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Well-Posedness of Passive Time-Varying Linear Cone Complementarity Systems

Quang Hung Pham ^{*} Bernard Brogliato [†] Félix A. Miranda-Villatoro [‡]

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Abstract

This article provides sufficient conditions to ensure the existence and uniqueness of absolutely continuous and bounded variation solutions of a family of time-dependent nonsmooth systems named time-varying linear cone complementarity systems. Passivity is a key property. Nonsmooth circuits illustrate the theoretical results.

Keywords: Linear complementarity system, time-varying linear cone complementarity system, first-order sweeping process, well-posedness, linear complementarity problem, set-valued mapping, maximal monotone operator, time-dependent differential inclusion, passive system.

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1 Introduction

Linear complementarity systems (LCS) have been extensively studied in Applied Mathematics, Automatic Control, and Circuits. They have broad applications across various fields including mechanics, electrical circuits, dynamic networks, optimization, economics, *etc*, see for instance [1, 2, 3, 4, 5]. A general form of LCS is given by:

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) + Eu(t), & (1a) \\ w(t, x) = Cx(t) + D\lambda(t) + Fu(t), & (1b) \\ 0 \leq \lambda(t) \perp w(t, x) \geq 0, & (1c) \end{cases}$$

where $x(t) \in \mathbb{R}^n$, $\lambda(t) \in \mathbb{R}^m$, $u(t) \in \mathbb{R}^u$, and the matrices A, B, C, D, E, F are constant with suitable dimensions. The concept and initial studies of (1) with $E = F = 0$, were introduced in the late 2000s in [6, 7]. Then, the existence and uniqueness of solutions (in the distributional sense) to this LCS were established in [8] under assumptions regarding the system's passivity, the minimality of (A, B, C) , and the full column rank of B . An extension is provided in [9], considering (1) with $E, F \neq 0$ (*i.e.*, input u occurs). It asserts that (1) always possesses a unique distributional solution if the quadruple (A, B, C, D) is passive with the storage function being positive definite, the matrix obtained by stacking B over $D + D^\top$ has full column rank, and the input $u(\cdot)$ is a piecewise Bohl function. To overcome limitations on the input, [10] provided sufficient conditions for (1) to have

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a unique locally absolutely continuous (AC) solution (resp. right-continuous and locally bounded variation (RCLBV) solution) in the case $D = 0$, where it is assumed that there exists $P = P^\top \succ 0$ such that $PB = C^\top$, $\text{rge}(C) - \mathbb{R}_+^m = \mathbb{R}^m$, where $\text{rge}(C)$ is the image of C , and $u(\cdot)$ is a locally AC (resp. RCLBV) function. The case of $D = \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix}$, denoted as $\text{diag}(D_1, 0)$, with $D_1 \succ 0$, and nonlinear complementarity systems is also tackled in [10, 11]. The well-posedness of Lur'e systems, an extension of (1) when (1c) is replaced by $\lambda_i \in -\partial\varphi_i(w_i)$ for all $i = \overline{1, m}$, where $\partial\varphi_i$ is the subdifferential of φ_i with $\varphi_i : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ being proper convex and lower semicontinuous functions, is studied in [12], generalizing [13]. It follows that if $D \succcurlyeq 0$ and φ_i is the indicator function of \mathbb{R}_+ for all $i = \overline{1, m}$, then this Lur'e system becomes (1) with $F = 0$, which has a unique Lipschitz continuous solution under assumptions that there exists $P = P^\top \succ 0$ such that $PB = C^\top$, $u(\cdot)$ is continuous with its derivative being locally integrable, and $\text{rge}(C) \cap \text{rint}(\text{Im}(\mathcal{N}_{\mathbb{R}_+^m} + D)) \neq \emptyset$, where $\text{rint}(\text{Im}(\mathcal{N}_{\mathbb{R}_+^m} + D))$ denotes the relative interior of the image of the set-valued mapping that is sum of the normal cone to \mathbb{R}_+^m and the linear map D . The well-posedness of this Lur'e system in cases $D = \text{diag}(0, D_1)$, where $D_1 \succ 0$, D_1 is a P-matrix (*i.e.*, the matrix has all its principal minors positive) and $D_1 \succcurlyeq 0$ under the extra conditions, is analyzed based on Kato's Theorem.

The well-posedness of linear cone complementarity systems (LCCS), an extension of (1) when (1c) is replaced by the constraint $\mathcal{K} \ni \lambda \perp w \in \mathcal{K}^*$, is analyzed in [14, 15] with \mathcal{K} being a polyhedral cone and \mathcal{K}^* is its dual cone, see also [11]. In particular, in [14], a forward solution (stronger than AC solution) exists and is unique if the quadruple (A, B, C, D) is passive with positive definite storage functions, $u(\cdot)$ is a Bohl function, $\mathcal{K} = \{M\xi \mid \forall \xi \geq 0\}$ for some matrix M , and $M^\top Fu(t) \in (\text{SOL}(0, M^\top DM))^* + \text{rge}(M^\top C)$ for all t , where $(\text{SOL}(0, M^\top DM))^*$ is the dual cone of the set of all solutions to the linear complementarity problem of the form $0 \leq \lambda \perp M^\top DM\lambda \geq 0$. An approach commonly used for studying LCSs (resp. LCCSs) is through a Lur'e differential inclusion (DI) of the form:

$$\dot{x}(t) \in Ax(t) + Eu(t) - B \left(\mathcal{N}_{\mathbb{R}_+^m - Fu(t)}^{-1} + D \right)^{-1} (Cx(t)). \quad (2)$$

This Lur'e DI is equivalent to (1) and is an extension of Moreau's sweeping process, see [16, 4]. This approach is used in [17], which took inspiration from [12, 14] to study the well-posedness of an extension of (1) where $F = 0$ and (1c) is replaced by $w \in \mathcal{M}(-\lambda)$ with \mathcal{M} being a set-valued mapping. In particular, this extended system is rewritten as the DI (2), in which the inverse mapping of the normal cone to the set $\mathbb{R}_+^m - Fu(t)$, that is, $\mathcal{N}_{\mathbb{R}_+^m - Fu(t)}^{-1}$ is replaced by \mathcal{M} . It has a unique locally AC solution if the quadruple (A, B, C, D) is passive (or passive by pole shifting, *i.e.*, the quadruple $(A - \alpha I, B, C, D)$ is passive for some $\alpha \geq 0$) with positive definite storage functions, \mathcal{M} is maximal monotone, $\text{rge}(C) \cap \text{rint}(\text{Im}(\mathcal{M} + D)) \neq \emptyset$, and $u(\cdot)$ is a locally integrable function. The case of positive semi-definite storage functions is also treated in [17].

As mentioned, the well-posedness of LCCS is tackled in [14], but the input $u(\cdot)$ is required to be a Bohl function, which is somewhat restrictive. This serves as motivation for results in [15]. They assert that the LCCS (*i.e.*, (2) with the substitution \mathbb{R}_+^m by \mathcal{K}^*) has a weak solution which is continuous (resp. right-continuous bounded variation (RCBV)) if $u(\cdot)$ is a locally AC (resp. RCLBV) function, $D \succcurlyeq 0$, $\ker(D + D^\top) \subseteq \ker(PB - C^\top)$ for some $P = P^\top \succ 0$, along with some additional non-trivial technical assumptions. An extension of [15] to obtain a strong solution is handled in [18], in which an extension of (2) when $\mathbb{R}_+^m - Fu(t)$ is replaced by a closed convex set $K(t, x(t))$, is considered and the existence of its Lipschitz continuous solutions is guaranteed. The uniqueness of the solution is tackled when $K(t, x(t)) = K_1(t) + h(t, x)$, where $K_1(\cdot)$ is a closed convex set-valued mapping, $h(t, \cdot)$ is a single-valued mapping along with additional assumptions.

In addition, when errors appear in the quadruple (A, B, C, D) , the well-posedness of LCCS with the time-varying cone $\mathcal{K}(t)$ and $F = 0$, is also analyzed as an application of main results. The well-posedness of a generalization of (2) when $\mathcal{N}_{\mathbb{R}_+^m - Fu(t)}(\cdot)$ is replaced by $F_{t,x(t)}(\cdot)$ the maximal monotone mapping, is provided in [19].

To the best of the author's knowledge, the most advanced result about the well-posedness of (1) is given in [20], an extension [17]. In there, (1) has a unique AC solution if the quadruple (A, B, C, D) is passive with the storage function $V(x) = \frac{1}{2}x^\top x$, $\text{rge}(C) \cap \text{rint}\left(\text{Im}(\mathcal{N}_{\mathbb{R}_+^m - Fu(t)}^{-1} + D)\right) \neq \emptyset$ for all t , and the input $u(\cdot)$ is an AC function. This result stems from the well-posedness of DIs of the form:

$$\dot{x}(t) \in -F(t, x(t)) + f(x(t)) + u(t),$$

where $F(t, \cdot)$ is a maximal monotone mapping for all $t \geq 0$, $f(\cdot)$ is a single-valued function and the input $u(\cdot)$ is a locally integrable function, is analyzed in [20].

It is undeniable that LCS and LCCS are important classes of non-smooth dynamical systems. However, in real-life scenarios, uncertainties always occur and impact the data within LCS (resp. LCCS) at arbitrary times. Therefore, considering an extended class of LCS and LCCS (*i.e.*, all matrices in (1) are time-dependent, (1c) is extended to the cone complementarity condition with time-varying cones), referred to as time-varying linear cone complementarity systems (TVLCCS), becomes essential. It should be noted that, while there have been articles addressing these systems (for instance, [1, 2] modeled a DC/DC Buck converter as TVLCCS), all of them have focused on modeling and numerical simulations. In this article, we concentrate on studying the existence and uniqueness of solutions to TVLCCS defined by:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)\lambda(t) + E(t)u(t) & \text{almost everywhere on } t \geq 0, & (3a) \\ w(t, x) = C(t)x(t) + D(t)\lambda(t) + F(t)u(t) & \text{for all } t \geq 0, & (3b) \\ \mathcal{K}(t) \ni \lambda(t) \perp w(t, x) \in \mathcal{K}^*(t) & \text{for all } t \geq 0, & (3c) \end{cases}$$

where $x(t) \in \mathbb{R}^n$, $\lambda(t) \in \mathbb{R}^m$, $u(t) \in \mathbb{R}^u$, $\mathcal{K}(t) \subseteq \mathbb{R}^m$, all matrix functions have compatible dimensions and are bounded on $[0, T]$ for some $T > 0$, and \mathcal{K}^* is the dual cone to \mathcal{K} . As we shall see, TVLCCS can be shown to be equivalent to a class of Lur'e DIs, which is an extension of (2). Hausdorff continuous time-varying convex nonempty compact sets $\mathcal{K}(t)$ may be considered for the analysis of Lur'e DIs.

The structure of this article is as follows. The main results are given in Sections 2.1, 2.2, 2.4, relying on and extending in a non-trivial way ideas and methods proposed in [15, 20, 10]. Section 2.3 provides relevant comments and nonsmooth circuits to illustrate the main results, and recalls results in [20, 14]. In Section 3, several examples of TVLCCS are provided to show the particular features of this class of nonsmooth systems and their relationships with Moreau's sweeping process. They are also analyzed based on the theoretical results in Section 2, thereby allowing us to raise open problems for future investigations. Conclusions end the article in Section 4.

Notations and Definitions: Let $S \subseteq \mathbb{R}^m$ be a nonempty set, S^0 is its least-norm element, S^* is its dual cone, $\text{int}(S)$ is its interior, $\text{rint}(S)$ is its relative interior, $\text{cl}(S)$ is its closure, $\text{aff}(S)$ is its affine hull, $\text{proj}_S(\cdot)$ is the projection map on S , and $\text{dist}(x, S)$ is the distance from a point $x \in \mathbb{R}^m$ to S . Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}^m$ be a function, $\text{dom } f$ is its domain, $f(t^+)$ and $f(t^-)$ are right and left limits of $f(\cdot)$ at $t \in I$, respectively. Denote by $AC([0, T], X)$ (resp. $L_1([0, T], X)$) the space of absolutely continuous (AC) (resp. Lebesgue integrable) functions $f : [0, T] \rightarrow X \subseteq \mathbb{R}^m$. Let $F : I \subseteq \mathbb{R} \rightrightarrows \mathbb{R}^m$

be a set-valued mapping, $\text{Dom } F$ is its domain, $\text{Im } F$ is its image, and $\text{Graph } F$ is its graph. Let S and S' be closed sets in \mathbb{R}^m , denote by $d_H(S, S')$ the Hausdorff distance between S and S' , defined by $d_H(S, S') := \max\{\sup_{x \in S} \text{dist}(x, S'), \sup_{y \in S'} \text{dist}(y, S)\} = \sup_{x \in \mathbb{R}^m} |\text{dist}(x, S) - \text{dist}(x, S')|$. Let $M \in \mathbb{R}^{m \times n}$ be a matrix, M^\top is its transpose, $\ker M$ is its null space, and $\text{rge}(M)$ is its image. Denote by $\text{diag}(d_1, d_2, \dots, d_m)$ the diagonal matrix in $\mathbb{R}^{m \times m}$ with d_1, d_2, \dots, d_m the elements on the diagonal. Let V be a linear subspace of \mathbb{R}^m , V^\perp is its orthogonal complement, and $\mathbb{R}^m = V \oplus V^\perp$, where \oplus is the direct sum. A matrix $D \in \mathbb{R}^{m \times m}$ is positive definite (resp. positive semi-definite), denoted by $D \succ 0$ (resp. $D \succeq 0$) if $x^\top D x > 0$ (resp. $x^\top D x \geq 0$) for all $x \in \mathbb{R}^m \setminus \{0\}$. D is copositive-plus if $x^\top D x \geq 0$ for all $x \in \mathbb{R}_+^m$ and the following implication holds: $[x^\top D x = 0, x \geq 0] \Rightarrow [(D + D^\top)x = 0]$. LCP and LCCP are for linear complementarity problems, and linear cone complementarity problems, respectively. Given $q \in \mathbb{R}^m$ and $D \in \mathbb{R}^{m \times m}$, denote by $\text{LCP}(q, D)$ the LCP which it finds λ such that $0 \leq \lambda \perp D\lambda + q \geq 0$, and $\text{SOL}(q, D)$ its set of all solutions. The LCCP has the form of $\mathcal{K} \ni \lambda \perp D\lambda + q \in \mathcal{K}^*$ for some cone \mathcal{K} , is an extended problem of $\text{LCP}(q, D)$. Denote by $\|\cdot\|$ the Frobenius norm, $\langle \cdot, \cdot \rangle$ the Euclidean inner product, which will be used throughout the article. Let $N_\infty := \{\text{subsequences of } N \text{ containing all } n \text{ beyond some } \hat{n}\}$. Let us recall some useful definitions, see [4, 21, 20, 5, 22, 23, 24, 25].

Definition 1. Let $S \subseteq \mathbb{R}^m$ be a closed convex nonempty set. The normal cone to S at $x \in S$ is defined by $\mathcal{N}_S(x) := \{p \in \mathbb{R}^m : \langle p, x - y \rangle \geq 0, \forall y \in S\}$.

If $S \subseteq \mathbb{R}^m$ is a closed convex nonempty set, then for any $x, y \in \mathbb{R}^m$, the following hold:

$$x - y \in -\mathcal{N}_S(x) \Leftrightarrow \langle y - x, x - u \rangle \geq 0, \forall u \in S \Leftrightarrow x = \text{proj}_S(y). \quad (4)$$

If $S \subseteq \mathbb{R}^m$ is a closed convex nonempty cone, then for any $x, y \in \mathbb{R}^m$:

$$S \ni x \perp y \in S^* \Leftrightarrow x \in -\mathcal{N}_{S^*}(y) \Leftrightarrow y \in -\mathcal{N}_S(x). \quad (5)$$

If $S := \{\gamma \in \mathbb{R}^m : A\gamma \geq b\}$ for some $A \in \mathbb{R}^{n \times m}$ and $b \in \mathbb{R}^n$, then for any $x \in S$, $\mathcal{N}_S(x)$ is given by:

$$\mathcal{N}_S(x) = \{p \in \mathbb{R}^m : p = -A^\top \lambda, 0 \leq \lambda \perp Ax - b \geq 0\}. \quad (6)$$

Definition 2. A set-valued mapping $F : I \subseteq \mathbb{R} \rightrightarrows \mathbb{R}^m$ is inner semicontinuous at $t^* \in I$ if $F(t^*) \subseteq \liminf_{t \rightarrow t^*} F(t)$, where

$$\liminf_{t \rightarrow t^*} F(t) := \{y \in \mathbb{R}^m : \forall t_n \xrightarrow{n \rightarrow \infty} t^*, \exists N \in N_\infty, \exists \{y_n\}_{n \in N} \subset F(t_n) \text{ s.t. } y_n \xrightarrow{n \rightarrow \infty} y\}.$$

We say that $F(\cdot)$ is inner semicontinuous on I if it is inner semicontinuous at every $t^* \in I$.

More general properties than inner and outer semicontinuity given in [21] are defined as the following.

Definition 3. A set-valued mapping $F : [0, T] \rightrightarrows \mathbb{R}^m$ is outer semicontinuous (resp. inner semicontinuous) from right at $t^* \in [0, T]$ if $\limsup_{t \searrow t^*} F(t) \subseteq F(t^*)$ (resp. $F(t^*) \subseteq \liminf_{t \searrow t^*} F(t)$), where

$$\limsup_{t \searrow t^*} F(t) := \{y \in \mathbb{R}^m : \exists (t_n, y_n)_{n \in \mathbb{N}} \subset [0, T] \times \mathbb{R}^m \text{ s.t. } y_n \in F(t_n), t_n \geq t^*, (t_n, y_n) \xrightarrow{n \rightarrow \infty} (t^*, y)\}$$

$$\liminf_{t \searrow t^*} F(t) := \{y \in \mathbb{R}^m : \forall t_n \xrightarrow{n \rightarrow \infty} t^*, t_n \geq t^*, \exists N \in N_\infty, \exists \{y_n\}_{n \in N} \subset F(t_n) \text{ s.t. } y_n \xrightarrow{n \rightarrow \infty} y\}.$$

We say that $F(\cdot)$ is outer semicontinuous (resp. inner semicontinuous) from right on $[0, T]$ if it is outer semicontinuous (resp. inner semicontinuous) from right at every $t^* \in [0, T]$.

Definition 4. A set-valued mapping $F : I \subseteq \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is said to be monotone on I if for every pairs $(x_1, y_1), (x_2, y_2) \in \text{Graph } F$, one has $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$. It is maximal monotone if its graph is not properly contained in the graph of any other monotone mapping.

The passivity of the quadruple $(A(t), B(t), C(t), D(t))$ plays an important role in this article.

Definition 5. The quadruple $(A(t), B(t), C(t), D(t))$ is passive if the following linear matrix inequality (LMI) has a differentiable solution $P(t) = P(t)^\top \succcurlyeq 0$ for all t , where \dot{P} is the derivative of P .

$$\begin{bmatrix} P(t)A(t) + A(t)^\top P(t) + \dot{P}(t) & P(t)B(t) - C(t)^\top \\ B(t)^\top P(t) - C(t) & -(D(t) + D(t)^\top) \end{bmatrix} \preccurlyeq 0. \quad (7)$$

Definition 6. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}^m$ be a function. Denote by \mathcal{P} the set of all partitions $\sigma = (x_0, x_1, \dots, x_{N_\sigma})$ of I such that $x_0 < x_1 < \dots < x_{N_\sigma} \in I$. The variation of $f(\cdot)$ over I is defined by:

$$V_I(f) = \sup_{\sigma \in \mathcal{P}} \sum_{i=0}^{N_\sigma-1} \|f(x_{i+1}) - f(x_i)\|.$$

$f(\cdot)$ is said to be of bounded variation (BV) on I if and only if $V_I(f) < +\infty$. We say that $f(\cdot)$ is of locally bounded variation (LBV) on I if it is BV on all compact subintervals of I . If $f(\cdot)$ is BV on I , its left and right limits always exist at every point in I . If f is right-continuous (RC) at every $t \in I$ and LBV (resp. BV) on I , then f is called right-continuous locally bounded variation (RCLBV) (resp. RCBV) on I . The concept of BV (resp. LBV) set-valued mappings $F : I \subseteq \mathbb{R} \rightrightarrows \mathbb{R}^m$ is obtained by substituting $d_H(F(x_{i+1}), F(x_i))$ for $\|f(x_{i+1}) - f(x_i)\|$.

Definition 7. A set-valued mapping $F : [0, T] \rightrightarrows \mathbb{R}^m$ is called AC on $[0, T]$ if there exists an AC function $v : [0, T] \rightarrow \mathbb{R}$ such that for every $x \in \mathbb{R}^m$:

$$|\text{dist}(x, F(t)) - \text{dist}(x, F(s))| \leq |v(t) - v(s)|, \quad \forall 0 \leq s \leq t \leq T.$$

F is called RCBV on $[0, T]$ if there exists a positive Radon measure μ on $[0, T]$ such that:

$$d_H(F(s), F(t)) \leq \mu((s, t]), \quad \forall 0 \leq s \leq t \leq T.$$

F is called locally AC (resp. RCLBV) on $[0, +\infty)$ if F is AC (resp. RCBV) on $[0, T]$ for all $T > 0$.

2 Well-posedness of the TVLCCS systems

This section is dedicated to deriving criteria that guarantee the existence and uniqueness of locally AC and RCLBV solutions to (3). This is done by using a classical result in [26] for ordinary differential equations (ODE) (Section 2.1), extending the results in [20] (Sections 2.2, 2.3) and in [10] (Section 2.4). The blanket assumption for (3) is made:

Assumption 1. The set $\mathcal{K}(t) \subseteq \mathbb{R}^m$ is a closed convex nonempty cone for each $t \in [0, T]$.

2.1 The case $D(t) \succ 0$ for all $t \geq 0$

Let us consider the TVLCCS (3). Using (5), the cone complementarity condition (3c) can be written equivalently as:

$$\lambda(t) \in -\mathcal{N}_{\mathcal{K}^*(t)}(C(t)x(t) + D(t)\lambda(t) + F(t)u(t)),$$

so that for all $t \geq 0$,

$$\lambda(t) \in - \left(\mathcal{N}_{\mathcal{K}^*(t)-F(t)u(t)}^{-1} + D(t) \right)^{-1} (C(t)x(t)). \quad (8)$$

The substitution of $\lambda(t)$ above back into (3a) yields

$$\dot{x}(t) \in A(t)x(t) + E(t)u(t) - B(t) \left(\mathcal{N}_{\mathcal{K}^*(t)-F(t)u(t)}^{-1} + D(t) \right)^{-1} (C(t)x(t)). \quad (9)$$

Note that (9) is also equivalent to:

$$\dot{x}(t) \in A(t)x(t) + E(t)u(t) - B(t) \left(\mathcal{N}_{\mathcal{K}^*(t)}^{-1} + D(t) \right)^{-1} (C(t)x(t) + F(t)u(t)). \quad (10)$$

The DI in (10) is named a Lur'e DI, because of its interpretation as the interconnection of a linear system and a set-valued mapping [13, 4, 12, 20]. When $\mathcal{K}^*(t)$ is not a cone but a compact nonempty set, the Lur'e DI does not stem from a TVLCCS as in (3). The following holds.

Proposition 1. *Consider the differential inclusion (10). Suppose that $A(\cdot), B(\cdot), C(\cdot), E(\cdot), F(\cdot), u(\cdot)$ are locally integrable functions, $B(\cdot), E(\cdot), F(\cdot)$ are uniformly bounded on $[0, T]$ for any $T > 0$, $\mathcal{K}^*(\cdot)$ is inner semicontinuous, and $D(\cdot)$ is continuous on $[0, +\infty)$ with $D(t) \succcurlyeq \rho I_m$, for all $t \geq 0$ and some $\rho > 0$. Then (10) possesses a unique locally AC solution on $[0, +\infty)$, for any $x(0) = x_0 \in \mathbb{R}^n$.*

Proof. It follows from [12, Proposition 1] that for each $t \geq 0$, the mapping $\left(\mathcal{N}_{\mathcal{K}^*(t)}^{-1} + D(t) \right)^{-1} (\cdot)$ is a single-valued function with domain equal to \mathbb{R}^m , and Lipschitz continuous with Lipschitz constant $\frac{2}{\lambda_{\min}(D(t) + D(t)^\top)}$, where $\lambda_{\min}(D(t) + D(t)^\top) \geq 2\rho$, is the smallest eigenvalue of $D(t) + D(t)^\top$. Hence, (10) is an ODE. To complete the proof, a classical result in [26] is used (see Theorem A1). It is verified that all conditions in Theorem A1 are satisfied on (10) under given assumptions. For each $t \geq 0$, set $g(t, C(t)x(t) + F(t)u(t)) := \left(\mathcal{N}_{\mathcal{K}^*(t)}^{-1} + D(t) \right)^{-1} (C(t)x(t) + F(t)u(t))$, $f(t, x(t)) := A(t)x(t) + E(t)u(t) - B(t)g(t, C(t)x(t) + F(t)u(t))$. Then, (10) is rewritten as: $\dot{x}(t) = f(t, x(t))$. Notice that, $f(t, \cdot)$ is Lipschitz continuous. Indeed, for each $t \geq 0$, for any $x, y \in \mathbb{R}^n$, using the Lipschitz continuity of $g(t, \cdot)$, we obtain that:

$$\begin{aligned} \|f(t, x) - f(t, y)\| &= \|A(t)(x - y) - B(t)(g(t, C(t)x + F(t)u(t)) - g(t, C(t)y + F(t)u(t)))\| \\ &\leq \left(\|A(t)\| + \frac{\|B(t)\| \|C(t)\|}{\rho} \right) \|x - y\|. \end{aligned}$$

Since $B(\cdot)$ is uniformly bounded on $[0, T]$ for any $T > 0$, $B(\cdot)$ is uniformly bounded on all compact subsets of $[0, +\infty)$. Moreover, $A(\cdot), B(\cdot), C(\cdot)$ are locally integrable on $[0, +\infty)$, then there exists a locally integrable function $\alpha(\cdot) : [0, +\infty) \rightarrow \mathbb{R}_+$, such that $\|f(t, x) - f(t, y)\| \leq \alpha(t)\|x - y\|$. Therefore, items 2, 3 in Theorem A1 hold. Next, it is proved that item 4 also holds. Indeed, take any $x_0 \in \mathbb{R}^n$ and fix it, then for each $t \geq 0$:

$$\|f(t, x_0)\| \leq \|A(t)\| \|x_0\| + \|E(t)u(t)\| + \|B(t)\| \|g(t, C(t)x_0 + F(t)u(t))\|,$$

in which the following holds using the Lipschitz continuity of $g(t, \cdot)$.

$$\begin{aligned} \|g(t, C(t)x_0 + F(t)u(t))\| &\leq \|g(t, C(t)x_0 + F(t)u(t)) - g(t, 0)\| + \|g(t, 0)\| \\ &\leq \frac{1}{\rho} (\|C(t)\| \|x_0\| + \|F(t)u(t)\|) + \|g(t, 0)\|. \end{aligned}$$

Because $g(t, 0) = \left(\mathcal{N}_{\mathcal{K}^*(t)}^{-1} + D(t) \right)^{-1} (0)$, it follows that $g(t, 0) \in \mathcal{N}_{\mathcal{K}^*(t)}(-D(t)g(t, 0))$, equivalently $\langle g(t, 0), D(t)g(t, 0) + z \rangle \leq 0$ for all $z \in \mathcal{K}^*(t)$. Since $0 \in \mathcal{K}^*(t)$ and $D(t) \succ \rho I_m$, it follows that $g(t, 0) = 0$. Now, recall that $A(\cdot), B(\cdot), C(\cdot), E(\cdot), F(\cdot), u(\cdot)$ are locally integrable and $B(\cdot), E(\cdot), F(\cdot)$ are uniformly bounded on $[0, T]$ for any $T > 0$ by assumption, there exists a locally integrable function $\beta(\cdot) : [0, +\infty) \rightarrow \mathbb{R}_+$, such that $\|f(t, x_0)\| \leq \beta(t)$, for all $t \geq 0$. The last part of this proof is dedicated to showing that item 1 in Theorem A1 is satisfied, that is, for each fixed $x \in \mathbb{R}^n$, $f(\cdot, x) = A(\cdot)x + E(\cdot)u(\cdot) - B(\cdot)g(\cdot, \xi(\cdot))$, is a measurable function, where $\xi(\cdot) : [0, +\infty) \rightarrow \mathbb{R}^m, \hat{t} \mapsto C(\hat{t})x + F(\hat{t})u(\hat{t})$. By assumption, $A(\cdot), B(\cdot), C(\cdot), E(\cdot), F(\cdot)$ and $u(\cdot)$ are measurable functions of time. Thus, it suffices to show that $g(\cdot, x)$ is also a measurable map (see *e.g.*, [26, p.474]). Let us fix $x \in \mathbb{R}^n$ and set $\hat{g}(\cdot) := g(\cdot, x)$. According to the definition of a measurable function provided in [26, Remark C.1.1], it suffices to show that for any open set $B \subseteq \mathbb{R}^m$,

$$\hat{g}^{-1}(B) := \{t \in [0, +\infty) : \hat{g}(t) \in B\} \in \mathfrak{B}([0, +\infty)), \quad (11)$$

where $\mathfrak{B}([0, +\infty))$ is the Borel σ -algebra of $[0, +\infty)$. A sufficient condition for (11) to hold is that $[0, +\infty) \setminus \hat{g}^{-1}(B)$ is a closed set (see details in Remark A1), which can be proven as follows. Let us take an arbitrary sequence $\{t_n\}_{n \in \mathbb{N}} \subset [0, +\infty) \setminus \hat{g}^{-1}(B)$ such that $\lim_{n \rightarrow \infty} t_n = t^* < +\infty$. Then $t^* \in [0, +\infty)$ and for each $n \in \mathbb{N}$, $\hat{g}(t_n) \in \mathbb{R}^m \setminus B$. Note that $\hat{g}(t_n)$ is bounded for all $n \in \mathbb{N}$. Indeed, as $g(t_n, 0) = 0$, so $\|\hat{g}(t_n)\| = \|g(t_n, x)\| \leq \|g(t_n, x) - g(t_n, 0)\| \leq \frac{1}{\rho} \|x\|$. Therefore, there exists a set $N_1 \subseteq \mathbb{N}$ such that $\{\hat{g}(t_n)\}_{n \in N_1} \subset \mathbb{R}^m \setminus B$ is convergent, say z^* is the limit point. Since B is an open set, $\mathbb{R}^m \setminus B$ is a closed set and thus $z^* \in \mathbb{R}^m \setminus B$. Recall that $\hat{g}(t_n) = \left(\mathcal{N}_{\mathcal{K}^*(t_n)}^{-1} + D(t_n) \right)^{-1} (x)$, which is equivalent to $\hat{g}(t_n) \in \mathcal{N}_{\mathcal{K}^*(t_n)}(x - D(t_n)\hat{g}(t_n))$. It follows that:

$$\langle \hat{g}(t_n), \gamma_n + D(t_n)\hat{g}(t_n) - x \rangle \leq 0, \quad \forall \gamma_n \in \mathcal{K}^*(t_n). \quad (12)$$

Now, we will prove that $\langle z^*, \gamma + D(t^*)z^* - x \rangle \leq 0$, for all $\gamma \in \mathcal{K}^*(t^*)$. Take any $\gamma \in \mathcal{K}^*(t^*)$. Since $\mathcal{K}^*(\cdot)$ is inner semicontinuous at t^* , there exist $N_2 \in \mathbb{N}_\infty$ and $\{\gamma_n\}_{n \in N_2} \subset \mathcal{K}^*(t_n)$ satisfies $\gamma_n \xrightarrow{n \rightarrow \infty} \gamma$. It follows from (12) that, for each $n \in N_1 \cap N_2$:

$$\langle \hat{g}(t_n), \gamma + D(t_n)\hat{g}(t_n) - x \rangle = \langle \hat{g}(t_n), \gamma - \gamma_n \rangle + \langle \hat{g}(t_n), \gamma_n + D(t_n)\hat{g}(t_n) - x \rangle \leq \|\hat{g}(t_n)\| \|\gamma - \gamma_n\|.$$

Notice that $N_1 \cap N_2$ coincides with N_1 beyond some position \hat{n} . It follows from the boundedness of $\{\hat{g}(t_n)\}_{n \in \mathbb{N}}$, and the continuity of the inner product $\langle \cdot, \cdot \rangle$ [27, p.4] and $D(\cdot)$ that the following holds.

$$\langle z^*, \gamma + D(t^*)z^* - x \rangle = \lim_{\substack{n \rightarrow \infty \\ n \in N_1 \cap N_2}} \langle \hat{g}(t_n), \gamma + D(t_n)\hat{g}(t_n) - x \rangle \leq 0.$$

Hence, $z^* \in \mathcal{N}_{\mathcal{K}^*(t^*)}(x - D(t^*)z^*)$, *i.e.*, $z^* = \left(\mathcal{N}_{\mathcal{K}^*(t^*)}^{-1} + D(t^*) \right)^{-1} (x) = \hat{g}(t^*)$. This implies that $\hat{g}(t^*) \in \mathbb{R}^m \setminus B$, *i.e.*, $t^* \in [0, +\infty) \setminus \hat{g}^{-1}(B)$. Thus, $[0, +\infty) \setminus \hat{g}^{-1}(B)$ is a closed set. \square

Note that the matrix $D(t)$ is not necessarily symmetric in Proposition 1. When Hausdorff's continuity is used instead of inner semicontinuity, it implies that if $\mathcal{K}^*(t)$ is a cone, then in general we have to impose $\mathcal{K}^*(t) = \mathcal{K}^*$, a constant convex nonempty cone. Otherwise if $\mathcal{K}^*(t)$ is a compact moving convex nonempty set, then the proposition applies to the Lur'e DI in (10), which no longer stems from the TVLCCS in (3). With a different approach, Corollary 1 in the next section provides sufficient conditions to ensure the existence and uniqueness of AC solutions to the TVLCCS (3) when $D(t) \succ 0$ for all $t \in [0, T]$, where it allows for the relaxation of the continuity of $D(\cdot)$ and the inner semicontinuity of $\mathcal{K}^*(\cdot)$.

