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## RESEARCH TYPE

# Stability analysis and integral extension of sub-optimal nonlinear damping control

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**Abstract**

The problems of robustness with respect to external inputs and of convergence rate estimation are considered for the so-called sub-optimal nonlinear damping control<sup>1</sup>. To this end, existence of solutions is justified, a strict Lyapunov function is found and its properties are investigated. The hyperexponential convergence of the unperturbed sub-optimal nonlinear damping control is demonstrated. Moreover, an integral extension is proposed to complement the controller with capability of compensation for static errors. The results are illustrated by numeric simulations and the control experiments.

**KEY WORDS**

Sub-optimal nonlinear damping control, Input-to-state stability, Integral control

## 1 | INTRODUCTION

Establishing robustness conditions with respect to external inputs and quantifying convergence performance are relevant for different types of control and estimation systems.

For linear systems, an internal stability and, correspondingly, input-to-state stability (ISS) in the presence of inputs are interrelated and can be assessed in Laplace domain by using the sensitivity function, which is composed of the transfer functions of the individual subsystems in the closed loop<sup>2,3</sup>. For nonlinear systems, appearance of perturbations may destroy the stability; then the ISS theory<sup>4,5</sup> builds one of the most popular tools for studying the stability and, thus, robustness in presence of external disturbances.

Another important performance characteristic in applications is the rate of convergence of trajectories to an equilibrium or a goal mode in the absence of uncertainty. For linear systems such a convergence is always exponential, but in the nonlinear setting many other kinds of decay can be observed, and finite-/fixed-time convergences are popular and useful examples<sup>6</sup>. In this work, for the case specific system, the hyperexponential convergence will be studied, which includes the latter ones as particular examples, and that implies approaching an equilibrium with the velocity higher than any exponential<sup>7,8</sup>. The systems admitting this kind of behavior demonstrate good robustness abilities being simple in discretization<sup>9</sup>.

The second-order nonlinear feedback control systems with the so-called sub-optimal damping, introduced and analyzed in<sup>1,10</sup>, are in focus of this work. The control method appears in equivalence to a standard PD (proportional derivative) controller, if some application related policies require maintaining the linear feedback term but allowing the design of a nonlinear damping for improving the convergence characteristics. Moreover, we address a possible extension by an integral feedback with capability of compensation for the static errors.

The rest of the paper is organized as follows. After the defined notations, the preliminaries to ISS are given in Section 2. The problem statement is provided in Section 3, followed by analysis of the robust stability in Section 4. In Section 5, we demonstrate the origins of the hyperexponential convergence of the unperturbed sub-optimal nonlinear damping control. The integral feedback extension is given in Section 6, while Section 7 provides both illustrative numerical and also experimental examples. Brief conclusions are drawn in Section 8. The auxiliary results are summarized in the Appendix.

## Notation

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ , where  $\mathbb{R}$  is the set of real numbers.
- $|\cdot|$  denotes the absolute value in  $\mathbb{R}$ ,  $\|\cdot\|$  is used for the Euclidean norm on  $\mathbb{R}^n$ .
- For a (Lebesgue) measurable function  $d : \mathbb{R}_+ \rightarrow \mathbb{R}^m$  define the norm  $\|d\|_{[0,t]} = \text{ess sup}_{s \in [0,t]} \|d(s)\|$ , and the set of  $d$  with the property  $\|d\|_\infty = \|d\|_{[0,+\infty)} < +\infty$  we further denote as  $\mathcal{L}_\infty^m$  (the set of essentially bounded measurable functions).
- A continuous function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{K}$  if  $\alpha(0) = 0$  and it is strictly increasing. The function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{K}_\infty$  if  $\alpha \in \mathcal{K}$  and it is increasing to infinity. A continuous function  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{KL}$  if  $\beta(\cdot, t) \in \mathcal{K}$  for each fixed  $t \in \mathbb{R}_+$  and  $\beta(s, \cdot)$  is decreasing to zero for each fixed  $s > 0$ .
- The *Young's inequality* claims that for any  $a, b \in \mathbb{R}_+$ <sup>11</sup>

$$ab \leq \frac{1}{p} a^p + \frac{p-1}{p} b^{\frac{p}{p-1}}$$

for any  $p > 1$ .

## 2 | PRELIMINARIES

Consider a dynamical system:

$$\dot{x}(t) = F(x(t), d(t)), \quad t \geq 0, \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the state and  $d(t) \in \mathbb{R}^m$  is the input,  $d \in \mathcal{L}_\infty^m$ ;  $F : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$  is continuous and it ensures forward existence and uniqueness of solutions of the system at least locally in time (e.g., it is locally Lipschitz continuous out the origin),  $F(0, 0) = 0$ . For any  $x_0 \in \mathbb{R}^n$  and  $d \in \mathcal{L}_\infty^m$  the respective solution is denoted by  $x(t, x_0, d)$  with  $x(0, x_0, d) = x_0$ .

The definitions and results given in this subsection follow from<sup>4,5</sup>.

**Definition 1.** The system (1) is called *ISS*, if there exist  $\beta \in \mathcal{KL}$  and  $\gamma \in \mathcal{K}$  such that

$$\|x(t, x_0, d)\| \leq \max\{\beta(\|x_0\|, t), \gamma(\|d\|_{[0,t]})\}, \quad \forall t \geq 0$$

for all  $x_0 \in \mathbb{R}^n$  and  $d \in \mathcal{L}_\infty^m$ .

The function  $\gamma$  from Definition 1 quantifies an *asymptotic gain* of (1):

$$\lim_{t \rightarrow +\infty} \|x(t, x_0, d)\| \leq \gamma(\|d\|_\infty)$$

satisfying for all  $x_0 \in \mathbb{R}^n$  and  $d \in \mathcal{L}_\infty^m$ .

If  $d \equiv 0$ , then from ISS we recover the conventional *global asymptotic stability at the origin*.

**Definition 2.** A continuously differentiable function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  is called *ISS-Lyapunov function* for the system (1) if there exist  $\alpha_1, \alpha_2, \eta \in \mathcal{K}_\infty$  and  $\varrho \in \mathcal{K}$  such that for all  $x \in \mathbb{R}^n$  and all  $d \in \mathbb{R}^m$ :

$$\begin{aligned} \alpha_1(\|x\|) &\leq V(x) \leq \alpha_2(\|x\|), \\ \frac{\partial V(x)}{\partial x} F(x, d) &\leq -\eta(\|x\|) + \varrho(\|d\|). \end{aligned}$$

**Theorem 1.** *The system (1) is ISS if and only if it admits an ISS-Lyapunov function.*

In the case that the system (1) is defined not on whole  $\mathbb{R}^n$  but only on its open subset or on a manifold, there exist the corresponding extensions of the ISS theory<sup>12,13</sup>.

