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# Finite-time convergent discrete-time algorithms: from explicit to backward schemes <sup>★</sup>

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## Abstract

Many continuous-time control laws are shown to achieve finite-time convergence. However, when discretized, such property is, most of the times, lost. In this article we study the conditions guaranteeing the finite-time convergence of discrete-time mappings and also investigate the cases when such maps can be interpreted as the backward-Euler discretization of system with a maximal monotone right-hand side. Finite-time stability is characterized. Many examples illustrate the theoretical developments.

*Key words:* Stability, Finite time stability, Discrete-time systems, Maximal monotone operators, Averaged maps.

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## 1 Introduction

Recently discrete-time algorithms with finite-time convergence properties have been studied, see, *e.g.*, [1,2,3,4,5,6]. These algorithms are given in an explicit single-valued form (as in (14) below), involving min and signum functions. Motivated by the fact that the backward (often called implicit) Euler discretization of sliding-mode systems enjoys finite-time convergence properties, see [7] and references therein, it is shown in [8] on two particular cases taken from [1] that it is possible to interpret some of these explicitly formulated algorithms as the backward Euler discretization of set-valued systems (differential inclusions). Maximal monotonicity is a crucial property in backward discretization to guarantee finite-time convergence. In parallel it is well-known in Optimization theory that proximal-point algorithms are the backward discretization of set-valued systems with maximal monotone right-hand side [9]. Recently the link between Optimization (especially proximal-point algorithms) and set-valued sliding-mode systems was made [10], relying on Convex Analysis, proximal mappings, maximal monotone mappings, Yosida approximations and resolvents of maximal monotone operators.

In this article the work in [8] is extended, and a general approach is provided to answer the question: when is an explicit discrete-time system (*i.e.*, a mapping) equal to the backward-Euler discretization of a set-valued system with maximal monotone right-hand side? In addition, when does it guarantee finite-time convergence?

This article is organized as follows: After introducing some notation and useful definitions, the problem to study is formally presented in Section 2. Next, the central property of averagedness and its relation with stability and resolvents of maximal monotone maps is presented in sections 3 and 4, respectively. Afterwards, the connection with subgradient vector fields and proximal-point algorithms is studied in Section 5. Further connections with the so called  $\theta$ -discretization and over-relaxations are shown in Section 6 and finite-time convergence is established in terms of inverses of the interaction map in Section 7. Finally, the paper ends with some conclusions in Section 8.

**Notation and definitions.** Let  $Y \neq \emptyset$  be a set. Then  $2^Y$  is the family of all subsets of  $Y$ . Given another set  $X$ , we say that the map  $\mathbf{M} : X \rightarrow 2^Y$  is *set-valued* or *multivalued*. Alternatively, we write  $\mathbf{M} : X \rightrightarrows Y$  to denote a multivalued map. The *domain* of  $\mathbf{M}$  is given by

$$\text{dom } \mathbf{M} = \{x \in \mathbb{R}^n \mid \mathbf{M}(x) \neq \emptyset\}. \quad (1)$$

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We are mainly concerned with the case  $X = Y = \mathbb{R}^n$ . The identity matrix in  $\mathbb{R}^{n \times n}$  is denoted as  $I$ . Given a positive definite matrix  $P \in \mathbb{R}^{n \times n}$  and two vectors  $x, y \in \mathbb{R}^n$ , we write

$$\langle x, y \rangle_P = x^\top P y, \quad (2)$$

for the *weighted inner product*. Accordingly, we will use  $\|x\|_P = \sqrt{x^\top P x}$  for the *weighted norm*. For simplicity, we will write  $\langle x, y \rangle$  and  $\|x\|$  in place of  $\langle x, y \rangle_I$  and  $\|x\|_I$ , respectively.  $\mathcal{B}_\varepsilon(x)$  denotes the *open ball* with center  $x \in \mathbb{R}^n$  and radius  $\varepsilon > 0$ ,

$$\mathcal{B}_\varepsilon(x) = \{\xi \mid \|x - \xi\| < \varepsilon\}. \quad (3)$$

We say that  $U$  is a *uniform neighborhood* of the subset  $S \subset \mathbb{R}^n$ , if there exists a positive number  $\varepsilon$  such that  $\mathcal{B}_\varepsilon(x) \subset U$  for all of  $x \in S$ .

A proper convex function  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  is *lower semicontinuous* if its epigraph is closed. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a proper, convex, and lower semicontinuous function. The convex *subdifferential* of  $f$  at the point  $x$  is given by

$$\partial f(x) = \{\eta \in \mathbb{R}^n \mid \langle \eta, \xi - x \rangle \leq f(\xi) - f(x), \text{ for all } \xi \in \text{dom } f\}. \quad (4)$$

The *set-valued derivative* of a proper, convex, and lower semicontinuous function  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  along the multivalued vector field  $\mathbf{M} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is defined as [11]

$$\mathcal{L}_{\mathbf{M}} f(x) = \{a \in \mathbb{R} \mid \exists \xi \in \mathbf{M}(x) \text{ such that: } \langle \eta, \xi \rangle = a, \text{ for all } \eta \in \partial f(x)\}. \quad (5)$$

A set-valued operator  $\mathbf{M} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is said to be *monotone* (respectively, *strongly monotone*) if, for any two pairs  $(x_i, y_i)$ ,  $x_i \in \mathbb{R}^n$ ,  $y_i \in \mathbf{M}(x_i)$ ,  $i = 1, 2$ ,

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0 \quad (6)$$

(respectively, if

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geq \gamma \|x_1 - x_2\|^2, \quad (7)$$

for some  $\gamma > 0$ ). The operator  $\mathbf{M}$  is *maximal monotone* if it is monotone and its graph is not strictly contained inside the graph of any other monotone operator. The bracket is defined as  $\lceil x \rceil^\gamma = |x|^\gamma \mathbf{sgn}(x)$ ,  $\gamma \in \mathbb{R}$ ,  $x \in \mathbb{R}$ .

## 2 Problem Statement

The point of departure is the following class of systems with multivalued right-hand sides:

$$\dot{x}(t) \in -\mathbf{M}(x(t)), \quad (8)$$

where  $x(t) \in \mathbb{R}^n$  is the state at time  $t \geq 0$  and  $\mathbf{M} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is a maximal monotone operator. Existence and uniqueness of absolutely continuous solutions [hold](#)

for (8) for all  $x(0) \in \text{dom}(\mathbf{M})$  [12]. This class encompasses many systems that exhibit sliding motions and are known to enjoy robust and finite-time convergence properties, as well as systems with state constraints [10]. The stability of the set of equilibria of (8) can be easily verified [12]. Indeed let

$$\text{Zero } \mathbf{M} := \mathbf{M}^{-1}(0) = \{x \in \mathbb{R}^n \mid 0 \in \mathbf{M}(x)\} \quad (9)$$

be the *set of zeros* of  $\mathbf{M}$ . Let  $x_*$  be an equilibrium of (8), *i.e.*,  $x_* \in \text{Zero } \mathbf{M}$ , and consider the Lyapunov candidate function  $V_{x_*}(x) = \frac{1}{2} \|x - x_*\|^2$ . Its set-valued derivative along the trajectories of (8) is

$$\mathcal{L}_{-\mathbf{M}} V_{x_*}(x) = \{\langle x - x_*, \xi \rangle \mid \xi \in -\mathbf{M}(x)\}. \quad (10)$$

By setting  $x_1 = x$ ,  $x_2 = x_*$ ,  $y_1 = \xi$ , and  $y_2 = 0$  in (6), we can readily see that  $\max \mathcal{L}_{-\mathbf{M}} V_{x_*}(x) \leq 0$ , so  $V_{x_*}$  is a Lyapunov function for the equilibrium  $x_*$  (see [11] for details on Lyapunov analysis using set-valued derivatives).

