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*Inria*

# Unique continuation theorem for symmetric harmonic hyperbolic systems with constant coefficients

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# Unique continuation theorem for symmetric harmonic hyperbolic systems with constant coefficients

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Project-Team Makutu

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**Abstract:** This document presents a simple proof of the unique continuation theorem for symmetric harmonic hyperbolic problems with constant coefficients. It only requires basic knowledge in functional analysis.

**Key-words:** Unique continuation, hyperbolic formulation, wave propagation, transport equations, acoustics, elasticity, Maxwell's equations.

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# **Théorème de prolongement unique pour les systèmes hyperboliques harmoniques symétriques à coefficients constants**

**Résumé :** Ce document propose une preuve simple du théorème de prolongement unique pour les problèmes hyperboliques harmoniques symétriques à coefficients constants. Celle-ci ne nécessite que des connaissances élémentaires en analyse fonctionnelle.

**Mots-clés :** Prolongement unique, formulation hyperbolique, propagation des ondes, équations de transport, acoustique, élasticité, équations de Maxwell.

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## 1 Introduction

The wave propagation problems are usually interpreted in a quasi-elliptic framework. Many authors resort to the Lax-Milgram Lemma, the Fredholm alternative [2] and the unique continuation theorem to prove the existence and uniqueness of their solutions [5, 9]. Numerically, continuous finite element methods are particularly appropriate to these applications. Yet, considering large domains leads to numerical dispersion problems, in addition to memory limitations for the storage of the matrix coming from the problem discretisation.

Many authors considered resorting to hyperbolic formulations and Discontinuous Galerkin-type approximations based on upwind fluxes, as they allow to use iterative resolution methods [3] and limit phase errors [1]. It is then necessary to plunge into the theoretical framework of hyperbolic equations to ensure existence and uniqueness of the solution of the studied problem.

This report aims at presenting a unique continuation principle which does not need any advanced tool of hyperbolic problem analysis, allowing computer engineers to lean on a theory that only asks for elementary knowledge of functional analysis. A reader who would like to specialise in these issues may refer to the works of Friedrichs [4], Rauch [11] and Hörmander [6], which focus on subelliptic operators.

## 2 The classical framework for hyperbolic systems

In this report, we focus on the time-harmonic version of the following partial derivative problems which have been extensively studied by Friedrichs [4] and Rauch [11] in the time-dependent framework. They take the following form:

**Problem 2.1.** Find  $\mathbf{Y} : \Omega \times \mathbb{R}^+ \longrightarrow \mathbb{R}^m$  such that

$$\begin{cases} \mathbf{M} \frac{\partial \mathbf{Y}}{\partial t}(\mathbf{x}, t) + \operatorname{div}(\mathbf{F}\mathbf{Y})(\mathbf{x}, t) = 0 & \text{for } \mathbf{x} \in \Omega \text{ and } t \geq 0, \\ \mathbf{F}_{\mathbf{n}_{\partial\Omega}(\mathbf{x}), \mathbf{x}}^- \mathbf{Y}(\mathbf{x}, t) = 0 & \text{for } \mathbf{x} \in \partial\Omega \text{ and } t \geq 0, \\ \mathbf{Y}(\mathbf{x}, 0) = \mathbf{Y}_0(\mathbf{x}) & \text{for } \mathbf{x} \in \Omega, \end{cases} \quad (1)$$

where the boundary of the domain  $\Omega \subset \mathbb{R}^k$  is denoted as  $\partial\Omega$  and the parameters of the PDE system verify the following Hypotheses.

**Hypothesis 2.2.** The domain  $\Omega$  is a polyhedral subset of  $\mathbb{R}^k$ , whose curved boundary  $\partial\Omega$  can be split into  $P$  faces, denoted as  $(\partial\Omega_j)_{j \in \llbracket 1, P \rrbracket}$ . Moreover, we suppose it has Lipschitz boundary in the sense of [10, Section 2.1.1], so as to fall within the scope of the divergence theorem [10, Section 3.1.2].

This includes a very large class of domains including polyhedral domains with curvilinear boundaries. Moreover, it implies that the mapping

$$\begin{aligned} \mathbf{n}_{\partial\Omega} &: \partial\Omega \rightarrow \mathbb{R}^k \\ \mathbf{x} &\mapsto \mathbf{n}_{\partial\Omega}(\mathbf{x}) \end{aligned} \quad , \quad (2)$$

which associates to any point of the boundary the unitary outward normal vector of  $\Omega$ , when restricted to a face  $\partial\Omega_j$  of the polyhedral, is of class  $C^\infty(\partial\Omega_j, \mathbb{R}^k)$ .

**Hypothesis 2.3.**  $\mathbf{F} := (\mathbf{F}^j)_{j \in \llbracket 1, k \rrbracket}$  is a matrix valued vector whose elements  $\mathbf{F}^j$  are  $k$  real space-independent symmetric matrices of  $\mathbb{R}^{m \times m}$ .

