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REGULAR PAPER

Robust Interval Predictive Tracking Control for Constrained and Perturbed Unicycle Mobile Robots

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Summary

In this paper, we propose the design of a robust control strategy for the trajectory tracking problem in perturbed unicycle mobile robots. The proposed strategy comprises the design of a robust control law, which is based on an Integral Sliding–Mode Control (ISM) approach together with an interval predictor–based state–feedback controller through a Model Predictive Control (MPC) scheme. The robust controller deals with some perturbations in the kinematic model, which represent slipping effects, and with state and input constraints that are related to restrictions on the workspace and saturated actuators, respectively. The proposed approach guarantees the exponential convergence to zero of the tracking error. The performance of the proposed approach is validated through some simulations tracking a lemniscate and a circumference curve.

KEYWORDS:

Unicycle Mobile Robots, Sliding–Mode Control, Model Predictive Control.

1 | INTRODUCTION

During the last decades, the unicycle mobile robots (UMRs) have been widely studied due to their capability of moving freely from one point to another one and to the broad diversity of possible real applications (see, *e.g.*,^{1, 2, and 3}).

The fact that the kinematic model of this class of systems does not fulfill Brockett’s necessary condition for smooth state–feedback stabilization⁴ represents one of the main difficulties regarding the control design⁵. Therefore, the design of non–smooth or time–varying feedback controllers is a requirement for this class of mobile robots. Moreover, as it is highlighted in^{6 and 7}, even if external forces cannot be considered in the kinematic model, there exist some other signals or non–modeled phenomena, *e.g.*, the skidding and slipping of the wheels and corrupt control signals, that must be taken into account for the design of a controller. Additionally, it is well–known that such UMRs must move in restricted work–spaces and have energy limitations. These effects can be seen as state and input constraints, and should also be considered in the control design. In this sense, the trajectory tracking control design, considering the non–holonomic constraints, external perturbations and system constraints, is still a challenging problem.

Regarding the literature taking into account external disturbances in the kinematic model, in⁸, the problem of finite–time posture stabilization is addressed for UMRs by means of sliding–mode control (SMC) and multi–rate output–feedback techniques. The proposed approach considers side–slipping effects characterized by additive perturbations but no disturbances are taken into account in the angular velocity. In⁹, a first–order SMC approach is proposed to deal with the trajectory tracking problem in perturbed UMRs. This approach considers some skidding and slipping effects on the wheels, input saturation constraints and

guarantees the asymptotic convergence to zero of the tracking error. A robust controller, based on the Super-Twisting algorithm, was presented in¹⁰, and guarantees asymptotic convergence of the tracking error to zero, despite the presence of some skidding and slipping effects. The proposed controller is continuous but local. In¹¹, the authors design an adaptive fractional order fuzzy PID controller to deal with the trajectory tracking problem in UMRs. The control design considers the dynamical model with several disturbances, *e.g.*, parametric uncertainties, noise, forced displacement, time delay, and uncertainty in the pose. In¹², the authors introduce a robust finite-time stabilizing controller for arbitrary-order non-holonomic systems. The proposed controller can be applied to a UMR but the considered external disturbances have no physical meaning for such a particular system. Recently, in¹³, a second-order sliding-mode controller is proposed for the trajectory tracking problem in UMRs. The proposed approach takes into account possible slipping of the wheels and guarantees global, uniform and finite-time convergence of the tracking error to zero. Similarly, in¹⁴, the authors deal with the tracking problem for UMRs with slipping effects and introduce a homogeneity-based control strategy, which provides Input-to-State stability (ISS), with respect to the slipping effects, for the tracking error dynamics. However, all of the above-mentioned works do not consider state constraints.

