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ALGEBRAS, GRAPHS AND ORDERED SETS – ALGOS 2020 & THE MATHEMATICAL CONTRIBUTIONS OF MAURICE POUZET

MIGUEL COUCEIRO AND DWIGHT DUFFUS

1. Introduction

The theory of ordered sets has its origins in logic, set theory, and algebra, particularly the study of lattices. The main questions studied and methods used in investigations of orders draw from these areas and have a clear combinatorial aspect. The theory is closely linked to graph theory, universal algebra and multiple-valued logic. It has applications to both classical topics ranging from set theory to formal calculus and to applied areas such as decision science, social networks, e-learning systems, security, and biomedical sciences.

The main scientific objective of the *First International Conference on Algebras, Graphs and Ordered Sets – ALGOS 2020* was to bring together specialists in graph theory, relational structures and ordered sets with computer scientists, have them present their most recent work, and discuss applications in related scientific fields. The hope is that these interactions will spawn new multidisciplinary projects.

ALGOS 2020 took place Wednesday August 26 through Friday August 28, 2021. Originally it was to happen at the Lorraine Research Laboratory in Computer Science and its Applications (LORIA), with support from LORIA, University of Lorraine, CNRS and INRIA. Due to the COVID-19 pandemic, the conference was held online with a platform provided by the Institut de Recherche en Informatique et Systèmes Aléatoires (IRISA). The organizing committee was comprised of ten mathematicians and chaired by Miguel Couceiro.

The conference provided an opportunity to honor Maurice Pouzet on the occasion of this seventy-fifth birthday. Thus, the conference welcomed submissions in areas closely related to Professor Pouzet's broad scientific interests: lattices and ordered sets; combinatorics and graph theory; set theory and the theory of relations; universal algebra and multi-valued logic; and, applications in the areas outlined above.

We first give a brief description of the conference with some details about plenary speakers and special sessions. This is followed by an overview of the main contributions of Maurice Pouzet to our discipline and to its community of scholars.

2. The ALGOS 2020 Conference

2.1. **Conference organization and proceedings.** Review of submissions to the conference was directed by a scientific committee of 35 scholars, with 27 contributions selected for presentation after a thorough review process. The contributions consisted of seven long papers, seven short papers, eight extended abstracts, and five abstracts for special sessions featuring open problems. These papers and 12 abstracts of the plenary talks can be found on the conference website hosted by LORIA (`https://algos2020.loria.fr`), where the conference program and the organizing and scientific committees are also posted.

An official conference proceedings, consisting of a selection of long and short conference papers, was edited by Miguel Couceiro, Pierre Monnin and Amedeo Napoli, and published in the *CEUR Workshop Proceedings*[1]. In addition, special issues of the *Journal of Multiple-Valued Logic and Soft Computing*, *Discrete Mathematics and Theoretical Computer Science*, and *Order* contain papers developed from conference presentations and articles devoted to closely related topics.

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[1] `http://ceur-ws.org/Vol-2925/`; see also `https://hal.archives-ouvertes.fr/hal-02918958` for the complete yet unpublished preproceedings.

2.2. **Program highlights.** Here are the plenary speakers and presentation titles:

Ron Aharoni, *The colorful world of rainbow sets*;

Youssef Boudabbous and Christian Delhommé, *F_3-reconstruction*;

Dwight Duffus, *Extremal problems for Boolean lattices and their quotients*;

Mirna Džamonja, *On logics that make a bridge from the discrete to the continuous*;

Michel Habib, *Graph searches and maximal cliques structure for cocomparability graphs*;

Joris van der Hoeven, *Ordering infinities*;

Claude Laflamme, *Maurice's siblings*;

Jaroslav Nešetřil, *EPPA & Ramsey*;

Max Pitz, *Applications of order trees in infinite graphs*;

Marc Pouzet, *Synchronous programming of real-time systems or turning mathematics into trustable code*;

Norbert Sauer, *Partitioning subgroups of the symmetric group* $\mathfrak{S}(U)$;

Stéphan Thomassé, *Twin-width.*

Following are the organizers of the special sessions dedicated to open problems:

Mohammad Alzohairi, Moncef Bouaziz, and Youssef Boudabbous, *Recursive construction of the minimal prime digraphs*;

Aymen Ben Amira, Jamel Dammak, and Hamza Si Kaddour, *Reconstruction of digraphs up to complementation*;

Jan Hubička, *Big Ramsey degrees of the universal homogeneous partial order are finite*;

Maurice Pouzet, *Well quasi-ordering and embeddability of relational structures*;

Nicolas M. Thiéry, *On relational structures with polynomial profile.*

ALGOS 2020 had a geographically diverse set of participants from Canada and the US, Africa and Europe, the Middle East and East Asia. Despite the conflicting time zones, there were at times around 200 participants connected simultaneously to conference presentations!