**Definition 2.4.** For a matrix valued vector  $\mathbf{F} = (\mathbf{F}^j)_{j \in \llbracket 1, k \rrbracket}$  which satisfies Hypothesis 2.3, we denote

- for all  $\mathbf{Y} \in \mathbb{C}^m$ ,  $\mathbf{F}\mathbf{Y}$  is a vector whose  $k$  elements  $\mathbf{F}^j\mathbf{Y}$  belong to  $\mathbb{C}^m$

$$\mathbf{F}\mathbf{Y} = (\mathbf{F}^1\mathbf{Y}, \dots, \mathbf{F}^k\mathbf{Y}). \quad (3)$$

- for all  $\mathbf{Y} \in \mathbb{C}^m$  and  $\mathbf{Y}' \in \mathbb{C}^m$ ,  $\mathbf{F}\mathbf{Y} \cdot \overline{\mathbf{Y}'}$  is a vector of  $\mathbb{C}^k$  defined as

$$\mathbf{F}\mathbf{Y} \cdot \overline{\mathbf{Y}'} = (\mathbf{F}^1\mathbf{Y} \cdot \overline{\mathbf{Y}'}, \dots, \mathbf{F}^k\mathbf{Y} \cdot \overline{\mathbf{Y}'}). \quad (4)$$

- for all  $\mathbf{Y} \in [\mathcal{D}'(\Omega)]^m$ ,  $\operatorname{div}(\mathbf{F}\mathbf{Y})$  is the divergence of  $\mathbf{F}\mathbf{Y}$  defined in the sense of generalised functions as

$$\operatorname{div}(\mathbf{F}\mathbf{Y}) = \sum_{j=1}^k \frac{\partial \mathbf{F}^j \mathbf{Y}}{\partial x_j}. \quad (5)$$

- for any  $\mathbf{n} = (n_1, \dots, n_k) \in \mathbb{R}^k$ ,  $\mathbf{F}_{\mathbf{n}} \in \mathbb{R}^{m \times m}$  is a symmetric real valued matrix defined as

$$\mathbf{F}_{\mathbf{n}} = \sum_{j=1}^k n_j \mathbf{F}^j. \quad (6)$$

**Hypothesis 2.5.** The real matrix  $\mathbf{M} \in \mathbb{R}^{m \times m}$  is symmetric positive definite and is independent of  $\mathbf{x} \in \Omega$ .

**Hypothesis 2.6.** For all  $\mathbf{d} \in \mathbb{R}^k$  and  $\mathbf{x} \in \partial\Omega$ , there exists a decomposition of  $\mathbf{F}_{\mathbf{d}}$

$$\mathbf{F}_{\mathbf{d}} = \mathbf{F}_{\mathbf{d}, \mathbf{x}}^+ + \mathbf{F}_{\mathbf{d}, \mathbf{x}}^-, \quad (7)$$

such that:

- $\mathbf{F}_{\mathbf{d}, \mathbf{x}}^+$  and  $\mathbf{F}_{\mathbf{d}, \mathbf{x}}^-$  are two positive and negative matrices of  $\mathbb{R}^{m \times m}$  respectively.
- the mapping

$$\begin{aligned} \partial\Omega &\rightarrow \mathbb{R}^{m \times m} \\ \mathbf{x} &\mapsto \mathbf{F}_{\mathbf{n}_{\partial\Omega}(\mathbf{x}), \mathbf{x}}^+, \end{aligned} \quad (8)$$

is measurable.

**Remark 2.7.** In particular, this definition falls within the framework of the semi-admissible BCs for Friedrichs systems (which are sufficient to show uniqueness in this framework). Indeed, for a given  $\mathbf{x} \in \partial\Omega$ , by introducing the positive boundary matrix  $\mathbf{M}_{\partial\Omega}(\mathbf{x}) = \mathbf{F}_{\mathbf{n}_{\partial\Omega}(\mathbf{x}), \mathbf{x}}^+ - \mathbf{F}_{\mathbf{n}_{\partial\Omega}(\mathbf{x}), \mathbf{x}}^-$ , the BC (1) can be written as:

$$\mathbf{F}_{\mathbf{n}_{\partial\Omega}(\mathbf{x}), \mathbf{x}}^- \mathbf{Y}(\mathbf{x}, t) = \mathbf{0} \iff (\mathbf{F}_{\mathbf{n}_{\partial\Omega}(\mathbf{x}), \mathbf{x}} - \mathbf{M}_{\partial\Omega}(\mathbf{x})) \mathbf{Y}(\mathbf{x}, t) = \mathbf{0}. \quad (9)$$

