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# On hyperexponential stabilization of a chain of integrators in continuous and discrete time

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## Abstract

A time-varying state feedback is presented that guarantees stability and a hyperexponential rate of convergence of all trajectories to the origin for a chain of integrators. It is shown that this convergence property is uniform with respect to matched bounded external disturbances. An implicit time-discretization scheme of the closed-loop system is given, which preserves all main properties of the continuous-time counterpart, and in addition has bounded errors with respect to the measurement noises. Based on this discretization, for sampled-and-hold implementation, a modified linear time-varying state feedback is proposed, which provides an accelerated rate of convergence to the continuous-time plant. The efficiency of the suggested control algorithms is illustrated through numeric experiments.

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## 1 Introduction

The problem of a state feedback design that ensures asymptotic stability of the origin for a continuous/discrete-time linear or nonlinear dynamical system belongs to the foundation of the control engineering [Utkin, 1992, Isidori, 1995, Khalil, 2002, Chernous'ko et al., 2008]. There are many methods solving this problem while providing different performance characteristics for the closed-loop dynamics, which include 1) the rate of convergence to the origin, that can be, for example, exponential or finite/fixed-time [Bhat and Bernstein, 2000, Polyakov, 2012]; 2) robustness with respect to the exogenous disturbances, which is frequently understood in the input-to-state stability sense [Sontag, 2007, Dashkovskiy et al., 2011]; 3) optimization of the measurement noise sensitivity (*i.e.*, increasing the gains leads to a faster convergence and better exogenous disturbance attenuation, however, at the price of the noise influence augmentation); and 4) a reliable discrete-time or sampled-and-hold implementation that preserves the main properties of the closed-loop system in the continuous-time (this can be a complex issue for highly nonlinear stabilizers [Efimov et al., 2017, Polyakov et al.,

2019]), in the sliding mode control systems it can be related with the chattering phenomenon [Shtessel et al., 2014]).

In various applications, an accelerated convergence of the trajectories to an equilibrium is required [Efimov and Polyakov, 2021], which is usually provided by discontinuous sliding mode or continuous finite/fixed-time control laws. Due to mentioned complexity of discrete-time implementation of a nonlinear feedback, their sampled/discrete-time realizations need a lot of effort. Tuning the control gains in nonlinear systems is also difficult. Recently, the prescribed-time stabilization framework (a special case of fixed-time convergence for time-varying or time-delay systems) gained popularity [Song et al., 2017, 2018, Holloway and Krstic, 2019], since it is based on utilization for control or estimation of linear dynamics, with time-varying gains growing to infinity in a fixed time, hence, definition of the solutions and conventional stability are problematic. This approach has another appealing advantage, providing a uniform convergence with respect to matched perturbations (similarly to sliding mode controls), but the price to pay is sensitivity to the measurement noises. There are also other concepts of accelerated convergence, and among them we would like to mention the *hyperexponential* one [Polyakov et al., 2015, Nekhoroshikh et al., 2022], which implies that the equilibrium can be reached asymptotically with a speed faster than in any exponentially converging dynamics.

The mentioned finite/fixed/prescribed-time convergence

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rates are popular examples of such a behavior, but there are also other cases as it is demonstrated by the following simple model of a linear time-varying system:

$$\dot{x}(t) = -(1+t)x(t) + d(t), \quad t \geq 0 \quad (1)$$

with  $x(t), d(t) \in \mathbb{R}$ , whose solutions admit an estimate:

$$|x(t)| \leq e^{-\frac{t^2}{2}-t}|x(0)| + \frac{2\|d\|_\infty}{1+t}, \quad t \geq 0$$

for any  $x(0) \in \mathbb{R}$  and  $d \in \mathcal{L}_\infty^1$ . Obviously, the convergence of  $e^{-\frac{t^2}{2}-t}$  is faster than  $e^{-\alpha t}$  for any  $\alpha > 0$ , moreover, the convergence to the origin is uniform over bounded inputs  $d$ . Such a feature of time-varying feedbacks in linear systems has been observed previously, see for example [Inoue et al., 2011, Wang and Song, 2019], and the main issue is again sensitivity to the measurement noises.

Motivated by this example, in [Efimov et al., 2023] a linear time-varying stabilizing feedback for a double integrator with convergence faster than any exponential was proposed (in the same spirit, a hyperexponentially decaying observer for a double integrator was designed in [Efimov et al., 2022]). The obtained convergence was shown to be uniform with respect to matched disturbances in the noise-free scenario (no knowledge of the perturbation's upper bound is required for the tuning), and admitting a simple digital implementation, which improves the sensitivity with respect to measurement noises. The gain selection guidelines were formulated using linear matrix inequalities. The present paper develops those results to the case of an arbitrary-order chain of integrators (generic form of the system requires a sophisticated analysis, while for a double integrator in [Efimov et al., 2022, 2023] everything was done by direct computations), in addition, the proof of accelerated convergence for the sampled-and-hold realization of the control is presented, the stability conditions are refined, and the simulation results are enlarged.

The paper is organized as follows. The brief preliminaries are given in Section 2. The problem statement is introduced in Section 3. The characteristics of the proposed control in continuous time are investigated in Section 4. The properties of its implicit Euler discretization and sample-and-hold implementation are discussed in Section 5. The results of simulations of the presented control are shown in Section 6.

### Notation

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ , where  $\mathbb{R}$  is the set of real numbers,  $\mathbb{Z}$  is the set of integer numbers,  $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$ .
- $|\cdot|$  denotes the absolute value in  $\mathbb{R}$ ,  $\|\cdot\|$  is used for the Euclidean norm on  $\mathbb{R}^n$  (and induced matrix norm).
- For a (Lebesgue) measurable function  $d : \mathbb{R}_+ \rightarrow \mathbb{R}^m$  define the norm  $\|d\|_\infty = \text{ess sup}_{t \geq 0} \|d(t)\|$  and the set of  $d$  with the property  $\|d\|_\infty < +\infty$  we further denote as  $\mathcal{L}_\infty^m$ .

- For a sequence  $d_k \in \mathbb{R}^m$ ,  $k \in \mathbb{Z}_+$  define the norm  $\|d\|_\infty = \sup_{k \in \mathbb{Z}_+} \|d_k\|$ , and the set of sequences with  $\|d\|_\infty < +\infty$  we further denote as  $\mathcal{L}_\infty^m$ .
- A continuous function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{K}$  if  $\alpha(0) = 0$  and it is strictly increasing. The function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{K}_\infty$  if  $\alpha \in \mathcal{K}$  and it is increasing to infinity. A continuous function  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{KL}$  if  $\beta(\cdot, t) \in \mathcal{K}$  for each fixed  $t \in \mathbb{R}_+$  and  $\beta(s, \cdot)$  is decreasing to zero for each fixed  $s > 0$ .
- Denote the identity matrix of dimension  $n \times n$  by  $I_n$  and the matrix of zeros of dimension  $m \times n$  by  $0_{m \times n}$ ,  $\text{diag}\{g\}$  represents a diagonal matrix of dimension  $n \times n$  with a vector  $g \in \mathbb{R}^n$  on the main diagonal.
- The relation  $P \prec 0$  ( $P \succeq 0$ ) means that a symmetric matrix  $P \in \mathbb{R}^{n \times n}$  is negative (positive semi-) definite,  $\lambda_{\min}(P)$  and  $\lambda_{\max}(P)$  denote the minimal and the maximal eigenvalues of such a matrix  $P$ .
- $\exp(1) = e$ .