A set  $\Omega$  is said to be *invariant* with respect to (8) if for every initial condition  $x_0 \in \Omega$ , there exists a maximal solution of (8) lying in  $\Omega$ . Let  $\mathcal{W}_{-\mathbf{M}} V_{x_*}$  be the largest invariant set contained in

$$\mathcal{Z}_{-\mathbf{M}} V_{x_*} := \text{Zero } \mathcal{L}_{-\mathbf{M}} V_{x_*}. \quad (11)$$

Note that, since  $\text{Zero } \mathbf{M}$  is invariant and  $\text{Zero } \mathbf{M} \subset \mathcal{Z}_{-\mathbf{M}} V_{x_*}$ , we have

$$\text{Zero } \mathbf{M} \subset \mathcal{W}_{-\mathbf{M}} V_{x_*} \subset \mathcal{Z}_{-\mathbf{M}} V_{x_*}. \quad (12)$$

By the Invariance Principle presented in [11, Theorem 3], the solutions of (8) converge to  $\mathcal{W}_{-\mathbf{M}} V_{x_*}$ .

Many of the advantageous properties mentioned above are best preserved when the differential inclusion (8) is discretised using the backward (also named *implicit* in the Control literature [13,14,7]) Euler method

$$x_{k+1} \in x_k - h \mathbf{M}(x_{k+1}) \quad (13)$$

with  $h > 0$  the sampling period. Such method produces the discrete-time system in explicit form

$$x_{k+1} = T(x_k), \quad (14)$$

where  $k \in \mathbb{N}$  is the discrete time and

$$T(x) = (\mathbf{I}_d + h \mathbf{M})^{-1}(x). \quad (15)$$

In this article, we are interested in the inverse problem, *i.e.*, in characterizing the discrete-time systems (14) that stem from a backward-Euler discretisation of (8). Note that, given an arbitrary map  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , we can always solve for  $\mathbf{M}$  such that (15) holds. Thus, we can further refine our question as follows.

**Problem 1** Under which conditions on  $T$  is the (possibly multivalued) map

$$\mathbf{M} = \frac{1}{h}(T^{-1} - \mathbf{I}_d) \quad (16)$$

maximal monotone?

When, as in (14),  $T$  is given but  $h$  is not specified, then there is a family of maximal monotone operators whose backward-Euler discretization results in  $T$ . The family is parameterized by  $h$  (note that the specific value of  $h$  has no effect on the maximal monotony of (16)). In other words, to uniquely characterize  $\mathbf{M}$ , it is necessary to specify the sampling period.

It turns out that the maximal monotone operators of the form (16) are completely characterized by the *firm non-expansiveness* of  $T$ , which is a particular instance of the property of *averageness* and plays an important role in the literature of numerical optimization [9,15]. These concepts and their relevance to discrete-time control will be discussed in more detail in the following sections.

**Remark 2** The explicit form (14) of the backward Euler discretization (13) is not to be confused with the forward (also named explicit in the literature [7]) Euler discretization of (8), which takes the form  $x_{k+1} \in x_k - h\mathbf{M}(x_k)$ , and which yields  $x_{k+1} \in (\mathbf{I}_d - h\mathbf{M})(x_k)$  instead of (14)-(15). As is known, the properties of the operator  $(\mathbf{I}_d - h\mathbf{M})$  and of  $T$  in (15) are quite different [10].

### 3 Averageness and asymptotic stability

In this section the notion of averageness is recalled. This property has the notable advantage of being easy to verify for concrete operators. The price to pay is conservativeness, in the sense that the property is stronger than asymptotic stability and asymptotic stability may be sufficient for many control purposes.

Allow us to recall a basic notion from fixed-point theory.

**Definition 3** Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . If, for any  $\xi, x \in \mathbb{R}^n$ :

$$\|T(\xi) - T(x)\|_P \leq \|\xi - x\|_P, \quad (17)$$

then  $T$  is said to be non-expansive with respect to the norm  $\|\cdot\|_P$ . If the inequality in (17) is strict for all  $x \neq \xi$ , then  $T$  is a contraction with respect to the norm  $\|\cdot\|_P$ . Whenever  $P = I$ , we simply say that  $T$  is non-expansive and, respectively, a contraction.

Contractive maps play a fundamental role in the study of stability of equilibria of linear discrete-time systems, since their global asymptotic stability (GAS) is completely characterized by the contraction property of the iteration map. That is, the origin of the linear system

$$x_{k+1} = \tilde{A}x_k \quad (18)$$

is GAS if and only if the map  $x \mapsto \tilde{A}x$  is a contraction with respect to some weighted norm  $\|\cdot\|_P$ . For the particular Euclidean norm  $\|\cdot\|$ , contractivity is equivalent to the induced norm of the operator being less than one, that is, to the condition  $\|\tilde{A}\|_{2,2} = \sigma_{\max}(\tilde{A}) < 1$  [16, Definition 4.1.2, Lemma 10.4.1]. Accordingly, the latter implies that the origin is GAS. However, requiring a contractive map in a general nonlinear setting may be too restrictive, as it is shown in the following simple example.

**Example 4** Consider the nonlinear system

$$\dot{x}(t) = -\gamma \text{sat}_\varepsilon(x(t)), \quad (19)$$

where  $\varepsilon > 0$  is a design parameter,  $\gamma > 0$  is the gain, and  $\text{sat}_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$  is the classical saturation function

$$\text{sat}_\varepsilon(x) = \begin{cases} -1, & \text{if } x < -\varepsilon \\ \frac{x}{\varepsilon}, & \text{if } -\varepsilon \leq x \leq \varepsilon \\ 1, & \text{if } \varepsilon < x \end{cases} \quad (20)$$

System (19) can be seen as a (somewhat arbitrary) regularization of the system

$$\dot{x}(t) \in -\gamma \text{sgn}(x(t)), \quad (21)$$

where

$$\text{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ [-1, 1] & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases} \quad (22)$$

is the signum multivalued function (to alleviate notation, we sometimes identify a singleton  $\{x\} \subset X$  with the element  $x \in X$ ). The system belongs to the class (8) with  $\mathbf{M} = \gamma \text{sgn}$ . The forward-Euler discretization of (19) has the form (14) with

$$T(x) = x - h\gamma \text{sat}_\varepsilon(x), \quad (23)$$

where  $h > 0$  is the sampling period. It is not difficult to see that (23) is Lipschitz continuous with constant  $L = \max\{1, |1 - \frac{h\gamma}{\varepsilon}|\}$ . The map  $T$  is at most non-expansive and never a global contraction. However, for the particular value of  $\varepsilon = h\gamma$ , the iteration (23) coincides with the backward-Euler discretization of (21). In such case, the origin is globally finite-time stable, as established in the literature on the implicit method [13,14].  $\triangle$

We can now articulate the main definition of this paper.

**Definition 5** Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a single-valued map. We say that  $T$  is  $\alpha$ -averaged,  $\alpha \in (0, 1)$ , if there exists a non-expansive map  $R : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , called the reflected resolvent, such that

$$T = (1 - \alpha)\mathbf{I}_d + \alpha R. \quad (24)$$

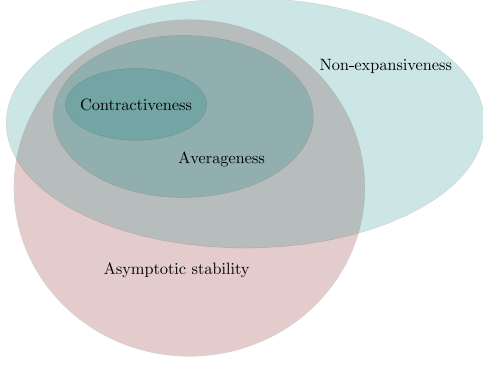


Figure 1. Every contractive operator is  $\alpha$ -averaged and every  $\alpha$ -averaged operator is non-expansive. The set of fixed points of an  $\alpha$ -averaged operator is globally asymptotically stable. There are operators that are non-expansive and, yet, their set of fixed points is not asymptotically stable. Conversely, not every asymptotically stable operator is non-expansive.

