



**HAL**  
open science

# Discretization of prescribed-time observers in the presence of noises and perturbations

Denis Efimov, Yuri V Orlov

► **To cite this version:**

Denis Efimov, Yuri V Orlov. Discretization of prescribed-time observers in the presence of noises and perturbations. *Systems & Control Letters*, In press. hal-04561768

**HAL Id: hal-04561768**

**<https://inria.hal.science/hal-04561768>**

Submitted on 27 Apr 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

# Discretization of prescribed-time observers in the presence of noises and perturbations

Denis Efimov<sup>1</sup>, Yury Orlov<sup>2</sup>

<sup>1</sup>*Inria, Univ. Lille, CNRS, UMR 9189 - CRISTAL, F-59000 Lille, France*

<sup>2</sup>*Mexican Scientific Research and Advanced Studies Center of Ensenada, Carretera Tijuana-Ensenada, B.C. 22860, Mexico*

---

## Abstract

An implicit Euler discretization scheme of the prescribed-time converging observer from Holloway and Krstic [6] is proposed, which preserves all main properties of the continuous-time counterpart, and it is investigated how to apply recursively this discretization on any interval of time. In addition, it is demonstrated that the estimation error is robustly stable with respect to the measurement noise with a linear gain (the property that does not hold in the continuous-time setting). The efficiency of the suggested discrete-time observer is illustrated through numeric experiments.

*Keywords:* Prescribed-time convergence, State observer, Differentiator, Discretization

---

## 1. Introduction

Estimation of the state vector for a dynamical system by utilizing the noisy measurements of a part of its components in real time is a well-known issue, which has many popular and well established solutions Holloway and Krstic [6], Khalil [8], Levant [10], Perruquetti et al. [17], Cruz-Zavala et al. [3], Lopez-Ramirez et al. [12], Orlov [15]. Nevertheless, due to its a widespread application in different domains of science and technology, it is still an active area of research. For instance, estimation of derivatives of a noisy signal can be presented as a state estimation problem Reichhartinger et al. [19], Orlov and Kairuz [16], Kairuz et al. [7].

The main performance characteristics demanded from an observer or a differentiator include (but not limited to): the time of convergence of the estimate to the true value, asymptotic precision in the perturbation-free case (the steady-state error), measurement noise sensitivity and the numeric implementation complexity (see a comparative survey Mojallizadeh et al. [14]). The existence of multiple estimation schemes is related with the fact that it is difficult to design an observer that can outperform others in all valuable characteristics.

A recent popular observer solution is based on the concept of prescribed-time stabilization Holloway and Krstic [6] (see also a recent survey on the topic Song et al. [21]), when an estimator with time-varying gains is designed in a way to guarantee that for any initial conditions, in the noise-free scenario, the estimation error vanishes exactly at the specified time instant, and this zero-settling property of the error is independent in the matched perturbations, which appear in the last equation (i.e., exact estimation of derivatives in a prescribed time). However, the noise sensitivity of estimation error for this state observer is not admissible, and special requirements are frequently imposed on the output perturbations in order to guarantee boundedness of the estimation error in the presence of measurement noise Steeves and Krstic [22], Li and Krstic [11]. Moreover, formally, the definition of solutions in such an observer after the settling time is problematic, and the authors do not consider the error behavior after the convergence (since the observer gains take infinite values at this instant of time). All these drawbacks are unusual and complicate the applicability of this observer.

Following Efimov et al. [5], where implicit discretization of a hyperexponentially (asymptotically) converging differentiator (an observer of second order) was studied, in this note we are going to analyze an implicit discretization scheme for the prescribed-time observer of a perturbed linear system in the canonical form. We show that such a discretization preserves the uniform prescribed-time convergence of the continuous-time counterpart, while being robust with respect to the measurement noise and avoiding the high-gain implementation problem. Moreover, introducing a mild modification the respective state estimates can be made well-posed in discrete time on the infinite horizon (that allows us to benefit from the superior disturbance compensation of the prescribed-time observer all the time). It is illustrated by simulations that the obtained observer has rather advantageous performance even in the case of a slow sampling.

The paper is organized as follows. The brief preliminaries are given in Section 2. The problem statement is introduced in Section 3. The properties of the considered observer in continuous time are recalled from Holloway and Krstic [6] in Section 4. The properties of its implicit Euler discretization are investigated in Section 5. The results of numeric simulation of the presented observer are shown in Section 6.

### Notation

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ , where  $\mathbb{R}$  is the set of real numbers;  $\mathbb{Z}$  is the set of integer numbers,  $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$ .
- $|\cdot|$  denotes the absolute value in  $\mathbb{R}$ ,  $\|\cdot\|$  is used for the Euclidean norm on  $\mathbb{R}^n$ .
- For a (Lebesgue) measurable function  $d : \mathbb{R}_+ \rightarrow \mathbb{R}^m$  define the norm  $\|d\|_\infty = \text{ess sup}_{t \in \mathbb{R}_+} \|d(t)\|$ , and the set of  $d$  with the property  $\|d\|_\infty < +\infty$  we further denote as  $\mathcal{L}_\infty^m$  (the set of essentially bounded measurable functions).
- For a sequence  $d_k \in \mathbb{R}^m$  with  $k \in \mathbb{Z}_+$  define its norm by  $|d|_\infty = \sup_{k \in \mathbb{Z}_+} \|d_k\|$  and the set of  $d$  with  $|d|_\infty < +\infty$  we denote by  $l_\infty^m$ .
- A continuous function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{K}$  if  $\alpha(0) = 0$  and it is strictly increasing. The function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{K}_\infty$  if  $\alpha \in \mathcal{K}$  and it is increasing to infinity.
- A finite series of integers  $1, 2, \dots, n$  is denoted by  $\overline{1, n}$ , and  $\{\overline{1, n}\} = \{1, 2, \dots, n\}$ .
- Denote the identity matrix of dimension  $n \times n$  by  $I_n$  and the matrix of zeros of dimension  $m \times n$  by  $0_{m \times n}$ .
- $\text{diag}\{g\}$  represents a diagonal matrix of dimension  $n \times n$  with a vector  $g \in \mathbb{R}^n$  on the main diagonal.
- The relation  $P \prec 0$  ( $P \succeq 0$ ) means that a symmetric matrix  $P \in \mathbb{R}^{n \times n}$  is negative (positive semi-) definite,  $\lambda_{\min}(P)$  denotes the minimal eigenvalue of such a matrix  $P$ .
- Denote  $\mathbf{e} = \exp(1)$ .

## 2. Preliminaries

The standard stability notions are used throughout and their definitions can be found in Khalil [9].

### 2.1. Uniform prescribed-time stability

Consider a non-autonomous differential equation:

$$\frac{dx(t)}{dt} = f(t, x(t), d(t)), \quad t \geq t_0, \quad t_0 \in \mathbb{R}_+, \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $d(t) \in \mathbb{R}^m$  is the vector of external disturbances and  $d \in \mathcal{L}_\infty^m$ ;  $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is a continuous function with respect to  $x$ ,  $d$  and piecewise continuous with respect to  $t$ ,  $f(t, 0, 0) = 0$  for all  $t \in \mathbb{R}_+$ . A solution of the system (1) for an initial condition  $x_0 \in \mathbb{R}^n$  at time instant  $t_0 \in \mathbb{R}_+$  and some  $d \in \mathcal{L}_\infty^m$  is denoted by  $X(t, t_0, x_0, d)$ , and we assume that  $f$  ensures existence and uniqueness of solutions  $X(t, t_0, x_0, d)$  at least locally in forward time.

The following definition is inspired by Holloway and Krstic [6], and it is specified for a control-free system (1) while also using the same prescribed-time-based terminology that the settling-time instant has been assigned at the designer will.