### 3 | PROBLEM STATEMENT

Consider a perturbed second-order feedback system with the so-called sub-optimal nonlinear damping<sup>1</sup> given by

$$\begin{aligned}\dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -kx_1(t) - \rho(x(t))x_2(t) + d(t), \\ \rho(x) &= \begin{cases} \frac{|x_2|}{|x_1|} & \text{if } x_1 \neq 0 \\ 0 & \text{otherwise} \end{cases},\end{aligned}\tag{2}$$

where  $k > 0$  is an arbitrary design parameter,  $x(t) = (x_1(t), x_2(t)) \in M = \{x \in \mathbb{R}^2 : x_1 \neq 0\} \cup \{0\}$  are the dynamic state variables,  $d \in \mathcal{L}_\infty$  is the external essentially bounded input,  $t \geq 0$ . Note that on  $M \setminus \{0\}$  the right-hand side of the system is locally Lipschitz continuous in  $x$  and  $d$ , hence, the solutions can be defined in the conventional Caratheodory sense. Moreover, (2) admits the zero solution at the origin if  $d \equiv 0$ . The detailed explanations why the solutions of (2) are well defined for any  $x(0) \in M$  and  $d \in \mathcal{L}_\infty$  are provided in the Appendix.

The idea behind the closed-loop dynamical system (2) is to increase the system damping in a vicinity of the output steady state, i.e., for small distance  $|x_1|$ , on the one hand, and simultaneously for high velocities, i.e., for large  $|x_2|$  values, on the other hand. This results in an energy dissipation rate which is cubic in  $|x_2|$  and hyperbolic (i.e., inverse) in  $|x_1|$ , cf.<sup>10</sup> and the time derivative of the Lyapunov function provided below in section 4. The sub-optimal nonlinear damping control (2) uses, in the same way as a standard PD-control, the feedback of state variables  $x_1$  and  $x_2$ . This makes the control structure of sub-optimal nonlinear damping control equivalent to that of the PD output feedback controller. Only the damping term  $-\rho(x)x_2$ , that corresponds to the differential (D) part  $-k_D x_2$  of a PD control with  $k_D > 0$ , is different and nonlinear. It is the key feature of the sub-optimal nonlinear damping control, that ensures providing a faster (hyperexponential in the case of  $d = 0$ ) convergence, see<sup>1, 10</sup>. Also a practical experimental study<sup>14</sup> revealed that the sub-optimal nonlinear damping control is mostly superior in comparison to its linear PD counterpart.

The exhaustive studies in<sup>1, 10</sup> demonstrated global asymptotic stability of (2) in  $M$  for  $d \equiv 0$ ; this result was obtained using a non-strict Lyapunov function. Also, it was shown that  $M$  is forward invariant for (2). Moreover, a hyperexponential convergence rate was observed in numeric experiments.

The goal of this work is to investigate the robust stability of (2) for  $d \neq 0$ , to provide explanations for the observed rate of convergence, and to design an integral extension for (2) that is able to compensate for any static input  $d(t) = \text{const}$ .

### 4 | ANALYSIS OF ROBUST STABILITY

First, let us briefly note that  $d \in \mathcal{L}_\infty$  does not break the invariance of  $M$ , as shown in details in the Appendix. Indeed, if  $x = 0$ , then any value of  $d \neq 0$  forces the system (2) to exit first in the domain where  $x_2 \neq 0$ , implying next that  $x_1 \neq 0$  (the system cannot move staying on the set  $x_1 = 0$ ). Next, for any  $x \in M$  with  $x_2 \neq 0$ , the term  $-\frac{|x_2|}{|x_1|}x_2$  dominates  $d \in \mathcal{L}_\infty$  while approaching the line  $x_1 = 0$  (a border of  $M$ ). Then, under the constraint

$$|d| \leq \frac{1}{2} \frac{|x_2|^2}{|x_1|},$$

which should be always satisfied for any  $d \in \mathcal{L}_\infty$  for almost all instants of time being sufficiently close to that border, the Lyapunov function, cf.<sup>10</sup>,

$$E(x) = kx_1^2 + x_2^2$$

admits the time derivative

$$\dot{E} = -2 \frac{|x_2|^3}{|x_1|} + 2|x_2||d| \leq -\frac{|x_2|}{|x_1|}x_2^2,$$

which implies boundedness of the state and that the surface  $x_1 = 0$  cannot be reached before the constraint  $x_2 = 0$  is satisfied. Note that on the line  $x_2 = 0$  the disturbances cannot shift the system out  $M$ . Hence, the trajectories stay always in  $M$ . Consequently, for  $d \in \mathcal{L}_\infty$  the solutions stay either in  $M \setminus \{0\}$  for all  $t \geq 0$  or settle in finite-time or asymptotically at the origin.

Having the solutions well defined, we can formulate the main result of this section:

**Theorem 2.** For the system (2), there exist  $\beta \in \mathcal{KL}$  and  $\gamma \in \mathcal{K}$  such that

$$\|x(t)\| \leq \max\{\beta(\|x(0)\|, t), \gamma(\|d\|_{[0,t]})\}, \quad \forall t \geq 0,$$

for all  $x(0) \in M$  and  $d \in \mathcal{L}_\infty$ .