Clearly,  $\alpha$ -averaged maps are non-expansive, although they are not necessarily contractions. Averaged maps play a fundamental role in general fixed-point iterations, particularly in proximal-point algorithms.

Let us define the set of *fixed points* of  $T$ ,  $\text{Fix } T \triangleq \{x \in \mathbb{R}^n \mid T(x) = x\}$ . The next result allows to calculate the convergence rate of a discrete-time system.

**Theorem 6** [15, Theorem 1] *Consider the discrete-time system  $x_{k+1} = T(x_k)$ , where  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is  $\alpha$ -averaged and  $\text{Fix } T \neq \emptyset$ . Then, the set  $\text{Fix } T$  is GAS and the iteration satisfies*

$$\|x_{k+1} - x_k\|^2 \leq \frac{\alpha}{(k+1)(1-\alpha)} \text{Dist}^2(x_0, \text{Fix } T). \quad (25)$$

Figure 1 provides a visual representation of the weakness–strength relation among the concepts discussed so far. As an illustration, consider again (14) with

$$T(x) = \tilde{a}x - \tilde{b}\gamma \text{sat}_\varepsilon(x), \quad (26)$$

$\tilde{a} \in \mathbb{R}$ ,  $\tilde{b} \in \mathbb{R}$ , and  $x \in \mathbb{R}$ . Similar to (19), the system can be regarded as the regularization followed by the forward-Euler discretization of a system with sliding modes. For regularization purposes, it has been suggested to choose  $\varepsilon$  small but arbitrary otherwise (see, e.g., [17]). This is probably not a good idea if the control is to be implemented in discrete-time, since the convergence of (14) with (26) is not guaranteed for all values of  $\varepsilon$ .

**Proposition 7** *The origin of (14) with (26) is GAS if and only if*

$$-\varepsilon(1 - \tilde{a}) < \tilde{b}\gamma < \varepsilon(1 + \tilde{a}). \quad (27)$$

**PROOF.** *Necessity:* Consider an arbitrary initial condition  $x_0 \in \mathcal{B}_\varepsilon \setminus \{0\}$ . Then, the iteration reduces to

$$x_{k+1} = \left( \tilde{a} - \frac{\tilde{b}\gamma}{\varepsilon} \right) x_k. \quad (28)$$

By contradiction, let us assume first that  $\tilde{b}\gamma \geq \varepsilon(1 + \tilde{a})$ , then

$$\tilde{a} - \frac{\tilde{b}\gamma}{\varepsilon} \leq \tilde{a} - \frac{\varepsilon(1 + \tilde{a})}{\varepsilon} = -1.$$

Thus, either  $x_{k+1} = -x_k$  for all  $k$  (in which case the solution does not converge to the origin) or  $|x_{k+1}| > |x_k|$  (in which case  $x_k$  eventually leaves  $\mathcal{B}_\varepsilon$ ). Clearly, the origin is not globally asymptotically stable. Similarly, for the case when  $\tilde{b}\gamma \leq -\varepsilon(1 - \tilde{a})$  we have that

$$\tilde{a} - \frac{\tilde{b}\gamma}{\varepsilon} \geq \tilde{a} + \varepsilon(1 - \tilde{a}) = 1.$$

Once again, the origin cannot be asymptotically stable as (28) is not a contraction in  $[-\varepsilon, \varepsilon]$  and the necessity of (27) follows.

*Sufficiency:* This part of the proof relies on showing that, whenever (27) holds, there exists  $\alpha \in (0, 1)$  such that the map  $T$  is  $\alpha$ -averaged. Then, GAS is ensured by Theorem 6. In view of Definition 5, it suffices to show that the map

$$R(x) = \frac{1-\alpha}{\alpha}x - \frac{1}{\alpha}T(x) = \left( \frac{1-\tilde{a}}{\alpha} - 1 \right)x + \frac{\tilde{b}\gamma}{\alpha} \text{sat}_\varepsilon(x), \quad (29)$$

is non-expansive for some  $\alpha \in (0, 1)$ . Note that  $R$  is piecewise affine, so it is enough to verify that the slope of  $R$  is smaller than 1 at each piece. We split the analysis into the following cases:

i)  $|x| \leq \varepsilon$ . In this interval,  $R$  is defined as

$$R(x) = \left( \frac{\varepsilon(1 - \tilde{a}) + \tilde{b}\gamma}{\varepsilon\alpha} - 1 \right)x. \quad (30)$$

Let  $\mu = \varepsilon(1 + \tilde{a}) - \tilde{b}\gamma$ . It follows from (27) that  $0 < \mu < 2\varepsilon$ . Hence,  $R$  is non-expansive if and only if

$$\left| \frac{\varepsilon(1 - \tilde{a}) + \tilde{b}\gamma}{\varepsilon\alpha} - 1 \right| = \left| \frac{2\varepsilon - \mu}{\varepsilon\alpha} - 1 \right| \leq 1;$$

that is, if and only if

$$0 \leq 1 - \frac{\mu}{2\varepsilon} \leq \alpha.$$

In other words, the restriction of  $R$  to  $\mathcal{B}_\varepsilon$  is non-expansive for any  $\alpha \in [1 - \frac{\mu}{2\varepsilon}, +\infty)$ .

ii)  $x > \varepsilon$ . In this case,  $R$  is given as

$$R(x) = \left( \frac{1 - \tilde{a}}{\alpha} - 1 \right) x + \frac{\tilde{b}\gamma}{\alpha}.$$

Thus,  $R$  is non-expansive if and only if

$$0 \leq \frac{1 - \tilde{a}}{2} \leq \alpha.$$

Hence, the restriction of  $R$  to the interval  $(\varepsilon, +\infty)$  is non-expansive for any  $\alpha \in \left[ \frac{1 - \tilde{a}}{2}, +\infty \right)$ .

The remaining case,  $x < -\varepsilon$  leads to the same conclusion as case ii) above. By putting all cases together, we can see that  $T$  is  $\alpha$ -averaged for any  $\alpha$  such that

$$\max \left\{ \frac{1 - \tilde{a}}{2}, \frac{1 - \tilde{a}}{2} + \frac{\tilde{b}\gamma}{2\varepsilon} \right\} \leq \alpha < 1, \quad (31)$$

and convergence towards the set of fixed points of  $T$  follows from Theorem 6. Finally, it is easy to verify that, under (27), the unique fixed point of  $T$  is the origin. This concludes the proof.

**Remark 8** Proposition 7 states a result which has apparently been ignored so far in the sliding-mode control literature, that the sample time  $h$  and the saturation width  $\varepsilon$  are intimately related: the sampling time has to be decreased when the saturation width is decreased. In general, tuning these parameters by trial and error may be tedious, even for simple systems [18]. This drawback is automatically avoided when implicit controllers, coming from backward-Euler discretizations, are used.  $\triangle$

In the following section, it will be shown that the family of operators  $T$  for which (16) is maximal monotone is a subclass of the  $\alpha$ -averaged operators.

#### 4 Resolvents and backward-Euler discretization

The *resolvent* is an essential construction from convex optimization.

**Definition 9** Given  $\mathbf{M} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ , the operator

$$J_h \mathbf{M} = (\mathbf{I}_d + h \mathbf{M})^{-1} \quad (32)$$

is called the *resolvent* of  $\mathbf{M}$  of index  $h > 0$ .

It is clear from (15) that the resolvent of an operator  $\mathbf{M}$  is precisely its backward-Euler discretization. We can now proceed to characterize the class of resolvents that stem from maximal monotone operators.

If  $\alpha = \frac{1}{2}$  in (24), then we say that  $T$  is *firmly non-expansive*.

**Proposition 10** [9, Proposition 23.8] The resolvent  $J_h \mathbf{M} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is firmly non-expansive if and only if  $\mathbf{M}$  is maximal monotone.

The following corollary is immediate though important.

**Corollary 11** The multivalued map  $\mathbf{M}$  in (16) is maximal monotone if and only if  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is firmly non-expansive. Thus, every firmly non-expansive map is the backward-Euler discretization of a differential inclusion of the form (8).