## 2 Preliminaries

The used standard stability notions and their definitions can be found in [Khalil, 2002].

### 2.1 Uniform hyperexponential stability [Efimov et al., 2022]

Consider a non-autonomous differential equation:

$$dx(t)/dt = f(t, x(t), d(t)), \quad t \geq t_0, \quad t_0 \in \mathbb{R}_+, \quad (2)$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $d(t) \in \mathbb{R}^m$  is the vector of external disturbances and  $d \in \mathcal{L}_\infty^m$ ;  $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is a continuous function with respect to  $x$ ,  $d$  and piecewise continuous with respect to  $t$ ,  $f(t, 0, 0) = 0$  for all  $t \in \mathbb{R}_+$ . A solution of the system (2) for an initial condition  $x_0 \in \mathbb{R}^n$  at time instant  $t_0 \in \mathbb{R}_+$  and some  $d \in \mathcal{L}_\infty^m$  is denoted as  $X(t, t_0, x_0, d)$ , and we assume that  $f$  ensures existence and uniqueness of solutions  $X(t, t_0, x_0, d)$  in forward time.

**Definition 1** For a given set  $\mathbb{D} \subset \mathcal{L}_\infty^m$ , the origin is called uniformly hyperexponentially stable for the system (2) if it is uniformly globally asymptotically stable, and for any  $\alpha > 0$  there exists  $\rho_\alpha \in \mathcal{K}$  and  $\beta_\alpha \in \mathcal{KL}$  such that for all  $x_0 \in \mathbb{R}^n$ ,  $t_0 \in \mathbb{R}_+$  and  $d \in \mathbb{D}$ , for all  $t \geq t_0$ :

$$\|X(t, t_0, x_0, d)\| \leq e^{-\alpha(t-t_0)} \rho_\alpha(\|x_0\|) + \beta_\alpha(\|d\|_\infty, t - t_0).$$

The system (1) is an illustration of the uniform hyperexponential stability with  $t_0 = 0$ ,  $\rho_\alpha(s) = e^{\frac{(\alpha-1)^2}{2}} s$  and  $\beta_\alpha(s, t) = \frac{2s}{1+t}$  for any  $\alpha > 0$ .

In this definition the hyperexponential rate of convergence is demanded only in initial conditions, while uniformity is understood in double meaning: as independence in the initial time  $t_0$  and in the input  $d \in \mathbb{D}$ . Despite it is assumed

that  $\mathbb{D} \subset \mathcal{L}_\infty^m$ , any other suitable class of inputs can be considered.

## 2.2 Auxiliary properties

We will use the following estimates:

**Lemma 1** [Wang et al., 2023] For all  $\tau \geq 0$  and all  $i \geq 1$  it holds:

$$r_i = (2i)^i \int_0^\tau \frac{e^{s-\tau}}{(2s+1)^i} ds \leq \frac{r_i}{(2\tau+1)^i},$$

$$r_i = (2i)^i \int_0^{\frac{2i-1}{2}} \frac{e^{s-\frac{2i-1}{2}}}{(2s+1)^i} ds + \left(\frac{2i+1}{2i}\right)^i (1 - e^{-0.5}) + (2i+1).$$

The following block matrix inversion formulas are used in the sequel:

$$\begin{aligned} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} &= \begin{bmatrix} A^{-1} + A^{-1}BS_1CA^{-1} & -A^{-1}BS_1 \\ -S_1CA^{-1} & S_1 \end{bmatrix} \\ &= \begin{bmatrix} S_2 & -S_2BD^{-1} \\ -D^{-1}CS_2 & D^{-1} + D^{-1}CS_2BD^{-1} \end{bmatrix}, \quad (3) \\ S_1 &= (D - CA^{-1}B)^{-1}, \quad S_2 = (A - BD^{-1}C)^{-1}, \end{aligned}$$

where  $A$ ,  $B$ ,  $C$  and  $D$  are matrices of appropriate dimensions ( $A$ ,  $D$ ,  $S_1$  and  $S_2$  should be nonsingular).

## 3 Problem statement

Consider a linear system in a canonical form with matched disturbances and noisy state measurements:

$$\dot{x}(t) = Ax(t) + b(u(t) + d(t)), \quad y(t) = x(t) + v(t), \quad (4)$$

where  $x(t) \in \mathbb{R}^n$  is the state,  $u(t) \in \mathbb{R}$  is the control to be synthesized,  $d \in \mathcal{L}_\infty^1$  and  $v \in \mathcal{L}_\infty^n$  are the exogenous disturbance and the measurement noise, respectively,  $y(t) \in \mathbb{R}^n$  is the signal available for feedback realization,

$$A = \begin{bmatrix} 0_{n-1 \times 1} & I_{n-1} \\ 0 & 0_{1 \times n-1} \end{bmatrix}, \quad b = \begin{bmatrix} 0_{n-1 \times 1} \\ 1 \end{bmatrix}.$$

It is required to design a state feedback  $u(t) = u(y(t), t)$  for this system, which stabilizes  $x(t)$  at the origin with a hyperexponential rate of convergence uniformly in  $d \in \mathbb{D} = \mathcal{L}_\infty^1$  while  $\|v\|_\infty = 0$ , and having a bounded regulation error for  $v \in \mathcal{L}_\infty^n$ . Note that any controllable single-input-single-output linear system can be presented in such a canonical form, and controllability is necessary for accelerated stabilization.

In the sequel, the problem will be solved, first, in the continuous time. Next, its discrete-time counterpart and the sampled-and-hold realization will be developed.

## 4 Control in continuous time

Let the desired control law for (4) be chosen in a linear time-varying form:

$$u(t) = K\Gamma(t)y(t), \quad \Gamma(t) = \text{diag}\{\varrho^n(t) \dots \varrho(t)\}^\top, \quad (5)$$

where  $K = [K_1 \dots K_n] \in \mathbb{R}^{1 \times n}$  is the controller gain that will be selected later,  $\varrho(t) = 1 + t$  is a strictly growing function of time. Similar control was used in [Chitour et al., 2020] for prescribed-time stabilization of (4) with a function  $\varrho(t)$  escaping to infinity in a finite time.

**Remark 1** Any other positive strictly growing integrable function of time  $\varrho(t)$  can be used in (5), e.g.,  $\varrho(t) = ae^{\alpha t}$  or  $\varrho(t) = (b+at)^\alpha$  for  $a > 0$ ,  $b > 0$  and  $\alpha > 0$ . The current choice is motivated by brevity of exposition.

