zeta) = u(\zeta_0) + (\zeta - \zeta_0)u'(\zeta_0) , \quad v(\zeta) = v(\zeta_0) + (\zeta - \zeta_0)v'(\zeta_0) \quad (7)$$

defined on a small right neighborhood of  $\zeta_0$ , where  $\lambda'(\zeta_0) \in \mathbb{R}$ ,  $u'(\zeta_0) \in \mathbb{R}^J$  and  $v'(\zeta_0) \in \mathbb{R}^I$  will be computed. Denoting  $I_j(\zeta_0) = \arg \min_{i \in I} (-\bar{a}_{ij}(\zeta_0) + v_i(\zeta_0))$  and  $J_i(\zeta_0) = \arg \max_{j \in J} (b_{ij}(\zeta_0) + u_j(\zeta_0))$ , we consider the non-linear eigenvalue problem

$$\begin{aligned} \min_{i \in I_j(\zeta_0)} -a'_{ij}(\zeta_0) + v'_i(\zeta_0) &= \lambda'(\zeta_0) + u'_j(\zeta_0) & \text{for all } j \in J \\ \max_{j \in J_i(\zeta_0)} b'_{ij}(\zeta_0) + u'_j(\zeta_0) &= v'_i(\zeta_0) & \text{for all } i \in I , \end{aligned} \quad (8)$$

where  $a'_{ij}$  and  $b'_{ij}$  denote the right derivatives of the piecewise affine functions  $\zeta \mapsto \bar{a}_{ij}(\zeta)$  and  $\zeta \mapsto b_{ij}(\zeta)$ . This problem can be solved by calling the strong mean payoff oracle, the payment matrices being given by  $a'_{ij}(\zeta_0)$  and  $b'_{ij}(\zeta_0)$ . We show that any solution  $\lambda'(\zeta_0), u'(\zeta_0), v'(\zeta_0)$  of (8) yields a solution  $\lambda(\zeta), u(\zeta), v(\zeta)$  of (6), as per (7), defined on a right neighborhood of  $\zeta_0$ . We now perform a *pivoting step*, similar in its principle to a pivoting in the simplex algorithm. We denote by  $\zeta_1$  the smallest value of  $\zeta > \zeta_0$  for which the Ansatz (7) is no longer a solution of (6). We reevaluate the sets of active constraints  $I_j(\zeta_1)$  and  $J_i(\zeta_1)$  at the new point  $u(\zeta_1)$  and  $v(\zeta_1)$ , and solve again (8).

Whereas the eigenvalue  $\lambda(\zeta)$  of the operator  $\bar{A}_\zeta^\sharp B_\zeta$  is unique for all values of  $\zeta$ , the eigenvector of this operator may not be unique (even up to an additive constant). In particular, the derivated eigenvalue problem (8) may have several solutions  $u'(\zeta), v'(\zeta)$ . However, we show that if  $A_0$  or  $B_0$  has *generic entries*, in the sense of not belonging to an explicit finite union of hyperplanes, then the eigenvector map  $\zeta \mapsto u(\zeta)$  becomes uniquely defined and then the number of pivoting steps is finite. The proof of this relies on a result of generic uniqueness of the eigenvector [5]. We resort to an unpleasant technicality for non-generic systems: we perturb explicitly the input to make it generic, at the cost of a dilation of  $W$  by a possibly large factor of at most  $(2|\mathcal{E}| + 1)^{|\mathcal{E}|}$  to ensure that the input remains integer. We denote by  $W' \leq (2|\mathcal{E}| + 1)^{|\mathcal{E}|} W$  this dilated cost.

Finally, from Theorem 7, we deduce that one can in fact retrieve the entire projection of the solution set by choosing  $\underline{\zeta} = -2n(2d)^{n-1}W'$  and  $\bar{\zeta} = 2n(2d)^{n-1}W'$ . Denoting by  $\text{piv}(|I|, |J|, W)$  the maximal number of pivoting steps when  $A_0, B_0$  ranges over the set of payment matrices with generic entries, we arrive at the following result.

**Theorem 10.** *The path following method computes the projection on the first coordinate of the solution set  $S = \{x \in \mathbb{R}^n : f_i^+(x) \geq f_i^-(x) \text{ for all } 1 \leq i \leq k\}$  in a number of*

$$\text{MPG}(\mathcal{O}(k|\mathcal{E}|), |\mathcal{E}|, 4|\mathcal{E}|(n(2d)^n + 1)W') + \text{MPG}(\mathcal{O}(k|\mathcal{E}|), |\mathcal{E}|, d) \text{piv}(\mathcal{O}(k|\mathcal{E}|), |\mathcal{E}|, W')$$

*arithmetic operations, where  $\mathcal{E}$  is a Canny-Emiris set as defined in Section 2. Moreover one has  $|\mathcal{E}| \leq \binom{n+N}{n}$  for  $N = \sum_{i=1}^k r_i d_i$ .*

When the solution set is finite, this method allows us to recover all solutions: for each element in the projection of  $S$  on the first coordinate, we substitute  $x_1$  to this value, and find the possible values of  $x_2$ , by considering a new spectral function, etc. We arrive at the following complexity.



