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# GEOMETRIC NEURAL NETWORK BASED ON PHASE SPACE FOR BCI DECODING

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## ABSTRACT

**Objective:** The integration of Deep Learning (DL) algorithms on brain signal analysis is still in its nascent stages compared to their success in fields like Computer Vision, especially in Brain-Computer Interface (BCI), where the brain activity is decoded to control external devices without requiring muscle control. Electroencephalography (EEG) is a widely adopted choice for designing BCI systems due to its non-invasive and cost-effective nature and excellent temporal resolution. Still, it comes at the expense of limited training data, poor signal-to-noise, and a large variability across and within-subject recordings. Finally, setting up a BCI system with many electrodes takes a long time, hindering the widespread adoption of reliable DL architectures in BCIs outside research laboratories. To improve adoption, we need to improve user comfort using, for instance, reliable algorithms that operate with few electrodes. **Approach:** Our research aims to develop a DL algorithm that delivers effective results with a limited number of electrodes. Taking advantage of the Augmented Covariance Method with SPDNet, we propose the SPDNet <sub>$\psi$</sub>  architecture and analyze its performance and computational impact, as well as the interpretability of the results. The evaluation is conducted on 5-fold cross-validation, using only three electrodes positioned above the Motor Cortex. The methodology was tested on nearly 100 subjects from several open-source datasets using the Mother Of All BCI Benchmark (MOABB) framework. **Main results:** The results of our SPDNet <sub>$\psi$</sub>  demonstrate that the augmented approach combined with the SPDNet significantly outperforms all the current state-of-the-art DL architecture in MI decoding. **Significance:** This new architecture is explainable, with a low number of trainable parameters and a reduced carbon footprint.

**Keywords** Brain-Computer Interfaces · Electroencephalography · Functional connectivity · SPD manifold · Riemannian optimization · Neural Network · Motor Imagery

## 1 Introduction

Brain-computer interface (BCI) technology allows direct communication between a user's brain activity and external devices. Originally designed to help people with disabilities [1], its uses are expanding to other fields, such as rehabilitation using virtual reality [2, 3]. Different signals acquired from brain activity can be used for such a technology, but Electroencephalography (EEG) is a widely adopted choice, as it is a non-invasive, portable, and inexpensive methodology with a very good temporal resolution. Motor imagery (MI) tasks are largely investigated among the brain activities considered when designing a BCI. As the subject is asked to mentally execute a movement without actually performing it, it provides an asynchronous and internal control scheme with no requirement for muscle capability.

The application of Deep Learning (DL) algorithms has garnered significant attention over several domains, ranging from natural language process [4] to protein structure prediction [5]. The field of EEG MI classification is no exception.

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32 However, DL algorithms have not yet established themselves in the BCI field as they have in other fields, such as  
 33 Computer Vision. This is due to several problems: limited data availability, low signal-to-noise ratio in EEG signals,  
 34 subject variability due to anatomical differences between individuals, and to session variability due to deviations in  
 35 electrode placement [6].

36 Unlike other fields where DL algorithms thrive on extensive data, in EEG applications, the emphasis is on enhancing  
 37 user comfort, resulting in smaller datasets with few trials and fewer electrodes. Large-scale EEG systems with numerous  
 38 electrodes not only require extended calibration periods, causing user tiredness but also introduce complexity, potentially  
 39 leading to increased error rates and heightened computational demands. Additionally, the increased cost associated  
 40 with extensive electrode setups deters the widespread development, deployment, and accessibility of BCI technology,  
 41 particularly for the patients who could benefit the most from it.

42 This research focuses on developing a novel DL architecture, SPDNet <sub>$\psi$</sub> , that outperforms the state-of-the-art classifi-  
 43 cation when using a limited number of electrodes. Building on the Augmented Covariance Method (ACM) [7] that  
 44 is an extension of the spatial covariance and the Symmetric Positive Definite (SPD) Neural Network -SPDNet [8],  
 45 we study the SPDNet <sub>$\psi$</sub>  impact on the performance when using a reduced number of electrodes. We also conducted  
 46 a comprehensive analysis of the model size, carbon emissions, and, ultimately, its explainability with respect to the  
 47 standard SPDNet.

48 Spatial covariance, however, is not the only one that can be extracted from the EEG signal. Another potential candidate  
 49 is coherence, which provides an alternative perspective on EEG signal characteristics. Historically, coherence has  
 50 proven to be an inherently unstable feature, which is challenging to compute accurately, resulting in less robust results  
 51 with respect to the spatial covariance. However, the use of information from both covariance and coherence has been  
 52 shown to increase performance as we are considering an estimator of the interactions between brain areas/ electrodes [9].  
 53 We are thus interested to study coherence in combination with SPDNet and especially to use the ACM methodology  
 54 adapted for the coherence matrix.

55 Our methodology is tested through a 5-fold cross-validation evaluation (*Within-Session*), utilizing only three electrodes  
 56 strategically positioned above the Motor Cortex. To validate our algorithm, we test our approach on almost 100 subjects  
 57 from openly available datasets using the Mother Of All BCI Benchmark (MOABB) framework [10]. This research  
 58 not only contributes to the advancement of EEG MI classification but also emphasizes the importance of developing  
 59 efficient, user-friendly algorithms with a minimal environmental impact in the broader context of BCI technology.  
 60 Additionally, our study places a strong emphasis on ensuring that our method is both reproducible and interpretable.

61 The article is organized as follows: In section 2, we delve into the current state-of-the-art in Deep Learning (DL) EEG  
 62 decoding, emphasizing distinctions from our approach. section 3 provides an overview of the theoretical foundations  
 63 of our model and the considered datasets. The obtained results from the Within-session evaluation are presented in  
 64 section 4. Subsequently, section 5 focuses on the method’s impact and current limitations, specifically focusing on its  
 65 explainability and environmental impact. Finally, section 6 summarizes the findings of our study.

## 66 2 Related Work

### 67 2.1 Machine Learning for EEG Decoding

68 In the domain of EEG decoding [11, 12], translating brain activity into meaningful data has become increasingly  
 69 dependent on machine learning (ML) methods [6, 9, 13–17]. However, here, we diverge by focusing on deep learning  
 70 techniques instead of a broad ML spectrum, moving beyond mere method comparison to introduce a novel EEG  
 71 decoding approach.

72 Schirrneister et al. [14] demonstrated that Deep Learning (DL) approaches, specifically *ShallowNet* and *DeepNet* can  
 73 perform on par with conventional machine learning in decoding raw EEG data and can be trained end-to-end, eliminating  
 74 several feature extraction steps. The application of DL algorithms enhances the generalization capability, enabling it  
 75 to handle the inherent variability present in EEG signals effectively. This applicability of DL for EEG data is further  
 76 corroborated by [15, 18–21]. Despite their effectiveness, these DL approaches demand extensive parameter tuning,  
 77 high power consumption, and large datasets for thorough evaluation. In contrast, our study proposes a streamlined  
 78 approach, leveraging Riemannian geometry and functional connectivity derivatives, which are less demanding in terms  
 79 of parameters, enhancing both efficiency and scalability.

### 80 2.2 Deep Riemannian Networks for EEG Decoding

81 Incorporating non-euclidean geometry, especially the Riemannian manifold, into EEG decoding has significantly  
 82 advanced the field [22–34]. A common practice is to compute spatial covariance matrices that capture signal features

83 with the structure of symmetric positive definite (SPD) matrices. These matrices not only enhance signal information  
 84 regarding topology and amplitude but also offer increased robustness to outliers and noise with Riemannian geometry,  
 85 maintaining invariance under linear transformations [24, 29].

86 Huang and Van Gool [22] work introduced *SPDNet*, a neural network that operates on SPD manifolds. This foundation  
 87 was used by Ju et al. [35], who applied SPDNet in bio-signal classification, enhancing transfer learning. Building on  
 88 this, Brooks et al. [36] adapted batch normalization for the SPD manifold, and Kobler et al. [25] further refined the batch  
 89 normalization component for EEG Decoding. Pan et al. [24] added to this perspective by proposing attention components  
 90 for the SPDNet zoo. Follow-up efforts included using residual layers [26, 31], filter bank inputs [23, 27, 30], mixing  
 91 traditional convolution by channels [34], and constraining diffusion models [37, 38]. Contrary to our approach, these  
 92 methods rely on adapting existing neural network components for the SPD manifold or preserving the dimensionality  
 93 assumptions, while our method combines the dimensionality expansion of the phase space reconstruction on the SPD  
 94 manifold.

### 95 2.3 Geometry Transformation in EEG data

96 Geometry Transformation (GT) in EEG analysis involves applying a transformation function to the data [6, 7, 15, 39].  
 97 This process commonly includes standard pre-processing transformations like resampling, band-passing, and filtering,  
 98 which have been proven to enhance model performance and biomarker discovery [16, 40–43]. Data augmentation, a  
 99 regularization geometry transformation, has been acknowledged for its role in enhancing brain decoding in EEG, with  
 100 its benefits varying depending on the task [15, 44]. While these transformations can improve the learning processing,  
 101 these steps usually do not consider the non-stationary dynamics properties of the biological signal. Our approach  
 102 broadens this scope by incorporating transformations that consider phase reconstruction in terms of its components of  
 103 nonlinear dynamics.

104 In the context of delay embedding phase geometry transformation, Chen et al. [45] demonstrated the effectiveness  
 105 of phase embedding in neural networks for one-dimensional bio-signal reconstruction, but without addressing the  
 106 non-euclidean nature of the signal. Our methodology, in contrast, considers the non-euclidean geometry, thereby  
 107 enriching the learning process. The studies by Carrara and Papadopoulo [7] and Zhou et al. [46] align with our  
 108 approach in their use of phase delays and (cross-)covariance matrices for EEG decoding. Our work, on the other hand,  
 109 differentiates by fully leveraging the capabilities of the SPD matrix in Riemannian Neural Networks, allowing for more  
 110 efficient, interpretable layers and achieving superior performance within challenging scenarios with a reduced number  
 111 of channels.