**Definition 1.** For given  $T > 0$  and a set  $\mathbb{D} \subset \mathcal{L}_\infty^m$ , the system (1) is called uniformly prescribed-time stable (uPTS) if there exist  $\sigma_1, \sigma_2 \in \mathcal{K}$  such that for all  $x_0 \in \mathbb{R}^n \setminus \{0\}$ ,  $t_0 \in \mathbb{R}_+$  and  $d \in \mathbb{D}$ :

$$\|X(t, t_0, x_0, d)\| \leq \max\{\sigma_1(\|x_0\|), \sigma_2(\|d\|_\infty)\}, \\ 0 < \|X(t, t_0, x_0, 0)\|$$

for all  $t \in [t_0, t_0 + T)$ , and

$$\lim_{t \rightarrow t_0 + T} \|X(t, t_0, x_0, d)\| = 0.$$

It is important to highlight that the boundedness of solutions is claimed on a finite interval  $[t_0, t_0 + T)$  only, and the solutions of (1) may be undefined for  $t > T$ . Hence, a uPTS system may do not demonstrate a Lyapunov stable behavior, and it is a variant of short-time stability (frequently also called finite-time one) as in Dorato [4]. In comparison with the concept of fixed-time stability Polyakov [18], an important feature of a prescribed-time stable system is that the settling time is the same for all initial conditions out of the origin in the disturbance-free setting. In this definition the uniformity of convergence is understood in double meaning: as independence in the initial time  $t_0$  and in the input  $d \in \mathbb{D}$ . Despite it is assumed that  $\mathbb{D} \subset \mathcal{L}_\infty^m$ , any other suitable class of inputs can be considered. The initial time  $t_0$  can also be fixed being not arbitrary in  $\mathbb{R}_+$  for many applications (the most common case is  $t_0 = 0$ ).

A simple scalar example of a uPTS system (1) for  $t_0 = 0$  is

$$\dot{x}(t) = -\frac{T}{T-t}x(t) + d(t), \quad t \in [0, T), \quad T > 0, \quad (2)$$

with  $x(t), d(t) \in \mathbb{R}$  (see also Song et al. [20]), whose solutions admit an estimate:

$$|x(t)| \leq \left(\frac{T-t}{T}\right)^T |x(0)| + \iota(t) \|d\|_\infty, \quad t \in [0, T)$$

for any  $x(0) \in \mathbb{R}$  and  $d \in \mathcal{L}_\infty^1$ , where

$$\iota(t) = \begin{cases} \frac{T(\frac{T-t}{T} - (\frac{T-t}{T})^T)}{T-1} & \text{if } T \neq 1, \\ (1-t) \ln \frac{1}{1-t} & \text{if } T = 1. \end{cases}$$

Hence,  $\sigma_1(s) = s$  and  $\sigma_2(s) = \iota_{\max} s$  with

$$\iota_{\max} = \begin{cases} \frac{T^{(1-T)^{-1}} - T^{(1-T)^{-T}}}{1-T^{-1}} & \text{if } T \neq 1, \\ e^{-1} & \text{if } T = 1. \end{cases}$$

## 2.2. Auxiliary property

The following block matrix inversion formulas are used in the sequel Lu and Shiu [13]:

$$\begin{aligned} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} &= \begin{bmatrix} A^{-1} + A^{-1}BS_1CA^{-1} & -A^{-1}BS_1 \\ -S_1CA^{-1} & S_1 \end{bmatrix} \\ &= \begin{bmatrix} S_2 & -S_2BD^{-1} \\ -D^{-1}CS_2 & D^{-1} + D^{-1}CS_2BD^{-1} \end{bmatrix}, \\ S_1 &= (D - CA^{-1}B)^{-1}, \quad S_2 = (A - BD^{-1}C)^{-1}, \end{aligned}$$

where  $A, B, C$  and  $D$  are matrices of appropriate dimensions ( $A, D, S_1$  and  $S_2$  should be nonsingular).

## 3. Problem statement

Assume that a signal  $\phi(t) \in \mathbb{R}$  is measured with a noise  $v(t) \in \mathbb{R}$ :

$$y(t) = \phi(t) + v(t),$$

where  $y(t) \in \mathbb{R}$ ,  $v \in \mathcal{L}_\infty^1$ , and  $\phi: \mathbb{R}_+ \rightarrow \mathbb{R}$  has  $n > 1$  derivatives  $\phi^{(i)}(t) = \frac{d^i \phi(t)}{dt^i}$  for  $i = \overline{1, n}$  with  $\phi^{(n)} \in \mathcal{L}_\infty^1$  (without a known constant upper bound).

It is required to estimate the derivatives  $\phi^{(i)}(t)$ ,  $i = \overline{1, n-1}$  of the signal  $\phi$  with a prescribed time of convergence  $T > 0$  and robustly with respect to the perturbation  $v$ .

The problem can be equivalently stated as the state estimation for the system

$$\dot{x}(t) = Ax(t) + b\phi^{(n)}(t), \quad y(t) = Cx(t) + v(t), \quad (3)$$

where  $x(t) \in \mathbb{R}^n$  is the state,  $x(0) = [\phi(0) \dots \phi^{(n-1)}(0)]^\top$ ,

$$A = \begin{bmatrix} 0_{n-1 \times 1} & I_{n-1} \\ 0 & 0_{1 \times n-1} \end{bmatrix}, \quad b = \begin{bmatrix} 0_{n-1 \times 1} \\ 1 \end{bmatrix}, \quad C = [1 \quad 0_{1 \times n-1}]$$

are in a canonical form, and  $\phi^{(n)} \in \mathcal{L}_\infty^1$  corresponds to an unknown external input. And further in this work an observer for this system will be considered, which estimates  $x(t)$  having uPTS estimation error for  $\phi^{(n)} \in \mathbb{D} = \mathcal{L}_\infty^1$  while  $\|v\|_\infty = 0$ , and possessing a bounded estimation error for  $v \in \mathcal{L}_\infty^1$  after a proper discretization.

#### 4. Estimation in continuous time

Following Holloway and Krstic [6], take the state observer for (3) in the form:

$$\begin{aligned}\dot{\hat{x}}(t) &= A\hat{x}(t) + D(t)L(y(t) - C\hat{x}(t)), \\ D(t) &= \text{diag}\{\varrho(t) \varrho^2(t) \dots \varrho^n(t)\}^\top,\end{aligned}\tag{4}$$

where  $\hat{x}(t) \in \mathbb{R}^n$  is the estimate of the state  $x(t)$ ,  $L = [L_1 \dots L_n]^\top \in \mathbb{R}^n$  is the observer gain that will be selected later,  $\varrho(t) = \frac{T}{T-t}$  is a strictly growing and unbounded function of time for  $t \in [0, T)$ .

*Remark 1.* Introducing mild modifications in the forthcoming analysis, any integrable strictly growing and unbounded (for  $t \rightarrow T$ ) function of time  $\varrho(t)$  can be used in (4).

*Remark 2.* As it is common for the prescribed-time converging systems Song et al. [21], in the presence of escaping to infinity at  $t = T$  gain  $\varrho(t)$ , the right-hand side of (4) is defined on a finite interval of time  $[0, T)$ . Due to this, for  $t \geq T$ , frequently, another stabilization or estimation algorithm is applied, since  $D(t)$  is not yet defined. Such a commutation is natural due to well-posedness of the system at  $t = T$ , and next for small regulation or observation errors other solutions can be used. However, such a switching does not take into account the advantageous uniformity of convergence realizable by prescribed-time controllers or estimators, which may be difficult to ensure through other methods. In this work, we will later consider another approach in the discrete-time setting by replacing  $t \in \mathbb{R}_+$  with  $\text{mod}(t, T) \in [0, T)$ . In such a case  $\varrho(\text{mod}(t, T))$  periodically ranges from 1 till  $+\infty$  while  $t$  passes from  $iT$  to  $(i+1)T$ , correspondingly, for any  $i \in \mathbb{Z}_+$ .

