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A Variational Approach to the Design of Discrete-time Super-twisting-like Algorithms

Félix A. Miranda-Villatoro

`felix.miranda-villatoro@inria.fr`

TRIPOP team at Centre Inria de l'Université Grenoble Alpes,

Grenoble INP, Laboratoire Jean Kuntzmann (LJK),

3800, Grenoble, France

Abstract

A family of discrete-time, super-twisting-like algorithms is presented. The algorithms are naturally vector-valued and are described in an implicit fashion, reminiscent of backward-Euler discretization schemes. The well-posedness of the closed-loop is established and the robust stability, against a family of external disturbances, is thoroughly studied. Implementation strategies, involving splitting-algorithms from convex optimization, are also discussed and compared. Finally, numerical simulations show the performance of the proposed schemes.

1 Introduction

The super-twisting algorithm (STA) [22] is one of the most popular schemes for control and observation in the family of high-order sliding mode methods. It has been vastly studied from a theoretical viewpoint (see *e.g.*, [13, 20, 23, 28, 31], and references therein), with recent extensions to stochastic systems [32] and infinite dimensional systems [3]. It is also used in a variety of engineering applications such as: control of mechanical systems [9], blood glucose regulation [19], trajectory tracking on quadrotor systems [10], reconstruction of atomic forces in probe microscopes [39], and control of wind energy conversion systems [12], to name just a few. Such success is due to its simple form, its robustness against external disturbances, and its intrinsic ability of diminishing the so called *chattering effect* in implementation. This reduction on chattering is mainly due to the fact that the control law is designed to be continuous, opposite to conventional first-order sliding controllers, where discontinuous signals drive the system's input directly [35, 36].

Nowadays, due to the versatility and the processing power of digital computers, it is common to carry out the control action in a discrete fashion. The resulting closed-loop, a discrete-time controller in feedback interconnection with a continuous-time plant, gives rise to an hybrid system [15, 24]. For this class of dynamical systems, guaranteeing the stability of closed-loop trajectories becomes a challenging task. A common partial solution relies on the *emulation* of the close-loop in continuous time. That is, the controller is designed in continuous-time, so that the closed-loop meets all performance specifications. Afterwards, the resulting control strategy is discretized using an specific scheme, (such as backward-Euler, forward-Euler, mid-point rule, exact discretization, *etc.*) together with a sufficiently small sampling time, for approaching the continuous-time behavior. It is well-known that emulation may work for linear systems [14], specially when the backward-Euler scheme is used. However, that is not necessarily the case for non-smooth systems. Indeed, for the case of super-twisting controllers, emulation does not always yields a good performance when disturbances are in place. Indeed, it has been shown in [2] that, the backward-Euler discretization of the super-twisting algorithm reported in [6], does not preserve the stability properties of the closed-loop, for any sampling time, when unbounded disturbances with bounded derivative are considered, whereas the continuous-time design does. This last fact motivates our departure from emulation, and leads us towards the context where the controller is conceived initially in a discrete-time setting, by considering either: a robust analysis on a suitable time-sampled model of the plant (such as exact discretization) or an hybrid approach. Here we focus on the former scheme.

Several discrete-time versions of the super-twisting algorithm have been recently proposed, [2, 6, 17, 20, 33, 38]. All of them dealing with the scalar case and in most cases only asymptotic convergence is ensured. In this work, a family of vector-valued discrete-time super-twisting-like algorithms is proposed. An important feature of the proposed algorithm is that it is grounded on the property of maximal

monotonicity of the maps involved, and no homogeneity property is used. Indeed, maximal monotonicity allows us to make use of tools from convex optimization for both, the stability analysis of the closed-loop and the numerical implementation of the control law. The STA was originally formulated in continuous-time, for the scalar case, and its analysis was first done via geometric arguments [22]. Recently, some multivariable continuous-time extensions have been proposed, see *e.g.*, [4, 26, 29, 30, 37], dealing mainly with a component-wise extension of the scalar case and the so called unitary control approach [35]. In this work, the generalization to the multivariable case appears naturally, as all of the proofs and algorithms apply directly to the case of multiple controllers. This generality covers the control strategies already reported in the literature and paves the path towards new approaches.

The paper is organized as follows. After presenting preliminary material and the notation used along the manuscript in the next subsection, the closed-loop setup and the proposed family of discrete-time super-twisting-like algorithms under study are presented in Section 2. Next, the well-posedness of the closed-loop is studied in Section 3 and the stability properties of the origin are thoroughly analyzed in Section 4. Afterwards, an splitting strategy for the computation of the controller values, inspired by the Douglas-Rachford algorithm, is studied in Section 5. Finally, the paper ends with some conclusions in Section 6.

Preliminaries and notations.

The inner product is denoted as $\langle \cdot, \cdot \rangle$ and the p -norm in \mathbb{R}^n as $\|\cdot\|_p$. The set \mathcal{B} denotes the open ball in \mathbb{R}^n . For any two sets $A, B \subset \mathbb{R}^n$, the set $A + B := \{a + b | a \in A, b \in B\}$. Let $\mathbf{M} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued map¹, the domain of \mathbf{M} and the graph of \mathbf{M} are the sets $\text{Dom } \mathbf{M} := \{v \in \mathbb{R}^n | \mathbf{M}(v) \neq \emptyset\}$ and $\text{Gph } \mathbf{M} := \{(v, w) \in \mathbb{R}^n \times \mathbb{R}^n | w \in \mathbf{M}(v)\}$, respectively. A set-valued map \mathbf{M} is *monotone* if for any two pairs $(v_i, w_i) \in \text{Gph } \mathbf{M}$, $i \in \{1, 2\}$,

$$0 \leq \langle v_1 - v_2, w_1 - w_2 \rangle. \quad (1)$$

It is strictly monotone if (1) holds with strict inequality for all $v_1 \neq v_2$. Further, \mathbf{M} is called *maximal monotone*, if it is monotone and its graph is not strictly contained inside the graph of any other monotone operator. In addition, \mathbf{M} is called θ -monotone if there exists a function $\theta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\theta(x, y) = \theta(y, x)$ and for any two pairs $(v_i, w_i) \in \text{Gph } \mathbf{M}$, $i \in \{1, 2\}$,

$$\theta(v_1, v_2) \|v_1 - v_2\|_2 \leq \langle v_1 - v_2, w_1 - w_2 \rangle. \quad (2)$$

Maximal θ -monotone operators are defined in an analogous way. It is clear that if $\theta \equiv 0$ in (2), then the map is monotone, whereas if $\theta(x, y) = \alpha \|x - y\|_2$, then \mathbf{M} is strongly monotone (resp. hypomonotone) if $\alpha > 0$ (resp. $\alpha < 0$), the reader is addressed to [21] for further details on θ -monotone maps.

A set-valued map \mathbf{M} is *outer semicontinuous* at $v_0 \in \text{Dom } \mathbf{M}$, if for any sequence $(v_n, w_n) \in \text{Gph } \mathbf{M}$ such that $v_n \rightarrow v$ and $w_n \rightarrow w$ then $(v, w) \in \text{Gph } \mathbf{M}$. It is well known that, in finite dimensions, maximal monotone operators are outer semicontinuous [7, Proposition 4.2.1-ii)].

Let $\Gamma_0(\mathbb{R}^n)$ denote the set of all convex, proper, lower-semicontinuous functions from \mathbb{R}^n to $\mathbb{R} \cup \{+\infty\}$. For a function $f \in \Gamma_0(\mathbb{R}^n)$, the convex subdifferential ∂f at a point $v \in \text{Dom } f$, is defined as the set-valued map

$$\partial f(v) = \{w \in \mathbb{R}^n | \langle w, s - v \rangle \leq f(s) - f(v) \text{ for all } s \in \mathbb{R}^n\}. \quad (3)$$

It is well known that the subdifferential of a function $f \in \Gamma_0(\mathbb{R}^n)$ is maximal monotone, and therefore, also outer semicontinuous at all points of its domain.

The *proximal map* to the function $f \in \Gamma_0(\mathbb{R}^n)$ at x is the map $\text{Prox} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfying

$$\text{Prox}_f(w) = \arg \min_{s \in \mathbb{R}^n} \left\{ f(s) + \frac{1}{2} \|s - w\|^2 \right\}.$$

It follows that for any $w \in \mathbb{R}^n$, see *e.g.*, [5, Proposition 16.34],

$$\text{Prox}_f(w) = (\text{Id} + \partial f)^{-1}(w), \quad (4)$$

where $\text{Id} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ denotes the identity map $x \mapsto x$. That is, $p = \text{Prox}_f(w)$ is the unique solution to the *generalized equation*

$$0 \in p - w + \partial f(p).$$

¹In this context set-valued means non-necessarily single-valued. Thus, single-valued functions are special cases of set-valued maps mapping points to singletons.

Moreover, Prox is a non-expansive map, *i.e.*, it is Lipschitz continuous with constant equal 1, (indeed, it is *firmly non-expansive* [5]).

The following lemma will be useful when studying the stability properties of the closed-loop.

Lemma 1.1. *Let $f \in \Gamma_0(\mathbb{R}^n)$ be such that $r\mathcal{B} \subset \partial f(0)$ for some $r > 0$. Then, for any $w \in \partial f(v)$ we have that $\|w\|_2 \geq r$ whenever $v \neq 0$.* \spadesuit

Proof. Let $(v, w) \in \text{Gph } \partial f$ be arbitrary. By assumption, $(0, w') \in \text{Gph } \partial f$ for any $w' \in r\mathcal{B}$, so that monotonicity of ∂f implies that

$$\langle v, w \rangle \geq \sup_{w' \in r\mathcal{B}} \langle v, w' \rangle = r \|v\|_2,$$

and the conclusion follows from the Cauchy-Schwarz inequality. This concludes the proof. \square

2 Set-valued super-twisting-like controller

Consider the following n -dimensional, perturbed integrator,

$$\dot{x}(t) = u(t) + d(t) \tag{5}$$

where $x(t), u(t), d(t) \in \mathbb{R}^n$, $d : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ is an unknown function accounting for time-dependant disturbances so that d is assumed independent of x with an uniformly bounded time-derivative so that

$$\|\dot{d}(t)\| \leq M. \tag{6}$$

The main target consists in driving the state of (5) to the origin in the presence of the disturbance d . To achieve that, we take inspiration from the following continuous-time super-twisting-like controller

$$u(t) \in -\gamma_1 K \partial f_1(x(t)) + \nu(t), \tag{7a}$$

$$\dot{\nu}(t) \in -\gamma_2 K \partial f_2(x(t)), \tag{7b}$$

where the matrix $K \in \mathbb{R}^{n \times n}$ is symmetric positive definite, for each $i \in \{1, 2\}$, $\gamma_i > 0$ is a constant gain and each function $f_i \in \Gamma_0(\mathbb{R}^n)$, so that each ∂f_i is a set-valued maximal monotone map. In the special case when $n = 1$, $f_1(\cdot) = \frac{2}{3} |\cdot|^{3/2}$, and $f_2(\cdot) = |\cdot|$, we retrieve the conventional super-twisting scheme. Note also that the controller (7) generalizes the conventional super-twisting algorithm in several directions. First, it is intrinsically multivariable. Second, it allows for general families of subdifferential maps ∂f_i , $i \in \{1, 2\}$, which may be either, single-valued or set-valued. This generalization departs from previous propositions [4, 26, 29, 30, 37], by identifying the maximal monotonicity of each subdifferential as the central property for studying the stability of the closed-loop, allowing to use powerful numerical schemes for the efficient numerical implementation, analysis, and design of the proposed super-twisting-like algorithm in a digital computer.

The following standing assumptions are considered along the manuscript,

Assumption 2.1. $\text{rint}(\text{Dom } f_1) \cap \text{rint}(\text{Dom } f_2) \neq \emptyset$.