After substitution of (5) in the right-hand side of the system (4), the closed-loop dynamics can be written as follows:

$$\dot{x}(t) = (A + bK\Gamma(t))x(t) + b(d(t) + K\Gamma(t)v(t)). \quad (6)$$

We are in position to formulate the main result of this section:

**Theorem 1** Let there exist  $P = P^\top \in \mathbb{R}^{n \times n}$ ,  $S = S^\top \in \mathbb{R}^{n \times n}$ ,  $U \in \mathbb{R}^{1 \times n}$ ,  $\gamma_1 > 0$  and  $\gamma_2 > 0$  such that the linear matrix inequalities are verified:

$$P^{-1} \succ 0, \quad S \succ 0, \quad Q \preceq 0, \\ HP^{-1} + P^{-1}H + S \succeq 0,$$

$$Q = \begin{bmatrix} Q_{11} & b & bU \\ b^\top & -\gamma_1 & 0_{1 \times n} \\ U^\top b^\top & 0_{n \times 1} & -\gamma_2 P^{-1} \end{bmatrix},$$

$$Q_{11} = P^{-1}A^\top + AP^{-1} + U^\top b^\top + bU + S + P^{-1},$$

$$H = \text{diag}\{[0, 1 \dots n-1]^\top\}.$$

Then for  $K = UP$  and any  $x(0) \in \mathbb{R}^n$ ,  $d \in \mathcal{L}_\infty^1$  and  $v \in \mathcal{L}_\infty^n$  in (6):

$$|x_i(t)| \leq \varrho^{i-1}(t) \mathcal{Q}(t, \|x(0)\|, \|v_1\|_\infty, \dots, \|v_n\|_\infty, \|d\|_\infty),$$

$$\mathcal{Q}(t, X_0, V_1, \dots, V_N, D) = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} e^{-\frac{t(t+2)}{4}} X_0$$

$$+ \sqrt{\frac{\gamma_1 r_n}{\lambda_{\min}(P)}} \frac{D}{\varrho^n(t)} + \sqrt{\gamma_2 \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \sum_{i=1}^n \frac{r_{i-1} V_i}{\varrho^{i-1}(t)}$$

for  $i = \overline{1, n}$  and for all  $t \geq 0$ , where  $r_i$  are defined in Lemma 1.

For  $\|v\|_\infty = 0$  the result of the theorem implies uniform hyperexponential stability of the origin in (4), (5) with  $d \in \mathcal{L}_\infty^1$ . This property means that all trajectories of (6) converge asymptotically to zero, and the class of disturbances  $d \in \mathcal{L}_\infty^1$ , for which the uniformity of the decay is kept, can be enlarged by ones growing in time *not faster than*  $\varrho(t)$  (i.e.,  $\limsup_{t \rightarrow +\infty} \frac{|d(t)|}{\varrho(t)} = 0$ ).

Note also that the given linear matrix inequalities are always feasible under the conditions of the theorem since the pair  $(A, b)$  is controllable.

**PROOF.** Define new auxiliary variable  $z(t) = \Lambda(t)x(t)$ , where

$$\Lambda(t) = \varrho^{-n}(t)\Gamma(t) = \text{diag}\{[1, \varrho^{-1}(t) \dots \varrho^{1-n}(t)]^\top\},$$

whose dynamics takes the form for  $A_K = A + bK$ :

$$\begin{aligned} \dot{z}(t) &= -\dot{\varrho}(t)\varrho^{-1}(t)Hz(t) + \Lambda(t)\dot{x}(t) \\ &= \varrho(t) \left( (A_K - \varrho^{-2}(t)H) z(t) + \varrho^{-n}(t)bd(t) + bK\Lambda(t)v(t) \right), \end{aligned}$$

where we used the relations

$$\Lambda(t) (A + bK\Gamma(t)) \Lambda^{-1}(t) = \varrho(t)(A + bK), \quad (7)$$

$$\Lambda(t)bK\Gamma(t) = \varrho(t)bK\Lambda(t), \quad \Lambda(t)bb^\top\Lambda(t) = \varrho^{-2}(t)bb^\top,$$

satisfying for all  $t \geq 0$ . Note that  $z(0) = x(0)$ . Finally, let us define a new time variable

$$\begin{aligned} \tau = \varphi(t) &= \frac{t(t+2)}{2}, \quad t = \varphi(\tau)^{-1} = \sqrt{2\tau+1} - 1, \\ d\tau &= (t+1)dt, \end{aligned}$$

where  $\varphi(\tau)^{-1}$  denotes the inverse of  $\varphi$ , then after the change of the time  $t \rightarrow \tau$ , the auxiliary variable  $z(\tau)$  dynamics is reduced to an equivalent useful representation (note that  $\tau \geq 0$ ):

$$\begin{aligned} \frac{dz(\tau)}{d\tau} &= (A_K - \varrho^{-2}(\tau)H) z(\tau) + \varrho^{-n}(\tau)bd(\tau) \\ &\quad + bK\Lambda(\tau)v(\tau), \end{aligned} \quad (8)$$

where  $d(\tau) = d(\varphi(\tau)^{-1})$  and  $v(\tau) = v(\varphi(\tau)^{-1})$  are the external signals in the new time,

$$\varrho(\tau) = \varrho(\varphi(\tau)^{-1}) = \sqrt{2\tau+1}, \quad \Lambda(\tau) = \Lambda(\varphi(\tau)^{-1}).$$

For analysis of stability properties in (8) let us choose a candidate Lyapunov function  $V(z) = z^\top Pz$  (that is positive definite since  $P \succ 0$  due to the conditions of the theorem), whose derivative in the new time  $\tau$  for the dynamics (8) can be written as follows:

$$\begin{aligned} \frac{dV(z(\tau))}{d\tau} &= z(\tau)^\top (A_K^\top P + PA_K) z(\tau) \\ &\quad + 2z(\tau)^\top P (\varrho^{-n}(\tau)bd(\tau) + bK\Lambda(\tau)v(\tau) - \varrho^{-2}(\tau)Hz(\tau)). \end{aligned}$$

Due to the introduced matrix inequalities we have  $PH + HP + X \succeq 0$ , where  $X = PSP$ , and

$$\begin{aligned} -\varrho^{-2}(\tau)PH - \varrho^{-2}(\tau)HP - \varrho^{-2}(\tau)(X - X) \\ \preceq \varrho^{-2}(\tau)X \preceq X \end{aligned}$$

for all  $\tau \geq 0$ . Therefore,

$$\begin{aligned} \frac{dV(z(\tau))}{d\tau} &\leq z(\tau)^\top (A_K^\top P + PA_K + X) z(\tau) \\ &\quad + 2z(\tau)^\top P (\varrho^{-n}(\tau)bd(\tau) + bK\Lambda(\tau)v(\tau)), \end{aligned}$$

which results in the following upper bound

$$\begin{aligned} \frac{dV(z(\tau))}{d\tau} &\leq \begin{bmatrix} z(\tau) \\ \varrho^{-n}(\tau)d(\tau) \\ \Lambda(\tau)v(\tau) \end{bmatrix}^\top \tilde{Q} \begin{bmatrix} z(\tau) \\ \varrho^{-n}(\tau)d(\tau) \\ \Lambda(\tau)v(\tau) \end{bmatrix} \\ &\quad -V(z(\tau)) + \gamma_1 \varrho^{-2n}(\tau)d^2(\tau) \\ &\quad + \gamma_2 v^\top(\tau)\Lambda(\tau)P\Lambda(\tau)v(\tau), \\ \tilde{Q} &= \begin{bmatrix} \tilde{Q}_{11} & Pb & PbK \\ b^\top P & -\gamma_1 & 0_{1 \times n} \\ K^\top b^\top P & 0_{n \times 1} & -\gamma_2 P \end{bmatrix}, \\ \tilde{Q}_{11} &= A_K^\top P + PA_K + X + P, \end{aligned}$$