**Corollary 11.** *A finite tropical semialgebraic set given by a collection of polynomial relations  $f_i^+ \triangleright_i f_i^-$  for  $1 \leq i \leq k$ , and  $f_i \nabla \mathbf{0}$  for  $k+1 \leq i \leq \ell$ , with  $\triangleright_i \in \{=, \geq\}$  for  $1 \leq i \leq k$  can be computed in*

$$\prod_{j \in [n]} \left[ \text{MPG} \left( \mathcal{O}(kM_j), \mathcal{O}(M_j), 4M_j(j(2d^{(j)})^j + 1)W' \right) \right. \\ \left. + \text{MPG} \left( \mathcal{O}(kM_j), \mathcal{O}(M_j), d^{(j)} \right) \text{piv} \left( \mathcal{O}(kM_j), \mathcal{O}(M_j), W' \right) \right]$$

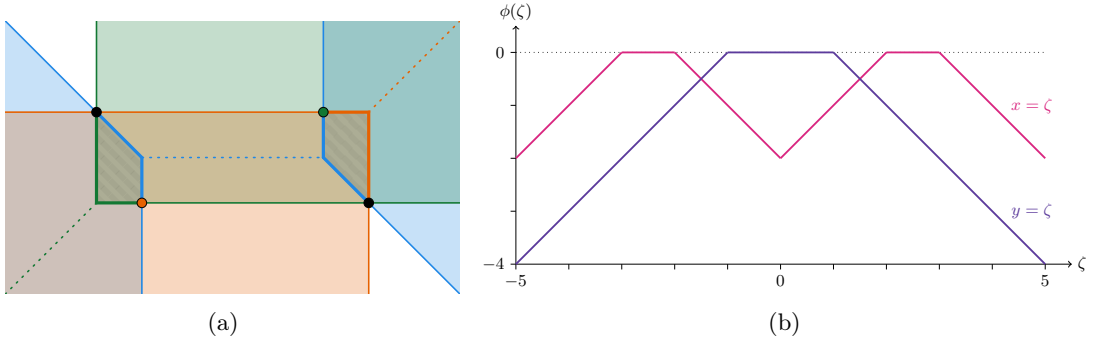
arithmetic operations, where  $M_j = \binom{j+N_j}{j}$  with  $N_j$  a integer linear combination of the degrees  $d_i^{(j)}$  in the last  $j$  variables of the polynomials  $f_i^+ \oplus f_i^-$  for  $1 \leq i \leq k$  and  $f_i$  for  $k+1 \leq i \leq \ell$ , that can be explicited using [3, Theorem 4.14] and  $d^{(j)} = \max_{1 \leq i \leq \ell} d_i^{(j)}$ .

*Example 12.* Consider the following system.

$$\begin{cases} 0 \oplus 0x^2y \geq 2x \oplus 2xy \\ 2xy \geq 1x \oplus -1y \\ 0 \geq -3x \oplus -1y \end{cases} \quad (9)$$

Observing the zero set of the two associated spectral functions displayed on Figure 1b, one finds that the solutions of (9) are included in  $([-3, -2] \sqcup [2, 3]) \times [-1, 1]$ , which is indeed confirmed by the representation of the solution set in Figure 1a.

Fig. 1: The collection of tropical semialgebraic sets arising from system (9) as well as the spectral functions obtained when specializing each variable respectively.



## 8 Numerical results

An implementation of algorithms to decide the feasibility of tropical polynomial systems is available at [11]. Experimental results obtained by the dichotomy method are displayed in Figure 2. The solvability of randomized instances of systems of tropical polynomial inequalities was examined twice. First, the polynomials were treated as full polynomials, with no consideration on their support. The resulting linearized system was thus given by a Macaulay matrix simply truncated to the according degree bound. This corresponds to the curves in red. Then, the sparsity of the

polynomials was used to obtain a much smaller linearized system, given by Macaulay matrices whose columns were this time obtained by computing the Minkowski sum of the supports of each polynomial. This corresponds to the curves in green.

The sheer size difference between the matrices in the first and second case is repercutted in the time difference for the policy iteration (solid line curves), which is naturally much faster in the case where the sparsity was exploited. How the bottleneck of the current implementation in this case can be observed for instances with a bigger number of variables, where the computation time of the Minkowski sum (dashed lines) to deal with sparse instances dominates the time of policy iteration.