2.2 Existence and uniqueness of AC solutions when $D(t) \succcurlyeq 0$

The well-posedness proof proposed in this section relies first on a transformation of the TVLCCS, similar to what is proposed in [15]. In a second step, the results in [20], which apply to a particular class of time-varying LCS, are extended to cope with more general TVLCCS in (3). Throughout this section, the following assumptions are made:

1. $D(t) \succcurlyeq 0$ (not necessarily symmetric) for all $t \in [0, T]$.
2. $\mathcal{K}(t) := \left\{ Q(t)^\top \xi, \text{ with } \xi \in \mathbb{R}_+^j \right\}$ is a polyhedral cone for all $t \in [0, T]$.

Note that when $D(t) \succcurlyeq 0$ then $\text{SOL}(0, Q(t)D(t)Q(t)^\top) = \left\{ \hat{Q}(t)^\top \xi, \text{ with } \xi \in \mathbb{R}_+^k \right\}$ is a polyhedral cone for all $t \in [0, T]$, where $\hat{Q}(t) \in \mathbb{R}^{k \times j}$ [20, p.1054]. Clearly $\mathcal{K}(t)$ satisfies Assumption 1.

2.2.1 State space transformation

Let us introduce the operator Φ defined as:

$$\begin{aligned} \Phi : [0, T] \times \mathbb{R}^m &\rightrightarrows \mathbb{R}^m \\ (t, v) &\mapsto \left(\mathcal{N}_{\mathcal{K}^*(t) - F(t)u(t)}^{-1} + D(t) \right)^{-1} (v). \end{aligned} \quad (13)$$

Therefore, (9) can be rewritten as the following DI:

$$\dot{x}(t) \in A(t)x(t) + E(t)u(t) - B(t)\Phi(t, C(t)x(t)). \quad (14)$$

Lemma 1. [15, Lemma 1] *For each $t \in [0, T]$, the operator $\Phi(t, \cdot)$ is maximal monotone.*

Let us present several lemmas that play an important role in the transformation of (14).

Lemma 2. *For all t , let $\mathcal{H}(t) := \{h \in \mathbb{R}^m : \hat{Q}(t)Q(t)h \geq 0\}$. Then, $\text{Dom } \Phi(t, \cdot) = \mathcal{H}(t) - F(t)u(t)$.*

Proof. For each $t \in [0, T]$, we have $\text{Dom } \Phi(t, \cdot) = \text{Im} \left(\mathcal{N}_{\mathcal{K}^*(t) - F(t)u(t)}^{-1} + D(t) \right)$. Let us prove that $\text{Im} \left(\mathcal{N}_{\mathcal{K}^*(t) - F(t)u(t)}^{-1} + D(t) \right) = \mathcal{H}(t) - F(t)u(t)$. Take any $p \in \text{Im} \left(\mathcal{N}_{\mathcal{K}^*(t) - F(t)u(t)}^{-1} + D(t) \right)$, then there exists $x \in \mathbb{R}^m$ such that $p \in \left(\mathcal{N}_{\mathcal{K}^*(t) - F(t)u(t)}^{-1} + D(t) \right) (x)$, i.e., $x \in \mathcal{N}_{\mathcal{K}^*(t) - F(t)u(t)}(p - D(t)x)$. Since $\mathcal{K}^*(t) - F(t)u(t) = \{\xi \in \mathbb{R}^m : Q(t)\xi \geq 0\} - F(t)u(t) = \{\xi \in \mathbb{R}^m : Q(t)\xi + Q(t)F(t)u(t) \geq 0\}$, using (6), then there exists $\lambda \in \mathbb{R}^j$ such that

$$\begin{cases} x = -Q(t)^\top \lambda, \\ w = Q(t)(p - D(t)x) + Q(t)F(t)u(t), \\ 0 \leq \lambda \perp w \geq 0, \end{cases} \quad \Leftrightarrow \quad \begin{cases} w = Q(t)D(t)Q(t)^\top \lambda + Q(t)p + Q(t)F(t)u(t), \\ 0 \leq \lambda \perp w \geq 0. \end{cases}$$

This implies that LCP $(Q(t)p + Q(t)F(t)u(t), Q(t)D(t)Q(t)^\top)$ is solvable. Since $Q(t)D(t)Q(t)^\top \succcurlyeq 0$ due to $D(t) \succcurlyeq 0$, it is copositive-plus [22, Exercise 3.12.1]. Then using [22, Corollary 3.8.10], we get $Q(t)p + Q(t)F(t)u(t) \in (\text{SOL}(0, Q(t)D(t)Q(t)^\top))^* = \{\xi \in \mathbb{R}^j : \hat{Q}(t)\xi \geq 0\}$. Therefore, $p \in \mathcal{H}(t) - F(t)u(t)$. Reversing the above process, we also obtain $p \in \text{Im} \left(\mathcal{N}_{\mathcal{K}^*(t) - F(t)u(t)}^{-1} + D(t) \right)$ for any $p \in \mathcal{H}(t) - F(t)u(t)$. The proof is complete. \square

Remark 1. *When $D(t) = D(t)^\top \succcurlyeq 0$ for all $t \in [0, T]$, $\text{Dom } \Phi(t, \cdot) = \mathcal{K}^*(t) + \text{rge}(D(t)) - F(t)u(t)$, see Appendix B. In general, this is not true when $D(t)$ is not symmetric, e.g., the circuit (64) in Section 2.3.4.*

Lemma 3. [15, Lemma 2] For a given $t \in [0, T]$ and $x \in \mathbb{R}^n$, consider $\lambda_1, \lambda_2 \in \Phi(t, C(t)x)$, then $\lambda_1 - \lambda_2 \in \ker(D(t) + D(t)^\top)$.

Lemma 4. Consider $t \in [0, T]$, $x \in \text{Dom } \Phi(t, C(t)\cdot)$ and fixed t, x . For each $\lambda_\alpha \in \Phi(t, C(t)x)$, let $\lambda^{\text{im}}(t, x) := \mathcal{P}_{D(t)}(\lambda_\alpha)$, which is the projection of λ_α on $\text{rge}(D(t) + D(t)^\top)$. Then $\lambda^{\text{im}}(t, x)$ is the same for any α .

Proof. Let any λ_1, λ_2 be in $\Phi(t, C(t)x)$. Set $V_1 := \text{rge}(D(t) + D(t)^\top)$ and $V_2 := \ker(D(t) + D(t)^\top)$. Since $V_1 = V_2^\perp$, using the Moreau's decomposition, we have: $\lambda_1 = \text{proj}_{V_1}(\lambda_1) + \text{proj}_{V_2}(\lambda_1) = \mathcal{P}_{D(t)}(\lambda_1) + \text{proj}_{V_2}(\lambda_1)$ and $\lambda_2 = \text{proj}_{V_1}(\lambda_2) + \text{proj}_{V_2}(\lambda_2) = \mathcal{P}_{D(t)}(\lambda_2) + \text{proj}_{V_2}(\lambda_2)$. Therefore, $V_1 \ni \mathcal{P}_{D(t)}(\lambda_1) - \mathcal{P}_{D(t)}(\lambda_2) = (\lambda_1 - \lambda_2) + (\text{proj}_{V_2}(\lambda_2) - \text{proj}_{V_2}(\lambda_1)) \in V_2$ by Lemma 3. It follows that $\mathcal{P}_{D(t)}(\lambda_1) - \mathcal{P}_{D(t)}(\lambda_2) \in V_1 \cap V_2 = \{0\}$, which gives $\mathcal{P}_{D(t)}(\lambda_1) = \mathcal{P}_{D(t)}(\lambda_2)$. \square

From Lemma 4, for each $\lambda_\alpha \in \Phi(t, C(t)x)$, λ_α can be written as $\lambda_\alpha := \lambda^{\text{im}}(t, x) + \lambda_\alpha^{\text{ker}}(t, x)$ for some $\lambda_\alpha^{\text{ker}}(t, x) \in \ker(D(t) + D(t)^\top)$. Therefore, we can rewrite (14) as $\dot{x}(t) = A(t)x(t) + E(t)u(t) - B(t)\lambda_\alpha$ for some $\lambda_\alpha \in \Phi(t, C(t)x(t))$, and this is equivalent to

$$\dot{x}(t) = A(t)x(t) + E(t)u(t) - B(t) \left(\lambda^{\text{im}}(t, x) + \lambda_\alpha^{\text{ker}}(t, x) \right). \quad (15)$$

In the following, an important assumption is introduced for the rest of Section 2.2.

Assumption 2. There is a continuously differentiable $P(t) = P(t)^\top \succ 0$ such that for all $t \in [0, T]$:

$$\ker(D(t) + D(t)^\top) \subseteq \ker(P(t)B(t) - C(t)^\top). \quad (16)$$

The inclusion (16) is implied by the LMI in (7), see [14, Lemma 1-(ii)]. Following a similar path as [15, (19)-(23), p.11], the differential inclusion (18) below is obtained. Let us rewrite a few steps to enhance the clarity of further developments. Let $R(t)$ be a square root of $P(t)$ such that $R(t) = R(t)^\top \succ 0$, for all $t \in [0, T]$. Let $z(t) = R(t)x(t)$, then we can rewrite (15) as follows

$$\dot{z}(t) = g(t, z(t)) + R(t)E(t)u(t) - R(t)B(t)\lambda^{\text{im}}(t, x) - R(t)B(t)\lambda_\alpha^{\text{ker}}(t, x), \quad (17)$$

where $g(t, z(t)) := \left(R(t)A(t)R(t)^{-1} + \dot{R}(t)R(t)^{-1} \right) z(t)$. Using Assumption 2, we obtain that $P(t)B(t)\lambda_\alpha^{\text{ker}}(t, x) = C(t)^\top \lambda_\alpha^{\text{ker}}(t, x)$, which implies $R(t)B(t)\lambda_\alpha^{\text{ker}}(t, x) = R(t)^{-1}C(t)^\top \lambda_\alpha^{\text{ker}}(t, x)$. Thus, (17) can be rewritten as:

$$\begin{aligned} \dot{z}(t) &= g(t, z(t)) + R(t)E(t)u(t) - R(t)B(t)\lambda^{\text{im}}(t, x) - R(t)^{-1}C(t)^\top \lambda_\alpha^{\text{ker}}(t, x) \\ &= g(t, z(t)) + \left(R(t)^{-1}C(t)^\top - R(t)B(t) \right) \lambda^{\text{im}}(t, x) - R(t)^{-1}C(t)^\top \lambda_\alpha + R(t)E(t)u(t) \\ &\in g(t, z(t)) + \left(R(t)^{-1}C(t)^\top - R(t)B(t) \right) \lambda^{\text{im}}(t, x) - R(t)^{-1}C(t)^\top \Phi(t, C(t)R(t)^{-1}z(t)) \\ &\quad + R(t)E(t)u(t). \end{aligned}$$

This is equivalent to the following differential inclusion:

$$\dot{z}(t) \in -\Psi(t, z(t)) + f(t, z(t)) + \bar{u}(t), \quad (18)$$

where $\Psi(t, z(t)) := R(t)^{-1}C(t)^\top \Phi(t, C(t)R(t)^{-1}z(t))$, $\bar{u}(t) := R(t)E(t)u(t)$, and

$$f(t, z(t)) := \left(R(t)A(t)R(t)^{-1} + \dot{R}(t)R(t)^{-1} \right) z(t) + \left(R(t)^{-1}C(t)^\top - R(t)B(t) \right) \lambda^{\text{im}}(t, R(t)^{-1}z(t)).$$

To sum up the above developments, let us state Proposition 2.

Proposition 2. *Let Assumptions 1, 2 hold with $P(t)$ the solution of (16), $D(t) \succcurlyeq 0$ for all $t \in [0, T]$. For each $t \in [0, T]$, set $z(t) = R(t)x(t)$ where $R(t)$ is a square root of $P(t)$ and $x(t)$ is the state in the system (3). Then (3) is equivalent to (18).*

Proof. Since Assumption 1 holds, the expressions (3b) and (3c) can be considered equivalent to (8), leading to the reformulation of (3) as (9). Using the definition of $\Phi(\cdot, \cdot)$ as described in (13), then (9) is replaced by (14). Applying Lemma 4 and the explanation as given immediately following it, (14) is equivalent to (15). Since Assumption 2 holds with $P(t)$ the solution of (16), therefore (15) can be rewritten equivalently as (17), where $z(t) = R(t)x(t)$ and $R(t)$ is a square root of $P(t)$. Because (16) holds, it can be concluded that (17) is equivalent to (18). The proof is complete. \square

2.2.2 Key lemmas for the main result

The following lemma confirms that [15, Assumption (A4)] (a key assumption in [15] which the necessity of it is presented in [15, Section 3.3.2]) always holds within the framework of Section 2.2.

Lemma 5. *For every $t \in [0, T]$, and each $v \in \text{rge}(C(t)) \cap \text{Dom } \Phi(t, \cdot)$, then $\text{rge}(D(t) + D(t)^\top) \cap \Phi(t, v) \neq \emptyset$.*

Proof. Let $t \in [0, T]$ and $v \in \text{rge}(C(t)) \cap \text{Dom } \Phi(t, \cdot)$, then there exists $x \in \mathbb{R}^n$ such that $v = C(t)x$. Following Lemma 4, $\lambda^{\text{im}}(t, x) := \mathcal{P}_{D(t)}(\lambda)$ is the same for all $\lambda \in \Phi(t, C(t)x)$. Using (4), the foregoing is equivalent to $\lambda - \lambda^{\text{im}}(t, x) \in \mathcal{N}_{\text{rge}(D(t)+D(t)^\top)}(\lambda^{\text{im}}(t, x))$, for all $\lambda \in \Phi(t, C(t)x)$. This can be rewritten as the following using Definition 1.

$$\langle \lambda - \lambda^{\text{im}}(t, x), \eta - \lambda^{\text{im}}(t, x) \rangle \leq 0, \quad \forall \eta \in \text{rge}(D(t) + D(t)^\top), \quad \forall \lambda \in \Phi(t, C(t)x).$$

Choosing $\eta = 0$, it holds that $\langle -\lambda^{\text{im}}(t, x), \lambda - \lambda^{\text{im}}(t, x) \rangle \leq 0$ for all $\lambda \in \Phi(t, C(t)x)$. Since $\Phi(t, \cdot)$ is maximal monotone due to Lemma 1, $\Phi(t, C(t)x)$ is a closed convex set [28, Proposition 20.31]. Therefore, $-\lambda^{\text{im}}(t, x) \in \mathcal{N}_{\Phi(t, C(t)x)}(\lambda^{\text{im}}(t, x))$, that is, $\lambda^{\text{im}}(t, x) = \text{proj}_{\Phi(t, C(t)x)}(0) \in \Phi(t, C(t)x)$ using (4). Hence, $\lambda^{\text{im}}(t, x) \in \text{rge}(D(t) + D(t)^\top) \cap \Phi(t, v)$. \square

From the proof of Lemma 5, for each $t \in [0, T]$, and $x \in \text{Dom } \Phi(t, C(t)\cdot)$, take $v := C(t)x \in \text{rge}(C(t)) \cap \text{Dom } \Phi(t, \cdot)$, then $\lambda^{\text{im}}(t, x) = \text{proj}_{\Phi(t, C(t)x)}(0)$. This means that $\lambda^{\text{im}}(t, x)$ is the least-norm element of $\Phi(t, C(t)x)$ which has actually been mentioned in the following lemma.

Lemma 6. [15, Lemma 3] *For each $t \in [0, T]$, $x \in \text{Dom } \Phi(t, C(t)\cdot)$, then $\lambda^{\text{im}}(t, x)$ is the least-norm element of the set $\Phi(t, C(t)x)$ and its dependence on x is Lipschitz continuous.*

Lemma 6 is a relaxed version of [15, Lemma 3] in the sense that the proof of the latter relies on the veracity of [15, Assumption (A4) and $J \geq 0$], whereas in this article, the former does not require [15, Assumption (A4)], as it is always valid due to Lemma 5. The next lemma extends Lemma 1 using [21, Theorem 12.43]. In order to accomplish this, the following assumption is considered.

Assumption 3. *For each $t \in [0, T]$, $\text{rge}(C(t)) \cap \text{rint}(\text{Dom } \Phi(t, \cdot)) \neq \emptyset$.*

Lemma 7. [15, Lemma 4] *Let Assumptions 1, 3 hold, and $D(t) \succcurlyeq 0$ for all $t \in [0, T]$. Then, $\Psi(t, \cdot)$ in (18) is a maximal monotone operator for each $t \in [0, T]$.*

The following lemmas will be used in the proof of Theorem 2 below.

Lemma 8. Let a matrix $C \in \mathbb{R}^{m \times n}$ and a vector $x \in \text{rge}(C^\top)$ be given. Then, there exists a positive number α (depending on C) satisfying $\|x\| \leq \alpha \|Cx\|$.

Proof. Since $x \in \text{rge}(C^\top)$, there exists a vector $v \in \mathbb{R}^m$ such that $x = C^\top v$. Therefore, $Cx = CC^\top v$. Using [29, Proposition 6.1.6-(xxv)] and triangle inequality of the norm, the following holds:

$$\|x\|^2 = \|C^\top v\|^2 = \|C^\dagger CC^\top v\|^2 \leq \|C^\dagger\|^2 \|CC^\top v\|^2 = \|C^\dagger\|^2 \|Cx\|^2,$$

where C^\dagger is a pseudo-inverse of C . Set $\alpha := \|C^\dagger\|$, then $\|x\| \leq \alpha \|Cx\|$. \square

Lemma 9. Let $C : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear operator, and $A \subseteq \mathbb{R}^m$, $B \subseteq \mathbb{R}^n$ be sets such that $B \supseteq C^{-1}(A) := \{x \in \mathbb{R}^n : Cx \in A\} \neq \emptyset$. Then, $A \subseteq C(B) := \{Cy, \forall y \in B\}$.

Proof. Take any $x \in A$. From the assumption, then $C^{-1}(x) = \{z \in \mathbb{R}^n : Cz = x\} \subseteq B$, i.e., there exists $B^o \subseteq B$ such that $C^{-1}(x) = B^o$. Therefore, $x \in C(B^o) \subseteq C(B)$. It can be concluded that $A \subseteq C(B)$. \square

2.2.3 An extended version of [20, Theorem 23]

In this subsection, we provide a slightly extended version of [20, Theorem 23] (recalled in Appendix E) which will play a core role in the proof of Theorem 2 below. Let us consider DIs of the form:

$$\dot{x}(t) \in -F(t, x(t)) + f(t, x(t)) + u(t), \quad x(0) = x_0, \quad (19)$$

where $F(t, \cdot) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, $f(\cdot, \cdot) : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $u : [0, T] \rightarrow \mathbb{R}^n$. We introduce the following assumptions [20, Section 4.1, p.1030], which shall be used for Theorem 1.

(A1) For each $t \in [0, T]$, the operator $F(t, \cdot)$ is maximal monotone.

(A2) The set-valued mapping $t \mapsto \text{Graph } F(t, \cdot)$ is outer semicontinuous from right on $[0, T]$.

(A3) There exists a nondecreasing function $\varphi \in AC([0, T], \mathbb{R})$ such that

$$\sup_{z \in \text{Dom } F(s, \cdot)} \text{dist}(z, \text{Dom } F(t, \cdot)) \leq \varphi(t) - \varphi(s), \quad \forall s, t \text{ with } 0 \leq s \leq t \leq T. \quad (20)$$

(LG) There exists $\gamma \in L_1([0, T], \mathbb{R}_+)$ such that for all $x \in \text{Dom } F(t, \cdot)$ and $t \in [0, T]$ then

$$\|F^0(t, x)\| \leq \gamma(t)(1 + \|x\|). \quad (21)$$

(YB) There exist a nondecreasing function $\theta : [0, T] \rightarrow \mathbb{R}$, and a scalar $\Lambda > 0$, such that for every $0 < \lambda < \Lambda$, then $\|\mathcal{Y}_\lambda(s + \lambda, x)\| \leq \theta(s + \lambda) - \theta(s) + \|F^0(s, x)\|$, for all s with $0 \leq s \leq T - \lambda$, and $x \in \text{Dom } F(s, \cdot)$ where $\mathcal{Y}_\lambda(t, \cdot)$ denotes the Yosida approximation of $F(t, \cdot)$.

Theorem 1. Consider the DI (19). Suppose that $u \in L_1([0, T], \mathbb{R}^n)$, $f(t, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a Lipschitz continuous function and there exist $M(T), N(T) > 0$ such that

$$\|f(t, z)\| \leq M(T) + N(T)\|z\|, \quad \forall z \in \text{dom } f(t, \cdot), \quad \forall 0 \leq t \leq T, \quad (22)$$

and $F(t, \cdot)$ satisfies assumptions **(A1)**, **(A2)**, and **(A3)** for all $t \in [0, T]$. If **(LG)** (resp. **(YB)**) holds, then for every $x_0 \in \text{cl}(\text{Dom } F(0, \cdot))$ (resp. $x_0 \in \text{Dom } F(0, \cdot)$), there exists a unique solution $x \in AC([0, T], \mathbb{R}^n)$.

Proof. See Appendix C. \square

2.2.4 Main result

The following result is a non-trivial adaptation of [20, Theorems 23, 24 and 26] to prove the well-posedness of TVLCCS systems. Recall that $\mathcal{H}(t) = \{h \in \mathbb{R}^m : \hat{Q}(t)Q(t)h \geq 0\}$ (see Lemma 2).

Theorem 2. *Consider the TVLCCS (3). Suppose that Assumption 2 with $P(t)$ the identity matrix, Assumption 3, and the following hold:*

1. *There exist $Q : [0, T] \rightarrow \mathbb{R}^{j \times m}$, $\hat{Q} : [0, T] \rightarrow \mathbb{R}^{k \times j}$ such that $\hat{Q}(\cdot)Q(\cdot)$ is bounded on $[0, T]$,*
 - (i) *for each $t \in [0, T]$, $\mathcal{K}(t) = \left\{ Q(t)^\top \xi, \text{ with } \xi \in \mathbb{R}_+^j \right\}$, for some j ,*
 - (ii) *for each $t \in [0, T]$, $\text{SOL}(0, Q(t)D(t)Q(t)^\top) = \left\{ \hat{Q}(t)^\top \xi, \text{ with } \xi \in \mathbb{R}_+^k \right\}$, for some k ,*
 - (iii) *$\mathcal{H}(s) \subseteq \mathcal{H}(t)$, for all $0 \leq s \leq t \leq T$.*
2. *Either $\text{rge}(C(t))$ or $\text{rge}(C(t)^\top)$ is a constant linear space for all $t \in [0, T]$.*
3. *$\text{rge}(C(s)) \cap (\mathcal{H}(s) - F(t)u(t)) \neq \emptyset$, for all $0 \leq s \leq t \leq T$.*
4. *$C(s)^{-1}(\text{Dom } \Phi(t, \cdot)) \subseteq C(t)^{-1}(\text{Dom } \Phi(t, \cdot))$, for all $0 \leq s \leq t \leq T$.*
5. *Mappings $C(\cdot), D(\cdot)$ are right-continuous, $\mathcal{K}^*(\cdot), \mathcal{H}(\cdot), (\text{rge}(C(\cdot)) + \text{aff}(\mathcal{H}(\cdot)))^\perp$ are inner semi-continuous from right, $F(\cdot)$ is AC and all matrix functions in (3) are bounded on $[0, T]$.*
6. *$u \in AC([0, T], \mathbb{R}^u)$.*

Then, there exists a unique solution $x \in AC([0, T], \mathbb{R}^n)$ of (3) for every $x_0 \in C(0)^{-1}(\text{Dom } \Phi(0, \cdot))$.

Proof. This proof is based on Theorem 1, an extended version of [20, Theorem 23]. The structure of this proof essentially consists of two main parts. The first part is converting (3) into a new system in the format of (19) by using the idea proposed in [15] (presented in Section 2.2.1). Subsequently, the second part is to apply Theorem 1 to this system.

• **First part.** Since Assumptions 1 and 2 hold with $P(t)$ the identity matrix, by Proposition 2, the system (3) is equivalent to the following differential inclusion:

$$\dot{x}(t) \in -\Psi(t, x(t)) + f(t, x(t)) + \bar{u}(t), \quad (23)$$

where for each $t \in [0, T]$, $\Psi(t, \cdot) = C(t)^\top \Phi(t, C(t)\cdot) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ (Φ is given in (13)), $f(t, \cdot) = A(t)(\cdot) + (C(t)^\top - B(t)) \lambda^{\text{im}}(t, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ (λ^{im} is given in Lemma 4), and $\bar{u}(\cdot) = E(\cdot)u(\cdot) : [0, T] \rightarrow \mathbb{R}^n$. Since Assumption 3 holds, for each $0 \leq t \leq T$:

$$\text{Dom } \Psi(t, \cdot) = \text{dom } f(t, \cdot) = \text{dom } \lambda^{\text{im}}(t, \cdot) = C(t)^{-1}(\text{Dom } \Phi(t, \cdot)) \neq \emptyset. \quad (24)$$

In addition, it can be verified that $\text{Dom } \Psi(t, \cdot)$ is a closed convex set for all $t \in [0, T]$.