*Proof.* Consider a strict Lyapunov function candidate:

$$V(x) = kx_1^2 + x_2^2 + \varepsilon x_1 x_2, \quad (3)$$

where  $\varepsilon > 0$  is a tuning parameter. Note that  $V$  is continuously differentiable, radially unbounded and positive definite provided that  $k > \frac{\varepsilon^2}{4}$ . The direct computations show that for  $x \neq 0$

$$\begin{aligned} \dot{V} &= \dot{x}_2(2x_2 + \varepsilon x_1) + \varepsilon x_2^2 + 2kx_1 x_2 \\ &= -2 \frac{|x_2|}{|x_1|} x_2^2 - k\varepsilon x_1^2 + \varepsilon x_2^2 (1 - \text{sign}(x_2) \text{sign}(x_1)) \\ &\quad + (2x_2 + \varepsilon x_1)d. \end{aligned}$$

Distinguishing between two cases we can calculate the desired upper bound on  $\dot{V}$ :

- If  $\text{sign}(x_2) = \text{sign}(x_1)$ , that corresponds to the state trajectories in the I or III quadrants, then using Young's inequality:

$$\begin{aligned} \varepsilon x_1 d &\leq 0.5\varepsilon \left( kx_1^2 + \frac{1}{k}d^2 \right), \\ 2|x_2||d| &\leq \frac{1}{3} \frac{|x_2|^3}{|x_1|} + \frac{2}{3} \sqrt{|x_1|} |d|^{1.5} \\ &\leq \frac{1}{3} \frac{|x_2|^3}{|x_1|} + \frac{k\varepsilon}{6} x_1^2 + \frac{1}{2\sqrt[3]{k\varepsilon}} d^2 \end{aligned}$$

we obtain:

$$\begin{aligned} \dot{V} &= -2 \frac{|x_2|}{|x_1|} x_2^2 - k\varepsilon x_1^2 + (2x_2 + \varepsilon x_1)d \\ &\leq -\frac{5}{3} \frac{|x_2|}{|x_1|} x_2^2 - \frac{k\varepsilon}{3} x_1^2 + \left( \frac{1}{\sqrt[3]{k\varepsilon}} + \frac{\varepsilon}{k} \right) \frac{d^2}{2}. \end{aligned}$$

- If  $\text{sign}(x_2) \neq \text{sign}(x_1)$ , that corresponds to the state trajectories in the II or IV quadrants, then

$$\begin{aligned} \dot{V} &\leq -2 \frac{|x_2|^3}{|x_1|} - k\varepsilon x_1^2 + 2\varepsilon x_2^2 + (2x_2 + \varepsilon x_1)d \\ &= -2x_2^2 \left( \frac{|x_2|}{|x_1|} - \varepsilon \right) - k\varepsilon x_1^2 + (2x_2 + \varepsilon x_1)d. \end{aligned}$$

Next, two scenarios are possible. First,  $\frac{|x_2|}{|x_1|} > \rho_1 \varepsilon$  for some  $\rho_1 > 1$ , which leads to the estimate

$$\begin{aligned} \dot{V} &\leq -2\varepsilon(\rho_1 - 1)x_2^2 - k\varepsilon x_1^2 + (2x_2 + \varepsilon x_1)d \\ &\leq -\varepsilon(\rho_1 - 1)x_2^2 - \frac{k\varepsilon}{2} x_1^2 + \left( \frac{\varepsilon^{-1}}{\rho_1 - 1} + \frac{\varepsilon}{2k} \right) d^2, \end{aligned}$$

where again Young's inequality was used in the last step. Otherwise,  $\frac{|x_2|}{|x_1|} \leq \rho_1 \varepsilon$  implies that  $x_2^2 \leq \rho_1^2 \varepsilon^2 x_1^2$ , which under the restriction

$$(1 - \rho_2)k \geq 2\rho_1^2 \varepsilon^2$$

for any  $\rho_2 \in (0, 1)$  and applying Young's inequality provides the estimates:

$$\begin{aligned}\dot{V} &\leq -2\frac{|x_2|^3}{|x_1|} - \rho_2 k \varepsilon x_1^2 - ((1 - \rho_2)k - 2\rho_1^2 \varepsilon^2) \varepsilon x_1^2 \\ &\quad + (2x_2 + \varepsilon x_1)d \\ &\leq -2\frac{|x_2|^3}{|x_1|} - \rho_2 k \varepsilon x_1^2 + (2x_2 + \varepsilon x_1)d \\ &\leq -\frac{5}{3}\frac{|x_2|^3}{|x_1|} - \frac{\rho_2 k \varepsilon}{3} x_1^2 + \left( \frac{1}{\sqrt[3]{k\varepsilon\rho_2}} + \frac{\varepsilon}{\rho_2 k} \right) \frac{d^2}{2}.\end{aligned}$$

Note that the right-hand side is unbounded in  $x_2$  due to the interrelation between the state variables in this scenario.

Combining both cases we derive the desired upper bound:

$$\begin{aligned}\dot{V} &\leq -\min \left\{ \frac{5}{3}\frac{|x_2|}{|x_1|}, \varepsilon(\rho_1 - 1) \right\} x_2^2 \\ &\quad - \min \left\{ \frac{1}{2}, \frac{\rho_2}{3} \right\} k \varepsilon x_1^2 + \sigma d^2, \\ \sigma &= \frac{1}{2} \max \left\{ \frac{\varepsilon}{\rho_2 k} + \frac{1}{\sqrt[3]{\rho_2 k \varepsilon}}, \frac{2}{\varepsilon(\rho_1 - 1)} + \frac{\varepsilon}{k} \right\}\end{aligned}$$

holding for any  $x \in M$  and  $d \in \mathbb{R}$ . Hence,  $V$  is an ISS-Lyapunov function for the system. Therefore, applying the comparison lemmas the desired result has been proven.  $\square$

The relation between  $k$  and  $\varepsilon$  reads that

$$k \geq \max \left\{ \frac{1}{4}, \frac{2\rho_1^2}{1 - \rho_2} \right\} \varepsilon^2 \quad (4)$$

for some values of  $\rho_1 > 1$  and  $\rho_2 \in (0, 1)$ . The values of parameters  $\rho_1$  and  $\rho_2$  can be selected to find a less conservative asymptotic gain with respect to  $d$ , or for adjustment of other quantities, see section 6 below.

## 5 | HYPEREXPONENTIAL CONVERGENCE

Developing the estimate for the time derivative of  $V$  derived above, it is possible to demonstrate existence of  $\rho_3 > 0$  such that

$$\dot{V} \leq -\rho_3 V + \sigma d^2$$

for all  $x \in M$  and  $d \in \mathbb{R}$ . Consequently, this Lyapunov function establishes an exponential convergence rate in the system, while the previous investigations discovered a much faster hyperexponential convergence<sup>14</sup>.