Assumption 2.1 guarantees that for any $K = K^\top \succ 0$, $\partial(f_1 + f_2) \circ K^{\frac{1}{2}} = \partial(f_1 \circ K^{\frac{1}{2}}) + \partial(f_2 \circ K^{\frac{1}{2}})$ and $K^{\frac{1}{2}} \circ \partial f_i \circ K^{\frac{1}{2}} = \partial(f_i \circ K^{\frac{1}{2}})$, $i \in \{1, 2\}$, see Theorem 16.37 in [5]. Assumption 2.1 is used below to guarantee the well-posedness of the proposed discrete-time model, as it is shown in the next section.

Throughout this paper we consider the case in which the control law (7) is designed in a discrete-time framework, having in mind that its implementation can be realized via a digital computer, whereas the state of the continuous-time plant (5) is sampled at constant time intervals of length $h := t_{k+1} - t_k > 0$. Figure 1 below depicts the resulting closed-loop.

Taking into account that the control signal $u(t)$ is piecewise constant, the following discrete-time model describes the dynamics of the perturbed plant at the sampling times $\{t_k\}_{k \in \mathbb{N}}$,

$$x_{k+1} = x_k + hu_k + h\bar{d}_k, \tag{8a}$$

$$\bar{d}_{k+1} = \bar{d}_k + h\Delta_k, \tag{8b}$$

where the ‘‘averaged perturbation’’ $\bar{d}_k = \frac{1}{h} \int_{t_k}^{t_k+h} d(\tau) d\tau$ is unknown and Δ_k is uniformly bounded. For the second equation, we applied the mean-value theorem for definite integrals to each component of the vector $d(t)$. Namely, component-wise we have that, for some $\tau_k^{(i)} \in [t_k, t_k + h]$

$$\bar{d}_{i,k} = \frac{1}{h} \int_{t_k}^{t_k+h} d_i(\tau) d\tau = d_i(\tau_k^{(i)}).$$

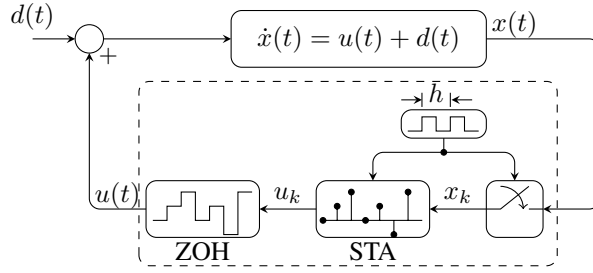


Figure 1: Closed-loop setup. The state of the continuous-time plant is sampled at regular intervals of length h . Such information is used to compute the control law at discrete-time intervals, and this last one is applied to the continuous-time plant via a zero-order-hold (ZOH) mechanism.

Hence, there exist $\tau_k^{(i)} \in [t_k, t_k + h]$ and $\tau_{k+1}^{(i)} \in [t_k + h, t_k + 2h]$ such that the following holds,

$$\bar{d}_{i,k+1} - \bar{d}_{i,k} = d_i(\tau_{k+1}^{(i)}) - d_i(\tau_k^{(i)}) = \tilde{h}_k^{(i)} \dot{d}_i(\theta_k^{(i)}),$$

for some $\theta_k^{(i)} \in [\tau_k^{(i)}, \tau_{k+1}^{(i)}]$, where $\tilde{h}_k^{(i)} := \tau_{k+1}^{(i)} - \tau_k^{(i)}$. Setting $\Delta_{i,k} = \frac{\tilde{h}_k^{(i)}}{h} \dot{d}_{i,k}$ lead us to (8b) and it follows from (6) that $\|\Delta_k\|_2 \leq 2M$.

Regarding the discrete-time controller, it has the following form

$$u_k = -K(\gamma_1 m_{1,k} + h\gamma_2 \rho m_{2,k}) + \nu_{k+1} \quad (9a)$$

$$\nu_{k+1} = \nu_k - h\gamma_2 K m_{2,k} \quad (9b)$$

$$m_{i,k} \in \partial f_i(\varrho(x_k, u_k)), \quad i \in \{1, 2\}, \quad (9c)$$

where $K = K^\top \succ 0$, $\rho > 0$, for $i \in \{1, 2\}$ $\gamma_i > 0$ and each $f_i \in \Gamma_0(\mathbb{R}^n)$ satisfies Assumption 2.1. Finally, $\varrho : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a known function that depends on the state x_k and the control u_k , whose explicit form is specified below. Thus, the controller (9) above is defined in an *implicit* fashion, as it takes the form

$$u_k \in \nu_{k+1} - K\partial(\gamma_1 f_1 + h\gamma_2 \rho f_2)(\tilde{x}_{k+1}), \quad (10a)$$

$$\nu_{k+1} \in \nu_k - h\gamma_2 K \partial f_2(\tilde{x}_{k+1}), \quad (10b)$$

$$\tilde{x}_{k+1} = \varrho(x_k, u_k). \quad (10c)$$

Conditions establishing the well-posedness (existence and uniqueness of solutions) of the close-loop (8)-(9) are presented in the following section. The main differences between the discrete-time controller (9) (equivalently, (10)) and those already reported in the state-of-the-art literature, (as for instance [6], [20] and [2]), consists on the choice of the map ϱ used here and the addition of the term $-h\gamma_2 \rho K m_{2,k}$ in (9a). Indeed, it is shown below that these differences allow us to establish the finite-time stability of the origin for the nominal case (notice that in [6], [2] and [20], only asymptotic stability is proved), as well as, to improve the robustness of the closed-loop when unbounded disturbances are considered.

In what follows it is shown that the closed-loop (8)-(9) is well-posed in a multivariable context and with general functions $f_i \in \Gamma_0(\mathbb{R}^n)$ satisfying Assumption 2.1. The main property that allows us to extend the design to more general cases than those reported in [6], [2], is the maximal monotonicity of the subdifferentials in (9). Such property turns out to be also central for the efficient computation of the control signals, as is shown in Section 5 below.

3 Well-posedness of the closed-loop

In this section the well-posedness (existence and uniqueness of solutions) of the scheme (8)-(9) is studied. We start by defining the function ϱ as

$$\varrho(x_k, u_k) = x_k + hu_k - h\nu_{k+1}, \quad (11)$$

so that the closed-loop system becomes

$$x_{k+1} = \tilde{x}_{k+1} + h(\nu_{k+1} + \bar{d}_k) \quad (12a)$$

$$\bar{d}_{k+1} = \bar{d}_k + h\Delta_k \quad (12b)$$

$$\nu_{k+1} = \nu_k - h\gamma_2 K m_{2,k} \quad (12c)$$

$$\tilde{x}_{k+1} = x_k - hK(\gamma_1 m_{1,k} + h\gamma_2 \rho m_{2,k}) \quad (12d)$$

$$m_{i,k} \in \partial f_i(\tilde{x}_{k+1}), \quad i \in \{1, 2\} \quad (12e)$$

It is worth to remark that the function ϱ in (11) differs from the one proposed in [6]. Indeed, in [6] the function considered is $\varrho_b(x_k, u_k) = x_k + hu_k$. Such choice is the result of the emulation of the continuous-time closed-loop system (5)-(7) using the backward Euler discretization. It is shown in [2] that the resulting controller is not robust with respect to unbounded disturbances with bounded derivative. In contrast, (11) does not show such an issue. Thus, preserving one characteristic feature of the continuous-time super-twisting algorithm.

Now, taking the change of coordinates $z_{1,k} = K^{-\frac{1}{2}}x_k$, $z_{2,k} = K^{-\frac{1}{2}}(\nu_k + \bar{d}_{k-1})$, (recalling that $K = K^\top \succ 0$, it follows that the proposed change of coordinates is well-defined), the closed-loop system (8)-(9) becomes

$$z_{1,k+1} = \tilde{z}_{1,k+1} + hz_{2,k+1} \quad (13a)$$

$$z_{2,k+1} = z_{2,k} - h\gamma_2 \hat{m}_{2,k} + h\hat{\Delta}_{k-1} \quad (13b)$$

$$\tilde{z}_{1,k+1} = z_{1,k} - h\gamma_1 \hat{m}_{1,k} - h^2\gamma_2 \rho \hat{m}_{2,k} \quad (13c)$$

$$\hat{m}_{i,k} \in \partial \hat{f}_i(\tilde{z}_{1,k+1}), \quad i \in \{1, 2\} \quad (13d)$$

where $\hat{m}_{i,k} = K^{\frac{1}{2}}m_{i,k}$, $\hat{\Delta}_k = K^{-\frac{1}{2}}\Delta_k$, $\hat{f}_i = f_i \circ K^{\frac{1}{2}} \in \Gamma_0(\mathbb{R}^n)$, for $i \in \{1, 2\}$. It is clear that the well-posedness of (13) is equivalent to that of (12). Thus, for the closed-loop (13) to be well-posed it must admit a unique solution. That is, for any initial condition $z_{1,0}, z_{2,0} \in \mathbb{R}^n$ there must exist unique sequences $\{z_{1,k}\}_{k \in \mathbb{N}}$, $\{z_{2,k}\}_{k \in \mathbb{N}}$, $\{\hat{m}_{1,k}\}_{k \in \mathbb{N}}$ and $\{\hat{m}_{2,k}\}_{k \in \mathbb{N}}$ satisfying (13) for all $k \in \mathbb{N}$. In addition, the selections $\hat{m}_{1,k} \in \partial \hat{f}_1(\tilde{z}_{1,k+1})$ and $\hat{m}_{2,k} \in \partial \hat{f}_2(\tilde{z}_{1,k+1})$ must depend only on data and system parameters available at time $t_k = kh$. The following theorem ensures that with the above assumptions the closed-loop (13) is well-posed.

Theorem 3.1. *Let $f_i \in \Gamma_0(\mathbb{R}^n)$, for $i \in \{1, 2\}$, such that Assumption 2.1 holds. In addition, let D_i be the set of points where \hat{f}_i is non-differentiable. Thus, if $D_1 \cap D_2 = \emptyset$, then for any initial condition $(z_{1,0}, z_{2,0}) \in \mathbb{R}^n \times \mathbb{R}^n$, (13) has a unique solution. \clubsuit*

Proof. Let $g = \gamma_1 \hat{f}_1 + h\gamma_2 \rho \hat{f}_2 \in \Gamma_0(\mathbb{R}^n)$, it follows from (13c) and (13d) that

$$z_{1,k} \in (\text{Id} + h\partial g)(\tilde{z}_{1,k+1}). \quad (14)$$

Hence, in view of (4), $\tilde{z}_{1,k+1}$ is given as

$$\tilde{z}_{1,k+1} = \text{Prox}_{hg}(z_{1,k}). \quad (15)$$

Since Prox_{hg} is a single-valued Lipschitz continuous function, then $\tilde{z}_{1,k+1}$ is uniquely determined by the function g , the time step h , and the value of the state $z_{1,k}$, all of them available at time $t_k = kh$. Now, the substitution of (15) back into (13c) yields

$$\gamma_1 \hat{m}_{1,k} + h\gamma_2 \rho \hat{m}_{2,k} = \frac{1}{h} (\text{Id} - \text{Prox}_{hg})(z_{1,k}) \in \partial g(\tilde{z}_{1,k+1}). \quad (16)$$