3. The Mathematical Contributions of Maurice Pouzet

3.1. **A brief mathematical biography.** Maurice Pouzet works in discrete mathematics with an emphasis on relational structures and, more specifically, (partially) ordered sets. His interests range over several areas, described briefly in this article, with substantial contributions to both finitary and infinitary relational theory. His advisor, Ernest Corominas, introduced him to ordered sets in 1968. After his PhD thesis in 1970 on algebras ordered by divisibility, a notion introduced by Graham Higman [39], he investigated applications of ordered sets to the theory of relations in the vein of Roland Fraïssé, and defended his thèse d'état in 1978. Since then, he has published 133 papers with 62 collaborators, colleagues and students. After Corominas

and Fraïssé, Ivo Rosenberg, Ivan Rival and Eric Milner had significant influence on his research. He supervised the doctoral theses of El Mostapha Jawhari, Nejib Zaguia, Mohamed Belhassan, Driss Misane, Faouzi Saïdane, Mustapha Kabil, Ilham Chakir, Mohamed Sobrani, Elias Tahhan-Bittar, Christian Delhommé, Stéphan Thomassé, Imed Zaguia, Olivier Marguin, Nicolas Thiéry, Bernard Roux, Eric San-Juan (co-direction with Luisa Iturrioz), Houmem Belkhechine (co-direction with Yousef Boudabbous) and Djamila Oudrar (co-direction with Moncef Abbas).

During his career, Professor Pouzet has been a central figure and resolute supporter of his research community. He has developed and maintained collaborations with colleagues around the globe, too extensive to enumerate fully. We highlight some here.

3.1.1. *France.* Maurice's influence in the French research community is extensive, and it is best illustrated by the fact that he was one of the key figures in the development of discrete mathematics in France. His impact can be measured by his contributions to the creation of various research groups and laboratories, in the organization of conferences and related events, as well as his role in several university programs in Lyon (his home town) and elsewhere. Again, our account is far from exhaustive and we simply mention a few key facts.

From 1982 to 1988 Maurice was the director of the Laboratory on Ordinal Algebra and Algorithms (LAOA) founded by Professor Corominas. The LAOA combined several research teams working in the field of ordered sets. With additional colleagues working in logic at the University Claude Bernard Lyon 1 he formed the LMDI group, which he headed until the recruitment of Jean-Louis Nicolas in 1988, and which led to the creation of the inter-institutional Center for Research in Computer Science and Logic Lyon St-Etienne (Cirille). During this period he also

- integrated with Ivo Rosenberg the Franco-Québécois project *Ordered Sets and their Applications*, from 1984 to 1986,
- constituted the research group *Order, Logic and Numbers in Computer Science* of the French PRC Mathematics and Computer Science in 1988, and
- participated in the development of a group in Lyon of the European network of Discrete Mathematics (DIMANET) in 1992, created by Walter Deuber, within the European program *Human Capital and Mobility*.

He then co-directed with André Goldman the Laboratory of Probability, Combinatorics and Statistics (LaPCS) from 1999 to 2003, that resulted from the merger of the teams Probability and Discrete Mathematics (two teams *d'accueil* of the French Ministry of Research). The teams on Discrete Mathematics included several members of the ENS Lyon Ecole Normale Supérieure de Lyon (ENS Lyon), including Adrian Bondy, Luisa Iturrioz, and Gerard Lopez.

Since then, Maurice has contributed to several international cooperative projects that supported many student exchanges, among which we emphasize:

- the Franco-Tunisian project *Mathematical Tools for Computer Science* (CMCU - 2005 to 2008), the French team led by Christian Mauduit (Marseille), the Tunisian team by Mohamed Mkaouar (Sfax);
- the INTAS project on *Universal Algebra and Lattice Theory* from 2004 to 2007; and
- the CMEP-Tassili (2009-2012) entitled *Interactions between Theoretical Computer Science and Combinatorics*, and the CHP-Maghreb (Algeria-Morocco-Tunisia) (2014-2016) entitled *Theoretical Computer Science in the Maghreb*, with Hacène Belbachir (Algeria) and Jean Gabriel Luque (France).