where  $\gamma_1, \gamma_2$  are positive parameters according to formulation of the theorem. Next, observe that

$$Q = T\tilde{Q}T,$$

where  $T = \text{diag}\{P^{-1}, 1, P^{-1}\}$  and  $U = KP^{-1}$ , hence, the requirement of the theorem  $Q \preceq 0$  implies that  $\tilde{Q} \preceq 0$  and

$$\begin{aligned} \frac{dV(z(\tau))}{d\tau} &\leq -V(z(\tau)) + \gamma_1 \varrho^{-2n}(\tau)d^2(\tau) \\ &\quad + \gamma_2 v^\top(\tau)\Lambda(\tau)P\Lambda(\tau)v(\tau) \leq -V(z(\tau)) + \gamma_1 \left( \frac{\|d\|_\infty}{\varrho^n(\tau)} \right)^2 \\ &\quad + \gamma_2 \lambda_{\max}(P) \sum_{i=1}^n \varrho^{2(1-i)}(\tau) \|v_i\|_\infty^2. \end{aligned}$$

Passing to the time domain we get an estimate:

$$\begin{aligned} V(z(\tau)) &\leq e^{-\tau}V(z(0)) + \gamma_1 \int_0^\tau e^{s-\tau} \left( \frac{\|d\|_\infty}{\varrho^n(s)} \right)^2 ds \\ &\quad + \gamma_2 \lambda_{\max}(P) \int_0^\tau e^{s-\tau} \sum_{i=1}^n \varrho^{2(1-i)}(s) \|v_i\|_\infty^2 ds \\ &= e^{-\tau}V(z(0)) + \gamma_1 \|d\|_\infty^2 \int_0^\tau \frac{e^{s-\tau}}{(2s+1)^n} ds \\ &\quad + \gamma_2 \lambda_{\max}(P) \sum_{i=1}^n \|v_i\|_\infty^2 \int_0^\tau \frac{e^{s-\tau}}{(2s+1)^{i-1}} ds \end{aligned}$$

for all  $\tau \geq 0$ . Using Lemma 1 and an obvious relation  $\int_0^\tau e^{s-\tau} ds \leq 1 = r_0$  for all  $\tau \geq 0$  we obtain:

$$\begin{aligned} V(z(\tau)) &\leq e^{-\tau}V(z(0)) + \gamma_1 r_n \left( \frac{\|d\|_\infty}{(2\tau+1)^n} \right)^2 \\ &\quad + \gamma_2 \lambda_{\max}(P) \sum_{i=1}^n \frac{r_{i-1} \|v_i\|_\infty^2}{(2\tau+1)^{i-1}} \\ &= e^{-\tau}V(z(0)) + \gamma_1 r_n \varrho^{-2n}(\tau) \|d\|_\infty^2 \\ &\quad + \gamma_2 \lambda_{\max}(P) \sum_{i=1}^n r_{i-1} \varrho^{2(1-i)}(\tau) \|v_i\|_\infty^2, \quad \tau \geq 0 \end{aligned}$$

and in the original time (using  $\tau = \varphi(t)$ )

$$\|z(t)\| \leq \mathcal{Q}(t, \|z(0)\|, \|v_1\|_\infty, \dots, \|v_2\|_\infty, \|d\|_\infty)$$

for all  $t \geq 0$ . Since  $z(t) = \Lambda(t)x(t)$ , the desired estimate on the behavior of  $x_i(t)$  for  $i = \overline{1, n}$  can be derived for all  $t \geq 0$  from this inequality.

The upper bound on the state of (4), (5) calculated in Theorem 1 implies that the gain of  $|x_i(t)|$  in  $v_i(t)$  is constant and the gains in  $v_j(t)$ ,  $j = \overline{1, i-1}$  or  $v_k(t)$ ,  $k = \overline{i+1, n}$  are growing or decaying in time  $t$ , respectively. For a small  $\|v\|_\infty$  and having the computations on a bounded interval of time it can be not a problem in application, but let us check that happens after a proper discretization of (6).

## 5 Discretization of the controlled system

Two cases are investigated in this section. First, a consistent method of discretization of (6) is presented. Second, the sample-and-hold implementation of the control (5) for (4) is considered based on the previously derived discretization.

### 5.1 Implicit Euler discretization of (6)

Note that (6) is modeled by a linear time-varying system with external inputs  $d(t)$  and  $v(t)$ . Since the time-varying control gain  $K\Gamma(t)$  is strictly growing to infinity, the explicit Euler discretization cannot be used for all  $t \geq 0$ , however, the implicit one can be effectively applied [Butcher, 2008, Brogliato and Polyakov, 2021]. Let  $h > 0$  be a constant discretization step, denote by  $t_k = hk$  for  $k \in \mathbb{Z}_+$  the discretization time instants, then application of the implicit Euler discretization method to (6) gives for  $k \in \mathbb{Z}_+$ :

$$\begin{aligned} \xi_{k+1} &= F^{-1}(t_{k+1})(\xi_k + hbd_{k+1} + L(t_{k+1})v_{k+1}), \\ F(t) &= I_n - h(A + bK\Gamma(t)), \quad L(t) = hbK\Gamma(t), \end{aligned} \quad (9)$$

where  $\xi_k \in \mathbb{R}^n$  is an approximation of  $x_k = x(t_k)$  (i.e.,  $\xi_k \rightarrow x_k$  as  $h \rightarrow 0$ ),  $v_k = v(t_k)$  and  $d_k = d(t_k)$ . In this work we will assume that  $v_k$  and  $d_k$  take finite values with bounded norms as before, i.e., in this section these sequences  $d \in L_\infty^1$  and  $v \in L_\infty^n$ .