**Remark 2.8.** A reasonable way to compute a decomposition as (7) is to consider an application

$$\begin{aligned} \widetilde{\mathbf{M}} &: \partial\Omega \rightarrow \mathbb{R}^{m \times m} \\ \mathbf{x} &\mapsto \widetilde{\mathbf{M}}(\mathbf{x}) \end{aligned} \quad (10)$$



that we suppose continuous on each one of the  $P$  faces  $\partial\Omega_j$ . Moreover, we suppose that  $\widetilde{\mathbf{M}}(\mathbf{x})$  is a symmetric positive definite matrix of  $\mathbb{R}^{m \times m}$  for all  $\mathbf{x} \in \partial\Omega$ . Thus, for a given  $\mathbf{d} \in \mathbb{R}^k$  and  $\mathbf{x} \in \partial\Omega$ , we can introduce the generalized eigenvalue problem

$$\mathbf{F}_{\mathbf{d}} \mathbf{Y} = \lambda \widetilde{\mathbf{M}}(\mathbf{x}) \mathbf{Y}, \quad (11)$$

and the inherent decomposition

$$\mathbf{F}_{\mathbf{d}} = \widetilde{\mathbf{M}}(\mathbf{x}) \mathbf{P}_{\mathbf{d},\mathbf{x}} \boldsymbol{\Lambda}_{\mathbf{d},\mathbf{x}} \mathbf{P}_{\mathbf{d},\mathbf{x}}^{-1} \quad \text{with} \quad \mathbf{P}_{\mathbf{d},\mathbf{x}}^{-1} = \mathbf{P}_{\mathbf{d},\mathbf{x}}^* \widetilde{\mathbf{M}}(\mathbf{x}), \quad (12)$$

where  $\mathbf{P}_{\mathbf{d},\mathbf{x}} \in \mathbb{R}^{m \times m}$  and  $\boldsymbol{\Lambda}_{\mathbf{d},\mathbf{x}} \in \mathbb{R}^{m \times m}$  respectively denote the matrix of the coordinates of the generalised eigenvectors and the diagonal matrix of the associated eigenvalues.

This naturally allows to define

$$\mathbf{F}_{\mathbf{d},\mathbf{x}}^+ = \widetilde{\mathbf{M}}(\mathbf{x}) \mathbf{P}_{\mathbf{d},\mathbf{x}} \boldsymbol{\Lambda}_{\mathbf{d},\mathbf{x}}^+ \mathbf{P}_{\mathbf{d},\mathbf{x}}^{-1} \quad \text{and} \quad \mathbf{F}_{\mathbf{d},\mathbf{x}}^- = \widetilde{\mathbf{M}}(\mathbf{x}) \mathbf{P}_{\mathbf{d},\mathbf{x}} \boldsymbol{\Lambda}_{\mathbf{d},\mathbf{x}}^- \mathbf{P}_{\mathbf{d},\mathbf{x}}^{-1}, \quad (13)$$

where  $\boldsymbol{\Lambda}_{\mathbf{d},\mathbf{x}}^+$  (respectively  $\boldsymbol{\Lambda}_{\mathbf{d},\mathbf{x}}^-$ ) denotes the diagonal matrices made of the positive (respectively negative) values of  $\boldsymbol{\Lambda}_{\mathbf{d},\mathbf{x}}$ .

This allows to define the naturally positive operator

$$\mathbf{M}_{\partial\Omega}(\mathbf{x}) = \mathbf{F}_{\mathbf{n}_{\partial\Omega}(\mathbf{x}),\mathbf{x}}^+ - \mathbf{F}_{\mathbf{n}_{\partial\Omega}(\mathbf{x}),\mathbf{x}}^- = \widetilde{\mathbf{M}}(\mathbf{x}) \mathbf{P}_{\mathbf{n}_{\partial\Omega}(\mathbf{x}),\mathbf{x}} |\boldsymbol{\Lambda}_{\mathbf{n}_{\partial\Omega}(\mathbf{x}),\mathbf{x}}| \mathbf{P}_{\mathbf{n}_{\partial\Omega}(\mathbf{x}),\mathbf{x}}^{-1}, \quad (14)$$

where  $|\boldsymbol{\Lambda}_{\mathbf{n}_{\partial\Omega}(\mathbf{x}),\mathbf{x}}| = \boldsymbol{\Lambda}_{\mathbf{n}_{\partial\Omega}(\mathbf{x}),\mathbf{x}}^+ - \boldsymbol{\Lambda}_{\mathbf{n}_{\partial\Omega}(\mathbf{x}),\mathbf{x}}^-$  stands for the absolute value of the diagonal matrix  $\boldsymbol{\Lambda}_{\mathbf{n}_{\partial\Omega}(\mathbf{x}),\mathbf{x}}$ .