Hence, the sum $\gamma_1 \hat{m}_{1,k} + h\gamma_2 \rho \hat{m}_{2,k}$ is also uniquely determined by the function g and the state $z_{1,k}$. However, note that (16) alone does not guarantee the uniqueness of each selection $\hat{m}_{i,k}$, $i \in \{1, 2\}$ in (13d). Now, if $D_1 \cap D_2 = \emptyset$, then for any $\tilde{z}_{1,k+1}$ either: $\partial \hat{f}_1(\tilde{z}_{1,k+1}) = \{\nabla \hat{f}_1(\tilde{z}_{1,k+1})\}$ or $\partial \hat{f}_2(\tilde{z}_{1,k+1}) = \{\nabla \hat{f}_2(\tilde{z}_{1,k+1})\}$. In the former case $\hat{m}_{1,k}$ is uniquely determined by (13d) and $\hat{m}_{2,k}$ is the unique point in $\partial \hat{f}_2(\tilde{z}_{1,k+1})$ satisfying (16), whereas in the latter case, $\hat{m}_{2,k}$ is uniquely determined by (13d) and $\hat{m}_{1,k}$ is the unique point in $\partial \hat{f}_1(\tilde{z}_{1,k+1})$ satisfying (16). Thus, in any case the values of $\hat{m}_{1,k}$ and $\hat{m}_{2,k}$ are uniquely determined by the state $z_{1,k}$ and known parameters. Finally, it is clear from (13b) that $z_{2,k+1}$ depends only of terms available at time t_k and the external disturbance $\hat{\Delta}_{k-1}$ so that the closed-loop (13) admits a unique solution. \square

The well-posedness of the original system (12) follows directly from Theorem 3.1. Notice that in (13) the state $z_{2,k}$ is unavailable for feedback, as it depends on the disturbance \bar{d}_k . Nevertheless, the control strategy u_k in (10)-(11) is implementable, since it depends on $z_{1,k} = K^{-\frac{1}{2}}x_{1,k}$, ν_k and known parameters. For conventional multivariable super-twisting algorithms, such as those proposed in [26, 30], the maps $f_1(\cdot) = \frac{2}{3}\|\cdot\|_2^{\frac{3}{2}}$ and $f_2(\cdot) = \|\cdot\|_2$ with $K = I_m$ are used, such that $\hat{f}_i = f_i$ for $i \in \{1, 2\}$. With such choice, Theorem 3.1 guarantees the well-posedness of the closed-loop as f_1 is continuously differentiable, so that $D_1 = \emptyset$. A less conventional choice consists in taking $f_1(\cdot) = \frac{2}{3}\|\cdot\|_2^{\frac{3}{2}} + \Psi_S(\cdot)$ where $S \subset \mathbb{R}^m$ is a closed, convex set, such that $0 \in \text{int} S$, and Ψ_S is the conventional indicator function of the set S mapping $x \mapsto 0$ if $x \in S$ and $x \mapsto +\infty$ otherwise. It is clear that with such f_1 we have $D_1 = \text{bdr} S$, whereas $D_2 = 0$, so that $D_1 \cap D_2 = \emptyset$ and the conclusion on Theorem 3.1 remains valid. The following example illustrates the control strategy.

Example 3.2 (Unitary super-twisting algorithm). *Consider the discrete-time controller (9) with $K = \kappa I_n \succ 0$, $f_1(\cdot) = \phi_1(\cdot) + \frac{\alpha_1}{2}\|\cdot\|_2^2$, $f_2(\cdot) = \phi_2(\cdot) + \frac{\alpha_2}{2}\|\cdot\|_2^2$, where $\phi_1(\cdot) = \frac{2}{3}\|\cdot\|_2^{\frac{3}{2}}$, $\phi_2(\cdot) = \|\cdot\|_2$, and $\alpha_i \geq 0$, $i \in \{1, 2\}$. Notice that, for the case when $\rho = 0$, we retrieve a discrete-time version of the multivariable unitary STA proposed in [30]. It is easy to verify that the associated functions $\hat{f}_i = f_i \circ K^{\frac{1}{2}}$, $i \in \{0, 1, 2\}$, satisfy all the hypothesis stated in Theorem 3.1. Thus, in order to compute the values for $\hat{m}_{1,k}$ and $\hat{m}_{2,k}$, the closed-loop (13) is rewritten equivalently as*

$$z_{1,k+1} = \tilde{z}_{1,k+1} + h z_{2,k+1} \quad (17a)$$

$$z_{2,k+1} = z_{2,k} - h\gamma_2\alpha_2\tilde{z}_{1,k+1} - h\gamma_2\hat{\mu}_{2,k} + h\hat{\Delta}_{k-1} \quad (17b)$$

$$\tilde{z}_{1,k+1} = \frac{1}{\beta}z_{1,k} - \frac{h}{\beta}(\gamma_1\hat{\mu}_{1,k} + h\gamma_2\rho\hat{\mu}_{2,k}) \quad (17c)$$

$$\hat{\mu}_{i,k} \in \partial\hat{\phi}_i(\tilde{z}_{1,k+1}), \quad i \in \{1, 2\} \quad (17d)$$

where $\beta = 1 + h\kappa(\alpha_1\gamma_1 + h\alpha_2\gamma_2\rho)$ and $\hat{\phi}_j = \phi_j \circ K^{\frac{1}{2}}$, $j \in \{1, 2\}$, so that $\hat{m}_{i,k} = \hat{\mu}_{i,k} + \alpha_i\kappa\tilde{z}_{1,k+1}$. It is noteworthy that, with the new structure (17), the computations of both, $\tilde{z}_{1,k+1}$ and the control values $\hat{m}_{i,k}$, $i \in \{1, 2\}$, is simplified. Concretely, setting $\Phi := \gamma_1\hat{\phi}_1 + h\gamma_2\rho\hat{\phi}_2$ it follows from (17c)-(17d) (and the definition of proximal map), that

$$\tilde{z}_{1,k+1} = \text{Prox}_{\frac{h}{\beta}\Phi} \left(\frac{1}{\beta}z_{1,k} \right), \quad (18)$$

$$\gamma_1\hat{\mu}_{1,k} + h\gamma_2\rho\hat{\mu}_{2,k} = \frac{\beta}{h} \left(\text{Id} - \text{Prox}_{\frac{h}{\beta}\Phi} \right) \left(\frac{1}{\beta}z_{1,k} \right), \quad (19)$$

Since $\hat{\phi}_1$ is continuously differentiable, then $\partial\hat{\phi}_1$ is single-valued and

$$\hat{\mu}_{1,k} = \nabla\hat{\phi}_1 \left(\text{Prox}_{\frac{h}{\beta}\Phi} \left(\frac{1}{\beta}z_{1,k} \right) \right) \quad (20a)$$

$$h\gamma_2\rho\hat{\mu}_{2,k} = \frac{\beta}{h} \left(\text{Id} - \text{Prox}_{\frac{h}{\beta}\Phi} \right) \left(\frac{1}{\beta}z_{1,k} \right) - \gamma_1\nabla\hat{\phi}_1 \left(\text{Prox}_{\frac{h}{\beta}\Phi} \left(\frac{1}{\beta}z_{1,k} \right) \right)$$

In terms of the original control variables, $\hat{m}_{i,k}$, it follows that,

$$\hat{m}_{i,k} = \hat{\mu}_{i,k} + \alpha_i\kappa\tilde{z}_{1,k+1} \in \partial\hat{f}_i(\tilde{z}_{1,k+1}), \quad i \in \{1, 2\}. \quad (21)$$

Thus, in order to be able to implement the controller (9), it remains to obtain an explicit expression for the proximal map $\text{Prox}_{\frac{h}{\beta}\Phi} = (\text{Id} + \frac{h}{\beta}\partial\Phi)^{-1}$ at the point $\frac{1}{\beta}z_{1,k}$. To that end we proceed as follows. First notice that $\tilde{z}_{1,k+1} = \text{Prox}_{\frac{h}{\beta}\Phi}(\frac{1}{\beta}z_{1,k})$ if, and only if, $\tilde{z}_{1,k+1}$ is the solution to the following generalized equation

$$\frac{1}{\beta}z_{1,k} \in \tilde{z}_{1,k+1} + \frac{h}{\beta} \left(\gamma_1\kappa^{\frac{3}{4}}\|\tilde{z}_{1,k+1}\|_2^{\frac{1}{2}} + h\gamma_2\rho\kappa^{\frac{1}{2}} \right) \partial\|\tilde{z}_{1,k+1}\|_2. \quad (22)$$

where,

$$\partial\|y\|_2 = \begin{cases} \frac{1}{\|y\|_2}y, & \text{if } y \neq 0 \\ \mathcal{B}, & \text{otherwise} \end{cases}. \quad (23)$$

From (22) it is clear that $\tilde{z}_{1,k+1} = 0$, if and only if, $z_{1,k} \in h^2\gamma_2\rho\kappa^{\frac{1}{2}}\mathcal{B}$. Indeed, if $\tilde{z}_{1,k+1} = 0$, then clearly (22)-(23) imply that $z_{1,k} \in h^2\gamma_2\rho\kappa^{\frac{1}{2}}\mathcal{B}$. On the contrary, if $\tilde{z}_{1,k+1} \neq 0$, then $z_{1,k}$ and $\tilde{z}_{1,k+1}$ are parallel,

since the substitution of (23) into (22) yields

$$\frac{z_{1,k}}{\beta} = \left(1 + \frac{h(\gamma_1 \kappa^{\frac{3}{4}} \|\tilde{z}_{1,k+1}\|_2^{\frac{1}{2}} + h\gamma_2 \rho \kappa^{\frac{1}{2}})}{\beta \|\tilde{z}_{1,k+1}\|_2} \right) \tilde{z}_{1,k+1}, \quad (24)$$

so that, the 2-norm of $z_{1,k}$ satisfies

$$\|z_{1,k}\|_2 = \beta \|\tilde{z}_{1,k+1}\|_2 + h\gamma_1 \kappa^{\frac{3}{4}} \|\tilde{z}_{1,k+1}\|_2^{\frac{1}{2}} + h^2 \gamma_2 \rho \kappa^{\frac{1}{2}}, \quad (25)$$

so that $z_{1,k} \notin h^2 \gamma_2 \rho \kappa^{\frac{1}{2}} \mathcal{B}$ and the conclusion follows. Moreover, solving (25) with respect to $\|\tilde{z}_{1,k+1}\|_2^{\frac{1}{2}}$, leads us to the expression

$$0 < \|\tilde{z}_{1,k+1}\|_2^{\frac{1}{2}} = \frac{-h\gamma_1 \kappa^{\frac{3}{4}}}{2\beta} + \frac{1}{2\beta} \sqrt{h^2 \gamma_1^2 \kappa^{\frac{3}{2}} + 4\beta \left(\|z_{1,k}\|_2 - h^2 \gamma_2 \rho \kappa^{\frac{1}{2}} \right)} =: q(\|z_{1,k}\|_2). \quad (26)$$

Hence, it follows from (24)-(26) that for $z_{1,k} \notin h^2 \gamma_2 \rho \kappa^{\frac{1}{2}} \mathcal{B}$,

$$\begin{aligned} \tilde{z}_{1,k+1} &= \left(1 + \frac{h \left(\gamma_1 \kappa^{\frac{3}{4}} q(z_{1,k}) + h\gamma_2 \rho \kappa^{\frac{1}{2}} \right)}{\beta q(z_{1,k})^2} \right)^{-1} \frac{z_{1,k}}{\beta} \\ &= \left(\frac{q(z_{1,k})^2}{\beta q(z_{1,k})^2 + h\gamma_1 \kappa^{\frac{3}{4}} q(z_{1,k}) + h^2 \gamma_2 \rho \kappa^{\frac{1}{2}}} \right) z_{1,k} \\ &= \frac{q(z_{1,k})^2}{\|z_{1,k}\|_2} z_{1,k}. \end{aligned} \quad (27)$$

Putting all together, the explicit expression for $\tilde{z}_{1,k+1}$ is obtained as

$$\begin{aligned} \tilde{z}_{1,k+1} &= \text{Prox}_{\frac{h}{\beta} \Phi} \left(\frac{z_{1,k}}{\beta} \right) \\ &= \begin{cases} 0, & \text{if } \|z_{1,k}\|_2 \leq h^2 \gamma_2 \rho \kappa^{\frac{1}{2}}; \\ \frac{q(z_{1,k})^2}{\|z_{1,k}\|_2} z_{1,k}, & \text{otherwise.} \end{cases} \end{aligned} \quad (28)$$