In the context of trajectory tracking control for constrained UMRs the MPC approach (see, *e.g.*,^{15, 16} and¹⁷) is quite popular. For instance, in¹⁸, a neural network-based robust MPC is proposed to stabilize a state and input constrained mobile robot in the presence of some additive disturbances. In the same sense, a couple of MPC approaches; namely, tube-MPC and nominal robust MPC, are proposed in¹⁹ for tracking unicycle robots. Such approaches are able to deal with input constraints and bounded additive disturbances. In²⁰, the authors present a self-triggered MPC for unicycle robots, which considers input constraint and bounded external disturbances. The proposed Lyapunov-based self-triggered mechanism allows to reduce the computation load of MPC. In²¹, the authors propose a trajectory tracking controller for UMRs based on an MPC approach with adaptive prediction horizon. The proposed controller is able to deal with some additive disturbances and system constraints. However, for all of the above-mentioned works, the considered additive disturbances lack physical meaning in UMRs since the skidding and slipping effects, or corrupt control signals, cannot be represented by additive disturbances, but by multiplicative ones. Moreover, all of the above-mentioned works only provide some ISS properties. Additionally, in the literature, there have been several proposals for the combination of MPC and ISMC. For instance, one can see^{22, 23, 24}, and²⁵. However, these works are not applied to UMRs and most of them only provide ISS properties.

Motivated by the above-mentioned issues, *i.e.*, meaningful external disturbances, robustness, and system constraints, in this paper, a robust control strategy is proposed to solve the trajectory tracking problem in perturbed unicycle mobile robots, taking into account state and input constraints. The proposed strategy comprises the design of a robust control law, which is based on an ISMC approach together with an interval predictor-based state-feedback controller through an MPC scheme. The robust controller deals with some perturbations in the kinematic model, and with state and input constraints that are related to restrictions on the workspace and saturated actuators, respectively. The proposed approach guarantees the exponential convergence to zero of the tracking error despite the considered perturbations, which represent slipping effects, and the system constraints; and moreover, the computational burden is relaxed due to the switching structure of the controller. A preliminary version of this paper is presented in²⁶. However, here, we provide detailed technical proofs of all results and also include additional simulation results.

The rest of the paper is organized as follows. The problem statement is given in Section 2. The proposed controller is presented in Section 5. In Section 6, we provide some simulation results. Concluding remarks are formulated in Section 7.

Notation: Denote the trigonometric functions $\sin(\theta)$, $\cos(\theta)$ and $\text{sinc}(\theta)$ as $s(\theta)$, $c(\theta)$ and $sc(\theta)$, respectively. Let us denote a sequence of integers $1, \dots, n$ as $\overline{1, n}$, for any $n \in \mathbb{N}$. The absolute value is represented as $|\cdot|$ while the Euclidean norm of a vector $z \in \mathbb{R}^n$ is denoted as $\|z\|$, and for a matrix $A \in \mathbb{R}^{m \times n}$, the induced norm is the spectral norm, *i.e.*, $\|A\| = \sqrt{\lambda_{\max}(A^T A)}$. The set of all inputs $u : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^p$ such that its L_∞ norm on $[0, \infty]$ is less than infinity, *i.e.*, $\|u\|_\infty := \|u\|_{[0, \infty]} = \text{esssup}_{t \geq 0} \|u(t)\| < \infty$, is denoted as \mathcal{L}_∞ . For a couple of vectors $x_1, x_2 \in \mathbb{R}^n$ and a couple of matrices $A_1, A_2 \in \mathbb{R}^{n \times n}$, the relations $x_1 \leq x_2$ and $A_1 \leq A_2$ are understood in the component-wise sense. In the same sense, for a matrix $A \in \mathbb{R}^{n \times n}$, define $A^+ = \max\{0, A\}$, $A^- = A^+ - A$ and $|A| = A^+ + A^-$, similarly for a vector. For a symmetric matrix $P \in \mathbb{R}^{n \times n}$, the notation $P < 0$ ($P \geq 0$) means that P is negative (positive) definite. A matrix $A \in \mathbb{R}^{n \times n}$ is called Metzler when all its non-diagonal elements are non-negative. The term $\text{He}(A)$ denotes $A + A^T$, for a matrix $A \in \mathbb{R}^{n \times n}$.