In addition to the semi-admissible conditions already satisfied in Remark 2.7, the following maximality property stands for this matrix

$$\ker(\mathbf{F}_{\mathbf{n}_{\partial\Omega}(\mathbf{x})} - \mathbf{M}_{\partial\Omega}(\mathbf{x})) \oplus \ker(\mathbf{F}_{\mathbf{n}_{\partial\Omega}(\mathbf{x})} + \mathbf{M}_{\partial\Omega}(\mathbf{x})) = \mathbb{R}^m. \quad (15)$$

Indeed, one has

$$\text{Im}(\mathbf{F}_{\mathbf{n}_{\partial\Omega}(\mathbf{x}),\mathbf{x}}^-) \cap \text{Im}(\mathbf{F}_{\mathbf{n}_{\partial\Omega}(\mathbf{x}),\mathbf{x}}^+) = \{\mathbf{0}\}, \quad (16)$$

where  $\text{Im}(A)$  stands for the image of a matrix  $A$ , and

$$\ker(\mathbf{F}_{\mathbf{n}_{\partial\Omega}(\mathbf{x}),\mathbf{x}}^-) \oplus \ker(\mathbf{F}_{\mathbf{n}_{\partial\Omega}(\mathbf{x}),\mathbf{x}}^+) = \mathbb{R}^m \quad (17)$$

since for a symmetric matrix  $A$ , one has  $\ker(A) = \text{Im}(A)^\perp$ . We conclude remarking that

$$\mathbf{F}_{\mathbf{n}_{\partial\Omega}(\mathbf{x})} - \mathbf{M}_{\partial\Omega}(\mathbf{x}) = 2 \mathbf{F}_{\mathbf{n}_{\partial\Omega}(\mathbf{x}),\mathbf{x}}^- \quad \text{and} \quad \mathbf{F}_{\mathbf{n}_{\partial\Omega}(\mathbf{x})} + \mathbf{M}_{\partial\Omega}(\mathbf{x}) = 2 \mathbf{F}_{\mathbf{n}_{\partial\Omega}(\mathbf{x}),\mathbf{x}}^+, \quad (18)$$

### 3 Time harmonic hyperbolic problem

In what follows, we will focus on the time-harmonic version of Problem 2.1.

To do so, we consider an angular frequency  $\omega > 0$  and we look for the solution  $\mathbf{Y} : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^m$  as

$$\mathbf{Y}(\mathbf{x}, t) = \Re(e^{i\omega t} \mathbf{Y}(\mathbf{x})), \quad (19)$$

with  $\mathbf{Y} : \Omega \rightarrow \mathbb{C}^m$ .

We introduce the Hilbert space of existence of  $\mathbf{Y}$

$$\mathcal{H} = \left\{ \mathbf{Y} \in [L^2(\Omega, \mathbb{C})]^m \text{ such that } \text{div}(\mathbf{F}\mathbf{Y}) \in [L^2(\Omega, \mathbb{C})]^m \right\}, \quad (20)$$

and then the time-harmonic problem

**Problem 3.1** (Time-harmonic hyperbolic problem). Find  $\mathbf{Y} \in [\mathbf{L}^2(\Omega, \mathbb{C})]^m$  such that

$$\begin{cases} i\omega \mathbf{M}\mathbf{Y}(\mathbf{x}) + \operatorname{div}(\mathbf{F}\mathbf{Y})(\mathbf{x}) = \mathbf{f}(\mathbf{x}) & \text{for } \mathbf{x} \in \Omega, \\ \mathbf{F}_{\mathbf{n}_{\partial\Omega}(\mathbf{x}), \mathbf{x}}^- \mathbf{Y}(\mathbf{x}) = \mathbf{0} & \text{for } \mathbf{x} \in \partial\Omega, \end{cases} \quad (21)$$

for a source term  $\mathbf{f} \in [\mathbf{L}^2(\Omega, \mathbb{C})]^m$ .

**Remark 3.2.** One will note that, under existence condition, both conditions  $\mathbf{Y} \in [\mathbf{L}^2(\Omega, \mathbb{C})]^m$  and  $\mathbf{f} \in [\mathbf{L}^2(\Omega, \mathbb{C})]^m$  imply that  $\operatorname{div}(\mathbf{F}\mathbf{Y}) \in [\mathbf{L}^2(\Omega, \mathbb{C})]^m$  and then  $\mathbf{Y} \in \mathcal{H}$ .

Well-posedness of Problem 3.1 is usually dealt thanks to regularisation techniques (see [8] for an example for the Maxwell's equations) and the Fredholm alternative, which ensures that the uniqueness of a solution implies its existence. Thus, we will focus on the uniqueness property, which is synthetised by the following theorem

**Theorem 3.3.** Let  $\mathbf{Y} \in [\mathbf{L}^2(\Omega, \mathbb{C})]^m$  such that

$$\begin{cases} i\omega \mathbf{M}\mathbf{Y}(\mathbf{x}) + \operatorname{div}(\mathbf{F}\mathbf{Y})(\mathbf{x}) = \mathbf{0} & \text{for } \mathbf{x} \in \Omega, \\ \mathbf{F}_{\mathbf{n}_{\partial\Omega}(\mathbf{x}), \mathbf{x}}^- \mathbf{Y}(\mathbf{x}) = \mathbf{0} & \text{for } \mathbf{x} \in \partial\Omega. \end{cases} \quad (22)$$

Then,  $\mathbf{Y} = \mathbf{0}$ .