Therefore, from (20), $\hat{\mu}_{1,k}$ and $\hat{\mu}_{2,k}$ are given as

$$\begin{aligned} \hat{\mu}_{1,k} &= \nabla \hat{\phi}_1(\tilde{z}_{1,k+1}) \\ &= \kappa^{\frac{3}{4}} \|\tilde{z}_{1,k+1}\|_2^{\frac{1}{2}} \partial \|\tilde{z}_{1,k+1}\|_2 \\ &= \begin{cases} 0, & \text{if } \|z_{1,k}\|_2 \leq h^2 \gamma_2 \rho \kappa^{\frac{1}{2}}; \\ \kappa^{\frac{3}{4}} \frac{q(z_{1,k})}{\|z_{1,k}\|_2} z_{1,k}, & \text{otherwise.} \end{cases} \end{aligned} \quad (29a)$$

$$\begin{aligned} \hat{\mu}_{2,k} &= \frac{\beta}{h^2 \gamma_2 \rho} (\text{Id} - \text{Prox}_{\frac{h}{\beta} \Phi}) \left(\frac{z_{1,k}}{\beta} \right) - \frac{\gamma_1}{h\gamma_2} \hat{\mu}_{1,k} \\ &= \begin{cases} \frac{1}{h^2 \gamma_2 \rho} z_{1,k}, & \text{if } \|z_{1,k}\|_2 \leq h^2 \gamma_2 \rho \kappa^{\frac{1}{2}}; \\ \frac{\kappa^{\frac{1}{2}}}{\|z_{1,k}\|_2} z_{1,k}, & \text{otherwise.} \end{cases} \end{aligned} \quad (29b)$$

In terms of the control variables, $\hat{m}_{i,k}$, $i \in \{1, 2\}$ we have that,

$$\hat{m}_{1,k} = \begin{cases} 0, & \text{if } \|z_{1,k}\|_2 \leq h^2 \gamma_2 \rho \kappa^{\frac{1}{2}}; \\ \frac{(\kappa^{\frac{3}{4}} + \alpha_1 \kappa q(z_{1,k})) q(z_{1,k})}{\|z_{1,k}\|_2} z_{1,k}, & \text{otherwise.} \end{cases} \quad (30a)$$

$$\hat{m}_{2,k} = \begin{cases} \frac{1}{h^2 \gamma_2 \rho} z_{1,k}, & \text{if } \|z_{1,k}\|_2 \leq h^2 \gamma_2 \rho \kappa^{\frac{1}{2}}; \\ \frac{\kappa^{\frac{1}{2}} + \alpha_2 \kappa q(z_{1,k})^2}{\|z_{1,k}\|_2} z_{1,k}, & \text{otherwise.} \end{cases} \quad (30b)$$

Finally, by setting $m_i = \kappa^{-\frac{1}{2}} \hat{m}_{i,k}$ and $z_{1,k} = \kappa^{\frac{1}{2}} z_{1,k}$ we retrieve the control law in terms of the original variables as

$$u_k = \nu_{k+1} - \kappa^{\frac{1}{2}} (\gamma_1 \hat{m}_{1,k} + h\gamma_2 \rho \hat{m}_{2,k}) \quad (31a)$$

$$\nu_{k+1} = \nu_k - h\kappa^{\frac{1}{2}} \gamma_2 \hat{m}_{1,k} \quad (31b)$$

It is noteworthy that, for the particular case when $n = 1$, $\rho = 1$, $\kappa = 1$, and $\alpha_1 = \alpha_2 = 0$, then the control (31) reduces to the one proposed in [2, Section III]. \clubsuit

4 Stability properties of the set-valued super-twisting-like controller

4.1 Nominal case ($\hat{\Delta}_k \equiv 0$)

Before studying the general case, we present the stability properties of the closed-loop (13) for the cases when $\Delta_k \equiv 0$.

Theorem 4.1. *Let all the assumptions of Theorem 3.1 hold. In addition, assume that \hat{f}_1 is continuously differentiable around the origin such that $\{0\} = \partial \hat{f}_1(0)$ and that there exists a class- \mathcal{K} function α such that $\langle \hat{m}_{1,k}, \hat{m}_{2,k} \rangle \geq \alpha(\tilde{z}_{1,k+1})$ for all $k \in \mathbb{N}$. Hence, if $\gamma_1 > 0$, $\gamma_2 > 0$, $\rho \geq 1$, and $\hat{\Delta}_k \equiv 0$ for all $k \in \mathbb{N}$, then the origin is globally asymptotically stable. \clubsuit*

Proof. It follows from (13a) and (13c) that

$$\tilde{z}_{1,k+1} = \tilde{z}_{1,k} + h z_{2,k} - h(\gamma_1 \hat{m}_{1,k} + h \gamma_2 \rho \hat{m}_{2,k}). \quad (32)$$

Let us consider the following Lyapunov candidate function

$$V_k = \gamma_2 \hat{f}_2(\tilde{z}_{1,k}) + \frac{1}{2} \|z_{2,k}\|_2^2.$$

Thus, from the definition of subdifferential and (13d) it follows that

$$\begin{aligned} V_{k+1} - V_k &\leq \gamma_2 \langle \hat{m}_{2,k}, \tilde{z}_{1,k+1} - \tilde{z}_{1,k} \rangle + \langle z_{2,k+1}, z_{2,k} - h \gamma_2 \hat{m}_{2,k} \rangle - \frac{1}{2} \|z_{2,k+1}\|_2^2 - \frac{1}{2} \|z_{2,k}\|_2^2 \\ &= \langle h \gamma_2 \hat{m}_{2,k}, z_{2,k+1} - \gamma_1 \hat{m}_{1,k} - h \gamma_2 (\rho - 1) \hat{m}_{2,k} \rangle - h \gamma_2 \langle \hat{m}_{2,k}, z_{2,k+1} \rangle - \frac{1}{2} \|z_{2,k+1} - z_{2,k}\|_2^2 \\ &= -h \gamma_1 \gamma_2 \langle \hat{m}_{1,k}, \hat{m}_{2,k} \rangle - h^2 \gamma_2^2 (\rho - 1) \|\hat{m}_{2,k}\|_2^2 - \frac{1}{2} \|z_{2,k+1} - z_{2,k}\|_2^2 \end{aligned} \quad (33)$$

Let $S_k = h \gamma_1 \gamma_2 \langle \hat{m}_{1,k}, \hat{m}_{2,k} \rangle + h^2 \gamma_2^2 (\rho - 1) \|\hat{m}_{2,k}\|_2^2 + \frac{1}{2} \|z_{2,k+1} - z_{2,k}\|_2^2$. Hence, $S_k \geq 0$ for all $k \in \mathbb{N}$ and $V_{k+1} - V_k \leq -S_k$. Moreover, $S_k \rightarrow 0$ as $k \rightarrow \infty$. Indeed, it follows from (33) that

$$\sum_{i=0}^k V_{i+1} - V_i = V_{k+1} - V_0 \leq - \sum_{i=0}^k S_i,$$

so that

$$\sum_{i=0}^k S_i \leq V_0 - V_{k+1} \leq V_0. \quad (34)$$

By taking the limit as $k \rightarrow \infty$ on both sides of (34), the convergence of the sequence $\{S_k\}_{k \in \mathbb{N}}$ towards zero follows. Now, it is clear that $S_k \rightarrow 0$, if and only if, *i*) $\tilde{z}_{1,k+1} \rightarrow 0$, *ii*) $\hat{m}_{2,k} \rightarrow 0$ and *iii*) $z_{2,k+1} - z_{2,k} \rightarrow 0$. Notice that, by assumption, $\tilde{z}_{1,k+1} \rightarrow 0$ implies $\hat{m}_{1,k} \rightarrow 0$. Thus, under these constraints, it follows from (13a) and (13c) that both $z_{1,k} \rightarrow 0$ and $z_{2,k} \rightarrow 0$. and the global asymptotic stability of the origin is concluded. The proof is complete. \square

Theorem 4.1 guarantees asymptotic convergence towards the origin. However, it is well-known that the classical super-twisting algorithm converges in finite time. Thus, to the end of ensuring finite-time convergence, let us introduce the following assumption

Assumption 4.2. $\hat{f}_2 \in \Gamma_0(\mathbb{R}^n)$ is such that $\text{int } \partial \hat{f}_2(0) \neq \emptyset$.

Assumption 4.2 excludes maps \hat{f}_2 that are continuously differentiable around zero, as it guarantees that $\partial \hat{f}_2(0)$ is not a singleton.

Proposition 4.3. *Let Assumption 4.2 and all the assumptions of Theorem 3.1 hold. In addition, let $\gamma_2 > 0$ be such that $\bigcup_{k \in \mathbb{N}} \{\hat{\Delta}_k\} \subset \gamma_2 \partial \hat{f}_2(0)$. If for some $k^* \in \mathbb{N}$, $z_{1,k^*-1}, z_{1,k^*} \in h \partial \hat{g}(0)$, then $\tilde{z}_{1,k^*+r} = 0$ for all $r \in \mathbb{N}$, whenever $\gamma_1 > 0$ and $\rho \geq 1$. \clubsuit*

Proof. If both $z_{1,k^*-1}, z_{1,k^*} \in h \partial \hat{g}(0)$, then it follows from (14) that $\tilde{z}_{1,k} = \tilde{z}_{1,k+1} = 0$. Hence, (32) implies that

$$z_{2,k^*} = \gamma_1 \hat{m}_{1,k^*} + h \gamma_2 \rho \hat{m}_{2,k^*},$$

and from (13a)-(13b) it follows that

$$\begin{aligned} z_{1,k^*+1} &= h(\gamma_1 \hat{m}_{1,k^*} + h\gamma_2(\rho - 1)\hat{m}_{2,k^*} + h^2 \hat{\Delta}_{k-1}) \\ &\in h\gamma_1 \partial \hat{f}_1(0) + h^2 \gamma_2(\rho - 1) \partial \hat{f}_2(0) + h^2 \gamma_2 \partial \hat{f}_2(0) \\ &\subset h\gamma_1 \partial \hat{f}_1(0) + h^2 \gamma_2 \rho \partial \hat{f}_2(0). \end{aligned}$$

Therefore $\tilde{z}_{1,k^*+2} = 0$, and the conclusion follows by using an induction argument. This concludes the proof. \square

Proposition 4.3 shows that in the cases when $\text{int } \partial \hat{g}(0) \neq \emptyset$, the invariance of $\{z \in \mathbb{R}^{2n} \mid \tilde{z}_{1,k} = 0\}$ is robust against the presence of disturbances $\hat{\Delta}_k$. Following [18] we say that the closed-loop (13) is in *discrete-time sliding-mode* whenever $\tilde{z}_{1,k} = 0$ for all $k \geq k^*$. It is shown in the next subsection that, during such stage, the control term $\hat{m}_{2,k}$ compensates for the disturbance $\hat{\Delta}_k$ with a delay of two time-steps.

Corollary 4.4. *Let Assumption 4.2 holds. Thus, under the assumptions of Theorem 4.1, the origin of (13) is finite-time stable whenever $\rho = 1$.* \clubsuit

Proof. It follows from Theorem 4.1 and Assumption 4.2, that there exists a finite $k^* \in \mathbb{N}$ such that $z_{1,k}$ and $z_{1,k+1} \in h\partial \hat{g}(0)$ for all $k \geq k^*$. Hence, from Proposition 4.3 it follows that for all $k \geq k^*$, $\tilde{z}_{1,k} = 0$ after a finite number of steps and $\tilde{z}_{1,k+r} = 0$ for all $r \in \mathbb{N}$. Thus, from (32), $z_{2,k} = h\gamma_2 \rho \hat{m}_{2,k} = h\gamma_2 \hat{m}_{2,k}$ so that (13b) implies that $z_{2,k+1} = 0$. Finally, it follows from (13a) that $z_{k+1} = h z_{2,k+1}$ and the claimed property follows. This concludes the proof. \square

The inner product condition $\langle \hat{m}_{1,k}, \hat{m}_{2,k} \rangle \geq \alpha(\tilde{z}_{1,k+1})$ is, in general, hard to verify in practice for vectors in \mathbb{R}^n . However, if for instance $\hat{f}_1 = \hat{f}_0 \circ \hat{f}_2$ for some continuously differentiable $\hat{f}_0 \in \Gamma_0(\mathbb{R})$ satisfying $0 = \nabla \hat{f}_0(0)$, then clearly $\partial \hat{f}_1(z) = \nabla \hat{f}_0(z) \partial \hat{f}_2(z)$ satisfies such condition. Notice that in the scalar case, the classical super-twisting algorithm satisfies such condition as $f_1(\cdot) = |f_2(\cdot)|^{\frac{3}{2}}$.