## 9 Concluding remarks

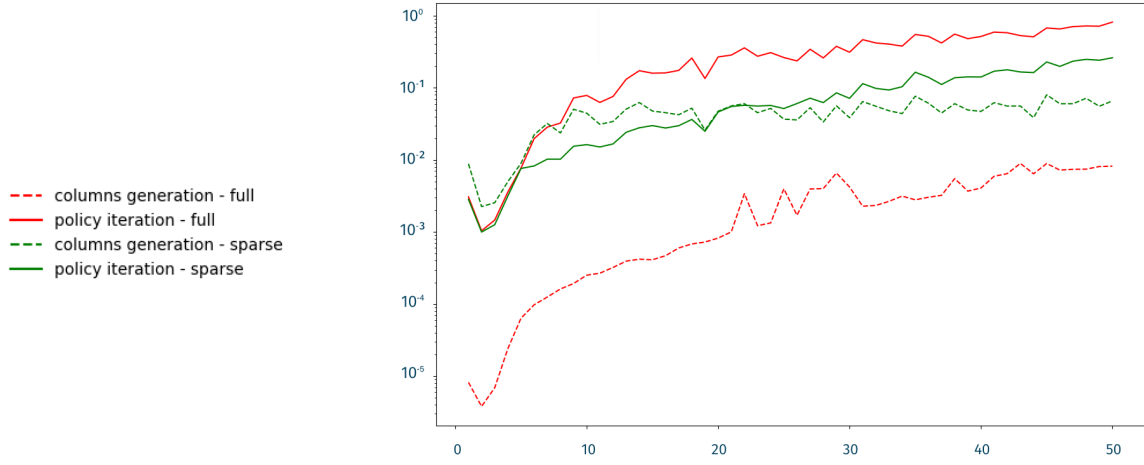
We developed an analogue of the eigenvalue method, allowing one to find solutions of systems of tropical equalities and inequalities. We focused on the problem of computing the projections of the solution set on the different coordinates. It would be interesting to extend the present methods to get additional informations on the solution set (like the number of connected components).

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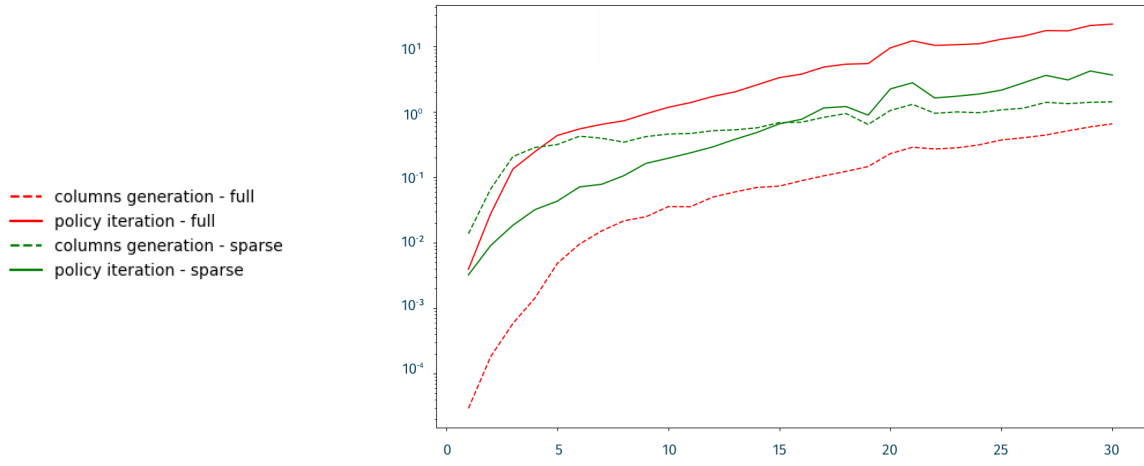
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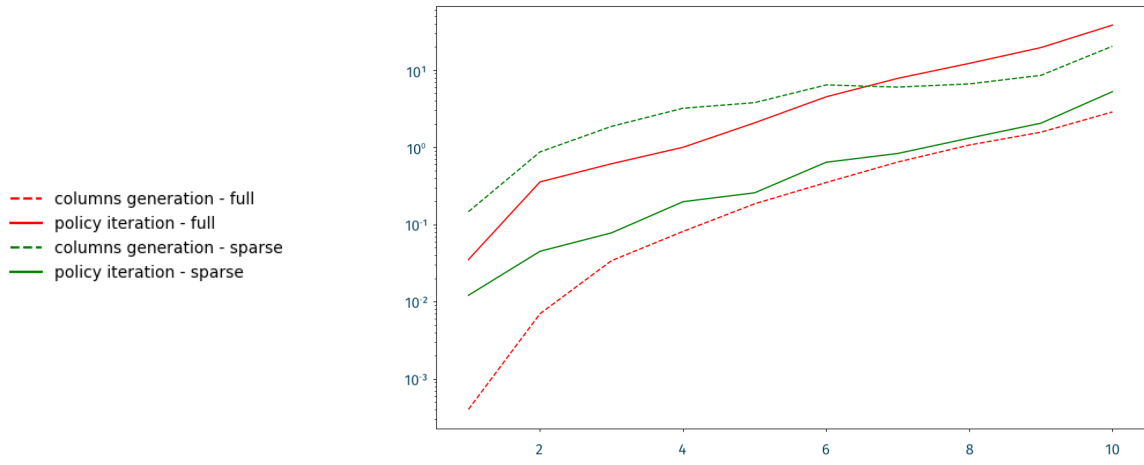
Fig. 2: Average CPU time in seconds on 10 random instances to decide the solvability of a system of  $k$  tropical polynomial inequalities in  $n$  variables in function of the degree



(a)  $n = 2$  and  $k = 3$



(b)  $n = 3$  and  $k = 4$



(c)  $n = 4$  and  $k = 5$