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On equivalence of Lyapunov-Razumikhin conditions and ISS for a class of time-delay systems

D. Efimov^{1,2}, A. Aleksandrov^{3,4}

Abstract—For a retarded nonlinear system with a Lipschitz nonlinearity on bounded sets outside the origin, this note proposes necessary conditions for application of the Lyapunov-Razumikhin method. The presented conditions are illustrated on the classes of linear and nonlinear homogeneous systems.

I. INTRODUCTION

There exist two generic frameworks for stability investigation in time-delay systems, which are based on analysis of properties of a Lyapunov-Razumikhin function or a Lyapunov-Krasovskii functional [13], [11] (see also a recent survey [2]). The latter method has been proven to be equivalent to the asymptotic and input-to-state stability (ISS) properties for many classes of the time-delay systems [17], [18], [7], and it can also be used to establish finite-time stability [15], [9]. The former approach is only sufficient for the asymptotic stability or ISS [13], [11], [21], but it can also be used to evaluate the rates of convergences in the retarded dynamics [6]. Both methods have extensions to distinguish delay-dependent and independent stability scenarios (i.e., separating the cases when the stable compartment is possible for restricted values of delays, or the stability conditions are uniform in the delay values) and can be used for small-gain analysis [4]. An advantage of Lyapunov-Razumikhin approach with respect to Lyapunov-Krasovskii one is that in many nonlinear cases it is more simple to find a Lyapunov-Razumikhin function than a Lyapunov-Krasovskii functional [21], [8], [10] (e.g., a Lyapunov function for the delay-free case can be tested looking for delay-dependent conditions).

The aim of this work is to complement the Lyapunov-Razumikhin function (LRF) approach by proposing necessary conditions of its applicability, and relating it with the existence of a Lyapunov function (LF) for the delay-free counterpart. The results are formulated in the ISS framework. The obtained outcomes will be illustrated on homogeneous time-delay models [8].

The paper is organized as follows. Preliminaries are given in Section II. The main results are formulated in Section III. The linear and homogeneous models are considered in Section IV. The final remarks and discussion are presented in Section V.

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II. PRELIMINARIES

The real numbers are denoted by \mathbb{R} , $\mathbb{R}_+ = \{s \in \mathbb{R} : s \geq 0\}$. Euclidean norm for a vector $x \in \mathbb{R}^n$ is defined as $|x|$, for a matrix $|\cdot|$ denotes the induced norm. We denote by $C_{[a,b]}$, $-\infty < a < b < +\infty$ the Banach space of continuous functions $\phi : [a, b] \rightarrow \mathbb{R}^n$ with the uniform norm $\|\phi\| = \sup_{a \leq \zeta \leq b} |\phi(\zeta)|$. For a Lebesgue measurable function of time $d : [a, b] \rightarrow \mathbb{R}^m$ define the norm $\|d\|_{[a,b]} = \text{ess sup}_{t \in [a,b]} |d(t)|$, then $\|d\|_\infty = \|d\|_{[0,+\infty)}$ and the space of d with $\|d\|_{[a,b]} < +\infty$ ($\|d\|_\infty < +\infty$) we further denote as $\mathcal{L}_{[a,b]}^m$ (\mathcal{L}_∞^m).

For a locally Lipschitz continuous function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ the upper directional Dini derivative is defined as follows:

$$D^+V(x)v = \limsup_{h \rightarrow 0^+} \frac{V(x + hv) - V(x)}{h}$$

for any $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$.

A continuous function $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to class \mathcal{K} if it is strictly increasing and $\sigma(0) = 0$; it belongs to class \mathcal{K}_∞ if it is also radially unbounded. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to class \mathcal{KL} if $\beta(\cdot, r) \in \mathcal{K}$ and $\beta(r, \cdot)$ is a strictly decreasing to zero for any fixed $r > 0$.

A. ISS for delay-free systems

Consider a dynamical system:

$$\dot{x}(t) = g(x(t), d(t)), \quad t \geq 0, \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state and $d(t) \in \mathbb{R}^m$ is the input, $d \in \mathcal{L}_\infty^m$; $g : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$ is a locally Lipschitz continuous function (that ensures forward existence and uniqueness of solutions of the system at least locally in time), $g(0, 0) = 0$. For any $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{L}_\infty^m$ the respective solution is denoted by $x(t, x_0, d)$ with $x(0, x_0, d) = x_0$.

The definitions and results given in this subsection follow [19], [5].

Definition 1. The system (1) is called practically ISS, if there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ and $q \geq 0$ such that

$$|x(t, x_0, d)| \leq \beta(|x_0|, t) + \gamma(\|d\|_{[0,t]}) + q, \quad \forall t \geq 0$$

for all $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{L}_\infty^m$.

If $q = 0$ then (1) is called ISS.

If the above estimate is satisfied for all $x_0 \in \mathbb{R}^n$ and all $d \in \mathcal{L}_\infty^m$ with $\max\{\|x_0\|, \|d\|_\infty\} \leq \rho$ for some $\rho > 0$, then (1) is called locally (practically) ISS.

If $d = 0$, then from ISS we recover the conventional global asymptotic stability property.

Definition 2. The system (1) is said to possess the practical asymptotic gain (AG) property, if there exist $\gamma \in \mathcal{K}$ and $q \geq 0$ such that

$$\limsup_{t \rightarrow +\infty} |x(t, x_0, d)| \leq \gamma(\|d\|_\infty) + q$$

for all $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{L}_\infty^m$. If $q = 0$ then (1) has an AG γ .

The system (1) with a locally Lipschitz continuous g admits (practical) AG if and only if it is (practical) ISS [5].