The following Theorem provides an alternative proof for the finite-time stability of the origin of (13), that does not uses the inner product condition. As before, we consider the case when $\hat{\Delta}_k \equiv 0$ for all $k \in \mathbb{N}$.

Theorem 4.5. *Let Assumption 4.2 and all the assumptions of Theorem 3.1 hold. In addition, let \hat{f}_1 be continuously differentiable around zero, so that $\partial \hat{f}_1 = \nabla \hat{f}_1$ is strictly monotone. Thus, if $\gamma_1 > 0$, $\gamma_2 > 0$, $\rho = 1$, and $\hat{\Delta}_k \equiv 0$ for all $k \in \mathbb{N}$, then the origin of (13) is finite-time stable.* \clubsuit

Proof. Let us consider the following Lyapunov candidate function

$$V_k = \gamma_2 \hat{f}_2(\tilde{z}_{1,k}) + \frac{1}{2} \|z_{2,k} - \gamma_1 \hat{m}_{1,k-1}\|_2^2. \quad (35)$$

It follows from the definition of subdifferential that

$$\begin{aligned} V_{k+1} - V_k &\leq \langle \gamma_2 \hat{m}_{2,k}, \tilde{z}_{1,k+1} - \tilde{z}_{1,k} \rangle + \langle z_{2,k+1} - \gamma_1 \hat{m}_{1,k}, z_{2,k} - h\gamma_2 \hat{m}_{2,k} - \gamma_1 \hat{m}_{1,k} \rangle \\ &\quad - \frac{1}{2} \|z_{2,k+1} - \gamma_1 \hat{m}_{1,k}\|_2^2 - \frac{1}{2} \|z_{2,k} - \gamma_1 \hat{m}_{1,k-1}\|_2^2 \\ &= \langle z_{2,k+1} - \gamma_1 \hat{m}_{1,k}, z_{2,k} - \gamma_1 \hat{m}_{1,k-1} \rangle - \gamma_1 \langle \hat{m}_{1,k} - \hat{m}_{1,k-1}, z_{2,k+1} - \gamma_1 \hat{m}_{1,k} \rangle \\ &\quad - \frac{1}{2} \|z_{2,k+1} - \gamma_1 \hat{m}_{1,k}\|_2^2 - \frac{1}{2} \|z_{2,k} - \gamma_1 \hat{m}_{1,k-1}\|_2^2 \\ &= -\frac{1}{2} \|z_{2,k+1} - \gamma_1 \hat{m}_{1,k} - (z_{2,k} - \gamma_1 \hat{m}_{1,k-1})\|_2^2 - \frac{\gamma_1}{h} \langle \hat{m}_{1,k} - \hat{m}_{1,k-1}, \tilde{z}_{1,k+1} - \tilde{z}_{1,k} \rangle \end{aligned} \quad (36)$$

Thus, by setting

$$\tilde{S}_k = \frac{1}{2} \|z_{2,k+1} - \gamma_1 \hat{m}_{1,k} - (z_{2,k} - \gamma_1 \hat{m}_{1,k-1})\|_2^2 + \frac{\gamma_1}{h} \langle \hat{m}_{1,k} - \hat{m}_{1,k-1}, \tilde{z}_{1,k+1} - \tilde{z}_{1,k} \rangle, \quad (37)$$

it follows that $V_{k+1} - V_k \leq \tilde{S}_k$. Using the same arguments as in the proof of Theorem 4.1, it follows that $\tilde{S}_k \rightarrow 0$ as $k \rightarrow \infty$. Now, using the strict monotonicity and continuity of $\nabla \hat{f}_1$, it follows that $\tilde{S}_k \rightarrow 0$, if and only if, *i*) $\tilde{z}_{k+1} - \tilde{z}_{1,k} \rightarrow 0$ and *ii*) $z_{2,k+1} - \gamma_1 \hat{m}_{1,k} - z_{2,k} + \gamma_1 \hat{m}_{1,k-1} \rightarrow 0$. Note that, as $\partial \hat{f}_1$ is single-valued and continuous, it follows from item *i*) that $\hat{m}_{1,k} - \hat{m}_{1,k-1} \rightarrow 0$, so that item *ii*) becomes *ii*)' $z_{2,k+1} - z_{2,k} \rightarrow 0$. Further, *ii*)' and (13b) imply that $\hat{m}_{2,k} \rightarrow 0$, and, from Lemma 1.1, it follows that $\tilde{z}_{1,k+1} \rightarrow 0$, and hence $\hat{m}_{1,k} \rightarrow 0$. Once more, from (13a) and (13c), it follows that $z_{1,k+1} \rightarrow 0$ and $z_{2,k+1} \rightarrow 0$. Finally, following the same arguments as in the proof of Corollary 4.4, we conclude that the origin is finite-time stable. The proof is complete. \square

4.2 Perturbed case

In what follows we present the stability properties of the origin for the perturbed case when $\hat{\Delta}_k$ is uniformly bounded. To that end, we introduce the virtual variable $\tilde{z}_{2,k+1} = z_{2,k} - h\gamma_2\hat{m}_{2,k}$ and set $\rho = 1$ in (13), so that the closed-loop (13) is rewritten as:

$$z_{1,k+1} = \tilde{z}_{1,k+1} + h\tilde{z}_{2,k+1} + h^2\hat{\Delta}_{k-1} \quad (38a)$$

$$z_{2,k+1} = \tilde{z}_{2,k+1} + h\hat{\Delta}_{k-1} \quad (38b)$$

$$\tilde{z}_{1,k+1} = z_{1,k} - h\gamma_1\hat{m}_{1,k} - h^2\gamma_2\hat{m}_{2,k} \quad (38c)$$

$$\tilde{z}_{2,k+1} = z_{2,k} - h\gamma_2\hat{m}_{2,k} \quad (38d)$$

$$\hat{m}_{i,k} \in \partial\hat{f}_i(\tilde{z}_{1,k+1}) \quad (38e)$$

Theorem 4.6. *Consider the closed-loop (38). Let Assumption 4.2 and all the assumptions of Theorem 3.1 hold. Let \hat{f}_1 be continuous differentiable so that $0 = \nabla\hat{f}_1(0)$ and $\nabla\hat{f}_1$ is θ -strongly monotone with $\theta(v_1, v_2) \geq 0$ for all $v_1, v_2 \in \mathbb{R}^n$. Hence, if $\bigcup_{k \in \mathbb{N}}\{\hat{\Delta}_k\} \subset \eta\mathcal{B}$ for some $0 < \eta$, then there exist sufficiently large gains $\gamma_1 > 0$, $\gamma_2 > \eta$, and a finite $k^* \in \mathbb{N}$, such that $\tilde{z}_{1,k+1} = 0$ and $\tilde{z}_{2,k+1} = 0$ for all $k \geq k^*$. In particular, $z_{1,k+1}$ and $z_{2,k+1}$ are ultimately bounded and satisfy $z_{1,k+1} \in h^2\eta\mathcal{B}$ and $z_{2,k+1} \in h\eta\mathcal{B}$ for all $k \geq k^*$. \clubsuit*

Proof. From (38) it is clear that

$$\tilde{z}_{1,k+1} = \tilde{z}_{1,k} + h\tilde{z}_{2,k+1} - h\gamma_1\hat{m}_{1,k}, \quad (39a)$$

$$\tilde{z}_{2,k+1} = \tilde{z}_{2,k} - h\gamma_2\hat{m}_{2,k} + h\hat{\Delta}_{k-2}. \quad (39b)$$

Let us consider the following positive definite function

$$V_k = \gamma_2\hat{f}_{2,k}(\tilde{z}_{1,k}) + \frac{1}{2}\|\tilde{z}_{2,k} - \gamma_1\hat{m}_{1,k-1}\|_2^2. \quad (40)$$

It follows from the definition of subdifferential in (3) and (39) that the mismatch $V_{k+1} - V_k$ satisfies

$$\begin{aligned} V_{k+1} - V_k &\leq -\frac{1}{2}\|\tilde{z}_{2,k+1} - \gamma_1\hat{m}_{1,k} - (\tilde{z}_{2,k} - \gamma_1\hat{m}_{1,k-1})\|_2^2 - \frac{\gamma_1}{h}\langle\hat{m}_{1,k} - \hat{m}_{1,k-1}, \tilde{z}_{1,k+1} - \tilde{z}_{1,k}\rangle \\ &\quad + h\langle\hat{\Delta}_{k-2}, \tilde{z}_{2,k+1} - \gamma_1\hat{m}_{1,k}\rangle. \end{aligned} \quad (41)$$

Recalling that by assumption $\partial\hat{f}_1$ is θ -strongly monotone, it follows that

$$\begin{aligned} V_{k+1} - V_k &\leq -\frac{1}{2}\|\tilde{z}_{2,k+1} - \gamma_1\hat{m}_{1,k} - (\tilde{z}_{2,k} - \gamma_1\hat{m}_{1,k-1})\|_2^2 - \frac{\gamma_1}{h}\theta(\tilde{z}_{1,k+1}, \tilde{z}_{1,k})\|\tilde{z}_{1,k+1} - \tilde{z}_{1,k}\| \\ &\quad + h\left\|\hat{\Delta}_{k-2}\right\|_2\|\tilde{z}_{2,k+1} - \gamma_1\hat{m}_{1,k}\|_2 \\ &= -\frac{1}{2}\|\tilde{z}_{2,k+1} - \gamma_1\hat{m}_{1,k} - (\tilde{z}_{2,k} - \gamma_1\hat{m}_{1,k-1})\|_2^2 - (\gamma_1\theta(\tilde{z}_{1,k+1}, \tilde{z}_{1,k}) - h\eta)\|\tilde{z}_{2,k+1} - \gamma_1\hat{m}_{1,k}\|. \end{aligned} \quad (42)$$

Thus, $V_{k+1} - V_k \leq 0$ whenever $\theta(\tilde{z}_{1,k+1}, \tilde{z}_{1,k}) \geq \frac{h}{\gamma_1}\eta$. Consequently, there exists $k^* \in \mathbb{N}$ and $\varepsilon_1(\gamma_1) > 0$ for which

$$\|\tilde{z}_{2,k+1} - \gamma_1\hat{m}_{1,k}\|_2 \leq h\varepsilon_1(\gamma_1) \text{ for all } k \geq k^*. \quad (43)$$

Hence, from (39) $\|\tilde{z}_{1,k+1} - \tilde{z}_{1,k}\|_2 \leq h^2\varepsilon_1(\gamma_1)$ and as $\partial\hat{f}_1 = \nabla\hat{f}_1$ is continuously differentiable, it follows that there exists $\delta_2(\gamma_1)$ such that

$$\|\hat{m}_{1,k} - \hat{m}_{1,k-1}\| \leq h\delta_2(\gamma_1) \text{ for all } k \geq k^*. \quad (44)$$

Note that, as γ_1 gets larger, both, $\varepsilon_1(\gamma_1)$ and $\delta_2(\gamma_1)$, can be chosen smaller. Now, by combining (43)-(44) it is inferred that

$$\|\tilde{z}_{2,k+1} - \tilde{z}_{2,k}\|_2 \leq h(2\varepsilon_1(\gamma_1) + \delta_2(\gamma_1)) \text{ for all } k \geq k^* + 1. \quad (45)$$

Therefore, from (39) we obtain the following upper bound for the term $\hat{m}_{2,k}$

$$\|\hat{m}_{2,k}\|_2 \leq \frac{2\varepsilon_1(\gamma_1) + \delta_2(\gamma_1) + \eta}{\gamma_2}. \quad (46)$$

Note that, for γ_1 and γ_2 sufficiently large, $\|\hat{m}_{2,k}\|_2 \leq r$ for all $k \geq k^* + 1$ and some $r > 0$ such that $r\mathcal{B} \subset \partial \hat{f}_2(0)$. Since $\hat{m}_{2,k} \in \partial \hat{f}_2(\tilde{z}_{1,k+1})$, it follows from Lemma 1.1 that $\tilde{z}_{1,k+1} = 0$ for all $k \geq k^* + 1$. Moreover, since $\nabla \hat{f}_1(0) = 0$, then $\hat{m}_{1,k} = 0$. Hence, (39) implies that $\tilde{z}_{2,k+1} = 0$ for all $k \geq k^* + 1$. Finally, the bounds on $z_{1,k+1}$ and $z_{2,k+1}$ follow directly from (38). This concludes the proof. \square

Notice that the ultimate bounds on $z_{1,k+1}$ and $z_{2,k+1}$ are independent of the gains γ_1 and γ_2 and depend only on the sampling time h and the bound on the disturbance term $\hat{\Delta}_k$. The following corollary brings some light into such independence property.