2 | PROBLEM STATEMENT

Consider the perturbed kinematic model of an UMR (see, Fig. 1)¹:

$$\dot{\theta} = [1 + d_1(t)]\omega, \quad (1a)$$

$$\dot{x} = [1 + d_2(t)]c(\theta)v, \quad (1b)$$

$$\dot{y} = [1 + d_2(t)]s(\theta)v, \quad (1c)$$

where $x \in \mathbb{R}$ and $y \in \mathbb{R}$ denote the midpoint between the wheels and $\theta \in \mathbb{R}$ represents the orientation angle of the UMR. The terms v and ω contain the linear and angular velocities of the UMR, and represent the control inputs. The terms d_1 and d_2 represent some time-varying perturbations, which are multiplicative to the inputs and that may come from the settling time of the internal controller that translates the velocity commands in current/voltage inputs and sends them to the motors²⁷ or non-modeled kinematics phenomena proportional to the control inputs, such as slipping of the wheels⁷. It is assumed that such time-varying perturbations $d_i(t)$ are unknown but bounded, *i.e.*, $-1 < d_i(t) \leq d_{\max} < 1$, for $i = 1, 2$, with a known positive constant d_{\max} . Note that the constraint $d_i(t) > -1$ ensures that the perturbations do not cause a change of sign in the control inputs.

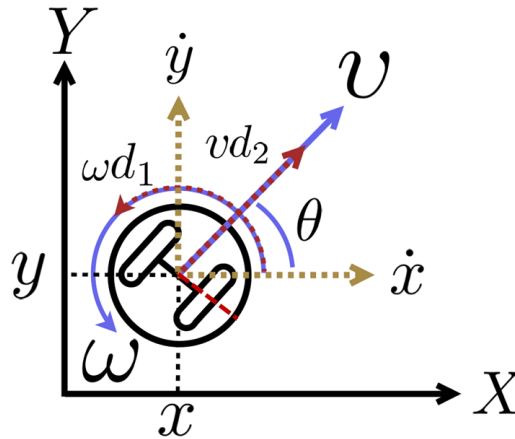


FIGURE 1 Schematic Diagram of the Perturbed UMR

The aim of this work is to design a trajectory tracking control for the UMR able to compensate some multiplicative perturbations and reach the desired trajectory taking into account some state and input constraints, *i.e.*, $x(t) \in \mathbb{X} = [x_{\min}, x_{\max}] \subset \mathbb{R}$, $y(t) \in \mathbb{Y} = [y_{\min}, y_{\max}] \subset \mathbb{R}$, $v(t) \in \mathbb{V} = [-v_{\max}, v_{\max}] \subset \mathbb{R}$ and $\omega(t) \in \mathbb{W} = [-\omega_{\max}, \omega_{\max}] \subset \mathbb{R}$, for all $t \geq t_0$, for some given sets \mathbb{X} , \mathbb{Y} , \mathbb{V} and \mathbb{W} .

3 | PRELIMINARIES

Consider the system

$$\dot{x} = f(t, x), \quad t \in \mathbb{R}_{\geq 0}, \quad x(0) = x_0, \quad (2)$$

where $x \in \mathbb{R}^n$ is the state vector; $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is assumed to be locally bounded uniformly in t . For f being discontinuous with respect to x , the solutions are understood in the sense of Filippov²⁸. Then, a solution of the system (2), for an initial condition $x_0 \in \mathbb{R}^n$ at time instant $t_0 \in \mathbb{R}_+$, is denoted as $x(t, t_0, x_0)$ and defined on some finite time interval $[t_0, t_0 + T)$ such that $0 \leq T < \infty$. Let Ω be open neighborhood of the origin in \mathbb{R}^n , $0 \in \Omega$.

Definition 1.²⁹ and³⁰. *At the steady state $x = 0$, the system (2) is said to be*

¹The perturbed kinematic model (1) and its description is taken from⁹,¹⁰, and¹³.