As for his role in discrete mathematics curricula, he contributed with Serge Grigorieff and Denis Richard to the creation in 1985 of the Master's programme in Discrete Mathematics (MMD) of the University Claude Bernard Lyon 1. During the few years of its existence, the training provided undergraduate students in mathematics career opportunities in computer science at many universities and companies. When the ENS Fontenay-Saint-Cloud moved to Gerland it became the ENS Lyon, and it opened a Master's program in Computer Science and Modeling, under the responsability of Grigorieff, which integrated the MMD. Several professors and lecturers in computer science from various universities (including Bordeaux, Caen, Lyon, Marseille, Montpellier, Nancy, Orsay, Paris and Sophia-Antipolis) came from these programs.

Maurice taught many graduate courses on ordered sets and the theory of relations at several universities, among them University Claude Bernard Lyon (1980 - 2003), the ENS Lyon (1989 - 1990), the University of Savoie Mont Blanc Chambery (1995-96), and the ENS Paris, the École Polytechnique and the University of Paris 6, 7 and 11 (1989 - 1995). He also taught at other research centers such as the CIRM Luminy (together with Michel Habib). Among many others, Jean-Yves Marion, now Professor of Computer Science and Director of the Lorraine Research Laboratory in Computer Science and its Applications (LORIA), Stephan Thomassé, currently Professor in Computer Science at the ENS Lyon, and Nicolas Thiery, now Professor in Computer Science at University Paris-Saclay, followed Maurice's courses. He also gave graduate and postgraduate courses around the world, for instance, in China at X'ian University (1983), in Venezuela at Simon Bolivar University (2005), in Algeria at University Oran-Essenia (2006), and in Morocco at University of Sfax (2007).

Maurice is well known for his hospitality and warmth. Many of us experienced heated discussions over a proof idea on the white board in his kitchen while he prepared an excellent dinner, or late night working sessions in his living room with chocolates and classical music. The latter also took place when Maurice was the invitee. On a personal note, the first-named author recalls a particular research visit of Maurice to Luxembourg in 2009. Maurice has this unique ability to motivate exchanges with colleagues of all backgrounds in mathematics and computer science. After a week of many encounters, Maurice was to leave the next morning and there was a paper to submit. It seemed unlikely that we would manage to finish it. As always, I invited him for dinner and when he arrived he had brought me a gift. I opened it and to my surprise it was a beautiful traditional coffee machine[2]. Needless to say, we spent the whole night drinking coffee and working on the paper, and by the time we had to stop and rush to his hotel for his belongings in order to make it to the train, the paper was complete!

3.1.2. *Canada.* Maurice Pouzet's many visits to Canada began in the late 1970's with Ivo Rosenberg in Montreal. He met Ivan Rival in 1979 at a combinatorics meeting organized by Rosenberg and Michel Deza. Several extended visits to Calgary and Montreal followed. By the time of the 1981 NATO Advanced Study Institute held at Banff, Canada, Maurice had already established productive collaborations with Rival and Eric Milner. Present at that meeting from Canada, in addition to Milner and Rival (who was the main workshop organizer) were Dwight Duffus, Richard Guy, Ivo Rosenberg, Bill Sands, Norbert Sauer, Mike Stone and Robert Woodrow, most of whom continue to collaborate with Maurice to this day. Maurice was also present at the second and third NATO Workshops in Banff, in 1984 and 1991, where new Canadian faces included Claude Laflamme and Nejib Zaguia who had just recently moved to Ottawa (although Nejib's collaboration with Maurice dates from several years earlier), Lucien Haddad and Imed Zaguia. In fact there have been many visits to Canada, including frequent stays in Calgary, Kingston,

[2]That is in constant use!

Montreal and Ottawa, which continue to this day.

Maurice would rent an apartment for a few months each summer in Calgary, and would be ready to take on mathematical problems over an espresso only a few minutes after his arrival and a short stop at the grocery store. These memorable summer days would be spent largely as a group working and arguing at the seminar room blackboard, with many projects continuing over several years. Visits in the fall term often give the Calgarians an opportunity to celebrate Maurice's birthday.