## 112 3 Material and Methods

113 In this section, we describe the EEG decoding problem, the phase space reconstruction transformation, the Riemannian  
 114 manifold properties, the Symmetric Positive Definite neural network components, and the datasets and baselines.

### 115 3.1 EEG Decoding

116 We consider EEG signals as real-valued matrices. Let us denote each bio-signal window captured during a cognitive task  
 117 on a given electrode by  $\mathbf{x} = x(t, t + T)$ , a time serie of  $T$  elements. We denote an epoch as  $\mathbf{X}_i = [\mathbf{x}_1, \dots, \mathbf{x}_C]$ , with  
 118  $\mathbf{X}_i \in \mathbb{R}^{C \times T}$  and the corresponding cognitive task is denoted by  $y_i \in \mathcal{Y}$ . Here,  $C$  represents the number of channels  
 119 (electrodes),  $T$  is the number of time points in the window, and  $i$  indexes the trial  $i = 1 \dots N$ , where  $N$  is the total  
 120 number of trials.

121 EEG decoding aims to construct a function  $f : \mathcal{X} \rightarrow \mathcal{Y}$ , which effectively maps each trial  $\mathbf{X}_i$  to its corresponding  
 122 label  $y_i$ . We can construct a neural network function represented as  $f_\theta : \mathcal{X} \rightarrow \mathcal{Y}$ , composing a sequence of  $l$  functions,  
 123 formulated as  $f_\theta = f_{\theta_1} \circ f_{\theta_2} \circ \dots \circ f_{\theta_l}$ . In this structure,  $\theta$  indicates the parameters describing each layer of the network.

### 124 3.2 Phase space reconstruction for the EEG signal

125 When using a reduced number of electrodes, the captured signals will contain only part of the real dynamics of the  
 126 brain. It is possible to recover part of this information using time delay embeddings based on the Takens’s theorem [47].  
 127 This methodology allows the reconstruction of the dynamical system in an alternative space, different from the sensors  
 128 one, but containing the same dynamical information as the brain.

129 Our method applies Takens’ theorem over the minimally pre-processed epoch-cropped time series, which allows for the  
 130 understanding of a system’s multi-variable dynamics through a single observable in a  $\Psi$ -dimensional space by employing

131 a phase embedding transformation as the initial function in our neural network [47–49]. This is made by constructing a  
 132 phase space using a delay vector constructed from the original signal, thereby enabling the reconstruction of the system  
 133 dynamics from these observations. Consider an embedding delay  $\tau$  and a delay function  $d_p(\mathbf{x}) = x(t - p\tau, t + T - p\tau)$ ,  
 134 the delay vector is defined as:

$$D_{\Psi-1}(\mathbf{X}) = [d_0(\mathbf{x}_1), d_1(\mathbf{x}_1), \dots, d_{\Psi-1}(\mathbf{x}_1), d_0(\mathbf{x}_2), \dots, d_{\Psi-1}(\mathbf{x}_2), \dots, d_0(\mathbf{x}_C), \dots, d_{\Psi-1}(\mathbf{x}_C)]^\top \quad (1)$$

135 where  $\Psi$  represents the embedding dimension, which dictates the order of magnitude of the phase space, and  $\tau$  is the  
 136 embedding delay, determining the temporal resolution of our analysis [50, 51]. As a result,  $D_{\Psi-1}(\mathbf{X})$  embodies an  
 137 embedding of the original phase space into a higher-dimensional space, enabling a detailed examination of the system’s  
 138 dynamics.

139 Consider EEG window signals captured during a cognitive task as  $\mathbf{X}_i$ , and consider this signal produced by a nonlinear  
 140 dynamical system. To create the higher dimensional space, we use the methodology  $f_{\text{delay}}(\mathbf{X})$  proposed by Carrara  
 141 and Papadopoulo [7], consisting of sliding measures over the observable with a subsequent concatenation. We define  
 142 the lag function  $s_p(\mathbf{x}) = x(t + p\tau, t + T - (\Psi - p)\tau)$ , and given the parameters  $\psi$  and  $\tau$ ,  $f_{\text{delay}}(\mathbf{X})$  is defined as:

$$f_{\text{delay}}(\mathbf{X}) = [s_0(\mathbf{x}_1), \dots, s_{\Psi-1}(\mathbf{x}_1), \dots, s_0(\mathbf{x}_C), \dots, s_{\Psi-1}(\mathbf{x}_C)]^\top \quad (2)$$

143 This function is applied on each window signal  $\mathbf{X}_i \in \mathcal{X} = \mathbb{R}^{C \times T}$ .

144 Selecting the optimal  $\tau$  and  $\psi$  values can be extremely time-consuming [7, 51]. To overcome this difficulty, we  
 145 use an adaption of Maximizing Derivatives On Projection (MDOP) [7, 52] method for the multi-epoch context of  
 146 EEG Decoding, shown in Algorithm 1. The algorithm relies on the function *mdop\_embedding* from the package  
 147 DelayEmbeddings to implement the MDOP procedure. The function aims to create a nonuniform embedding that  
 148 generates a vector of different lags  $\tau_i$ .

---

#### Algorithm 1 MDOP for EEG Decoding

---

```

using DelayEmbeddings: mdop_embedding
2: function MDOP FOR EPOCHS( $X$ )
     $\tau = 0$ 
4:    $\psi = 0$ 
    for  $i = 1 : \text{\#epochs}$  do
6:      $\tau_i = \text{mdop\_embedding}(X_i)$ 
      $\tau = \tau + \lfloor \tau_i \rfloor$ 
8:      $\psi = \psi + |\text{len}(\tau_i)|$ 
    end for
10:   $\bar{\tau}_{Epoch} = \lfloor \frac{\tau}{|\text{\#epochs}|} \rfloor$ 
      $\bar{\psi}_{Epoch} = \lfloor \frac{\psi}{|\text{\#epochs}|} \rfloor$ 
12: return  $\bar{\tau}_{Epoch}, \bar{\psi}_{Epoch}$ 
end function
    
```

---

149 The MDOP algorithm emerged as a novel geometric for analyzing dynamical systems, as opposed to conventional  
 150 statistical and information-theoretic methods, such as mutual information and continuity statistics. Central to MDOP  
 151 is optimizing an embedding to ensure the reconstructed attractor is as expanded as possible while simultaneously  
 152 reducing redundancy among the delay components. The algorithm employs a recursive approach, wherein each step of  
 153 the embedding cycle identifies the lag  $\tau$  that yields the highest beta statistics. This chosen lag is then utilized in the  
 154 subsequent reconstruction phase. This iterative process continues until the algorithm obtains satisfying embedding  
 155 dimensions  $\psi$ , selected using the method of false nearest neighbors.

### 156 3.3 Riemannian Manifold

157 Symmetric Positive Definite (SPD) matrices have begun to play a key role in several applications, ranging from  
 158 brain imaging to Computer Vision [53, 54]. In particular, the classification approach of SPD matrices based on the  
 159 Riemannian distance algorithm is the current state-of-the-art in the BCI-MI classification [43, 55].

160 Let’s define  $\mathcal{M}_n$  the space of real square matrix and  $\mathcal{S}^n$  the space of symmetric matrix, where  $\mathcal{S}^n = \{S^n \in \mathcal{M}_n \mid$   
 161  $S^\top = S\}$ . It is now possible to define the space of Symmetric Positive Definite (SPD) matrices as

$$\mathcal{S}_{++}^n = \{S \in \mathcal{S}^n \mid \mathbf{x}^\top S \mathbf{x} > 0 \quad \forall \mathbf{x} \in \mathbb{R}^n\} \quad (3)$$

162 This formulation allows us to represent matrices belonging to  $\mathcal{S}_{++}^n$  as points on a Riemann manifold with a dimension  
 163 of  $n(n+1)/2$ . The space of SPD matrices forms a manifold with negative curvature [56, 57], so Euclidean geometry  
 164 concepts do not apply.

165 It is possible to define several distances between two SPD matrices  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , generally depending on the length of the  
 166 geodesic connecting  $\mathcal{S}_1$  and  $\mathcal{S}_2$  on the Riemann manifold. Often, the affine-invariant metric is used in the context of BCI.  
 167 We proceed, however, to give a mathematical formulation based on the Log-Euclidean metric [58]. This formulation  
 168 reduces the computational burden associated with the affine-invariant framework while preserving robust theoretical  
 169 properties [58].

170 The space of Symmetric Positive Definite (SPD) matrices can be endowed with a Lie group structure. For a comprehen-  
 171 sive understanding, please refer to [58]. It becomes possible to define a distance metric between two SPD matrices  $\mathcal{S}_1$   
 172 and  $\mathcal{S}_2$  as the bi-variate metric on the Lie group of SPD matrices

$$d(\mathcal{S}_1, \mathcal{S}_2) = \|\log(\mathcal{S}_2) - \log(\mathcal{S}_1)\| \quad (4)$$

173 where  $\|\cdot\|$  is the norm associated with the metric, and  $\log$  is the matrix logarithm. The Log-Euclidean metric on the  
 174 Lie group of SPD matrices corresponds to an Euclidean metric within the logarithmic domain of the SPD matrices.