Definition 3. A locally Lipschitz continuous function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is called practical ISS-LF for the system (1) if there exist $\alpha_1, \alpha_2, \eta \in \mathcal{K}_\infty$, $\varrho \in \mathcal{K}$ and $r \geq 0$ such that for all $x \in \mathbb{R}^n$ and all $d \in \mathbb{R}^m$:

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|),$$

$$V(x) \geq \max\{\varrho(|d|), r\} \implies D^+V(x)g(x, d) \leq -\eta(|x|).$$

If $r = 0$, then such a function V is called ISS-LF.

Theorem 1. [20] *The system (1) is (practically) ISS if and only if it admits a (practical) ISS-LF.*

Remark 1. If a (practical) ISS-LF verifies the properties of Definition 3 only for $\max\{|x|, |d|\} \leq \rho$ with some $\rho > 0$, then it is equivalent to local (practical) ISS.

Remark 2. Existence of a practical ISS-LF implies that the system (1) admits $\alpha_1^{-1} \circ \varrho$ as an AG. Indeed, the interval of definition of solutions $[0, +\infty)$ can be decomposed on two subsets $\mathcal{T}_1 \cup \mathcal{T}_2 = [0, +\infty)$:

$$t \in \mathcal{T}_1 : \max\{\varrho(\|d\|_\infty), r\} > V(x(t)),$$

$$t \in \mathcal{T}_2 : \max\{\varrho(\|d\|_\infty), r\} \leq V(x(t)).$$

For any non-empty connected interval $[t_s, t_e] \subset \mathcal{T}_2$ and $t \in [t_s, t_e]$ we get

$$\begin{aligned} \dot{V}(t) &= D^+V(x(t))g(x(t), d(t)) \leq -\eta(|x(t)|) \\ &\leq -\eta \circ \alpha_2^{-1}(V(x(t))), \end{aligned}$$

which by comparison lemmas [19] implies existence of $\beta \in \mathcal{KL}$ such that

$$V(x(t)) \leq \beta(V(x(t_s)), t - t_s)$$

for all $t \in [t_s, t_e]$ and $\beta(\zeta, 0) = \zeta$ (since $V(x(t))$ is strictly decreasing on these time intervals). Due to commutations between \mathcal{T}_1 and \mathcal{T}_2 , $V(x(t_s))$ has two admissible values: $V(x(0))$ and $t_s = 0$ or $V(x(t_s)) = \max\{\varrho(\|d\|_\infty), r\}$ and $t_s > 0$. Therefore,

$$\begin{aligned} V(x(t)) &\leq \max\{\beta(V(x(0)), t), \beta(\max\{\varrho(\|d\|_\infty), r\}, 0), \\ &\quad \max\{\varrho(\|d\|_\infty), r\}\} \\ &= \max\{\beta(V(x(0)), t), \varrho(\|d\|_\infty), r\} \end{aligned}$$

for all $t \geq 0$, which recalling Definition 2 gives the desired result.

In addition, existence of an ISS-LF leads to ISS property in (1) for simply continuous g or even for a discontinuous one [1]. We will need the following result:

Lemma 1. *Consider a dynamical system:*

$$\dot{x}(t) = \ell(x(t), d(t), u(t)), \quad t \geq 0, \quad (2)$$

where $x(t) \in \mathbb{R}^n$ is the state, and $d(t) \in \mathbb{R}^m$, $u(t) \in \mathbb{R}^p$ are the inputs, $d \in \mathcal{L}_\infty^m$, $u \in \mathcal{L}_\infty^p$; $\ell : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}^n$ is a continuous map. Assume that for $u \equiv 0$ it admits a (practical) ISS-LF and for any $\psi_0 > 0$ there exist $\psi_0 \in [0, \psi_0)$, $\psi_1 \in \mathbb{R}_+$ and $\psi \in \mathcal{K}$ such that

$$|\ell(x, d, u) - \ell(x, d, 0)| \leq \psi_0 + (\psi_1 + \psi(\max\{|x|, |d|, |u|\}))|u|$$

for any $x \in \mathbb{R}^n$, $d \in \mathbb{R}^m$ and $u \in \mathbb{R}^p$. Then, there exists a continuous matrix function $B : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^{p \times p}$ with continuous inverse such that (2) is locally practically ISS for $u(t) = B(x(t), d(t))v(t)$ with respect to the new input $v \in \mathcal{L}_\infty^p$ and d .

If ℓ is locally Lipschitz continuous then $\psi_0 = 0$, while $\psi_0 > 0$ covers the case with lack of this property at the origin (e.g., a homogeneous system of negative degree).

Proof. According to conditions of the lemma, there exist a locally Lipschitz continuous function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ and $\alpha_1, \alpha_2, \eta \in \mathcal{K}_\infty$, $\varrho \in \mathcal{K}$ with $r \geq 0$ such that for all $x \in \mathbb{R}^n$ and all $d \in \mathbb{R}^m$:

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|),$$

$$V(x) \geq \max\{\varrho(|d|), r\} \implies D^+V(x)\ell(x, d, 0) \leq -\eta(|x|).$$