Corollary 4.7. *Let all assumptions of Theorem 4.6 hold. After some finite-time $\tilde{z}_{1,k+1} = \tilde{z}_{2,k+1} = 0$ and the control law (9) is insensitive against increments on the values of the gains γ_1 and γ_2 .* \clubsuit

Proof. The proof follows by noticing that $\tilde{z}_{1,k+1} = \tilde{z}_{2,k+1} = 0$ for all $k > k^*$, implies that *i)* $\hat{m}_{1,k} = 0$, and *ii)* $z_{2,k} = h\gamma_2 \hat{m}_{2,k}$ for all $k > k^*$. Thereby, as $z_{2,k} = K^{-\frac{1}{2}}(\nu_{k+1} + \bar{d}_{k-1})$ and recalling that we have set $\rho = 1$, it is inferred that in such situation $u_k = -K(\gamma_1 m_{1,k} + h\gamma_2 \rho m_{2,k}) + \nu_{k+1} = \nu_{k+1} - h\gamma_2 K m_{2,k} = -\bar{d}_{k-1}$ for all $k > k^*$. Therefore, the controller is independent of the explicit value of the gains. \square

Corollary 4.7 is akin to first-order set-valued sliding-mode controllers [18], [27], where the explicit values of the gain only affects the reaching phase. During sliding phase, the precise value of the gain is irrelevant, just a minimum bound has to be respected to maintain the sliding motion. Thus, in that case, any increment in the values of the gains will have no affection on the closed-loop, as the controller is *automatically* compensating for the disturbance with one-step delay. In other words, over-estimation of the control gains has no effect on the closed-loop once the sliding surface has been reached. To the best of author's knowledge, this property of gain invariance is unique to selection-based controllers and it represents an important advantage against implementations based on arbitrary graph approximations and *ad-hoc* regularizations of the non-smooth maps involved.

Regarding the θ -strong monotonicity property, it is not difficult to show that functions of the sort $\|\cdot\|^\alpha$ are indeed θ monotone, as the following example shows.

Proposition 4.8. *Let $p \in \{1, 2\}$ and let $\varphi_p \in \Gamma_0(\mathbb{R}^n)$ mapping $v \mapsto \|v\|_p^{1+\alpha}$. If $\alpha > \frac{1}{2}$, then the maps $\partial \varphi_p$ are θ_p -strongly monotone, with*

$$\theta_1(v_a, v_b) = \frac{(\|v_a\|_1 - \|v_b\|_1)^2}{(\|v_a\|_1^{1-\alpha} + \|v_b\|_1^{1-\alpha}) \|v_a - v_b\|_2}, \quad (47a)$$

$$\theta_2(v_a, v_b) = \frac{\|v_a - v_b\|_2}{\|v_a\|_2^{1-\alpha} + \|v_b\|_2^{1-\alpha}}. \quad (47b)$$

\clubsuit

Proof. Let $(v_i, w_i) \in \text{Gph } \partial f_p$ for $i \in \{a, b\}$, then $w_i = \|v_i\|_p^\alpha w'_i$, where $w'_i \in \partial \|v_i\|_p$, and simple computations show that

$$\begin{aligned} \langle v_a - v_b, w_a - w_b \rangle &= \frac{\langle v_a - v_b, \|v_a\|_p w'_a - \|v_b\|_p w'_b \rangle}{\|v_a\|_p^{1-\alpha} + \|v_b\|_p^{1-\alpha}} \\ &\quad + \frac{\|v_a\|_p^{1-\alpha} \|v_b\|_p^{1-\alpha}}{\|v_a\|_p^{1-\alpha} + \|v_b\|_p^{1-\alpha}} \langle v_a - v_b, \|v_a\|_p^{2\alpha-1} w'_a - \|v_b\|_p^{2\alpha-1} w'_b \rangle \end{aligned} \quad (48)$$

Noting that $\|v_i\|_p^{2\alpha-1} w'_i \in \partial \|v_i\|_p^{2\alpha}$ for $i \in \{a, b\}$, and as $\alpha \geq \frac{1}{2}$, then $\partial \|\cdot\|_p^{2\alpha}$ is maximal monotone. Thus, (48) satisfies

$$\langle v_a - v_b, w_a - w_b \rangle \geq \frac{\langle v_a - v_b, \|v_a\|_p w'_a - \|v_b\|_p w'_b \rangle}{\|v_a\|_p^{1-\alpha} + \|v_b\|_p^{1-\alpha}} \quad (49)$$

Now, let $p = 2$. In this case, it is clear from (23) that, for any $w'_i \in \partial \|v_i\|_2$, $\|v_i\|_2 w'_i = v_i$, for $i \in \{a, b\}$. Thus, (49) becomes

$$\langle v_a - v_b, w_a - w_b \rangle \geq \frac{\|v_a - v_b\|_2}{\|v_a\|_2^{1-\alpha} + \|v_b\|_2^{1-\alpha}} \|v_a - v_b\|_2. \quad (50)$$

and θ -strong monotonicity of $\partial\varphi_2$ follows. On the other hand, if $p = 1$, then

$$\begin{aligned} \langle v_a - v_b, \|v_a\|_1 w'_a - \|v_b\|_1 w'_b \rangle &= \sum_{i=1}^n (\|v_a\|_1 \operatorname{sgn}([v_a]_i) - \|v_b\|_1 \operatorname{sgn}([v_b]_i)) ([v_a]_i - [v_b]_i) \\ &\geq \|v_a\|_1^2 + \|v_b\|_1^2 - \sum_{i=1}^n (\|v_a\|_1 |[v_b]_i| + \|v_b\|_1 |[v_a]_i|) \\ &= (\|v_a\|_1 - \|v_b\|_1)^2. \end{aligned} \quad (51)$$

Hence, (49) becomes

$$\langle v_a - v_b, w_a - w_b \rangle \geq \frac{(\|v_a\|_1 - \|v_b\|_1)^2}{(\|v_a\|_1^{1-\alpha} + \|v_b\|_1^{1-\alpha}) \|v_a - v_b\|_2} \|v_a - v_b\|_2. \quad (52)$$

This concludes the proof. \square

Example 4.9. Let us consider the problem of attitude stabilization of a fully actuated rigid body. The attitude of a rigid body, relative to a given reference frame, is represented uniquely by an orthonormal matrix $R \in SO(3) := \{R \in \mathbb{R}^{3 \times 3} \mid \det(R) = 1, \text{ and } R^{-1} = R^\top\}$. Thus, the kinematics of the rigid-body attitude is described by

$$\frac{d}{dt} R(t) = R(t)\omega(t)^\times, \quad (53)$$

where $\omega(t)$ denotes the angular velocity of the body, relative to a reference frame, $(\cdot)^\times$ denotes the conventional isomorphism between $\mathfrak{so}(3)$ (the tangent space to $SO(3)$ at $R = I$) and \mathbb{R}^3 , that is

$$\omega^\times = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}.$$

Assuming that the control action provides torques along the principal edges of the rigid body, the attitude dynamics is then expressed in the body's frame via Euler's equation

$$J \frac{d}{dt} \omega(t) = (J\omega(t))^\times \omega(t) + \tau(t) + d(t), \quad (54)$$

where $J = J^\top \succ 0$ denotes the inertia matrix, $\tau(t) \in \mathbb{R}^3$ is the control action, and $d(t)$ denotes the external disturbances. Given an initial attitude $R_0 \in SO(3)$ and initial angular velocity $\omega_0 \in \mathbb{R}^3$, the objective consists in designing $\tau(t)$ so that the full state is regulated at $R_d = I \in SO(3)$, $\omega_d = 0$ regardless of the disturbance $d(t)$.

For designing the discrete-time controller, we consider the following discrete-time approximation to (53)-(54)

$$R_{k+1} = R_k \Gamma_k, \quad (55a)$$

$$J\omega_{k+1} = J\omega_k + h(J\omega_k)^\times \omega_k + h(\tau_k + d_k), \quad (55b)$$

where $\Gamma_k = (I + \frac{h}{2}\omega_k^\times)(I - \frac{h}{2}\omega_k^\times)^{-1}$. The discrete model (55) is the result of a mid-point discretization of $R(t)$ together with a forward Euler discretization of $\omega(t)$ in (53)-(54). It is not difficult to see that the mid-point discretization with respect to R ensures the positive invariance of $SO(3)$ with respect to (55a). Following the design guidelines in [16], let us consider the sliding variable $\sigma_k = \omega_{e,k} + \frac{1}{2} \operatorname{vex}(R_{e,k} - R_{e,k}^\top)$, where $\omega_{e,k} = \omega_k - R_{e,k}^\top \omega_d$, $R_{e,k} = R_d^\top R_k$, and $\operatorname{vex} : \mathfrak{so}(3) \rightarrow \mathbb{R}^3$ is the inverse map to $(\cdot)^\times$. The discrete-time dynamics of σ_k is thus described as

$$J\sigma_{k+1} = J\sigma_k + h(J\omega_k)^\times \omega_k + h(\tau_k + d_k) + \frac{h}{4} J \operatorname{vex}(R_k(\Gamma_k + I)\omega_k^\times + \omega_k^\times(\Gamma_k + I)^\top R_k^\top). \quad (56)$$

Setting

$$\tau_k = Ju_k - (J\omega_k)^\times \omega_k - \frac{1}{4} J \operatorname{vex}(R_k(\Gamma_k + I)\omega_k^\times + \omega_k^\times(\Gamma_k + I)^\top R_k^\top) \quad (57)$$

lead us to the dynamics

$$\sigma_{k+1} = \sigma_k + hu_k + h\bar{d}_k, \quad (58)$$

where $\bar{d}_k = J^{-1}d_k$. In order to drive σ_k towards the origin in finite time, the super-twisting-like algorithm (31) developed in Example 3.2 is considered with $z_{1,k} = \sigma_k$. It was shown in [16] that $R = I$, $\omega = 0$ is an almost globally stable equilibrium of (53)-(54) on $\mathcal{D} := \{(R, \omega) \in SO(3) \times \mathbb{R}^3 \mid \sigma(R, \omega) = 0\}$. Regarding the parameters of the plant, we take $J = \text{diag}(3, 4, 5)$ with initial conditions

$$R_0 = \begin{bmatrix} 0.6862 & 0.1905 & 0.7019 \\ 0.7261 & -0.2363 & -0.6456 \\ 0.0428 & 0.9527 & -0.3005 \end{bmatrix}, \text{ and } \omega_0 = 0,$$

whereas the parameters of the controller (31) are as shown in Table 1.

$$\frac{\kappa = 1 \quad \gamma_1 = 5 \quad \gamma_2 = 15}{\rho = 1 \quad a_1 = 0 \quad a_2 = 0}$$

Table 1: Parameter values for the super-twisting-like controller (31)

For the simulation of the closed-loop, we consider two sampling times, denoted as h_p and h_c , so that $h_p \ll h_c$. The sampling time $h_p = 50\mu\text{s}$ is used for integrating the dynamics (55b), simulating a continuous time process in a digital computer, whereas h_c is used for updating the values of the controller τ_k . Thereby, the hybrid setup depicted in Figure 1 is approximated. We consider three cases for the updating time of the controller. Concretely, i) $h_c = 500\text{ms}$, ii) $h_c = 50\text{ms}$, and iii) $h_c = 5\text{ms}$. For comparison purposes, we also consider the forward-Euler discretization of the multivariable super-twisting controller proposed in [30], that is

$$u_k = \nu_k - \gamma_1 \frac{z_{1,k}}{\|z_{1,k}\|_2^{\frac{1}{2}}}, \quad (59a)$$

$$\nu_{k+1} = \nu_k - h\gamma_2 \frac{z_{1,k}}{\|z_{1,k}\|_2}. \quad (59b)$$

where the gains γ_1 and γ_2 are as in Table 1. Finally, we take $d(t) = [\sin(5t) \cos(\sqrt{3}t) \quad \cos(\pi t/2)^2 \quad \sin(\sqrt{5}t)]$ in all simulations.