175 Distances, geodesics, and Riemannian means exhibit a more straightforward formulation within the Log-Euclidean  
 176 metric than the affine-invariant case, maintaining comparable invariance properties. However, this simplification comes  
 177 at the cost of more intricate formulations for exponential and logarithmic mapping. The mapping of  $\mathcal{S}_2$  respect to  $\mathcal{S}_1$  is  
 178 defined as

$$\begin{aligned} \log_{\mathcal{S}_1}(\mathcal{S}_2) &= D_{\log(\mathcal{S}_1)} \exp \cdot (\log(\mathcal{S}_2) - \log(\mathcal{S}_1)), \\ \exp_{\mathcal{S}_1}(\mathcal{S}_2) &= \exp(\log(\mathcal{S}_1) + D_{\mathcal{S}_1} \log \cdot \mathcal{S}_2). \end{aligned} \quad (5)$$

179 where  $D_{\log(\mathcal{S}_1)} \exp$  represent the differential at point  $\mathcal{S}_1$  of the exponential function and similarly for  
 180  $(D_{\log(\mathcal{S}_1)} \exp)^{-1} = D_{\mathcal{S}_1} \log$ . This formulation can be simplified if the reference matrix for the mapping is the  
 181 identity matrix.

182 In BCI, the spatial covariance is estimated from the pre-processed EEG signal  $\mathbf{X} \in \mathbb{R}^{C \times T}$ . Several estimators can be  
 183 used to estimate covariance [59], but the most popular is the sample covariance matrix

$$f_{\text{Cov}}(\mathbf{X}_i) = \frac{1}{T-1} \sum_{i=1}^T \mathbf{X}_i \mathbf{X}_i^\top \quad (6)$$

### 184 3.3.1 Symmetric Positive Definite Neural Networks Components

185 In the context of geometry neural networks, Huang and Van Gool [8] proposed the neural network SPDNet with three  
 186 SPD layers:

187 **BiMap Layer:** The bi-linear mapping level aims at creating more compact and discriminating SPD matrices just like  
 188 the convolutional level, with the complication, however, that SPD matrices live in a Riemannian space. We can describe  
 189 the equation as

$$f_{\text{BiMap}}(\mathbf{Z}_{k-1}) = \mathbf{Z}_k = \mathbf{W}_k \mathbf{Z}_{k-1} \mathbf{W}_k^\top, \quad (7)$$

190 where  $\mathbf{Z}_{k-1} \in \mathcal{S}_{d_{k-1}}^{++}$  and  $\mathbf{W}$  is, for this layer, the learnable parameter. Because the output  $\mathbf{Z}_k$  must belong to the SPD  
 191 space, we require that  $\mathbf{W}_k \in \mathbb{R}^{(d_k, d_{k-1})}$  belongs to a compact Stiefel  $St(d_k, d_{k-1})$  manifold.

192 **ReEig Layer:** This layer introduces a non-linearity with a similar approach to a Rectified Linear Unit (ReLU) level.  
 193 In practice, it rectifies SPD matrices by thresholding small eigenvalues to  $\varepsilon$ .

$$f_{\text{ReEig}}(\mathbf{Z}_{k-1}) = \mathbf{Z}_k = \mathbf{U}_{k-1} \max(\varepsilon \mathbf{I}, \Sigma_{k-1}) \mathbf{U}_{k-1}^\top, \quad (8)$$

194 where  $\mathbf{U}_{k-1}$  and  $\Sigma_{k-1}$  are not learned but obtained using eigenvalue decomposition of the previous layer,  $\mathbf{Z}_{k-1} =$   
 195  $\mathbf{U}_{k-1} \Sigma_{k-1} \mathbf{U}_{k-1}^\top$ . This layer does not present any trainable parameter.

196 **LogEig:** This layer aims to transport the SPD matrix obtained from the previous layers from a Riemannian space to  
 197 an Euclidean one using the Log-Euclidean metric [58]. This is formally the expression (5) considering the mapping  
 198 with respect to the identity, so formally, the layer is defined as

$$f_{\text{LogEig}}(\mathbf{Z}_{k-1}) = \mathbf{Z}_k = \log(\mathbf{Z}_{k-1}) = \mathbf{U}_{k-1} \log(\Sigma_{k-1}) \mathbf{U}_{k-1}^\top \quad (9)$$

where again, the  $\mathbf{U}_{k-1}$  and  $\Sigma_{k-1}$  are not learned but obtained using eigenvalue decomposition. This layer not present any trainable parameter. This logarithmic mapping is applied using the Identity as a reference matrix. Once this layer is applied, we can use the classical deep learning method in the Euclidean space as a Multi-Layer Perceptron.

Given the framework, we define our neural network, denoted as  $f$ , as a composition of sequential transformations:

$$f_{\text{SPDNet}\psi} = f_{\text{delay}} \circ f_{\text{Cov}} \circ f_{\text{BiMap}} \circ f_{\text{ReEig}} \circ f_{\text{LogEig}} \circ f_{\text{MLP}}, \quad (10)$$

where each  $f_i$  represents a specific transformation or layer within the network. The learning process is formalized as the mapping  $f_{\text{SPDNet}\psi} : \mathcal{X} \rightarrow \mathcal{Y}$ , which operates on the training dataset.  $\theta$  denotes the set of parameters within the parameter space  $\Theta$ . The optimization objective is to minimize an average loss  $\ell$  over the training dataset, defined as:

$$\min_{\theta} \frac{1}{N} \sum_{i=1}^N \ell(f_{\text{SPDNet}\psi}(\mathbf{X}_i), \mathbf{y}_i). \quad (11)$$

The model’s generalization capability is further assessed using an independent test set.

In a DL typical training process, optimization methods like Adam [60] are normally used for implementing the back-propagation procedure. However, in the context of SPDNet, we must the weights of  $W$  of the BiMap layer in such a way that the new weights still belong to the Stiefel manifolds, i.e., they are still orthonormal matrices. To solve the problem, we use the RiemannAdam [61] optimization from the geopt [62] library. In addition to the optimization strategy, we use the standard cross-entropy loss function  $\ell$  as a loss function, a well-established and widely used objective function in classification tasks.

To summarize, the resulting architecture SPDNet $_{\psi}$  uses enriched SPD matrices that encapsulate a broader spectrum of information than their traditional counterparts. In addition, such an approach allows, through the use of nonlinear systems theory, the reconstruction of a phase space that contains more information with respect to the one extracted from the original signal, allowing the use of fewer electrodes. The augmented SPD matrices require an SPDNet with a larger number of parameters, which is partly counteracted by using fewer electrodes. In particular, this representation accentuates how the input SPD matrix is modified from the initial dimension  $\mathcal{S}_C^{++}$  to  $\mathcal{S}_{C \times \psi}^{++}$ .

Overall, this enrichment of the information contained in the SPD matrices not only expands the range of discriminative information that can be extracted but also forces the network to adapt to more intricate and fuzzy patterns in the data. Figure 1 provides a graphical picture of our methodology.

In the case of SPDNet $_{\psi}$ , we set the subspace dimension of the BiMap layer at half of the input dimension [8]. However, for standard SPDNet, we maintained the subspace dimension at the same value as the original input dimension (although in this case the BiMap layer acts only as a rotation) in order to proceed with subsequent analyses concerning the explainability of the model. Anyway, diminishing the subspace dimension of half in the case of SPDNet consistently led to a reduction in performance. Note that the current implementation of SPDNet has a ReEig layer that is not scale-independent because of the parameter  $\varepsilon = 10^{-4}$ . To enhance the architecture’s scale independence, we have introduced a standardization procedure for the raw signal, bringing every channel to a zero mean and a unit standard deviation.

### 3.4 Datasets

In order to assess the replicability of our approach, we used six different open datasets from MOABB [10] consisting of almost 100 subjects as shown in Table 1.

It turns out that it is essential to employ different datasets to obtain reliable results, especially when evaluating different ML techniques in EEG decoding, in order to avoid the bias of a single data set [13].

Dataset	Subjects	Channels	Sampling Rate (Hz)	Sessions	Tasks	Trials/Class	Epoch (s)
BNCI2014001 [63]	9	22	250	2	4	144	[2, 6]
BNCI2014004 [64]	9	3	250	5	2	360	[3, 7.5]
Cho2017 [65]	52	64	512	1	2	100	[0, 3]
Schirrneister2017 [66]	14	128	500	1	4	120	[0, 4]
Weibo2014 [67]	10	60	200	1	7	80	[3, 7]
Zhou2016 [68]	4	14	250	3	3	160	[0, 5]

Table 1: Motor Imagery datasets considered during this study

We employed standard pre-processing steps designed for all datasets described in [10]. We applied band-pass filtering with the overlap-add method between [8 – 32] Hz, artifact rejection, and electrode standardization, bringing every