For showing the hyperexponential convergence of the system (2) in the case  $d = 0$ , introduce the new auxiliary variables  $z_1 = x_1$  and  $z_2 = x_2 + ktz_1$ . Then, away from the origin the transformed closed-loop dynamic system takes the form:

$$\dot{z}_1 = z_2 - ktz_1, \quad (5)$$

$$\dot{z}_2 = \left( kt - \frac{|z_2 - ktz_1|}{|z_1|} \right) (z_2 - ktz_1). \quad (6)$$

Distinguishing the sign of the numerator inside of the first brackets, (6) can be rewritten as

$$\dot{z}_2 = \begin{cases} \left( \frac{kt(|z_1| + z_1) - z_2}{|z_1|} \right) (z_2 - ktz_1), & \text{if } z_2 > ktz_1, \\ \left( \frac{kt(|z_1| - z_1) + z_2}{|z_1|} \right) (z_2 - ktz_1), & \text{if } z_2 \leq ktz_1. \end{cases} \quad (7)$$

Now, when further distinguishing between the sign of  $z_1$ , (6) results in

$$\dot{z}_2 = -\frac{|z_2 - ktz_1|}{|z_1|} z_2 \quad (8)$$

if  $z_2 > ktz_1 \wedge z_1 \leq 0 \vee z_2 \leq ktz_1 \wedge z_1 \geq 0$ .

Transforming the above condition back into the initial state space, one can recognize that these constraints are verified in II or IV quadrant, *i.e.*, for  $x_1 x_2 < 0$ . Note that this subspace of the state variables is sufficient for analysis since all trajectories of (2) are unambiguously running into these quadrants for  $d \equiv 0$ , see<sup>1,10</sup>. Therefore, (8) is nothing but

$$\dot{z}_2 + \alpha(t)z_2 = 0, \quad \text{if } x_1 x_2 < 0, \quad (9)$$

where  $\alpha(t) = \frac{|x_2(t)|}{|x_1(t)|} > 0$ . Thus, the homogeneous solution of (9), correspondingly (8), has the form

$$z_2(t) = \exp\left(-\int_0^t \alpha(s)ds\right) z_2(0). \quad (10)$$

Substituting back  $z_2$ -transformation into (10), results in

$$\dot{x}_1 + ktx_1 = \exp\left(-\int_0^t \frac{|x_2(s)|}{|x_1(s)|} ds\right) x_2(0). \quad (11)$$

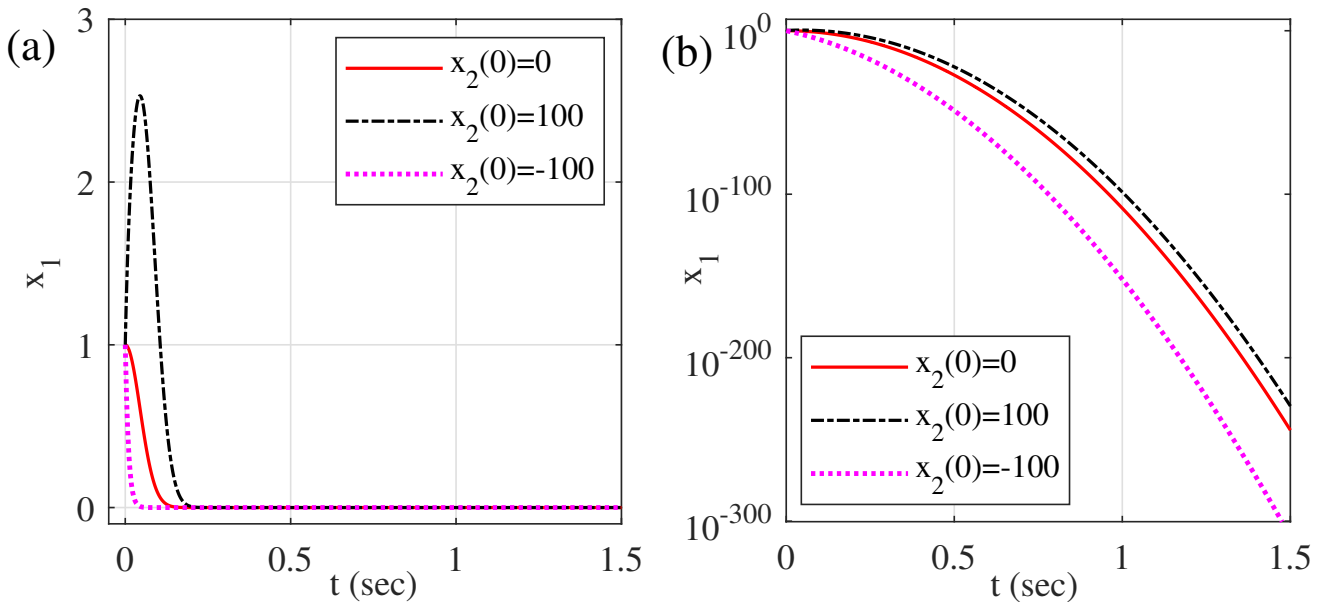
Since  $x_1(t)$  is globally convergent, see previous section, the right-hand side of (11) is decreasing for any  $x_2(0)$ . Analyzing the homogeneous solution of (11), *i.e.*, of

$$\dot{x}_1 + ktx_1 = 0,$$

it is easy to show that

$$x_1(t) = \exp\left(-\frac{k}{2}t^2\right) x_1(0), \quad (12)$$

which converging faster than any exponential due to presence of the quadratic term in the exponent. This provides an intuition behind the reasons of the hyperexponential convergence of trajectories in vicinity to the origin, *i.e.*, as the right-hand side of (11)  $\rightarrow 0$ . Exemplary convergence of (11) is visualized in Fig. 1 from the numerical simulation with  $k = 500$  and  $x_1(0) = 1$ , and different initial conditions  $x_2(0) \in \{-100, 0, 100\}$ .



**FIGURE 1** Response of (11) for  $x_1(0) = 1$ ,  $x_2(0) \in \{-100, 0, 100\}$ , and  $k = 500$ , linear scale in (a) and logarithmic scale in (b).