**Corollary 12** If  $T$  is  $\alpha$ -averaged with  $\alpha \in (0, \frac{1}{2}]$ , then the multivalued map  $\mathbf{M}$  in (16) is maximal monotone.

**PROOF.** The proof follows from Remark 4.27 in [9], as an  $\alpha_1$ -averaged map is also  $\alpha_2$ -averaged for all  $0 < \alpha_1 \leq \alpha_2 < 1$ .

**Example 13** Consider (14) with  $T : \mathbb{R} \rightarrow \mathbb{R}$  the projection on the interval  $[a, b]$ ,

$$T(x) = \begin{cases} a & \text{if } x < a \\ x & \text{if } x \in [a, b] \\ b & \text{if } x > b \end{cases}. \quad (33)$$

Being a projection,  $T$  is non-expansive. Let us solve for the reflected resolvent in (24) with  $\alpha = \frac{1}{2}$ ,

$$R(x) = 2T(x) - x = \begin{cases} 2a - x & \text{if } x < a \\ x & \text{if } x \in [a, b] \\ 2b - x & \text{if } x > b \end{cases}. \quad (34)$$

The operator is clearly non-expansive (it is piecewise affine with slope  $\pm 1$  at each piece), so  $T$  is actually firmly non-expansive. It follows from Proposition 10 that  $\mathbf{M} = \frac{1}{h}(T^{-1} - \mathbf{I}_d)$  is maximal monotone. Indeed, as expected,  $\mathbf{M}$  is the normal cone mapping

$$\mathbf{M}(x) = \mathcal{N}_{[a,b]}(x) = \begin{cases} \mathbb{R}_- & \text{if } x = a \\ \{0\} & \text{if } x \in (a, b) \\ \mathbb{R}_+ & \text{if } x = b \\ \emptyset & \text{otherwise} \end{cases}, \quad (35)$$

which is known to define a maximal monotone mapping (see Example 20 below). In analogy with its discrete-time counterpart, the vector field (8) with  $\mathbf{M}$  as in (35) constraints the state to the interval  $[a, b]$ . Also, the set of equilibria of the continuous system, again the interval  $[a, b]$ , coincides exactly with the set of fixed points of the discrete-time system.  $\triangle$

**Example 14** Consider (14) with

$$T(x) = \left( \max \left\{ 0, |x|^{p/q} - c^{p/q} \right\} \right)^{q/p} \text{sgn}(x), \quad (36)$$

where  $c = \left(h\gamma\frac{p}{q}\right)^{q/p}$  and  $p, q \in \mathbb{N}$ ,  $p < q$ . It is the exact discrete model of

$$\dot{x}(t) = -\tilde{\mathbf{M}}(x(t)) \quad (37)$$

with  $\tilde{\mathbf{M}}(x) = \gamma[x]^{1-\frac{p}{q}}$ . Here, we have defined  $[x]^r = |x|^r \mathbf{sgn}(x)$  for any  $r \in \mathbb{R}$ . By exact discretization it is meant that, if the ODE has the solution  $x(t) = \varphi(t, x_0)$ , then  $T(x) = \varphi(h, x)$ .

For  $|x| \leq c$  we have  $T(x) = 0$ , so the associated reflected resolvent, with  $\alpha = \frac{1}{2}$ , is  $R(x) = -x$  and  $\frac{d}{dx}R(x) = -1$ . For  $|x| > c$ , we have

$$\frac{d}{dx}R(x) = 2 \left(1 - \left(\frac{c}{|x|}\right)^{\frac{p}{q}}\right)^{\frac{q}{p}-1} - 1. \quad (38)$$

Clearly,  $|\frac{d}{dx}R(x)| < 1$  for any  $x$ , so  $R$  is a contraction and, accordingly,  $T$  is a firmly non-expansive operator. The operator (36) is also the backward-Euler discretization of (8) with

$$\begin{aligned} \mathbf{M}(x) &= \frac{1}{h}(T^{-1}(x) - x) \\ &= \frac{1}{h} \left[ \left( |x|^{\frac{p}{q}} + c^{\frac{p}{q}} \right)^{\frac{q}{p}} \cdot \mathbf{sgn}(x) - x \right]. \end{aligned} \quad (39)$$

By Corollary 11,  $\mathbf{M}$  is maximal monotone.  $\triangle$

It follows from Proposition 10 that, if  $\mathbf{M}$  is maximal monotone, then the resolvent (32) is  $\frac{1}{2}$ -averaged and the corresponding reflected resolvent is defined implicitly by

$$J_h \mathbf{M}(x) = \frac{1}{2}(\mathbf{I}_d + R)(x), \quad (40)$$

see [15]. The reflected resolvent is also called the *Cayley operator*. It is similar to the usual Cayley operator widely used in Automatic Control, which allows to transform positive real systems into bounded real systems, and *vice-versa* [19]. Thus, in this case the fixed-point iteration  $x_{k+1} = J_h \mathbf{M}(x_k)$  takes another equivalent form, a form that belongs to the class of Krasnosel'skiĭ-Mann iterations,

$$x_{k+1} = \frac{1}{2}x_k + \frac{1}{2}R(x_k), \quad (41)$$

which are known to converge to  $\text{Fix } R$  [15].

In the following section we will focus on the case in which the maximal monotone operator can be expressed as the subdifferential of a convex function.

## 5 Discretization of gradient systems

Allow us to state a result derived from [11, Theorem 3].

**Theorem 15** Consider the sub-gradient system

$$\dot{x}(t) \in -\partial f(x(t)), \quad (42)$$

where  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  is a proper lower-semicontinuous convex function. The solutions of the differential inclusion (42) converge to  $\text{Zero } \partial f$  (whenever nonempty). If in addition  $f$  is radially unbounded [20], convergence is global inside  $\text{dom}(\partial f)$ .

**PROOF.** Note that  $f$  is a candidate Lyapunov function for  $\text{Zero } \partial f$ . Its set-valued derivative is

$$\begin{aligned} \mathcal{L}_{-\partial f} f(x) &= \{a \in \mathbb{R} \mid \exists v \in -\partial f(x) \\ &\quad \text{such that } \langle v, \xi \rangle = a, \text{ for all } \xi \in \partial f(x)\}. \end{aligned} \quad (43)$$

Since the condition  $\langle v, \xi \rangle = a$  must hold in particular for  $\xi = -v$ , we conclude that

$$\mathcal{L}_{-\partial f} f(x) \subset \{-\|v\|^2 \mid v \in \partial f(x)\}, \quad (44)$$

so either  $\mathcal{L}_{-\partial f} f(x) = \emptyset$  or  $\max \mathcal{L}_{-\partial f} f(x) \leq 0$ . The exposition is simplified by agreeing that  $\max \emptyset = -\infty$ , in which case, we can simply write  $\max \mathcal{L}_{-\partial f} f(x) \leq 0$  and conclude that  $f$  is a Lyapunov function. Moreover, it follows from (44) that

$$\mathcal{Z}_{-\partial f} f(x) = \{x \in \mathbb{R}^n \mid 0 \in \mathcal{L}_{-\partial f} f(x)\} \subset \text{Zero } \partial f. \quad (45)$$

On the other hand, since  $\text{Zero } \partial f$  is invariant and  $\text{Zero } \partial f \subset \mathcal{Z}_{-\partial f} f(x)$ , we conclude that the largest invariant set contained in  $\mathcal{Z}_{-\partial f} f(x)$  is  $\text{Zero } \partial f$ .

In the context of Optimisation, the relevance of the previous theorem is revealed by noting that  $\text{Zero } \partial f$  is equal to the set of minimizers of  $f$ .

The operator

$$\text{Prox}_f = (\mathbf{I}_d + \partial f)^{-1} \quad (46)$$

is called the *proximal mapping* of  $f$ . It is well-known that, when  $f$  satisfies the above-mentioned conditions, then  $\partial f$  is a maximal monotone operator [21].