• **Second part.** Let us focus on (23). Since items 5 and 6 hold, $E(\cdot)$ and $u(\cdot)$ are bounded on $[0, T]$, which implies $\bar{u} \in L_1([0, T], \mathbb{R}^n)$. For each $t \in [0, T]$, the operator $\Psi(t, \cdot)$ is maximal monotone (*i.e.*, $\Psi(t, \cdot)$ satisfies **(A1)** in Theorem 1) due to Lemma 7, and the function $f(t, \cdot)$ is Lipschitz continuous due to Lemma 6. Following this, let us prove that for all $t \in [0, T]$ and for all $z \in \text{dom } f(t, \cdot)$, then $f(t, z)$ satisfies (22). Indeed, let any $t \in [0, T]$, $z \in \text{dom } f(t, \cdot)$, then for any $z^*(t) \in \text{dom } f(t, \cdot)$, it

is inferred from the Lipschitz continuity of $f(t, \cdot)$ that:

$$\begin{aligned} \|f(t, z)\| &\leq \|f(t, z^*(t))\| + \|f(t, z) - f(t, z^*(t))\| \\ &\leq \left(\|A(t)\| \|z^*(t)\| + \|C(t)^\top - B(t)\| \|\lambda^{\text{im}}(t, z^*(t))\| \right) + L_f(t) \|z - z^*(t)\| \\ &\leq (\|A(t)\| + L_f(t)) \|z^*(t)\| + \|C(t)^\top - B(t)\| \|\lambda^{\text{im}}(t, z^*(t))\| + L_f(t) \|z\|, \end{aligned}$$

where $L_f(t)$ is a (bounded) Lipschitz constant. For each $t \in [0, T]$, we can choose $z^*(t) \in \text{dom } f(t, \cdot)$ such that $\|z^*(t)\| < +\infty$ and $\|\lambda^{\text{im}}(t, z^*(t))\| < +\infty$. Moreover, since the matrix functions are bounded from item 5, and $L_f(\cdot)$ is bounded on $[0, T]$, there exist $M(T), N(T) > 0$ (depending on T) such that $\|f(t, z)\| \leq M(T) + N(T)\|z\|$. In the remaining part of the proof, we shall prove that for each $t \in [0, T]$, the operator $\Psi(t, \cdot)$ satisfies **(A2)**, **(A3)** and **(LG)** in Theorem 1, under the provided assumptions.

[Proof that (LG) in (21) holds]: For each $t \in [0, T]$, since $\Psi(t, \cdot)$ is maximal monotone, $\Psi(t, x)$ is a closed convex set for all $x \in \text{Dom } \Psi(t, \cdot)$ [28, Proposition 20.31]. So, for any $x \in \text{Dom } \Psi(t, \cdot)$, there always exists a least-norm element of the set $\Psi(t, x)$, denoted as $\Psi^0(t, x)$. Therefore, what has to be done is to show that there exists $\gamma \in L_1([0, T], \mathbb{R}_+)$ such that (21) holds, that is,

$$\|\Psi^0(t, x)\| \leq \gamma(t) (1 + \|x\|), \quad \forall x \in \text{Dom } \Psi(t, \cdot), \quad \forall t \in [0, T].$$

For each $t \in [0, T]$, let $x \in \text{Dom } \Psi(t, \cdot)$. Then $x \in \text{dom } \lambda^{\text{im}}(t, \cdot)$ from (24) and we have

$$\|\Psi^0(t, x)\| = \min_{\lambda \in \Psi(t, x)} \|\lambda\| \leq \|C(t)^\top \lambda^{\text{im}}(t, x)\| \leq \|C(t)^\top\| \|\lambda^{\text{im}}(t, x)\|. \quad (25)$$

The first inequality in (25) is obtained because $\lambda^{\text{im}}(t, x) \in \Phi(t, C(t)x)$ using Lemma 6, which implies $C(t)^\top \lambda^{\text{im}}(t, x) \in \Psi(t, x)$. Besides that, $\lambda^{\text{im}}(t, \cdot)$ is a Lipschitz continuous function by Lemma 6. Therefore, for any $x^*(t) \in \text{dom } \lambda^{\text{im}}(t, \cdot)$, it holds that:

$$\|\lambda^{\text{im}}(t, x) - \lambda^{\text{im}}(t, x^*(t))\| \leq L_\lambda(t) \|x - x^*(t)\|,$$

where $L_\lambda(t) > 0$ is a (bounded) Lipschitz constant. Using the triangle inequality of the norm, we get

$$\|\lambda^{\text{im}}(t, x)\| \leq \|\lambda^{\text{im}}(t, x) - \lambda^{\text{im}}(t, x^*(t))\| + \|\lambda^{\text{im}}(t, x^*(t))\| \leq L_\lambda(t) \|x - x^*(t)\| + \|\lambda^{\text{im}}(t, x^*(t))\|.$$

Accordingly, it follows that:

$$\|\lambda^{\text{im}}(t, x)\| \leq L_\lambda(t) \|x\| + L_\lambda(t) \|x^*(t)\| + \|\lambda^{\text{im}}(t, x^*(t))\|. \quad (26)$$

From (25) and (26), it is inferred that:

$$\|\Psi^0(t, x)\| \leq \|C(t)^\top\| (L_\lambda(t) \|x\| + L_\lambda(t) \|x^*(t)\| + \|\lambda^{\text{im}}(t, x^*(t))\|) \leq \gamma(t) (1 + \|x\|),$$

where $\gamma(\cdot) : [0, T] \rightarrow \mathbb{R}_+$, $t \mapsto \gamma(t) := \|C(t)^\top\| \max(L_\lambda(t), L_\lambda(t) \|x^*(t)\| + \|\lambda^{\text{im}}(t, x^*(t))\|)$ for some $x^*(t) \in \text{dom } \lambda^{\text{im}}(t, \cdot)$, and $\gamma \in L_1([0, T], \mathbb{R}_+)$ since $L_\lambda(\cdot), C(\cdot)$ are bounded on $[0, T]$ using item 5.

[Proof that (A3) in (20) holds]: Let $s, t \in [0, T]$ be such that $s \leq t$. Let $x \in C(s)^{-1}(\text{Dom } \Phi(s, \cdot))$. Recall that $C(t)^{-1}(\text{Dom } \Phi(t, \cdot))$ is a closed convex set, so there exists a unique projection of x onto it, denoted as $y = \text{proj}_{C(t)^{-1}(\text{Dom } \Phi(t, \cdot))}(x)$. From Lemma 2, for each $h \in [0, T]$, the domain of $\Phi(h, \cdot)$ is defined by

$$\text{Dom } \Phi(h, \cdot) = \mathcal{H}(h) - F(h)u(h) = \{\beta \in \mathbb{R}^m : \hat{Q}(h)Q(h)\beta + \hat{Q}(h)Q(h)F(h)u(h) \geq 0\}. \quad (27)$$

In addition, it follows from item 1 (iii) that:

$$\text{Dom } \Phi(t, \cdot) \supseteq \mathcal{H}(s) - F(t)u(t) = \{\beta \in \mathbb{R}^m : \hat{Q}(s)Q(s)\beta + \hat{Q}(s)Q(s)F(t)u(t) \geq 0\}. \quad (28)$$

The proof is divided into two cases, depending on which case item 2 holds.

• In the first case, we assume that $\text{rge}(C(h))$ is a constant linear space for all $h \in [0, T]$. Let us start similarly to [20, p.1050]. Let $z = \text{proj}_{\text{rge}(C(t)) \cap \text{Dom } \Phi(t, \cdot)}(C(s)x)$, then there exists $\xi \in \mathbb{R}^m$ such that $z = C(t)\xi$. Using [29, (2.4.12)-p.95], we obtain $\ker(C(s))^\perp = \text{rge}(C(s)^\top)$. It follows from [30, p.29] that $\mathbb{R}^m = \ker C(s) \oplus \text{rge}(C(s)^\top)$. Let $x = x_1 + x_2$ where $x_1 \in \ker C(s)$, $x_2 \in \text{rge}(C(s)^\top)$, then we obtain:

$$\text{dist}(C(s)x, \text{rge}(C(t)) \cap \text{Dom } \Phi(t, \cdot)) = \|C(s)x_2 - C(t)\xi\|. \quad (29)$$

Proceeding as in [20, p.1055], for each $h \in \{s, t\}$, let $G(h)$ be a matrix such that $\ker G(h) = \text{rge}(C(h))$. Since $\text{rge}(C(s)) = \text{rge}(C(t))$, we choose $G(s) = G(t) = G$, a constant matrix. It follows from (27) that:

$$\text{rge}(C(h)) \cap \text{Dom } \Phi(h, \cdot) = \{\beta \in \mathbb{R}^m : G\beta = 0, \hat{Q}(h)Q(h)\beta + \hat{Q}(h)Q(h)F(h)u(h) \geq 0\}, \quad \forall h \in \{s, t\}.$$

Using (28), the following holds:

$$\begin{aligned} \text{rge}(C(t)) \cap \text{Dom } \Phi(t, \cdot) &= \text{rge}(C(s)) \cap \text{Dom } \Phi(t, \cdot) \supseteq \text{rge}(C(s)) \cap (\mathcal{H}(s) - F(t)u(t)) \\ &= \{\beta \in \mathbb{R}^m : G\beta = 0, \hat{Q}(s)Q(s)\beta + \hat{Q}(s)Q(s)F(t)u(t) \geq 0\}. \end{aligned}$$

It is inferred that $\text{dist}(C(s)x, \text{rge}(C(t)) \cap \text{Dom } \Phi(t, \cdot)) \leq \text{dist}(C(s)x, \text{rge}(C(s)) \cap (\mathcal{H}(s) - F(t)u(t)))$ since item 3 holds. Mimicking [20, equations (78), (79)], since $C(s)x \in \text{rge}(C(s)) \cap \text{Dom } \Phi(s, \cdot)$, so $GC(s)x = 0$ and $-\hat{Q}(s)Q(s)C(s)x \leq \hat{Q}(s)Q(s)F(s)u(s)$. Using Hoffman's bound on polyhedral sets (see Lemma A1), there exists a positive number $\alpha(s)$ (depending only on $C(s)$ and $\hat{Q}(s)Q(s)$) such that:

$$\begin{aligned} \text{dist}(C(s)x, \text{rge}(C(s)) \cap (\mathcal{H}(s) - F(t)u(t))) &\leq \alpha(s) \|\max(0, -\hat{Q}(s)Q(s)C(s)x - \hat{Q}(s)Q(s)F(t)u(t))\| \\ &\leq \alpha(s) \|\max(0, \hat{Q}(s)Q(s)F(s)u(s) - \hat{Q}(s)Q(s)F(t)u(t))\| \\ &\leq \alpha(s) \|\hat{Q}(s)Q(s)\| \|F(t)u(t) - F(s)u(s)\| \\ &\leq \bar{M}(T) \|F(t)u(t) - F(s)u(s)\|, \end{aligned}$$

where $\bar{M}(T)$ is a bound of $\alpha(\cdot) \|\hat{Q}(\cdot)Q(\cdot)\|$ on $[0, T]$, and it exists since $\hat{Q}(\cdot)Q(\cdot)$ is bounded due to item 1 holds. Since $F(\cdot)$ and $u(\cdot)$ are AC due to items 5 and 6, so is $F(\cdot)u(\cdot)$. Denote by $(\dot{F}u)$ the derivative of the product of two functions $F(\cdot)$ and $u(\cdot)$. Proceeding as in [20, p.1056], it is inferred that:

$$\text{dist}(C(s)x, \text{rge}(C(t)) \cap \text{Dom } \Phi(t, \cdot)) \leq \text{dist}(C(s)x, \text{rge}(C(s)) \cap (\mathcal{H}(s) - F(t)u(t))) \leq \theta(t) - \theta(s), \quad (30)$$

where $\theta(\cdot) := \bar{M}(T) \int_0^{(\cdot)} \|(\dot{F}u)(\tau)\| d\tau$, is nondecreasing and AC on $[0, T]$. From (29), it is concluded that:

$$\|C(s)x_2 - C(t)\xi\| \leq \theta(t) - \theta(s). \quad (31)$$

Since $\text{rge}(C(s)) = \text{rge}(C(t))$, there exists $\bar{\xi} \in \text{rge}(C(s)^\top)$ such that $C(t)\xi = C(s)\bar{\xi}$. Since $x_2 - \bar{\xi} \in$

$\text{rge}(C(s)^\top)$, there exists a positive number $\bar{\alpha}(s)$ (depending on $C(s)$) satisfying

$$\|x_2 - \bar{\xi}\| \leq \bar{\alpha}(s) \|C(s)x_2 - C(s)\bar{\xi}\| = \bar{\alpha}(s) \|C(s)x_2 - C(t)\xi\| \leq \bar{\alpha}(s)(\theta(t) - \theta(s)), \quad (32)$$

in which the first inequality is derived similarly to [20, p.1050] using Lemma 8, and the last inequality is due to (31). The inequality (32) can be rewritten as follows

$$\|x - (\bar{\xi} + x_1)\| \leq \bar{\alpha}(s)(\theta(t) - \theta(s)).$$

Using item 4, since $(\bar{\xi} + x_1) \in C(s)^{-1}(\text{Dom } \Phi(t, \cdot)) \subseteq C(t)^{-1}(\text{Dom } \Phi(t, \cdot))$, it implies that:

$$\|x - y\| \leq \|x - (\bar{\xi} + x_1)\| \leq \varphi_1(t) - \varphi_1(s), \quad (33)$$

where $\varphi_1(\cdot) := \bar{N}(T)\theta(\cdot)$ with $\bar{N}(T)$ is a bound of $\bar{\alpha}(\cdot)$ on $[0, T]$. Clearly $\varphi_1(\cdot)$ is nondecreasing and AC on $[0, T]$. Then, (33) is equivalent to $\text{dist}(x, C(t)^{-1}(\text{Dom } \Phi(t, \cdot))) \leq \varphi_1(t) - \varphi_1(s)$. Hence,

$$\sup_{x \in C(s)^{-1}(\text{Dom } \Phi(s, \cdot))} \text{dist}(x, C(t)^{-1}(\text{Dom } \Phi(t, \cdot))) \leq \varphi_1(t) - \varphi_1(s), \quad \forall 0 \leq s \leq t \leq T. \quad (34)$$

Because (24) holds, so (34) implies that **(A3)** in Theorem 1 is satisfied.

• In the second case, we assume that $\text{rge}(C(h)^\top)$ is a constant linear space for all $h \in [0, T]$. Since item 3 holds, $C(s)^{-1}(\mathcal{H}(s) - F(t)u(t)) \neq \emptyset$. Using item 1 (iii) and item 4, we obtain the following:

$$\emptyset \neq C(s)^{-1}(\mathcal{H}(s) - F(t)u(t)) \subseteq C(s)^{-1}(\mathcal{H}(t) - F(t)u(t)) = C(s)^{-1}(\text{Dom } \Phi(t, \cdot)) \quad (35)$$

$$\subseteq C(t)^{-1}(\text{Dom } \Phi(t, \cdot)) = \text{Dom } \Psi(t, \cdot). \quad (36)$$

It follows from (36) and Lemma 9 that $\mathcal{H}(s) - F(t)u(t) \subseteq C(s)(\text{Dom } \Psi(t, \cdot))$, which is equivalent to:

$$\text{Dom } \Phi(s, \cdot) \subseteq C(s)(\text{Dom } \Psi(t, \cdot)) + (F(t)u(t) - F(s)u(s)), \quad (37)$$

Since $x \in C(s)^{-1}(\text{Dom } \Phi(s, \cdot))$, $C(s)x \in \text{Dom } \Phi(s, \cdot)$. Therefore, (37) implies that there exists $\bar{y} \in \text{Dom } \Psi(t, \cdot)$ such that $C(s)x = C(s)\bar{y} + F(t)u(t) - F(s)u(s)$ and moreover, the following holds:

$$\text{dist}(C(s)x, C(s)(\text{Dom } \Psi(t, \cdot))) = \text{dist}(C(s)\bar{y} + F(t)u(t) - F(s)u(s), C(s)(\text{Dom } \Psi(t, \cdot))) \quad (38)$$

$$\leq \text{dist}(C(s)\bar{y}, C(s)(\text{Dom } \Psi(t, \cdot))) + \|F(t)u(t) - F(s)u(s)\| \quad (39)$$

$$= \|F(t)u(t) - F(s)u(s)\|. \quad (40)$$

In addition, there exists $z \in \text{Dom } \Psi(t, \cdot)$ such that:

$$\text{dist}(C(s)x, C(s)(\text{Dom } \Psi(t, \cdot))) = \|C(s)x - C(s)z\|. \quad (41)$$

Since $\mathbb{R}^n = \ker C(s) \oplus \text{rge}(C(s)^\top)$, there exist $x_1, z_1 \in \ker C(s)$ and $x_2, z_2 \in \text{rge}(C(s)^\top)$ that satisfy $x = x_1 + x_2$ and $z = z_1 + z_2$. Then, from (41) and (40), it is inferred that:

$$\|C(s)x_2 - C(s)z_2\| \leq \|F(t)u(t) - F(s)u(s)\|. \quad (42)$$

Since $x_2 - z_2 \in \text{rge}(C(s)^\top)$, using Lemma 8, there exists a positive number $\bar{\alpha}(s)$ (as defined in (32)) satisfying $\|x_2 - z_2\| \leq \bar{\alpha}(s)\|C(s)x_2 - C(s)z_2\|$ which can be rewritten as follows:

$$\|x - (z_2 + x_1)\| \leq \bar{\alpha}(s)\|C(s)x_2 - C(s)z_2\|. \quad (43)$$

Since $\text{rge}(C(s)^\top) = \text{rge}(C(t)^\top)$, so $\ker(C(s))^\perp = \ker(C(t))^\perp$, that is, $\ker C(s) = \ker C(t)$. Therefore, $x_1, z_1 \in \ker C(t)$. Since $z \in \text{Dom } \Psi(t, \cdot)$, $C(t)z \in \text{Dom } \Phi(t, \cdot)$. It can be inferred that $C(t)(z_2 + x_1) \in \text{Dom } \Phi(t, \cdot)$, *i.e.*, $(z_2 + x_1) \in \text{Dom } \Psi(t, \cdot)$. From (42) and (43), and recalling that $y = \text{proj}_{C(t)^{-1}(\text{Dom } \Phi(t, \cdot))}(x) = \text{proj}_{\text{Dom } \Psi(t, \cdot)}(x)$ as defined at the beginning of the proof of **(A3)**, we obtain:

$$\text{dist}(x, \text{Dom } \Psi(t, \cdot)) = \|x - y\| \leq \|x - (z_2 + x_1)\| \leq \bar{N}(T)\|F(t)u(t) - F(s)u(s)\|, \quad (44)$$

where $\bar{N}(T)$ is defined after (33) (since $\bar{\alpha}(\cdot)$ is defined exactly as in (32)). Proceeding as in [20, p.1056], it is implied that: $\text{dist}(x, \text{Dom } \Psi(t, \cdot)) \leq \varphi_2(t) - \varphi_2(s)$, with $\varphi_2(\cdot) := \bar{N}(T) \int_0^{(\cdot)} \|(F'u)(\tau)\| d\tau$, is nondecreasing and AC on $[0, T]$. Thus,

$$\sup_{x \in \text{Dom } \Psi(s, \cdot)} \text{dist}(x, \text{Dom } \Psi(t, \cdot)) \leq \varphi_2(t) - \varphi_2(s), \quad \forall 0 \leq s \leq t \leq T,$$

that is, **(A3)** in Theorem 1 is satisfied.

[Proof that (A2) holds]: Let us start similarly to [20, p.1051]. Let $\{t_n, x_n, y_n\}_{n \in \mathbb{N}} \subseteq [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$ be sequences such that $y_n \in \Psi(t_n, x_n)$, $t_n \geq t$, and $\lim_{n \rightarrow \infty} (t_n, x_n, y_n) = (t, x, y)$, for some $t \in [0, T]$ and $x, y \in \mathbb{R}^n$. From Definition 3, we need to prove that $y \in \Psi(t, x)$. Since $y_n \in \Psi(t_n, x_n)$, for each $n \in \mathbb{N}$, there exists

$$z_n \in \Phi(t_n, C(t_n)x_n) \quad (45)$$

such that $y_n = C(t_n)^\top z_n$. By the initial assumption, $\{y_n\}_{n \in \mathbb{N}}$ converges, therefore $\{C(t_n)^\top z_n\}_{n \in \mathbb{N}}$ converges. Since $\Phi(t_n, \cdot)$ is maximal monotone (see Lemma 1), mimicking [20, p.1051] to obtain (46) and (47), with the note that in this case, \mathcal{W} and $\ker B$ in [20] are replaced by $W(t_n)$, the subspace parallel to the affine hull of $\text{Dom } \Phi(t_n, \cdot)$ and $\ker(C(t_n)^\top)$, respectively. In particular, it follows from Lemma 2 that $W(t_n) = \text{aff}(\mathcal{H}(t_n)) \ni 0$ and is thus a vector subspace of \mathbb{R}^m . Then for each $n \in \mathbb{N}$, we get $z_n = z_n^{(1)} + z_n^{(2)}$ where,

$$z_n^{(1)} \in \ker(C(t_n)^\top) \cap W(t_n)^\perp \quad \text{and} \quad z_n^{(2)} \in \left(\ker(C(t_n)^\top) \cap W(t_n)^\perp \right)^\perp, \quad (46)$$

and the following holds:

$$z_n^{(2)} \in \Phi(t_n, C(t_n)x_n). \quad (47)$$

Next, we shall prove that $\{z_n^{(2)}\}_{n \in \mathbb{N}}$ is bounded. Mimicking [20, p.1052], suppose on the contrary

that $\{z_n^{(2)}\}_{n \in \mathbb{N}}$ is unbounded and let us consider the sequence $\left\{ \frac{z_n^{(2)}}{\|z_n^{(2)}\|} \right\}_{n \in \mathbb{N}}$. Since this sequence

is bounded, there exists $N_1 \subseteq \mathbb{N}$ such that $\left\{ \frac{z_n^{(2)}}{\|z_n^{(2)}\|} \right\}_{n \in N_1}$ converges. Define

$$\xi_\infty = \lim_{\substack{n \rightarrow \infty \\ n \in N_1}} \frac{z_n^{(2)}}{\|z_n^{(2)}\|}, \quad \|\xi_\infty\| = 1. \quad (48)$$

Follows from (46) and the fact that $\{C(t_n)^\top z_n^{(2)}\}_{n \in \mathbb{N}}$ converges, then $\lim_{n \rightarrow \infty} C(t_n)^\top z_n^{(2)} = \gamma$ for some $\gamma \in \mathbb{R}^n$. Because $\|z_n^{(2)}\|$ is unbounded and $C(\cdot)$ is right-continuous due to item 5, so we have

$$C(t)^\top \xi_\infty = \left(\lim_{\substack{n \rightarrow \infty \\ n \in N_1}} C(t_n)^\top \right) \left(\lim_{\substack{n \rightarrow \infty \\ n \in N_1}} \frac{z_n^{(2)}}{\|z_n^{(2)}\|} \right) = \lim_{\substack{n \rightarrow \infty \\ n \in N_1}} \frac{C(t_n)^\top z_n^{(2)}}{\|z_n^{(2)}\|} = 0.$$

Thus, it is inferred:

$$\xi_\infty \in \ker (C(t)^\top). \quad (49)$$

Using Lemma 2, the inner semicontinuity from right of $\mathcal{H}(\cdot)$, and right-continuity of $F(\cdot), u(\cdot)$ on $[0, T]$ derived from items 5, 6, then $\text{Dom } \Phi(\cdot, \cdot) : [0, T] \rightrightarrows \mathbb{R}^m, \hat{t} \mapsto \text{Dom } \Phi(\hat{t}, \cdot)$, is inner semicontinuous from right on $[0, T]$. From this, let us prove that

$$\langle \xi_\infty, \eta \rangle \leq 0, \quad \forall \eta \in \text{Dom } \Phi(t, \cdot). \quad (50)$$

Indeed, take any $\eta \in \text{Dom } \Phi(t, \cdot)$. Then, there exists a set $N_2 \in N_\infty$, and a sequence $\{\eta_n\}_{n \in N_2} \subset \text{Dom } \Phi(t_n, \cdot)$ such that $\lim_{n \rightarrow \infty} \eta_n = \eta$. There always exists a bounded sequence $\{\xi_n\}_{n \in N_2} \subset \Phi(t_n, \eta_n)$. Since $\Phi(t_n, \cdot)$ is maximal monotone (see Lemma 1), and (47) holds, then for each $n \in N_2$:

$$\left\langle \frac{z_n^{(2)} - \xi_n}{\|z_n^{(2)}\|}, C(t_n)x_n - \eta_n \right\rangle \geq 0. \quad (51)$$

Let $n \in N_1 \cap N_2$ tend to infinity for both sides of (51), notice that $N_1 \cap N_2$ coincides with N_1 beyond some position. Using the continuity of the inner product $\langle \cdot, \cdot \rangle$ [27, p.4], the right-continuity of $C(\cdot)$, the boundedness of $\{\xi_n\}_{n \in N_2}$, and (49), then $\langle \xi_\infty, \eta \rangle \leq 0$. Since η is any choice in $\text{Dom } \Phi(t, \cdot)$, hence (50) holds. The expressions (49), (50) and $\text{Dom } \Phi(t, \cdot)$ act as [20, (71), (72) and $\text{im}(\mathcal{M} + D) - v(t)$], respectively. Proceeding as in [20, p.1053] with note that Assumption 3 holds, we get

$$\xi_\infty \in \ker (C(t)^\top) \cap W(t)^\perp, \quad (52)$$

where $W(t) := \text{aff}(\mathcal{H}(t))$. Using [29, (2.4.14), p.95] and item 5, then $\ker (C(\cdot)^\top) \cap W(\cdot)^\perp \equiv (\text{rge}(C(\cdot)) + \text{aff}(\mathcal{H}(\cdot)))^\perp$ is inner semicontinuous from right on $[0, T]$. Therefore, it follows from (46) and (48) that:

$$\xi_\infty \in \left(\ker (C(t)^\top) \cap W(t)^\perp \right)^\perp. \quad (53)$$

Indeed, let $\nu \in \ker (C(t)^\top) \cap W(t)^\perp$. There exists $N_3 \in N_\infty$, and $\{\nu_n\}_{n \in N_3} \subset \ker (C(t_n)^\top) \cap W(t_n)^\perp$ such that $\lim_{n \rightarrow \infty} \nu_n = \nu$. Notice that $N_1 \cap N_3$ coincides with N_1 beyond some position. Therefore, it is inferred that:

$$\langle \xi_\infty, \nu \rangle = \left\langle \lim_{\substack{n \rightarrow \infty \\ n \in N_1 \cap N_3}} \frac{z_n^{(2)}}{\|z_n^{(2)}\|}, \lim_{\substack{n \rightarrow \infty \\ n \in N_1 \cap N_3}} \nu_n \right\rangle = \lim_{\substack{n \rightarrow \infty \\ n \in N_1 \cap N_3}} \left\langle \frac{z_n^{(2)}}{\|z_n^{(2)}\|}, \nu_n \right\rangle = 0, \quad (54)$$

where the first, second, and last equality comes from (48), the continuity of $\langle \cdot, \cdot \rangle$, and (46), respectively. In view of (52), (53) and (48), we get a contradiction. Thus, $\{z_n^{(2)}\}_{n \in \mathbb{N}}$ is bounded. There exists a set $N_4 \subseteq \mathbb{N}$ such that $\{z_n^{(2)}\}_{n \in N_4}$ converges, say to z . It follows from (46), the initial assumption, and the right-continuity of $C(\cdot)$ that:

$$y = \lim_{\substack{n \rightarrow \infty \\ n \in N_4}} y_n = \lim_{\substack{n \rightarrow \infty \\ n \in N_4}} C(t_n)^\top z_n = \lim_{\substack{n \rightarrow \infty \\ n \in N_4}} C(t_n)^\top z_n^{(2)} = C(t)^\top z.$$

Notice that, if $z \in \Phi(t, C(t)x)$ then $y \in C(t)^\top \Phi(t, C(t)x) = \Psi(t, x)$ and the proof is complete. So, it remains to prove that $z \in \Phi(t, C(t)x)$. Indeed, from (47) and (13), then for each $n \in \mathbb{N}$:

$$z_n^{(2)} \in \left(\mathcal{N}_{\mathcal{K}^*(t_n) - F(t_n)u(t_n)} + D(t_n) \right)^{-1} (C(t_n)x_n), \quad (55)$$

which is rewritten as $z_n^{(2)} \in \mathcal{N}_{\mathcal{K}^*(t_n)-F(t_n)u(t_n)} \left(C(t_n)x_n - D(t_n)z_n^{(2)} \right)$. Following Definition 1, this is equivalent to:

$$\left\langle z_n^{(2)}, \zeta_n - H(t_n) \right\rangle \leq 0, \quad \forall \zeta_n \in \mathcal{K}^*(t_n), \quad (56)$$

where $H(t_n) := C(t_n)x_n - D(t_n)z_n^{(2)} + F(t_n)u(t_n)$. Let us prove that $\langle z, \zeta - H(t) \rangle \leq 0, \forall \zeta \in \mathcal{K}^*(t)$ where $H(t) := C(t)x - D(t)z + F(t)u(t)$. Take any $\zeta \in \mathcal{K}^*(t)$. Since $\mathcal{K}^*(\cdot)$ is inner semicontinuous from right on $[0, T]$ due to item 5, there exists $N_5 \in \mathbb{N}_\infty$, and $\{\zeta_n\}_{n \in N_5} \subset \mathcal{K}^*(t_n)$ such that $\lim_{n \rightarrow \infty} \zeta_n = \zeta$. Since $\langle \cdot, \cdot \rangle$ is continuous, $C(\cdot), D(\cdot), F(\cdot), u(\cdot)$ are right-continuous from items 5, 6, and (56) holds, then the following holds:

$$\langle z, \zeta - H(t) \rangle = \lim_{\substack{n \rightarrow \infty \\ n \in N_4 \cap N_5}} \left\langle z_n^{(2)}, \zeta_n - H(t_n) \right\rangle \leq 0.$$

Since ζ is any choice in $\mathcal{K}^*(t)$, so $\langle z, \zeta - H(t) \rangle \leq 0, \forall \zeta \in \mathcal{K}^*(t)$, i.e., $z \in \mathcal{N}_{\mathcal{K}^*(t)-F(t)u(t)}(C(t)x - D(t)z)$. Proceeding as (55), it is concluded that $z \in \Phi(t, C(t)x)$. \square

Remark 2. Consider the item 2 in Theorem 2. It can be removed while still ensuring the validity of Theorem 2 if the inclusion in item 4 turns into an equality, i.e., $C(s)^{-1}(\text{Dom } \Phi(t, \cdot)) = C(t)^{-1}(\text{Dom } \Phi(t, \cdot))$, for all $0 \leq s \leq t \leq T$. Because in this case, the value $z_2 + x_1$ in (44) is still ensured to belong to $\text{Dom } \Psi(t, \cdot)$. Indeed, let us return to the part immediately before (44). Since $z \in \text{Dom } \Psi(t, \cdot)$, $z \in C(t)^{-1}(\text{Dom } \Phi(t, \cdot)) = C(s)^{-1}(\text{Dom } \Phi(t, \cdot))$. Therefore, $C(s)(z_2 + x_1) = C(s)z \in \text{Dom } \Phi(t, \cdot)$, which implies $z_2 + x_1 \in C(s)^{-1}(\text{Dom } \Phi(t, \cdot)) = C(t)^{-1}(\text{Dom } \Phi(t, \cdot)) = \text{Dom } \Psi(t, \cdot)$.