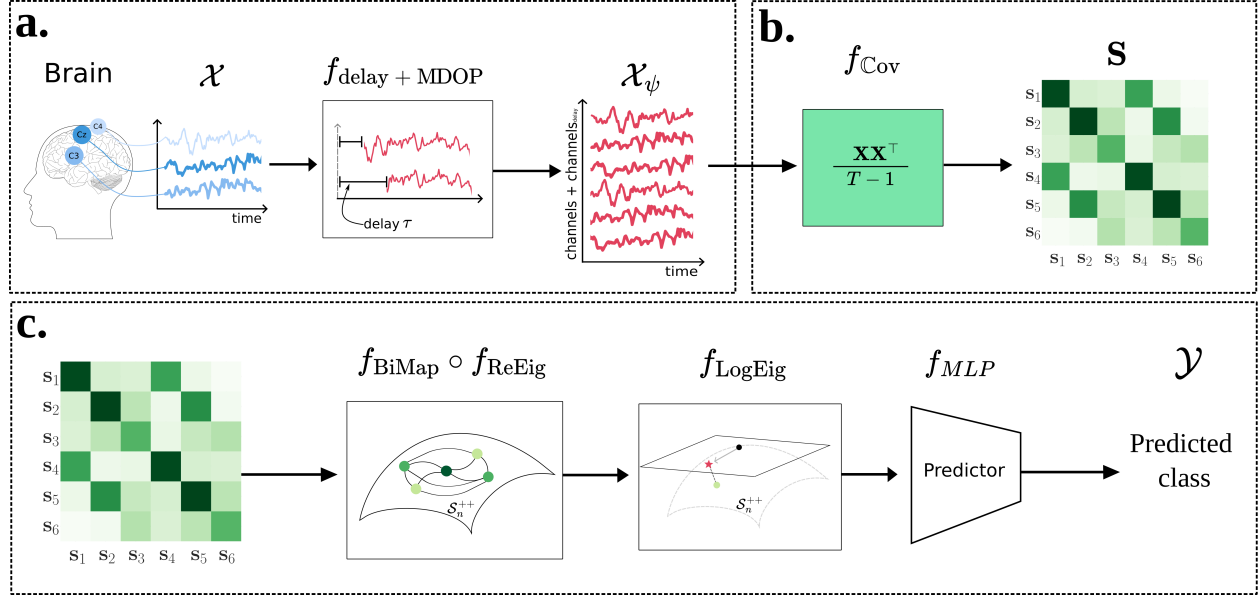


Figure 1: *Overview of our approach.* **A.** Phase space reconstruction for the EEG signal. For each trial from  $\mathbf{X}$ , we apply the function  $f_{delay}$  to reconstruct the phase space. During each evaluation fold, we adopted the MDOP algorithm to estimate suitable function parameters for embedding dimension  $\psi$  and the embedding delay  $\tau$ . For illustrative purposes, this figure highlights the central cerebral cortex electrodes. **B.** Conversion of the phase space time-series with  $f_{Cov}$  for the covariance representation of the Symmetric Positive Definite Space  $\mathcal{S}_{++}^n$ , represented in green. **C.** The covariance feature space is then used to train the Symmetric Positive Definite Neural Network with BiMap, ReEig, LogEig, and MLP layers. Once the representation returns to the Euclidean space (red), after the LogEig, we adjust a fully connected layer to predict the associate label  $\mathcal{Y}$  for each trial.

237 channel to a zero mean and a unit standard deviation. Additionally, we opted to use the complete epoch duration,  
 238 although this duration varies among datasets, as shown in Table 1.

239 We concentrate on a binary classification task, aiming to discern between imagined movements of the right and left  
 240 hand using an intra-subject/within-session evaluation, which is based on a 5-fold cross-validation is conducted in each  
 241 session.

242 Additionally, we focus on investigating our algorithm’s robustness with fewer electrode scenes. For this purpose, we  
 243 selected a consistent set of three electrodes (C3, Cz, and C4) across datasets, all adhering to the 10-20 standard montage.  
 244 Importantly, these electrodes are situated directly above the motor cortex, an area concentrating the most relevant  
 245 information for MI-BCI classification [69].

### 246 3.5 Baseline Comparison

247 We compare the performance of our pipeline to that of several state-of-the-art DL Neural Networks used for BCI. All  
 248 algorithms in our study were tested using the same set of three electrodes to maintain uniformity in our experimental  
 249 approach. To standardize the models’ parameters, we initially resample the input time series to align the signal with the  
 250 state of the art in order to use the parameter’s architecture of the original paper.

- 251 1. **ShallowNet** [14], a neural network architecture with independent spatial and temporal convolution steps with  
 252 Relu activation, with standardized and re-sampled EEG signal at 250Hz.
- 253 2. **DeepNet** [14], an approach similar to ShallowNet with a deeper linear layer, with standardized and re-sampled  
 254 EEG signal at 250Hz.
- 255 3. **EEGNet** [18], have a depth-wise convolutional layer functions as a spatial filter across channels, complemented  
 256 by a separable convolution layer for features extraction designed for EEG with a sample frequency of 128Hz.
- 257 4. **EEGTCNet** [19], is a neural network that wrapper the EEGNet and includes a Temporal Convolution Network  
 258 over the embedded representation, with standardized and re-sampled EEG signal at 250Hz.



259 5. **EEGITNet** [20], a neural network inspired by InceptionNet with parallel convolution layers with different  
 260 scales. The parameters network has been designed to EEG signals at 128Hz.

261 6. **EEGNeX** [21], a neural network inspired by EEGNet incorporation the key components from ConvNeXt. The  
 262 parameters network has been designed to EEG signals at 128Hz.

263 These methods were implemented and trained using MOABB version 1.0 [10]. For in-depth information about the  
 264 deep learning hyper-parameters, please consult Table A2. In order to obtain a fair time comparison, all the following  
 265 results are computed on the same hardware, a Dell C6420 dual-Xeon Cascade Lake SP Gold 6240 @ 2.60GHz.

## 266 4 Results

267 In the following section, we analyze the results produced using within-session (WS) evaluation. This method is based  
 268 on a 5-fold cross-validation performed on each session independently. In particular, we report two separate analyses.

269 The first analysis is dedicated to contrasting the performance of  $\text{SPDNet}_\psi$ , with a specific focus on the exploitation of  
 270 the covariance feature, against the state-of-the-art techniques in DL applied in EEG. The subsequent analysis, reported  
 271 in the Appendix, shifts the focus towards the exploration of coherence as a feature for classification. Despite the ongoing  
 272 investigations into coherence, we deliberately separated this analysis due to its comparatively lesser performance  
 273 against the benchmarks achieved with covariance features. Nevertheless, the augmentation procedure is a significant  
 274 methodology to improve the performance of both feature sets.

275 For the first analysis, we tested the performance of  $\text{SPDNet}_\psi$  against the baseline comparison defined in 3.5 across  
 276 a broad spectrum of datasets. The results are listed in Table 2, while the detailed statistical analysis can be found in  
 277 Figure 2.

278 Table 2 shows ROC AUC results for right-hand versus left-hand classification, where bold numbers represent the best  
 279 score in each dataset, revealing a consistent trend wherein our algorithm  $\text{SPDNet}_\psi$  surpasses the DL state-of-the-art in  
 280 the context of reduced datasets. This behavior is true for all datasets, with the notable exception of the Schirrmeister2017  
 281 dataset, where it is marginally surpassed by ShallowNet and DeepNet. It is important to note that the results of the latter  
 282 two, however, are not statistically significant for this dataset. As the number of subjects per dataset is too small to reach  
 283 decent statistical power for investigating the differences between pipelines, the results across datasets are aggregated to  
 284 conduct a meta-analysis. Our  $\text{SPDNet}_\psi$  yields statistically better results than all other approaches as shown in Fig. 2  
 285 (b), where only significant interactions are displayed ( $p > 0.05$ ), the reported values and color show the  $t$ -values.

286 The second relevant point is to see the impact of the augmentation procedure compared to the standard SPDNet. Figure 2  
 287 (a) elucidates this aspect by illustrating the relative performance enhancements of various models against the SPDNet  
 288 standard. Notably, ShallowConvNet and DeepConvNet exhibit positive performance increments across the majority of  
 289 the datasets examined. However, the augmentation procedure, as implemented by  $\text{SPDNet}_\psi$ , not only consistently yields  
 290 positive relative improvement against the standard SPDNet across all datasets but also the improvement introduced  
 291 by the augmentation procedure turns out to be so significant that it even outperforms methods that initially surpassed  
 292 the SPDNet benchmark. This improvement is not only demonstrated by the relative improvement with respect to the  
 293 SPDNet standard but also is evident through the meta-analysis comparison (refer to Fig 2 (c), (d) and (e)).

294 Ultimately, the statistical analysis conducted using Figure 2 (b) allows us to demonstrate that  $\text{SPDNet}_\psi$  outperforms the  
 295 state-of-the-art DL methodology in a reduced dataset context. Its versatility and reliability, particularly in data-limited  
 296 scenarios, position the method as a promising solution for real-world applications.

Table 2: Average AUC-ROC (%) on six datasets (BCNI2014-01, BCNI2014-04, Cho2017, Schirrmeister2017, Weibo2014, and Zhou2016), for a left vs. right motor imagery task. The highest performances are in bold-face.

Models	BCNI2014001	BCNI2014004	Cho2017	Schirrmeister2017	Weibo2014	Zhou2016
DeepNet [14]	75.80 ± 15.45	72.80 ± 19.48	63.13 ± 14.45	73.04 ± 15.67	73.97 ± 18.07	91.74 ± 7.00
ShallowNet [14]	75.85 ± 15.41	72.17 ± 18.61	64.14 ± 13.03	<b>73.59 ± 15.19</b>	75.36 ± 15.69	88.03 ± 8.55
EEGNet [18]	70.64 ± 19.87	70.27 ± 18.91	60.23 ± 14.98	69.80 ± 16.51	71.94 ± 17.80	88.95 ± 7.84
EEGTCNet [19]	65.98 ± 17.25	66.86 ± 18.41	56.43 ± 12.31	67.87 ± 17.36	65.94 ± 15.69	81.47 ± 11.66
EEGITNet [20]	66.64 ± 13.98	64.93 ± 14.49	54.68 ± 11.97	62.98 ± 16.19	56.97 ± 17.66	72.82 ± 12.78
EEGNeX [21]	68.86 ± 17.27	68.29 ± 17.85	56.64 ± 12.83	64.26 ± 17.58	58.73 ± 19.40	80.25 ± 15.55
SPDNet [8]	71.02 ± 15.64	70.15 ± 16.85	59.95 ± 12.61	67.40 ± 13.01	67.04 ± 17.62	88.85 ± 8.05
<b>SPDNet<sub>ψ</sub> (Our)</b>	<b>75.98 ± 17.00</b>	<b>80.46 ± 16.64</b>	<b>66.00 ± 13.29</b>	72.02 ± 13.07	<b>78.01 ± 19.64</b>	<b>94.92 ± 3.34</b>