The algorithm  $x_{k+1} = J_h \mathbf{M}(x_k)$  is equivalent to any of the following iterations:

$$x_{k+1} = \text{Prox}_{hf}(x_k), \quad (47)$$

$$x_{k+1} - x_k \in -h\partial f(x_{k+1}). \quad (48)$$

It can be readily seen from (48) that the proximal-point algorithm corresponds to the backward-Euler discretization of (42). The algorithm is known to be more efficient and have better approximation properties than its



forward-Euler counterpart [22]. It follows from Theorem 6 and Corollary 11 that the solutions of the discrete-time system (48) also converge to the set of minimizers of  $f$ .

A natural refinement of Problem 1 is the following.

**Problem 16** *Under which conditions on  $T$  does there exist a proper, lower-semicontinuous, and convex function  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  such that  $T = \text{Prox}_{hf}$ ?*

A similar question has been posed in the literature of convex optimization. Before stating the next result, let us introduce a well-known notion.

**Definition 17** [21] *A mapping  $\mathbf{M} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is cyclically monotone if*

$$\langle x_1 - x_0, y_0 \rangle + \langle x_2 - x_1, y_1 \rangle + \cdots + \langle x_0 - x_m, y_m \rangle \leq 0 \quad (49)$$

for any set of pairs  $(x_i, y_i)$ ,  $i = 1, \dots, m$ ,  $m \in \mathbb{N}$  arbitrary, and  $y_i \in \mathbf{M}(x_i)$ . If, furthermore, the graph of  $\mathbf{M}$  cannot be properly contained in the graph of any other cyclically monotone mapping, we say that  $\mathbf{M}$  is maximal cyclically monotone.

**Theorem 18** [23, Theorem 12.25] *A mapping  $\mathbf{M} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  has the form  $\mathbf{M} = \partial f$  for some proper lower-semicontinuous convex function  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  if and only if  $\mathbf{M}$  is maximal cyclically monotone. Then  $f$  is determined by  $\mathbf{M}$  uniquely up to an additive constant.*

A scalar maximal monotone mapping  $\mathbf{M} : \mathbb{R} \rightrightarrows \mathbb{R}$  is always maximal cyclically monotone and thus can be integrated [23].

**Example 19** *We have seen in Example 14 that  $T$  in (36) is the exact discretization of (37) with  $\tilde{\mathbf{M}}(x) = \gamma [x]^{1-\frac{p}{q}}$ . Since  $\tilde{\mathbf{M}}$  is cyclically monotone, it can be integrated. In other words,  $T$  is the exact discretization of the gradient system*

$$\dot{x}(t) = -\partial \tilde{f}(x(t)), \quad (50)$$

where

$$\tilde{f}(x) = \gamma \frac{q}{2q-p} |x|^{2-\frac{p}{q}}. \quad (51)$$

By Proposition 10,  $T$  is also the backward-Euler discretization of (8) with  $\mathbf{M}$  as in (39). The operator can be integrated as  $\mathbf{M} = \partial f$ , so that  $T = \text{Prox}_{hf}$  with

$$f(x) = c|x|F\left(-\frac{q}{p}, \frac{q}{p}, \frac{q+p}{q}, -\left|\frac{x}{c}\right|^{\frac{p}{q}}\right) - \frac{1}{2}x^2 \quad (52)$$

Here,  $F$  is the hypergeometric function [24]

$$F(a, b, c, x) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{x^m}{m!} \quad (53)$$

and

$$(x)_m = \begin{cases} 1 & m = 0 \\ x \cdot (x+1) \cdots (x+m-1) & m > 0 \end{cases} \quad (54)$$

In the simple case  $p = 1$ ,  $q = 2$ , we have

$$\tilde{f}(x) = \gamma \frac{2}{3} |x|^{3/2} \quad (55)$$

and  $f$  reduces to a finite expression,

$$f(x) = \gamma \frac{2}{3} |x|^{3/2} + \gamma^2 \frac{1}{4} h|x|. \quad (56)$$

In general,  $f$  is an infinite series, the first four terms of which are given by

$$\begin{aligned} f(x) = & -\frac{1}{2}x^2 + (\gamma h)^{\frac{q-p}{p}} \left(\frac{q}{p}\right)^{\frac{3p-q}{p}} \frac{p}{p+q} |x|^{\frac{p+q}{q}} + \\ & \left(\frac{p}{q}\gamma h\right)^{\frac{q}{p}} |x| + \\ & (\gamma h)^{\frac{q-2p}{p}} \left(\frac{q}{p}\right)^{\frac{4p-q}{p}} \frac{q^2 - p^2}{4p^2 + 6pq + 2q^2} |x|^{\frac{2p+q}{q}} + \cdots \end{aligned} \quad (57)$$

△

**Example 20** *The normal cone (35) can be integrated as  $\mathbf{M} = \partial f$  with  $f$  the indicator function*

$$f(x) = \begin{cases} 0 & \text{if } x \in [a, b] \\ +\infty & \text{otherwise} \end{cases}. \quad (58)$$

△

## 6 General averaged operators and variations of the Euler method

It has been established in Theorem 6 that the iteration of every operator  $T$  of the form (24) with  $R$  non-expansive and  $\alpha \in (0, 1)$  converges to  $\text{Fix} T$ . If, furthermore,  $\alpha = \frac{1}{2}$ , then  $T$  is the backward-Euler discretization of (8) with  $\mathbf{M}$  maximal monotone. A natural question is: what type of discretization corresponds to  $\alpha$ -averaged operators when  $\alpha \neq \frac{1}{2}$ ? This question is tackled from the perspectives of the  $\theta$ -discretization method and of operator relaxation.

### 6.1 The $\theta$ -discretization method

Recall that the usual  $\theta$ -method for the discretization of (8) is given implicitly by

$$x_{k+1} = x_k - h \mathbf{M}(\theta x_{k+1} + (1-\theta)x_k) \quad (59)$$

with  $\theta \in [0, 1]$  a constant parameter. The values  $\theta = 1$  and  $\theta = 0$  correspond, respectively, to the backward and the forward Euler methods. Thus, with the iteration (59)



it is possible to choose a compromise between the forward and backward Euler methods. The iteration can be written in the explicit form (14). The value  $\theta = 0$  yields the (possibly multivalued) operator  $T = \text{Id} - h\mathbf{M}$ , whereas  $\theta > 0$  yields the single-valued operator

$$T = \left(1 - \frac{1}{\theta}\right) \text{Id} + \frac{1}{\theta} (\text{Id} + h\theta \mathbf{M})^{-1}. \quad (60)$$

In other words, the convex combination  $T_\theta = \theta T + (1 - \theta) \text{Id}$  is the resolvent of index  $h\theta$  of the operator  $\mathbf{M}$ ,

$$T_\theta = (\text{Id} + h\theta \mathbf{M})^{-1}. \quad (61)$$

Similarly to (16),  $\mathbf{M}$  is computed as

$$\mathbf{M} = \frac{1}{h\theta} (T_\theta^{-1} - \text{Id}). \quad (62)$$

**Corollary 21** *Given  $\theta > \frac{1}{2}$ , the multivalued map  $\mathbf{M}$  in (62) is maximal monotone if and only if  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is  $\frac{1}{2\theta}$ -averaged. Thus, every  $\alpha$ -averaged map is the  $\theta$ -discretization of a differential inclusion of the form (8).*

**PROOF.** Let  $\mathbf{M}$  be maximal monotone, then  $T_\theta$  in (61) is firmly non-expansive, so there exists a non-expansive operator  $R_\theta$  such that (see Definition 5)

$$T_\theta = \frac{1}{2} (\text{Id} + R_\theta), \quad (63)$$

that is,

$$T = \left(1 - \frac{1}{\theta}\right) \text{Id} + \frac{1}{2\theta} (\text{Id} + R_\theta) = \left(1 - \frac{1}{2\theta}\right) \text{Id} + \frac{1}{2\theta} R_\theta. \quad (64)$$

Hence,  $T$  is  $\frac{1}{2\theta}$ -averaged. Conversely, if  $T$  is  $\frac{1}{2\theta}$ -averaged, then there exists a non-expansive operator  $R$  such that

$$\left(1 - \frac{1}{\theta}\right) \text{Id} + \frac{1}{\theta} (\text{Id} + h\theta \mathbf{M})^{-1} = \left(1 - \frac{1}{2\theta}\right) \text{Id} + \frac{1}{2\theta} R. \quad (65)$$

This implies that

$$(\text{Id} + h\theta \mathbf{M})^{-1} = \frac{1}{2} \text{Id} + \frac{1}{2} R. \quad (66)$$

It follows from Proposition 10 that  $\mathbf{M}$  is maximal monotone. This concludes the proof.