In such a case there exist $\rho_0 \in \mathbb{R}_+$ and $\rho \in \mathcal{K}_\infty$ such that

$$|D^+V(x)| \leq \rho_0 + \rho(|x|)$$

for all $x \in \mathbb{R}^n$. Hence, for $V(x) \geq \max\{\varrho(|d|), r\}$ we get:

$$\begin{aligned} D^+V(x)\ell(x, d, B(x, d)v) &= D^+V(x)\ell(x, d, 0) \\ &\quad + D^+V(x)(\ell(x, d, B(x, d)v) - \ell(x, d, 0)) \\ &\leq -\eta(|x|) + |D^+V(x)| |\ell(x, d, B(x, d)v) - \ell(x, d, 0)| \\ &\leq -\eta(|x|) + (\rho_0 + \rho(|x|)) (\psi_0 + (\psi_1 \\ &\quad + \psi(\max\{|x|, |d|, |B(x, d)v|\}))|B(x, d)v|) \end{aligned}$$

for any $x \in \mathbb{R}^n$, $d \in \mathbb{R}^m$ and $v \in \mathbb{R}^p$. For any given $R > 0$, let us select a sufficiently small $\psi_0 \geq 0$ such that the inequalities $3\psi_0(\rho_0 + \rho(R)) < \eta(R)$ and $\max\{r, \alpha_2 \circ \eta^{-1}(3\psi_0(\rho_0 + \rho(R)))\} \leq r$ for $r > 0$ are verified (or just take $r = \alpha_2 \circ \eta^{-1}(3\psi_0(\rho_0 + \rho(R)))$ if V is an ISS-LF and $r = 0$), then for all $|x| \leq R$, $d \in \mathbb{R}^m$, $v \in \mathbb{R}^p$ and $V(x) \geq \max\{\varrho(|d|), r\}$ we have:

$$\begin{aligned} D^+V(x)\ell(x, d, B(x, d)v) &\leq -\frac{2}{3}\eta(|x|) + (\rho_0 + \rho(|x|)) (\psi_1 \\ &\quad + \psi(\max\{|x|, |d|, |B(x, d)v|\}))|B(x, d)v|. \end{aligned}$$

Construct a continuous function B such that

$$|B(x, d)| \leq \frac{1}{3 + (\rho_0 + \rho(|x|))^2(\psi_1 + \psi(\max\{|x|, |d|\}))^2}$$

for all $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$, then

$$\begin{aligned} D^+V(x)\ell(x, d, B(x, d)v) &\leq -\frac{2}{3}\eta(|x|) \\ &\quad + \frac{(\rho_0 + \rho(|x|))(\psi_1 + \psi(\max\{|x|, |d|\}))}{3 + (\rho_0 + \rho(|x|))^2(\psi_1 + \psi(\max\{|x|, |d|\}))^2} |v| \\ &\leq -\frac{1}{3}\eta(|x|) \end{aligned}$$

for all $|x| \leq R$, $d \in \mathbb{R}^m$, $v \in \mathbb{R}^p$ and $V(x) \geq \max\{\varrho'(|v|), \varrho(|d|), r\}$, where $\varrho'(s) = \alpha_2 \circ \max\{s, \eta^{-1}(s)\}$. From this estimate the local practical ISS property can be derived. \square

A variant of this lemma has been proven in [3] for a smooth V and $\psi_0 = 0$. As in [3] it can be observed that if $\varrho \in \mathcal{K}_\infty$ then B can be chosen to be independent of d . According to the proof, the constant R can be selected arbitrary large, and $R = +\infty$ for the case $\psi_0 = 0$; the parameter r can be kept zero if $\psi_0 = 0$.

B. ISS for time-delay systems

Consider a generic functional differential equation of retarded type [14]:

$$\dot{x}(t) = f(x_t, d(t)), \quad t \geq 0, \quad (3)$$

where $x(t) \in \mathbb{R}^n$ and $x_t \in C_{[-\tau, 0]}$ is the state function, $x_t(s) = x(t+s)$, $-\tau \leq s \leq 0$ and $\tau > 0$ is a finite delay (the main results will be presented for a particular example of this model with pointwise delay (4)); $d(t) \in \mathbb{R}^m$ is the external input, $d \in \mathcal{L}_\infty^m$; $f : C_{[-\tau, 0]} \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a continuous function, $f(0, 0) = 0$, and it provides (at least local) existence and uniqueness of solutions in forward time for the system (3). Denote such a unique solution satisfying the initial condition $x_0 \in C_{[-\tau, 0]}$ with the input $d \in \mathcal{L}_\infty^m$ by $x(t, x_0, d)$, and $x_t(s, x_0, d) = x(t+s, x_0, d)$ for $-\tau \leq s \leq 0$, which is defined on some time interval $[-\tau, T)$ for $T \in (0, +\infty)$.

Definition 4. [16], [12] The system (3) is called practically ISS, if there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ and $q \geq 0$ such that

$$|x(t, x_0, d)| \leq \beta(\|x_0\|, t) + \gamma(\|d\|_\infty) + q, \quad \forall t \geq 0$$

for all $x_0 \in C_{[-\tau, 0]}$ and all $d \in \mathcal{L}_\infty^m$.

If $q = 0$ then (3) is called ISS.

If the above estimate is satisfied for all $x_0 \in C_{[-\tau, 0]}$ and all $d \in \mathcal{L}_\infty^m$ with $\max\{\|x_0\|, \|d\|_\infty\} \leq \rho$ for some $\rho > 0$, then (3) is called locally (practically) ISS.

There exist two methods evaluating ISS property of the system (3) based on a LRF or a Lyapunov-Krasovskii functional. The former approach can be formulated as follows [21]:

Definition 5. A locally Lipschitz continuous function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is called practical ISS-LRF if

(i) for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and all $x \in \mathbb{R}^n$:

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|);$$

(ii) for some $\alpha, \gamma_x, \gamma_d \in \mathcal{K}$ and $r \geq 0$, with $\gamma_x(s) < s$ for all $s > 0$:

$$\max\{\gamma_x \left(\max_{\theta \in [-\tau, 0]} V(\varphi(\theta)) \right), \gamma_d(|d|), r\} \leq V(\varphi(0)) \Rightarrow D^+V(\varphi(0)) f(\varphi, d) \leq -\alpha(|\varphi(0)|)$$

for all $\varphi \in C_{[-\tau, 0]}$ and all $d \in \mathbb{R}^m$.

If $r = 0$, then such a function V is called ISS-LRF.