In order to calculate the expression of  $F^{-1}(t)$  and to further analyze its properties, let us use the first block matrix inversion formula given in (3). To this end, represent this matrix as follows

$$F(t) = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C}(t) & \mathcal{D}(t) \end{bmatrix},$$

where

$$\mathcal{A} = I_{n-1} - h \begin{bmatrix} 0_{n-2 \times 1} & I_{n-2} \\ 0 & 0_{1 \times n-2} \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 0_{n-2 \times 1} \\ -h \end{bmatrix},$$

$$\mathcal{C}(t) = h [-K_1 \varrho^n(t) \cdots -K_{n-1} \varrho^2(t)], \quad \mathcal{D}(t) = 1 - hK_n \varrho(t),$$

then

$$F^{-1}(t) = \frac{W(t)}{O(t)},$$

$$O(t) = \mathcal{D}(t) - \mathcal{C}(t)\mathcal{A}^{-1}\mathcal{B} = 1 - \sum_{i=1}^n K_i h^{n-i+1} \varrho^{n-i+1}(t),$$

$$W(t) = \begin{bmatrix} \mathcal{A}^{-1}(O(t)I_{n-1} + \mathcal{B}\mathcal{C}(t)\mathcal{A}^{-1}) - \mathcal{A}^{-1}\mathcal{B} \\ -\mathcal{C}(t)\mathcal{A}^{-1} & 1 \end{bmatrix}.$$

Moreover, the direct computations show that

$$\begin{aligned} \mathcal{A}^{-1} &= \begin{bmatrix} 1 & h & \cdots & h^{n-2} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & h \\ 0 & \cdots & 0 & 1 \end{bmatrix}, \quad \mathcal{A}^{-1}\mathcal{B} = - \begin{bmatrix} h^{n-1} \\ h^{n-2} \\ \vdots \\ h \end{bmatrix}, \\ \mathcal{C}(t)\mathcal{A}^{-1} &= [(\mathcal{C}(t)\mathcal{A}^{-1})_1 \cdots (\mathcal{C}(t)\mathcal{A}^{-1})_{n-1}], \end{aligned}$$

where

$$(\mathcal{C}(t)\mathcal{A}^{-1})_j = - \sum_{i=1}^j K_i h^{j-i+1} \varrho^{n-i+1}(t), \quad j = 1, \dots, n-1.$$

As we can conclude performing these computations, the discrete state transition matrix  $F^{-1}(t)$  is nonsingular (since  $O(t) > 0$  for all  $t \geq 0$  due to  $K_i < 0$ ,  $i = \overline{1, n}$  for a stabilizing control gain  $\bar{K}$ ) and elementwise bounded for all  $t \geq 0$  (since  $O(t)$  and  $\mathcal{C}(t)\mathcal{A}^{-1}$  are polynomial functions of  $\varrho(t)$  of order  $n$ , and  $O(t)$  appears as the common denominator of  $F^{-1}(t)$ ). Moreover,

$$\lim_{t \rightarrow +\infty} F^{-1}(t) = - \begin{bmatrix} 0 & 0 & \cdots & 0 \\ h^{-1} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ h^{1-n} & \cdots & h^{-1} & 0 \end{bmatrix},$$

which implies that asymptotically the own dynamics of (9) approaches a static linear system with a nilpotent matrix of index  $n$ . The measurement noise  $v$  gain matrix

$$F^{-1}(t)L(t) = -h \begin{bmatrix} h^{n-1} \\ \vdots \\ h \\ 1 \end{bmatrix} K \frac{\Gamma(t)}{O(t)}$$

is also elementwise bounded with

$$\lim_{t \rightarrow +\infty} F^{-1}(t)L(t) = - \begin{bmatrix} 1 & 0 & \cdots & 0 \\ h^{-1} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ h^{1-n} & 0 & \cdots & 0 \end{bmatrix},$$

which evaluates the asymptotic noise sensitivity (note that if  $h \rightarrow 0$ , then the discretized system (9) has to approach

the behaviour of the original dynamics (6), which is not robust in  $v$ , consequently, here we observe the asymptotic dependence of the gain of order  $h^{-1}$ ). And, obviously, the effect of the disturbance  $d$  is still asymptotically annihilated by the control with

$$hF^{-1}(t)b = \frac{1}{O(t)} \begin{bmatrix} h^n \\ \vdots \\ h \end{bmatrix}.$$

It remains to demonstrate that the hyperexponential rate of convergence is also preserved in (9).

**Theorem 2** *Let for some  $\phi > 0$ ,  $\psi > 0$  and  $\sigma > 0$  there exist  $P = P^\top \in \mathbb{R}^{n \times n}$ ,  $K \in \mathbb{R}^{1 \times n}$  such that the matrix inequalities are verified:*

$$\begin{aligned} P > 0, \quad (A + bK)P^{-1} + P^{-1}(A + bK)^\top &\preceq 0, \\ HP^{-1} + P^{-1}H &\succeq 0, \\ (A + bK)P^{-1}(A + bK)^\top - \psi^{-1}P^{-1} \\ -\phi^{-1}bKK^\top b^\top - \sigma^{-1}bb^\top &\succeq 0, \end{aligned}$$

where  $H = \text{diag}\{[0, 1 \dots n-1]^\top\}$ . Then for any  $\xi_0 \in \mathbb{R}^n$ ,  $v \in L_\infty^n$  and  $d \in L_\infty^1$  in (9):

$$\begin{aligned} \|\xi_k\|^2 &\leq \varrho^{2(n-1)}(t_k) \left(\frac{\psi}{h^2}\right)^k \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \|\xi_0\|^2 \prod_{i=0}^{k-1} \varrho^{-2}(t_{i+1}) \\ &+ \frac{\varrho^{2(n-1)}(t_k)}{\lambda_{\min}(P)} \sum_{i=0}^{k-1} \left(\frac{\psi}{h^2}\right)^{k-1-i} \left[\phi \sum_{l=1}^n \varrho^{2(1-l)}(t_{i+1}) v_{l,i+1}^2 \right. \\ &\quad \left. + \sigma \varrho^{-2n}(t_{i+1}) d_{i+1}^2\right] \prod_{j=i+1}^{k-1} \varrho^{-2}(t_{j+1}) \end{aligned}$$

for all  $k \geq 1$ .

Note that the first two matrix inequalities of Theorem 2 are verified for  $P$  and  $K$  found in Theorem 1, the last matrix inequality is always satisfied in this case for some  $\phi > 0$ ,  $\psi > 0$  and  $\sigma > 0$ . Then the only additional condition to check is the third inequality, whose more relaxed version with the matrix  $S$  has been also introduced in Theorem 1.

**PROOF.** To investigate stability and the rate of convergence in (9), let us define a time-varying Lyapunov function candidate (the same was used before):

$$V_k = \xi_k^\top \Pi_k \xi_k, \quad \Pi_k = \Lambda(t_k) P \Lambda(t_k),$$

where  $P = P^\top > 0$  has been introduced in the conditions of the theorem. For  $\gamma > 0$ ,  $\psi > 0$  and  $\sigma > 0$ , defining time-varying parameters  $\psi_k = h^{-2} \psi \varrho^{-2}(t_{k+1})$ ,  $\sigma_k = \sigma \varrho^{-2n}(t_{k+1})$  and a matrix  $\Phi_k = \phi \Lambda^2(t_{k+1})$  we obtain

for all  $k \in \mathbb{Z}_+$ :

$$\begin{aligned} V_{k+1} - \psi_k V_k &= \begin{bmatrix} \xi_k \\ d_{k+1} \\ v_{k+1} \end{bmatrix}^\top Q_k \begin{bmatrix} \xi_k \\ d_{k+1} \\ v_{k+1} \end{bmatrix} \\ &+ v_{k+1}^\top \Phi_k v_{k+1} + \sigma_k d_{k+1}^2 \end{aligned}$$

for

$$\begin{aligned} Q_k &= \begin{bmatrix} I_n \\ hb^\top \\ L^\top(t_{k+1}) \end{bmatrix} (F^{-1}(t_{k+1}))^\top \Pi_{k+1} F^{-1}(t_{k+1}) \\ &\times \begin{bmatrix} I_n \\ hb^\top \\ L^\top(t_{k+1}) \end{bmatrix}^\top - \begin{bmatrix} \psi_k \Pi_k & 0 & 0 \\ 0 & \sigma_k & 0 \\ 0 & 0 & \Phi_k \end{bmatrix}. \end{aligned}$$