3.1.3. *North Africa.* Since 1980 the postgraduate mathematics program at the University Claude Bernard has welcomed many students from the Maghreb and neighboring countries such as Libya, Lebanon and Syria. Ernest Corominas introduced Maurice Pouzet to that program. With the help of Roland Assous and Christine Charretton, Maurice supervised several students. In fact, his first doctoral students were Nejib Zaguia (Tunisia), El Mostapha Jawhari, Driss Misane and Mohamed Belhassan (Morocco). Nejib defended his thesis in 1983 on infinite orders and his *thèse d'état* in Lyon in 1989 on algorithmic aspects of orders and graphs. He became a professor in Computer Science at the University of Ottawa. El Mostapha and Driss defended their theses in 1983 and 1984 on a metric point of view on orders and graphs as well as on fixed point problems. The former became a lecturer in Marrakech and the latter, a professor in Rabat. Mohamed Belhassan defended his thesis in 1984 under the co-supervision of Robert Bonnet, both on Boolean algebra problems and on Fodor and Alexandroff properties for ordered sets. He is currently an aggregated professor in Lyon.

At this point, Mustapha Kabil, Ilham Chakir and Mohamed Sobrani (Morocco) joined Maurice in Lyon. They defended their theses in 1992, respectively on injective envelopes of graphs, transition systems and word ideals; on chains of ideals and the algebraic dimension of distributive lattices; on the order structure of the collection of ages of relations. After their PhD theses in Morocco, they became university professors: Mustapha Kabil in Mohammedia, Ilham Chakir in Settat, Mohamed Sobrani in Fes. Despite not being their supervisor, Maurice remained close to several Moroccan students such as Mohammed Bekkali and Fathia Erraji (who did their theses with Robert Bonnet), Abdelmajid Hilali (who did his thesis with Vincent Bouchitté), and also Abderrhaim Boussairi (who did his thesis with Gérard Lopez, then a Professor in Chambéry and a member of the discrete mathematics laboratory of Lyon). The latter is currently a university Professor in Casablanca.

Coming from Tunisia, Faouzi Saidane defended his thesis in 1991 on the study of transition systems from a metric point of view, and he became an assistant Professor in Gafsa. Imed Zaguia defended his thesis on perpendicular orders in 1997, and he is currently a Professor at the Royal Military College of Canada in Kingston, Ontario. Maurice also had many collaborations with Youssef Boudabbous and his students in Sfax, and also his brother Imed and Moncef Bouaziz. Youssef did his thesis with Gérard Lopez in 1994 and later, Maurice and Youssef, co-supervised the thesis of Houmem Belkhechine (2009).

The collaboration of Maurice with Algeria started with a symposium on discrete mathematics organised at the University of Oran Essenia in 2006, by Abderrahmane Smail who defended his thesis in the 80's on dynamical systems. But, in fact, Maurice collaborated some years earlier with Hamza Si Kaddour from Oran and was a student of Robert Bonnet, and is currently Associate Professor in University Claude Bernard. Also, at the University of Oran, Maurice supervised three master's students: Nacera Souddi (2009), Zoubidda Bouazza (2010) and Zohra Tadj (with Hamza Si Kaddour in 2011). All are currently faculty members in Algerian

universities. He was also part of two international agreements involving Algeria and France: the CMEP-Tassili (2009-2012), entitled *Interactions between Theoretical Computer Science and Combinatorics*, and the CHP-Maghreb (Algebra-Morocco-Tunisia 2014-2016), entitled *Theoretical Computer Science in the Maghreb*.

Still concerning Algeria, he co-supervised with Mohammed Abbas the PhD thesis of Djamila Oudrar on the enumeration of discrete structures via relational theory. Djamila is now an associate professor at the USTHB. He made several visits to the USTHB where he gave courses and lectures, and was a member of several thesis juries. He was and is, to a large extent, an important driving force in the programs initiated by Hacène Belbachir in combinatorics. Maurice was a main organizer of conferences including ECSYMATH'06 (Oran, Algeria), ROGICS'08 (Mehdia, Tunisia), DIMACOS'11 (Mohammadia, Morocco), DIMACOS'12 (Beirut, Lebanon), DIMACOS'15 (Sidi Bel Abbès, Algeria), DIMACOS'19 (Hammamet, Tunisia).