Theorem 2. *If for the system (3) there exists a (practical) ISS-LRF, then it is (practically) ISS.*

If the conditions of Theorem 2 are verified only for $\max\{\|\varphi\|, |d|\} \leq \rho$ with some $\rho > 0$ (and $\gamma_x(s) < s$ for $s \in (0, \rho)$), then it implies local (practical) ISS.

III. MAIN RESULT

In this note we will consider a particular example of the system (3):

$$\dot{x}(t) = F(x(t), x(t-\tau), d(t)), \quad (4)$$

where $f(\phi, d) = F(\phi(0), \phi(-\tau), d)$ is a continuous function ensuring existence and uniqueness of solutions at least locally in forward time, and the remaining notation stays the same, $F(0, 0, 0) = 0$. We will need a local boundedness of F in the sequel:

Assumption 1. *For any $R > 0$ and any $\varepsilon > 0$ there exist $\varepsilon \in [0, \varepsilon)$ and $L_R > 0$ such that*

$$|F(x, z, d) - F(\xi, \zeta, \delta)| \leq L_R(\varepsilon + |x - \xi| + |z - \zeta| + |d - \delta|)$$

for all $x, \xi, z, \zeta \in \mathbb{R}^n$ and $d, \delta \in \mathbb{R}^m$ with $\max\{|x|, |\xi|, |z|, |\zeta|, |d|, |\delta|\} \leq R$.

The case with $\varepsilon > 0$ corresponds to a non-Lipschitz F at the origin. Our first simple observation connects the existence of an ISS-LRF for (4) with ISS of the delay-free system:

$$\dot{x}(t) = F(x(t), x(t), d(t)). \quad (5)$$

Proposition 1. *Let there exist $\bar{\tau} > 0$ such that for any $\tau \in [0, \bar{\tau}]$, (4) admits a (practical) ISS-LRF V as in Definition 5, then (5) is (practically) ISS with (practical) ISS-LF V and AG $\alpha_1^{-1} \circ \gamma_d$.*

Proof. In such a case, there exists a locally Lipschitz continuous (practical) ISS-LRF $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that for any $\tau \in [0, \bar{\tau}]$ for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and all $x \in \mathbb{R}^n$:

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|),$$

and for some $\alpha, \gamma_x, \gamma_d \in \mathcal{K}$ and $r \geq 0$, with $\gamma_x(s) < s$ for all $s > 0$:

$$\max\{\gamma_x \left(\max_{\theta \in [-\tau, 0]} V(\varphi(\theta)) \right), \gamma_d(|d|), r\} \leq V(\varphi(0)) \Rightarrow D^+V(\varphi(0)) F(\varphi(0), \varphi(-\tau), d) \leq -\alpha(|\varphi(0)|)$$

for all $\varphi \in C_{[-\tau, 0]}$ and all $d \in \mathbb{R}^m$. If $\tau \rightarrow 0$, then (4) can be reduced to (5), while $\gamma_x(\max_{\theta \in [-\tau, 0]} V(\varphi(\theta)))$ is transformed to $\gamma_x(V(\varphi(0)))$, which is always smaller than $V(\varphi(0))$ due to the properties of γ_x . Therefore, in the limit case with $\tau = 0$ (by replacing the norm for the state from $C_{[-\tau, 0]}$ with the respective one in \mathbb{R}^n) we obtain:

$$\max\{\gamma_d(|d|), r\} \leq V(x) \Rightarrow D^+V(x) F(x, x, d) \leq -\alpha(|x|)$$

for any $x \in \mathbb{R}^n$ and any $d \in \mathbb{R}^m$, which due to Theorem 1 implies (practical) ISS of (5). The AG of the system can be estimated from the last relation using standard arguments (see Remark 2) and it is equal to $\alpha_1^{-1} \circ \gamma_d$. \square

Remark 3. In conditions of Proposition 1, according to Lemma 1, the system $\dot{x}(t) = F(x(t), x(t) + B(x(t), d(t))u(t), d(t))$ is locally practically ISS with respect to both inputs u

and d with the gain $\gamma_u(s) = \alpha_2 \circ \max\{s, \alpha^{-1}(s)\}$ and some continuous matrix function B with continuous inverse. Since $F(x(t), x(t - \tau), d(t)) = F(x(t), x(t) + B(x(t), d(t))B^{-1}(x(t), d(t))(x(t - \tau) - x(t)), d(t))$, with respect to solely $x(t - \tau)$ the Lyapunov-Razumikhin condition locally can be written as $\gamma_u(\kappa|x(t - \tau)|) \leq V(x(t))$ for some $\kappa > 0$, which is implied by $\gamma_u(\kappa\alpha_1^{-1}(V(x(t - \tau)))) \leq V(x(t))$. Therefore, since for a local ISS-LRF the relation between $V(x(t - \tau))$ and $V(x(t))$ is formulated through γ_x , we obtain that

$$\gamma_u(\kappa\alpha_1^{-1}(s)) \leq \gamma_x(s) < s, \quad \forall s \in (0, \rho]$$

for some $\rho > 0$ provided that the gain γ_u has been calculated in a tight non-restrictive way. The last property can be equivalently rewritten as follows:

$$\alpha_1^{-1} \circ \gamma_u(\kappa s) < s, \quad \forall s \in (0, \rho], \quad (6)$$

which corresponds to a contraction constrain on AG in (5) with respect to an auxiliary input in the second argument of the function F .

Surprisingly, a converse result for Proposition 1 can be obtained under mild auxiliary hypotheses:

Proposition 2. *Let Assumption 1 be satisfied, (5) possess a local practical ISS-LF V as in Definition 3 and for $\varrho'(s) = \alpha_2 \circ \max\{s, \eta^{-1}(s)\}$ AG*

$$\alpha_1^{-1} \circ \varrho'(\kappa s) < s, \quad \forall s \in (0, \rho] \quad (7)$$

with some $\kappa > 0$ and $\rho > 0$, then there exists $\bar{\tau} > 0$ such that (4) admits V as a local practical ISS-LRF for all $\tau \in [0, \bar{\tau}]$.