We need to find the restrictions on  $P$  and the gains  $\psi$ ,  $\phi$ ,  $\sigma$  such that  $Q_k \preceq 0$  for all  $k \geq 0$ . To this end, note that  $-Q_k$  is the Schur complement of

$$\tilde{Q}_k = \begin{bmatrix} \psi_k \Pi_k & 0 & 0 & I_n \\ 0 & \sigma_k & 0 & hb^\top \\ 0 & 0 & \Phi_k & L^\top(t_{k+1}) \\ I_n & hb & L(t_{k+1}) & F(t_{k+1}) \Pi_{k+1}^{-1} F^\top(t_{k+1}) \end{bmatrix},$$

then we seek for conditions guaranteeing  $\tilde{Q}_k \succeq 0$ , which calculating another Schur complement of  $\tilde{Q}_k$  are equivalent to

$$\begin{aligned} \hat{Q}_k &= \begin{bmatrix} \Phi_k & L^\top(t_{k+1}) \\ L(t_{k+1}) & F(t_{k+1}) \Pi_{k+1}^{-1} F^\top(t_{k+1}) \end{bmatrix} \\ &- \begin{bmatrix} 0 & 0 \\ I_n & hb \end{bmatrix} \begin{bmatrix} \psi_k^{-1} \Pi_k^{-1} & 0 \\ 0 & \sigma_k^{-1} \end{bmatrix} \begin{bmatrix} 0 & I_n \\ 0 & hb^\top \end{bmatrix} \\ &= \begin{bmatrix} \Phi_k & L^\top(t_{k+1}) \\ L(t_{k+1}) & Z_k - h^2 \sigma_k^{-1} bb^\top \end{bmatrix} \succeq 0, \\ Z_k &= F(t_{k+1}) \Pi_{k+1}^{-1} F^\top(t_{k+1}) - \psi_k^{-1} \Pi_k^{-1}. \end{aligned}$$

Finally, computing the Schur complement of  $\hat{Q}_k$  we get an equivalent condition to check:

$$Z_k - h^2 \sigma_k^{-1} bb^\top - L(t_{k+1}) \Phi_k^{-1} L^\top(t_{k+1}) \succeq 0.$$

Multiplying it from both sides on  $\Lambda(t_{k+1})$  and taking into account (7) lead to the following matrix inequality:

$$\begin{aligned} 0 &\preceq \Lambda(t_{k+1}) (Z_k - h^2 \sigma_k^{-1} bb^\top - \Phi_k^{-1}) \Lambda(t_{k+1}) \\ &= P^{-1} - \psi_k^{-1} \Lambda(t_{k+1}) \Lambda^{-1}(t_k) P^{-1} \Lambda^{-1}(t_k) \Lambda(t_{k+1}) \\ &\quad + h^2 \varrho^2(t_{k+1}) (A + bK) P^{-1} (A + bK)^\top \\ &\quad - h \varrho(t_{k+1}) \left( (A + bK) P^{-1} + P^{-1} (A + bK)^\top \right) \\ &\quad - h^2 \varrho^2(t_{k+1}) b K \Lambda(t_{k+1}) \Phi_k^{-1} \Lambda(t_{k+1}) K^\top b^\top \\ &\quad - h^2 \sigma_k^{-1} \varrho^{2(1-n)}(t_{k+1}) bb^\top. \end{aligned}$$

The direct computations show that

$$P^{-1} \succeq \Lambda(t_{k+1})\Lambda^{-1}(t_k)P^{-1}\Lambda^{-1}(t_k)\Lambda(t_{k+1})$$

provided that

$$HP^{-1} + P^{-1}H \succeq 0.$$

Indeed, for an arbitrary vector  $w \in \mathbb{R}^n$  define a scalar function  $q(t) = w^\top \Lambda^{-1}(t)P^{-1}\Lambda^{-1}(t)w$ , whose time derivative can be written as  $\dot{q}(t) = w^\top \dot{\Lambda}^{-1}(t)P^{-1}\Lambda^{-1}(t)w + w^\top \Lambda^{-1}(t)P^{-1}\dot{\Lambda}^{-1}(t)w = \varrho^{-1}(t)w^\top \Lambda^{-1}(t)(HP^{-1} + P^{-1}H)\Lambda^{-1}(t)w \geq 0$  for any  $t \geq 0$  implying that  $q(t)$  is a non-decreasing function of time, which due to arbitrariness of  $w$  can be equivalently stated as  $\Lambda^{-1}(t_{k+1})P^{-1}\Lambda^{-1}(t_{k+1}) \succeq \Lambda^{-1}(t_k)P^{-1}\Lambda^{-1}(t_k)$  leading to the desired relation. Therefore, the property  $Q_k \preceq 0$  for all  $k \geq 0$  follows from

$$\begin{aligned} 0 \leq & P^{-1} - \psi_k^{-1}P^{-1} + h^2\varrho^2(t_{k+1})(A + bK)P^{-1}(A + bK)^\top \\ & - h\varrho(t_{k+1})\left((A + bK)P^{-1} + P^{-1}(A + bK)^\top\right) \\ & - h^2\varrho^2(t_{k+1})bK\Lambda(t_{k+1})\Phi_k^{-1}\Lambda(t_{k+1})K^\top b^\top \\ & - h^2\sigma_k^{-1}\varrho^{2(1-n)}(t_{k+1})bb^\top \end{aligned}$$

and substituting the expressions for  $\psi_k$ ,  $\sigma_k$  and  $\Phi_k$  we get

$$\begin{aligned} 0 \leq & P^{-1} - h\varrho(t_{k+1})\left((A + bK)P^{-1} + P^{-1}(A + bK)^\top\right) \\ & + h^2\varrho^2(t_{k+1})\left[(A + bK)P^{-1}(A + bK)^\top - \psi^{-1}P^{-1}\right. \\ & \left. - \phi^{-1}bKK^\top b^\top - \sigma^{-1}bb^\top\right] \end{aligned}$$

which follows from the inequality

$$\begin{aligned} & h^2\varrho^2(t_{k+1})\left[(A + bK)P^{-1}(A + bK)^\top - \psi^{-1}P^{-1}\right. \\ & \left. - h^{-1}\varrho^{-1}(t_{k+1})\left((A + bK)P^{-1} + P^{-1}(A + bK)^\top\right)\right. \\ & \left. - \phi^{-1}bKK^\top b^\top - \sigma^{-1}bb^\top\right] \succeq 0. \end{aligned}$$

If  $(A + bK)P^{-1} + P^{-1}(A + bK)^\top \preceq 0$ , then the latter inequality can be simplified:

$$\begin{aligned} & (A + bK)P^{-1}(A + bK)^\top - \psi^{-1}P^{-1} \\ & - \phi^{-1}bKK^\top b^\top - \sigma^{-1}bb^\top \succeq 0. \end{aligned}$$