Remark 3. When $\text{Dom } \Psi(t, \cdot)$ (defined in (24)) is a constant set for all $t \in [0, T]$, then assumptions on $F(\cdot), u(\cdot)$ in items 5, 6 in Theorem 2 are relaxed to $F(\cdot)$ being right-continuous and bounded on $[0, T]$, $u(\cdot)$ being right-continuous on $[0, T]$ and locally integrable, and the boundedness of $\hat{Q}(\cdot)Q(\cdot)$ in item 1 is lifted. Because the AC of $F(\cdot), u(\cdot)$ (resp. the boundedness of $\hat{Q}(\cdot)Q(\cdot)$) is only used in the part ‘‘Proof that (A3) in (20) holds’’ (specifically, in (30) and after (44) (resp. immediately before (30))), where (A3) always holds when $\text{Dom } \Psi(t, \cdot)$ is constant. With the properties of $F(\cdot)$ and $u(\cdot)$ as given, the remaining relevant parts in Theorem 2 are still correct.

In Proposition 1 in Section 2.1, sufficient conditions are provided to ensure the well-posedness of (3) when $D(t) \succcurlyeq \rho I_m$, for some $\rho > 0$, and Assumption 1 holds. The following result is a consequence of Theorem 2, particularly useful when $D(\cdot)$ is discontinuous or $\mathcal{K}^*(\cdot)$ is not inner semicontinuous on $[0, T]$. It is very close to Proposition 1.

Corollary 1. Consider the TVLCCS (3). Assume that $\mathcal{K}(t)$ is a polyhedral cone for all $t \in [0, T]$. Assume further that all data $(A(t), B(t), C(t), D(t), E(t), F(t))$ in (3) are bounded on $[0, T]$ with $D(t) \succcurlyeq \rho I_m$ for all $t \in [0, T]$ and some $\rho > 0$, $u(\cdot) \in L_1([0, T], \mathbb{R}^u)$, and mappings $C(\cdot), D(\cdot), F(\cdot), u(\cdot)$ are right-continuous, $\mathcal{K}^*(\cdot)$ is inner semicontinuous from right on $[0, T]$. Then, there exists a unique solution $x \in AC([0, T], \mathbb{R}^n)$ to (3) for every $x(0) = x_0 \in \mathbb{R}^n$.

Proof. Let us examine conditions in Theorem 2. For each $t \in [0, T]$, since $D(t) \succcurlyeq \rho I_m$, so $D(t) + D(t)^\top \succcurlyeq 2\rho I_m$ which implies that Assumption 2 holds. Since $\mathcal{K}(t)$ is a polyhedral cone, there exists $Q(t) \in \mathbb{R}^{j \times m}$ such that $\mathcal{K}(t) = \{Q(t)^\top \xi \mid \xi \in \mathbb{R}_+^j\}$. It can be verified that $\text{SOL}(0, Q(t)D(t)Q(t)^\top) = \ker(Q(t)^\top) \cap \mathbb{R}_+^j$. Therefore, $(\text{SOL}(0, Q(t)D(t)Q(t)^\top))^* = (\ker(Q(t)^\top) \cap \mathbb{R}_+^j)^* = \text{rge}(Q(t)) + \mathbb{R}_+^j$ using [21, Corollary 11.25, Example 6.23], [29, (2.4.14)-p95]. Hence, we get $\mathcal{H}(t) = \{\xi \in \mathbb{R}^m : Q(t)\xi \in (\text{SOL}(0, Q(t)D(t)Q(t)^\top))^*\} = \mathbb{R}^m$ for all $t \in [0, T]$. In addition, it follows from Lemma 2 that $\text{Dom } \Phi(t, \cdot) = \mathbb{R}^m = \mathcal{H}(s) - F(t)u(t)$, for all $0 \leq s \leq t \leq T$. Therefore, Assumption 3,

items 3 and 4 hold. Moreover, since item 4 still holds when the inclusion becomes equality (*i.e.*, $C(s)^{-1}(\text{Dom } \Phi(t, \cdot)) = C(t)^{-1}(\text{Dom } \Phi(t, \cdot)) = \mathbb{R}^n$, $\forall 0 \leq s \leq t \leq T$), according to Remark 2, item 2 becomes irrelevant. Since $C(t)^{-1}(\text{Dom } \Phi(t, \cdot)) = \mathbb{R}^n$ for all $t \in [0, T]$, using Remark 3, then items 1, 5 and 6 are guaranteed by the given assumptions and $\mathcal{H}(t) = \mathbb{R}^m$ for all $t \in [0, T]$. \square

2.3 Comments and Examples

2.3.1 Comments on the assumptions in Theorem 2

This section provides analysis to offer a clearer understanding of assumptions made in Theorem 2.

(a) Let us consider an example $(D(t), \mathcal{K}(t))$ such that items 1, 5 in Theorem 2 hold. Let $D(t) = \text{diag}(2t + 1, 0) \succcurlyeq 0$, and $\mathcal{K}(t) = \{[x_1 \ x_2]^\top \in \mathbb{R}^2 : 3x_2 \geq 2tx_1, x_1 + x_2 \geq 0\}$ for all $t \in [0, T]$. It can be checked that item 1 is satisfied by the following:

$$\mathcal{H}(t) = \left\{ x \in \mathbb{R}^2 : \hat{Q}(t)Q(t)x \geq 0 \right\} = \{[x_1 \ x_2]^\top \in \mathbb{R}^2 : x_2 \geq 0\}, \quad \forall 0 \leq t \leq T,$$

where $Q(t) = \begin{bmatrix} 1 & \frac{2t}{3} \\ -1 & 1 \end{bmatrix}$, $\hat{Q}(t) = \begin{bmatrix} 1 & 1 \end{bmatrix}$ for all $t \in [0, T]$, and $\hat{Q}(\cdot)Q(\cdot)$ is bounded on $[0, T]$. Clearly $(\text{rge}(C(t)) + \text{aff}(\mathcal{H}(t)))^\perp = 0$ for all $C(t)$, and $\mathcal{K}^*(t) = \{x \in \mathbb{R}^2 : Q(t)x \geq 0\}$, for all $t \in [0, T]$. For any $t^* \in [0, T]$, any $x^* \in \mathcal{K}^*(t^*)$ and any sequence $\{t_n\}_{n \in \mathbb{N}} \subset [0, T]$ such that $\lim_{n \rightarrow \infty} t_n = t^*$,

set $x_n := Q(t_n)^{-1}Q(t^*)x^*$ where $Q(t_n)^{-1} = \begin{bmatrix} \frac{3}{2t_n+3} & \frac{-2t_n}{2t_n+3} \\ \frac{3}{2t_n+3} & \frac{2t_n+3}{2t_n+3} \end{bmatrix}$, then the sequence $\{x_n\}_{n \in \mathbb{N}}$ satisfies $x_n \in \mathcal{K}^*(t_n)$ for all $n \in \mathbb{N}$, and $\lim_{n \rightarrow \infty} x_n = x^*$. It follows that $\mathcal{H}(\cdot)$, $(\text{rge}(C(\cdot)) + \text{aff}(\mathcal{H}(\cdot)))^\perp$, and $\mathcal{K}^*(\cdot)$ are inner semicontinuous on $[0, T]$. Since $D(\cdot)$ is bounded and continuous on $[0, T]$, assumptions on $(D(t), \mathcal{K}(t))$ in item 5 in Theorem 2 are satisfied.

(b) Let us consider an example $(C(t), D(t), \mathcal{K}(t), F(t)u(t))$ such that $\text{rge}(C(t))$ is not constant, but $\text{rge}(C(t)^\top)$ is a constant linear space and Assumption 3, items 1, 3 and 4 in Theorem 2 hold. Let

$$C(t) = \begin{bmatrix} 1 & 0 \\ \sin(t) & \sin(t) \\ 0 & 1 \end{bmatrix}, \quad D(t) = \text{diag}(t+1, 1, 0), \quad \mathcal{K}(t) = \mathbb{R}_+^3, \quad \text{and } F(t)u(t) = \begin{bmatrix} a(t) \\ b(t) \\ c(t) \end{bmatrix} \text{ for all } t \in [0, T],$$

for some functions $a(\cdot), b(\cdot), c(\cdot) : [0, T] \rightarrow \mathbb{R}$ such that $c(t) > 0$ for all $t \in [0, T]$. We obtain:

$$\text{rge}(C(t)) = \begin{cases} \{[x \ 0 \ z]^\top, \forall x, z \in \mathbb{R}\}, & \text{if } t = k\pi, \text{ for some } k \in \mathbb{N} \\ \{[x \ y \ z]^\top \in \mathbb{R}^3 : y = \sin(t)(x + z)\}, & \text{otherwise} \end{cases},$$

is not a constant linear space for all $t \in [0, T]$. However, $\text{rge}(C(t)^\top) = \mathbb{R}^2$ for all $t \in [0, T]$. Since $D(t) = D(t)^\top \succcurlyeq 0$ for all $t \in [0, T]$, using Remark 1, it follows that for all $0 \leq s \leq t \leq T$:

$$\text{Dom } \Phi(t, \cdot) = (\mathcal{K}(t) \cap \ker D(t))^* - F(t)u(t) = \{[x \ y \ z]^\top \in \mathbb{R}^3 : z \geq -c(t)\} = \mathcal{H}(s) - F(t)u(t),$$

where $\mathcal{H}(t) = (\mathcal{K}(t) \cap \ker D(t))^* = [\mathbb{R} \ \mathbb{R} \ \mathbb{R}_+]^\top$ is constant for all $t \in [0, T]$. It can be concluded that Assumption 3, and items 1, 3 hold. Moreover, item 4 is satisfied since for all $0 \leq s \leq t \leq T$:

$$C(s)^{-1}(\text{Dom } \Phi(t, \cdot)) = \{[x_1 \ x_2]^\top \in \mathbb{R}^2 : x_2 \geq -c(t)\} = C(t)^{-1}(\text{Dom } \Phi(t, \cdot)).$$

(c) Assumption 2 is implied by the passivity of the considered system. Indeed, suppose that (3) is passive with the storage function $V(t, x) = \frac{1}{2}x^\top P(t)x$, for some continuously differentiable function $P(t) = P(t)^\top \succ 0$. Using Definition 5, this is equivalent to the LMI in (7), which implies that $\ker(D(t) + D(t)^\top) \subseteq \ker(P(t)B(t) - C(t)^\top)$, for all $t \in [0, T]$ [14, Lemma 1-(ii)].

(d) In Theorem 2, Assumption 2 is supposed to be true with $P(t)$ the identity matrix. This is without loss of generality. Indeed, suppose that the TVLCCS (3) satisfies Assumption 2 with $P(t)$ being a continuously differentiable function such that $P(t) = P(t)^\top \succ 0$. Let $z(t) = R(t)x(t)$, where $R(t)$ is a square root of $P(t)$. Then (3) can be written equivalently as:

$$\begin{cases} \dot{z}(t) = \bar{A}(t)z(t) + R(t)B(t)\lambda(t) + R(t)E(t)u(t), \\ w(t, z) = C(t)R(t)^{-1}z(t) + D(t)\lambda(t) + F(t)u(t), \\ \mathcal{K}(t) \ni \lambda(t) \perp w(t, z) \in \mathcal{K}^*(t), \end{cases} \quad (57)$$

where $\bar{A}(t) := \left(R(t)A(t)R(t)^{-1} + \dot{R}(t)R(t)^{-1} \right)$. Since Assumption 2 holds, it follows from (16) that:

$$\ker \left(D(t) + D(t)^\top \right) \subseteq \ker \left(P(t)B(t) - C(t)^\top \right) \subseteq \ker \left(R(t)B(t) - (C(t)R(t)^{-1})^\top \right).$$

This means that (57) satisfies Assumption 2 where an identity matrix is a solution of (16). Using Theorem 2, the well-posedness of (57) is guaranteed, and thus, so is the well-posedness of (3).

(e) Assumption 3 is the standard condition that appears in most results of well-posedness of DIs.

2.3.2 Discussion on the extension and differences with respect to [20], [14]

Theorem 2 is an extension of the well-posedness proof in the seminal work [20], which is well suited to some classes of time-varying LCS, see [4, Example 9] (contrarily to other articles which rely on Vladimirov's pseudistance [31, 32]). This section will clarify the difference and the extension of Theorem 2 compared to [20, Theorems 24, 26], [14, Theorem 10], which are presented in Section 1.

(a) Theorem 2 is the extension of [20, Theorem 26] and [14, Theorem 10]: Firstly, in the framework of [20], all matrix functions in (3) are constants and $\mathcal{K}(t) = \mathbb{R}_+^m$ for all $t \geq 0$. Therefore, items 1, 2, 4, 5 in Theorem 2 hold trivially. In addition, it can be noted that if Assumption 3 holds then item 3 holds since $\mathcal{H}(\cdot)$ is a constant set in this case. As mentioned in Section 2.3.1, Assumption 2 is a consequence of the passivity of the quadruple (A, B, C, D) , which can be used to replace Assumption 2. Hence, assumptions in Theorem 2 are simplified to assumptions in [20, Theorem 26]. Using the same explanation, we can also recover [14, Theorem 10] from Theorem 2 with very similar assumptions. Particularly, with this approach, the input u is required to be an AC function, which is less restrictive compared to the condition of u being a Bohl function in [14].

(b) The approach to the problem is different: Let us consider (3) where all matrix functions and the cone \mathcal{K} are constant. In [20, Theorems 24, 26], the author's approach is to consider the DI:

$$\dot{x}(t) \in -H(t, x(t)) + Eu(t),$$

where $H(t, x) := -Ax + B \left(\mathcal{N}_{\mathcal{K}^*}^{-1} + D \right)^{-1} (Cx + Fu(t))$, for all $t \geq 0$. After that, assumptions are given such that $H(t, \cdot)$ satisfies assumptions **(A1)**, **(A2)**, **(A3)** and **(LG)** in [20, Theorem 22]. However, in Theorem 2, the author's approach is to consider the DI:

$$\dot{x}(t) \in -\Psi(t, x(t)) + f(t, x(t)) + E(t)u(t),$$

using the method proposed in [15], where $\Psi(t, x) := C(t)^\top \left(\mathcal{N}_{\mathcal{K}^*(t) - F(t)u(t)}^{-1} + D(t) \right)^{-1} (C(t)x)$, $f(t, x) := A(t)x + (C(t)^\top - B(t)) \lambda^{\text{im}}(t, x)$ (λ^{im} is given in Lemma 4) for all $t \geq 0$. After that, assumptions are made such that $\Psi(t, \cdot)$ satisfies assumptions **(A1)**, **(A2)**, **(A3)** and **(LG)** in Theorem 1, a (slightly) extended version of [20, Theorem 23].

(c) Extend the correct domain for the hypothesis in [20, Theorem 26]: To confirm the validity of the assumption **(LG)** (see in (21)), the main idea in Theorem 2 comes from Lemma 6, that is, $\lambda^{\text{im}}(t, \cdot)$ is a Lipschitz continuous function for all $t \in [0, T]$. However, in [20, Theorems 24, 26], the main idea comes from the assumption that: There exists $\alpha > 0$ such that

$$\|B((\mathcal{N}_{\mathcal{K}^*}^{-1} + D)^{-1})^0(\eta)\| \leq \alpha(1 + \|\eta\|), \quad \forall \eta \in \text{Dom}(\mathcal{N}_{\mathcal{K}^*}^{-1} + D)^{-1}.$$

In [20, Theorem 26], this assumption holds when $\mathcal{K} = \mathbb{R}_+^m$. Moreover, we can prove that this assumption is also true when \mathcal{K} is a closed convex nonempty cone, by using the idea proposed in [15]. Indeed, for each $\eta \in \text{Dom}(\mathcal{N}_{\mathcal{K}^*}^{-1} + D)^{-1}$:

$$(\mathcal{N}_{\mathcal{K}^*}^{-1} + D)^{-1}(\eta) = (\mathcal{N}_{\mathcal{K}^* - Fu(t)}^{-1} + D)^{-1}(\eta - Fu(t)) = \Phi(t, \eta - Fu(t)) = \Phi(t, \bar{\eta}),$$

where $\bar{\eta} = \eta - Fu(t) \in \text{Dom} \Phi(t, \cdot)$. Therefore, we have

$$\|B((\mathcal{N}_{\mathcal{K}^*}^{-1} + D)^{-1})^0(\eta)\| = \|B\Phi^0(t, \bar{\eta})\|. \quad (58)$$

Using the same proof as [15, Lemmas 2, 3] (see Lemmas 3, 6), we get the following results:

1. For each $t \in [0, T]$ and $x \in \text{Dom} \Phi(t, \cdot)$, consider $\lambda_1, \lambda_2 \in \Phi(t, x)$, then $\lambda_1 - \lambda_2 \in \ker(D + D^\top)$.
2. Due to item 1 as aforementioned, for each $t \in [0, T]$ and $x \in \text{Dom} \Phi(t, \cdot)$, let $\bar{\lambda}^{\text{im}}(t, x) = \mathcal{P}_D(\Phi(t, x))$ then $\bar{\lambda}^{\text{im}}(t, x)$ is the least-norm element of the set $\Phi(t, x)$ and its dependence on x is Lipschitz continuous.

Note that, item 2 of the foregoing holds under assumptions that $D \succcurlyeq 0$ and for every $t \in [0, T]$, and each $x \in \text{Dom} \Phi(t, \cdot)$ then $\text{rge}(D + D^\top) \cap \Phi(t, x) \neq \emptyset$. As shown in Lemma 5, this assumption is always true. Thus, we obtain:

$$\|B\Phi^0(t, \bar{\eta})\| = \|B\bar{\lambda}^{\text{im}}(t, \bar{\eta})\| \leq \|B\| \|\bar{\lambda}^{\text{im}}(t, \bar{\eta})\|. \quad (59)$$

The boundedness of $\bar{\lambda}^{\text{im}}(t, \bar{\eta})$ is shown as:

$$\begin{aligned} \|\bar{\lambda}^{\text{im}}(t, \bar{\eta})\| &\leq \|\bar{\lambda}^{\text{im}}(t, \bar{\eta}) - \bar{\lambda}^{\text{im}}(t, \bar{\eta}^*(t))\| + \|\bar{\lambda}^{\text{im}}(t, \bar{\eta}^*(t))\| \\ &\leq L(t)\|\bar{\eta}\| + L(t)\|\bar{\eta}^*(t)\| + \|\bar{\lambda}^{\text{im}}(t, \bar{\eta}^*(t))\|, \end{aligned} \quad (60)$$

for some $\bar{\eta}^*(t) \in \text{Dom} \Phi(t, \cdot)$, $L(t)$ is a (bounded) Lipschitz constant. It is verified that there exists $\eta^* \in \text{Dom}(\mathcal{N}_{\mathcal{K}^*}^{-1} + D)^{-1}$ satisfying the following:

$$\bar{\eta}^*(t) = \eta^* - Fu(t) \quad \text{and} \quad \bar{\lambda}^{\text{im}}(t, \bar{\eta}^*(t)) = \Phi^0(t, \bar{\eta}^*(t)) = ((\mathcal{N}_{\mathcal{K}^*}^{-1} + D)^{-1})^0(\eta^*). \quad (61)$$

From (58), (59), (60) and (61), it is concluded that:

$$\begin{aligned} \|B((\mathcal{N}_{\mathcal{K}^*}^{-1} + D)^{-1})^0(\eta)\| &\leq \|B\| (L^*\|\eta - Fu(t)\| + L^*\|\eta^* - Fu(t)\| + \|((\mathcal{N}_{\mathcal{K}^*}^{-1} + D)^{-1})^0(\eta^*)\|) \\ &\leq \|B\| (L^*\|\eta\| + 2L^*\|Fu(t)\| + L^*\|\eta^*\| + \|((\mathcal{N}_{\mathcal{K}^*}^{-1} + D)^{-1})^0(\eta^*)\|) \\ &\leq \alpha(1 + \|\eta\|), \end{aligned}$$

where $\alpha := \|B\| \max\{L^*, 2cL^* + L^*\|\eta^*\| + \|((\mathcal{N}_{\mathcal{K}^*}^{-1} + D)^{-1})^0(\eta^*)\|\}$ is a positive number, and L^*, c are bounds of $L(\cdot)$ and $Fu(\cdot)$ on $[0, T]$, respectively.

2.3.3 Nonsmooth time-varying circuit example

Recall that the constitutive law for an *ideal* time-dependent capacitor $\mathbf{C}(t)$ is, $\mathbf{Q}(t) = \mathbf{C}(t)\mathbf{V}(t)$, where $\mathbf{Q}(t)$, $\mathbf{C}(t)$, and $\mathbf{V}(t)$ denote, respectively, the charge, the capacitance, and the voltage across the capacitor at time t . By definition, the derivative of the electric charge is the current flowing through the capacitor. Hence, for a time-dependent capacitor $\mathbf{I}(t) = \dot{\mathbf{Q}}(t) = \dot{\mathbf{C}}(t)\mathbf{V}(t) + \mathbf{C}(t)\dot{\mathbf{V}}(t)$.

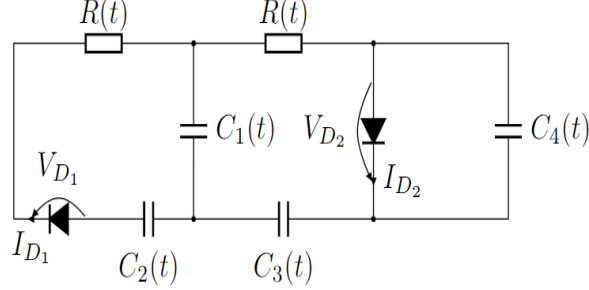


Figure 1: Time-varying RC circuit with ideal diodes.

Let us consider as state variables the vector $x(t) := [x_1(t), \dots, x_4(t)]^\top$, where x_1, \dots, x_3 denote the electrical charges in the capacitors \mathbf{C}_1 , \mathbf{C}_2 and \mathbf{C}_3 , x_4 denotes the voltage across the capacitor \mathbf{C}_4 , respectively. Then, assuming that $\mathbf{C}_i(t) = \mathbf{C}_j(t) > 0$ for all $i, j \in \{1, \dots, 4\}$ and $\mathbf{R}(t) > 0$ for all $t \geq 0$, the equations describing the time evolution of the circuit in Figure 1 are given as:

$$\left\{ \begin{array}{l} \begin{array}{l} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{array} = \underbrace{\begin{bmatrix} -\frac{2}{\mathbf{R}(t)\mathbf{C}(t)} & -\frac{1}{\mathbf{R}(t)\mathbf{C}(t)} & \frac{1}{\mathbf{R}(t)\mathbf{C}(t)} & -\frac{1}{\mathbf{R}(t)} \\ -\frac{1}{\mathbf{R}(t)\mathbf{C}(t)} & -\frac{1}{\mathbf{R}(t)\mathbf{C}(t)} & 0 & 0 \\ \frac{1}{\mathbf{R}(t)\mathbf{C}(t)} & 0 & -\frac{1}{\mathbf{R}(t)\mathbf{C}(t)} & \frac{1}{\mathbf{R}(t)} \\ -\frac{1}{\mathbf{R}(t)\mathbf{C}(t)^2} & 0 & \frac{1}{\mathbf{R}(t)\mathbf{C}(t)^2} & -\left(\frac{1}{\mathbf{R}(t)\mathbf{C}(t)} + \frac{\dot{\mathbf{C}}(t)}{\mathbf{C}(t)}\right) \end{bmatrix}}_{A(t)} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{\mathbf{R}(t)} & 0 \\ \frac{1}{\mathbf{R}(t)} & 0 \\ 0 & 0 \\ 0 & \frac{1}{\mathbf{C}(t)} \end{bmatrix}}_{B(t)} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}, \\ \\ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{1}{\mathbf{R}(t)\mathbf{C}(t)} & -\frac{1}{\mathbf{R}(t)\mathbf{C}(t)} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{C(t)} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{\mathbf{R}(t)} & 0 \\ 0 & 0 \end{bmatrix}}_{D(t)} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}, \\ \mathbb{R}_+^2 \ni \lambda \perp w \in \mathbb{R}_+^2, \end{array} \right. \quad (62)$$

where $\lambda_1 = -V_{D_1}$, $w_1 = I_{D_1}$, $\lambda_2 = I_{D_2}$ and $w_2 = -V_{D_2}$. Since $D(t) \succ 0$, in the following, we shall use Theorem 2 to show the existence and uniqueness of an AC solution to (62). Firstly, the matrix $P(t) = P(t)^\top = \text{diag}(1, 1, 1, \mathbf{C}(t)) \succ 0$ satisfies Assumption 2 if and only if $\mathbf{C}(\cdot)$ is a continuously differentiable function with $\mathbf{C}(t) > 0$ for all $t \in [0, T]$. Then, we can rewrite equivalently (62) as:

$$\left\{ \begin{array}{l} \dot{z}(t) = \left(\mathbf{R}(t)\mathbf{A}(t)\mathbf{R}(t)^{-1} + \dot{\mathbf{R}}(t)\mathbf{R}(t)^{-1} \right) z(t) + \mathbf{R}(t)\mathbf{B}(t)\lambda(t), \\ w(t, z) = \underbrace{\mathbf{C}(t)\mathbf{R}(t)^{-1}}_{\bar{\mathbf{C}}(t)} z(t) + \mathbf{D}(t)\lambda(t), \\ 0 \leq \lambda(t) \perp w(t, z) \geq 0, \end{array} \right. \quad (63)$$

where $z(t) = R(t)x(t)$ and $R(t) = R(t)^\top = \sqrt{P(t)} = \text{diag}(1, 1, 1, \sqrt{\mathbf{C}(t)}) \succ 0$. Now, we shall check that (63) satisfies all assumptions in Theorem 2.

- a. It is verified that (63) satisfies Assumption 2 with a solution being an identity matrix if $\mathbf{C}(\cdot)$ is continuously differentiable, and $\mathbf{C}(t) > 0$ for all $t \in [0, T]$ (see **(d)** in Section 2.3.1).
- b. For each $t \geq 0$, there exist $Q(t) = \text{diag}(1, 1)$ and $\hat{Q}(t) = [0 \ 1]$ such that items 1 (i), 1 (ii) in Theorem 2 hold. Since $D(t) = D(t)^\top \succ 0$, using Remark 1 then the following is obtained, and thus item 1 (iii) holds.

$$\mathcal{H}(t) = (\mathcal{K}(t) \cap \ker D(t))^* = (\{[0 \ 1]^\top \xi, \text{ with } \xi \in \mathbb{R}_+\})^* = \{[x \ y]^\top \in \mathbb{R}^2 : y \geq 0\}, \forall t \geq 0.$$

- c. For each $t \geq 0$, $\text{Dom } \Phi(t, \cdot) = \mathcal{H}(t) = \{[x \ y]^\top \in \mathbb{R}^2 : y \geq 0\}$, and $\text{rge}(\bar{C}(t)) = \mathbb{R}^2$. So, $\text{rge}(\bar{C}(t)) \cap \text{rint}(\text{Dom } \Phi(t, \cdot)) \neq \emptyset$. Hence, Assumption 3 and items 2, 3 in Theorem 2 hold.
- d. The correctness of item 4 is inferred from the following:

$$\bar{C}(s)^{-1}(\text{Dom } \Phi(t, \cdot)) = \{[x_1 \ x_2 \ x_3 \ x_4]^\top \in \mathbb{R}^4 : x_4 \geq 0\} = \bar{C}(t)^{-1}(\text{Dom } \Phi(t, \cdot)), \forall 0 \leq s \leq t.$$

- e. Functions $\bar{C}(\cdot)$, $D(\cdot)$ are right-continuous if $\mathbf{C}(\cdot)$, $\mathbf{R}(\cdot)$ are right-continuous. The coefficient matrices of (63) are bounded on $[0, T]$ if $\frac{1}{\mathbf{R}(\cdot)}$, $\frac{1}{\mathbf{C}(\cdot)}$ and $\frac{\dot{\mathbf{C}}(\cdot)}{\mathbf{C}(\cdot)}$ are bounded on $[0, T]$. Since $\mathcal{K}^*(\cdot)$, $\mathcal{H}(\cdot)$ are constant mappings and $(\text{rge}(C(\cdot)) + \text{aff}(\mathcal{H}(\cdot)))^\perp \equiv \{0\}$, item 5 in Theorem 2 holds.

We conclude that, for any $T > 0$, if $\mathbf{C}(\cdot)$ is a continuously differentiable function, $\mathbf{R}(\cdot)$ is a right-continuous function and $\mathbf{C}(t), \mathbf{R}(t) > 0$ for all $t \in [0, T]$, then (63) satisfies all assumptions in Theorem 2. Hence, there exists a unique solution $z \in AC([0, T], \mathbb{R}^n)$ to (63) for every $z_0 \in \bar{C}(0)^{-1}(\text{Dom } \Phi(0, \cdot))$. This is equivalent to there exists a unique solution $x \in AC([0, T], \mathbb{R}^n)$ to the system (62) for every $x_0 \in C(0)^{-1}(\text{Dom } \Phi(0, \cdot)) = \{[x_1 \ x_2 \ x_3 \ x_4]^\top \in \mathbb{R}^4 : x_4 \geq 0\}$.