Proof. In such a case, according to Definition 3 and Remark 1, for (5) there exist V with some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty, \eta, \varrho \in \mathcal{K}$ and $r \geq 0$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|),$$

$$V(x) \geq \max\{\varrho(|d|), r\} \implies D^+V(x)F(x, x, d) \leq -\eta(|x|)$$

for all $x \in \mathbb{R}^n$ and all $d \in \mathbb{R}^m$ with $\max\{|x|, |d|\} \leq R$ for some $R > 0$. Since all conditions of Lemma 1 are satisfied (the arguments can be repeated for a local practical ISS-LF), there are $\eta' \in \mathcal{K}$ and a continuous $B : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^{n \times n}$ with continuous inverse such that for the system (4) with a given $\tau > 0$ we obtain:

$$V(\phi(0)) \geq \max\{\varrho'(|B^{-1}(\phi(0), d)|\phi(-\tau) - \phi(0)|), \varrho(|d|), r\} \implies D^+V(\phi(0))F(\phi(0), \phi(-\tau), d) \leq -\eta'(|\phi(0)|)$$

for any $\phi \in C_{[-2\tau, 0]}$ (for the sequel we need to consider the state from a lifted space on the interval $[-2\tau, 0]$, which is a technical modification) and all $d \in \mathbb{R}^m$ satisfying restriction $\max\{|\phi(0)|, |d|\} \leq R$ (decreasing the value of ε the bounds r and R can be preserved), where $\varrho'(s) = \alpha_2 \circ \max\{s, \eta^{-1}(s)\}$. Denote $\tilde{B} = \sup_{\max\{|\phi(0)|, |d|\} \leq R} |B^{-1}(\phi(0), d)|$. Further consider only behavior of V obtained on the solutions of (4), i.e., when the effect of initial conditions passes through F for

$t \geq \tau$. Take $\|\phi\| \leq R$ and $\|d\|_\infty \leq R$, then due to Assumption 1 there is $L_R > 0$ such that

$$\begin{aligned} |\phi(-\tau) - \phi(0)| &= \left| \int_{-\tau}^0 F(\phi(s), \phi(s - \tau), d(s)) ds \right| \\ &\leq L_R \int_{-\tau}^0 (\varepsilon + |\phi(s)| + |\phi(s - \tau)| + |d(s)|) ds \\ &\leq L_R \tau (\varepsilon + 2\|\phi\| + \|d\|_{[-\tau, 0]}) \\ &\leq L_R \tau (\varepsilon + 2\|\phi\| + \|d\|_\infty). \end{aligned}$$

Therefore,

$$V(\phi(0)) \geq \max\{\varrho'(\bar{L}_R \tau (\varepsilon + R) + 2\bar{L}_R \tau \|\phi\|), \varrho(|d|), r\} \implies D^+V(\phi(0))F(\phi(0), \phi(-\tau), d) \leq -\eta'(|\phi(0)|)$$

for any $\|\phi\| \leq R$ and $\|d\|_\infty \leq R$, where $\bar{L}_R = L_R \tilde{B}$. Since $\varrho'(a + b) \leq \varrho'(2 \max\{a, b\})$ for any $a, b \in \mathbb{R}_+$, the last property follows after

$$V(\phi(0)) \geq \max\{\varrho'(4\bar{L}_R \tau \|\phi\|), \varrho(|d|), \varrho'(2\bar{L}_R \tau (\varepsilon + R)), r\} \implies D^+V(\phi(0))F(\phi(0), \phi(-\tau), d) \leq -\eta'(|\phi(0)|).$$

According to (7), there exists $\gamma_x \in \mathcal{K}$ such that $\varrho'(\kappa s) \leq \gamma_x \circ \alpha_1(s)$ and $\gamma_x(s) < s$ for all $s \in (0, \rho]$ (recall, (7) can be rewritten as $\varrho'(\kappa\alpha_1^{-1}(s)) < s$). Note that if $4\frac{\bar{L}_R \tau}{\kappa} \|\phi\| \leq \rho$ and $4\frac{\bar{L}_R \tau}{\kappa} \leq 1$, then $\varrho'(4\bar{L}_R \tau \|\phi\|) \leq \gamma_x \circ \alpha_1(4\frac{\bar{L}_R \tau}{\kappa} \|\phi\|) \leq \gamma_x \circ \alpha_1(\|\phi\|) \leq \gamma_x(\max_{\theta \in [-2\tau, 0]} V(\phi(\theta)))$, consequently,

$$\begin{aligned} V(\phi(0)) &\geq \max\left\{\gamma_x\left(\max_{\theta \in [-2\tau, 0]} V(\phi(\theta))\right), \varrho(|d|), \right. \\ &\quad \left. \varrho'(2\bar{L}_R \tau (\varepsilon + R)), r\right\} \implies \\ D^+V(\phi(0))F(\phi(0), \phi(-\tau), d) &\leq -\eta'(|\phi(0)|) \end{aligned}$$

for all $\|\phi\| \leq \tilde{R}$ and all $|d| \leq R$ with $\tilde{R} = \min\{R, \frac{\kappa}{4\bar{L}_R \tau} \rho\}$. Define $\bar{\tau} = \min\{\frac{\kappa}{4\bar{L}_R R} \rho, \frac{1}{2\bar{L}_R(\varepsilon + R)} \varrho'^{-1}(r)\}$, then the last implication can be simplified:

$$\begin{aligned} V(\phi(0)) &\geq \max\left\{\gamma_x\left(\max_{\theta \in [-2\tau, 0]} V(\phi(\theta))\right), \varrho(|d|), r\right\} \implies \\ D^+V(\phi(0))F(\phi(0), \phi(-\tau), d) &\leq -\eta'(|\phi(0)|) \end{aligned}$$

for all $\max\{\|\phi\|, |d|\} \leq R$ provided that $\tau \in [0, \bar{\tau}]$. Hence, V is a local practical ISS-LRF, and according to Theorem 2 the system (4) is locally practically ISS. \square

In the case of ISS property of (5) (i.e., $r = 0$) for $\varepsilon = 0$ the analysis as above can be repeated considering perturbation d_t (not as a pointwise input, but as a function on the interval $[t - \tau, t]$), then ISS can be preserved for (4), see the next section for an example.