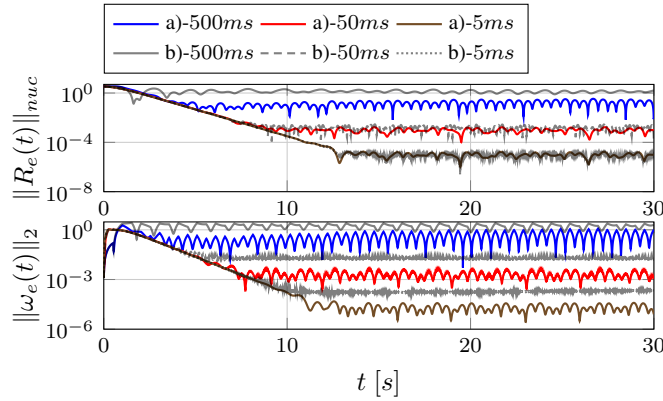


Figure 2: Time trajectories of the error norms $\|R_e(t)\|_{nuc}$ and $\|\omega_e(t)\|_2$ for three different sampling times (500ms, 50ms, and 5ms). Case a) corresponds to trajectories with the feedback (31), whereas case b) corresponds to trajectories with the feedback (59). In all cases, the system is affected by the disturbance $d(t) = [\sin(5t) \cos(\sqrt{3}t) \quad \cos(\pi t/2)^2 \quad \sin(\sqrt{5}t)]$.

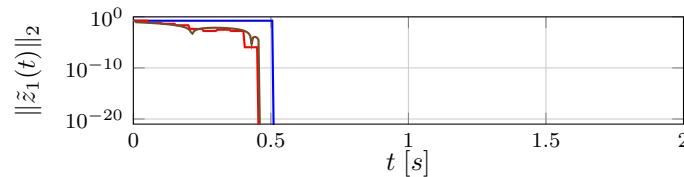


Figure 3: Time evolution of $\|z_1(t)\|_2$ for three different sampling times (500ms (blue line), 50ms (red line) and 5ms (brown line)). In all cases, $z_1(t)$ converges to zero in finite time.

samples/sec	$\gamma_1 = 5, \gamma_2 = 15$				$\gamma_1 = 10, \gamma_2 = 30$				$\gamma_1 = 50, \gamma_2 = 150$			
	2	5	10	50	2	5	10	50	2	5	10	50
$\ u_a\ _{L_2}$	73.11	13.64	11.39	9.94	73.11	17.17	13.12	11.61	73.11	17.17	22.32	18.98
$\ u_b\ _{L_2}$	112.87	52.39	30.80	11.43	277.26	84.60	43.34	13.99	–	1.27×10^6	2402.02	388.02
$\ e_a\ _{L_\infty} \times 10^{-3}$	338.88	23.72	6.59	0.27	338.88	23.72	6.59	0.27	338.88	23.72	6.59	0.27
$\ e_b\ _{L_\infty} \times 10^{-3}$	1908.15	121.43	15.73	0.30	3149.47	245.64	31.26	0.67	–	3999.99	2961.79	56.31

Table 2: Performance measures for controllers (31) ($\|u_a\|_{L_2}, \|e_a\|_{L_\infty}$) and (59) ($\|u_b\|_{L_2}, \|e_b\|_{L_\infty}$) at different sampling frequencies and for different values of gains. Note that the precision achieved with (31) is robust to variations on the gain values.

The time trajectories of the norm of the error signal $R_e(t) = R_d - R(t)$ are plotted in Figure 2. It can be shown that, with the selected gains, the controller (31) achieves a superior performance for low frequency sampling, when compared to the explicit discretization (59), whereas both controllers become alike for high frequencies. Nevertheless, notice that the forward controller (59) shows a chattering level higher than that of (31). The time evolution of $\|\tilde{z}_1(t)\|_2$ is shown in Figure 3 for the three sampling steps discussed above. It is shown that, in all cases, $\tilde{z}_{1,k}$ goes to zero after a finite number of steps as stated in Theorem 4.6. In order to compare the performance of both controllers, we consider the L_2 -norm of the control input and the L_∞ -norm of its associated error signal, once the ultimate bound is reached². The former performance index quantifies the energy expended by the controller, whereas the latter quantifies the precision achieved. Table 2 shows the obtained performance measures, for sampling frequencies in the set $\{2, 5, 10, 50\}$ samples/second and for three cases a) $\gamma_1 = 5, \gamma_2 = 15$; b) $\gamma_1 = 10, \gamma_2 = 30$; and c) $\gamma_1 = 50, \gamma_2 = 150$. Once more, the proposed controller shows a remarkable performance at low frequencies and, as the sampling frequency increases, the L_2 -norm of both controllers becomes similar. In a similar fashion, (31) achieves a better precision than (59) and the gap reduces as the frequency upraises. Note that the controller (31) has a strong insensitivity to changes on the gains as it shows almost no change on its L_2 -norm and maintains practically the same precision as predicted by Corollary 4.7. On the contrary, the controller (59) shows an important degradation of performance to the same change. Such invariance of performance with respect large gains for (31) is due to the fact that the controller (31) is the result of a selection process. This is reminiscent of 1st-order set-valued sliding mode controllers, see e.g., [18, 27]. This robustness property and the improved performance, with respect to (59), become important in applications with limited resources, such as satellite attitude control in spatial applications.

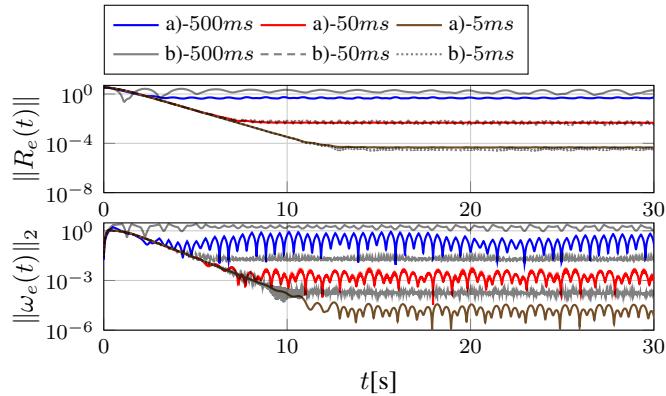


Figure 4: Time trajectories of the error norms $\|R_e(t)\|_{nuc}$ and $\|\omega_e(t)\|_2$ for three different sampling times (500ms, 50ms, and 5ms). Case a) corresponds to trajectories with the feedback (31), whereas case b) corresponds to trajectories with the feedback (59). In all cases, the system is affected by the disturbance $d(t) = [\sin(5t) \cos(\sqrt{3}t) \quad \cos(\pi t/2)^2 \quad 5t]$.

Finally, it is worth to emphasize that the control scheme (10) does not suffer the stability issues of emulation-based controllers, such as [6], for the case of unbounded disturbances with uniformly bounded derivative. Indeed, as an illustration, let us consider the unbounded disturbance

$$d(t) = [\sin(5t) \cos(\sqrt{3}t) \quad \cos(\pi t/2)^2 \quad 5t] ,$$

and the controller (31) with the same parameters as in Table 1. Figure 4 shows the closed-loop error

²With the range of sampling frequencies studied here, the ultimate bound is reached after $t > 15s$.

trajectories for the same sampling times as before. It is clear that trajectories remain bounded and the control objective is reached. \clubsuit

5 Approximation of the control law via proximal splitting algorithms

The explicit structure of the proposed super-twisting-like algorithm (10) always involve the proximal map of the sum of at least two convex functions, (cf., (16)). In general, such proximal maps are difficult to compute. In such situations, proximal splitting algorithms [8, 34] become a central tool, providing a sequence that approximates the controller and that involves the proximal maps of simpler functions. In this section we study the implementation of the controller via a modified version of the celebrated Douglas-Rachford splitting.

Let us consider the following control strategy in the style of (9)

$$u_{c,k} = -K(\gamma_1 m_{1,k} + h\gamma_2 \rho m_{2,k}) + \nu_{k+1} \quad (60a)$$

$$\nu_{k+1} = \nu_k - h\gamma_2 K m_{2,k} \quad (60b)$$

$$m_{1,k} \in \partial f_1(\varrho(x_k, u_k) + h^2 \gamma_2 \rho(m_{2,k} - m_{2,k-1})) \quad (60c)$$

$$m_{2,k} \in \partial f_2(\varrho(x_k, u_k)) \quad (60d)$$

As above, we take $z_{1,k} = K^{-\frac{1}{2}}x_k$, $z_{2,k} = K^{-\frac{1}{2}}(\nu_{k+1} + \bar{d}_k)$, $\hat{m}_{i,k} = K^{\frac{1}{2}}m_{i,k}$, $\hat{f}_i = f_i \circ K^{\frac{1}{2}}$, $i \in \{1, 2\}$, and $\tilde{z}_{1,k+1} = \varrho(x_k, u_k) = z_{1,k} - h(\gamma_1 \hat{m}_{1,k} + h\gamma_2 \rho \hat{m}_{2,k})$, so that the closed-loop (8), (60) becomes

$$z_{1,k+1} = \tilde{z}_{1,k+1} + h z_{2,k+1} \quad (61a)$$

$$z_{2,k+1} = z_{2,k} - h\gamma_2 \hat{m}_{2,k} + h\hat{\Delta}_{k-1} \quad (61b)$$

$$\tilde{z}_{1,k+1} = z_{1,k} - h(\gamma_1 \hat{m}_{1,k} + h\gamma_2 \rho \hat{m}_{2,k}) \quad (61c)$$

$$\hat{m}_{1,k} = \partial \hat{f}_1(\tilde{z}_{1,k+1} + h^2 \gamma_2 \rho(\hat{m}_{2,k} - \hat{m}_{2,k-1})) \quad (61d)$$

$$\hat{m}_{2,k} \in \partial \hat{f}_2(\tilde{z}_{1,k+1}) \quad (61e)$$

Proposition 5.1. *The closed-loop (61) is well-posed and the selection of values $\hat{m}_{1,k}$ and $\hat{m}_{2,k}$ are given by*

$$\hat{m}_{1,k} = \frac{1}{h\gamma_1} \left(\text{Id} - \text{Prox}_{h\gamma_1 \hat{f}_1} \right) \circ \left(z_{1,k} - \left(\text{Id} - \text{Prox}_{h^2 \gamma_2 \rho \hat{f}_2} \right) (\vartheta_k) \right), \quad (62a)$$

$$\hat{m}_{2,k} = \frac{1}{h^2 \gamma_2 \rho} \left(\text{Id} - \text{Prox}_{h^2 \gamma_2 \rho \hat{f}_2} \right) (\vartheta_{k+1}), \quad (62b)$$

where ϑ_k satisfies the recursive equation

$$\vartheta_{k+1} = \text{Prox}_{h\gamma_1 \hat{f}_1} \left(z_{1,k} - \left(\text{Id} - \text{Prox}_{h^2 \gamma_2 \rho \hat{f}_2} \right) (\vartheta_k) \right) + \left(\text{Id} - \text{Prox}_{h^2 \gamma_2 \rho \hat{f}_2} \right) (\vartheta_k), \quad (63)$$

where $\vartheta_0 \in \mathbb{R}^n$ is arbitrary.