2.3.4 An example of the DC/DC Buck converter

Let us consider the DC/DC Buck converter reported in [1, p.1822]. Its circuit scheme is provided in [1], and recalled in Figure 2, where ED1 and ED2 represent electronic devices (ED), specifically, ED1 is an antiparallel connection of an ideal diode (ID) and a MOSFET, ED2 is an ID.

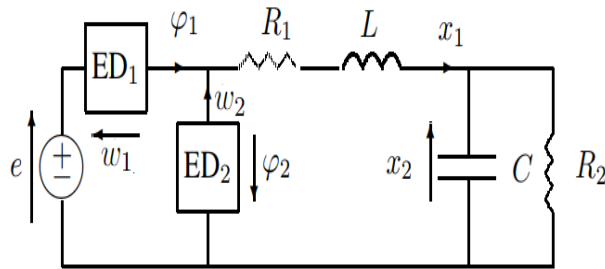


Figure 2: Circuit scheme of a DC/DC buck converter [1, Fig. 11].

Let $x = [x_1 \ x_2]^\top$, where x_1 and x_2 are the current, the voltage through the inductor and capacitor, respectively. Assume that $R_1, R_2, L, C > 0$. Then, the time evolution of this circuit is described by

the following equations:

$$\left\{ \begin{array}{l} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{R_2 C} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{L} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1(t) \\ \lambda_2(t) \end{bmatrix}, \\ \begin{bmatrix} w_1(t, x) \\ w_2(t, x) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{C(t)=C} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{D(t)=D} \begin{bmatrix} \lambda_1(t) \\ \lambda_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{F(t)=F} \underbrace{e}_{u(t)=u}, \\ \mathcal{K}(t) \ni \lambda(t) \perp w(t, x) \in \mathcal{K}^*(t), \end{array} \right. \quad (64)$$

where $\lambda_1 = -\varphi_1$, $\lambda_2 = -\varphi_2$, and e is a positive number, $\mathcal{K}(t) = [\mathbb{R} \ \mathbb{R}_+]^\top$ if the switch ED1 is ON and $\mathcal{K}(t) = [\mathbb{R}_+ \ \mathbb{R}_+]^\top$ if the switch ED1 is OFF. Let us assume that ED1 is either ON or OFF on closed intervals of the form $I = [a, b] \subseteq \mathbb{R}$, for some $0 \leq a < b$. Since $P(t) = \text{diag}(L, L) \succ 0$ is a solution of (16), using the comment **(d)** in Section 2.3.1, we can rewrite (64) equivalently as:

$$\left\{ \begin{array}{l} \dot{z}(t) = \begin{bmatrix} -\frac{R_1}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{R_2 C} \end{bmatrix} z(t) + \begin{bmatrix} 0 & \frac{1}{\sqrt{L}} \\ 0 & 0 \end{bmatrix} \lambda(t), \\ w(t, z) = \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{\bar{C}(t)=\bar{C}} z(t) + \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{D(t)=D} \lambda(t) + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{F(t)=F} \underbrace{e}_{u(t)=u}, \\ \mathcal{K}(t) \ni \lambda(t) \perp w(t, z) \in \mathcal{K}^*(t). \end{array} \right. \quad (65)$$

Obviously, (65) is a TVLCCS which satisfies $D(t) = D \succneq 0$. Let us now examine whether (65) satisfies assumptions of Theorem 2. Let $T > 0$.

- a. It is evident that (65) satisfies Assumption 2 with the identity matrix being a solution to (16) and items 2, 4, 6 in Theorem 2 hold trivially.
- b. For each $t \in [0, T]$, there exist bounded mappings $Q(\cdot)$, $\hat{Q}(\cdot)$ satisfying items 1 (i), 1 (ii) in Theorem 2, defined by:

$$Q(t) = \begin{cases} \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}, & \text{if ED1 is ON,} \\ \text{diag}(1, 1), & \text{if ED1 is OFF,} \end{cases} \quad \text{and} \quad \hat{Q}(t) = \begin{cases} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 \end{bmatrix}, & \text{if ED1 is ON,} \\ \begin{bmatrix} 1 & 0 \end{bmatrix}, & \text{if ED1 is OFF.} \end{cases}$$

Moreover, $\mathcal{H}(t) = \{[x \ y]^\top \in \mathbb{R}^2 : x \geq 0\}$ is a constant set for all $t \in [0, T]$. Thus, item 1 (iii) in Theorem 2 holds.

- c. Since all matrices in (65) are constants, $\mathcal{H}(\cdot)$ is a constant function, and $(\text{rge}(C(\cdot)) + \text{aff}(\mathcal{H}(\cdot)))^\perp$ is the zero function, so item 5 is satisfied if $\mathcal{K}^*(\cdot)$ is inner semicontinuous from right on $[0, T]$. This is obtained by the initialization convention that $\mathcal{K}(\cdot)$ is constant on intervals $[a, b] \subseteq \mathbb{R}$.
- d. We have $\text{rge}(\bar{C}(t)) = [0 \ \mathbb{R}]^\top$ and $\text{Dom} \Phi(t, \cdot) = \mathcal{H}(t) - F(t)u(t) = [(\mathbb{R}_+ - e) \ \mathbb{R}]^\top$ for all $t \in [0, T]$. Therefore, Assumption 3 and item 3 hold.

From all the above, (65) satisfies all assumptions of Theorem 2. Hence, (65) has a unique AC solution $z(\cdot)$ on $[0, T]$ for any initial state $z_0 \in \mathbb{R}^2$. Since (65) is equivalent to (64), there exists a unique AC solution $x(\cdot)$ to the system (64) on $[0, T]$ for any initial state $x_0 \in \mathbb{R}^2$.

To the best of the author's knowledge, this example is introduced in [1], with a primary focus on modelling and numerical simulation, see in [1, p.1823]. However, from an analytical standpoint, there is a notable absence of results in prior literature to show the existence of solutions for (64). This vulnerability is resolved as above. The well-posedness of converters proposed in [2, 3] (*e.g.*, DC/DC Boost converter, three-phase power converter) can also be verified through Theorem 2. When considering ED1 in Figure 2 as an ideal switch (IS), the DC/DC Buck converter is modelled identically to (64), but with a distinct $\mathcal{K}(t)$. Specifically, $\mathcal{K}(t) = [0 \ \mathbb{R}_+]^\top$ if ED1 is ON and $\mathcal{K}(t) = [\mathbb{R} \ \mathbb{R}_+]^\top$ if ED1 is OFF. In this case, it can be checked that $\mathcal{H}(t) = \{[x \ y]^\top \in \mathbb{R}^2 : y \geq 0\}$ if ED1 is ON and $\mathcal{H}(t) = \{[x \ y]^\top \in \mathbb{R}^2 : x \geq 0\}$ if ED1 is OFF. If ED1 changes its state then item 1 (iii) in Theorem 2 is not satisfied, thus the existence of AC solutions to this system is not guaranteed by Theorem 2. However, if ED1 changes its state only once from ON to OFF, the well-posedness of this system is ensured by Theorem 1, where the assumption **(A3)** holds and the remaining assumptions are verified by the correctness of all assumptions in Theorem 2 except for item 1 (iii).

2.4 The case $D(t) \equiv 0$ (RCLBV solutions)

The foregoing sections are dedicated to the study of (3) when either $D(t) \succ 0$, or when $D(t) \succcurlyeq 0$ for all $t \geq 0$. In both cases, AC solutions are sought. Because the TVLCCS (3) with $D(t) = 0$ has various applications, it deserves to be considered separately. Firstly, let us introduce and recall some important assumptions used throughout this section.

Assumption 4. *The set $\mathcal{K}(t) \subseteq \mathbb{R}^m$ is a nonempty closed convex polyhedral cone for all $t \geq 0$.*

Assumption 2. *There is a continuously differentiable $P(t) = P(t)^\top \succ 0$ such that for all $t \geq 0$:*

$$P(t)B(t) = C(t)^\top. \quad (66)$$

Now, using the equivalence of (3) and (57), a consequence of Theorem 2 when $D(t) = 0$ is provided below, where $R(t)$ is a square root of $P(t)$ in (66), and $\mathcal{H}(t) = \mathcal{K}^*(t)$ for all $t \in [0, T]$ (see Remark 1).

Corollary 2. *Consider the TVLCCS (3). Suppose that Assumptions 2, 3, 4, and the following hold:*

1. *For each $t \in [0, T]$, $\mathcal{K}(t) \subseteq \mathcal{K}(s)$, for all $s \in [0, t]$.*
2. *Either $\text{rge}(C(t))$ or $\text{rge}(R(t)^{-1}C(t)^\top)$ is a constant linear space for all $t \in [0, T]$.*
3. *$\text{rge}(C(s)) \cap (\mathcal{K}^*(s) - F(t)u(t)) \neq \emptyset$, for all $0 \leq s \leq t \leq T$.*
4. *$(C(s)R(s)^{-1})^{-1}(\mathcal{K}^*(t) - F(t)u(t)) \subseteq (C(t)R(t)^{-1})^{-1}(\mathcal{K}^*(t) - F(t)u(t))$, for all $0 \leq s \leq t \leq T$.*
5. *$C(\cdot)R(\cdot)^{-1}$ is right-continuous, $\mathcal{K}^*(\cdot)$, $(\text{rge}(C(\cdot)) + \text{aff}(\mathcal{K}^*(\cdot)))^\perp$ are inner semicontinuous from right, $F(\cdot)$, $u(\cdot)$ are AC and all matrix functions in (3) are bounded on $[0, T]$.*

Then, (3) has a unique solution $x \in AC([0, T], \mathbb{R}^n)$, for every $x(0) \in C(0)^{-1}(\mathcal{K}^(0) - F(0)u(0))$.*

In the upcoming section, Assumption 3 and all items in Corollary 2 are relaxed in special cases, and RCLBV solutions with possible jumps are studied, under the condition that $D(t) = 0$ for all $t \geq 0$.

2.4.1 Existence and uniqueness of RCLBV solutions

Let us consider the system (3). It can be equivalently rewritten under Assumptions 2 and 4 using Proposition 2, as follows:

$$\dot{z}(t) \in -R(t)^{-1}C(t)^\top \mathcal{N}_{S(t)}(C(t)R(t)^{-1}z(t)) + f(t, z(t)), \quad (67)$$

in which $z(t) := R(t)x(t)$, where $R(t)$ is a square root of $P(t)$ given in (66), $S(t) := \mathcal{K}^\star(t) - F(t)u(t)$, and $f(t, z(t)) := \left(R(t)A(t)R(t)^{-1} + \dot{R}(t)R(t)^{-1}\right)z(t) + R(t)E(t)u(t)$. For each $t \geq 0$, $S(t)$ is a closed convex polyhedral set from Assumption 4, and $C(t)R(t)^{-1}$ defines a linear operator. Therefore, (67) is rewritten as the first-order sweeping process (FOSwP):

$$\dot{z}(t) \in -\mathcal{N}_{\phi(t)}(z(t)) + f(t, z(t)), \quad (68)$$

where $\phi(t) := (C(t)R(t)^{-1})^{-1}S(t)$ for each $t \geq 0$, using Proposition A1. Hence $\phi(t)$ is given by:

$$\phi(t) = \{x \in \mathbb{R}^n : C(t)R(t)^{-1}x + F(t)u(t) \in \mathcal{K}^\star(t)\}. \quad (69)$$

The well-posedness of the FOSwP (68) and its extensions in Hilbert space has been studied in [33, 25, 34, 35], where the absolute continuity and bounded variability of $\phi(\cdot)$ play a crucial role. At this step, the equivalence between the FOSwP and the TVLCCS formalisms has just been shown formally in the sense that both right-hand sides in (3a) and in (68) are the same. To ensure $\phi(\cdot)$ is RCLBV or locally AC, [10, Proposition 3.2] (this proposition states that when all matrices in (3) are constant and $\mathcal{K}(t) = \mathbb{R}_+^m$, if $Fu(t)$ is RCLBV (resp. locally AC) then $\phi(\cdot)$ is RCLBV (resp. locally AC) using Lemma A2) cannot be applied for this purpose. Because in [10], the time-variation of $\phi(\cdot)$ depends only on $Fu(\cdot)$, while in (69) it depends on $C(\cdot)$, $R(\cdot)^{-1}$, $\mathcal{K}^\star(\cdot)$ and $F(\cdot)u(\cdot)$, therefore Lemma A2 is not generally applicable for evaluating $d_H(\phi(s), \phi(t))$ for some $s, t \in \mathbb{R}_+$. However, in certain special cases of (3), it can be used to establish sufficient conditions to guarantee the RCLBV or locally AC of $\phi(\cdot)$. The next proposition is an extension of [10, Proposition 3.2] for TVLCCSs.

Proposition 3. *Consider the system (3) and its equivalent (68). Suppose that Assumptions 2, 4 and the following hold:*

1. *There is $Q : \mathbb{R}_+ \rightarrow \mathbb{R}^{j \times m}$ such that $\mathcal{K}^\star(t) = \{\xi \in \mathbb{R}^m : Q(t)\xi \geq 0\}$, for all $t \geq 0$ and some j .*
2. *$Q(t)C(t)R(t)^{-1}$ is a constant matrix for all $t \geq 0$.*
3. *$F(t)u(t) \in \text{rge}(C(t)) + \mathcal{K}^\star(t)$, for all $t \geq 0$.*

Then, $\phi(\cdot)$ in (69) is locally AC (resp. RCLBV) on $[0, +\infty)$ if $Q(\cdot)F(\cdot)u(\cdot)$ is a locally AC function (resp. RCLBV function) on $[0, +\infty)$.

Proof. For each $t \geq 0$, $\phi(t)$ in (69) can be rewritten as follows:

$$\phi(t) = \{x \in \mathbb{R}^n : Q(t)C(t)R(t)^{-1}x + Q(t)F(t)u(t) \in \mathbb{R}_+^j\} = \{x \in \mathbb{R}^n : Kx + Q(t)F(t)u(t) \in \mathbb{R}_+^j\},$$

where $K \in \mathbb{R}^{j \times n}$ is a constant matrix from item 2. Since $\text{rge}(C(t)) = \text{rge}(C(t)R(t)^{-1})$ (see [29, Proposition 2.6.3]) and item 3 holds, there exists $\gamma \in \mathbb{R}^n$ satisfying $C(t)R(t)^{-1}\gamma + F(t)u(t) \in \mathcal{K}^\star(t)$, which implies that $\phi(t)$ is a nonempty set for all $t \geq 0$. Let s, t be such that $0 \leq s \leq t$, then there exists a constant γ_K that depends only on K such that

$$d_H(\phi(s), \phi(t)) \leq \gamma_K \|Q(t)F(t)u(t) - Q(s)F(s)u(s)\|, \quad (70)$$

using Robinson's bound in Lemma A2. It follows from (70) that $\phi(\cdot)$ is locally AC on $[0, +\infty)$ if $Q(\cdot)F(\cdot)u(\cdot)$ is locally AC on $[0, +\infty)$. To prove that $\phi(\cdot)$ is RCLBV on $[0, +\infty)$, the existence of a positive Radon measure μ on $[0, T]$ satisfying $d_H(\phi(s), \phi(t)) \leq \mu((s, t])$ for all $0 \leq s \leq t \leq T$, for all $T > 0$, must be ensured. Indeed, for each $T > 0$, since $Q(\cdot)F(\cdot)u(\cdot)$ is RCBV on $[0, T]$, there exists a vector measure (or known as differential measure) d_{QFu} on $[0, T]$ associated with $Q(\cdot)F(\cdot)u(\cdot)$ such that

$$Q(t)F(t)u(t) - Q(s)F(s)u(s) = d_{QFu}((s, t]), \quad \forall 0 \leq s \leq t \leq T, \quad (71)$$

see [34, p.140], [36, 37]. Let $\tilde{\mu} := |d_{QFu}|$ be the variation measure of d_{QFu} , then $\tilde{\mu}$ is a positive Radon measure on $[0, T]$, see [38, p.383], [33]. From (71), it is inferred that:

$$\|Q(t)F(t)u(t) - Q(s)F(s)u(s)\| = \|d_{QFu}((s, t])\| \leq \tilde{\mu}((s, t]), \quad \forall 0 \leq s \leq t \leq T, \quad (72)$$

where the inequality is obtained from the definition of variation measures [39, p.2]. Let $\mu := \gamma_K \tilde{\mu}$, it can be deduced from (70) and (72) that $\phi(\cdot)$ is RCBV on $[0, T]$, is thus RCLBV on $[0, +\infty)$. \square

As mentioned in the proof of Proposition 3, if $Q(\cdot)F(\cdot)u(\cdot)$ is RCBV on $[0, T]$, then there exists a positive Radon measure μ such that $d_H(\phi(s), \phi(t)) \leq \mu((s, t])$, for all $0 \leq s \leq t \leq T$. Moreover, there always exists a nondecreasing RCBV function $\tilde{u} : [0, T] \rightarrow [0, +\infty)$ such that $\mu((s, t]) = \tilde{u}(t) - \tilde{u}(s)$ for all $0 \leq s \leq t \leq T$. Indeed, let $\tilde{u}(t) := \mu((0, t])$ for all $t \in [0, T]$, then $\tilde{u}(\cdot)$ is nondecreasing and right-continuous on $[0, T]$. Since $\mu = \gamma_K |d_{QFu}|$ and $Q(\cdot)F(\cdot)u(\cdot)$ is BV on $[0, T]$,

$$\mu((0, T]) = \gamma_K \sup_{0=t_0 < t_1 < \dots < t_k=T} \sum_{i=0}^{k-1} \|d_{QFu}((t_i, t_{i+1}])\| = \gamma_K V_{[0, T]}(QFu) < +\infty.$$

Hence, $V_{[0, T]}(\tilde{u}) = \tilde{u}(T) - \tilde{u}(0) = \mu((0, T]) < +\infty$, that is, $\tilde{u}(\cdot)$ is BV on $[0, T]$. Therefore, it is implied that [4, Assumption **(A3)**] holds, which is an important assumption when studying FOSwPs. The next proposition allows for the relaxation of item 2 in Proposition 3, at the price of assuming bounded sets $\phi(t)$. It can be observed that computing $\phi(t)$ is straightforward, but finding a mapping $Q(\cdot)$ that satisfies item 2 in Proposition 3 is generally not easy, or even impossible. The following result is useful in these cases when $\phi(t)$ is a bounded set for all $t \geq 0$ (it is a moving polytope).

Proposition 4. *Consider the system (3) and its equivalent (68). Suppose that Assumptions 2, 4 and the following hold:*

1. *There is $Q : \mathbb{R}_+ \rightarrow \mathbb{R}^{j \times m}$ such that $\mathcal{K}^*(t) = \{\xi \in \mathbb{R}^m : Q(t)\xi \geq 0\}$, for all $t \geq 0$ and some j .*
2. *$\phi(t)$ is bounded for all $t \geq 0$.*
3. *$F(t)u(t) \in \text{rge}(C(t)) + \mathcal{K}^*(t)$, for all $t \geq 0$.*

Then, $\phi(\cdot)$ in (69) is locally AC (resp. RCLBV) on $[0, +\infty)$ if $Q(\cdot)C(\cdot)R(\cdot)^{-1}$ and $Q(\cdot)F(\cdot)u(\cdot)$ are locally AC (resp. RCLBV) on $[0, +\infty)$.

Proof. Clearly $\phi(t) = \{x \in \mathbb{R}^n : Q(t)C(t)R(t)^{-1}x + Q(t)F(t)u(t) \in \mathbb{R}_+^j\}$, which is nonempty for all $t \geq 0$ due to item 3. Since $\phi(t)$ is bounded for all $t \geq 0$ (item 2), there exists $M > 0$ such that:

$$\|x\| \leq M, \quad \forall x \in \phi(t), \quad \forall t \geq 0.$$

To simplify the writing, set $\tilde{A}(t) := Q(t)C(t)R(t)^{-1} \in \mathbb{R}^{j \times n}$, $b(t) := Q(t)F(t)u(t) \in \mathbb{R}^j$ for all $t \geq 0$. Let s, t such that $0 \leq s \leq t$. Using Corollary A1, $\text{dist}(x, \phi(s)) \leq M(\|\tilde{A}(t) - \tilde{A}(s)\| + \|b(t) - b(s)\|)$,

for all $x \in \phi(t)$. This implies $\sup_{x \in \phi(t)} \text{dist}(x, \phi(s)) \leq M(\|\tilde{A}(t) - \tilde{A}(s)\| + \|b(t) - b(s)\|)$. By a symmetric argument, the right-hand side of this inequality is also a bound of $\sup_{x \in \phi(s)} \text{dist}(x, \phi(t))$. Hence, the following holds:

$$d_H(\phi(s), \phi(t)) \leq M(\|\tilde{A}(t) - \tilde{A}(s)\| + \|b(t) - b(s)\|). \quad (73)$$

For each $h \geq 0$, for each $i \in \{1, \dots, n\}$, let $a_i(h) \in \mathbb{R}^j$ denote the columns of $\tilde{A}(h)$. Recall that the norm being used is the Frobenius norm. Using [29, Fact 9.8.12-(vii)], we obtain:

$$\begin{aligned} \|\tilde{A}(t) - \tilde{A}(s)\| + \|b(t) - b(s)\| &\leq \sqrt{n} \max_{1 \leq i \leq n} (\|a_i(t) - a_i(s)\|_1) + \|b(t) - b(s)\|_1 \\ &\leq \sqrt{n} \left(\sum_{i=1}^n \|a_i(t) - a_i(s)\|_1 \right) + \|b(t) - b(s)\|_1, \end{aligned} \quad (74)$$

where $\|\cdot\|_1$ is the maximum absolute column sum norm. Suppose that $\tilde{A}(\cdot)$ and $b(\cdot)$ are locally AC, then for all $i \in \{1, \dots, n\}$ there exist derivatives of $a_i(\cdot)$ and $b(\cdot)$, denoted as \dot{a}_i and \dot{b} respectively, such that $a_i(t) - a_i(s) = \int_s^t \dot{a}_i(\tau) d\tau$, and $b(t) - b(s) = \int_s^t \dot{b}(\tau) d\tau$. It follows that:

$$\sqrt{n} \left(\sum_{i=1}^n \|a_i(t) - a_i(s)\|_1 \right) + \|b(t) - b(s)\|_1 \leq \int_s^t \left[\sqrt{n} \left(\sum_{i=1}^n \|\dot{a}_i(\tau)\|_1 \right) + \|\dot{b}(\tau)\|_1 \right] d\tau. \quad (75)$$

It follows from (73), (74), and (75) that $d_H(\phi(s), \phi(t)) \leq v(t) - v(s)$, where $v(\cdot)$ is a nondecreasing locally AC function and defined by:

$$v(\cdot) : [0, +\infty) \rightarrow \mathbb{R}_+, \quad \hat{t} \mapsto v(\hat{t}) = \int_0^{\hat{t}} M \left[\sqrt{n} \left(\sum_{i=1}^n \|\dot{a}_i(\tau)\|_1 \right) + \|\dot{b}(\tau)\|_1 \right] d\tau.$$

Thus $\phi(\cdot)$ is locally AC. Let us now prove that $\phi(\cdot)$ is RCLBV if $\tilde{A}(\cdot)$ and $b(\cdot)$ are RCLBV. Indeed, for each $T > 0$, proceeding as the proof of Proposition 3, there exist positive Radon measures $\tilde{\mu}_i$ on $[0, T]$ for $i = 1, \dots, n+1$ such that:

$$\begin{aligned} M\sqrt{n} \left(\sum_{i=1}^n \|a_i(t) - a_i(s)\|_1 \right) + M\|b(t) - b(s)\|_1 &\leq M\sqrt{n} \left(\sum_{i=1}^n \tilde{\mu}_i((s, t]) \right) + M\tilde{\mu}_{n+1}((s, t]) \\ &= \mu((s, t]), \quad \forall 0 \leq s \leq t \leq T, \end{aligned} \quad (76)$$

where $\mu := M\sqrt{n} \left(\sum_{i=1}^n \tilde{\mu}_i \right) + M\tilde{\mu}_{n+1}$. Using [40, Theorem 7.1.7], μ is a positive Radon measure on $[0, T]$. It follows from (73), (74), (76) that $\phi(\cdot)$ is RCBV on $[0, T]$, is thus RCLBV on $[0, +\infty)$. \square

An appropriate concept for the solutions of the FOSwP (68) when the moving set $\phi(\cdot)$ is RCLBV, is proposed in [34] and used in [10, 4], see Definition A1. Under this definition, the well-posedness of (68) has been shown in [10, 4], which is recalled here:

Theorem 3. [10, Theorem 3.5] *Assume that $u \in L_1([0, +\infty), \mathbb{R}^u)$, $A(\cdot)$ and $E(\cdot)$ are measurable and uniformly bounded on $[0, T]$ for any $T > 0$, and the set-valued mapping $\phi(\cdot)$ is locally AC (resp. RCLBV) with non-empty values on $[0, +\infty)$. Then the FOSwP (68) has one and only one locally AC (resp. RCLBV) solution $z(\cdot)$ on $[0, +\infty)$, for every initial condition $z(0) = z_0 \in \phi(0)$.*

Remark 4. In Propositions 3, 4, item 3 plays a role in ensuring the non-emptiness of $\phi(t)$ for all $t \geq 0$. Additionally, an equivalent form of item 3 is obtained as follows:

$$\text{rge}(C(t)) \cap (\mathcal{K}^*(t) - F(t)u(t)) \neq \emptyset, \quad \forall t \geq 0. \quad (77)$$

When $D(\cdot) \equiv 0$, then (77) is a relaxed form of Assumption 3. There are cases where Assumption 3 is false but (77) is true. For instance, $C(t) = [1 \ -1]^\top$, $F(t)u(t) = 0$ and $\mathcal{K}^*(t) = \mathbb{R}_+^2$, for all $t \geq 0$. In addition, all items in Corollary 2 are also relaxed by Propositions 3, 4, and Theorem 3.

Remark 5. In [10], the matrices $C(t), R(t)^{-1}, F(t)$ are constant and $\mathcal{K}^*(t) = \mathcal{K}(t) = \mathbb{R}_+^m$, for all $t \geq 0$. Therefore, to ensure the non-emptiness of $\phi(t)$, the constraint qualification (CQ): $\text{rge}(C) - \mathbb{R}_+^m = \mathbb{R}^m$ is used in [10, p.8] (see its statement in Appendix D). Moreover, this CQ is indeed derived from:

$$-Fu(t) - f \in \text{rge}(C) - \mathbb{R}_+^m, \quad (78)$$

where f is a vector in the complementarity condition. Because in [10], they aim to avoid imposing assumptions on the vector f (i.e., treating f as an arbitrary vector in \mathbb{R}^m), so the CQ as previously is implicitly understood as a sufficient condition for (78) holding for every $f \in \mathbb{R}^m$. Thus, item 3 in Propositions 3, 4 is an extension of (78) when $f = 0$.

The well-posedness of (3) is ensured by combining Propositions 3, 4 and Theorem 3. Intuitively, this may seem unrelated to Theorem 2 or Corollary 2, but in fact, they are very close. Indeed, (3) is equivalent to (68) under Assumptions 2, 4. Assume that $\mathcal{K}^*(t) := \{\xi \in \mathbb{R}^m : Q(t)\xi \geq 0\}$ for some matrix $Q(t) \in \mathbb{R}^{j \times m}$, for all $t \geq 0$ and for some $j \in \mathbb{N}$ (i.e., item 1 in Proposition 3 holds). Then $\phi(t) = \{x \in \mathbb{R}^n : Q(t)C(t)R(t)^{-1}x + Q(t)F(t)u(t) \geq 0\}$ and (68) is rewritten as the following TVLCCS using (6), where $z(t) \in \mathbb{R}^n$ and $\hat{\lambda}(t) \in \mathbb{R}^j$.

$$\begin{cases} \dot{z}(t) = \left(R(t)A(t)R(t)^{-1} + \dot{R}(t)R(t)^{-1} \right) z(t) + (Q(t)C(t)R(t)^{-1})^\top \hat{\lambda}(t) + R(t)E(t)u(t), \\ \hat{w}(t, z) = Q(t)C(t)R(t)^{-1}z(t) + Q(t)F(t)u(t), \\ 0 \leq \hat{\lambda}(t) \perp \hat{w}(t, z) \geq 0. \end{cases} \quad (79)$$

Here, we can easily see that the conditions to guarantee Proposition 3 and Theorem 3 being true are very close to the conditions ensuring Theorem 2 is satisfied on the system (79).