**Corollary 22** *Let  $\mathbf{M}$  be maximal monotone. The  $\theta$ -discretizations of (8) with  $\theta > \frac{1}{2}$  produce an operator  $T$  for which the set of fixed points is GAS.*

**PROOF.** Note that, if  $\mathbf{M}$  is maximal monotone, then  $\text{Zero } \mathbf{M}$  is convex [9]. By Remark 27 below, we have  $\text{Fix } T = \text{Zero } \mathbf{M}$ , so the set of zeros of  $T$  is convex. The results then follows from Theorem 6.

The analysis can be pushed further towards what convergence properties are obtained for (60) when  $\theta \notin [0, 1]$ . It is noteworthy that, because of Corollary 21,  $T$  is  $\alpha$ -averaged even for  $\theta > 1$ , in which case  $\alpha \in (0, \frac{1}{2})$ . Since by Corollary 12 the operator  $T$  is firmly non-expansive, the same  $T$  can be either seen as the result of a  $\theta$ -discretization of a maximal monotone operator  $\mathbf{M}$  or as the backward discretization of the maximal monotone operator

$$\tilde{\mathbf{M}} = \left( \left(1 - \frac{1}{\theta}\right) \text{Id} + \frac{1}{\theta} (\text{Id} + h\theta \mathbf{M})^{-1} \right)^{-1} - \text{Id}, \quad (67)$$

where  $\theta \geq 1$ .

## 6.2 Relaxation

Relaxation has been studied in the context of proximal-point algorithms as a way of improving the rate of convergence of numerical schemes [25].

**Definition 23** *Let  $T$  be a firmly-nonexpansive map. The map  $\tilde{T} : \mathbb{R}^n \rightarrow \mathbb{R}^n$*

$$\tilde{T} = (1 - \rho) \text{Id} + \rho T, \quad (68)$$

*is an under-relaxation of  $T$  if  $\rho \in (0, 1)$  and an over-relaxation if  $\rho > 1$ .*

**Proposition 24** *Let  $\tilde{T}$  be  $\alpha$ -averaged. If  $\alpha \in (0, \frac{1}{2})$  then  $\tilde{T}$  is an under-relaxation of a firmly non-expansive map. On the other hand, if  $\alpha \in (\frac{1}{2}, 1)$  then  $\tilde{T}$  is an over-relaxation.*

**PROOF.** Let  $\tilde{T}$  be  $\tilde{\alpha}$ -averaged, then there exists a non-expansive map  $R$  such that

$$\begin{aligned} \tilde{T} &= (1 - \tilde{\alpha}) \text{Id} + \tilde{\alpha} R \\ &= \left(1 - \tilde{\alpha} + \frac{\tilde{\alpha}}{\alpha} - \frac{\tilde{\alpha}}{\alpha}\right) \text{Id} + \frac{\tilde{\alpha}}{\alpha} \alpha R \\ &= \left(1 - \frac{\tilde{\alpha}}{\alpha}\right) \text{Id} + \frac{\tilde{\alpha}}{\alpha} ((1 - \alpha) \text{Id} + \alpha R) \end{aligned} \quad (69)$$

By setting  $\rho = \frac{\tilde{\alpha}}{\alpha}$ , it follows that

$$\tilde{T} = (1 - \rho) \text{Id} + \rho T, \quad (70)$$

where  $T = (1 - \alpha) \text{Id} + \alpha R$  is  $\alpha$ -averaged. Thus, for  $\alpha = \frac{1}{2}$  it follows that  $\rho = 2\tilde{\alpha} \in (0, 2)$  and the conclusion follows.

**Corollary 25** *Every  $\alpha$ -averaged map is either an under-relaxed, an over-relaxed, or a pure backward-Euler discretization of a differential inclusion of the form (8).*

By combining Proposition 24 and Corollary 21, we can see that the  $\theta$ -method applied to (8) with  $\theta \in (\frac{1}{2}, 1)$  and  $\theta > 1$  leads, respectively, to an over-relaxation and an under-relaxation of the resolvent.

In practice over-relaxation is used for improving the speed of convergence of proximal-point algorithms in Optimization [25], whereas the properties of under-relaxed iterations have been less explored due to its opposite effect in the speed of convergence.

## 7 Finite-time convergence

In continuous time, finite-time convergence is established either by invoking homogeneity properties, by further applying Lyapunov analysis, or both [26]. In discrete time, it is sometimes possible to establish this property simply by analyzing the inverse discrete-time map.

**Proposition 26** *Consider the discrete-time system (14) and suppose that  $\text{Fix } T$  is GAS. If, in addition, there exists a uniform neighborhood  $\bar{U}$  of  $\text{Fix } T$  such that*

$$\bar{U} \subset T^{-m}(\text{Fix } T) \quad (71)$$

for some  $m \in \mathbb{N}$ , then  $\text{Fix } T$  is globally finite-time stable.

**PROOF.** Suppose that  $\text{Fix } T$  is GAS. Then, for every initial condition  $x_0 \in \mathbb{R}^n$  and every uniform neighborhood  $U$  of  $\text{Fix } T$ , there exists a natural number  $K(U, x_0)$  such that  $k \geq K(U, x_0)$  implies  $x_k \in U$ . By applying  $T^m$  to both sides of (71), we can see that  $x_k \in \text{Fix } T$  for every  $k \geq K(\bar{U}, x_0) + m$ .

**Remark 27** *Recall that, if  $\alpha = \frac{1}{2}$ , then  $T = J_h \mathbf{M}$ . This shows that, under this condition, the discrete-time system (14) is the backward Euler discretization of the differential inclusion (8). In this case, the ingredients in (71) can be easily computed. We have*

$$\text{Fix } T = \text{Zero } \mathbf{M} \quad (72)$$

and, if  $\text{Fix } T = \{0\}$ , then

$$T^{-1}(\text{Fix } T) = h \mathbf{M}(0). \quad (73)$$

△

**Remark 28** *It is a well-known fact that, for continuous-time systems, non-Lipschitz vector fields are necessary for finite-time stability of trajectories. However, in view of Proposition 26, such necessity vanishes in the discrete-time setting. As it is shown below in the examples, finite-time stability can be achieved with Lipschitz continuous maps.*

**Example 29** *The results in Propositions 7 and 26 can be used for the analysis and design of the so-called reaching laws in discrete-time sliding mode control [27,28,29]. As a simple illustration, let us consider the following reaching law proposed in [27]:*

$$x_{k+1} = (1 - h\gamma_1)x_k - h\gamma_2 \text{sgn}(x_k). \quad (74)$$

*In this case,  $x_k \mapsto T(x_k)$  is discontinuous. Therefore,  $T$  is not averaged and finite-time stability of the origin is not ensured. Indeed, as stated in [27], after some finite time, the state will oscillate around the origin, switching signs at each time-step. In order to change the oscillatory behavior for exact finite-time stability of the origin, it is necessary to modify the map  $T(x_k)$ . There is an endless list of maps that achieve such task. Indeed, as stated in Proposition 6, any averaged map satisfying the conditions of Proposition 26 will work. A powerful, yet simple, approach consists in redefining the reaching law (74) in an implicit way by using backward terms. We consider the following two options:*

$$x_{k+1} \in (1 - h\gamma_1)x_k - h\gamma_2 \text{sgn}(x_{k+1}), \quad (75a)$$

and

$$x_{k+1} \in x_k - h\gamma_1 x_{k+1} - h\gamma_2 \text{sgn}(x_{k+1}). \quad (75b)$$