Proof. Let $\vartheta_{k+1} = \tilde{z}_{1,k+1} + h^2 \gamma_2 \rho \hat{m}_{2,k}$. Thus, from (61e) it follows that

$$\tilde{z}_{1,k+1} = \text{Prox}_{h^2 \gamma_2 \rho \hat{f}_2}(\vartheta_{k+1}), \quad (64)$$

and consequently $\hat{m}_{2,k}$ satisfies (62b). On the other hand, let $\xi_{k+1} = \tilde{z}_{1,k+1} + h^2 \gamma_2 \rho(\hat{m}_{2,k} - \hat{m}_{2,k-1})$. It thus follows from (61c), together with (62b), that

$$\begin{aligned} \xi_{k+1} + h\gamma_1 \hat{m}_{1,k} &= \tilde{z}_{1,k+1} + h^2 \gamma_2 \rho(\hat{m}_{2,k} - \hat{m}_{2,k-1}) + h\gamma_1 \hat{m}_{1,k} \\ &= z_{1,k} - h^2 \gamma_2 \rho \hat{m}_{2,k-1} \\ &= z_{1,k} - \left(\text{Id} - \text{Prox}_{h^2 \gamma_2 \rho \hat{f}_2} \right) (\vartheta_k) \end{aligned} \quad (65)$$

and (61d) leads us to

$$\xi_{k+1} = \text{Prox}_{h\gamma_1 \hat{f}_1} \left(z_{1,k} - \left(\text{Id} - \text{Prox}_{h^2 \gamma_2 \rho \hat{f}_2} \right) (\vartheta_k) \right). \quad (66)$$

Now, from (65) and (66) the expression for $\hat{m}_{1,k}$ in (62a) is obtained. In addition, from the definitions of ϑ_{k+1} and ξ_{k+1} we have that

$$\begin{aligned}\vartheta_{k+1} &= \tilde{z}_{1,k+1} + h^2\gamma_2\rho\hat{m}_{2,k} \\ &= \xi_{k+1} + h^2\gamma_2\rho\hat{m}_{2,k-1},\end{aligned}\tag{67}$$

so that (63) follows from (62b) and (66). The well-posedness of the closed-loop (61) is immediate, as all the variables depend on known quantities at time $t_k = kh$. This concludes the proof. \square

Remark. The scheme (61c)-(61e) is reminiscent of the famous Douglas-Rachford splitting applied to the generalized equation

$$z_2 \in \gamma_1\partial\hat{f}_1(z_1) + h\gamma_2\rho\partial\hat{f}_2(z_1).\tag{68}$$

Indeed, it follows from (61a) and from (64) that

$$z_{1,k} = \text{Prox}_{h^2\gamma_2\rho\hat{f}_2}(\vartheta_k) + hz_{2,k}.$$

Thus, (63) becomes

$$\vartheta_{k+1} = \text{Prox}_{h\gamma_1\hat{f}_1}\left(2\text{Prox}_{h^2\gamma_2\rho\hat{f}_2}(\vartheta_k) - \vartheta_k + hz_{2,k}\right) + (\text{Id} - \text{Prox}_{h^2\gamma_2\rho\hat{f}_2})(\vartheta_k),\tag{69}$$

which is the classical Douglas-Rachford splitting associated with (68), see [8, 11, 25] for further details and properties about the Douglas-Rachford splitting. \clubsuit

Corollary 5.2. *Consider the closed-loop (61) and let all assumptions of Theorem 4.6 hold. If in addition, $\partial\hat{f}_1$ is locally bounded and $\partial\hat{f}_2$ is uniformly bounded. Then, there exist $h > 0$ sufficiently small and gains $\gamma_1 > 0$, $\gamma_2 > \eta$ sufficiently large, such that $\tilde{z}_{1,k+1}$ reaches the origin after a finite-number of steps. \clubsuit*

Proof. The proof follows the same steps as the proof of Theorem 4.6 together with the use of the outer semicontinuity of $\partial\hat{f}_1$. Indeed, as $\hat{m}_{2,k}$ is uniformly bounded by assumption, then $\tilde{z}_{1,k+1} + h^2\gamma_2\rho(\hat{m}_{2,k} - \hat{m}_{2,k-1})$ is in a $\mathcal{O}(h^2)$ neighborhood of $\tilde{z}_{1,k+1}$. It follows from (61d) that, for any $\varepsilon > 0$, there is $h > 0$, and sufficiently small, such that $\hat{m}_{1,k} \in \partial\hat{f}_1(\tilde{z}_{1,k+1}) + \varepsilon\mathcal{B}$ [7, Proposition 2.5.24-i)]. Thus, as $\partial\hat{f}_1$ is a θ -monotone map, it follows that

$$\langle \hat{m}_{1,k} - \hat{m}_{1,k-1}, \tilde{z}_{1,k+1} - \tilde{z}_{1,k} \rangle \geq (\theta(\tilde{z}_{1,k+1}, \tilde{z}_k) - \varepsilon)\|\tilde{z}_{1,k+1} - \tilde{z}_{1,k}\|.\tag{70}$$

The rest of the proof follows as the proof of Theorem 4.6 *mutatis mutandis*. \square

Example 5.3. *Let us consider once more the attitude stabilization problem described in Example 4.9. For the controller, let us take this time (60) with $f_1(\cdot) = \frac{2}{3}\|\cdot\|_1^{\frac{3}{2}}$, $f_2(\cdot) = \|\cdot\|_2$. Notice that, in this case, an explicit expression for the proximal map Prox_{hg} , with $g = \gamma_1\hat{f}_1 + h\gamma_2\rho\hat{f}_2$, is hard to obtain. Nevertheless, the proposed Douglas-Rachford-based splitting allows us to compute an approximation in an iterative way. Thus, following a procedure similar to the one presented in Example 3.2, the proximal maps to $h\gamma_1\hat{f}_1$ and $h^2\gamma_2\rho\hat{f}_2$ are computed as*

$$[\text{Prox}_{h\gamma_1\hat{f}_1}(p)]_i = \begin{cases} 0, & \text{if } |p_i| \leq h\gamma_1\kappa^{\frac{3}{4}}\tilde{q}(\|p\|_1); \\ p_i - h\gamma_1\kappa^{\frac{3}{4}}\tilde{q}(\|p\|_1)\text{sgn}(p_i), & \text{otherwise.} \end{cases}\tag{71a}$$

$$\text{Prox}_{h^2\gamma_2\rho\hat{f}_2}(p) = \begin{cases} 0, & \text{if } \|p\|_2 \leq h^2\gamma_2\rho\kappa^{\frac{1}{2}}; \\ p - \frac{h^2\gamma_2\rho\kappa^{\frac{1}{2}}}{\|p\|_2}p, & \text{otherwise.} \end{cases}\tag{71b}$$

where, $p \in \mathbb{R}^n$ and

$$\tilde{q}(s) = \frac{-h\gamma_1\kappa^{\frac{3}{4}}m + \sqrt{(h\gamma_1\kappa^{\frac{3}{4}}m)^2 + 4s}}{2}.\tag{72}$$

As a comparison, we also consider the calculation of the controller (9) by using the numerical solver CLARABEL [1]. That is, at each sampling interval, we solve the optimization problem

$$\text{Prox}_{hg}(z_{1,k}) = \arg \min_{w \in \mathbb{R}^n} \left\{ \gamma_1\hat{f}_1(w) + h\gamma_2\rho\hat{f}_2(w) + \frac{1}{2h}\|w - z_{1,k}\|_2^2 \right\}.\tag{73}$$

So that, with the help of (15) and (16), the controller (9) becomes

$$u_{d,k} = \nu_{k+1} - \frac{\kappa^{\frac{1}{2}}}{h} (\text{Id} - \text{Prox}_{hg})(z_{1,k}), \quad (74a)$$

$$\nu_{k+1} = \begin{cases} \nu_k - \frac{\kappa^{\frac{1}{2}}}{h} z_{1,k}, & \text{if } \|\text{Prox}_{hg}(z_{1,k})\| \leq 1 \times 10^{-9}; \\ \nu_k - \frac{h\gamma_2\kappa^{\frac{1}{2}}}{\|\text{Prox}_{hg}(z_{1,k})\|_2} \text{Prox}_{hg}(z_{1,k}), & \text{otherwise.} \end{cases} \quad (74b)$$

Figure 5 shows the resulting closed-loop error trajectories for the system with feedback (60) and (74). Even though the errors are very similar, it is worth to mention that the calculation of (73)-(74) takes between 1.9ms to 2.0ms at each iteration, whereas the calculation of (60), via (62) and (71)-(72), takes between 165 μ s to 200 μ s at each iteration. So that the proposed splitting is approximately 10 times faster than the numeric approach (73)-(74). In any case, both approaches are entirely feasible with the range of sampling frequencies considered here. It is worth to mention that, in this example and the previous one, all the simulations were coded in Python 3 and all of them were performed on an standard laptop (8 GB of RAM memory, 4-cores CPU running at 2.8GHz). \clubsuit

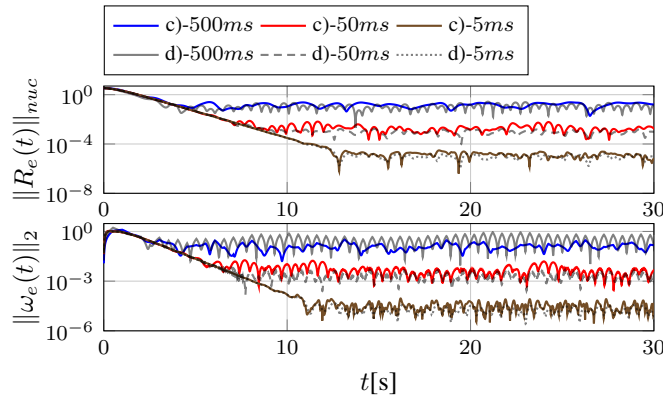


Figure 5: Time trajectories of the error norms $\|R_e(t)\|_{nuc}$ and $\|\omega_e(t)\|_2$ for three different sampling times (500ms, 50ms, and 5ms). Case c) corresponds to trajectories with the feedback (60), whereas case d) corresponds to trajectories with the feedback (73)-(74). In all cases, the system is affected by the disturbance $d(t) = [\sin(5t) \cos(\sqrt{3}t) \quad \cos(\pi t/2)^2 \quad \sin(\sqrt{5}t)]$.

6 Conclusions

A family of discrete-time super-twisting-like algorithms for control was introduced and its well-posedness, as well as, its stability and robustness properties were analyzed. Contrary to most of the literature on super-twisting algorithms, the presented theoretical analysis makes no use of homogeneity properties of the vector field, but instead it is based on the maximal monotonicity of the maps defining the controller. This approach opens an alternative way for studying higher order sliding-mode controllers based on variational methods. Moreover, an adaptation of the Douglas-Rachford splitting, for the implementation of the control law, was also studied. Numerical simulations illustrate how the proposed family of controllers surpass the performance of conventional forward-Euler emulation-based designs.

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