Define the estimation error as  $e(t) = x(t) - \hat{x}(t)$ , whose dynamics can be written as follows:

$$\dot{e}(t) = (A - D(t)LC)e(t) + b\phi^{(n)}(t) - D(t)Lv(t).\tag{5}$$

Define new auxiliary variable  $\epsilon(t) = \Gamma(t)e(t)$ , where

$$\begin{aligned}\Gamma(t) &= \varrho(t)D^{-1}(t) \\ &= \text{diag}\{[1 \quad \varrho^{-1}(t) \dots \varrho^{1-n}(t)]^\top\},\end{aligned}$$

note that  $\epsilon(0) = e(0)$ , whose dynamics takes the form:

$$\begin{aligned}\dot{\epsilon}(t) &= -\varrho(t)\Delta\Gamma(t)\epsilon(t) + \Gamma(t)\dot{e}(t) \\ &= \varrho(t) \left( (A_L - \Delta)\epsilon(t) + \varrho^{-n}(t)b\phi^{(n)}(t) - Lv(t) \right), \\ A_L &= A - LC, \quad \Delta = T^{-1}\text{diag}\{[0 \ 1 \dots n-1]^\top\},\end{aligned}\tag{6}$$

where we used the identities:

$$\begin{aligned}\Gamma(t)(A - D(t)LC)\Gamma^{-1}(t) &= \varrho(t)(A - LC), \\ \Gamma(t)b &= \varrho^{1-n}(t)b.\end{aligned}\tag{7}$$

Next, let us define a new time variable Holloway and Krstic [6], Chitour et al. [2]

$$\begin{aligned}\tau = T \ln \frac{T}{T-t} &\Rightarrow t = T(1 - e^{-T^{-1}\tau}), \\ d\tau &= \varrho(t)dt,\end{aligned}$$

or

$$\tau = \varphi(t) = T \ln \varrho(t), \quad t = \varphi^{-1}(\tau) = T(1 - e^{-T^{-1}\tau}),$$

note that  $\tau \in \mathbb{R}_+$ , then after the change of the time  $\varrho(t) = \varrho(\varphi^{-1}(\tau)) = e^{T^{-1}\tau}$ , and the auxiliary error  $\epsilon(t)$  dynamics is reduced to an equivalent useful representation:

$$\frac{d\epsilon(\tau)}{d\tau} = (A_L - \Delta)\epsilon(\tau) + e^{-nT^{-1}\tau}b\phi^{(n)}(\tau) - Lv(\tau),\tag{8}$$

where  $\phi^{(n)}(\tau) = \phi^{(n)}(\varphi^{-1}(\tau))$  and  $v(\tau) = v(\varphi^{-1}(\tau))$  are the external signals in the new time. For analysis of stability properties in (8) let us choose a candidate Lyapunov function  $V(\epsilon) = \epsilon^\top P \epsilon$  with some positive definite matrix  $P = P^\top \succ 0$  specified later, whose derivative in the new time  $\tau$  for the dynamics (8) can be written as follows:

$$\begin{aligned} \frac{dV(\epsilon(\tau))}{d\tau} &= \epsilon(\tau)^\top \left( (A_L - \Delta)^\top P + P(A_L - \Delta) \right) \epsilon(\tau) \\ &\quad + 2\epsilon(\tau)^\top P \left( e^{-nT^{-1}\tau} b \phi^{(n)}(\tau) - Lv(\tau) \right) \\ &= \begin{bmatrix} \epsilon(\tau) \\ \frac{\phi^{(n)}(\tau)}{e^{nT^{-1}\tau}} \\ v(\tau) \end{bmatrix}^\top Q \begin{bmatrix} \epsilon(\tau) \\ \frac{\phi^{(n)}(\tau)}{e^{nT^{-1}\tau}} \\ v(\tau) \end{bmatrix} - V(\epsilon(\tau)) \\ &\quad + \gamma_1 e^{-2nT^{-1}\tau} \left( \phi^{(n)}(\tau) \right)^2 + \gamma_2 v^2(\tau), \\ Q &= \begin{bmatrix} (A_L - \Delta)^\top P + P(A_L - \Delta) + P & Pb & -PL \\ b^\top P & -\gamma_1 & 0 \\ -L^\top P & 0 & -\gamma_2 \end{bmatrix} \end{aligned} \quad (9)$$

for any  $\gamma_1 > 0$  and  $\gamma_2 > 0$ . Due to observability of the pair  $(A, C)$ , it is easy to see that there exist  $P, L, \gamma_1$  and  $\gamma_2$  such that  $Q \preceq 0$  and

$$\begin{aligned} \frac{dV(\epsilon(\tau))}{d\tau} &\leq -V(\epsilon(\tau)) + \gamma_1 e^{-2nT^{-1}\tau} \left( \phi^{(n)}(\tau) \right)^2 + \gamma_2 v^2(\tau) \\ &\leq -V(\epsilon(\tau)) + \gamma_1 \left( \frac{\|\phi^{(n)}\|_\infty}{e^{nT^{-1}\tau}} \right)^2 + \gamma_2 \|v\|_\infty^2. \end{aligned}$$

Passing to the  $\tau$ -time domain we get an estimate:

$$\begin{aligned} V(\epsilon(\tau)) &\leq e^{-\tau} V(\epsilon(0)) + \gamma_1 e^{-\tau} \int_0^\tau e^s \left( \frac{\|\phi^{(n)}\|_\infty}{e^{nT^{-1}s}} \right)^2 ds \\ &\quad + \gamma_2 e^{-\tau} \int_0^\tau e^s \|v\|_\infty^2 ds \end{aligned}$$

for all  $\tau \geq 0$ . Straightforward computations show that

$$\begin{aligned} \gamma_1 e^{-\tau} \|\phi^{(n)}\|_\infty^2 \int_0^\tau e^{(1-2nT^{-1})s} ds &\leq \gamma_1 \|\phi^{(n)}\|_\infty^2 \tilde{\iota}(\tau), \\ \tilde{\iota}(\tau) &= \begin{cases} \frac{e^{-2nT^{-1}\tau} - e^{-\tau}}{1-2nT^{-1}} & \text{if } T \neq 2n, \\ \tau e^{-\tau} & \text{if } T = 2n \end{cases} \end{aligned}$$

and obviously  $\gamma_2 e^{-\tau} \int_0^\tau e^s \|v\|_\infty^2 ds \leq \gamma_2 \|v\|_\infty^2$  for all  $\tau \geq 0$ , then

$$V(\epsilon(\tau)) \leq e^{-\tau} V(\epsilon(0)) + \gamma_1 \tilde{\iota}(\tau) \|\phi^{(n)}\|_\infty^2 + \gamma_2 \|v\|_\infty^2, \quad \tau \geq 0,$$

and in the original time

$$V(\epsilon(t)) \leq \varrho^{-T}(t) V(\epsilon(0)) + \gamma_1 \tilde{\iota}(t) \|\phi^{(n)}\|_\infty^2 + \gamma_2 \|v\|_\infty^2$$

for all  $t \in [0, T)$ , where

$$\tilde{\iota}(t) = T \begin{cases} \frac{\varrho^{-2n}(t) - \varrho^{-T}(t)}{T-2n} & \text{if } T \neq 2n, \\ \varrho^{-2n}(t) \ln \varrho(t) & \text{if } T = 2n. \end{cases}$$

Since  $e(t) = \Gamma^{-1}(t)\epsilon(t)$ , the desired estimate on the behavior of  $e(t)$  can be derived for all  $t \geq 0$  from this inequality, and we can repeat the result of Holloway and Krstic [6] for the considered uPTS scenario:

**Theorem 1.** Let there exist  $P = P^\top \in \mathbb{R}^{n \times n}$ ,  $U \in \mathbb{R}^n$ ,  $\gamma_1 > 0$  and  $\gamma_2 > 0$  such that the linear matrix inequalities are verified:

$$P \succ 0, \begin{bmatrix} Q_{11} & Pb & -U \\ b^\top P & -\gamma_1 & 0 \\ -U^\top & 0 & -\gamma_2 \end{bmatrix} \preceq 0,$$

$$Q_{11} = (A - \Delta)^\top P + P(A - \Delta) - UC - C^\top U^\top + P.$$

Then for  $L = P^{-1}U$  and any  $e(0) \in \mathbb{R}^n$ ,  $\phi^{(n)} \in \mathcal{L}_\infty^1, v \in \mathcal{L}_\infty^1$  in (3), (4):

$$\sqrt{\lambda_{\min}(P)} \begin{bmatrix} |e_1(t)| \\ |e_2(t)| \\ \vdots \\ |e_n(t)| \end{bmatrix} \leq \begin{bmatrix} 1 \\ \varrho(t) \\ \vdots \\ \varrho^{n-1}(t) \end{bmatrix}$$

$$\times (\varrho^{-T/2}(t) \sqrt{e(0)^\top P e(0)} + \sqrt{\gamma_1 \bar{i}(t)} \|\phi^{(n)}\|_\infty + \sqrt{\gamma_2} \|v\|_\infty)$$

for all  $t \in [0, T)$ .