## 4 The hyperbolic integration by parts formula

The framework we introduced in Section 2 allows to consider the hyperbolic integration by parts Theorem:

**Theorem 4.1.** Let  $\Omega$  be a domain of  $\mathbb{R}^k$  which verifies Hypothesis 2.2, and  $\mathbf{Y}, \mathbf{Y}' \in \mathcal{H}$  (see (20)). Let's consider a set  $(\mathbf{F}^j)_{j \in \llbracket 1, k \rrbracket}$  of  $k$  real symmetric matrices of  $\mathbb{R}^{m \times m}$ . Then one has

$$\int_{\Omega} \operatorname{div}(\mathbf{F}\mathbf{Y}) \cdot \overline{\mathbf{Y}'} \, d\mathbf{x} + \int_{\Omega} \mathbf{Y} \cdot \operatorname{div}(\overline{\mathbf{F}\mathbf{Y}'}) \, d\mathbf{x} = \int_{\partial\Omega} \mathbf{F}_{\mathbf{n}_{\partial\Omega}} \mathbf{Y} \cdot \overline{\mathbf{Y}'} \, d\boldsymbol{\sigma}. \quad (23)$$

*Proof.* First, let's consider  $\mathbf{Y}, \mathbf{Y}' \in [\mathcal{C}^\infty(\Omega, \mathbb{C})]^m$ . Let  $I$  denote the left term of (23) :

$$I = \sum_{j=1}^k \int_{\Omega} \frac{\partial \mathbf{F}^j \mathbf{Y}}{\partial x_j} \cdot \overline{\mathbf{Y}'} + \mathbf{Y} \cdot \frac{\partial \overline{\mathbf{F}^j \mathbf{Y}'}}{\partial x_j} \, d\mathbf{x}. \quad (24)$$

As the matrices  $\mathbf{F}^j$  are real symmetric and space-independent, one has

$$I = \sum_{j=1}^k \int_{\Omega} \frac{\partial [\mathbf{F}^j \mathbf{Y} \cdot \overline{\mathbf{Y}'}]}{\partial x_j} \, d\mathbf{x} = \int_{\Omega} \operatorname{div}(\mathbf{f}) \, d\mathbf{x}, \quad (25)$$

where we denote

$$\mathbf{f} = \mathbf{F}\mathbf{Y} \cdot \overline{\mathbf{Y}'} \in [\mathcal{C}^\infty(\Omega, \mathbb{C})]^k. \quad (26)$$

Thus, by using [10, Section 3.1.2], one finally gets

$$I = \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{n}_{\partial\Omega} \, d\boldsymbol{\sigma} = \int_{\partial\Omega} \sum_{j=1}^k n_{\partial\Omega}^j \mathbf{F}^j \mathbf{Y} \cdot \overline{\mathbf{Y}'} \, d\boldsymbol{\sigma} = \int_{\partial\Omega} \mathbf{F}_{\mathbf{n}_{\partial\Omega}} \mathbf{Y} \cdot \overline{\mathbf{Y}'} \, d\boldsymbol{\sigma}. \quad (27)$$

We can conclude by density of  $[\mathcal{C}^\infty(\Omega, \mathbb{C})]^m$  in  $[\mathbf{L}^2(\Omega, \mathbb{C})]^m$  and then in  $\mathcal{H}$ .  $\square$

## 5 The unique continuation principle in $\mathbb{R}^k$

Let's consider matrices  $\mathbf{M}$  and  $(\mathbf{F}^j)_{j \in \llbracket 1, k \rrbracket} \in \mathbb{R}^{m \times m}$  which verify the Hypotheses of Section 2. Then one has

**Theorem 5.1.** *Let  $\mathbf{Y} \in [L^2(\mathbb{R}^k, \mathbb{C})]^m$  such that*

$$i\omega \mathbf{M} \mathbf{Y} + \operatorname{div}(\mathbf{F} \mathbf{Y}) = \mathbf{0}. \quad (28)$$

Then,  $\mathbf{Y} = \mathbf{0}$ .

*Proof.* Let  $\widehat{\mathbf{Y}} \in [L^2(\mathbb{R}^k, \mathbb{C})]^m$  be the Fourier transform of  $\mathbf{Y}$  defined as

$$\forall \boldsymbol{\xi} \in \mathbb{R}^k, \widehat{\mathbf{Y}}(\boldsymbol{\xi}) = \int_{\mathbb{R}^k} \mathbf{Y}(\mathbf{x}) e^{-i\mathbf{x} \cdot \boldsymbol{\xi}} d\mathbf{x}. \quad (29)$$

Thus, it is solution of

$$\forall \boldsymbol{\xi} \in \mathbb{R}^k, i\omega \mathbf{M} \widehat{\mathbf{Y}}(\boldsymbol{\xi}) + \sum_{j=1}^k i\xi_j \mathbf{F}^j \widehat{\mathbf{Y}}(\boldsymbol{\xi}) = \mathbf{0}. \quad (30)$$