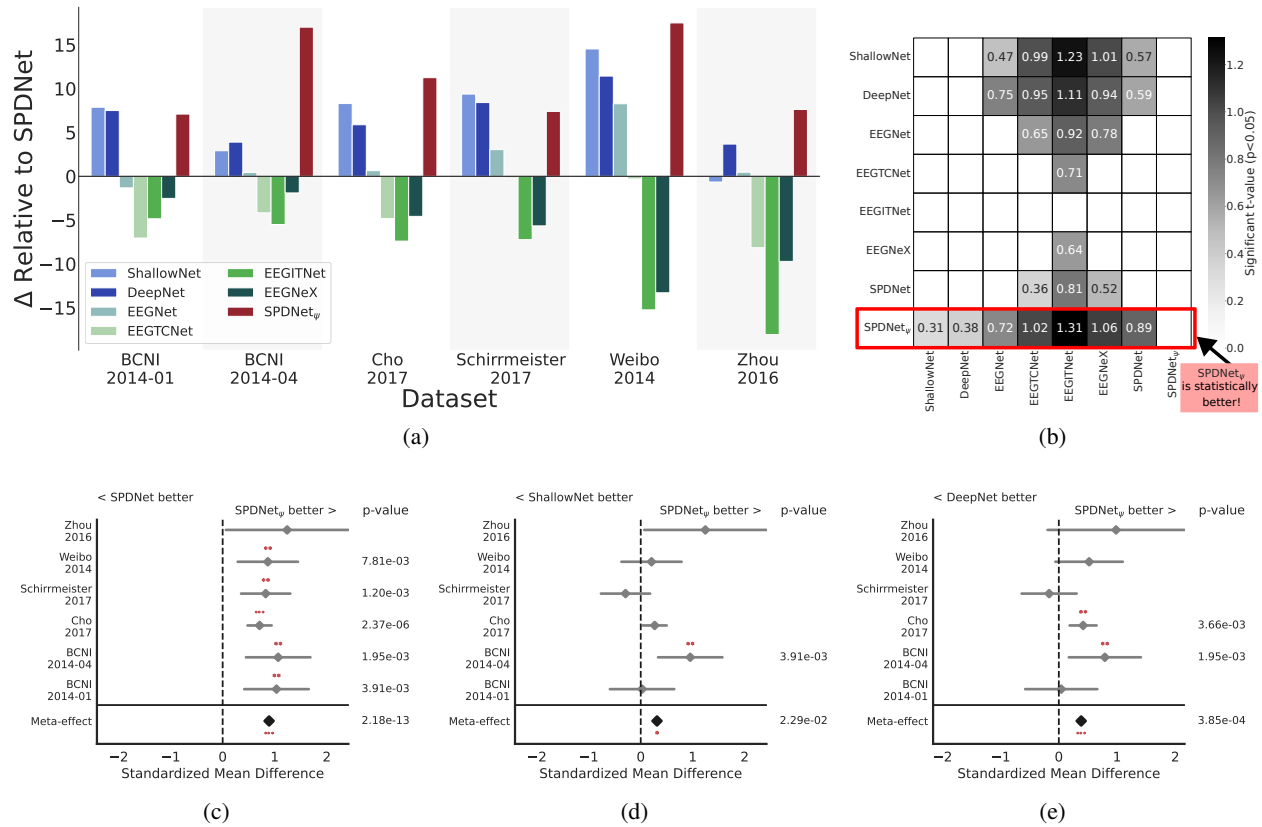


Figure 2: Results for Right vs Left-hand classification, using Within-Session evaluation. Plot (a) provides the relative improvement of the AUC-ROC in percentage of the method considered with respect to the standard SPDNet of the different pipelines considered. Plot (b) shows a combined meta-analysis (over all datasets) of the different pipelines. It shows the significance of the algorithm on the y-axis being better than the one on the x-axis. The gray level represents the significance level of the ROC-AUC difference in terms of t-values. We only show significant interactions ( $p < 0.05$ ). Plots (c), (d), and (e) show the meta-analysis of SPDNet $_{\psi}$  against SPDNet, ShallowNet, and DeepNet, respectively. We show the standardized mean differences of p-values computed as a one-tailed Wilcoxon signed-rank test for the hypothesis given in the plot title. The gray bar denotes the 95% interval. \* stands for  $p < 0.05$ , \*\* for  $p < 0.01$ , and \*\*\* for  $p < 0.001$ .

297 **5 Discussion**

298 **5.1 Impact of augmentation on most and least responsive 5 subject of Cho2017**

299 In the previous section, we explored the outcome obtained by SPDNet $_{\psi}$  within the context of a reduced number of  
 300 electrodes. Now, our attention shifts to a deeper exploration of this new algorithm, with a specific emphasis on the  
 301 interpretability and computational performance of the method. To facilitate this comprehensive analysis, still in the  
 302 context of three electrodes, we have chosen to concentrate on the examination of the five most and least responsive  
 303 subjects within the Cho2017 dataset (the wider dataset in terms of the number of subjects), selected using the Minimum  
 304 Distance to the Mean (MDM) algorithm [55] with covariance as a feature.

305 To analyze the performance, we use different pipelines, MDM [55], AugMDM [7], SPDNet, and SPDNet $_{\psi}$ , each  
 306 offering a distinct perspective on the impact of augmentation techniques on both traditional Machine Learning (ML)  
 307 and advanced Deep Learning (DL) algorithms. Furthermore, we assess the performance using three different SPD  
 308 features: covariance, instantaneous coherence, and imaginary coherence (Figure 3).

309 For the five most responsive subjects, covariance stands out as the top-performing feature as it could be expected.  
 310 However, in the case of the five least responsive subjects, this dominance of covariance is less clear when considering  
 311 the state-of-the-art methodology.

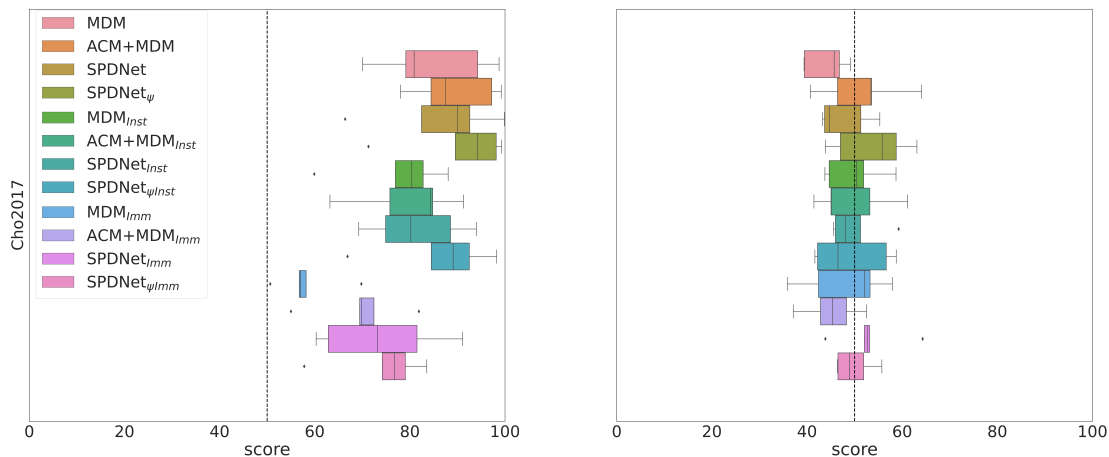


Figure 3: (a) Plot showing the box plot of the five most responsive subjects (3, 14, 35, 41, 43) of Cho2017. We see that the augmentation procedure consistently leads to notable improvements in the classification performance for all the SPD estimators considered. (b) Plot showing the box plot of the least responsive five subjects (2, 7, 29, 34, 50) of Cho2017. We see that the augmentation procedure increases the performance for the covariance feature, while the improvement is less clear for the FC estimator.

312 The outcomes of our study are in concordance with the state-of-the-art [9]: results achieved using Covariance as SPD  
 313 feature outperform those obtained with the Functional Connectivity (FC) estimator.

314 It becomes evident that the augmentation procedure consistently leads to notable improvements in the classification  
 315 performance for the top-performing five subjects. However, the situation becomes less definitive for the least responsive  
 316 five subjects, where the augmentation effect is not as pronounced or evident. In this case, their signals exhibit a lower  
 317 SNR, which can substantially impact the MDOP algorithm, leading to a poor estimation of the hyper-parameter. In  
 318 this case, the augmentation procedure achieves a very positive impact on covariance while there is even a decrease in  
 319 performance using coherence.

## 320 5.2 Interpretability

321 Creating explainable algorithms is crucial in the field of DL, and it holds even greater significance in healthcare data. In  
 322 healthcare, understanding why an algorithm makes specific decisions is of great importance for clinicians and patients,  
 323 ensuring transparency, accountability, bias mitigation, education, and trust-building in the collaboration between AI  
 324 systems and medical experts.

325 For analyzing the explainability of the  $\text{SPDNet}_\psi$  we use the GradCam++ algorithm [70], a technique designed for  
 326 visualizing and explaining the decision-making process of neural networks. The GradCam++ output is essentially a  
 327 visual representation that reveals which elements within the data are the most important for the model. GradCam++ has  
 328 proven to be a robust method for spatial interpretability of EEG signals in a class-agnostic manner [71].

329 In our analysis, we focus on Subject 41 from the Cho2017 dataset, which is notable for being among the top 5 subjects  
 330 that exhibit the most substantial performance improvement through the augmentation procedure. Similar findings also  
 331 hold for other subjects. In particular, using covariance as a feature, we show a sample of the test dataset that was  
 332 correctly classified by the  $\text{SPDNet}_\psi$  algorithm (left hand classified as left hand) but was misclassified by SPDNet 4  
 333 using the same feature. Similar findings have been observed for the FC estimators. This figure represents the output of  
 334 the GradCam++ applied to the output of the ReEig layer.