3.2. **Selected mathematical topics.** Turning to mathematics, Maurice's research divides almost equally between ordered sets and relational structures. We have selected five topics that illustrate his extensive and diverse contributions.

3.2.1. *Cofinality.* The *cofinality* of a poset P, denoted $cf(P)$, is the least cardinal κ such that P contains a cofinal subset of size κ. As is well-known, each chain P contains a well-ordered cofinal subset, hence the cofinality of P is a regular cardinal. In 1979, Pouzet proved that

if P has no infinite antichain, then P contains a cofinal subset which is a direct sum of finitely many posets, each isomorphic to a finite product of regular ordinals.

In particular, the cofinality of P is a regular cardinal. The result was motivated by a question of Galvin linking the notions of *well-quasi-order* (wqo) and of *better-quasi-order* (bqo). A poset P is wqo if it contains neither an infinite antichain nor an infinite descending sequence, while a poset is bqo if the collection $P^{<\omega_1}$ of countable ordinal sequences with values in P is wqo [63]. These classes of ordered sets play a fundamental role in the theory of relations, notably the solution by Laver [52] of Fraïssé's conjectures on countable chains. The question of Galvin was: does every wqo have a bqo cofinal subset? Since being a bqo is preserved under finite unions and products, Maurice's result answered Galvin's question postively. While he did not publish his result, it was featured in two editions of Fraïssé's book [32] and appears in several papers.

Pouzet's result prompted Milner and Sauer to conjecture in 1981 [59] that a poset P with singular cofinality κ contains an antichain of size $cf(\kappa)$. If $cf(\kappa)$ is uncountable, the answer is not known without set theoretical hypotheses. Building on Hajnal and Sauer [38], Milner and Pouzet showed that $\kappa^{<cf(\kappa)} = \kappa$ suffices [60]. Later, Gorelic [36] and Rinot [88] improved this result. The theme of the relationship between the structure of cofinal subsets of a poset and the size of antichains was later extended to closure systems, for instance, see [34] and [61]. Recently, with Abraham, Bonnet and Dzamonja, Maurice is investigating a question of Abraham, Kubis, Bonnet [1]: is every wqo a countable union of bqos?

3.2.2. *Hereditary classes. Hereditary classes* of relational structures are those closed with respect to embeddings, in the following sense. A relational structure R is *embeddable* in a structure R' if R is isomorphic to an induced substructure of R'. The embeddability relation is a quasi-order on the class $\tilde{\Omega}_\mu$ of relational structures of a given arity μ. Fraïssé saw that basic notions about ordered sets like initial segments, ideals, chains and antichains have direct counterparts in terms of relational structures. For example, a class $\mathfrak{C}$ of structures is *hereditary* if it contains every structure which can be embedded in some member of $\mathfrak{C}$. Hence, hereditary classes are initial

segments of $\tilde{\Omega}_\mu$ quasi-ordered by embeddability. If R is a relational structure, the *age* of R is the set $Age(R)$ of finite restrictions of R considered up to isomorphism. This is an ideal of the poset Ω_μ of finite relational structures of arity μ considered up to isomorphism and ordered via embeddability. Fraïssé (1954) showed that the converse holds for countable ideals.

Hereditary classes of finite graphs and posets abound in combinatorics. For the logician, hereditary classes of finite structures of a given finite arity are classes of finite models of *universal theories* – theories axiomatized by universal sentences. Such classes have a finite axiomatization if and only they have, up to isomorphism, only finitely many bounds, by the Tarski-Vaught theorem (1957). A *bound* of a hereditary class $\mathcal{C}$ of finite relational structures is a finite relational structure S that is minimal, with respect to embeddability, in the complement of the class. The set $Forb(\mathcal{C})$ of bounds determines $\mathcal{C}$. Since the bounds form an antichain, if $\mathcal{C}$ has no infinite antichain with respect to embeddability then obviously every hereditary class included in $\mathcal{C}$ has only finitely bounds belonging to $\mathcal{C}$. But $\mathcal{C}$ itself may have infinitely many bounds – a simple example is the age of an infinite path.