*Remark 3.* For  $\|v\|_\infty = 0$  the result of the theorem implies uPTS of the estimation error  $e(t)$  in (3), (4) with  $\phi^{(n)} \in \mathcal{L}_\infty^1$  provided that  $T > 2n - 2$  (note that any arbitrary time of convergence can be assigned by substituting  $\varkappa t \rightarrow t$  in  $\varrho$  for any  $\varkappa > 1$ ). This property means that (4) is a prescribed-time exact differentiator (as the super-twisting algorithm Levant [10] in finite time), and the class of derivatives  $\phi^{(n)} \in \mathcal{L}_\infty^1$ , for which the uniformity of the estimates is kept, can be enlarged by ones satisfying the constraint

$$\lim_{t \rightarrow T} \phi^{(n)}(t) \varrho^{n-1}(t) \sqrt{\bar{i}(t)} = 0,$$

where

$$\varrho^{n-1}(t) \sqrt{\bar{i}(t)} = \varrho^{-1}(t) \sqrt{T} \begin{cases} \sqrt{\frac{1 - \varrho^{2n-T}(t)}{T - 2n}} & \text{if } T \neq 2n, \\ \sqrt{\ln \varrho(t)} & \text{if } T = 2n, \end{cases}$$

hence, it is bounded and decaying to zero for  $t \rightarrow T$  provided that  $T > 2n - 2$ .

Clearly, the given linear matrix inequalities are always feasible (they are similar to ones used in a Luenberger observer design for (3)). Moreover, due to the form of  $\Delta$ , the elements  $L_i$  of  $L$  for  $i \geq 2$  can be selected zero, which is unusual for an observer design.

The upper bound on the estimation error of (3), (4) calculated in Theorem 1 implies that the gain of  $|e_i(t)|$  in  $v$  is proportional to  $\varrho^{i-1}(t)$  for  $i \in \overline{1, n}$ , hence, it becomes infinite at  $t = T$  for  $i > 1$ . It is a serious drawback for a non-vanishing noise  $v$  often met in practical implementation. Frequently, to avoid this issue, the precision is sacrificed by stopping the growth of observer/controller gains slightly before the prescribed time instant  $T$  (it actually implies that the system loses the prescribed-time convergence quality).

Let us investigate what happens after discretization of (4).

## 5. Estimation in discrete time

Note that (4) is modeled by a linear time-varying system with an external known input  $y(t)$ . Since the time-varying gain  $D(t)$  is strictly growing to infinity, the explicit Euler discretization cannot be used for all  $t \in [0, T]$ , however, the implicit one can be effectively applied Butcher [1].

Let  $h > 0$  be constant discretization step, denote by  $t_k = hk$  for  $k \in \mathbb{Z}_+$  the discretization time instants, and slightly loosing generality assume that there exists  $N \in \mathbb{Z}_+$  such that  $T = Nh$ , then application of the implicit Euler discretization method to (4) gives for  $k \in \{0, N - 1\}$ :

$$\hat{x}_{k+1} = Z(t_{k+1}) (\hat{x}_k + hD(t_{k+1})Ly_{k+1}), \quad (10)$$

where  $\hat{x}_k$  is an approximation of  $\hat{x}(t_k)$ ,  $y_k = y(t_k)$ ,

$$Z(t) = (I_n - h(A - D(t)LC))^{-1}.$$

Note that

$$I_n - h(A - D(t)LC) = \begin{bmatrix} 1 + L_1 h \varrho(t) & \mathcal{B} \\ \mathcal{C}(t) & \mathcal{D}^{-1} \end{bmatrix}, \quad (11)$$

where

$$\begin{aligned} \mathcal{B} &= [-h \ 0 \ \dots \ 0], \\ \mathcal{D} &= \left( I_{n-1} - h \begin{bmatrix} 0_{n-2 \times 1} & I_{n-2} \\ 0 & 0_{1 \times n-2} \end{bmatrix} \right)^{-1} = \\ &= \begin{bmatrix} 1 & h & \dots & h^{n-2} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & h \\ 0 & \dots & 0 & 1 \end{bmatrix}, \quad \mathcal{C}(t) = h \begin{bmatrix} L_2 \varrho^2(t) \\ L_3 \varrho^3(t) \\ \vdots \\ L_n \varrho^n(t) \end{bmatrix}, \end{aligned}$$

then using the second block inversion formula given above we derive

$$\begin{aligned} Z(t) &= \frac{K(t)}{O(t)}, \quad K(t) = \begin{bmatrix} 1 & -\mathcal{B}\mathcal{D} \\ -\mathcal{D}\mathcal{C}(t) & \mathcal{D}O(t) + \mathcal{D}\mathcal{C}(t)\mathcal{B}\mathcal{D} \end{bmatrix}, \\ O(t) &= 1 + \sum_{i=1}^n L_i h^i \varrho^i(t). \end{aligned} \quad (12)$$

It is straightforward to check that  $O(t) \geq 1 + \sum_{i=1}^n L_i h^i > 0$  for all  $t \geq 0$  provided that the matrix  $A - LC$  is Hurwitz (in such a case  $L_i > 0$  for all  $i = \overline{1, n}$ , and note that this is an additional constraint to the conditions of Theorem 1, where it is enough to have Hurwitz property for the matrix  $A - LC - \Delta$ ). Since

$$\begin{aligned} \mathcal{B}\mathcal{D} &= -[h \dots h^{n-1}], \\ \mathcal{D}\mathcal{C}(t) &= [(\mathcal{D}\mathcal{C}(t))_1 \dots (\mathcal{D}\mathcal{C}(t))_{n-1}]^\top, \\ (\mathcal{D}\mathcal{C}(t))_j &= \sum_{i=j+1}^n L_i h^{i-j} \varrho^i(t), \quad j = 1, \dots, n-1, \end{aligned}$$

as we can conclude from these expressions, the discrete state transition matrix  $Z(t)$  is nonsingular and elementwise bounded for all  $t \in [0, T]$ : indeed, the elements of  $K(t)$  are either constants or polynomial functions of  $\varrho(t)$  of orders  $\{\overline{1, n}\}$ , the denominator  $O(t)$  is also a polynomial of  $\varrho(t)$  of order  $n$  (we assume that  $L_n > 0$  due to necessary Hurwitz property of  $A - LC$ ) separated with zero; the function  $\varrho(t)$  is monotonously growing from 1 to  $+\infty$  while  $t$  passes from 0 to  $T$ , then boundedness of  $Z(t)$  is obvious for  $t \in [0, T)$  and, in addition, the direct computations show

$$\lim_{t \rightarrow T} Z(t) := Z(T) = - \begin{bmatrix} 0 & 0 & \dots & 0 \\ h^{-1} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ h^{1-n} & \dots & h^{-1} & 0 \end{bmatrix},$$

which is a nilpotent matrix of order  $n$  (all eigenvalues of  $Z(T)$  are zeros,  $Z^n(T) = 0_{n \times n}$ , and a linear discrete-time system with such a state transition matrix,  $x_{k+1} = Z(T)x_k$ , demonstrates a fixed-time convergence to the origin in at least  $n$  steps).