Remark 6. When $z(\cdot)$ is an RCLBV solution of (68), it may possess jumps. Inspired by the state-jump model provided in [41, p.167], [42, p.91], [5, p.62], at time of jumps t , (68) can be rewritten as:

$$z(t^+) - z(t^-) \in -\mathcal{N}_{\phi(t^+)} \left(\frac{z(t^+) + ez(t^-)}{1+e} \right), \quad (80)$$

where $e \in [0, 1]$ is the restitution coefficient (see e.g., [10, Remark 3.7]). Suppose that items 1, 3 in Proposition 3 (resp. Proposition 4) hold. Using (4), then (80) is equivalent to $z(t^+) = (1+e) \text{proj}_{\phi(t^+)}(z(t^-)) - ez(t^-)$, and $z(t^+)$ exists and is unique since $\phi(t^+)$ (given in (69)) is a nonempty closed convex set. However, in many cases, it is not easy to compute this projection. Therefore, another way to calculate $z(t^+)$ is based on the equivalence of (80) with the following:

$$\begin{cases} z(t^+) - z(t^-) = (Q(t^+)C(t^+)R(t^+)^{-1})^\top \hat{\lambda}, \\ 0 \leq \hat{\lambda} \perp \hat{w} = Q(t^+)C(t^+)R(t^+)^{-1} \left(\frac{z(t^+) + ez(t^-)}{1+e} \right) + Q(t^+)F(t^+)u(t^+) \geq 0. \end{cases} \quad (81)$$

$$\Leftrightarrow \begin{cases} \hat{w} = \underbrace{\frac{1}{1+e} Q(t^+) C(t^+) P(t)^{-1} C(t^+)^{\top} Q(t^+)^{\top} \hat{\lambda}}_K + \underbrace{Q(t^+) C(t^+) R(t)^{-1} z(t^-) + Q(t^+) F(t^+) u(t^+)}_q, \\ 0 \leq \hat{\lambda} \perp \hat{w} \geq 0. \end{cases} \quad (82)$$

by using (6). Clearly $\text{LCP}(q, K)$ is solvable, where the symmetric matrix K is positive semi-definite. Therefore, it can be solved using programming languages such as Python or Matlab to get the value of $\hat{\lambda}$, and then $z(t^+)$. Note that when $D(\cdot) \equiv 0$, under Assumptions 2, 4 then (3) is equivalent to (68), and thus at time of jumps, (3) is equivalent to (82) where $z(t^-) = R(t)x(t^-)$.

Remark 7. So far, to obtain the existence and uniqueness of an AC solution to (3) when $D(\cdot) \equiv 0$, either a combination of Proposition 3 (resp. Proposition 4) and Theorem 3, or Corollary 2 can be used. If all matrices in (3) are constant and $\mathcal{K}(t) = \mathbb{R}_+^m$ for all $t \geq 0$, then these approaches (except Proposition 4) become equivalent. However, in the general case, they are different and can be used selectively based on each specific situation. This will be illustrated by examples in the next section.

2.4.2 Examples

Let us consider the circuit in Figure 3. Denote by λ the current through the diode, and w the voltage across the diode, $x(t) := [x_1(t) \ x_2(t)]^{\top}$, where x_1, x_2 denote the charge in the capacitor and the current through the inductor, respectively. The dynamics of this circuit is given by:

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -\frac{1}{\mathbf{R}(t)\mathbf{C}(t)} & -1 \\ \frac{1}{\mathbf{L}(t)\mathbf{C}(t)} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \lambda(t) + \begin{bmatrix} -\frac{1}{\mathbf{R}(t)} \\ 0 \end{bmatrix} u(t), \\ w(t, x) = \begin{bmatrix} 1 \\ \mathbf{C}(t) \end{bmatrix} x(t) + u(t), \\ 0 \leq \lambda(t) \perp w(t, x) \geq 0. \end{cases}$$

Let $\mathbf{C}(t) = \frac{t^2+2}{t^2+1}$, $u(t) = 1 + \sin(t)$ and $\mathbf{R}(t), \mathbf{L}(t) > \rho$, for all $t \geq 0$, for some $\rho > 0$ such that $\mathbf{R}(\cdot), \mathbf{L}(\cdot)$ are measurable functions. It can be checked that Assumptions 4, 3, 2 hold with $P(t) = \text{diag}\left(\frac{1}{\mathbf{C}(t)}, 1\right)$ is a solution of (66), and items 1, 2, 3, 5 in Corollary 2 hold. However, this system does not satisfy item 4 in Corollary 2. Indeed, let $s = 0$ and $t = 1$, we obtain:

$$\begin{aligned} (C(s)R(s)^{-1})^{-1} (\mathbb{R}_+ - u(t)) &= \left\{ [\xi_1 \ \xi_2]^{\top} \in \mathbb{R}^2 : \xi_1 \geq -\sqrt{2}(1 + \sin(1)) \right\} \\ &\not\subseteq \left\{ [\xi_1 \ \xi_2]^{\top} \in \mathbb{R}^2 : \xi_1 \geq -\sqrt{\frac{3}{2}}(1 + \sin(1)) \right\} \\ &= (C(t)R(t)^{-1})^{-1} (\mathbb{R}_+ - u(t)). \end{aligned}$$

Therefore, the existence of an AC solution to this system is not guaranteed by Corollary 2. On the other hand, this system satisfies items 1, 2, and 3 in Proposition 3 with $Q(t) = \sqrt{\mathbf{C}(t)}$, for all $t \geq 0$. Since $Q(\cdot)u(\cdot) = (1 + \sin(\cdot))\sqrt{\frac{(\cdot)^2+2}{(\cdot)^2+1}}$ is a locally AC function on $[0, +\infty)$, $\phi(\cdot)$ is locally AC using Proposition 3. Using Theorem 3, this system has a unique locally AC solution on $[0, +\infty)$ for every $x(0) \in \{[\xi_1 \ \xi_2]^{\top} \in \mathbb{R}^2 : \xi_1 \geq -2\}$.

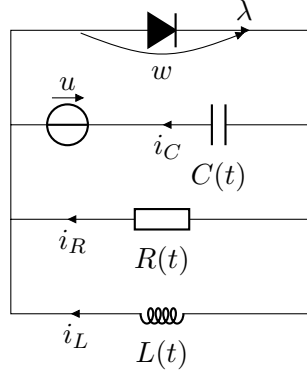


Figure 3: Time-varying RLC circuit with an ideal diode.

The next example illustrates a TVLCCS system that possesses a unique RCLBV solution. The following system is considered where $x(t) \in \mathbb{R}$ and $\lambda(t) \in \mathbb{R}^2$.

$$\begin{cases} \dot{x}(t) = x(t) + B(t)\lambda(t), \\ 0 \leq \lambda(t) \perp w(t, x) = C(t)x(t) + u(t) \geq 0, \end{cases} \quad (83)$$

in which, $B(t) = \begin{bmatrix} 1 & -1 \end{bmatrix}$, $C(t) = \begin{bmatrix} t+1 \\ -(t+1) \end{bmatrix}$, and $u(t) = \begin{cases} [0 \ \exp(2t)]^\top, & \text{if } t < 2 \\ [1 \ 0]^\top, & \text{if } t \geq 2 \end{cases}$, for all $t \geq 0$.

The system (83) is a TVLCCS and satisfies all assumptions in Proposition 3. Indeed,

1. Obviously, Assumptions 4, 2 hold where $P(t) = t + 1$ is a solution of (66).
2. We have $\mathcal{K}^*(t) = \{\xi \in \mathbb{R}^2 : Q(t)\xi \geq 0\}$, where $Q(t) = \text{diag}\left(\frac{1}{\sqrt{t+1}}, \frac{1}{\sqrt{t+1}}\right) \in \mathbb{R}^{2 \times 2}$. Moreover, $Q(t)C(t)R(t)^{-1} = [1 \ -1]^\top$ is a constant matrix for all $t \geq 0$. Hence, items 1, 2 in Proposition 3 are validated.
3. Clearly $\mathcal{K}^*(t) - u(t) = \begin{cases} \mathbb{R}_+^2 - [0 \ \exp(2t)]^\top, & \text{if } t < 2 \\ \mathbb{R}_+^2 - [1 \ 0]^\top, & \text{if } t \geq 2 \end{cases}$, and $\text{rge}(C(t)) = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \xi, \forall \xi \in \mathbb{R} \right\}$ is a constant linear space for all $t \geq 0$. Therefore, $\text{rge}(C(t)) \cap (\mathcal{K}^*(t) - u(t)) \neq \emptyset$ for all $t \geq 0$. Using Remark 4, then item 3 in Proposition 3 holds.

4. $Q(t)u(t) = \begin{cases} \begin{bmatrix} 0 & \frac{\exp(2t)}{\sqrt{t+1}} \end{bmatrix}^\top, & \text{if } t < 2 \\ \begin{bmatrix} \frac{1}{\sqrt{t+1}} & 0 \end{bmatrix}^\top, & \text{if } t \geq 2 \end{cases}$, which yields $Q(\cdot)u(\cdot)$ is a RCLBV function on $[0, +\infty)$.

Indeed, first, $Q(\cdot)u(\cdot)$ is obviously right-continuous at $t = 2$. Let any $T > 0$ and $(Qu)(t) := Q(t)u(t)$ for all $t \in [0, T]$, then the variation of $Q(\cdot)u(\cdot)$ over $[0, T]$ is given by:

$$\begin{aligned} V_{[0, T]}(Qu) &= \sup_{\sigma \in \mathcal{P}} \sum_{i=0}^{N_\sigma-1} \|(Qu)(t_{i+1}) - (Qu)(t_i)\| \leq \sup_{\sigma \in \mathcal{P}} \left(\sum_{i=0}^{i^*-1} \|(Qu)(t_{i+1}) - (Qu)(t_i)\| \right. \\ &\quad \left. + \|(Qu)(t_{i^*+1}) - (Qu)(t_{i^*})\| + \sum_{i=i^*+1}^{N_\sigma-1} \|(Qu)(t_{i+1}) - (Qu)(t_i)\| \right) \\ &= \sup_{\sigma \in \mathcal{P}} \left(\frac{\exp(2t_{i^*})}{\sqrt{t_{i^*}+1}} - \frac{\exp(2t_0)}{\sqrt{t_0+1}} + \sqrt{\frac{1}{t_{i^*+1}+1} + \frac{\exp(4t_{i^*})}{t_{i^*}+1}} + \frac{1}{\sqrt{t_{i^*+1}+1}} - \frac{1}{\sqrt{t_{N_\sigma}+1}} \right), \end{aligned}$$

where \mathcal{P} is the set of all partition of $[0, T]$ and i^* be an index such that $t_{i^*} < 2$ and $t_{i^*+1} \geq 2$. It follows that $V_{[0, T]}(Qu) < (\exp(4) - 1) + \left(\sqrt{1 + \exp(8)}\right) + \left(1 - \frac{1}{\sqrt{T+1}}\right) < +\infty$. This implies that $Q(\cdot)u(\cdot)$ is BV on $[0, T]$ for all $T > 0$, is hence LBV on $[0, +\infty)$.

Thus, $\phi(\cdot)$ defined in (69) is RCLBV on $[0, +\infty)$. Using Theorem 3, the system (83) has a unique RCLBV solution on $[0, +\infty)$, for every $x(0) \in [0, 1]$. In this case, the moving set $\phi(t)$ and the Hausdorff distance of $\phi(\cdot)$ at two arbitrary times are stated as follows:

$$\phi(t) = \begin{cases} \left[0, \frac{\exp(2t)}{\sqrt{t+1}}\right], & \text{if } t < 2 \\ \left[\frac{-1}{\sqrt{t+1}}, 0\right], & \text{if } t \geq 2 \end{cases}, \quad d_H(\phi(t), \phi(s)) = \begin{cases} \frac{\exp(2t)}{\sqrt{t+1}} - \frac{\exp(2s)}{\sqrt{s+1}}, & \text{if } s \leq t < 2 \\ \max\left\{\frac{\exp(2s)}{\sqrt{s+1}}, \frac{1}{\sqrt{t+1}}\right\}, & \text{if } s < 2 \leq t \\ \frac{1}{\sqrt{s+1}} - \frac{1}{\sqrt{t+1}}, & \text{if } 2 \leq s \leq t \end{cases}.$$

This implies that the Robinson's bound (70) in Proposition 3 holds with $\gamma_K = 1$. Using Remark 6, at jump time $t = 2$ then (83) is equivalent to the following:

$$\begin{cases} x(t^+) - x(t^-) = \left[\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}}\right] \hat{\lambda}, \\ \hat{w} = \frac{1}{1+e} \left[\sqrt{3} & -\sqrt{3}\right]^\top (x(t^+) + ex(t^-)) + \left[\frac{1}{\sqrt{3}} & 0\right]^\top, \\ 0 \leq \hat{\lambda} \perp \hat{w} \geq 0. \end{cases}$$

It is rewritten as: $0 \leq \hat{\lambda} \perp \hat{w} = \frac{1}{1+e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \hat{\lambda} + \begin{bmatrix} \sqrt{3} \\ -\sqrt{3} \end{bmatrix} x(t^-) + \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \end{bmatrix} \geq 0$, where $e \in [0, 1]$ is the restitution coefficient. To obtain a numerical simulation of (83), let us time-discretize this system. The following scheme is interpreted as utilizing a state jump model with $e = 0$.

A Direct Backward (Implicit) Euler Scheme

Let $[0, T]$, $T > 0$ be the interval that we need to obtain the solution. Imitating the ideas for discretizing the LCS systems proposed in [5], the time-discretization of (83) is:

$$\begin{cases} x_{k+1} - x_k = hx_{k+1} + hB_{k+1}\lambda_{k+1}, \\ 0 \leq \lambda_{k+1} \perp w_{k+1} = C_{k+1}x_{k+1} + u_{k+1} \geq 0, \end{cases} \quad (84)$$

where (t_0, t_1, \dots, t_n) is a partition of $[0, T]$ with $t_{k+1} - t_k = h > 0$ the time step, $n = \frac{T}{h}$, and $x_k := x(t_k)$, $\lambda_k := \lambda(t_k)$, $w_k := w(t_k, x_k)$, $B_{k+1} := B(t_{k+1})$, $C_{k+1} := C(t_{k+1})$, $u_{k+1} := u(t_{k+1})$, for all $k = \overline{0, n-1}$. With h small enough, after some algebraic manipulations, (84) is equivalent to:

$$\begin{cases} x_{k+1} = \frac{1}{1-h} (x_k + hB_{k+1}\lambda_{k+1}), \\ 0 \leq \lambda_{k+1} \perp (hC_{k+1}B_{k+1})\lambda_{k+1} + C_{k+1}x_k + (1-h)u_{k+1} \geq 0, \end{cases} \quad (85)$$

where for each $k \in \{0, 1, \dots, n-1\}$, the second line of (85) represents the LCP($q(t_{k+1}), M$) defined by:

$$0 \leq \lambda_{k+1} \perp \underbrace{\left(h \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)}_{M(t_{k+1})=M \succcurlyeq 0} \lambda_{k+1} + \underbrace{\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} x_k + (1-h) \frac{u_{k+1}}{t_{k+1}+1}\right)}_{q(t_{k+1})} \geq 0.$$

Since item 3 holds, $q(t_{k+1}) \in \text{rge}(C_{k+1}) + \mathbb{R}_+^2 = (\text{SOL}(0, M))^*$ for all $k = \overline{0, n-1}$. Clearly $M = M^\top \succcurlyeq 0$, is thus copositive-plus [22, Exercise 3.12.1]. Therefore, for each $k \in \{0, 1, \dots, n-1\}$, the LCP($q(t_{k+1}), M$) has a solution, using [22, Theorem 3.8.6]. Moreover, the LCP($q(t_{k+1}), M$) is completely equivalent to the following quadratic program (QP) [22, p.5]:

$$\begin{aligned} & \text{minimize} && f(\lambda) = \frac{1}{2} \lambda^\top M \lambda + q(t_{k+1})^\top \lambda \\ & \text{subject to} && \lambda \geq 0. \end{aligned}$$

This QP is solved by CVXPY (a domain-specific language for convex optimization) in Python. Take $h = 5 \times 10^{-5} s$ and $x(0) = 0.5$, then the numerical simulation of (83) is depicted in Figure 4.

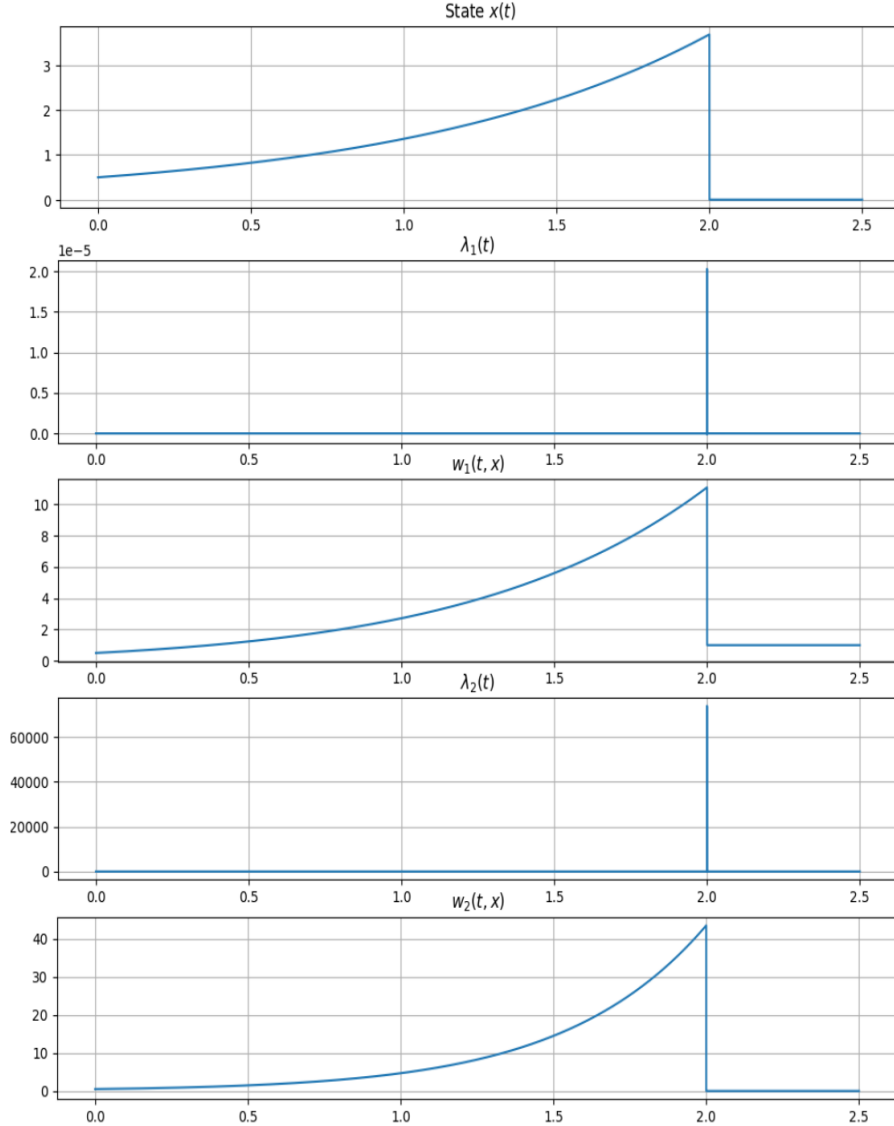


Figure 4: Simulation of (83) with Backward Euler algorithm, time step $h = 5 \times 10^{-5} s$, the initial condition $x(0) = 0.5$.

3 Further examples and future works

3.1 Simple examples

Let us illustrate some peculiarities of TVLCCS (3) compared to LCS (1), especially concerning state jumps (LCS possess state jumps at the initial time, or may jump at times when $Fu(t)$ is discontinuous [9, 10]). To this aim, several scalar examples are considered, to which the foregoing theoretical developments do not apply, and may thus motivate further investigations.

3.1.1 Example 1: State jumps in TVLCCS with constant $\mathcal{K}(t)$

The following scalar system is considered where $x(t) \in \mathbb{R}$ and $\lambda(t) \in \mathbb{R}$.

$$\begin{cases} \text{(a)} & \dot{x}(t) = 0.05x(t) + \frac{t^2+1}{t^2+2} \sin(t+1)\lambda(t), \\ \text{(b)} & 0 \leq \lambda(t) \perp w(t, x(t)) = \sin(t+1)x(t) \geq 0, \quad x(0) = 1. \end{cases} \quad (86)$$

The system (86) is a special case of (3) when $A(t) = 0.05$, $B(t) = \frac{t^2+1}{t^2+2} \sin(t+1)$, $C(t) = \sin(t+1)$, $D(t) = E(t) = F(t) = 0$ and $\mathcal{K}(t) = \mathcal{K}^*(t) = \mathbb{R}_+$, for all $t \geq 0$. Let us analyze solutions of (86).

Necessity of state jumps: From (86)(b), we get $\lambda(0) = 0$ since $w(0, x(0)) = \sin(1) > 0$. It is deduced from (86)(a) that $\dot{x}(0) = 0.05x(0)$. We conclude that for each $t \in [0, \pi - 1)$ then $\lambda(t) = 0$ and $\dot{x}(t) = 0.05x(t) > 0$, so a solution is given by $x(t) = \exp(0.05t)$ on $[0, \pi - 1)$. Therefore, $x((\pi - 1)^-) > 1$. Obviously, $w((\pi - 1)^+, x((\pi - 1)^+)) \geq 0$ if and only if $x((\pi - 1)^+) \leq 0$. Hence, for the function $w(t, x(t))$ not to violate the unilateral constraint, the time-function $x(\cdot)$ must jump from a positive value greater than one to a non-positive value at the time $t = \pi - 1$. Let us try $x((\pi - 1)^+) = 0$. It can be checked that for each $t \geq \pi - 1$, $x(t) = 0$ and $\lambda(t) = 0$ satisfy (86) and thus $x(t) = 0$ is a global solution on $[\pi - 1, +\infty)$. This short analysis shows that the trajectory of (86) cannot be continuous (*i.e.*, a state jump has to occur) at $t = \pi - 1$.

FOSwP interpretation, and time-discretization: Embedding (86) into a FOSwP helps to understand the underlying mechanisms. In order to clarify the above choice (*i.e.*, $x((\pi - 1)^+) = 0$), let us use the transformation proposed in [13] [10, Section 3.1] to rewrite (86) equivalently as the FOSwP:

$$\dot{z}(t) - H(t)z(t) \in -\mathcal{N}_{\phi(t)}(z(t)), \quad (87)$$

where $z(t) = R(t)x(t)$, $H(t) = (\dot{R}(t)R(t)^{-1} + 0.05)$ with $R(t) = \sqrt{\frac{t^2+2}{t^2+1}}$, \dot{R} is the derivative of R , and $\phi(t) = \{x \in \mathbb{R} : C(t)R(t)^{-1}x \geq 0\}$. Let us propose a time-discretization of (87). Moreau's catching-up algorithm [16] (which can be viewed as an extension of the backward (implicit) Euler algorithm) for (87) reads as: $z_{k+1} \in z_k + hH_{k+1}z_{k+1} - \mathcal{N}_{\phi_{k+1}}(z_{k+1})$, where $z_k := z(t_k)$, $H_{k+1} := H(t_{k+1})$, $\phi_{k+1} := \phi(t_{k+1})$ and h is a sufficiently small time step. Set $F(z_{k+1}) := (1 - hH_{k+1})z_{k+1} - z_k$, then the previous expression is equivalent to $0 \in F(z_{k+1}) + \mathcal{N}_{\phi_{k+1}}(z_{k+1})$. Since $(1 - hH_{k+1})$ is positive for sufficiently small $h > 0$, $F(\cdot)$ is strongly monotone for any given z_k . In addition, since ϕ_{k+1} is a closed convex set for all $k \geq 0$, the generalized equation $0 \in F(z_{k+1}) + \mathcal{N}_{\phi_{k+1}}(z_{k+1})$ has a unique solution (see *e.g.*, [5, Theorem 2.35]). However, we could not use the expression $0 \in F(z_{k+1}) + \mathcal{N}_{\phi_{k+1}}(z_{k+1})$ to numerically simulate (86), but must rewrite it in the form $z_{k+1} = \text{proj}_{\phi_{k+1}}((1 - hH_{k+1})^{-1}z_k)$ using (4). A numerical simulation of (86) is shown in Figure 5 and it demonstrates the existence of a jump at $t = \pi - 1$, as predicted.

Let us now analyze solutions of (87), then solutions of (86). Because $\phi(0) = \mathbb{R}_+$ and $z(0) = \sqrt{2} > 0$, then $\dot{z}(0) = H(0)z(0) > 0$ is obtained by (87). It can be checked that $\dot{z}(t) = H(t)z(t)$ on $[0, \pi - 1)$, so a solution is given by $z(t) = c \exp\left(\int_0^t H(v) dv\right)$ for some $c \in \mathbb{R}$. Since $z(0) = \sqrt{2}$, this implies that $c = \sqrt{2}$, and $z(t) = R(t) \exp(0.05t)$. Since $\phi((\pi - 1)^-) = \mathbb{R}_+$ and $\phi((\pi - 1)^+) = \mathbb{R}_-$, the set $\phi(t)$ jumps at the time $t = \pi - 1$. This explains why the state jump occurs. Using the rule of state jump which is proposed in [16] and used in [10], [5, Section 2.4.1], and [4, Section 5.2], then at time $t = \pi - 1$, the state jump is represented by:

$$z(t^+) - z(t^-) \in -\mathcal{N}_{\phi(t^+)}(z(t^+)). \quad (88)$$

This implies that $z(t^+) = \text{proj}_{\phi(t^+)}(z(t^-))$ using (4). Since $\phi((\pi - 1)^+) = \mathbb{R}_-$ and $z((\pi - 1)^-) > 0$, the unique post-jump solution is $z((\pi - 1)^+) = 0$. On $[\pi - 1, +\infty)$, it can be verified that $z(t) = 0$ is the unique solution of (87). This leads to the same result as the analysis above that the global solution is given by $x(t) = \exp(0.05t)$ if $t \in [0, \pi - 1)$, and $x(t) = 0$ if otherwise, is unique in the sense that if there is another global solution $\bar{x}(t)$ then $x(t) = \bar{x}(t)$ almost everywhere on $[0, +\infty)$.

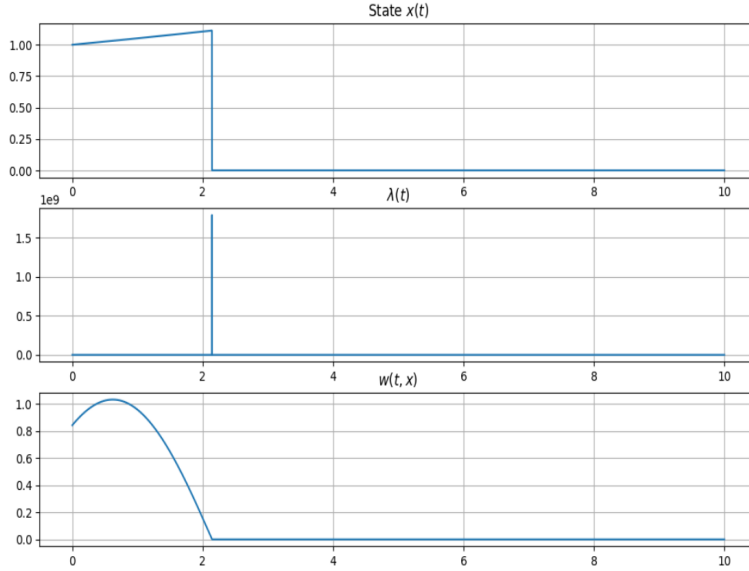


Figure 5: Simulation of (86) using the catching-up algorithm with time step $h = 10^{-4}$ s.

A more general state-jump rule with nonzero restitution: Following [41], let us now consider a second model of state jump defined by:

$$z(t^+) - z(t^-) \in -\mathcal{N}_{\phi(t^+)}\left(\frac{z(t^+) + ez(t^-)}{1 + e}\right), \quad (89)$$

where $e \in [0, 1]$ plays the role of a restitution coefficient, familiar in Contact Mechanics [43]. It follows that $z(t^+) = (1 + e) \text{proj}_{\phi(t^+)}(z(t^-)) - ez(t^-)$ using (4). This model becomes (88) if $e = 0$. Let us analyze solutions of (87) in the cases $e = 1$ and $e = 0.7$. When $e = 1$, similarly as above, $z(t) = R(t) \exp(0.05t)$ is a solution on $[0, \pi - 1)$. It follows from (89) that at time $t = \pi - 1$, $z((\pi - 1)^+) = -z((\pi - 1)^-) < 0$. Then, it can be checked that $z(t) = -R(t) \exp(0.05t)$ is a solution on $[\pi - 1, 2\pi - 1)$. Repeating this process, the state jump occurs at times $t = k\pi - 1$, for any $k \in \mathbb{N} \setminus \{0\}$, and it is concluded that $z(t) = R(t) \exp(0.05t)$ if $t \in [k\pi - 1, (k + 1)\pi - 1)$ for some

$k \in \mathbb{N}$, k is even, and $z(t) = -R(t) \exp(0.05t)$ if otherwise. Thus, the global solution is defined by $x(t) = \exp(0.05t)$ if $t \in [k\pi - 1, (k+1)\pi - 1)$ for some $k \in \mathbb{N}$, k is even, and $x(t) = -\exp(0.05t)$ if otherwise. When $e = 0.7$, doing the same analysis as above then a global solution is given by:

$$x(t) = \begin{cases} (0.7)^k \exp(0.05t), & \text{if } t \in [k\pi - 1, (k+1)\pi - 1) \text{ with } k \in \mathbb{N}, k \text{ is even,} \\ -(0.7)^k \exp(0.05t), & \text{if } t \in [k\pi - 1, (k+1)\pi - 1) \text{ with } k \in \mathbb{N}, k \text{ is odd.} \end{cases}$$

To numerically simulate (86) in the cases $e = 1$ or 0.7 , we shall discretize the following measure differential inclusion (MDI) [16, 5, 42]:

$$dz - H(t)z(t)dt \in -\mathcal{N}_{\phi(t^+)} \left(\frac{z(t^+) + ez(t^-)}{1+e} \right), \quad (90)$$

where dz is the differential measure associated with the function $z(\cdot)$, which is expected to be LBV on $[0, +\infty)$ based on the mentioned analysis. Clearly (90) is a combination of (87) and (89). If the state is continuous at t then $z(t^+) = z(t^-) = z(t)$, $\frac{dz}{dt} = \dot{z}(t)$, and (90) becomes (87). Otherwise, that is, the state jump occurs at t , then (90) is equivalent to the MDI:

$$dz(\{t\}) \in -\mathcal{N}_{\phi(t^+)} \left(\frac{z(t^+) + ez(t^-)}{1+e} \right). \quad (91)$$

Since $dz(\{t\}) = z(t^+) - z(t^-)$, (90), (91) become (89). Let us consider (90) on the interval $(t_k, t_{k+1}]$:

$$dz((t_k, t_{k+1}]) - \int_{t_k}^{t_{k+1}} H(t)z(t)dt \in -\mathcal{N}_{\phi(t_{k+1})} \left(\frac{z(t_{k+1}^+) + ez(t_{k+1}^-)}{1+e} \right), \quad (92)$$

which is approximated by:

$$z(t_{k+1}) - z(t_k) - hH(t_{k+1})z(t_{k+1}) \in -\mathcal{N}_{\phi(t_{k+1})} \left(\frac{z(t_{k+1}) + ez(t_k)}{1+e} \right), \quad (93)$$

where $h := t_{k+1} - t_k$ is the time step. Set $z(t_k) := z_k, z(t_{k+1}) := z_{k+1}, \phi(t_{k+1}) := \phi_{k+1}$, and $H(t_{k+1}) := H_{k+1}$, then (93) can be rewritten as: $(1 - hH_{k+1})z_{k+1} - z_k \in -\mathcal{N}_{\phi_{k+1}} \left(\frac{z_{k+1} + ez_k}{1+e} \right)$.