Since all required matrix inequalities have been introduced in the formulation of the theorem, it is shown above that

$$\begin{aligned} V_{k+1} & \leq \psi_k V_k + v_{k+1}^\top \Phi_k v_{k+1} + \sigma_k d_{k+1}^2 \\ & = \frac{\psi}{h^2} \varrho^{-2}(t_{k+1}) V_k + \phi \sum_{i=1}^n \varrho^{2(1-i)}(t_{k+1}) v_{i,k+1}^2 \\ & \quad + \sigma \varrho^{-2n}(t_{k+1}) d_{k+1}^2 \end{aligned} \quad (10)$$

for all  $k \in \mathbb{Z}_+$ . Hence,

$$\begin{aligned} V_k & \leq \left(\frac{\psi}{h^2}\right)^k V_0 \prod_{i=0}^{k-1} \varrho^{-2}(t_{i+1}) \\ & + \sum_{i=0}^{k-1} \left(\frac{\psi}{h^2}\right)^{k-1-i} \left[\phi \sum_{l=1}^n \varrho^{2(1-l)}(t_{i+1}) v_{l,i+1}^2\right. \\ & \quad \left. + \sigma \varrho^{-2n}(t_{i+1}) d_{i+1}^2\right] \prod_{j=i+1}^{k-1} \varrho^{-2}(t_{j+1}) \end{aligned}$$

for all  $k \in \mathbb{Z}_+$ , which gives the required estimate.

As we can see from the obtained results, the convergence in the initial error is faster than any exponential and there is no asymptotic dependence in  $d$ , while the noise  $v$  gain admits a static linear upper bound.

## 5.2 Sample-and-hold implementation of (5)

Assume that in (4) the control  $u(t) = u(t_k) = u_k$  stays constant on the interval  $[t_k, t_{k+1})$  for all  $k \in \mathbb{Z}_+$ , where as before  $t_k = kh$  with  $h > 0$  being the sampling period. Then using the formula for solutions of a linear system we obtain for all  $k \in \mathbb{Z}_+$ :

$$x_{k+1} = A_h x_k + b_h u_k + \delta_k, \quad (11)$$

where

$$\begin{aligned} A_h & = I_n + \sum_{i=1}^{n-1} \frac{h^i}{i!} A^i, \quad b_h = \begin{bmatrix} \frac{h^n}{n!} \\ \vdots \\ h \end{bmatrix}, \\ \delta_k & = \int_0^h \begin{bmatrix} \frac{(h-s)^{n-1}}{(n-1)!} \\ \vdots \\ 1 \end{bmatrix} d(t_k + s) ds, \end{aligned}$$

and  $x_k = x(t_k)$  (it is not an approximation of solution in (4), but its exact value).

In order to find the values of  $u_k$ , as in [Polyakov et al., 2022], we look for a solution of the following minimization problem:

$$u(t_k) = \arg \min_{u_k \in \mathbb{R}, \|v\|_\infty=0, \|d\|_\infty=0, \xi_k=x_k} \|x_{k+n} - \xi_{k+n}\|,$$

*i.e.*, in the perturbation-free case the control  $u_k$  has to ensure closeness of  $x_{k+n}$  to  $\xi_{k+n}$  from (9) having the same initial conditions with  $\xi_k = x_k$ . Hence, we look for a solution of the following equation:

$$\prod_{j=0}^{n-1} F^{-1}(t_{k+1} + jh)x_k = A_h^n x_k + C_h \begin{bmatrix} u_{k+n-1} \\ \vdots \\ u_k \end{bmatrix},$$

where the controllability matrix  $C_h = [b_h \ A_h b_h \ \dots \ A_h^{n-1} b_h]$  is not singular due to controllability of the pair  $(A_h, b_h)$ , and there is an exact solution

$$\begin{bmatrix} u_{k+n-1} \\ \vdots \\ u_k \end{bmatrix} = C_h^{-1} \left( \prod_{j=0}^{n-1} F^{-1}(t_{k+1} + jh) - A_h^n \right) x_k.$$

Hence, applying consecutively  $u_{k+j}(x_k)$  from this formula for  $j = 0, n-1$  we can ensure that  $x_{k+n} = \xi_{k+n}$ , but such

a control is not a feedback in the conventional sense. In [Polyakov et al., 2022], also the first step feedback

$$u_k = \Upsilon(t_{k+1})x_k, \quad (12)$$

where

$$\Upsilon(t) = \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} C_h^{-1} \left( \prod_{j=0}^{n-1} F^{-1}(t+jh) - A_h^n \right),$$

was proposed for all  $k \in \mathbb{Z}_+$ , and its properties have been analyzed in the context of finite-time stabilization by a nonlinear control in the continuous time. It is obvious that  $\Upsilon(t)$  is a bounded matrix-valued function of time since  $F^{-1}(t)$  possesses this property.

In our case, the closed-loop system (11), (12) takes the form:

$$\begin{aligned} x_{k+1} &= (A_h + b_h \Upsilon(t_{k+1}))x_k + b_h \Upsilon(t_{k+1})v_k + \delta_k \quad (13) \\ &= \left( N_h + G_h \prod_{j=0}^{n-1} F^{-1}(t_{k+1} + jh) \right) x_k + b_h \Upsilon(t_{k+1})v_k + \delta_k \\ &= (N_h + \Pi_k)x_k + \tilde{\delta}_k, \quad \tilde{\delta}_k = b_h \Upsilon(t_{k+1})v_k + \delta_k, \\ G_h &= b_h \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} C_h^{-1}, \quad N_h = A_h - G_h A_h^n, \\ \Pi_k &= G_h \prod_{j=0}^{n-1} F^{-1}(t_{k+1} + jh), \end{aligned}$$

where the matrix  $N_h$  is nilpotent of index  $n$  (i.e.,  $N_h^n = 0$ ) and

$$\lim_{k \rightarrow +\infty} \Pi_k = 0$$

since  $F^{-1}(t)$  is nilpotent of index  $n$ ; the new input  $\tilde{\delta}$  is bounded. The sampled-and-hold control system (13) inherits the properties of (9):

**Theorem 3** *Let the control gain  $K$  be chosen to verify the conditions of Theorem 2, then the system (13) is input-to-state stable<sup>1</sup> and in the disturbance-free case the rate of convergence to the origin is faster than any exponential.*

**PROOF.** Since  $N_h$  is nilpotent (all eigenvalues are zero), there exist symmetric and positive definite matrices  $P', Q', S' \in \mathbb{R}^{n \times n}$  such that  $N_h^\top P' N_h - P' = -Q'$  and for a Lyapunov function  $V'_k = x_k^\top P' x_k$ , the inequality

$$V'_{k+1} - V'_k \leq -0.5x_k^\top Q' x_k + x_k^\top \Pi_k^\top S' \Pi_k x_k + \tilde{\delta}_k^\top S' \tilde{\delta}_k$$

for all  $k \in \mathbb{Z}_+$  is respected (i.e.,  $V'$  is an input-to-state stable Lyapunov function Angeli [1999], Jiang and Wang [2001] for the nominal system  $x_{k+1} = N_h x_k + (\Pi_k x_k + \tilde{\delta}_k)$  by considering the term in the brackets as a perturbation).