335 In all three scenarios, the standard SPDNet places more emphasis on the diagonal terms of SPD matrices. On the  
 336 other side, when employing  $\text{SPDNet}_\psi$ , a more comprehensive picture unfolds. While diagonal terms continue to play a  
 337 pivotal role, what becomes increasingly evident is the significant contribution of the off-diagonal terms. In essence, this  
 338 observation emphasizes that the  $\text{SPDNet}_\psi$  model leverages both individual electrode characteristics (diagonal terms)  
 339 and the interplay between electrodes in time and space (off-diagonal terms) as crucial elements in its classification  
 340 decision-making process.

341 To delve deeper into the interpretability of our proposed network, we employ a paired t-test to highlight statistically  
 342 significant differences in the augmented covariance matrix (ACM) between left and right hands. Our analysis centers

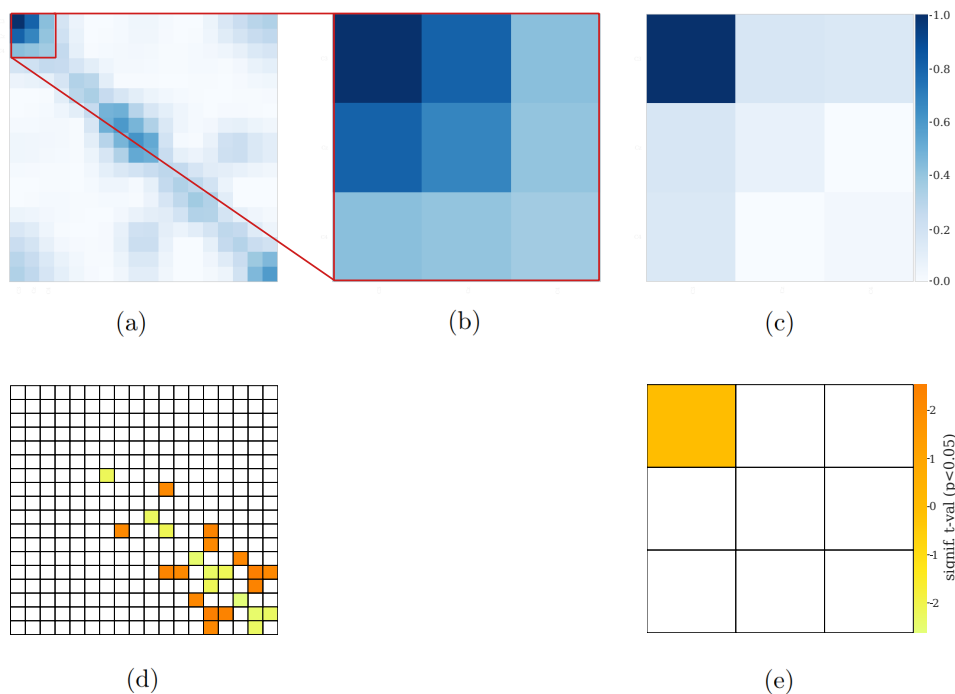


Figure 4: Interpretability parallel between GradCam++ and t-test between right hand left-hand classification for subject 41 of Cho2017. Plot (a) is the GradCam++ visualization for Augmented covariance. Plot (b) is a zoom in the region corresponding to normal Covariance. Plot (c) is the GradCam++ visualization for covariance. Plot (d) is the t-test of the Augmented covariance matrix. Plot (e) is the t-test of the covariance matrix. The t-test visualization represents only the significant interactions ( $p < 0.05$ ).

343 on subject 41 from Cho2017, specifically on the test dataset of the initial fold of the cross-validation, with consistent  
 344 findings observed in the other folds. Given that 20 samples are available for each imagined movement, we are in the  
 345 domain of application of the t-test.

346 Examining Figure 4 (d) and (e), it is evident that SPDNet places emphasis primarily on the statistically significant pair  
 347 of electrodes, showing a clear parallel between the GradCam++ visualization and the t-test. In contrast, SPDNet <sub>$\psi$</sub>  not  
 348 only considers significant interactions but also extends its focus to nearly significant interactions.

### 349 5.3 Convergence Behavior

350 To analyze convergence behavior, we again consider subject 41 of the Cho2017 dataset. Again, to get a fair comparison,  
 351 all the following results are calculated on the same hardware, a Dell C6420 dual-Xeon Cascade Lake SP Gold 6240 @  
 352 2.60GHz.

353 We examine the outcomes concerning both training and validation metrics, including loss and Receiver Operating  
 354 Characteristic Area Under the Curve (ROC AUC) scores. The consolidated results are visually presented in Figure 5,  
 355 wherein the graphs depict the average values across the considered 5-fold cross-validation.

356 Notably, our observations reveal that the loss function of SPDNet <sub>$\psi$</sub>  converges to zero at a notably accelerated pace  
 357 and exhibits a considerably more stable trajectory compared to standard SPDNet. Moreover, while the loss function of  
 358 SPDNet <sub>$\psi$</sub>  steadily approaches zero, the standard SPDNet experiences a more gradual descent, deviating from the path  
 359 to complete convergence but ultimately plateauing around 0.4. These findings highlight that the capacity of SPDNet is  
 360 limited and that the algorithm stops learning at a certain point.

361 Interestingly, when examining the ROC AUC graph, we observe that the validation scores of SPDNet <sub>$\psi$</sub>  converge  
 362 consistently to higher values. Notably, this convergence is achieved even faster compared to the SPDNet standard,  
 363 requiring only 94% of the iterations. This implies that SPDNet <sub>$\psi$</sub>  attains a comparable performance to SPDNet but with  
 364 a significantly reduced number of iterations, making it an efficient and effective choice for real-life applications.

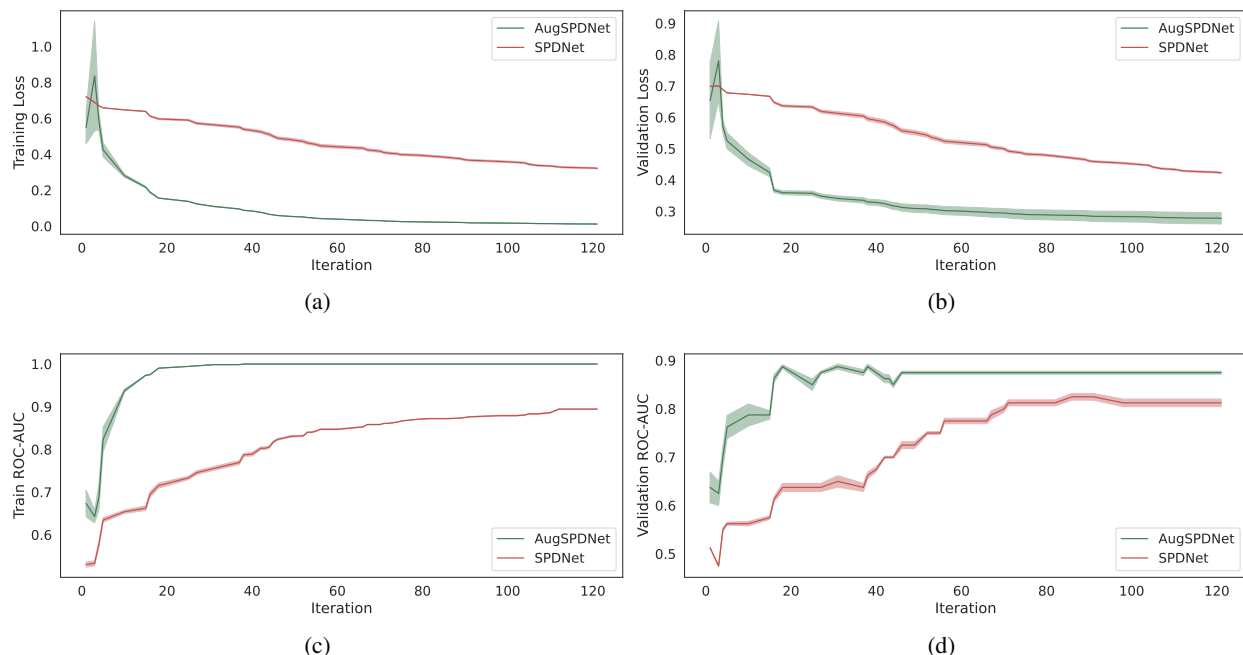


Figure 5: Convergence behavior for right-hand vs. left-hand classification using WS evaluation. The plot uses subject 41 of the Cho2017 dataset, showing the mean value for each plot computed over the 5-Fold considered in the WS evaluation. Plot (a) provides the training loss, while plot (b) is the validation one. Plot (c) provides the ROC-AUC for the training phase, while plot (d) the one for the validation.

### 365 5.4 Computational Performances

366 In this section, we are interested in the analysis of computational performance, with a specific focus on two key aspects:  
 367 execution time and environmental impact. The results on the best 5 subjects of Cho2017 (subjects 3, 14, 35, 41, 43) are  
 368 presented in Table 3.

369 Recently, much attention has begun to be given to the sustainability of ML and AI algorithms [72]. Since more and more  
 370 models continue to proliferate and demand substantial computational resources, it is crucial to estimate and mitigate the  
 371 energy consumption and emissions associated with their training and deployment. To assess the environmental impact  
 372 of our algorithm, we use the CodeCarbon [73] library. CodeCarbon is an open-source library that enables developers to  
 373 monitor carbon dioxide ( $CO_2$ ) emissions, offering a way to analyze the carbon footprint of our work.

374 Interestingly our analysis shows that the SPDNet<sub>q</sub> exhibits the longer computational time among the evaluated algorithms  
 375 while simultaneously standing out as one of the most environmentally friendly in terms of its  $CO_2$  impact (see Fig 6).  
 376 This intriguing juxtaposition can be attributed to the fact that, during each fold of the cross-validation process, there's a  
 377 necessity to optimize the model's hyper-parameters, which naturally augments the computational time. However, the  
 378 MDOP algorithm consumes fewer resources than gradient descent procedures, thus resulting in a comparatively smaller  
 379 environmental impact. In fact, the use of augmented procedure introduces the complication of two hyper-parameters but  
 380 nevertheless manages to keep the number of trainable parameters of the model contained (see Table 4). It's plausible  
 381 that our algorithm has yet to reach its optimal performance. Further investigation is needed to improve the optimization  
 382 strategies.