Moreover, the input  $y_{k+1}$  gain matrix, which evaluates the noise sensitivity of (10), is also bounded for all  $t \in [0, T]$ :

$$\begin{aligned} F(t) &= Z(t)D(t)L = \frac{1}{O(t)} [\tilde{F}_1(t) \dots \tilde{F}_n(t)]^\top, \\ \tilde{F}_j(t) &= \sum_{i=j}^n L_i h^{i-j} \varrho^i(t), \quad j = 1, \dots, n, \end{aligned} \quad (13)$$

where the derivations are done using the given expression for  $K(t)$  and following the observations that  $F_1(t) = L_1 \varrho(t) - \mathcal{B}\mathcal{D}\Lambda(t)$ , where

$$\Lambda(t) = [(D(t)L)_2 \dots (D(t)L)_n]^\top = h^{-1}\mathcal{C}(t),$$

then recognizing that  $O(t) = L_1 h \varrho(t) + 1 - \mathcal{B}\mathcal{D}\mathcal{C}(t)$  we get:

$$\begin{aligned} \begin{bmatrix} \tilde{F}_2(t) \\ \vdots \\ \tilde{F}_n(t) \end{bmatrix} &= -\mathcal{D}\mathcal{C}(t)L_1 \varrho(t) + (\mathcal{D}O(t) + \mathcal{D}\mathcal{C}(t)\mathcal{B}\mathcal{D})\Lambda(t) \\ &= \mathcal{D}((O(t) - L_1 h \varrho(t))I_{n-1} + \mathcal{C}(t)\mathcal{B}\mathcal{D})h^{-1}\mathcal{C}(t) \\ &= \mathcal{D}((1 - \mathcal{B}\mathcal{D}\mathcal{C}(t))I_{n-1} + \mathcal{C}(t)\mathcal{B}\mathcal{D})h^{-1}\mathcal{C}(t) = h^{-1}\mathcal{D}\mathcal{C}(t). \end{aligned}$$

Consequently, all elements of  $F(t)$  are polynomial functions of  $\varrho(t)$  of the order  $n$  in numerator and denominator, while

$$\begin{aligned} &h \lim_{t \rightarrow T} F(t) \\ &= \lim_{t \rightarrow T} \frac{1}{\sum_{i=1}^n L_i h^i \varrho^i(t) + 1} \begin{bmatrix} \sum_{i=1}^n L_i h^i \varrho^i(t) \\ \vdots \\ \sum_{i=n}^n L_i h^{i-n+1} \varrho^i(t) \end{bmatrix} \\ &= [1 \ h^{-1} \dots h^{-n+1}]^\top =: hF(T), \end{aligned}$$

which evaluates the noise sensitivity at  $t = T$ . Therefore, as before for  $K(t)$ ,  $F(t)$  is bounded for all  $t \in [0, T]$ .

Consider a discrete-time model representing the solutions of (3), which can be approximated using the same implicit Euler method:

$$\begin{aligned} x_{k+1} &= Z(t_{k+1}) \left( x_k + hb\phi_{k+1}^{(n)} + hD(t_{k+1})L(y_{k+1} - v_{k+1}) \right), \\ y_k &= Cx_k + v_k, \end{aligned} \quad (14)$$

for  $k \in \mathbb{Z}_+$ , where  $x_k$  should approach to  $x(t_k)$  as  $h$  converges to zero,  $\phi_k^{(n)} = \phi^{(n)}(t_k)$  and  $v_k = v(t_k)$  (formally this perturbation  $v_k$  is different from the one used in (3) since it should also include the discretization error, but with a light ambiguity in notation we will continue to use the same symbol). In this work we will assume that  $v_k$  and  $\phi_k^{(n)}$  take finite values with bounded norms as before, i.e.,  $v \in l_\infty^1$  and  $\phi^{(n)} \in l_\infty^1$ .

To analyze the properties of (10) we will consider the discretization error  $e_k = x_k - \hat{x}_k$ , whose stability and convergence rate have been evaluated in Theorem 1 for the continuous-time scenario. The direct computations show that

$$e_{k+1} = Z(t_{k+1}) \left( e_k + hb\phi_{k+1}^{(n)} - hD(t_{k+1})Lv_{k+1} \right). \quad (15)$$

To investigate stability and the rate of convergence in (15), we will consider first a finite interval of time with  $k \in \{0, \overline{N-2}\}$ , and next possible extensions for  $k \in \mathbb{Z}_+$ .

### 5.1. Analysis for $k \in \{0, \overline{N-2}\}$

Let us define a time-varying Lyapunov function candidate (the same was used before):

$$V_k = e_k^\top \Pi_k e_k, \quad \Pi_k = \Gamma(t_{k+1})P\Gamma(t_{k+1}), \quad \forall k \in \{0, \overline{N-2}\}, \quad (16)$$

where  $P = P^\top \succ 0$  is as in Theorem 1 (more precise requirements will be defined below). Note that for  $k = N-1$ , by definition  $t_{k+1} = T$ , hence,  $\Gamma(T) = \text{diag}\{[1 \ 0 \dots 0]^\top\}$  and  $\Gamma(T)P\Gamma(T)$  is a singular matrix, then such a choice of  $\Pi_k$  is admissible for  $k \in \{0, \overline{N-2}\}$  only (it also explains our division of the stability analysis on two cases: for  $k \leq N-2$  and  $k = N-1$ ).

For time-varying parameters  $\alpha_k > 0$ ,  $\gamma_k > 0$  and  $\sigma_k > 0$  determined later for  $k \in \{0, \overline{N-2}\}$  we obtain

$$V_{k+1} - \alpha_k V_k = \begin{bmatrix} e_k \\ \phi_{k+1}^{(n)} \\ v_{k+1} \end{bmatrix}^\top Q_k \begin{bmatrix} e_k \\ \phi_{k+1}^{(n)} \\ v_{k+1} \end{bmatrix} + \gamma_k \left( \phi_{k+1}^{(n)} \right)^2 + \sigma_k v_{k+1}^2$$

for

$$Q_k = \begin{bmatrix} I_n \\ hb^\top \\ -hL^\top D(t_{k+1}) \end{bmatrix} Z^\top(t_{k+1}) \Pi_{k+1} Z(t_{k+1}) \\ \times \begin{bmatrix} I_n \\ hb^\top \\ -hL^\top D(t_{k+1}) \end{bmatrix}^\top - \begin{bmatrix} \alpha_k \Pi_k & 0 & 0 \\ 0 & \gamma_k & 0 \\ 0 & 0 & \sigma_k \end{bmatrix}.$$

We need to find the restrictions on  $P$  and the gains  $\alpha_k, \gamma_k, \sigma_k$  such that  $Q_k \succeq 0$ . To this end, using the Schur complement of  $Q_k$  in  $Z^\top(t_{k+1}) \Pi_{k+1} Z(t_{k+1})$  Zhang [23] and multiplying the obtained matrix from right and left sides on  $\text{diag}\{Z^\top(t_{k+1}), Z^{-1}(t_{k+1})\Gamma^{-1}(t_{k+1}), 1, 1\}$  and its transpose, respectively, the latter property is equivalent to

$$\tilde{Q}_k \succeq 0, \tilde{Q}_k = \begin{bmatrix} \tilde{Q}_{11}^k & \tilde{Q}_{12}^k \\ (\tilde{Q}_{12}^k)^\top & \tilde{Q}_{22}^k \end{bmatrix}, \\ \tilde{Q}_{11}^k = \begin{bmatrix} \Pi_{k+1}^{-1} & \Gamma^{-1}(t_{k+1}) \\ \Gamma^{-1}(t_{k+1}) & \alpha_k R(t_{k+1}) \end{bmatrix}, \\ \tilde{Q}_{12}^k = h \begin{bmatrix} Z(t_{k+1})b & -F(t_{k+1}) \\ 0 & 0 \end{bmatrix}, \tilde{Q}_{22}^k = \begin{bmatrix} \gamma_k & 0 \\ 0 & \sigma_k \end{bmatrix},$$

where

$$R(t_{k+1}) = \Gamma^{-1}(t_{k+1}) Z^{-\top}(t_{k+1}) \Pi_k Z^{-1}(t_{k+1}) \Gamma^{-1}(t_{k+1}).$$

Noting that

$$R(t) = \Gamma^{-1}(t) (I_n - h(A - D(t)LC))^\top \Gamma(t) P \\ \times \Gamma(t) (I_n - h(A - D(t)LC)) \Gamma^{-1}(t)$$

and recalling (7) we obtain

$$R(t) = P - h\varrho(t) ((A - LC)^\top P + P(A - LC)) \\ + h^2 \varrho^2(t) (A - LC)^\top P (A - LC),$$

which implies that this matrix is positive definite for any  $t \geq 0$  under the restriction

$$(A - LC)^\top P + P(A - LC) \prec 0, \quad (17)$$

i.e., the matrix  $A - LC$  is Hurwitz (as we required previously) and the matrix  $P$  corresponds to the solution of the respective Lyapunov equation.