Let's consider  $\boldsymbol{\xi} \in \mathbb{R}^k$  and denote  $\mathbf{d}_{\boldsymbol{\xi}} = \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \in \mathcal{S}_k$  the unitary vector along the direction of  $\boldsymbol{\xi}$ , with  $\mathcal{S}_k$  the unitary sphere of  $\mathbb{R}^k$  for the norm  $|\cdot|$ . Then, (30) can be rewritten as

$$\omega \mathbf{M} \widehat{\mathbf{Y}}(\boldsymbol{\xi}) + |\boldsymbol{\xi}| \mathbf{F}_{\mathbf{d}_{\boldsymbol{\xi}}} \widehat{\mathbf{Y}}(\boldsymbol{\xi}) = \mathbf{0} \quad (31)$$

Yet, as  $\mathbf{F}_{\mathbf{d}_{\boldsymbol{\xi}}}$  is a real symmetric matrix, we can introduce the generalised eigenvalue problem

$$\mathbf{F}_{\mathbf{d}_{\boldsymbol{\xi}}} \mathbf{w} = \lambda \mathbf{M} \mathbf{w}, \quad (32)$$

and the associated decomposition

$$\mathbf{F}_{\mathbf{d}_{\boldsymbol{\xi}}} = \mathbf{M} \mathbf{Q}(\mathbf{d}_{\boldsymbol{\xi}}) \boldsymbol{\Lambda}(\mathbf{d}_{\boldsymbol{\xi}}) \mathbf{Q}^{-1}(\mathbf{d}_{\boldsymbol{\xi}}), \quad (33)$$

where  $\mathbf{Q}(\mathbf{d}_{\boldsymbol{\xi}}) \in \mathbb{R}^{m \times m}$  is a matrix of eigenvector coordinates and  $\boldsymbol{\Lambda}(\mathbf{d}_{\boldsymbol{\xi}}) \in \mathbb{R}^{m \times m}$  is a real diagonal matrix of eigenvalues, that we denote as

$$\lambda_1(\mathbf{d}_{\boldsymbol{\xi}}) \leq \dots \leq \lambda_m(\mathbf{d}_{\boldsymbol{\xi}}). \quad (34)$$

In particular, functions  $\mathbf{d}_{\boldsymbol{\xi}} \mapsto \lambda_j(\mathbf{d}_{\boldsymbol{\xi}})$  are continuous for any  $j \in \llbracket 1, m \rrbracket$  (see [7, Chapter 2]). The invertibility of  $\mathbf{M}$  and  $\mathbf{Q}(\mathbf{d}_{\boldsymbol{\xi}})$  then allows to rewrite (31), by denoting  $\widehat{\mathbf{W}}(\boldsymbol{\xi}) = \mathbf{P}^{-1}(\mathbf{d}_{\boldsymbol{\xi}}) \widehat{\mathbf{Y}}(\boldsymbol{\xi})$ , as

$$\mathbf{A}(\boldsymbol{\xi}) \widehat{\mathbf{W}}(\boldsymbol{\xi}) = \mathbf{0}, \quad (35)$$

where  $\mathbf{A}(\boldsymbol{\xi}) = \omega \mathbf{I}_m + |\boldsymbol{\xi}| \boldsymbol{\Lambda}(\mathbf{d}_{\boldsymbol{\xi}})$  and  $\mathbf{I}_m \in \mathbb{R}^{m \times m}$  stands from the identity matrix of  $\mathbb{R}^{m \times m}$ .

Obtaining that  $\mathbf{A}(\boldsymbol{\xi})$  is invertible for almost every  $\boldsymbol{\xi} \in \mathbb{R}^k$  will then imply that  $\widehat{\mathbf{Y}} = \mathbf{0}$  almost everywhere. Yet, (35) can be rewritten as

$$\forall j \in \llbracket 1, m \rrbracket, [\omega + |\boldsymbol{\xi}| \lambda_j(\mathbf{d}_{\boldsymbol{\xi}})] \widehat{W}_j(\boldsymbol{\xi}) = 0. \quad (36)$$

Then, the fact that  $\mathbf{A}(\boldsymbol{\xi})$  is not invertible corresponds to

$$\exists j \in \llbracket 1, m \rrbracket, \omega + |\boldsymbol{\xi}| \lambda_j(\mathbf{d}_{\boldsymbol{\xi}}) = 0 \quad (37)$$

or else

$$\boldsymbol{\xi} \in \bigcup_{j \in [1, m]} \Gamma_j, \quad (38)$$

where  $\Gamma_j$  stands for

$$\Gamma_j = \{ \boldsymbol{\xi} \in \mathbb{R}^k, \exists \mathbf{d} \in \mathcal{S}_k, \boldsymbol{\xi} = |\boldsymbol{\xi}| \mathbf{d} \text{ and } \omega + |\boldsymbol{\xi}| \lambda_j(\mathbf{d}) = 0 \}. \quad (39)$$

As all the sets  $\Gamma_j$  have a null measure in  $\mathbb{R}^k$ , one obtains that  $\widehat{\mathbf{Y}} = \mathbf{0}$  almost everywhere, which implies the expected result by inverse Fourier transform.  $\square$

**Remark 5.2.** *The eigenvectors associated to (32) correspond to the plane waves.*

## 6 The unique continuation principle in a bounded domain

Such a result can also be derived in the case of a bounded domain, with a null normal flux boundary condition.