In 1972 [68] Pouzet proved that

$\mathcal{C}$ of finite arity has finitely many bounds if the class $\mathcal{C}_p$ of $(S, U_1, \ldots, U_p)$ where $S \in \mathcal{C}$, and $U_1, \ldots, U_p$ are unary relations, has no infinite antichain, this for every integer p.

This result, which he considers as his most original contribution, has several striking consequences. First, the age of an infinite monomorphic relational structure of finite arity has finitely many bounds and these structures are *chainable*, that is, admit a linear order of the ground set simplifying definable predicates. Another consequence is that a relational structure definable by a poset whose age is included in the age of a series-parallel poset has finitely many bounds. Pouzet's work on bounds for hereditary classes continues to the present: recent work with I. Zaguia [84] concerns the hereditary class $\mathcal{C}$ of finite bipartite permutation graphs and the characterization of more restrictive *hereditary* wqo's.

Pouzet, often with his students, has done a great deal of work on the order structure of hereditary classes and ages as well as initial segments and ideals of posets ([85, 14, 13, 81]). He has considered wqo ages and their variants, particularly ages for which the collection of sub-ages is well-founded (it is not known if they are wqo) [71]. The use of a partial order to describe classes of structures is not limited to the study of relational structures. In [11, 30, 67], the authors study the set of equational classes of Boolean functions by means of the minor ordering, and investigate its structural properties [15] as well as of its dual topological space [23].

3.2.3. *The profile.* The *profile* of a relational structure R is the function φ_R defined as follows: for each positive integer n, $\varphi_R(n)$ is the number of n-element substructures of R, up to isomorphism. In 1971, he proved that

the profile $\varphi_R(n)$ of an infinite relational structure is nondecreasing in n,

a result that was conjectured with Fraïssé [33]. In 1976, he improved this, showing that $\varphi_R(n) \leq \varphi_R(n+p)$ provided that the domain of R, possibly finite, has at least $2n+p$ elements. His proof [69] is based on the non-degeneracy of the Kneser matrix of the n-element subsets of a $2n+p$ element set, a result obtained independently by Gottlieb (1966) and Kantor (1972). He applied the same result with Rosenberg [77], proving that the collection of set-orbits of a permutation group action on a finite set has the strong Sperner property. This required constructing a matching from p-element to $(n+p)$-element structures. A deep generalization

to matroids was obtained with Aharoni [2]. Extension of the profile to enumerating infinite substructures is studied with Gibson and Woodrow [35]. In his thèse d'état [70], Pouzet proves

profiles of relational structures with μ finite have either polynomial growth or their growth is faster than every polynomial.

The growth of φ_R is a *polynomial of degree k* if $a \leq \frac{\varphi_R(n)}{n^k} \leq b$ where a and b are non-negative constants. The growth is faster than every polynomial if $\frac{\varphi_R(n)}{n^k}$ goes to infinity with n, for every k. For further details, see [72, 73]; see also [75] for a survey.

Maurice conjectured that the profile has polynomial growth of degree k if and only if the generating series of the profile $\mathcal{H}_{\varphi_R} := \sum \varphi_R(n)$ is a rational fraction of the form

$$\frac{P(Z)}{(1-Z)(1-Z^2)\cdots(1-Z^{k+1})} \ .$$

where P is a polynomial with integer coefficients (possibly negative). Equivalently, φ_R is, for a sufficiently large n, a *quasi-polynomial*, that is, a polynomial in n with periodic coefficients. Since φ_R is nondecreasing this would give that $\varphi_R(n) \sim an^k$, as yet unknown.

A positive answer has been obtained for relational structures admitting a finite decomposition into monomorphic components. Several important structures, for instance, tournaments with polynomial profiles [82], have this property, but not graphs. The case of graphs was settled independently by Balogh *et al.* [5]. They prove also that if the profile is not polynomial then it is greater than the partition function. Balogh, Bollobás and Morris [6] proved that if $\mathfrak{C}$ is a hereditary class of finite ordered graphs then its profile $\varphi_{\mathfrak{C}}$ (by extension, the function that counts, for every integer n, the number of n-element structures in that class) is either polynomial or is ranked by the Fibonacci functions (see Klazar [49] for recent developments). Using the notion of monomorphic decomposition, Oudrar [64] proved that the class $\mathfrak{S}$ of ordered binary structures that do not have a finite monomorphic decomposition has a finite basis (a subset $\mathfrak{B}$ such that every member of $\mathfrak{S}$ embeds some member of $\mathfrak{B}$). From this result, the following dichotomy property holds for every hereditary class $\mathfrak{C}$ of finite ordered binary structures of a given finite type.