## 6 | INTEGRAL CONTROL EXTENSION

Now, assuming that the nominal system dynamics is affected by some unknown constant disturbance term  $d = \text{const} \neq 0$ , the control system (2) is extended by an additional state variable  $x_3 \in \mathbb{R}$ , which is responsible for the integral term and compensation of  $d$ , thus resulting in

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= -kx_1 - \rho(x)x_2 + d + x_3,\end{aligned}\tag{13}$$

where the dynamics of  $x_3$  has to be designed. To this end, let us extend the Lyapunov function (3) as:

$$V(x) = kx_1^2 + x_2^2 + \varepsilon x_1 x_2 + 0.5\gamma^{-1}(d + x_3)^2,$$

where  $\gamma > 0$  is a tuning gain. As before, this Lyapunov function is radially unbounded and positive definite in  $(x_1, x_2) \times (d + x_3) \in M \times \mathbb{R}$  provided that  $k > \frac{\varepsilon^2}{4}$ , and following the previous computations

$$\begin{aligned}\dot{V} &= -2\frac{|x_2|}{|x_1|}x_2^2 - k\varepsilon x_1^2 + \varepsilon x_2^2(1 - \text{sign}(x_2)\text{sign}(x_1)) \\ &\quad + (2x_2 + \varepsilon x_1)(d + x_3) + \gamma^{-1}(d + x_3)\dot{x}_3.\end{aligned}$$

Take (in such a case the total input  $d + x_3$  is well defined for (2))

$$\dot{x}_3 = -\gamma(2x_2 + \varepsilon x_1),\tag{14}$$

then

$$\dot{V} = -2\frac{|x_2|}{|x_1|}x_2^2 - k\varepsilon x_1^2 + \varepsilon x_2^2(1 - \text{sign}(x_2)\text{sign}(x_1))$$

and as it has been proven above, the right-hand side of this expression can be upper bounded by a negative definite function of  $x_1$  and  $x_2$  under the constraint (4) (hence, the solutions are defined for all  $t \geq 0$ ). The global asymptotic convergence to the origin follows LaSalle arguments<sup>15, Theorem 10</sup>. Indeed, the term  $\rho(x)x_2$  stays bounded on the trajectories of the system, and there is  $c > 0$  such that:

$$\int_0^{+\infty} (x_1^2(t) + x_2^2(t))dt \leq c^{-1}V(0),$$

which by<sup>15</sup> implies that trajectories of the system asymptotically approach the largest invariant subset inside  $x_1 = x_2 = 0$ , i.e.,  $x_3 = -d$ . The following result has been proven:

**Theorem 3.** *For the system (13), (14), the equilibrium  $x_1 = x_2 = 0$ ,  $x_3 = -d$  is globally asymptotically stable in  $M \times \mathbb{R}$  provided that (4) holds.*

Now (4) and the parameters  $\rho_1 > 1$  and  $\rho_2 \in (0, 1)$  can be used to tune the gains  $k$  and  $\gamma$ . For the sake of simplicity, consider a less conservative assumption

$$k > \frac{\varepsilon^2}{4},$$

cf. (4), and suppose  $0 < k \ll 2\gamma$  without loss of generality. Then, for the integral feedback gain  $k_i \equiv \gamma\varepsilon$ , cf. (14), one can write

$$k > \frac{1}{4} \frac{k_i^2}{\gamma^2}.\tag{15}$$

Since the output feedback gain is essentially captured in (14), one can substitute the left-hand-side of (15) by  $2\gamma$  and obtain  $\gamma > (1/8)^{1/3} k_i^{2/3}$ . Since the output feedback gain equals  $2\gamma$  for the large part, one obtain a less conservative parametric condition

$$k > k_i^{2/3}.\tag{16}$$



## 7 | SIMULATIONS AND EXPERIMENTS

### 7.1 | Numerical simulation

Numerical simulations of the closed-loop control system (13), (14) are performed with the fixed-step 0.00001 and automatically selected solver of Matlab/Simulink. Two integral control gain values  $k_i = \{50, 1000\}$  are assigned, for which the boundary condition  $k = k_i^{2/3}$ , cf. (16), results in  $k = \{13.57, 100\}$ , respectively. Then, another pair of the feedback gain values  $k = \{9, 70\}$ , that is violating (16) for the assigned pair of  $k_i$ , is also used to evaluate an unstable control response. Here the disturbance value is set to zero, *i.e.*,  $d \equiv 0$ . The asymptotically stable and unstable control responses are shown in Fig. 2, for  $k_i = 50$  above and for  $k_i = 1000$  below.