**Theorem 6.1.** *Let  $\mathbf{Y} \in [L^2(\Omega, \mathbb{C})]^n$  such that*

$$\begin{cases} i\omega \mathbf{M}\mathbf{Y}(\mathbf{x}) + \operatorname{div}(\mathbf{F}\mathbf{Y})(\mathbf{x}) = \mathbf{0} & \text{for } \mathbf{x} \in \Omega, \\ \mathbf{F}_{\mathbf{n}_{\partial\Omega}(\mathbf{x})}\mathbf{Y}(\mathbf{x}) = \mathbf{0} & \text{for } \mathbf{x} \in \partial\Omega. \end{cases} \quad (40)$$

*Then,  $\mathbf{Y} = \mathbf{0}$ .*

*Proof.* Let  $\widetilde{\mathbf{Y}} : \mathbb{R}^k \rightarrow \mathbb{C}^m$  be the continuation of  $\mathbf{Y}$  by  $\mathbf{0}$  out of  $\Omega$ , defined by

$$\widetilde{\mathbf{Y}} = \mathbf{Y} \quad \text{in } \Omega \quad \text{and} \quad \widetilde{\mathbf{Y}} = \mathbf{0} \quad \text{in } \mathbb{R}^k \setminus \Omega. \quad (41)$$

In particular, one naturally has  $\widetilde{\mathbf{Y}} \in [L^2(\mathbb{R}^k, \mathbb{C})]^m$ , as the continuation of  $\mathbf{Y} \in [L^2(\Omega, \mathbb{C})]^m$  by  $\mathbf{0}$  out of  $\Omega$ .

Let  $\phi \in [\mathcal{D}(\mathbb{R}^k)]^m$ . According to the definition (41) of  $\widetilde{\mathbf{Y}}$  and by definition of the divergence in the sense of generalised functions, one has

$$\begin{aligned} \int_{\mathbb{R}^k} \operatorname{div}(\mathbf{F}\widetilde{\mathbf{Y}}) \cdot \bar{\phi} \, d\mathbf{x} &= - \int_{\mathbb{R}^k} \sum_{j=1}^k \mathbf{F}^j \widetilde{\mathbf{Y}} \cdot \frac{\partial \bar{\phi}}{\partial x_j} \, d\mathbf{x} \\ &= - \int_{\Omega} \sum_{j=1}^k \mathbf{F}^j \mathbf{Y} \cdot \frac{\partial \bar{\phi}}{\partial x_j} \, d\mathbf{x}. \end{aligned} \quad (42)$$

As  $\mathbf{F}^j$  is a real symmetric matrix, we obtain

$$\int_{\mathbb{R}^k} \operatorname{div}(\mathbf{F}\widetilde{\mathbf{Y}}) \cdot \bar{\phi} \, d\mathbf{x} = - \int_{\Omega} \mathbf{Y} \cdot \operatorname{div}(\overline{\mathbf{F}\phi}) \, d\mathbf{x}, \quad (43)$$

and Theorem 4.1 then leads to

$$\int_{\mathbb{R}^k} \operatorname{div}(\mathbf{F}\widetilde{\mathbf{Y}}) \cdot \bar{\phi} \, d\mathbf{x} = \int_{\Omega} \operatorname{div}(\mathbf{F}\mathbf{Y}) \cdot \bar{\phi} \, d\mathbf{x} - \int_{\partial\Omega} \mathbf{F}_{\mathbf{n}_{\partial\Omega}} \mathbf{Y} \cdot \bar{\phi} \, d\sigma. \quad (44)$$

Thus, by using the equations (40), one gets

$$\int_{\mathbb{R}^k} \operatorname{div}(\mathbf{F}\widetilde{\mathbf{Y}}) \cdot \bar{\phi} \, d\mathbf{x} = - \int_{\Omega} i\omega \mathbf{M}\mathbf{Y} \cdot \bar{\phi} \, d\mathbf{x}, \quad (45)$$

and the definition of  $\tilde{\mathbf{Y}}$  allows to conclude that

$$\int_{\mathbb{R}^k} \left[ i\omega \mathbf{M} \tilde{\mathbf{Y}} + \operatorname{div}(\mathbf{F} \tilde{\mathbf{Y}}) \right] \cdot \bar{\phi} \, d\mathbf{x}. \quad (46)$$

The previous theorem then allows to conclude that  $\mathbf{Y} = \mathbf{0}$ .  $\square$

We can finally consider classic Friedrichs boundary conditions, characterising a null ingoing flux.