Either there is an integer ℓ such that every member of $\mathfrak{C}$ has a monomorphic decomposition into at most ℓ blocks and in this case the profile of $\mathfrak{C}$ is bounded by a polynomial of degree $\ell - 1$ (in fact, it is a polynomial), or $\mathfrak{C}$ contains the age of a structure that does not have a finite monomorphic decomposition, in which case the profile of $\mathfrak{C}$ is bounded below by the Fibonacci function.

This result was extended in [66] to ordered relational structures.

Binary relational structures, notably the ordered ones, can be reconstructed by means of indecomposable structures, those with no nontrivial modules, also known as intervals and autonomous sets. (See [29] for the general theory and [24] for infinite binary structures.) Extending a result of Albert and Atkinson [3], Oudrar and Pouzet [65] prove that if a hereditary class of finite ordered binary relational structures contains only finitely many indecomposable members, the generating series of its profile is algebraic. These classes are hereditary wqo. Pouzet and Zaguia [83] considered infinite indecomposable structures which do not embed strictly smaller infinite indecomposable structures: again, their age is wqo.

According to Peter Cameron, the profile of a relational structure R can be viewed as the Hilbert function of a graded algebra, the *age-algebra* of $\mathcal{A}(R)$. Hence, via the profile, there is an interesting link between algebra and the theory of relations. An important conjecture of Cameron about the integrality of this algebra is solved in [76]. On the one hand, if the age-algebra is finitely generated, then the Hilbert series has the form given above, but in general the converse does not hold. On the other hand, Falque and Thiéry [31] proved that if the age-algebra is the orbit algebra of a permutation group, then the converse is true, answering a longstanding question of Macpherson [58].

3.2.4. *Metric spaces.* These can be generalized by enlarging the eligible measures of distance. Motivated by the classification scheme of Duffus and Rival [28], based on retracts and products, and inspired by the use of Helly's property by Quilliot [86, 87], Pouzet considered ordered sets and graphs as types of metric spaces for which the values of the distance function belong to an ordered monoid $\mathcal{H}$ equipped with an involution and satisfying a distributivity condition [74]. Such an ordered monoid (called an *involutive Heyting algebra*) is the dual or opposite of an *involutive quantale*, a notion introduced during the same period by Mulvey [62]. With Jawhari and Misane [41], then his students, Pouzet characterized the absolute retracts with respect to non-expansive mappings in the same way as Aronszajn and Panitchpakdi [4] and obtained the existence of an injective envelope as Isbell [40] did for ordinary metric spaces. Here is their result.

In the category of metric spaces over an involutive Heyting algebra $\mathcal{H}$, hyperconvex, injective and absolute retracts coincide and are retracts of some powers of $\mathcal{H}$ equipped with an appropriate metric. Furthermore, every metric space has an injective envelope.

In the case of ordered sets, the injective objects are complete lattices and the injective envelope is obtained with the MacNeille completion [7]. With Bandelt and Saïdane [8] he proved that in the category of reflexive, oriented graphs, injective and absolute retracts coincide with retracts of powers of the MacNeille completion of the free monoid on the two letter alphabet equipped with a suitable metric, and are retracts of products of oriented zigzags, and hence acyclic.

With Rosenberg [78], Maurice extended his approach to transition systems. In this case, the distance between two states x, y is the language accepted by the automaton with initial state x and final state y. With Kabil [44], he gave a description of the injective envelope of 2-element metric spaces in the category of directed graphs and in some categories of transition systems (for further results, see [45]). Using properties of the injective envelope of 2-element generalized metric spaces, Kabil, Pouzet and Rosenberg [47] proved that

the monoid of nonempty final segments of the free monoid generated by a poset is free.

The notions of retraction and the fixed point property are naturally linked together, absolute retracts being natural candidates for the fixed point property. The Tarski fixed point theorem for complete lattices and the Sine-Soardi fixed point theorem for bounded hyperconvex ordinary metric spaces were extended in [41]. Revisiting generalized metric spaces and generalizing those which are bounded and hyperconvex with the notion of relational structures with compact normal structure, he proved with Khamsi [48] that

every commuting family of endomorphisms of a relational structure R made of binary relations has a common fixed point whenever R has a compact and normal structure.