<sup>1</sup> There exist functions  $\rho \in \mathcal{KL}$  and  $\sigma \in \mathcal{K}_\infty$  such that  $\|x_k\| \leq \rho(\|x_0\|, k) + \sigma(|\tilde{\delta}|_\infty)$  for all  $k \in \mathbb{Z}_+$ , for all  $x_0 \in \mathbb{R}^n$  and  $\tilde{\delta} \in L_\infty^n$  Angeli [1999], Jiang and Wang [2001].

Recalling the convergence property of  $\Pi_k$ , there is  $\hat{k} \geq 0$  such that

$$V'_{k+1} - V'_k \leq -0.25x_k^\top Q' x_k + \tilde{\delta}_k^\top S' \tilde{\delta}_k$$

for all  $k \geq \hat{k}$ . Moreover,  $\Pi_k$  is a bounded matrix and the system has linear dynamics, then there is  $c' > 0$  such that  $\|x_{\hat{k}}\| \leq c'(\|x_0\| + |\tilde{\delta}|_\infty)$ . Combining this bound with the properties of  $V'$ , the input-to-state stability of (13) follows (with an exponential convergence rate). If  $|\tilde{\delta}|_\infty = 0$ , then

$$x_{k+s} = N_h^s x_k + \sum_{i=0}^{s-1} N_h^{s-1-i} \Pi_{k+i} x_{k+i}$$

for any  $k, s \in \mathbb{Z}_+$ , and for  $s \geq n$ ,

$$\begin{aligned} x_{k+s} &= \sum_{i=s-n}^{s-1} N_h^{s-1-i} \Pi_{k+i} x_{k+i} \\ &= \sum_{j=0}^{n-1} N_h^{n-1-j} \Pi_{k+j+s-n} x_{k+j+s-n}, \end{aligned}$$

and the current value of the state  $x_{k+s}$  is always a function of the previous  $n$  elements  $x_{k+j+s-n}$  with time-varying gains  $N_h^{n-1-j} \Pi_{k+j+s-n}$  for  $j = \overline{0, n-1}$ . Due to stability property established previously,  $\|x_k\| \leq C' \|x_0\|$  for all  $k \in \mathbb{Z}_+$  and  $x_0 \in \mathbb{R}^n$  for some  $C' > 0$ , while the gains  $N_h^{n-1-j} \Pi_{k+j+s-n}$  have an upper bound converging to zero faster than any exponential (it can be extracted from the estimates for  $\xi_k$  given in Theorem 2), then

$$\|x_{k+s}\| \leq C' \|x_0\| \sum_{j=0}^{n-1} \|N_h^{n-1-j} \Pi_{k+j+s-n}\|,$$

which implies the hyperexponential convergence for the state of (13).

## 6 Simulations

Let us illustrate performance of the proposed hyperexponential stabilizer (5) in numeric experiments. Take  $n = 4$ , then for

$$K = -[384 \ 400 \ 140 \ 20]$$

there exist solutions of linear matrix inequalities from theorems 1 and 2. In order to compare the proposed control with existing solutions, a linear control  $u_k = K(x_k + v_k)$  is applied to (11), and a sliding mode bounded control based on implicit Lyapunov function approach from [Polyakov et al., 2015, subsection 6.1] is utilized. For  $h = 0.1$ , the results of simulations for  $v_k = 0$  and  $d_k = 0$  for all  $k \in \mathbb{Z}_+$  are presented in Fig. 1. On the top plots, the state trajectories are shown versus the time for (9) and (13), on the bottom plot, the state norms (in logarithmic scale) for all control algorithms are given. Due to the shape of decay for the norm, we can conclude that for (9) and (13) the convergence to the origin is hyperexponential (it would be a straight line for the case of an exponential convergence of the state; for (13) the convergence is slower than for (9)). To demonstrate robustness, take the perturbation

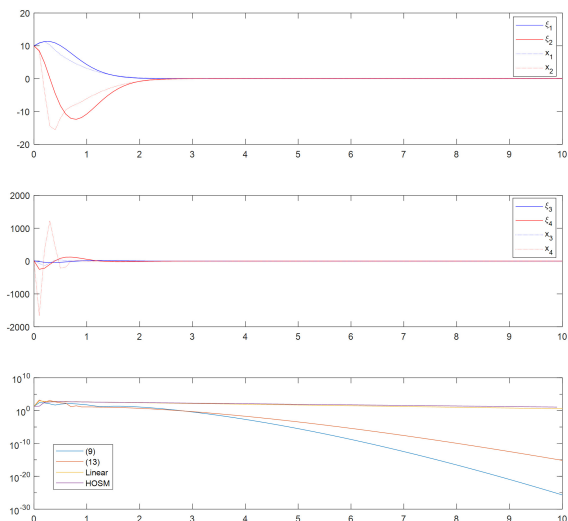


Fig. 1. The results of simulation in the noise-free case

(as it follows from Theorem 1, disturbance cannot grow faster than  $\varrho(t)$ ):

$$d(t) = \left( \sqrt{t} + \sin(5t) + \text{rnd}(1) \right) / 30$$

and the noise

$$v(t) = 0.01 (\text{rnd}(n) + \sin(\omega t)),$$

where  $\text{rnd}(n)$  denotes uniformly distributed in the interval  $[0, 1]^n$  random numbers,  $\omega = [11, \dots, 10 + n]$ , then the respective simulation results are given in Fig. 2. As we can conclude, both systems, (9) and (11), (12), demonstrate a similar behavior (again the robustness of (9) is better). They also demonstrate a faster convergence in comparison with linear or sliding mode controls.

## 7 Conclusion

A new control law is proposed, which has a simple discrete-time implementation, guaranteeing a hyperexponential rate of convergence of the state trajectories to the origin uniformly in the external matched inputs in the noise-free case. It has also certain robustness with respect to the measurement noise. The tuning rules are formulated using feasible linear matrix inequalities. The results of simulation demonstrate a good performance of the proposed control.

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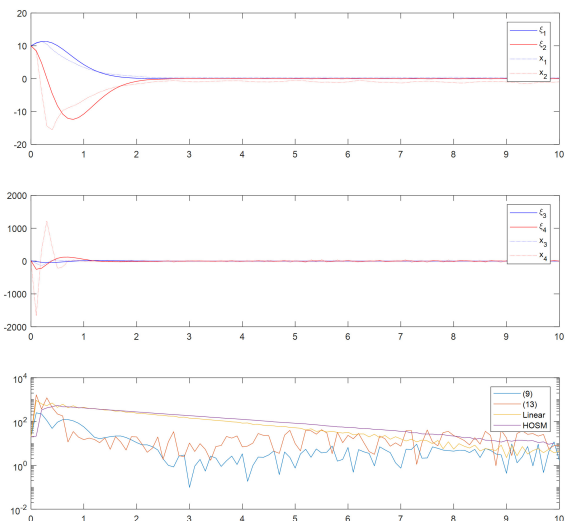


Fig. 2. The results of simulation with perturbations

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