To formulate the conditions implying  $\tilde{Q}_k \succeq 0$ , first, let us investigate the restrictions for  $\tilde{Q}_{11}^k \succ 0$ . Note that

$$P \succeq \Gamma^{-1}(t_{k+1}) \Gamma(t_{k+2}) P \Gamma(t_{k+2}) \Gamma^{-1}(t_{k+1}) \quad (18)$$

for any  $k \in \overline{\{0, N-2\}}$  provided that  $HP + PH \succeq 0$ , where

$$H = \text{diag}\{[0 \ 1 \ 2 \ \cdots \ n-1]^\top\}.$$

Indeed, let  $w \in \mathbb{R}^n$  be an arbitrary nonzero vector, denote  $q(t) = w^\top \Gamma(t) P \Gamma(t) w$ , then

$$\frac{dq(t)}{dt} = w^\top \frac{d\Gamma(t)}{dt} P \Gamma(t) w + w^\top \Gamma(t) P \frac{d\Gamma(t)}{dt} w \\ = -\varrho^{-1}(t) (w^\top \Gamma(t) H P \Gamma(t) w + w^\top \Gamma(t) P H \Gamma(t) w) \\ = -\varrho^{-1}(t) w^\top \Gamma(t) (HP + PH) \Gamma(t) w \leq 0.$$

Hence,  $t_{k+1} < t_{k+2}$  implies

$$\Gamma(t_{k+1}) P \Gamma(t_{k+1}) \succeq \Gamma(t_{k+2}) P \Gamma(t_{k+2}),$$

leading to the desired property (18). Calculating the Schur complement of  $\tilde{Q}_{11}^k$  in  $\Pi_{k+1}^{-1}$  we get an inequality:

$$S_k \succ 0, \\ S_k = \alpha_k R(t_{k+1}) - \Gamma^{-1}(t_{k+1})\Gamma(t_{k+2})P\Gamma(t_{k+2})\Gamma^{-1}(t_{k+1}),$$

and since the inequality (18) is satisfied for  $k \in \overline{\{0, N-2\}}$ , the property  $\tilde{Q}_{11}^k \succ 0$  follows (17) for  $\alpha_k = a$  or  $\alpha_k = \frac{a}{h\varrho(t_{k+1})}$  for some  $a \geq 1$  (in the latter case there exists  $a \geq 1$  such that  $a[(A-LC)^\top P + P(A-LC)] \succ P$ ). If  $\alpha_k = \frac{a}{h^2\varrho^2(t_{k+1})}$  with  $a > 0$ , then an auxiliary linear matrix inequality should be verified:

$$a(A-LC)^\top P(A-LC) - P \succ 0,$$

and, again under (17) (the matrix  $A-LC$  is nonsingular), always there is  $a > 0$  such that it is true. Returning back to verification of  $\tilde{Q}_k \succeq 0$ , and having  $\tilde{Q}_{11}^k \succ 0$  for such a choice of  $\alpha_k$ , we can also use the Schur complement of  $\tilde{Q}_k$  in  $\tilde{Q}_{11}^k$  to check the desired property:

$$\tilde{Q}_{22}^k - (\tilde{Q}_{12}^k)^\top (\tilde{Q}_{11}^k)^{-1} \tilde{Q}_{12}^k \succeq 0.$$

To simplify this expression, denote  $T_k = (\tilde{Q}_{11}^k)^{-1}$ , which can be calculated using the first block inversion formula given in the preliminaries with the first block element

$$T_k^{11} = \Pi_{k+1} + \Pi_{k+1}\Gamma^{-1}(t_{k+1})S_k^{-1}\Gamma^{-1}(t_{k+1})\Pi_{k+1},$$

and it has been shown above that  $S_k$  is nonsingular and bounded for selected  $\alpha_k = \frac{a}{h^2\varrho^2(t_{k+1})}$  with  $a > 0$  for any  $k \in \overline{\{0, N-2\}}$ . This notation leads to

$$\tilde{Q}_{22}^k - (\tilde{Q}_{12}^k)^\top (\tilde{Q}_{11}^k)^{-1} \tilde{Q}_{12}^k = \begin{bmatrix} \gamma_k & 0 \\ 0 & \sigma_k \end{bmatrix} - h^2 \times \\ \begin{bmatrix} b^\top Z^\top(t_{k+1})T_k^{11}Z(t_{k+1})b & -b^\top Z^\top(t_{k+1})T_k^{11}F(t_{k+1}) \\ -F^\top(t_{k+1})T_k^{11}Z(t_{k+1})b & F^\top(t_{k+1})T_k^{11}F(t_{k+1}) \end{bmatrix}.$$

Using the block form of  $Z(t)$ , it is straightforward to check that

$$Z(t)b = \frac{1}{O(t)} \begin{bmatrix} \vdots \\ h^{n-j} \left( 1 + \sum_{i=1}^{j-1} L_i h^i \varrho^i(t) \right) \\ \vdots \end{bmatrix}, \\ j = 1, \dots, n$$

is of order  $\varrho^{-1}(t)$  (the highest order of numerator of each element is  $\varrho^{j-1}(t)$  for each  $j = 1, \dots, n$ , while denominator is of order  $\varrho^n(t)$ ), while it has been discussed that  $F(t)$  is globally bounded for all  $t \in [0, T]$ . Note that  $T_k^{11}$  is of the order  $\Gamma^2(t_{k+2})$  ( $S_k$  can be considered to be bounded, as well the product  $\Gamma(t_{k+2})\Gamma^{-1}(t_{k+1})$ ), hence bounded for  $k \in \overline{\{0, N-2\}}$ . Therefore, there exists  $g > 0$  and  $s > 0$  such that for  $\gamma_k = \frac{h^2 g}{\varrho^2(t_{k+1})}$  and  $\sigma_k = h^2 s$  the property  $\tilde{Q}_k \succeq 0$  is verified for all  $k \in \overline{\{0, N-2\}}$ , which leads to the estimate:

$$V_{k+1} \leq \frac{a}{h^2\varrho^2(t_{k+1})} V_k + \frac{gh^2|\phi^{(n)}|_\infty^2}{\varrho^2(t_{k+1})} + sh^2|v|_\infty^2 \quad (19)$$

for all  $k \in \overline{\{0, N-2\}}$ . Therefore, the following result can be formulated:

**Theorem 2.** *Let there exist  $P = P^\top \in \mathbb{R}^{n \times n}$ ,  $U \in \mathbb{R}^n$  such that the linear matrix inequalities are verified:*

$$P \succ 0, \quad A^\top P + PA - C^\top U^\top - UC \prec 0, \\ HP + PH \succeq 0.$$

Then for  $L = UP^{-1}$  there exist  $a > 0$ ,  $g > 0$  and  $s > 0$  such that for any  $e_0 \in \mathbb{R}^n$  in (15):

$$\begin{aligned} \lambda_{\min}(P)\|e_k\|^2 &\leq \varrho^{2(n-1)}(t_{k+1}) \left[ \prod_{i=0}^{k-1} \frac{a}{h^2 \varrho^2(t_{i+1})} e_0^\top P e_0 \right. \\ &\quad + gh^2 |\phi^{(n)}|_\infty^2 \sum_{i=0}^{k-1} \frac{1}{\varrho^2(t_{i+1})} \prod_{\ell=0}^{k-i-2} \frac{a}{h^2 \varrho^2(t_{\ell+1})} \\ &\quad \left. + sh^2 |v|_\infty^2 \sum_{i=0}^{k-1} \prod_{\ell=0}^{k-i-2} \frac{a}{h^2 \varrho^2(t_{\ell+1})} \right] \end{aligned}$$

for all  $k \in \overline{\{0, N-2\}}$ .