*These two implicit laws stem, respectively, from the forward-backward-Euler (semi-implicit) and the backward-Euler (implicit) discretisations of the differential inclusion  $\dot{x}(t) \in -\gamma_1 x(t) - \gamma_2 \text{sgn}(x(t))$ . After solving for  $x_{k+1}$ , (75a) and (75b) are written, respectively, as*

$$\begin{aligned} x_{k+1} &= \text{Prox}_{h\gamma_2|\cdot|}((1 - h\gamma_1)x_k) \\ &= (1 - h\gamma_1)x_k - h\gamma_2 \min \left\{ \frac{|(1 - h\gamma_1)x_k|}{h\gamma_2}, 1 \right\} \text{sgn}(x_k) \\ &= (1 - h\gamma_1)x_k - h\gamma_2 \text{sat}_{\frac{h\gamma_2}{1-h\gamma_1}}(x_k) \\ &=: T_a(x_k), \end{aligned} \quad (76a)$$

and

$$\begin{aligned} x_{k+1} &= \text{Prox}_{\frac{h\gamma_2}{1+h\gamma_1}}(x_k) \\ &= x_k - \frac{h\gamma_2}{1+h\gamma_1} \min \left\{ \frac{(1+h\gamma_1)|x_k|}{h\gamma_2}, 1 \right\} \text{sgn}(x_k) \\ &= x_k - h \frac{\gamma_2}{1+h\gamma_1} \text{sat}_{\frac{h\gamma_2}{1+h\gamma_1}}(x_k) \\ &=: T_b(x_k). \end{aligned} \quad (76b)$$

*Under the assumption that both gains  $\gamma_1$  and  $\gamma_2$  are strictly positive, it follows from the first equality in (76b) that  $T_b$  is a  $\frac{1}{2}$ -averaged map, whereas  $T_a$  is  $\alpha$ -averaged*

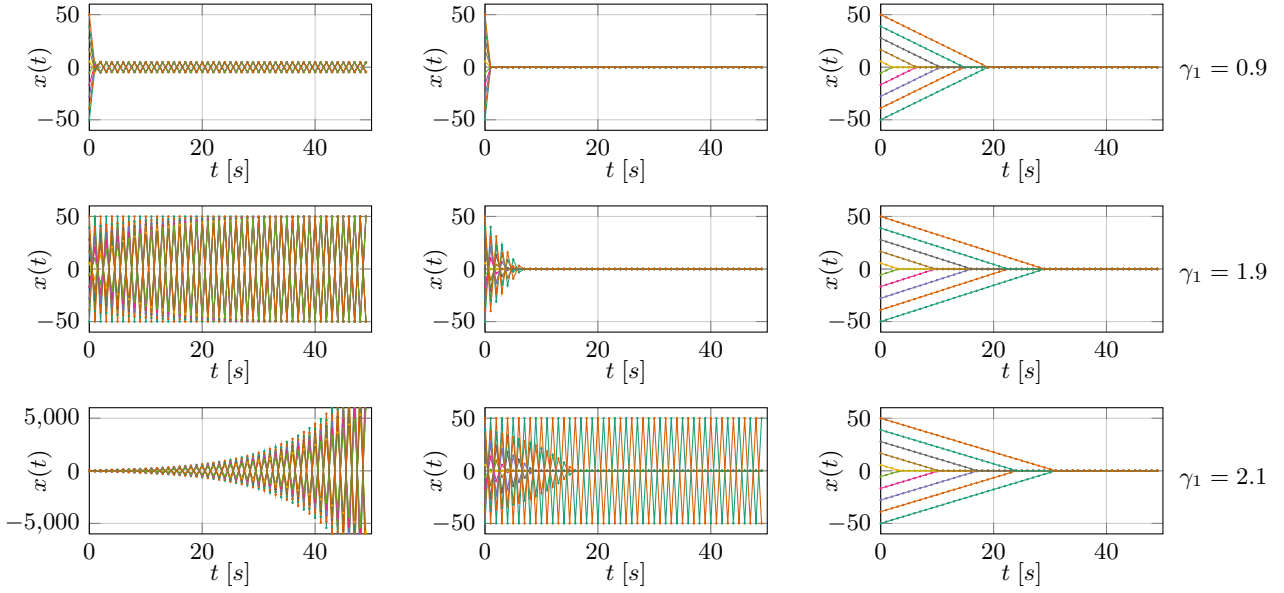


Figure 2. Time evolution of trajectories for the reaching laws (74) (left column), (75a) (central column) and (75b) (right column) for three cases: *i*)  $\gamma_1 = 0.9$  (top row); *ii*)  $\gamma_1 = 1.9$  (middle row); and *iii*)  $\gamma_1 = 2.1$  (bottom row). The design (75b) converges towards zero in finite-time for any combination of positive gains, whereas the other two laws have a constraint in the value of the gain  $\gamma_1$ . For the reaching law (74), the origin is never finite-time stable.

whenever

$$0 < \frac{h\gamma_1}{2} < \alpha < 1, \quad 0 < \gamma_2. \quad (77)$$

Thus, in both cases the origin is GAS in view of Theorem 6. Finally, it follows from direct computations that

$$T_a^{-1}(0) = \left[ -\frac{h\gamma_2}{1-h\gamma_1}, \frac{h\gamma_2}{1-h\gamma_1} \right], \quad (78a)$$

$$T_b^{-1}(0) = \left[ -\frac{h\gamma_2}{1+h\gamma_1}, \frac{h\gamma_2}{1+h\gamma_1} \right]. \quad (78b)$$

Therefore, in both cases, the origin is globally finite-time stable according to Proposition 26. To conclude this example, we point out that the conditions for set-stability of the boundary layer induced by the reaching law (74) are  $0 < 1 - h\gamma_1$  and  $0 < \gamma_2$ , see [27]; whereas for the reaching law (76a) the conditions guarantee global finite-time stability of the origin and they are relaxed as (77); and for (76b) the only condition is the positive sign of both gains, also guaranteeing the global finite-time stability of the origin. Figure 2 depicts the time evolution of trajectories for the three different reaching laws with parameters:  $h = 1s$ ,  $\gamma_1 \in \{0.9, 1.9, 2.1\}$ , and  $\gamma_2 = 5$ , showing the superiority of the proposed designs.

**Example 30** Consider again the discrete-time system of Example 19. Since

$$\text{Fix } T = \mathbf{M}^{-1}(0) = \{0\} \quad (79)$$

and

$$T^{-1}(0) = h\mathbf{M}(0) = [-c, c], \quad (80)$$

the origin is globally finite-time stable by Proposition 26.  $\triangle$

**Example 31** Allow us to revisit Example 3.2 in [1]. Consider (14) with

$$T(x) = x - h\gamma \text{sgn}(x) \min \left\{ \frac{|x|}{h\gamma}, |x|^{1-\frac{p}{q}} \right\}. \quad (81)$$

It has already been shown in [8] that it is the resolvent of a maximal monotone operator. A shorter proof can be given in terms of Corollary 11: The operator can be equivalently written as

$$T(x) = \begin{cases} 0 & \text{if } |x| \leq (h\gamma)^{\frac{q}{p}} \\ x - \gamma h |x|^{1-\frac{p}{q}} & \text{if } |x| > (h\gamma)^{\frac{q}{p}} \end{cases}. \quad (82)$$