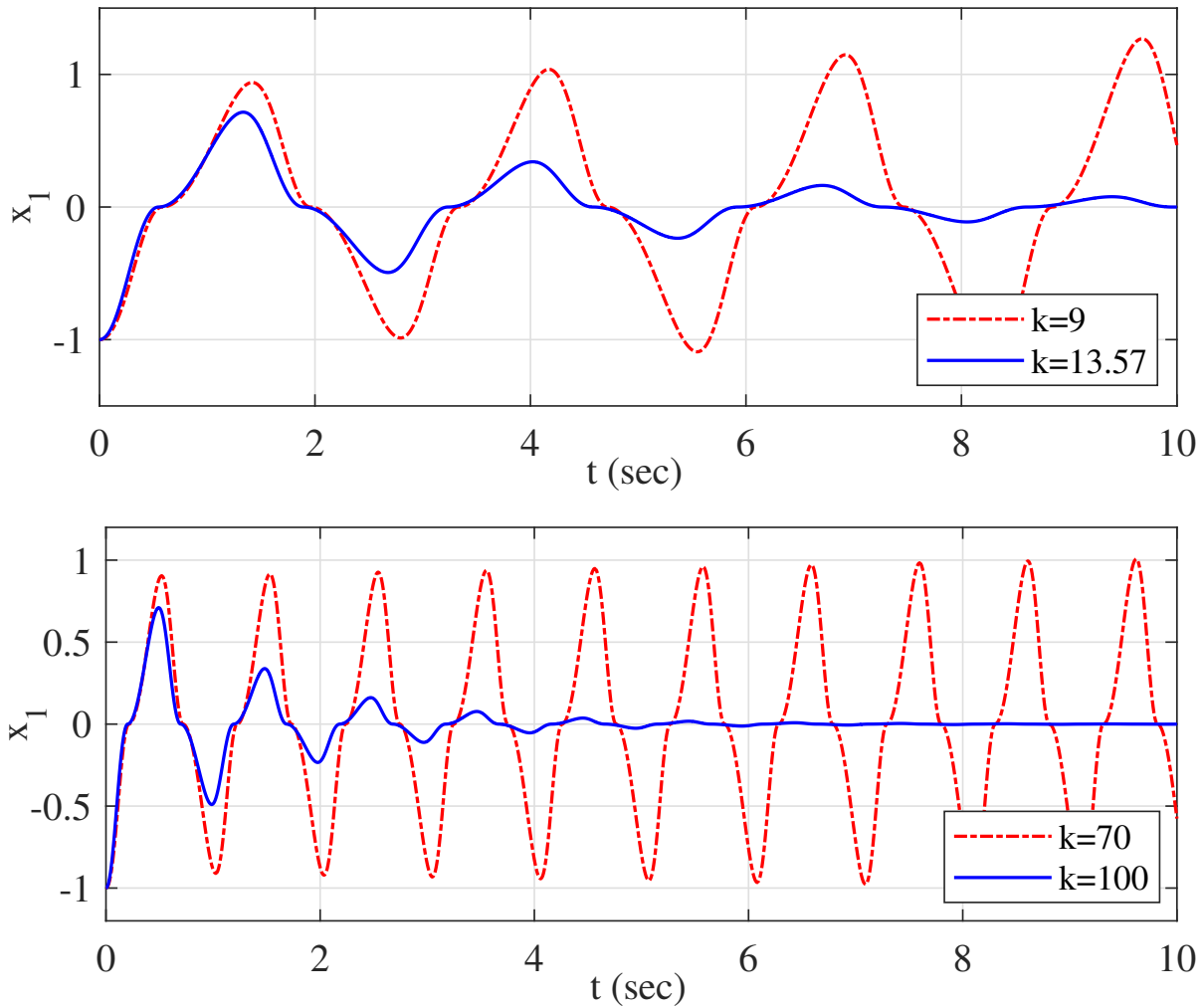
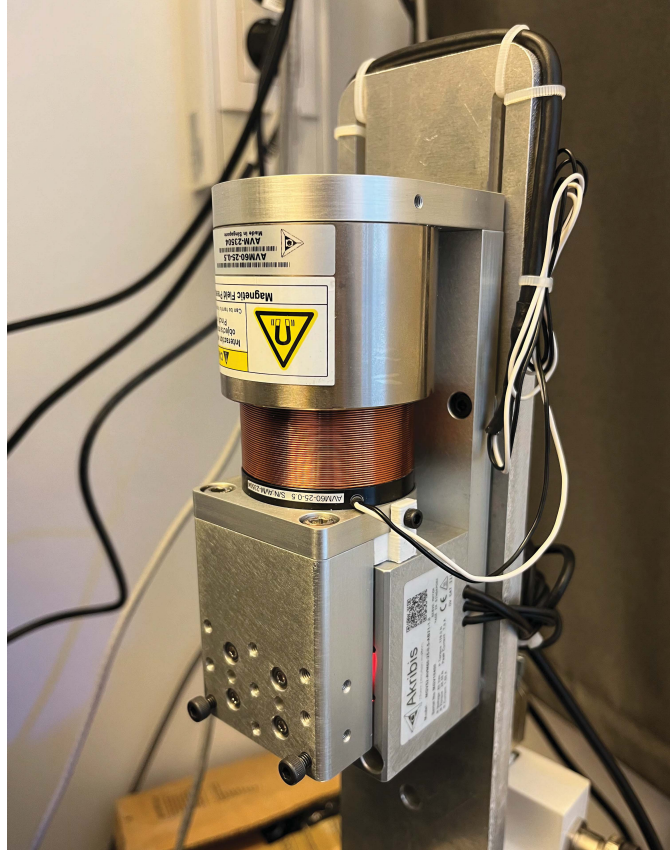


FIGURE 2 Controlled response for  $k_i = 50$  and  $k = \{13.57, 9\}$  above, and for  $k_i = 1000$  and  $k = \{100, 70\}$  below.

### 7.2 | Experimental example

Experimental evaluation of the sub-optimal nonlinear damping control system (2), and its integral extension (13), (14), is performed on a laboratory setup of the voice-coil-based actuator with one degree of freedom, see Fig. 3. The sampling rate is 10

kHz, and the low-level compiled numerical solver uses the most simple first-order forward Euler scheme. The  $x_1(t)$  output state is measured by a contactless inductive displacement sensor, with a nominal repeatability of  $\pm 12$  micrometers and a relatively large level of the measurement noise. The total relative displacement range is limited to about 13 millimeters in total. More details on the experimental system and its modeling can be found, *e.g.*, in<sup>14</sup>.



**FIGURE 3** Experimental setup of voice-coil-based actuator.

Note that the bounded linear damping effects, *i.e.*, proportional to  $x_2$ , are neglected. On the contrary, the stiction nonlinearities due to Coulomb friction and stator-mover force ripples are pre-compensated by a low-amplitude and high-frequency jitter injection, *cf.*<sup>14</sup>. Then, the external essentially bounded input disturbance  $d(t) \simeq \text{const}$  is mainly due to the gravity force. This way, the system under control is close to a perturbed double integrator, *cf.* (2). Note that for this is valid, all below given control parameters are scaled by  $\alpha = Km^{-1}$ , where  $K$  is the overall actuator gain and  $m$  is the overall lumped moving mass. The evaluated sub-optimal nonlinear damping control with  $k = 625\alpha$  is shown in Fig. 4 for different initial output values  $x_{1,0} = \{-0.005, -0.008\}$  meter. One can recognize that a constant disturbance  $d$  leads to an approximately the same residual error  $\approx 0.003$  meter, which confirms the result of Theorem 2. The sub-optimal nonlinear damping control extended by the integral feedback action with  $\varepsilon\gamma \equiv k_i$ , *cf.* (14), is evaluated once for  $k_i = 6000\alpha$  and once for  $k_i = 27000\alpha$ . Both measured responses are shown in Fig. 5. For the assigned former  $k_i$  value, the static error is compensated, thus verifying all conditions of Theorem 3. At the same time, the latter  $k_i$  value is violating the condition (16) and the system response yields as not asymptotically stable, that is inline with Theorem 3.