**Corollary 6.2.** *Let  $\mathbf{Y} \in [L^2(\Omega, \mathbb{C})]^m$  such that*

$$\begin{cases} i\omega \mathbf{M} \mathbf{Y}(\mathbf{x}) + \operatorname{div}(\mathbf{F} \mathbf{Y})(\mathbf{x}) = \mathbf{0} & \text{for } \mathbf{x} \in \Omega, \\ \mathbf{F}_{\mathbf{n}_{\partial\Omega}(\mathbf{x}), \mathbf{x}}^- \mathbf{Y}(\mathbf{x}) = \mathbf{0} & \text{for } \mathbf{x} \in \partial\Omega. \end{cases} \quad (47)$$

where  $\mathbf{F}_{\mathbf{n}_{\partial\Omega}(\cdot), \cdot}^- : \partial\Omega \rightarrow \mathbb{R}^{m \times m}$  is associated to a decomposition of  $\mathbf{F}_{\mathbf{n}_{\partial\Omega}}$  which verifies Hypothesis 2.6 of Section 2.

Then,  $\mathbf{Y} = \mathbf{0}$ .

*Proof.* We will refer to the previous Theorem by showing that  $\mathbf{F}_{\mathbf{n}_{\partial\Omega}} \mathbf{Y} = \mathbf{0}$  on  $\partial\Omega$ . To do so, let's test (47) by  $\mathbf{Y}$  and integrate over  $\Omega$ :

$$\int_{\Omega} i\omega \mathbf{M} \mathbf{Y} \cdot \bar{\mathbf{Y}} + \operatorname{div}(\mathbf{F} \mathbf{Y}) \cdot \bar{\mathbf{Y}} \, d\mathbf{x} = 0. \quad (48)$$

In particular, Theorem 4.1 leads to

$$\int_{\Omega} i\omega \mathbf{M} \mathbf{Y} \cdot \bar{\mathbf{Y}} - \mathbf{Y} \cdot \operatorname{div}(\mathbf{F} \bar{\mathbf{Y}}) \, d\mathbf{x} + \int_{\partial\Omega} \mathbf{F}_{\mathbf{n}_{\partial\Omega}} \mathbf{Y} \cdot \bar{\mathbf{Y}} \, d\sigma = 0. \quad (49)$$

As the matrices  $\mathbf{F}^j$  are real, one has  $\operatorname{div}(\mathbf{F} \bar{\mathbf{Y}}) = \overline{\operatorname{div}(\mathbf{F} \mathbf{Y})}$ , and by using (47), one gets

$$\int_{\Omega} i\omega \mathbf{M} \mathbf{Y} \cdot \bar{\mathbf{Y}} - \mathbf{Y} \cdot \overline{-i\omega \mathbf{M} \mathbf{Y}} \, d\mathbf{x} + \int_{\partial\Omega} \mathbf{F}_{\mathbf{n}_{\partial\Omega}} \mathbf{Y} \cdot \bar{\mathbf{Y}} \, d\sigma = 0. \quad (50)$$

As  $\mathbf{M}$  is a real symmetric matrix, this finally gives

$$\int_{\partial\Omega} \mathbf{F}_{\mathbf{n}_{\partial\Omega}} \mathbf{Y} \cdot \bar{\mathbf{Y}} \, d\sigma = 0. \quad (51)$$

Moreover, the BC  $\mathbf{F}_{\mathbf{n}_{\partial\Omega}(\mathbf{x}), \mathbf{x}}^- \mathbf{Y}(\mathbf{x}) = \mathbf{0}$  implies that

$$\int_{\partial\Omega} \mathbf{F}_{\mathbf{n}_{\partial\Omega}(\mathbf{x}), \mathbf{x}}^+ \mathbf{Y}(\mathbf{x}) \cdot \bar{\mathbf{Y}}(\mathbf{x}) \, d\sigma = 0. \quad (52)$$

Yet, this integral is well-defined on each one of the  $P$  faces by measurability, and the positivity Hypothesis 2.6 implies  $\mathbf{F}_{\mathbf{n}_{\partial\Omega}(\mathbf{x}), \mathbf{x}}^+ \mathbf{Y}(\mathbf{x}) = \mathbf{0}$  for almost every  $\mathbf{x} \in \partial\Omega$ , and then

$$\mathbf{F}_{\mathbf{n}_{\partial\Omega}(\mathbf{x})} \mathbf{Y}(\mathbf{x}) = \mathbf{F}_{\mathbf{n}_{\partial\Omega}(\mathbf{x}), \mathbf{x}}^- \mathbf{Y}(\mathbf{x}) + \mathbf{F}_{\mathbf{n}_{\partial\Omega}(\mathbf{x}), \mathbf{x}}^+ \mathbf{Y}(\mathbf{x}) = \mathbf{0}. \quad (53)$$

We can finally refer to Theorem 6.1 and conclude.  $\square$

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