a) Uniformly Stable (US) if for any $\epsilon > 0$ there is $\delta(\epsilon)$ such that for any $x_0 \in \Omega$, if $\|x_0\| \leq \delta(\epsilon)$; then, $\|x(t, t_0, x_0)\| \leq \epsilon$, for all $t \geq t_0$ and any $t_0 \in \mathbb{R}_+$;

b) Uniformly Exponentially Stable (UES) if it is US and exponentially converging from Ω , i.e., for any $x_0 \in \Omega$, there exist $k, \gamma > 0$ such that $\|x(t, t_0, x_0)\| \leq k\|x_0\|e^{-\gamma(t-t_0)}$, for all $t \geq t_0$ and any $t_0 \in \mathbb{R}_+$;

c) Uniformly Finite-Time Stable (UFTS) if it is US and finite-time converging from Ω , i.e., for any $x_0 \in \Omega$, there exists $0 \leq T_{x_0} < +\infty$ such that $x(t, t_0, x_0) = 0$, for all $t \geq t_0 + T_{x_0}$ and any $t_0 \in \mathbb{R}_+$. The function $T_0(x_0) = \inf\{T_{x_0} \geq 0 : x(t, t_0, x_0) = 0 \forall t \geq t_0 + T_{x_0}\}$ is called the settling-time of the system (2).

The following lemma, related to interval bounds, is used.

Lemma 1.³¹ Let $z \in \mathbb{R}^n$ be a varying vector such that $\underline{z} \leq z \leq \bar{z}$, for some $\underline{z}, \bar{z} \in \mathbb{R}^n$. If $A \in \mathbb{R}^{m \times n}$ is a constant matrix; then

$$A^+ \underline{\xi} - A^- \bar{\xi} \leq A \xi \leq A^+ \bar{\xi} - A^- \underline{\xi}.$$

If $A \in \mathbb{R}^{m \times n}$ is a variable matrix such that $\underline{A} \leq A \leq \bar{A}$, for some $\underline{A}, \bar{A} \in \mathbb{R}^{m \times n}$; then

$$\underline{A}^+ \underline{\xi}^+ - \bar{A}^+ \bar{\xi}^- - \underline{A}^- \bar{\xi}^+ + \bar{A}^- \underline{\xi}^- \leq A \xi \leq \bar{A}^+ \bar{\xi}^+ - \underline{A}^+ \bar{\xi}^- - \bar{A}^- \underline{\xi}^+ + \underline{A}^- \underline{\xi}^-.$$

4 | LPV TRACKING ERROR DYNAMICS

Let us define the tracking errors as follows

$$e_1 = \theta_d - \theta, \tag{3a}$$

$$e_2 = c(\theta)(x_d - x) + s(\theta)(y_d - y), \tag{3b}$$

$$e_3 = c(\theta)(y_d - y) - s(\theta)(x_d - x), \tag{3c}$$

where x_d , y_d and θ_d come from a reference kinematic model for the UMR, i.e.,

$$\dot{\theta}_d = \omega_d, \tag{4a}$$

$$\dot{x}_d = c(\theta_d)v_d, \tag{4b}$$

$$\dot{y}_d = s(\theta_d)v_d, \tag{4c}$$

where v_d and ω_d are the linear and angular reference velocities, respectively. These are assumed continuous and bounded by some positive constants $\underline{v}_d, \bar{v}_d$ and $\bar{\omega}_d$, i.e., $0 < \underline{v}_d < v_d(t) \leq \bar{v}_d$, and $\|\omega_d\|_\infty \leq \bar{\omega}_d$, and such that $v_d(t) \in \mathbb{V}$ and $\omega_d(t) \in \mathbb{W}$, for all $t \geq 0$. Moreover, the trajectories of the reference model also hold the state constraints, i.e., $x_d(t) \in \mathbb{X}$ and $y_d(t) \in \mathbb{Y}$, for all $t \geq 0$.

Therefore, the tracking error dynamics can be calculated as

$$\dot{e}_1 = -\omega d_1(t) + \tau_1, \tag{5a}$$

$$\dot{e}_2 = [1 + d_1(t)]\omega e_3 - v d_2(t) + \tau_2, \tag{5b}$$

$$\dot{e}_3 = -[1 + d_1(t)]\omega e_2 + v_d s(e_1), \tag{5c}$$

with the virtual control inputs τ_1 and τ_2 satisfying

$$\tau_1 = \omega_d - \omega, \tag{6a}$$

$$\tau_2 = v_d c(e_1) - v. \tag{6b}$$

Note that the tracking error dynamics (5) can be rewritten as follows:

$$\dot{e} = A(\rho_1)e + B[\tau + F(\rho_2)d(t)], \tag{7}$$

where $e = (e_1, e_2, e_3)^\top \in \mathbb{R}^3$, $\tau = (\tau_1, \tau_2)^\top \in \mathbb{R}^2$, $d(t) = (d_1(t), d_2(t))^\top \in \mathbb{R}^2$ and

$$A(\rho_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & [1 + d_1(t)]\omega \\ v_d s c(e_1) & -[1 + d_1(t)]\omega & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F(\rho_2) = \begin{pmatrix} -\omega & 0 \\ 0 & -v \end{pmatrix},$$

with the vectors of scheduling variables composed by $\rho_1 = (v_d s c(e_1), [1 + d_1(t)]\omega)^\top \in \mathbb{R}^2$ and $\rho_2 = (v, \omega)^\top \in \mathbb{R}^2$, which are bounded due to the physical and actuator limitations, and the fact that $0 < \underline{v}_d < v_d(t) \leq \bar{v}_d$ and $\|\omega_d\|_\infty \leq d_{\max} < 1$. It is clear

that system (7) is in LPV form and the input and state constraint sets are given now as follows:

$$\mathbb{E} = \{e \in \mathbb{R}^3 : (e_1, e_2, e_3) \in \mathbb{R} \times [-\bar{x}_y, \bar{x}_y] \times [-\bar{y}_y, \bar{y}_y]\}, \quad (8)$$

$$\mathbb{U} = \{\tau \in \mathbb{R}^2 : (\tau_1, \tau_2) \in [-\bar{\tau}_1, \bar{\tau}_1] \times [-\bar{\tau}_2, \bar{\tau}_2]\}, \quad (9)$$

where $\bar{x}_y = (x_{\max} - x_{\min}) + (y_{\max} - y_{\min})$, $\bar{\tau}_1 = \bar{\omega}_d + \omega_{\max}$ and $\bar{\tau}_2 = \bar{v}_d + v_{\max}$. Moreover, note that there always exist a Metzler matrix $A_0 \in \mathbb{R}^{3 \times 3}$, and some matrices $A_j \in \mathbb{R}^{3 \times 3}$, for $j = 1, 4$, such that the following equations

$$A(\rho_1) = A_0 + \sum_{j=1}^4 \alpha_j(\rho_1) A_j, \quad (10a)$$

$$\sum_{j=1}^4 \alpha_j(\rho_1) = 1, \quad \alpha_j(\rho_1) \in [0, 1], \quad (10b)$$

hold for the system (7). Therefore, the problem now is to design a robust control law τ such that the trajectories of the system (7) converge to zero despite the disturbances d and taking into account the state and input constraints (8) and (9), *i.e.*, $e(t) \in \mathbb{E}$ and $\tau(t) \in \mathbb{U}$, for all $t \geq 0$.

Thus, the idea is to design a controller in the following form:

$$\tau = u_0 + u_1. \quad (11)$$

The controller (11) is divided into two parts. The nonlinear element u_1 is the ISMC that will compensate the effect of the matched disturbances from the beginning, *i.e.*, from $t = 0$, without reaching phase, taking into account only the input constraints; whereas u_0 is the nominal control part, which is composed of an interval predictor–based state–feedback controller and the MPC scheme that will deal with the state and input constraints.

According to (9), $\|\tau\| \leq \tau_{\max} = \sqrt{(\bar{\omega}_d + \omega_{\max})^2 + (\bar{v}_d + v_{\max})^2}$, for given $\omega_{\max}, v_{\max} > 0$; then, a specific control effort can be assigned to each part of the controller (11), *i.e.*, $\|u_0\| \leq u_{0\max}$ and $\|u_1\| \leq u_{1\max}$, with some constants $u_{0\max}, u_{1\max} > 0$ such that $u_{0\max} + u_{1\max} \leq \tau_{\max}$.