## 8 | CONCLUSION

In this paper, we analyzed ISS-like properties of the sub-optimal nonlinear damping control under investigation. We extended the nonlinear closed-loop dynamic system<sup>1</sup> by the matched essentially bounded disturbance input and showed the global

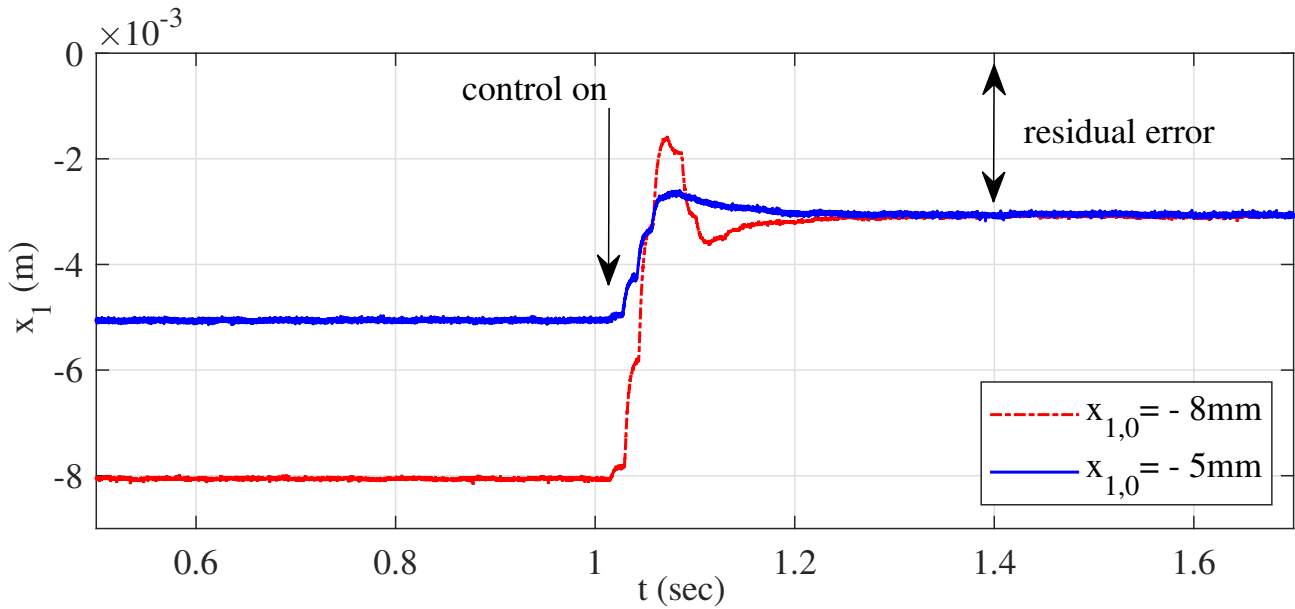


FIGURE 4 Sub-optimal nonlinear damping control for different initial output values.

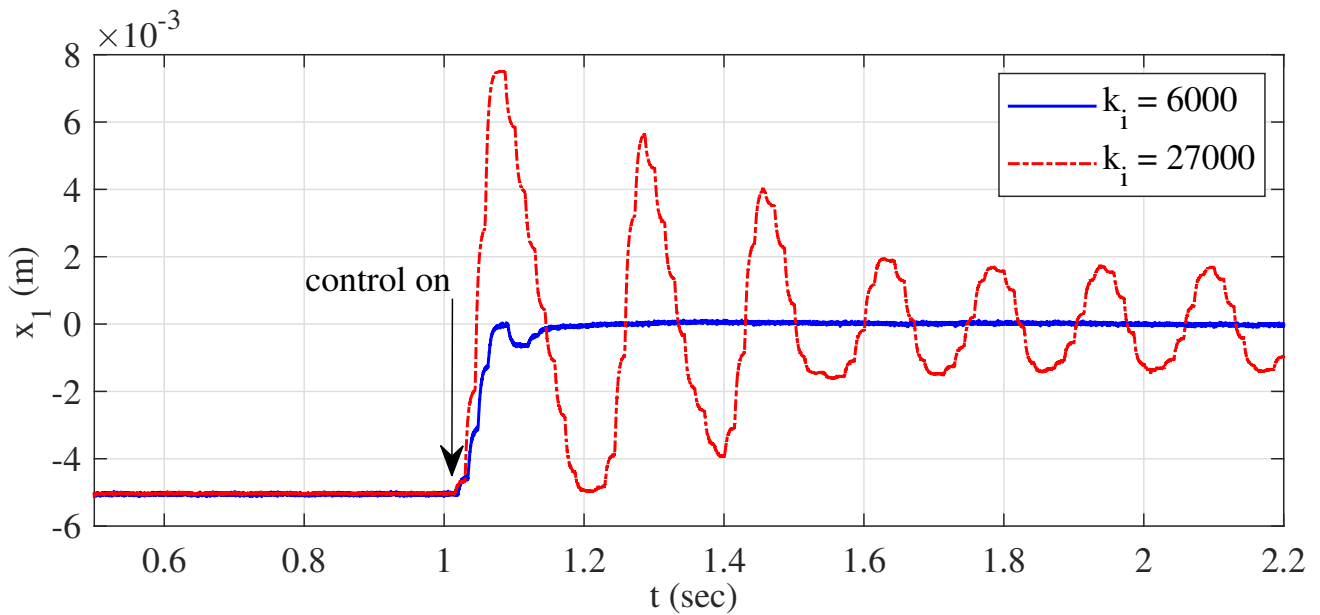


FIGURE 5 Sub-optimal nonlinear damping control extended by integral feedback for  $k_i = 6000$  and  $k_i = 27000$ .

robust stability with respect to that. Furthermore, we justified the hyperexponential convergence of the unperturbed system, and provided a possible integral control extension for compensating the stationary output errors. Based of Theorem 3, we derived a less conservative and practical tuning rule for proportional and integral control gains. Here it is worth recalling that the nonlinear damping term does not require any additional gain parameter to be designed, cf. <sup>1</sup>. For confirming and visualizing the developed stability analysis and parametric control conditions, we showed both the numerical and experimental laboratory examples. The simulations also demonstrated that the introduction of the integral term slows down the convergence rates in the system (the hyperexponential decay is lost), while improper tuning of the integral part may result in transient oscillations. Another observed issue is the sensitivity to the measurement noises that needs to be evaluated. Future works will also include the analysis and

optimization of the transient performance in case of a disturbed system, investigation of regularizations at the origin, and further alternative strategies for the integral control extension.