Remark 4. Note that in the proof, the domain of attraction and the amplitude of admissible inputs are characterized by R and the set of ultimate boundedness is governed by r , which can be chosen arbitrary large or small, respectively, if V is a global ISS-LF. However, the growth of R or decay of r decrease the obtained value of $\bar{\tau}$.

The condition (7) serves to guarantee that the offset r and the upper bound R stay in (4) the same as in (5), and without (7) the values of r and R cannot be preserved in general (see Theorem 3 of [21]). The restriction on AG (7) is verified if

the functions α_1 and ϱ' are of the same order locally at the origin, then playing with κ it is possible to ensure the required relation.

The result of Proposition 2 can be interpreted as follows: if (5) is practically ISS, then (4) is locally practically ISS (there exists a respective ISS-LRF) for any sufficiently small delay, where domain of stability can be arbitrary enlarged at the price of decay of the maximal admissible delay $\bar{\tau}$, provided that Assumption 1 is satisfied with the restriction (7) imposed on the AG of (5) with respect to an additive input in the second argument of F . Proposition 1 provides a kind of converse statement, that existence of practical ISS-LRF for (4) guarantees practical ISS property for (5) (the constraint (7) to the respective AG can be obtained from (6) since the functions γ_u and ϱ' are connected). Therefore, existence of an ISS-LRF for (4) can be related with ISS property of the delay-free system (5), which is the main finding of this work. In particular, if F is locally Lipschitz continuous everywhere, for the case of local practical (lp) ISS the following implications are satisfied:

$$\begin{array}{ccc} (4) \text{ is lp-ISS } \forall \tau \in [0, \bar{\tau}] & \xrightarrow{\text{Prop. 1}} & (5) \text{ is lp-ISS} & \xrightarrow{\text{Rem. 3}} & (6) \\ \uparrow & & \downarrow & & \\ \exists \text{lp-ISS-LRF } \forall \tau \in [0, \bar{\tau}] & \xleftarrow{\text{Prop. 2}} & \exists \text{lp-ISS-LF} & & \end{array} \quad \begin{array}{c} \text{+ (7)} \\ \text{Prop. 2} \end{array}$$

where gray arrows correspond to existing results given in theorems 1 and 2.

IV. THE CASES OF LINEAR AND HOMOGENEOUS SYSTEMS

In this section two examples are given, when the conditions of propositions 1 and 2 (together with the constraints on AG (6) and (7)) are satisfied.

A. Linear systems

Consider the linear counterpart of (4):

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau) + d(t),$$

where $A_0 \in \mathbb{R}^{n \times n}$ and $A_1 \in \mathbb{R}^{n \times n}$ are constant matrices. The delay-free system (5) can be written as follows:

$$\dot{x}(t) = (A_0 + A_1)x(t) + d(t),$$

and it is ISS if and only if there exist matrices $P, Q \in \mathbb{R}^{n \times n}$, such that $P = P^\top > 0$, $Q = Q^\top > 0$ and

$$(A_0 + A_1)^\top P + P(A_0 + A_1) = -2Q.$$

In such a case an ISS-LF can be chosen as $V(x) = x^\top P x$ whose derivative takes the form:

$$\dot{V} = -2x^\top Q x + 2x^\top P d \leq -x^\top Q x + d^\top P Q^{-1} P d$$

for any $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^n$, which leads to the estimates given in Definition 3 with $\alpha_1(s) = \lambda_{\min}(P)s^2$, $\alpha_2(s) = \lambda_{\max}(P)s^2$, $\eta(s) = 0.5\lambda_{\min}(Q)s^2$ and $\varrho(s) = 2\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}\lambda_{\max}(PQ^{-1}P)s^2$, hence $\varrho'(s) = 2\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}\lambda_{\max}(PQ^{-1}P)|A_1|^2s^2$, where $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ denote the minimum and maximum

eigenvalues of a symmetric matrix P . Obviously, the condition (7) is verified globally:

$$\alpha_1^{-1} \circ \varrho'(\kappa s) = |A_1| \sqrt{2\frac{\lambda_{\max}(P)\lambda_{\max}(PQ^{-1}P)}{\lambda_{\min}(P)\lambda_{\min}(Q)}} \kappa s < s$$

for $\kappa < |A_1|^{-1} \sqrt{\frac{\lambda_{\min}(P)\lambda_{\min}(Q)}{2\lambda_{\max}(P)\lambda_{\max}(PQ^{-1}P)}}$.