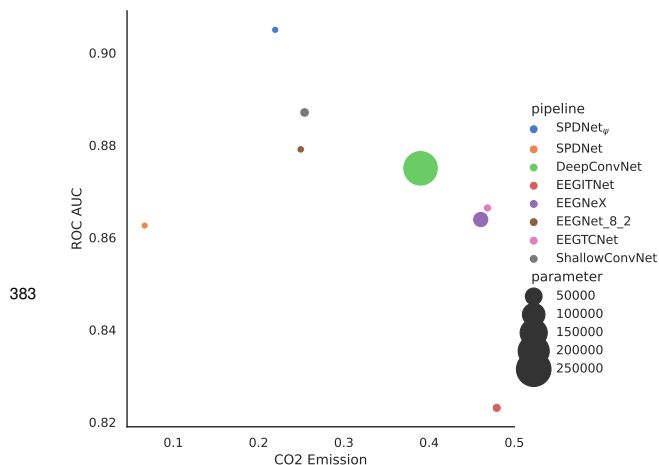


Figure 6: Performance (ROC AUC) versus  $CO_2$  consumption (g  $CO_2$  equivalent). The size of the blob indicates the number of trainable parameters.

	Avg. Time (s)	$CO_2$ Emission (g $CO_2$ )
DeepNet	21.77	0.39
ShallowNet	15.10	0.25
EEGNet-8,2	14.96	0.25
EEGITNet	28.44	0.48
EEGTCNet	28.83	0.47
EEGNeX	28.15	0.46
SPDNet	6.80	0.07
SPDNet $_{\psi}$	30.16	0.22

Table 3: Average computational time and  $CO_2$  equivalent consumption over the best five subjects of Cho2017

Stage	Layer	Output	Parameters
Input		[1, 3, 1537]	0
Augmentation	AugmentedDataset	[1, 18, 1393]	0
Covariance	Covariances	[1, 18, 18]	0
SPDNet	BiMap	[1, 9, 9]	162
	ReEig	[1, 9, 9]	0
	LogEig	[1, 45]	0
Classification	Linear	[1, 2]	45
Total			207

Table 4: Architecture of SPDNet $_{\psi}$  with parameter number depending on  $\phi$ . Here, it is reported for the specific case of order  $\phi = 6$ .

## 5.5 Limitation and Future Directions

The existing SPDNet $_{\psi}$ , as employed thus far, utilizes MDOP for hyper-parameter selection, offering computational efficiency compared to grid-search methods [7]. However, the solutions obtained with MDOP were sub-optimal compared with using a more extensive hyper-parameter selection. A potential avenue for future exploration involves streamlining and accelerating the grid-search process to harness the capabilities of the SPDNet $_{\psi}$  architecture fully.

Further enhancements could be achieved by adopting more intricate SPDNet architectures, such as incorporating batch normalization, exploring different BiMap layers, or experimenting with a Bottleneck architecture.

An intriguing direction arises from the insights gained in the explainability study, revealing that SPDNet $_{\psi}$  predominantly focuses on diagonal elements. This suggests the possibility of employing a Region of Interest approach, where the network initially prioritizes the main diagonals while treating the remainder of the matrix as background.

Continued improvements could be realized through comprehensive algorithm testing across diverse datasets, tasks, and evaluation procedures. This approach allows for a better understanding of performance in more complex scenarios, particularly those posed by intra- and inter-subject variabilities.

Using three electrodes only is advantageous for practical scenarios. In fact, this decision was strategically made to shorten the length of the experiment, thereby minimizing patient fatigue and maintaining their attention throughout the session.

Another challenge was the difficulty in classifying data from less responsive subjects. Despite our efforts, there remains a need for improved methods better to capture the subjects' intentions and their specific characteristics.

## 6 Conclusion

This study focuses on integrating the Augmented Covariance Matrix (ACM) with SPDNet to develop a practical algorithm intended for real-world applications, with a deliberate focus on scenarios involving a limited number of electrodes. Specifically, we restricted our electrode usage to just 3, strategically placed on the motor cortex.

In fact, the augmented methodology, based on Takens theorem, demonstrates a particularly effective performance when applied with a reduced number of electrodes. However, it is important to note that using only three electrodes can impact the results. For instance, in the BNCI2014001 dataset, employing just three electrodes led to a performance reduction of about 10% in the ShallowNet architecture compared to using all available electrodes.

Validation of our approach involves leveraging nearly 100 subjects from several open datasets, a task facilitated by the MOABB framework. The resulting SPDNet $_{\psi}$  algorithm outperforms existing DL algorithms in BCI-EEG classification and offers some explainability elements through GradCam++ visualization. Remarkably, this algorithm requires a modest number of trainable parameters and exhibits a lower environmental impact, measured by  $CO_2$  equivalent emissions.

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## 8 Open-source availability

The code will be made public during publication time under the BSD-3 License.

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## 619 A Results using Coherence as feature

620 In this section, we shift our focus to reporting results due to the use of Imaginary and Instantaneous coherence features  
 621 as input. Specifically, we aim to draw a comparison between the  $\text{SPDNet}_\psi$  results and the non DL state-of-the-art  
 622 pipelines (usually created for covariance but here applied on coherence). It’s worth noting that, in the subsequent  
 623 analysis, we abstain from comparing these results with the performance of state-of-the-art DL models, given that  
 624 the employment of coherence-based features typically yields lower results across the board. Nevertheless, we find it  
 625 particularly intriguing to present the outcomes achieved by  $\text{SPDNet}_\psi$  in the context of coherence-based features since  
 626 we observe that  $\text{SPDNet}_\psi$  demonstrates a statistically significant improvement. This shows the potential and significance  
 627 of  $\text{SPDNet}_\psi$  in the broader landscape of signal processing and pattern recognition.

- 628 1. **CohCSP+LDA** [74], a combination of the Common spatial pattern (CSP) algorithm followed by a classification  
 629 performed on a shrinkage Linear Discriminant Analysis (LDA).
- 630 2. **Coh + TS + EN** [9], a Riemannian classification method in the tangent space using an Elastic Network as  
 631 classifier.
- 632 3. **CohFgMDM** [75], a Riemannian classification method by Minimum Distance to the Mean after having  
 633 applied a geodesic filtering.
- 634 4. **CohMDM** [55], a Riemannian classification method by Minimum Distance to the Mean.
- 635 5. **Coh + TS + SVM** [55], a Riemannian classification method in the tangent space using a Support Vector  
 636 Machine (SVM).

637 Results can be found in Table A1 while the detailed statistical analysis can be found in Figures A1 and A2.

Table A1: Summary of performances via average AUC-ROC (%) on six datasets (BCNI2014-01, BCNI2014-04, Cho2017, Schirrneister2017, Weibo2014, and Zhou2016), for a left vs. right motor imagery task. Bold numbers represent the best score in each dataset. Estimators Imaginary Coherence and Instantaneous Coherence

Models	Estimator	BCNI2014001	BCNI2014004	Cho2017	Schirrneister2017	Weibo2014	Zhou2016
SPDNet	Imm	56.55±7.41	58.12±12.20	56.16±8.94	59.15±7.84	60.59±10.78	62.89±8.97
CSP+LDA	Imm	57.66±7.19	53.09±8.55	51.63±6.30	53.06±5.20	56.77±7.62	64.47±7.47
TANG+SVM	Imm	59.39±8.86	53.66±8.26	52.45±6.24	52.93±4.83	56.09±8.89	64.19±9.49
FgMDM	Imm	60.19±7.72	52.83±8.90	51.93±6.39	53.99±4.57	57.50±7.92	65.52±8.72
MDM	Imm	60.30±7.83	54.04±9.00	52.09±6.67	53.36±5.34	57.31±7.33	66.27±8.60
Cov+EN	Imm	60.71±7.92	52.67±8.96	52.07±6.70	54.42±4.07	57.21±8.23	64.58±8.41
$\text{SPDNet}_\psi$	Imm	<b>60.89±10.17</b>	<b>65.70±14.06</b>	<b>56.51±8.60</b>	<b>61.69±9.81</b>	<b>64.75±11.33</b>	<b>72.16±7.40</b>
SPDNet	Inst	63.42±5.76	63.71±12.88	57.61±11.09	59.73±6.89	66.11±12.02	75.07±7.63
MDM	Inst	69.74±14.19	59.89±10.72	58.76±9.65	52.01±5.77	63.24±14.25	81.94±8.91
TANG+SVM	Inst	69.82±14.76	60.44±9.76	58.86±10.11	53.31±5.42	63.10±15.51	85.02±8.29
CSP+LDA	Inst	70.49±14.80	60.13±10.23	59.23±10.40	51.94±6.06	64.32±15.92	84.95±8.94
Cov+EN	Inst	71.01±14.33	59.34±10.89	59.37±9.86	52.06±5.39	64.67±14.79	84.53±8.81
FgMDM	Inst	71.61±13.86	60.64±10.24	59.45±10.29	52.53±5.32	65.09±14.73	85.17±8.36
$\text{SPDNet}_\psi$	Inst	<b>71.84±16.75</b>	<b>68.83±17.41</b>	<b>62.04±12.02</b>	<b>65.37±12.91</b>	<b>68.31±14.17</b>	<b>87.03±6.13</b>

638 In this study, the situation is even rosier:  $\text{SPDNet}_\psi$ , used with either imaginary or instantaneous coherence features,  
 639 emerges as the best pipeline across all datasets. When utilizing imaginary coherence,  $\text{SPDNet}$  alone already achieves  
 640 statistically superior results compared to state-of-the-art approaches employing the same features. However, the  
 641 augmentation procedure introduces a significant performance boost, a finding supported by the meta-effect analysis  
 642 shown in Fig A1 (c).