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□

## APPENDIX

### WELL-DEFINITENESS OF SOLUTIONS

First, consider an auxiliary system

$$\begin{aligned}\dot{x}_1(t) &= |x_1(t)|x_2(t), \\ \dot{x}_2(t) &= -k|x_1(t)|x_1(t) - |x_2(t)|x_2(t) + |x_1(t)|d(t),\end{aligned}\tag{1}$$

where  $x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2$ ,  $k > 0$  and  $d(t) \in \mathbb{R}$  (with  $d \in \mathcal{L}_\infty$ ) have the same meaning as in the system (2). One can see that the solutions of (1) are well defined for all  $x(0) \in \mathbb{R}^2$  and  $d \in \mathcal{L}_\infty$  at least locally in the forward time (the right-hand side of (1) is locally Lipschitz continuous for all values of the arguments, in addition, it is standard homogeneous of positive degree<sup>6</sup>). Moreover, the origin is the unique equilibrium, and  $x(t) = 0$  for all  $t \geq 0$  uniformly in any  $d \in \mathcal{L}_\infty$  (the set  $x_1 = 0$  is invariant for any  $d \in \mathcal{L}_\infty$ ). Now, we want to show that the solutions of (1) are globally defined for all  $t \geq 0$ . Consider a strict Lyapunov function candidate (3),

$$V(x) = kx_1^2 + x_2^2 + \varepsilon x_1 x_2,$$

which is positive definite for  $k > \frac{\varepsilon^2}{4}$  and where  $\varepsilon > 0$  is a tuning gain. Using Young's inequality, the time derivative of  $V$  for (1) can be upper bounded as follows (the details of computations are similar to ones given in Section 4):

$$\dot{V} \leq -\frac{1}{3}|x_2|^3 - \frac{1}{4}\varepsilon k|x_1|^3 + \frac{4}{3k} \left( \frac{\varepsilon}{k} + \frac{1}{\varepsilon} \right) |d|^3$$

provided that  $\frac{1}{8}\sqrt{\frac{3k}{2}} \geq \varepsilon$ . Therefore, according to Theorem 1, the system (1) is ISS, and its solutions are well defined for all  $t \geq 0$ , for any  $x(0) \in \mathbb{R}^2$  and any  $d \in \mathcal{L}_\infty$ .

Now, define  $\tau \in \mathbb{R}_+$  as

$$\tau = \int_0^t |x_1(t)| dt, \quad (2)$$

which is a new (independent) time variable scaled by the state norm and has the corresponding derivative

$$d\tau = |x_1(t)| dt. \quad (3)$$

This change of the time is well defined for all initial conditions  $x_1(0) \neq 0$  or the inputs  $d \in \mathcal{L}_\infty$  that do not force the solutions to reach the line  $x_1 = 0$  in a finite time (if  $\lim_{t \rightarrow +\infty} x_1(t) = 0$  then the variable  $\tau$  is well-defined for all  $t \in \mathbb{R}_+$ ; a trajectory of (1) cannot enter and next leave the set where  $x_1 = 0$  due to its invariance uniformly in  $d$ ). In such a case, in the new time  $\tau$  the system (1) takes the form (2), hence, (2) and (1) have the same solutions for  $x(0) \in M$  and respective  $d \in \mathcal{L}_\infty$ , which are solely differing in time by the corresponding scaling (2). This concludes the existence and uniqueness of solutions for (2) as it is shown above for (1).

It is left to prove that  $d \in \mathcal{L}_\infty$  cannot ensure attractiveness of the line  $x_1 = 0$  in a finite time. To this end, proceed by contradiction: assume that there exists a  $d$  that forces the coordinate  $x_1$  to reach zero in a finite time. Then, it should exist also some auxiliary (virtual) control function  $c(x_1)$  which can drive the dynamic subsystem  $\dot{x}_1 = |x_1|x_2 = -|x_1|c(x_1)$ , cf. (1), to  $x_1 = 0$  in a finite time (the same result can be obtained considering also time dependence of  $c$ , which is omitted here for brevity). Note that the finite time convergence of  $x_1$  is inevitable here for oversteering, otherwise, the system (1) demonstrates the asymptotic convergence rates proved above. Next, consider a dynamic variable

$$r = x_2 + c(x_1)$$

which should converge to zero, also in a finite time, in order to allow for the above virtual control being finite-time stabilizing for  $x_1$ . Evaluating its time derivative one obtains

$$\dot{r} + |x_2|r = |x_1| \left( d - kx_1 + \frac{\partial c(x_1)}{\partial x_1} x_2 + \frac{c(x_1)}{|x_1|} |x_2| \right). \quad (4)$$

Therefore, to guarantee finite-time convergence of  $r$  the disturbance  $d$  has to contain the terms proportional to  $-\frac{|r|^\alpha \text{sign}(r)}{|x_1|}$  for some  $\alpha \in [0, 1)$ . Analyzing the right-hand-side of (4), one can see that in the limit mode  $\dot{r} = r = 0$  and outside of  $x_1 = 0$ , the following should hold

$$d = kx_1 - \frac{\partial c(x_1)}{\partial x_1} x_2 - \frac{c(x_1)}{|x_1|} |x_2|. \quad (5)$$

Since  $x_2 = 0$  for  $x_1 \neq 0$  cannot be a solution providing the required convergence, and due to (5) has singularity at  $x_1 = 0$  that may violate  $d \in \mathcal{L}_\infty$  assumption, one can try to equate the second and third summands in (5) for  $x_2 \neq 0$ , which results in the differential equation

$$c(x_1) + \text{sign}(x_2)|x_1| \frac{\partial c(x_1)}{\partial x_1} = 0$$

that has the solution  $c(x_1) = x_1^{-\text{sign}(x_2)\text{sign}(x_1)}$  being discontinuous at  $x_1 = 0$  (the dependence on  $x_2$  can be hidden in each particular quadrant). Nevertheless, for  $x_1 x_2 > 0$  the solution  $c(x_1) = x_1^{-1}$  provides the finite time stability of the system  $\dot{x}_1 = -|x_1|c(x_1) = -\text{sign}(x_1)$ , however, in such a case  $-\frac{|r|^\alpha \text{sign}(r)}{|x_1|}$  goes to infinity as  $x_1$  approaches zero. Hence we conclude that there is no such a disturbance signal  $d \in \mathcal{L}_\infty$  satisfying (4), which can drive the system (1) to  $x_1 = 0$  in finite time, this way breaking the invariance of  $M$  for (2).