*Proof.* Under the introduced restrictions, the property (17) is verified for given  $P$  and  $L$ , and

$$\begin{aligned} V_k &\leq \prod_{i=0}^{k-1} \frac{a}{h^2 \varrho^2(t_{i+1})} V_0 \\ &\quad + gh^2 |\phi^{(n)}|_\infty^2 \sum_{i=0}^{k-1} \frac{1}{\varrho^2(t_{i+1})} \prod_{\ell=0}^{k-i-2} \frac{a}{h^2 \varrho^2(t_{\ell+1})} \\ &\quad + sh^2 |v|_\infty^2 \sum_{i=0}^{k-1} \prod_{\ell=0}^{k-i-2} \frac{a}{h^2 \varrho^2(t_{\ell+1})} \end{aligned}$$

for all  $k \in \overline{\{0, N-2\}}$ , which gives the required estimate.  $\square$

As we can see from the obtained results, the convergence in the initial error is close to prescribed-time one for  $t \in [0, T)$ , and the dependence in  $\phi^{(n)}$  becomes infinitesimal at  $k = N-2$ , while the noise gain admits a static linear upper bound.

### 5.2. Analysis for $k \geq N-1$

For  $k = N-1$ ,  $t_{k+1} = T$  then all eigenvalues of the matrix  $Z(T)$  are zero, hence, there is  $\bar{P} = \bar{P}^\top \succ 0$  such that  $Z^\top(T)\bar{P}Z(T) \prec \beta\bar{P}$  for some  $\beta \in (0, 1)$ . If we would like to extend the analysis to  $k > N-1$  we need to introduce a definition of  $Z(t)$  and  $F(t)$  in (10) for  $t > T$ . A possible approach is just to take

$$Z(t) = Z(T), \quad F(t) = F(T), \quad \forall t \geq T.$$

In such a case let us modify our time-varying Lyapunov function  $V_k = e_k^\top \Pi_k e_k$  as follows:

$$\Pi_k = \begin{cases} \Gamma(t_{k+1})P\Gamma(t_{k+1}) & \text{if } k \in \overline{\{0, N-2\}} \\ \bar{P} & \text{if } k \geq N-1 \end{cases}, \quad (20)$$

then it has been already established the shape of convergence of the estimation error for  $k \in \overline{\{0, N-2\}}$  in Theorem 2, while for  $k \geq N-1$  we get the error dynamics (note that  $Z(T)b = 0$ )

$$e_{k+1} = Z(T)e_k - hF(T)v_{k+1},$$

which is independent in  $\phi^{(n)}$ . Moreover, in the noise-free case  $e_{N+n-1} = 0$  (since  $Z^n(T) = 0_{n \times n}$ ), and we recover the prescribed-time convergence uniformly in  $\phi^{(n)}$  as in the continuous time, while also obtaining the bounded noise gains. However, it is easy to see that the observer (10) in such a situation corresponds to the Euler approximation of the derivatives:

$$\begin{aligned} \begin{bmatrix} \hat{x}_{1,k+1} \\ \vdots \\ \hat{x}_{n,k+1} \end{bmatrix} &= Z(T) \begin{bmatrix} \hat{x}_{1,k} \\ \vdots \\ \hat{x}_{n,k} \end{bmatrix} + hF(T)y_{k+1} \\ &= \begin{bmatrix} y_{k+1} \\ h^{-1}(y_{k+1} - y_k) \\ h^{-2}(y_{k+1} - 2y_k + y_{k-1}) \\ \vdots \\ h^{1-n}(y_{k+1} - \hat{x}_{1,k} - \dots - h^{n-2}\hat{x}_{n-1,k}) \end{bmatrix}, \end{aligned}$$

whose performance is well-known Reichhartinger et al. [19].

Another approach is to recursively apply the observer (10) always staying onto the interval  $k \in \overline{\{0, N-2\}}$  for the values of matrices  $Z$  and  $F$ :

$$\begin{aligned}\hat{x}_{k+1} &= Z(\tilde{t}_{k+1})\hat{x}_k + hF(\tilde{t}_{k+1})y_{k+1}, \\ \tilde{t}_{k+1} &= \text{mod}(t_{k+1}, T)\end{aligned}\tag{21}$$

for all  $k \in \mathbb{Z}_+$ . Then, the prescribed-time convergence may be sacrificed and the dependence on initial conditions is almost eliminated at  $t_k = T$ , while for  $t_k > T$  such an observer filters the noise and estimate the derivatives following the advantageous performance evaluated in Theorem 2.

*Remark 4.* Recalling Remark 2, note that in the continuous-time noise-free case, a similar modification can be applied to (4) extending its application to all  $t \geq 0$ , but in the presence of an arbitrary bounded noise  $v$  the estimate  $\hat{x}$  may become infinite while  $\text{mod}(t, T) \rightarrow T$ . Moreover, definition of solutions and analysis of well-posedness for  $t \geq 0$  in a noise-free scenario is a subject of a separate research. In the discrete-time setting, (21) is naturally well-posed since  $Z$  and  $F$  are continuous functions on  $[0, T)$ .

Let us illustrate the efficiency of the proposed iterative scheme in simulations.

## 6. Simulations

For simulations, let us take  $n = 4$  and

$$\phi(t) = 0.1t^5 - 100t + \sin(7t) + 1000$$

and the noise

$$v(t) = 10h^3(\text{rnd}(1) + \sin(35t)),$$

where  $\text{rnd}(1)$  generates a uniformly distributed in the interval  $[0, 1]$  random number. Since

$$\begin{aligned}\dot{\phi}(t) &= 0.5t^4 - 100 + 7\cos(7t), \\ \ddot{\phi}(t) &= 2t^3 - 49\sin(7t), \\ \phi^{(3)}(t) &= 6t^2 - 343\cos(7t), \\ \phi^{(4)}(t) &= 12t + 2401\sin(7t),\end{aligned}$$

the perturbation  $\phi^{(n)}$  is an unbounded and linearly growing function of time, which blocks application of many existing differentiation solutions on an infinite interval of time, whose utilization is based on assumption of boundedness of  $\phi^{(n)}$  (e.g., the linear high-gain observer Khalil [8] or the super-twisting differentiator Levant [10]), which is however not an issue for the considered uPTS observer, as it is explained in Remark 3. The linear matrix inequalities from Theorem 2 are satisfied for

$$L = [46\ 791\ 6026\ 17160]^\top,$$

and take  $T = 2.5$ . For comparison purposes, the linear high-gain observer Khalil [8] will be applied with the gains

$$L_{HG} = D(0.5T)L$$

under similar implicit Euler discretization (this value of  $L_{HG}$  was chosen to have a similar convergence time for the output estimation error), together with a linear fixed-time observer to which (10) is reduced at  $t_{k+1} = T$ :

$$\hat{\xi}_{k+1} = Z(T)\left(\hat{\xi}_k + hD(T)Ly_{k+1}\right),$$

where  $\hat{\xi}_k \in \mathbb{R}^n$  is the vector of estimates ( $Z(T)$  is a nilpotent matrix). The results of simulations are presented in figures 1 and 2 for  $h = 0.1$  and  $h = 0.01$ , respectively. On the top plots a norm of the estimation error for  $\phi$ ,  $\dot{\phi}$ ,  $\ddot{\phi}$  and  $\phi^{(3)}$  is shown for all observers: red solid lines for uPTS, dot blue lines for high-gain and dash green lines for linear fixed-time observers. On the bottom plots examples of estimation are given for  $\phi^{(3)}$  (dash-dot violet lines) for all observers using the same colors. As we can conclude from these results, the error of proposed observer quickly converges to a vicinity of the origin, with the discrepancy proportional to the noise amplitude. Moreover, comparison of the integral estimation errors for all observers, on the final time interval  $t \in [3T, 4T]$ , shows that (21) outperforms the linear static analogues as it is shown in Table 1.