643 In the case of instantaneous coherence, on the other hand, the impact of the augmentation procedure is even more  
 644 significant. In fact, with this feature, the standard  $\text{SPDNet}$  fails to exhibit statistically superior performance in  
 645 comparison to the state-of-the-art pipelines.  $\text{SPDNet}_\psi$ , on the other hand, stands out as the top-performing pipeline,  
 646 showing that the augmentation procedure brings a statistically significant influence on classification outcomes (see  
 647 Fig A2 (a)).

## 648 B Additional Results

Table A2: Parameters common to all DL pipelines.

Parameter	Value
Epoch	300
Batch Size	64
Validation Split	0.1
Loss	Sparse Categorical Crossentropy
Optimizer	Adam Learning Rate = 0.001
Callbacks ES	Early Stopping Patience = 75 Monitor = Validation Loss
Callbacks LR	ReduceLROnPlateau Patience = 75 Monitor = Validation Loss Factor = 0.5

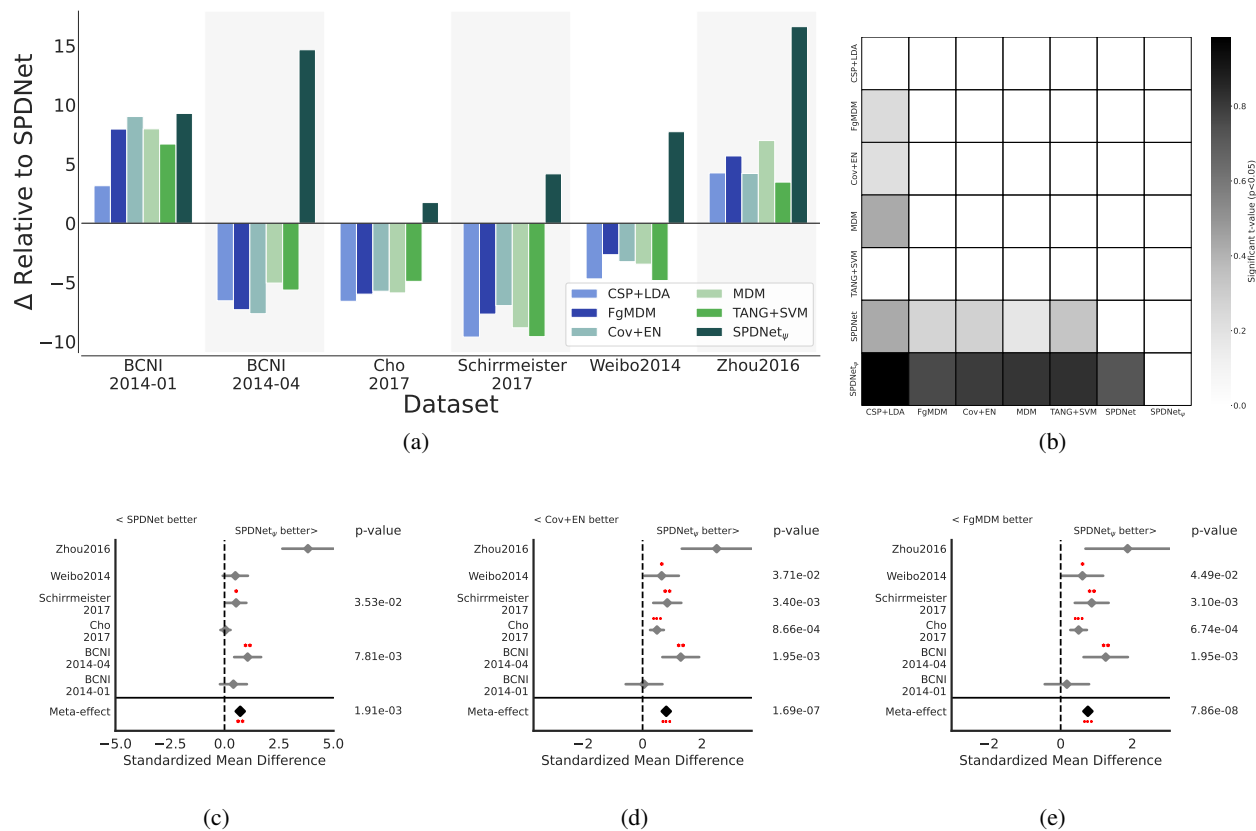


Figure A1: Result for right hand vs left hand classification, using within-session evaluation for state-of-the-art imaginary coherence pipelines. Plot (a) provides the relative improvement of the method considered with respect to the standard SPDNet of the different pipelines considered. Plot (b) shows a combined meta-analysis (over all datasets) of the different pipelines. It shows the significance of the algorithm on the y-axis being better than the one on the x-axis. The gray level represents the significance level of the ROC-AUC difference in terms of t-values. We only show significant interactions ( $p < 0.05$ ). Plots (c), (d), and (e) show the meta-analysis of SPDNet $_{\psi}$  against SPDNet, COV+EN, and FgMDM, respectively. We show the standardized mean differences of p-values computed as a one-tailed Wilcoxon signed-rank test for the hypothesis given in the plot title. The gray bar denotes the 95% interval. \* stands for  $p < 0.05$ , \*\* for  $p < 0.01$ , and \*\*\* for  $p < 0.001$ .

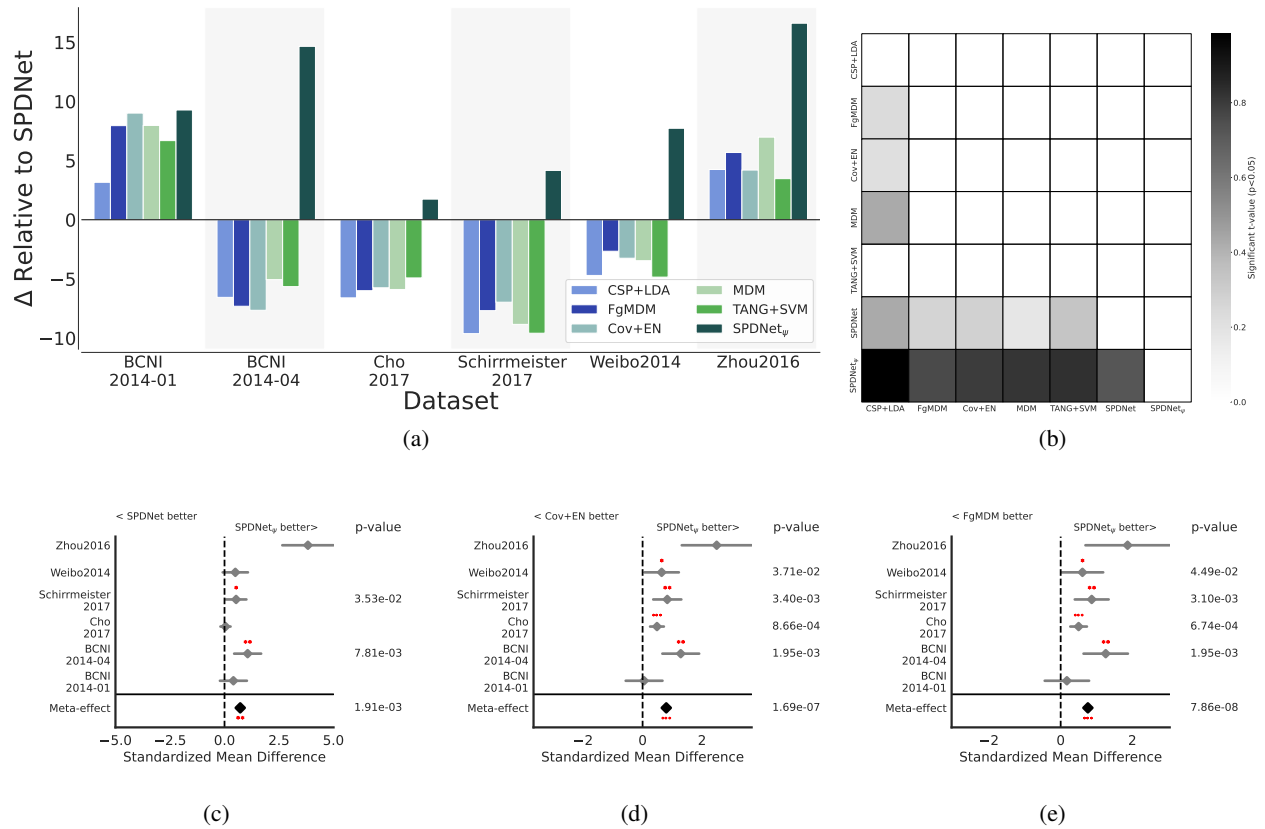


Figure A2: Result for right hand vs left hand classification, using within-session evaluation for state-of-the-art instantaneous coherence pipelines. Plot (a) provides the relative improvement of the method considered with respect to the standard SPDNet of the different pipelines considered. Plot (b) shows a combined meta-analysis (over all datasets) of the different pipelines. It shows the significance of the algorithm on the y-axis being better than the one on the x-axis. The gray level represents the significance level of the ROC-AUC difference in terms of t-values. We only show significant interactions ( $p < 0.05$ ). Plots (c), (d), and (e) show the meta-analysis of SPDNet $_{\psi}$  against SPDNet, COV+EN, and FgMDM, respectively. We show the standardized mean differences of p-values computed as a one-tailed Wilcoxon signed-rank test for the hypothesis given in the plot title. The gray bar denotes the 95% interval. \* stands for  $p < 0.05$ , \*\* for  $p < 0.01$ , and \*\*\* for  $p < 0.001$ .