This is equivalent to $z_{k+1} = (1+e)\text{proj}_{\phi_{k+1}} \left(\frac{(1 - hH_{k+1})e + 1}{(1+e)(1 - hH_{k+1})} z_k \right) - ez_k$ using (4). A numerical simulation of (86) when using the model (89) with $e = 1$ or 0.7 is presented in Figures 6.

Calculation of the sets and solutions of local and global variations: Let us consider the variation of the moving set $\phi(t)$ and state $x(t)$ of the system (86) on $[0, +\infty)$. Denote by \mathcal{P} the set of all partitions of $[0, T]$ for some $T > 0$.

(a) The moving set $\phi(\cdot)$ is not LBV, hence not BV on $[0, +\infty)$.

Let us choose a partition $\sigma = (t_0, \dots, t_{N_\sigma}) \in \mathcal{P}$ such that $0 = t_0 < t_1 < \dots < t_k = \pi - 1 < \dots < t_{N_\sigma} = T$ for some $T > \pi - 1$. Then we have

$$V_{[0, T]}(\phi) \geq \sum_{i=0}^{N_\sigma-1} d_H(\phi(t_{i+1}), \phi(t_i)) \geq d_H(\mathbb{R}_+, \mathbb{R}) \geq \sup_{a \in \mathbb{R}} \text{dist}(a, \mathbb{R}_+) = +\infty.$$

This implies that $\phi(t)$ is not BV on $[0, T]$, and therefore is not LBV on $[0, +\infty)$.

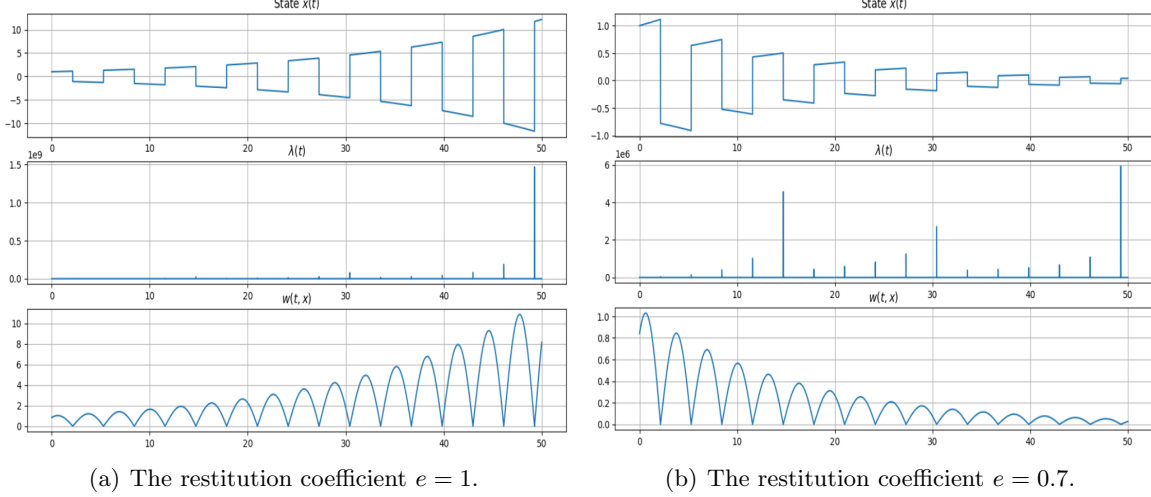


Figure 6: Simulation of (86) using the state jump model (89) with time step $h = 5 \times 10^{-4}$ s.

(b) The solution $x(\cdot)$ without restitution is BV on $[0, +\infty)$ (see Figure 5).

For each $T > 0$ and partition $\sigma = (t_0, \dots, t_{N_\sigma}) \in \mathcal{P}$, denote by $S_\sigma^{[0, \pi-1)}$, e_{max}^σ the set of all index i where $t_i \in [0, \pi - 1)$, the largest index in the set $S_\sigma^{[0, \pi-1)}$, respectively. Then we have

$$\begin{aligned} V_{[0, +\infty)}(x) &= \lim_{T \rightarrow \infty} V_{[0, T]}(x) = \lim_{T \rightarrow \infty} \left(\sup_{\sigma \in \mathcal{P}} \sum_{i=0}^{N_\sigma-1} |x(t_{i+1}) - x(t_i)| \right) = \sup_{\sigma \in \mathcal{P}} \sum_{i \in S_\sigma^{[0, \pi-1)}} |x(t_{i+1}) - x(t_i)| \\ &= \sup_{\sigma \in \mathcal{P}} (2x(t_{e_{max}^\sigma}) - x(t_0)) < \exp(0.05(\pi - 1)) - 1 < +\infty. \end{aligned}$$

(c) The solution $x(\cdot)$ with restitution $e = 1$ is not BV, but LBV on $[0, +\infty)$ (see Figure 6).

Firstly, $x(\cdot)$ is not BV on $[0, +\infty)$ since it is unbounded on $[0, +\infty)$. Now, let us fix $T > 0$ and take $h = \lceil \frac{T+1}{\pi} \rceil$. Then, the variation of $x(\cdot)$ on $[0, T]$ is given by:

$$V_{[0, T]}(x) = \sup_{\sigma \in \mathcal{P}} \left(\sum_{j=0}^{h-1} \sum_{\substack{i \in S_\sigma^{j+1} \\ i \neq e_{max}^{\sigma, j+1}}} |x(t_{i+1}) - x(t_i)| + \sum_{j=1}^{h-1} |x(t_{e_{min}^{\sigma, j+1}}) - x(t_{e_{max}^{\sigma, j}})| \right),$$

with respect to a partition $\sigma \in \mathcal{P}$, $S_\sigma^{j+1} := S_\sigma^{[j\pi-1, (j+1)\pi-1)}$ is the set of all indices i such that $t_i \in [j\pi - 1, (j+1)\pi - 1)$, the values $e_{min}^{\sigma, j+1}$, $e_{max}^{\sigma, j+1}$ are the smallest and largest indices of S_σ^{j+1} for all $j = 0, h-1$, respectively, and N_σ is the number of elements of σ . By some basic calculations, we obtain:

$$\begin{aligned} V_{[0, T]}(x) &= \sup_{\sigma \in \mathcal{P}} \left(\sum_{j=1}^h (-1)^{j+1} (x(t_{e_{max}^{\sigma, j}}) - x(t_{e_{min}^{\sigma, j}})) + \sum_{j=1}^{h-1} (-1)^j (x(t_{e_{min}^{\sigma, j+1}}) - x(t_{e_{max}^{\sigma, j}})) \right) \\ &= \sup_{\sigma \in \mathcal{P}} \left(\left(2 \sum_{j=1}^{h-1} (-1)^{j+1} x(t_{e_{max}^{\sigma, j}}) \right) + (-1)^{h+1} x(t_{e_{max}^{\sigma, h}}) - x(t_{e_{min}^{\sigma, 1}}) \right) \\ &\leq \sup_{\sigma \in \mathcal{P}} \left(\left(2 \sum_{j=1}^h (-1)^{j+1} x(t_{e_{max}^{\sigma, j}}) \right) - x(t_{e_{min}^{\sigma, 1}}) \right). \end{aligned}$$

In this case, we have $|x(t_{e_{max}^{\sigma,j}})| \leq \exp(0.05(j\pi - 1))$ and $x(t_{e_{min}^{\sigma,1}}) = x(t_0) = 1$, for all $j = \overline{1, h}, \sigma \in \mathcal{P}$. Hence the boundedness of $V_{[0,T]}(x)$ is obtained:

$$V_{[0,T]}(x) \leq 2 \sum_{j=1}^h \exp(0.05(j\pi - 1)) - 1 = 2 \exp(0.05(\pi - 1)) \frac{1 - \exp(0.05h\pi)}{1 - \exp(0.05\pi)} - 1 < +\infty.$$

Therefore, the solution $x(\cdot)$ is BV on $[0, T]$ for any $T > 0$ and thus, it is LBV on $[0, +\infty)$.

(d) The solution $x(\cdot)$ with restitution $e = 0.7$ is BV on $[0, +\infty)$ (see Figure 6).

Using the same idea as above, the variation of $x(\cdot)$ on $[0, T]$ is evaluated as:

$$V_{[0,T]}(x) \leq \sup_{\sigma \in \mathcal{P}} \left(\left(2 \sum_{j=1}^h (-1)^{j+1} x(t_{e_{max}^{\sigma,j}}) \right) - x(t_{e_{min}^{\sigma,1}}) \right),$$

where $h = \lceil \frac{T+1}{\pi} \rceil$. In this case, $|x(t_{e_{max}^{\sigma,j}})| \leq (0.7)^{j-1} \exp(0.05(j\pi - 1))$ for all $j = \overline{1, h}, \sigma \in \mathcal{P}$. Therefore, the previous expression implies that:

$$\begin{aligned} V_{[0,+\infty)}(x) &= \lim_{T \rightarrow \infty} V_{[0,T]}(x) \leq \lim_{h \rightarrow \infty} \left(2 \sum_{j=1}^h (0.7)^{j-1} \exp(0.05(j\pi - 1)) \right) - 1 \\ &= \frac{2 \exp(0.05(\pi - 1))}{1 - 0.7 \exp(0.05\pi)} - 1 < +\infty. \end{aligned}$$

In general, let us consider the system (86) where $A(t) = a$ is a constant. We point out now the relation between the coefficient a and the restitution coefficient e such that the solution $x(\cdot)$ is BV or LBV. Clearly, for each value a and e we get a unique solution $x(\cdot)$ of the form:

$$x(t) = \begin{cases} e^k \exp(at), & \text{if } t \in [k\pi - 1, (k+1)\pi - 1) \text{ with } k \in \mathbb{N}, k \text{ is even,} \\ -e^k \exp(at), & \text{if } t \in [k\pi - 1, (k+1)\pi - 1) \text{ with } k \in \mathbb{N}, k \text{ is odd.} \end{cases}$$

From the above analysis, only three types of solutions exist, as depicted in Figures 5, 6. Therefore, the solution $x(\cdot)$ is BV on $[0, +\infty)$ if and only if the following constraint holds:

$$e^k \exp(a((k+1)\pi - 1)) > e^{k+2} \exp(a((k+3)\pi - 1)), \forall k \in \mathbb{N}. \quad (94)$$

Simplifying these constraints, then (94) is equivalent to: $e \exp(a\pi) < 1$. It follows that:

- If $a < 0$ then $x(\cdot)$ is BV on $[0, +\infty)$ for all $e \in [0, 1]$.
- If $a > 0$ and $e \geq \frac{1}{\exp(a\pi)}$ then $x(\cdot)$ is LBV, but it is not BV on $[0, +\infty)$.
- If $a > 0$ and $e < \frac{1}{\exp(a\pi)}$ then $x(\cdot)$ is BV on $[0, +\infty)$.
- If $a = 0$ and $e = 1$ then $x(\cdot)$ is LBV, but it is not BV on $[0, +\infty)$.
- If $a = 0$ and $e \in [0, 1)$ then $x(\cdot)$ is BV on $[0, +\infty)$.

It is concluded that in the framework of the system (86) where the coefficient $A(t)$ is a constant, the solution $x(\cdot)$ is always RCLBV on $[0, +\infty)$, even though the set-valued mapping $\phi(\cdot)$ is not RCLBV.

Existence of finite accumulations of state jumps: Let $C(t) = \sin\left(\frac{1}{t-1}\right)$ in (3) (which is not BV on $[0, 1]$). Intuitively, there should exist a finite accumulation of state jumps at $t = 1$, because such $C(t)$ makes the admissible domain $C(t)x \geq 0$ jump with accumulation. Let us investigate this fact for the following TVLCCS:

$$\begin{cases} \dot{x}(t) = 0.05x(t) + \frac{t^2+1}{t^2+2} \sin\left(\frac{1}{t-1}\right)\lambda(t), \\ 0 \leq \lambda(t) \perp w(t, x(t)) = \sin\left(\frac{1}{t-1}\right)x(t) \geq 0, \quad x(0) = -1. \end{cases} \quad (95)$$

In the FOSwP format, the system (95) can be rewritten equivalently as:

$$\dot{z}(t) - H(t)z(t) \in -\mathcal{N}_{\phi(t)}(z(t)), \quad (96)$$

where $z(t) = R(t)x(t)$, $H(t) = (\dot{R}(t)R(t)^{-1} + 0.05)$, and $\phi(t) = \{x \in \mathbb{R} : C(t)R(t)^{-1}x \geq 0\}$ with $R(t) = \sqrt{\frac{t^2+2}{t^2+1}}$. After few manipulations, $\phi(t)$ is calculated as:

$$\phi(t) = \begin{cases} \mathbb{R}_+, & \text{if } t \in \left(1 + \frac{1}{k\pi}, 1 + \frac{1}{(k-1)\pi}\right) \cup \left(1 + \frac{1}{\pi}, +\infty\right), \text{ for some } k \in \mathbb{Z} \setminus \{0, 1\} \text{ is odd,} \\ \mathbb{R}, & \text{if } t = 1 + \frac{1}{k\pi}, \text{ for some } k \in \mathbb{Z} \setminus \{0\}, \\ \mathbb{R}_-, & \text{otherwise.} \end{cases}$$

Clearly $\phi(t)$ will switch an infinite number of times in the interval $[0, 2]$. In this example, to indicate finite accumulation, we consider the model of state jumps with restitution $e \in [0, 1)$ as in (89). At time $t = 1 + \frac{1}{k\pi}$ for some $k \in \mathbb{Z} \setminus \{0\}$, (96) can be rewritten as $z(t^+) = (1+e) \text{proj}_{\phi(t^+)}(z(t^-)) - ez(t^-)$, in view of (89). The unique solution to (95) is defined by:

$$x(t) = \begin{cases} -\exp(0.05t), & \text{if } t \in \left[0, 1 - \frac{1}{\pi}\right), \\ (-1)^{k+1}e^k \exp(0.05t), & \text{if } t \in \left[1 - \frac{1}{k\pi}, 1 - \frac{1}{(k+1)\pi}\right), \text{ for some } k \in \mathbb{N} \setminus \{0\}, \\ 0, & \text{if } t \in [1, +\infty). \end{cases} \quad (97)$$

The numerical simulations of (95) when $e = 0.98$, time steps $h = 3 \times 10^{-4}s, 3 \times 10^{-5}s, 3 \times 10^{-6}s$ are depicted in Figures 8, 9. If we decrease time step h , then the discrete-time solution approaches the analytical solution as in (97) with an accumulation time of jumps at $t = 1s$. The numerical simulations of (95) in the interesting cases $e = 0$ and 1 with time step $h = 3 \times 10^{-4}s$ are also shown in Figures 7. If we consider (95) with the model of state jump when $e \in [0, 1)$, there always exists an RCBV solution on $[0, +\infty)$. Indeed, first, the solution $x(\cdot)$ in (97) is evidently right-continuous. Let $T > 0$ and we consider the variation of $x(\cdot)$:

$$V_{[0, T]}(x) = \sup_{\sigma \in \mathcal{P}} \sum_{i=0}^{N_\sigma-1} |x(t_{i+1}) - x(t_i)|.$$

For each partition $\sigma \in \mathcal{P}$, let $S_\sigma^{-1}, S_\sigma^0, S_\sigma^k$, be the sets of all indices i such that $t_i \geq 1, t_i \in [0, 1 - \frac{1}{\pi})$, and $t_i \in [1 - \frac{1}{k\pi}, 1 - \frac{1}{(k+1)\pi})$, the values $e_{min}^{\sigma, k}, e_{max}^{\sigma, k}$ are the smallest and largest indices of S_σ^k for all $k \in \{-1, 0, 1, \dots\}$, and $e^{\sigma, \infty}$ is the index such that $t_{e^{\sigma, \infty}} < 1$ and $t_{e^{\sigma, \infty}+1} \geq 1$, respectively. Hence,

we obtain the following assessment:

$$\begin{aligned}
V_{[0,T]}(x) &= \sup_{\sigma \in \mathcal{P}} \left(\sum_{k=-1}^{\infty} \sum_{i \in S_{\sigma}^k, i \neq e_{\max}^{\sigma,k}} |x(t_{i+1}) - x(t_i)| + \sum_{k=0}^{\infty} |x(t_{e_{\min}^{\sigma,k+1}}) - x(t_{e_{\max}^{\sigma,k}})| + |x(t_{e_{\min}^{\sigma,-1}}) - x(t_{e^{\sigma,\infty}})| \right) \\
&= \sup_{\sigma \in \mathcal{P}} \left(\sum_{k=0}^{\infty} \sum_{i \in S_{\sigma}^k, i \neq e_{\max}^{\sigma,k}} |x(t_{i+1}) - x(t_i)| + \sum_{k=0}^{\infty} |x(t_{e_{\min}^{\sigma,k+1}}) - x(t_{e_{\max}^{\sigma,k}})| + |x(t_{e^{\sigma,\infty}})| \right) \\
&= \sup_{\sigma \in \mathcal{P}} \left(\underbrace{\sum_{k=0}^{\infty} |x(t_{e_{\min}^{\sigma,k}}) - x(t_{e_{\max}^{\sigma,k}})|}_{P_1} + \underbrace{\sum_{k=0}^{\infty} |x(t_{e_{\min}^{\sigma,k+1}}) - x(t_{e_{\max}^{\sigma,k}})|}_{P_2} + \underbrace{|x(t_{e^{\sigma,\infty}})|}_{P_3} \right). \tag{98}
\end{aligned}$$

Since $t_{e^{\sigma,\infty}} < 1$, there exists $k \in \mathbb{N}$ such that $t_{e^{\sigma,\infty}} \in S_{\sigma}^k$, then $P_3 = e^k \exp(0.05t_{e^{\sigma,\infty}}) < \exp(0.05)$. Moreover, we also have:

$$P_1 \leq \sum_{k=0}^{\infty} |e^k (\exp(0.05t_{e_{\min}^{\sigma,k}}) - \exp(0.05t_{e_{\max}^{\sigma,k}}))| \leq 2 \exp(0.05) \sum_{k=0}^{\infty} e^k = \frac{2 \exp(0.05)}{1 - e}.$$

And similarly, the following is verified.

$$P_2 = \sum_{k=0}^{\infty} |e^{k+1} \exp(0.05t_{e_{\min}^{\sigma,k+1}}) - e^k \exp(0.05t_{e_{\max}^{\sigma,k}})| \leq \exp(0.05) \sum_{k=0}^{\infty} (e^{k+1} + e^k) = \frac{(e+1) \exp(0.05)}{1 - e}.$$

From bounds of P_1, P_2 , and P_3 , it implies that $V_{[0,+\infty)}(x) = \lim_{T \rightarrow \infty} V_{[0,T]}(x) < \frac{4 \exp(0.05)}{1 - e} < +\infty$. That is, the solution $x(\cdot)$ is BV on $[0, +\infty)$.

Remark 8. *This conclusion holds for any $e \in [0, 1)$. The crucial property in the above example is the existence of an accumulation of jumps which implies that the state vanishes at the ‘singular’ time $t = 1s$. In general TVLCCS the existence of such accumulation cannot be shown. The merit of this example is to prove that some TVLCCS are equivalent to an FOSwP with varying sets that are not RCLBV, but which have a unique RCBV solution with an accumulation of state jumps.*

Remark 9. *When $e = 1$, it is difficult to give a meaning to the solution at $t = 1$. The solution is BV on interval $[0, 1 - \epsilon]$ and $[1 + \epsilon, \epsilon']$ with $\epsilon' > 1 + \epsilon, \epsilon > 0$. But the variation of $x(\cdot)$ on such interval tends to infinity when ϵ tends to zero. Also, it is unclear how to ‘cross’ $t = 1$. Figure 7(b) illustrates this fact. The catching-up algorithm goes through the ‘singular time’ $t = 1$, if there is no time step k such that $t_k = 1$.*

Remark 10. *In Figures 8, we observe that $\lambda(t)$ and $w(t, x)$ may take negative values when $t \geq 1$ which violates the linear complementarity conditions stated in the second line of (95). This is because the obtained solution is an approximate solution due to the discretization of the system, and therefore it and complementarity variables (i.e., λ, w) become inaccurate when the time step h is large. For instance, at the jump times t then $w(t^+, x(t^+)) = \sin\left(\frac{1}{t^+-1}\right) x(t^+)$ is inaccurate because the value $x(t^+)$ is incorrect when $t \geq 1$. Here, as observed, when we decrease the time step from $h = 3 \times 10^{-4}s$ (or $h = 3 \times 10^{-5}s$) to $h = 3 \times 10^{-6}s$, this issue is improved. Therefore, if we consider a sufficiently small time step h , this issue will disappear. A similar issue can be found in [5, p.19].*

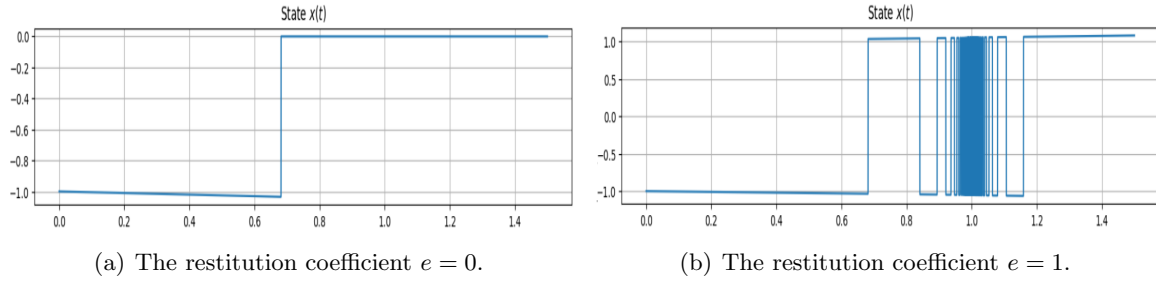


Figure 7: Simulation of (95) using the state jump model (89) with time step $h = 3 \times 10^{-4} s$.

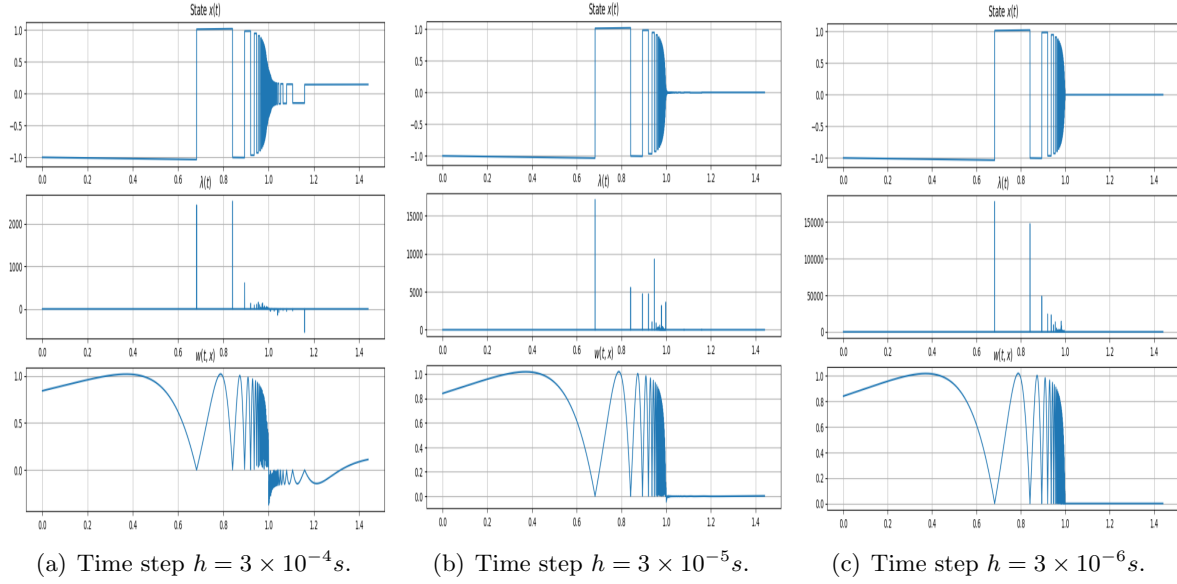


Figure 8: Simulation of (95) using the state jump model (89) with the restitution coefficient $e = 0.98$.

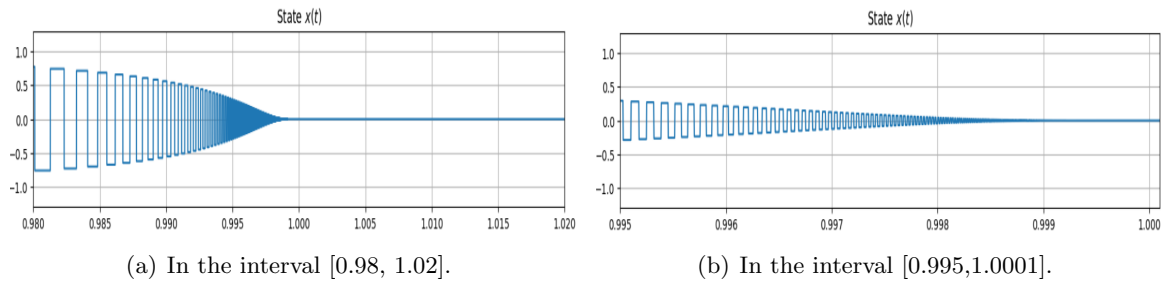


Figure 9: Simulation of (95) using the state jump model (89) with $e = 0.98$, time step $h = 3 \times 10^{-6} s$.

3.1.2 Example 2: State jumps in TVLCCS with time-varying $\mathcal{K}(t)$

Let us consider the following two-dimensional TVLCCS:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)\lambda(t), \\ \mathcal{K}(t) \ni \lambda(t) \perp w(t, x(t)) = C(t)x(t) \in \mathcal{K}^*(t), \\ x(0) = [1 \ 2]^\top, \end{cases} \quad (99)$$

where $x(t), \lambda(t) \in \mathbb{R}^2$, $A(t) = \text{diag}(a_1, a_2)$ for some $a_1, a_2 \in \mathbb{R}$, and $\mathcal{K}(t) = [\mathbb{R}_+ \ (t_0 - t)\mathbb{R}_+]^\top$, $B(t) = C(t) = \text{diag}(b_1(t), b_2(t))$ such that on $[0, t_0)$, $b_1(t) > 0, b_2(t) = 0$, on $(t_0, +\infty)$, $b_1(t) = 0, b_2(t) > 0$, and $b_1(t_0) = b_2(t_0) = 0$ for some $t_0 \in \mathbb{R}_+$. Let us analyze solutions of (99). Firstly, we can rewrite (99) in the FOSwP format as:

$$\dot{x}(t) - A(t)x(t) \in -\mathcal{N}_{\phi(t)}(x(t)), \quad (100)$$

where $\phi(t) = \{x \in \mathbb{R}^2 : C(t)x(t) \in \mathcal{K}^*(t)\}$. It follows that $\phi(t) = [\mathbb{R}_+ \ \mathbb{R}]^\top$ if $t < t_0$, $\phi(t) = \mathbb{R}^2$ if $t = t_0$, and $\phi(t) = [\mathbb{R} \ \mathbb{R}_-]^\top$ if $t > t_0$. Obviously, $\phi(t)$ jumps at time t_0 , and the Hausdorff distance $d_H(\phi(t_0^-), \phi(t_0^+)) = +\infty$. On $[0, t_0)$, since $x(t) \in \text{int}(\phi(t))$, (100) can be rewritten as $\dot{x}(t) = A(t)x(t)$, which implies that $x(t) = [\exp(a_1 t) \ 2 \exp(a_2 t)]^\top > 0$. Since $x(t_0^-) > 0$ and $\phi(\cdot)$ switches at time $t = t_0$, the second component of the state $x(\cdot)$ jumps at $t = t_0$. Using the model of state jump with restitution coefficient $e \in [0, 1]$ as in (89), we obtain:

$$x(t_0^+) - x(t_0^-) \in -\mathcal{N}_{\phi(t_0^+)}\left(\frac{x(t_0^+) + ex(t_0^-)}{1 + e}\right).$$

Therefore, $x(t_0^+) = (1 + e)\text{proj}_{\phi(t_0^+)}(x(t_0^-)) - ex(t_0^-) = [\exp(a_1 t_0) \ -2e \exp(a_2 t_0)]^\top$. On $[t_0, +\infty)$, proceeding as above, $x(t)$ has the form $x(t) = [\exp(a_1 t) \ -2e \exp(a_2 t)]^\top$. From all of the above, a global solution of the system (99) is given by $x(t) = [\exp(a_1 t) \ 2 \exp(a_2 t)]^\top$ if $t < t_0$, and $x(t) = [\exp(a_1 t) \ -2e \exp(a_2 t)]^\top$ if $t \geq t_0$. Choosing $a_1 = a_2 = 0.05, t_0 = 4, e = 0$ and $b_1(\cdot), b_2(\cdot)$ to be any functions that satisfy conditions set out at the beginning, and the time step $h = 5 \times 10^{-4}s$. Then a numerical simulation of (99) is depicted in Figure 10. From the form of the solution $x(\cdot)$, clearly $x(\cdot)$ is always right-continuous, and in this case, it is not BV, but LBV on $[0, +\infty)$. Consider the general cases of a_1, a_2 and e , we conclude that:

- If $a_1, a_2 \in (-\infty, 0]$ then $x(t)$ is BV on $[0, +\infty)$ for all $e \in [0, 1]$.
- If $a_1 > 0$ or $a_2 > 0$ then $x(t)$ is not BV, but LBV on $[0, +\infty)$ for all $e \in [0, 1]$.