See [46] for a survey of the metric approach and see [27] for a study of endomorphisms. Notions in the theory of relations, like *indivisibility* and *homogeneity*, are particularly interesting in the case of metric spaces and their generalizations. See [26] and [79] for further details.

3.2.5. *Isomorphism and equimorphism.* These are measures of similarity between relational structures. A *sibling* of a relational structure R is any R' such that R and R' are embeddable in each other, that is, are equimorphic. In [91] (p.2, Conjecture 2), Thomassé proposed

Conjecture 1. *Every countable relation R has, up to isomorphism, 1, $\aleph_0$ or $2^{\aleph_0}$ siblings.*

A positive answer was given for chains in [50] using some results of Jullien's thesis [42]. It is not even known if R has one or infinitely many siblings, even if R is a countable loopless undirected graph. In fact, this is unsolved for countable trees (not ordered trees but connected graphs with no cycles). The *tree alternative property* holds for a tree T if either every tree equimorphic to T is isomorphic to T, or there are infinitely many pairwise nonisomorphic trees that are equimorphic to T. Bonato and Tardif [9] conjectured that the tree alternative property holds for every tree and proved that it holds for rayless trees (for further background, see [92, 10]). Laflamme, Pouzet and Sauer [51] proved that it holds for scattered trees (trees in which no subdivision of the binary tree can be embedded). But they could not settle the conjecture for the complete ternary tree with some leaves attached. As it turns out, induced subgraphs equimorphic to these trees are connected; hence for such trees the tree alternative property amounts to the fact that every tree has one or infinitely many siblings.

Graphs, and more generally binary structures, can be decomposed into simpler pieces via labelled trees, the bricks being the indecomposable structures. In [37] Hahn, Pouzet and Woodrow stated

Conjecture 2. *A countable graph G has 1, $\aleph_0$ or $2^{\aleph_0}$ siblings if every induced indecomposable subgraph has 1, $\aleph_0$ or $2^{\aleph_0}$ siblings.*

If their conjecture holds true, then Thomassé's conjecture must hold true for countable *cographs*, that is, graphs with no induced subgraph isomorphic to the 4-vertex path. As it is well known, no induced subgraph of a cograph with more than two vertices can be indecomposable (see [89] for finite graphs, [43] for infinite graphs). They proved a weaker version of Thomassé's conjecture:

a countable cograph has either one or infinitely many siblings.

The proof relies on tree decompositions of a cograph and Thomassé's theorem on series-parallel posets [90].

Maurice Pouzet has several other results, in particular, on the reconstruction of relations, with Hamza Si Kaddour, Gérard Lopez and his students [80, 25]. On a similar topic, it is noteworthy to recall the contributions of Maurice to the study of minor quasi-ordering of functions [23], and its join-irreducible functions [11]. The former led to the notion of *arity gap* [16] and several variants [17, 55, 56, 57], as well as to several decomposition results pursued by other authors [18, 19, 20]. The latter motivated a functional variant of the reconstruction problem [21, 22, 53, 54]. These constitute some examples of the many active research topics that are rooted in Maurice's work.

4. Conclusion and acknowledgements

We have briefly surveyed some of Maurice Pouzet's main contributions. It would be hard, if not senseless, to give a wider account since Maurice remains extremely active (despite his many comments otherwise) and his work underlies ongoing research too diverse to track. This explains

why some topics that deserve a place in this note remained unmentioned, for which both authors apologize.

However, we hope to have given a fair account of the work of our dear friend Maurice to whom we (the many colleagues, young researchers, students, visitors and friends) are truly thankful. His generosity, kindness, and friendship, both in research and in everyday life, have made working with him a true pleasure, a lesson, a great model of research and life. Luckily, we still have plenty of work left unfinished, Maurice, but this would not be the right place to recall the missed and upcoming deadlines!

Last, we wish to express our deepest gratitude to all those who contributed to this short note. We are particularly indebted to Youssef Boudabbous, Hamza Si Kaddour, Hacene Belbachir, Claude Laflamme, Robert Woodrow and, of course, Maurice Pouzet!

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