The same V can be considered as an ISS-LRF for the original system [11]. First, to formulate delay-independent stability conditions: in such a case if there exist $\epsilon \in (0, 1)$, the symmetric matrices $P, R, S \in \mathbb{R}^{n \times n}$ and a scalar $\omega_d > 0$ such that

$$\begin{array}{l} \epsilon P \geq S > 0, \quad \epsilon R \geq P, \\ \left[\begin{array}{ccc} A_0^\top P + P A_0 + R & P A_1 & P \\ A_1^\top P & -S & 0 \\ P & 0 & -\omega_d I_n \end{array} \right] \leq 0 \end{array}$$

all conditions of Definition 5 can be verified for $\alpha_1(s) = \lambda_{\min}(P)s^2$, $\alpha_2(s) = \lambda_{\max}(P)s^2$, $\gamma_d(s) = \frac{2\epsilon\omega_d}{1-\epsilon}s^2$, then $\gamma_u(s) = \frac{2\epsilon\omega_d}{1-\epsilon}|A_1|^2s^2$, and $\gamma_x(s) = \epsilon s$. Clearly, the condition (6) is satisfied for some $\kappa > 0$. Second, to relax these constraints, the delay-dependent stability conditions can be derived by repeating the steps of the proof in Proposition 2: for $\epsilon \in (0, 1)$, find the symmetric matrices $P, R \in \mathbb{R}^{n \times n}$ and scalars $\omega_d, \sigma > 0$ such that

$$\begin{array}{l} \epsilon R \geq P > 0, \\ \left[\begin{array}{ccc} Z & P A_1 & P \\ A_1^\top P & -\sigma I_n & 0 \\ P & 0 & -\omega_d I_n \end{array} \right] \leq 0, \\ Z = (A_0 + A_1)^\top P + P(A_0 + A_1) + R, \end{array}$$

then all conditions of Definition 5 are verified with respect to d_t for $\alpha_1(s) = \lambda_{\min}(P)s^2$, $\alpha_2(s) = \lambda_{\max}(P)s^2$, $\gamma_d(s) = \frac{3\epsilon\omega_d}{1-\epsilon}s^2$, with $\gamma_u(s) = \frac{3\epsilon\omega_d}{1-\epsilon}|A_1|^2s^2$, and $\gamma_x(s) = \epsilon s$ provided that $0 \leq \tau \leq \bar{\tau} = \frac{\min\{\sqrt{\lambda_{\min}(P)\epsilon}, \sqrt{\omega_d}\}}{2\sqrt{\sigma} \max\{1, |A_0|, |A_1|\}}$. Again, the condition (6) can be satisfied.

Thus, the results of propositions 1 and 2 can be illustrated on linear systems, where (7) is always globally verified, and it is a well-known observation that if a linear delay-free system is globally asymptotically stable (for linear systems it implies ISS), then for a sufficiently small delay the system is also ISS [11], and above it is shown that in addition it possesses an ISS-LRF.

B. Homogeneous systems

Linear systems are homogeneous of zero degree [9], let us consider a nonlinear homogeneous system (4) with delays and inputs appearing in the same channel

$$\dot{x}(t) = F(x(t), x(t - \tau) + d(t)).$$

In such a case, there exist $r_i > 0$, $i = 1, \dots, n$ called weights (denote $r = [r_1 \dots r_n]^\top$) and $\nu > -\min_{i=1 \dots n} r_i$ verifying the equality

$$F(\Lambda_r(\lambda)x, \Lambda_r(\lambda)z) = \lambda^\nu \Lambda_r(\lambda)F(x, z)$$

for any $\lambda > 0$ and any $x, z \in \mathbb{R}^n$, where $\Lambda_r(\lambda) = \text{diag}\{\lambda^{r_i}\}_{i=1}^n$ is a diagonal matrix having on the main diagonal elements λ^{r_i} (it is called the dilation matrix) [8], [10]. Following Lemma 2 of [10] or Lemma 4 of [23], assume that the delay-free counterpart (5) is globally asymptotically stable for $d = 0$, then there is a continuously differentiable and homogeneous LF $V(x)$ for (5) such that

$$V(\Lambda_r(\lambda)x) = \lambda^\mu V(x), \quad (8)$$

$$\theta_1 |x|_r^\mu \leq V(x) \leq \theta_2 |x|_r^\mu, \quad D^+V(x)F(x, x) \leq -\theta_3 |x|_r^{\nu+\mu}$$

for all $x \in \mathbb{R}^n$, all $\lambda > 0$ and any $\mu > \max_{i=1 \dots n} r_i$ (the homogeneity degree of V), where

$$\theta_1 = \min_{|y|_r=1} V(y), \quad \theta_2 = \max_{|y|_r=1} V(y), \\ \theta_3 = \min_{|y|_r=1} -D^+V(y)F(y, y)$$

are all positive constants, and

$$|x|_r = \left(\sum_{i=1}^n |x_i|^{\frac{\chi}{r_i}} \right)^{\frac{1}{\chi}}$$

is a homogeneous norm for some $\chi \geq \max_{i=1 \dots n} r_i$ [9] (it is not a norm in the usual sense since it does not satisfy the triangle inequality, but it gives a distance equivalent to $|\cdot|$, hence, it can be used for the stability analysis; in addition, by scaling the weights r and the choice of χ it can be made locally Lipschitz continuous [9]). Moreover, V is also an ISS-LF for (5). Indeed, for all $x, d \in \mathbb{R}^n$ we obtain

$$D^+V(x)F(x, x+d) \leq -\theta_3 |x|_r^{\nu+\mu} \\ + D^+V(x)(F(x, x+d) - F(x, x)) \\ \leq -\theta_3 |x|_r^{\nu+\mu} + |x|_r^{\nu+\mu} \theta_4 |F(y, y + \Lambda_r^{-1}(|x|_r)d) - F(y, y)|,$$

where $y = \Lambda_r^{-1}(|x|_r)x$ (note that $|y|_r = 1$) and $\theta_4 = \max_{|y|_r=1} |D^+V(y)|$. Assuming continuity of F in its arguments, there is $\delta > 0$ such that

$$|\Lambda_r^{-1}(|x|_r)d|_r \leq \delta \Rightarrow$$

$$\max_{|y|_r=1} |F(y, y + \Lambda_r^{-1}(|x|_r)d) - F(y, y)| \leq \frac{\theta_3}{2\theta_4},$$