Thus the system (99) always has a RCLBV solution for all $a_1, a_2 \in \mathbb{R}$ and for all $e \in [0, 1]$.

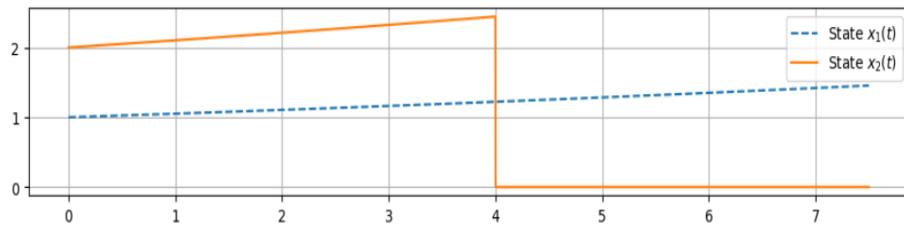


Figure 10: Simulation of (99) using the state jump model (89) with $e = 0$, time step $h = 5 \times 10^{-4}s$, $a_1 = a_2 = 0.05$, and $t_0 = 4$.

3.1.3 Recapitulation

These examples have shown important differences concerning the ability to transform TVLCCS into FOSwP, as well as the jump times of the state in TVLCCS, compared to LCS, which have been studied in [10, 4, 9, 8, 44]. Firstly, it is possible to transform from TVLCCS to FOSwP and *vice versa*, even when $\phi(\cdot)$ (*i.e.*, time-varying set-valued mapping in FOSwP) is not LBV, which violates a usual condition in the analysis of FOSwP used in Theorem 3, and in [4, Section 5.2, (A3)], [45, 10]. This is impossible for LCS. Actually, using the same reasoning as in [10, Proposition 3.2],

it can be proved that a time-varying $C(t)$ is necessary to pass from the TVLCCS to a FOSwP with such “unbounded” set jump. Indeed a TVLCCS with constant C (and zero D) and constant \mathcal{K} yields for each t a polyhedral set $\phi(t)$ where the time-dependency stems solely from $Fu(t)$ in (3b). Moreover $d_H(\phi(t), \phi(s)) \leq \gamma \|Fu(t) - Fu(s)\|$ for some $\gamma > 0$ and all $s, t \in \mathbb{R}_+$ (using Lemma A2 which compiles previous results by Walkup and Wets, and by Robinson). Thus, provided $u(\cdot)$ is a bounded function, it is impossible to obtain an FOSwP as above if C and $\mathcal{K} = \mathbb{R}_+^m$ are constant. For instance, the FOSwP: $-\dot{x}(t) \in \mathcal{N}_{\phi(t)}(x(t))$, $x(0) = x_0 \in \phi(0)$, $\phi(t) = \mathbb{R}_+$ if $t \in [0, t_1)$, $\phi(t) = \mathbb{R}_-$ if $t \in [t_1, +\infty)$, cannot be shown to be equivalent to any LCS (1) with $w(t, x) = Cx + Fu(t)$. The key to this issue lies in the time variation of $C(\cdot)$. The matrix $C(t)$ in (3) plays a significant role in the study of TVLCCS. Secondly, when LCS (1) is autonomous (*i.e.*, $E = F = 0$), results in [8, 44] have shown that state jumps occur only at the initial time due to inconsistent initialization, *i.e.*, $Cx(0) \notin (\text{SOL}(0, D))^*$. When LCS (1) has time-varying external inputs, the results in [9] pointed out that state jumps occur at arbitrary time t where $Fu(t)$ is discontinuous (*i.e.*, $\{t : x(t^+) \neq x(t^-)\} \subseteq \{t : Fu(t^+) \neq Fu(t^-)\}$) or at the initial time if $Cx(0) + Fu(0) \notin (\text{SOL}(0, D))^*$. In other words, if $Fu(\cdot)$ is continuous then $x(\cdot)$ is continuous (excepted possibly at the initial time). However, in the examples above, even though $F(\cdot)u(\cdot) = 0$ is a continuous function, the state $x(\cdot)$ is discontinuous at $t = \pi - 1$ in (86), and at $t = t_0$ in (99) even if $C(\cdot)$ is a continuous function and at $t = 1 - \frac{1}{\pi}$ in (95). This is a new finding when studying TVLCCS.

3.2 Analysis of the causes of state jumps in Section 3.1

In this section, we will use the theoretical results presented in Section 2.4 to clarify the reason why state jumps appear in examples in Section 3.1. Firstly, let us come back to the system (86). Note that state jumps occur when $T > \pi - 1$. It can be checked that (86) does not satisfy Assumption 3 and items 2, 4 in Corollary 2, where $P(t) = \frac{t^2+2}{t^2+1}$ is a solution of (66). Indeed, we have:

$$\text{rge}(C(t)) = \text{rge}(R(t)^{-1}C(t)^\top) = \begin{cases} 0, & \text{if } t = k\pi - 1 \text{ for some } k \in \mathbb{N} \setminus \{0\}, \\ \mathbb{R}, & \text{otherwise.} \end{cases}$$

Thus, item 2 fails. Let $s = \frac{\pi-1}{2}$, $t = \frac{3\pi-3}{2}$, then $s < t$ and $(C(s)R(s)^{-1})^{-1}(\mathbb{R}_+) = \mathbb{R}_+ \not\subseteq \mathbb{R}_- = (C(t)R(t)^{-1})^{-1}(\mathbb{R}_+)$. So item 4 fails. Finally, Assumption 3 fails due to the following:

$$\text{rge}(C(t)) \cap \text{rint}(\mathbb{R}_+) = \begin{cases} \emptyset, & \text{if } t = k\pi - 1 \text{ for some } k \in \mathbb{N} \setminus \{0\}, \\ \mathbb{R}_+ \setminus \{0\}, & \text{otherwise.} \end{cases}$$

Therefore, the existence of an AC solution to (86) is not guaranteed by Corollary 2. In addition, it can also be verified that there is no mapping $Q(\cdot)$ that satisfies assumptions in Proposition 3. Proposition 4 can not be used either since $\phi(t)$ in the FOSwP (87), which is equivalent to (86), is not bounded. Hence, we may expect that state jumps will be necessary, and they will be analyzed by certain conditions beyond the results in this article. Secondly, let us consider the system (95) (*resp.* (99)) and note that state jumps occur when $T > 1 - \frac{1}{\pi}$ (*resp.* $T > t_0$). Similar to the above analysis, it is verified that (95) does not satisfy Assumption 3, items 2, 4, and 5 in Corollary 2, since $C(\cdot)$ is not right-continuous at $t = 1s$ (*resp.* (99) does not satisfy Assumption 3, items 1, 2, 4, and 5 in Corollary 2, since $C(\cdot)$ is not right-continuous and $\mathcal{K}^*(\cdot)$ is not inner semicontinuous from right at t_0). In addition, these examples also do not satisfy assumptions in Propositions 3, 4. Thus state jumps (as described in Section 3.1) can be expected for these system.

3.3 Further comments

Some assumptions which are made in the foregoing theoretical results deserve to be relaxed. It would be interesting to relax items 1 (iii) and 2 in Theorem 2. As the examples in Section 3.1 demonstrate, the role played by $C(\cdot)$ is prominent in TVLCCS. It yields equivalent FOSwP with switching unbounded sets for which the Hausdorff distance is unsuitable, and with Zeno behavior (infinitely many switches in a finite time). In this article, TVLCCS are studied through the analysis of Lur'e DIs as in (10). As pointed out after the proof of Proposition 1, one can study Lur'e DIs which do not stem from TVLCCS. In this setting, extending the results in [19, 18] to varying matrices and/or cones which depend also on the state, certainly is of interest. The above examples show that modifying the state jump rule can significantly influence the solution's behavior. Extensions to higher dimensions could be tackled.

4 Conclusions

This article deals with the well-posedness of finite-dimensional time-varying linear cone complementarity systems. All the system's data are allowed to depend on time, including the cones defining the complementarity constraints. The passivity of a suitable operator is a key assumption. It allows us to transform the dynamics into a suitable form known as Lur'e differential inclusion. Various cases are tackled depending on the system's data, with absolutely continuous or right-continuous of local bounded variation solutions. Examples from electrical circuits illustrate the theoretical developments.

Appendices

A Some useful definitions and results

Theorem A1. [26, Proposition C.3.8] *Consider the following nonlinear differential equation:*

$$\dot{x}(t) = f(t, x(t)), \quad t \geq 0; \quad x(0) = x_0, \quad (\text{i1})$$

where $x(t) \in \mathbb{R}^n$ and $f : [0, +\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. Assume that the following conditions hold:

1. $f(\cdot, x) : [0, +\infty) \rightarrow \mathbb{R}^n$ is measurable for each fixed x .
2. $f(t, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous for each fixed t .
3. There is a locally integrable function $\alpha : [0, +\infty) \rightarrow \mathbb{R}_+$ such that for each $t \geq 0$:

$$\|f(t, x) - f(t, y)\| \leq \alpha(t)\|x - y\|, \quad \forall x, y \in \mathbb{R}^n.$$

4. For each fixed $x_0 \in \mathbb{R}^n$, there is a locally integrable function $\beta : [0, +\infty) \rightarrow \mathbb{R}_+$ such that:

$$\|f(t, x_0)\| \leq \beta(t), \quad \text{for almost all } t.$$

Then, there exists a unique locally AC solution to (i1) on $[0, +\infty)$, for any $x_0 \in \mathbb{R}^n$.

Remark A1. Denote by $\mathfrak{B}(\mathbb{R})$, $\mathfrak{B}([0, +\infty))$ the Borel σ -algebra of \mathbb{R} and $[0, +\infty)$, respectively [40, Definition 1.2.10]. Since $\mathfrak{B}(\mathbb{R})$ contains open and closed sets in \mathbb{R} , using [40, Lemma 6.2.4] then $\mathfrak{B}([0, +\infty)) = \{A \in \mathfrak{B}(\mathbb{R}) : A \subset [0, +\infty)\}$. Clearly if $f^{-1}(B) \in \mathfrak{B}([0, +\infty))$, then $f^{-1}(B)$ is a Lebesgue measurable set. A sufficient condition for this is that $[0, +\infty) \setminus f^{-1}(B)$ is a closed set. Indeed, it is obtained by the following:

$$f^{-1}(B) = \underbrace{[0, +\infty)}_{\in \mathfrak{B}(\mathbb{R})} \cap \underbrace{\mathbb{R} \setminus ([0, +\infty) \setminus f^{-1}(B))}_{\in \mathfrak{B}(\mathbb{R})} \in \mathfrak{B}(\mathbb{R}).$$

Lemma A1. (Hoffman's bound) [46, Lemma 3.2.3] For any matrices $C \in \mathbb{R}^{l \times n}$ and $A \in \mathbb{R}^{m \times n}$, there exists a constant $c > 0$ (depending only on C and A) such that for all vectors $d \in \mathbb{R}^l$ and $b \in \mathbb{R}^m$ for which $P := \{x \in \mathbb{R}^n : Cx = d, Ax \leq b\}$ is nonempty,

$$\text{dist}(x, P) \leq c(\|Cx - d\| + \|\max(0, Ax - b)\|), \quad \forall x \in \mathbb{R}^n.$$

Denote by $P(A, b) := \{x \in \mathbb{R}^n : Ax \leq b\}$ a convex polyhedral set, for some $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

Corollary A1. [46, Corollary 3.2.5] Let the matrix $A \in \mathbb{R}^{m \times n}$ be given. Then, for every bounded subset S of \mathbb{R}^n , there exists a constant $L > 0$ such that for every vector $b \in \mathbb{R}^m$ for which $P(A, b)$ is nonempty and for every $x \in S \cap P(A', b')$,

$$\text{dist}(x, P(A, b)) \leq L(\|A - A'\| + \|b - b'\|).$$

Note that L is a bound of S .

Proposition A1. (Chain rule) [28, 21, 4] Let $f : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semicontinuous function, and $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear operator. Assume that either the function $f(\cdot)$ is polyhedral or for some x_0 with $Ax_0 \in \text{dom}(f) : \text{rge}(A) - \mathbb{R}_+(\text{dom}(f) - Ax_0)$ is a vector subspace of \mathbb{R}^m , where $\text{dom}(f) := \{y \in \mathbb{R}^m : f(y) < +\infty\}$. Then the subdifferential in the sense of convex analysis of the composite functional $f \circ A : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is given by:

$$\partial(f \circ A)(x) = A^\top \partial f(Ax), \quad \forall x \in \mathbb{R}^n.$$

Lemma A2. (Robinson's bound) [47, p.760] Let $\mathcal{K} := \{y \in \mathbb{R}^m : Dy \geq 0\}$ be a closed convex polyhedral cone, for some matrix D . Let $S := \{x \in \mathbb{R}^n : Ax - b \in \mathcal{K}\}$, and $S' := \{x \in \mathbb{R}^n : Ax - b' \in \mathcal{K}\}$, where $A \in \mathbb{R}^{m \times n}$ is a matrix, and $b, b' \in \mathbb{R}^m$ are vectors. Suppose that S and S' are nonempty, then there exist a constant $\gamma_{A, \mathcal{K}}$ (depending only on A and \mathcal{K}) such that:

$$d_H(S, S') \leq \gamma_{A, \mathcal{K}} \|D\| \|b - b'\|.$$

Definition A1. [10, 34, 4] Consider the following FOSwP:

$$\dot{z}(t) \in -\mathcal{N}_{\phi(t)}(z(t)) + f(t, z(t)), \quad z(0) = z_0 \in \phi(0). \quad (\text{i2})$$

Assume that $\phi(\cdot)$ is RCLBV on $[0, +\infty)$, that is, for each $T > 0$, there exists a positive Radon measure μ on $[0, T]$ such that $d_H(\phi(s), \phi(t)) \leq \mu((s, t])$, for all $0 \leq s \leq t \leq T$. Assume further that there exists a locally integrable function $\beta(\cdot) : [0, +\infty) \rightarrow \mathbb{R}_+$ such that $\|f(t, z)\| \leq \beta(t)(1 + \|z\|)$ for all $z \in \mathbb{R}^n$, for all $t \geq 0$. A mapping $z : [0, +\infty) \rightarrow \mathbb{R}^n$ is a solution of (i2) if:

(i) $z(\cdot)$ is RCLBV on $[0, +\infty)$ and satisfies $z(0) = z_0$, $z(t) \in \phi(t)$ for all $t \geq 0$;

(ii) there exists a positive Radon measure ν AC equivalent to the measure $\mu + \lambda$ and with respect to which the differential measure dz of $z(\cdot)$ is AC with the density $\frac{dz}{d\nu}$ is a locally integrable function from $[0, +\infty)$ to \mathbb{R}^n with respect to the measure ν , and the following holds:

$$\frac{dz}{d\nu}(t) \in -\mathcal{N}_{S(t)}(z(t)) + f(t, z(t))\frac{d\lambda}{d\nu}(t), \quad \nu - \text{almost everywhere on } [0, +\infty),$$

where λ is a one-dimensional Lebesgue measure, and $\frac{d\lambda}{d\nu}(\cdot)$ is the density relative to ν of λ .

B The proof of Remark 1 when $D(t) = D(t)^\top \succcurlyeq 0$

Firstly, for each $t \in [0, T]$, since $\text{rge}(D(t)) = (\ker D(t)^\top)^\perp$ [29, (2.4.14)-p.95], using the symmetry of $D(t)$ and [21, Example 6.23], we get $\text{rge}(D(t)) = (\ker D(t))^*$. Therefore, $\mathcal{K}^*(t) + \text{rge}(D(t)) = (\mathcal{K}(t) \cap \ker D(t))^*$ [21, Corollary 11.25]. So now, let us prove that $\mathcal{H}(t) = (\mathcal{K}(t) \cap \ker D(t))^*$, for all $t \in [0, T]$. Indeed, let any $t \in [0, T]$.

[\subseteq] Let any $y \in \mathcal{H}(t)$, then $\hat{Q}(t)Q(t)y \geq 0$, that is, $Q(t)y \in (\text{SOL}(0, Q(t)D(t)Q(t)^\top))^*$. This implies that

$$\langle Q(t)y, z \rangle \geq 0, \quad \forall z \in \text{SOL}(0, Q(t)D(t)Q(t)^\top). \quad (\text{ii1})$$

Take any $\alpha \in \mathcal{K}(t) \cap \ker D(t)$, then there exists $\xi \in \mathbb{R}_+^j$ such that $\alpha = Q(t)^\top \xi$ and $D(t)\alpha = 0$. Therefore, we obtain $Q(t)D(t)Q(t)^\top \xi = 0$, and thus $\xi \in \text{SOL}(0, Q(t)D(t)Q(t)^\top)$. It follows from (ii1) that $\langle y, \alpha \rangle = \langle y, Q(t)^\top \xi \rangle = \langle Q(t)y, \xi \rangle \geq 0$. Because α is chosen arbitrarily, $\langle y, \alpha \rangle \geq 0$ for all $\alpha \in \mathcal{K}(t) \cap \ker D(t)$. This means $y \in (\mathcal{K}(t) \cap \ker D(t))^*$.

[\supseteq] Let any $h \in (\mathcal{K}(t) \cap \ker D(t))^*$, then $\langle h, z \rangle \geq 0$ for all $z \in \mathcal{K}(t) \cap \ker D(t)$. To obtain $h \in \mathcal{H}(t)$, we shall prove that $Q(t)h \in (\text{SOL}(0, Q(t)D(t)Q(t)^\top))^*$. Indeed, take any $\alpha \in \text{SOL}(0, Q(t)D(t)Q(t)^\top)$, then we have $\alpha \in \mathbb{R}_+^j$ and $\langle \alpha, Q(t)D(t)Q(t)^\top \alpha \rangle = 0$. Since $D(t) = D(t)^\top \succcurlyeq 0$, there exists $\tilde{D}(t) = \tilde{D}(t)^\top \succcurlyeq 0$ such that $\tilde{D}(t)^2 = D(t)$. Moreover, it follows from the symmetry of $\tilde{D}(t)$ that $\ker D(t) = \ker \tilde{D}(t)$. Therefore, we get $0 = \langle \alpha, Q(t)D(t)Q(t)^\top \alpha \rangle = \langle \tilde{D}(t)Q(t)^\top \alpha, \tilde{D}(t)Q(t)^\top \alpha \rangle = \|\tilde{D}(t)Q(t)^\top \alpha\|^2$. Hence, we get $Q(t)^\top \alpha \in \ker \tilde{D}(t) = \ker D(t)$. Alternatively, since $\alpha \in \mathbb{R}_+^j$, $Q(t)^\top \alpha \in \mathcal{K}(t)$. Therefore, $Q(t)^\top \alpha \in \mathcal{K}(t) \cap \ker D(t)$ and thus, $\langle h, Q(t)^\top \alpha \rangle \geq 0$, i.e., $\langle Q(t)h, \alpha \rangle \geq 0$.

C The proof of Theorem 1 in Section 2.2.3

The proof is the same as the proof of [20, Theorem 23], with some small changes in steps 1 and 2 as shown below. *In step 1:* At the beginning of the proof, using **(A1)** and the Lipschitzness of $f(t, \cdot)$, then for all $t \in [0, T]$ we also obtain:

$$\begin{aligned} \langle \dot{x}_{l+1}(t) - \dot{x}_l(t), x_{l+1}(t) - x_l(t) \rangle &\leq \langle f(t, x_l(t)) - f(t, x_{l-1}(t)), x_{l+1}(t) - x_l(t) \rangle \\ &\leq L_f(t) \|x_l(t) - x_{l-1}(t)\| \|x_{l+1}(t) - x_l(t)\| \\ &\leq L_f \|x_l(t) - x_{l-1}(t)\| \|x_{l+1}(t) - x_l(t)\|. \end{aligned}$$

where $L_f(t)$ is a (bounded) Lipschitz constant and $L_f := \max_{t \in [0, T]} (L_f(t))$. *In step 2:* At the beginning of the case where F satisfies (LG) (see [20, p.1047]), let $\varphi_l(t) := \int_0^t \|f(\tau, x_{l-1}(\tau))\| + \|u(\tau)\| d\tau$ for all $t \in [0, T]$ and $l \geq 1$. By assumption, $\|f(t, z)\| \leq M(T) + N(T)\|z\|$ for all $z \in \text{dom } f(t, \cdot)$ and for all $t \in [0, T]$. Therefore, the following holds:

$$\varphi_l(\bar{\tau}) - \varphi_l(\underline{\tau}) \leq (N(T)C_1 + M(T))(\bar{\tau} - \underline{\tau}) + \int_{\underline{\tau}}^{\bar{\tau}} \|u(\tau)\| d\tau, \quad \forall 0 \leq \underline{\tau} < \bar{\tau} \leq T,$$

where C_1 is a bound of $x_l(\cdot)$ on $[0, T]$ for all $l \geq 1$, $N(T), M(T)$ coincide with L_f and $\|f(t, 0)\|$ in [20, Theorem 23], respectively. The rest of the proof is the same as the proof of [20, Theorem 23].

D Constraint qualification [10, Equation (22)]

For each $t \in [0, +\infty)$, let us consider the moving set $\phi(t) := \{x \in \mathbb{R}^n : CR^{-1}x + Fu(t) + f \geq 0\}$, where $C \in \mathbb{R}^{m \times n}$, $F \in \mathbb{R}^{m \times u}$, $R^{-1} \in \mathbb{R}^{n \times n}$, $f \in \mathbb{R}^m$ are the constant matrices and vector, $u(t) \in \mathbb{R}^u$. Then, $\phi(t)$ is nonempty for all $t \geq 0$ if the following (*i.e.*, [10, Equation (22)]) holds:

$$\text{rge}(C) - \mathbb{R}_+^m = \mathbb{R}^m.$$

This means for all $y \in \mathbb{R}^m$, there exist $y_1 \in \text{rge}(C)$ and $y_2 \in \mathbb{R}_+^m$ such that $y = y_1 - y_2$.

E Summary of main results in [20]

The paper [20] has focused on investigating the existence of solutions to the following differential inclusion:

$$\dot{x}(t) \in -F(t, x(t)), \quad \forall t \in [0, T]; \quad x(0) = x_0, \quad (\text{v1})$$

where $F(t, \cdot) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is maximal monotone for each $t \in [0, T]$. We say that x is an absolutely continuous solution of (v1) if $x \in AC([0, T], \mathbb{R}^n)$, and $x(t) \in \text{Dom } F(t, \cdot)$ and x satisfies (v1) almost everywhere on $[0, T]$. The following theorem represents the primary result of the paper. Additionally, the paper presents several results concerning broader classes and corollaries when applied to systems consisting of differential equations, where certain variables are combined using a maximal monotone operator (LCS for instance). It is important to emphasize that the assumptions **(A1)**, **(A2)**, **(A3)**, **(LG)**, and **(YB)** in Section 2.2.3 will be put to use in the subsequent part.

Theorem E1. [20, Theorem 11] *Consider (v1) and assume that **(A1)**-**(A3)** hold. If the linear growth assumption **(LG)** holds, then there exists a unique AC solution of (v1) for every $x_0 \in \text{cl}(\text{Dom } F(0, \cdot))$. In case the Yosida approximation bound assumption **(YB)** holds, then there exists a unique AC solution of (v1) for every $x_0 \in \text{Dom } F(0, \cdot)$.*

E.1 Extensions of Theorem E1

The first extension is provided when considering a non-autonomous DI in the form of:

$$\dot{x}(t) \in -F(t, x(t)) + u(t), \quad \forall t \in [0, T]; \quad x(0) = x_0, \quad (\text{v2})$$

where $F(t, \cdot) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is maximal monotone for all $t \geq 0$ and $u \in L_1([0, T], \mathbb{R}^n)$.

Theorem E2. [20, Theorem 22] *Consider (v2) and suppose that $u \in L_1([0, T], \mathbb{R}^n)$ and $F(t, \cdot)$ satisfies assumptions **(A1)**-**(A3)**. If **(LG)** holds, then for every $x_0 \in \text{cl}(\text{Dom } F(0, \cdot))$, there exists a unique solution $x \in AC([0, T], \mathbb{R}^n)$. If **(YB)** holds, then for every $x_0 \in \text{Dom } F(0, \cdot)$, there exists a unique solution $x \in AC([0, T], \mathbb{R}^n)$.*

The second extension is presented when a Lipschitz perturbation appears. Let us consider DIs of the form:

$$\dot{x}(t) \in -F(t, x(t)) + f(x(t)) + u(t), \quad \forall t \in [0, T]; \quad x(0) = x_0, \quad (\text{v3})$$

where $F(t, \cdot) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is maximal monotone for all $t \in [0, T]$, $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a function and $u \in L_1([0, T], \mathbb{R}^n)$. Based on Theorem E2, the following result guarantees the existence and uniqueness of a solution to (v3).

Theorem E3. [20, Theorem 23] *Consider (v3) and suppose that $u \in L_1([0, T], \mathbb{R}^n)$, $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a Lipschitz continuous function, and $F(t, \cdot)$ satisfies assumptions **(A1)**-**(A3)**. If **(LG)** holds, then for every $x_0 \in \text{cl}(\text{Dom } F(0, \cdot))$, there exists a unique solution $x \in AC([0, T], \mathbb{R}^n)$. If **(YB)** holds, then for every $x_0 \in \text{Dom } F(0, \cdot)$, there exists a unique solution $x \in AC([0, T], \mathbb{R}^n)$.*

E.2 A corollary of Theorem E2

A corollary of Theorem E2 is derived when we apply it to a class of systems created by linear passive systems and the maximal monotone relationships among certain variables in those systems. In particular, let us consider the following system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bz(t) + u(t), & \text{(v4a)} \\ w(t) = Cx(t) + Dz(t) + v(t), & \text{(v4b)} \\ w(t) \in \mathcal{M}(-z(t)), & \text{(v4c)} \end{cases}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^n$, $v \in \mathbb{R}^m$ are external inputs and $\mathcal{M} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is some set-valued map. From (v4b) and (v4c), we get $z(t) \in -(\mathcal{M} + D)^{-1}(Cx(t) + v(t))$ and substituting it into (v4a), then (v4) is equivalent to the following, where it takes the form of (v2).

$$\dot{x}(t) \in -H(t, x(t)) + u(t),$$

where

$$H(t, x(t)) = -Ax + B(\mathcal{M} + D)^{-1}(Cx + v(t)) \quad \text{(v5)}$$

and $\text{Dom } H(t, \cdot) = C^{-1}(\text{Im}(\mathcal{M} + D) - v(t))$. The next theorem specifies the conditions under which $H(\cdot, \cdot)$, defined in (v5), satisfies the assumptions **(A1)**-**(A3)** and **(LG)** in Theorem E2.

Theorem E4. [20, Theorem 24] *Let $T > 0$. Suppose that*

(i) *The system (A, B, C, D) is passive with the storage function $x \mapsto \frac{1}{2}x^\top x$.*

(ii) *\mathcal{M} is maximal monotone.*

(iii) *For all $t \in [0, T]$, we have $\text{rge}(C) \cap \text{rint}(\text{Im}(\mathcal{M} + D) - v(t)) \neq \emptyset$.*

(iv) *v is bounded on $[0, T]$.*

(v) *There exist an AC nondecreasing function $\theta : [0, T] \rightarrow \mathbb{R}$ such that for all s, t with $0 \leq s \leq t \leq T$:*

$$\sup_{w \in \text{rge}(C) \cap (\text{Im}(\mathcal{M} + D) - v(s))} \text{dist}(w, \text{rge}(C) \cap (\text{Im}(\mathcal{M} + D) - v(t))) \leq \theta(t) - \theta(s).$$

(vi) *There exists a positive number α such that for all $\eta \in \text{Dom}(\mathcal{M} + D)^{-1}$, then*

$$\|B((\mathcal{M} + D)^{-1})^0(\eta)\| \leq \alpha(1 + \|\eta\|).$$

*Then, $H(\cdot, \cdot)$ satisfies assumptions **(A1)**, **(A2)**, **(A3)**, and **(LG)**.*

E.3 A corollary of Theorem E4

In this section, the linear complementarity systems with time-dependent external inputs are considered as a special case of (v4) if $\mathcal{M} \equiv \mathcal{P} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is the maximal monotone set-valued mapping defined by:

$$\mathcal{P}(\xi) := \{\eta \in \mathbb{R}^m : \eta \geq 0, \xi \leq 0, \text{ and } \langle \eta, \xi \rangle = 0\}, \text{ for any } \xi \in \mathbb{R}^m.$$

The aforementioned LCS systems have the form:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bz(t) + u(t), \\ w(t) = Cx(t) + Dz(t) + v(t), \\ 0 \leq z(t) \perp w(t) \geq 0, \end{cases} \quad (\text{v6})$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ are external inputs. The following theorem is a corollary of Theorem E4, obtained by directly applying Theorem E4 to (v6).

Theorem E5. [20, Theorem 26] *Let $T > 0$. Suppose that*

- (i) *The system (A, B, C, D) is passive with the storage function $x \mapsto \frac{1}{2}x^\top x$.*
- (ii) *$\text{rge}(C) \cap \text{rint}(\text{Im}(\mathcal{P} + D) - v(t)) \neq \emptyset$ for all $t \in [0, T]$.*
- (iii) *$v \in AC([0, T], \mathbb{R}^m)$.*

Then, $H(\cdot, \cdot)$ (defined by (v5) with $\mathcal{M} \equiv \mathcal{P}$) satisfies assumptions (A1), (A2), (A3), and (LG).

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