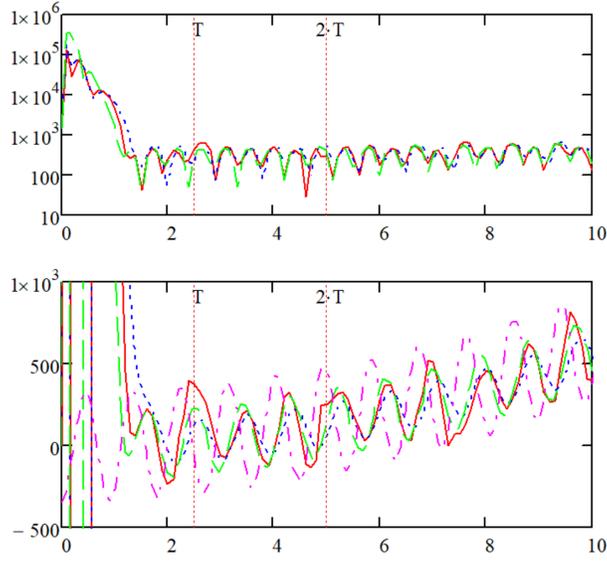


Figure 1: The norms of estimation error and the estimates of  $\phi^{(3)}$  (dash-dot violet lines) for uPTS (red solid lines), high-gain (blue dash lines) and fixed-time (dash green lines) observers versus time  $t$ ,  $h = 0.1$

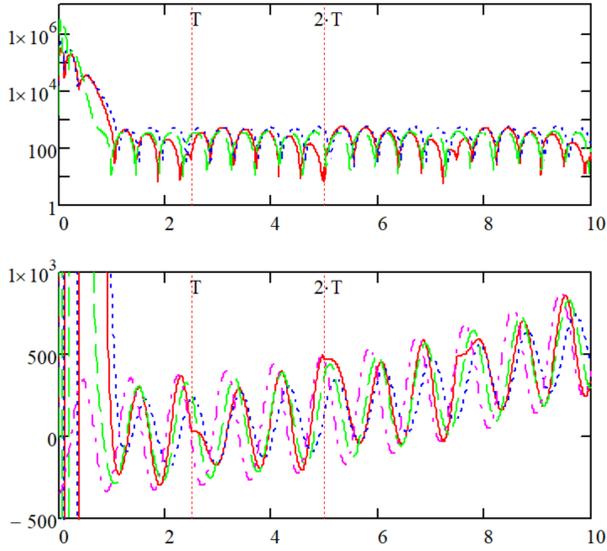


Figure 2: The norms of estimation error and the estimates of  $\phi^{(3)}$  (dash-dot violet lines) for uPTS (red solid lines), high-gain (blue dash lines) and fixed-time (dash green lines) observers versus time  $t$ ,  $h = 0.01$

	(21)	High-gain	Fixed-time
$h = 0.1$	4998	5501	5364
$h = 0.01$	2094	4213	2834

Table 1: The integral of the error norm for all observers on the interval  $[3T, 4T)$

## 7. Conclusion

A new discretization scheme for a simple implementation of the prescribed time observer from Holloway and Krstic [6] is proposed, which guarantees an accelerated rate of convergence of the estimation error in discrete time, while remaining prescribed-time exact in the noise-free case. It has also certain robustness with respect to the measurement noise. The tuning rules are formulated using feasible linear matrix inequalities. Supporting simulation results demonstrate a good performance of the proposed discrete-time realization of the observer. Extension of this discretization scheme to the case of a prescribed-time stabilizing controller can be considered as a direction of future research.

## References

- [1] Butcher, J.C., 2008. Numerical Methods for Ordinary Differential Equations. 2nd ed., John Wiley & Sons, New York.
- [2] Chitour, Y., Ushirobira, R., Bouhemou, H., 2020. Stabilization for a perturbed chain of integrators in prescribed time. *SIAM Journal on Control and Optimization* 58, 1022–1048. doi:10.1137/19M1285937.
- [3] Cruz-Zavala, E., Moreno, J.A., Fridman, L.M., 2011. Uniform robust exact differentiator. *IEEE Trans. Automat. Control* 56, 2727–2733.
- [4] Dorato, P., 2006. An Overview of Finite-Time Stability. Birkhäuser Boston, Boston, MA. pp. 185–194.
- [5] Efimov, D., Polyakov, A., Zimenko, K., Wang, J., 2022. An exact robust hyperexponential differentiator, in: Proc. 61th IEEE CDC, Cancun.
- [6] Holloway, J., Krstic, M., 2019. Prescribed-time observers for linear systems in observer canonical form. *IEEE Transactions on Automatic Control* 64, 3905–3912.
- [7] Kairuz, R., Orlov, Y., Aguilar, L., 2022. Robust observer design with prescribed settling-time bound and finite varying gains. *European Journal of Control* 68, 100667.
- [8] Khalil, H., 2017. High-Gain Observers in Nonlinear Feedback Control. *Advances in Design and Control*, SIAM, Philadelphia.
- [9] Khalil, H.K., 2002. *Nonlinear Systems*. third ed., Prentice Hall.
- [10] Levant, A., 1998. Robust exact differentiation via sliding modes techniques. *Automatica* 34, 379–384.
- [11] Li, W., Krstic, M., 2023. Prescribed-time mean-nonovershooting control under finite-time vanishing noise. *SIAM Journal on Control and Optimization* 61, 1187–1212. doi:10.1137/22M1471171.
- [12] Lopez-Ramirez, F., Polyakov, A., Efimov, D., Perruquetti, W., 2018. Finite-time and fixed-time observer design: Implicit Lyapunov function approach. *Automatica* 87, 52–60.
- [13] Lu, T.T., Shiou, S.H., 2002. Inverses of  $2 \times 2$  block matrices. *Computers & Mathematics with Applications* 43, 119–129. doi:https://doi.org/10.1016/S0898-1221(01)00278-4.
- [14] Mojallizadeh, M.R., Brogliato, B., Polyakov, A., Selvarajan, S., Michel, L., Plestan, F., Ghanes, M., Barbot, J.P., Aoustin, Y., 2023. A survey on the discrete-time differentiators in closed-loop control systems: Experiments on an electro-pneumatic system. *Control Engineering Practice* 136, 105546. doi:https://doi.org/10.1016/j.conengprac.2023.105546.
- [15] Orlov, Y., 2022. Time space deformation approach to prescribed-time stabilization: Synergy of time-varying and non-Lipschitz feedback designs. *Automatica* 144, 110485.
- [16] Orlov, Y., Kairuz, R., 2022. Autonomous output feedback stabilization with prescribed settling-time bound. *IEEE Transactions on Automatic Control* doi:10.1109/TAC.2022.3173988.
- [17] Perruquetti, W., Floquet, T., Moulay, E., 2008. Finite-time observers: application to secure communication. *IEEE Transactions on Automatic Control* 53, 356–360.

- [18] Polyakov, A., 2012. Nonlinear feedback design for fixed-time stabilization of linear control systems. *IEEE Transactions on Automatic Control* 57(8), 2106–2110.
- [19] Reichhartinger, M., Efimov, D., Fridman, L., 2018. Special issue on differentiators. *International Journal of Control* 91, 1980–1982. doi:10.1080/00207179.2018.1487580.
- [20] Song, Y., Wang, Y., Holloway, J., Krstic, M., 2017. Time-varying feedback for regulation of normal-form nonlinear systems in prescribed finite time. *Automatica* 83, 243–251.
- [21] Song, Y., Ye, H., Lewis, F.L., 2023. Prescribed-time control and its latest developments. *IEEE Transactions on Systems, Man, and Cybernetics: Systems* 53, 4102–4116. doi:10.1109/TSMC.2023.3240751.
- [22] Steeves, D., Krstic, M., 2022. Prescribed-time stabilization robust to measurement disturbances, in: *Proc. American Control Conference (ACC)*, pp. 3873–3878. doi:10.23919/ACC53348.2022.9867221.
- [23] Zhang, F. (Ed.), 2005. *The Schur Complement and Its Applications*. volume 4 of *Numerical Methods and Algorithms*. Springer. doi:doi:10.1007/b105056.