Again, let us compute the reflected resolvent  $R = 2T - \mathbf{I}_d$ . For  $|x| \leq (h\gamma)^{\frac{q}{p}}$ ,  $R(x) = -x$ . For  $|x| > (h\gamma)^{\frac{q}{p}}$ ,  $T(x)$  is equal to the forward-Euler discretization of (37). For the latter values of  $x$ , we have

$$\frac{d}{dx}R(x) = 1 - \left(1 - \frac{p}{q}\right) \frac{h\gamma}{|x|^{\frac{p}{q}}} \quad (83)$$

and  $|\frac{d}{dx}R(x)| < 1$ , so  $T$  is firmly non-expansive. Since  $\text{Fix } T = \{0\}$ , the origin is globally asymptotically stable by Theorem 6. It can be readily seen from (82) that

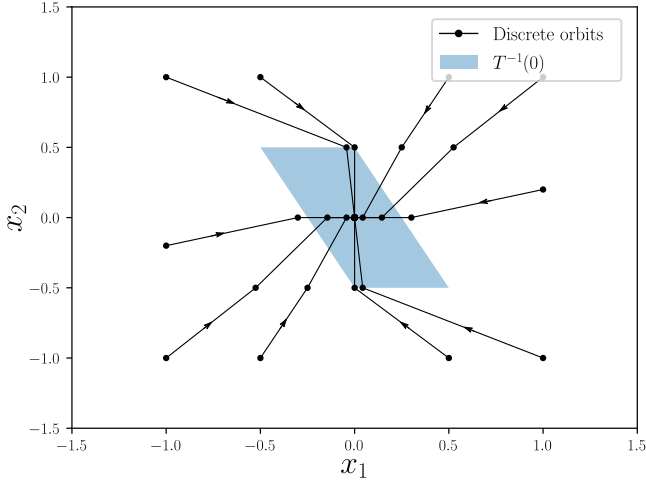


Figure 3. Discrete orbits of the iteration  $x_{k+1} = T(x_k)$  with  $T$  defined by (84) with  $\gamma_1 = 2$ ,  $\gamma_2 = 1$ , and  $h = 0.5$ . The zero set  $T^{-1}(0)$  (light blue), contains a neighborhood of the origin. By definition, an orbit (black) that reaches  $T^{-1}(0)$  reaches the origin on the following step.

$T^{-1}(0) = [-(h\gamma)^{\frac{q}{p}}, (h\gamma)^{\frac{q}{p}}]$ , which of course contains a neighborhood of the origin. Thus, as an alternative to [1, Theorem 4.1], Proposition 26 provides a way for showing the finite-time stability of the origin.  $\triangle$

The role of  $m$  in Proposition 26 is similar to that of the *sliding-order* in continuous-time sliding-mode control. The following example allows to better appreciate this.

**Example 32** Consider the cascade system

$$\begin{aligned} x_{1,k+1} &= \text{Prox}_{hf_1}(x_{1,k} + hx_{2,k}), \\ x_{2,k+1} &= \text{Prox}_{hf_2}(x_{2,k}) \end{aligned} \quad (84)$$

where

$$\begin{aligned} f_1(x_1) &= \gamma_1 \frac{2}{3} |x_1|^{3/2} + \gamma_1^2 \frac{1}{4} h |x_1|, \\ f_2(x_2) &= \gamma_2 |x_2| \end{aligned} \quad (85)$$

The global finite-time stability of the origin follows trivially from the finite-time stability of the origin of the sub-systems  $x_{i,k+1} = \text{Prox}_{hf_i}(x_{i,k})$ ,  $i = 1, 2$ . Allow us however to study this property under the light of Proposition 26. The system has the form (14) and the preimage of 0 under  $T$  is the parallelogram

$$T^{-1}(0) = \{(x_1, x_2) \in \mathbb{R}^2 \mid |x_1 + hx_2| \leq c_1, |x_2| \leq c_2\}, \quad (86)$$

where  $c_1 = (\gamma_1 \frac{h}{2})^2$  and  $c_2 = \gamma_2 h$ . Clearly, the parallelogram has non-empty interior, so the condition (71) holds with  $m = 1$ . Figure 3 shows the set  $T^{-1}(0)$  along with some discrete orbits of (84) for  $\gamma_1 = 2$ ,  $\gamma_2 = 1$ , and  $h = 0.5$ .

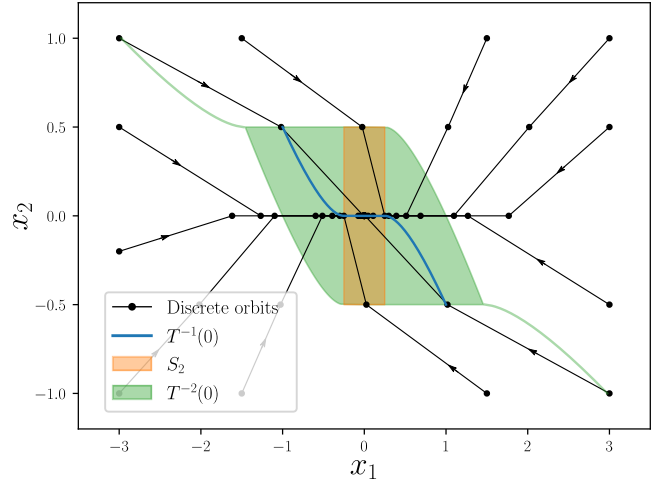


Figure 4. Discrete orbits of the iteration  $x_{k+1} = T(x_k)$  with  $T$  defined by (87) with  $\gamma_1 = 2$ ,  $\gamma_2 = 1$ , and  $h = 0.5$ . The zero set  $T^{-1}(0)$  (light blue) does not contain a neighborhood of the origin. The rectangle  $S_2$  (light orange) and its superset  $T^{-2}(0)$  (light green) do contain a neighborhood of the origin. An orbit (black) that reaches  $T^{-2}(0)$  reaches  $T^{-1}(0)$  on the following step, and hence the origin on at most two more steps.

On the other hand, for the system

$$\begin{aligned} x_{1,k+1} &= \text{Prox}_{hf_1}(x_{1,k}) + hx_{2,k} \\ x_{2,k+1} &= \text{Prox}_{hf_2}(x_{2,k}) \end{aligned} \quad (87)$$

we have

$$T^{-1}(0) = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid x_2 = -\frac{1}{h} \text{Prox}_{hf_1}(x_1), |x_2| \leq c_2 \right\}, \quad (88)$$

the interior of which is empty (see Figure 4), so the condition (71) does not hold with  $m = 1$ . Let us now test the condition with  $m = 2$ . It is difficult to write the second preimage explicitly, but it is relatively straightforward to show that it contains a neighborhood of the origin. Consider the interval  $S_1 = [-c_1, c_1]$  and note that, since  $S_1 \times \{0\} \subset T^{-1}(0)$ , we have  $T^{-1}(S_1 \times \{0\}) \subset T^{-2}(0)$ . For our purposes, it will suffice to show that the preimage of  $S_1 \times \{0\}$  contains a neighborhood of the origin. It is clear that  $|x_1| \leq c_1$  and  $|x_2| \leq \min\{\frac{c_1}{h}, c_2\}$  imply that  $|\text{Prox}_{hf_1}(x_1) + hx_2| \leq c_1$  and  $\text{Prox}_{hf_2}(x_2) = 0$ , so the rectangle

$$S_2 = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid |x_1| \leq c_1, |x_2| \leq \min\left\{\frac{c_1}{h}, c_2\right\} \right\} \quad (89)$$

is a subset  $T^{-1}(S_1 \times \{0\})$ . Since  $S_2$  has nonempty interior, we can conclude the finite-time convergence to the origin. Figure 4 also shows the sets  $S_2$  and  $T^{-2}(0)$  along with some discrete orbits of (87) for the same parameters as before.  $\triangle$

## 8 Conclusions

In this article it is shown that the maps  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  that can be written as the backward-Euler discretization of a maximally monotone operator  $\mathbf{M} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  are precisely the firmly non-expansive operators. Several implications follow. First, the backward-Euler discretization of a maximally monotone operator is always globally asymptotically stable and the convergence rate can be estimated. Second, the stability of a discrete map can be investigated by testing for firm non-expansiveness, a property that is straightforward to verify and obviates the need to perform Lyapunov analysis.

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