which implies

$$|x|_r \geq \delta^{-1} |d|_r \Rightarrow D^+V(x)F(x, x+d) \leq -0.5\theta_3 |x|_r^{\nu+\mu}$$

for all $x, d \in \mathbb{R}^n$. The condition (7),

$$\alpha_1^{-1} \circ \varrho'(\kappa s) = \left(\frac{\theta_2}{\theta_1} \right)^{\mu-1} \delta^{-1} \kappa s < s, \quad \forall s > 0,$$

is true for any $\kappa < \delta \left(\frac{\theta_1}{\theta_2} \right)^{\mu-1}$. Finally, Lemma 2 in [10] or Lemma 4 [23] (see also [22] for the case $\nu = 0$) are the analogues of Proposition 2 in this case, and repeating the steps of its proof we can show that V is an ISS-LRF for a homogeneous system (4) with respect to the input d_t :

Corollary 1. *If (5) is a homogeneous system of degree $\nu > -\min_{i=1 \dots n} r_i$, which is asymptotically stable at the origin for $d = 0$ with a respective homogeneous LF V of degree μ satisfying (8), then for any $r \geq 0$ and $R > 0$ ($r = 0$ for $\nu \geq 0$)*

and $R = +\infty$ for $\nu = 0$) there exists $\bar{\tau} > 0$ such that in (4) for all $\tau \in [0, \bar{\tau}]$ and any $\max\{\|\phi\|, \|d\|_{[-\tau, 0]}\} < R$ (recall that $\phi \in C_{[-2\tau, 0]})$

$$V(\phi(0)) \geq \max\{\epsilon \max_{\theta \in [-2\tau, 0]} V(\phi(\theta)), \theta_5 \|d\|_{[-\tau, 0]}^\mu, r\} \Rightarrow \\ D^+V(\phi(0))F(\phi(0), \phi(-\tau) + d) \leq -0.5\theta_3 |\phi(0)|_r^{\nu+\mu}$$

for some $\epsilon \in (0, 1)$ and $\theta_5 > 0$.

Proof. Let us just explain the differences with respect to the proof of Proposition 2: here we replace Assumption 1 by the homogeneity condition. Denote $r_{\max} = \max_{i=1 \dots n} r_i$ and $r_{\min} = \min_{i=1 \dots n} r_i$, by scaling weights we can always guarantee that $r_{\max} = 1$, then we can take $\chi = 1$ and the following properties are satisfied for the homogeneous norm:

$$|x + z|_r \leq 2^{\frac{1}{r_{\min}} - 1} (|x|_r + |z|_r), \quad |\lambda x|_r \leq \max\{\lambda, \lambda^{\frac{1}{r_{\min}}}\} |x|_r; \\ |F(x, z)|_r \leq M \max\{\lambda_{xz}^{\nu+1}, \lambda_{xz}^{\frac{\nu}{r_{\min}} + 1}\}, \quad \lambda_{xz} = \max\{|x|_r, |z|_r\}, \\ M = \max_{|\xi|_r \leq 1, |\zeta|_r \leq 1} |F(\xi, \zeta)|_r$$

for any $x, z \in \mathbb{R}^n$ and $\lambda > 0$. Note that for any $\varepsilon \geq 0$ ($\varepsilon = 0$ for $\nu \geq 0$) and $R > 0$ there is $\mathcal{L}_R > 0$ such that $\max\{\lambda_{xz}^{\nu+1}, \lambda_{xz}^{\frac{\nu}{r_{\min}} + 1}\} \leq \mathcal{L}_R(\lambda_{xz} + \varepsilon)$ provided that $\lambda_{xz} < R$, and for $\nu = 0$ the natural choice is $\mathcal{L}_R = 1$ with $R = +\infty$. Next,

$$|\phi(-\tau) - \phi(0)|_r = \left| \int_{-\tau}^0 F(\phi(s), \phi(s-\tau) + d(s)) ds \right|_r \\ \leq \max\{1, \tau^{\frac{1}{r_{\min}} - 1}\} \int_{-\tau}^0 |F(\phi(s), \phi(s-\tau) + d(s))|_r ds \\ \leq 2^{\frac{1-r_{\min}}{r_{\min}}(1+\frac{\nu}{r_{\min}})} M \max\{\tau, \tau^{\frac{1}{r_{\min}}}\} \max\{\max\{\|\phi\|_r, \\ \|\phi\|_r + \|d\|_r\}^{\nu+1}, \max\{\|\phi\|_r, \|\phi\|_r + \|d\|_r\}^{\frac{\nu}{r_{\min}} + 1}\} \\ \leq K_R \max\{\tau, \tau^{\frac{1}{r_{\min}}}\} (2\|\phi\|_r + \|d\|_r + \varepsilon), \\ K_R = 2^{\frac{1-r_{\min}}{r_{\min}}(1+\frac{\nu}{r_{\min}})} M \mathcal{L}_R$$

for $\max\{\|\phi\|_r, \|d\|_r\} < R$, where

$$\|\phi\|_r = \sup_{\theta \in [-2\tau, 0]} |\phi(\theta)|_r,$$

and the remaining steps are similar. \square

Proposition 1 in such a case is obviously satisfied together with (6).

V. CONCLUSIONS

For a nonlinear continuous-time system with pointwise delay, it is shown how the existence of a LRF can be related with ISS of the delay-free model and delay-dependent ISS of the system itself, which makes the Lyapunov-Razumikhin approach necessary and sufficient for such a robust stability property under mild conditions. The main technical restriction consists in the contraction property of the AG of the system. The obtained results are illustrated on linear and nonlinear homogeneous systems.

Enlarging the class of applicable models and relaxing the technical constraints imposed on AG can be considered as directions of further researches.

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