5 | ROBUST CONTROL DESIGN

The proposed controller comprises the design of a robust control law, which is based on an ISMC approach together with an interval predictor–based state–feedback controller and the MPC, which deals with state and input constraints. Recall that the ISMC approach allows us to independently design the nonlinear control law u_1 and the nominal control law u_0 providing a separate theoretical analysis (for more details, see³²).

The proposed approach will guarantee exponential stability for the tracking error dynamics. In the following sections, we describe the proposed methodology.

5.1 | Integral Sliding–Mode Control Design

Let us define the following sliding variable

$$s(e(t)) = G[e(t) - e(0)] - G \int_0^t [A_0 e(\sigma) + B u_0(\sigma)] d\sigma, \quad (12)$$

where $G \in \mathbb{R}^{2 \times 3}$ is such that $\det(GB) \neq 0$. The optimal way to design G is $G = B^\top$ or $G = (B^\top B)^{-1} B^\top$, for more details see *e.g.*,³³ and³⁴. Note that the dynamics of the sliding variable satisfies

$$\dot{s} = GB[u_1 + F(\rho_2)d(t)] + G \sum_{j=1}^4 \alpha_j(\rho_1) A_j e. \quad (13)$$

Then, the ISMC u_1 is proposed as

$$u_1 = -\zeta(e) \frac{(GB)^\top s}{\|(GB)^\top s\|}, \quad (14)$$

with some positive gain $\zeta(e) > 0$, for all $e \in \mathbb{E}$. The following lemma provides the conditions to ensure the finite–time convergence of the sliding variable to zero fulfilling the input constraint.

Lemma 2. Let the ISMC (14), with $G = B^\top$, be applied to the system (13), for a given $u_{1\max} > 0$. If the gain $\zeta(e)$ is selected as

$$\zeta(e) = \gamma + F_{\max} + A_{\max} \|e\|, \quad (15)$$

with

$$F_{\max} = d_{\max} \max(v_{\max}, \omega_{\max}),$$

$$A_{\max} = \sum_{j=1}^4 \alpha_j^* \|A_j\|, \text{ with } \alpha^* = \arg \max_{\alpha \in \Gamma} \sum_{j=1}^4 \alpha_j \|A_j\|,$$

with $\Gamma \subset \mathbb{R}^4$ being the set of all convex weight vectors $\alpha = (\alpha_1, \dots, \alpha_4)^\top \in \mathbb{R}^4$ such that (10b) holds, and some $\gamma > 0$ such that

$$0 < \gamma \leq u_{1\max} - F_{\max} - A_{\max} e_{\max}, \quad (16)$$

with $e_{\max} \leq \sqrt{4\pi^2 + 2\bar{x}\bar{y}^2}$, is satisfied for a given $u_{1\max} > 0$; then, $s = 0$ is UFTS.

Proof. Let us take a candidate Lyapunov function $V_s = (s^\top s)/2$. Therefore, the time derivative of V_s , along the trajectories of the system (13), taking into account (14) and (12), satisfies

$$\dot{V}_s = -\zeta(e) s^\top G B \frac{(GB)^\top s}{\|(GB)^\top s\|} + s^\top G B F(\rho_2) d(t) + s^\top G \sum_{j=1}^4 \alpha_j(\rho_1) A_j e.$$

Note that, if $G = B^\top$; then, we have $GB = I_2$ and $\|G\| = 1$. Therefore, the time derivative of V_s is upper bounded as

$$\dot{V}_s \leq -\zeta(e) \|s\| + \|s\| (\|F(\rho_2)d\|_\infty + A_{\max} \|e\|),$$

where $A_{\max} = \sum_{j=1}^4 \alpha_j^* \|A_j\|$, with $\alpha^* = \arg \max_{\alpha \in \Gamma} \sum_{j=1}^4 \alpha_j \|A_j\|$. Moreover, by definition, $\|F(\rho_2)d\|_\infty$ satisfies

$$\|F(\rho_2)d\|_\infty \leq d_{\max} \max(v_{\max}, \omega_{\max}) = F_{\max}.$$

Hence, if the gain $\zeta(e)$ is designed according to (15), i.e., $\zeta(e) = \gamma + F_{\max} + A_{\max} \|e\|$; and γ satisfies the constraint (16), which implies that $\|u_1\| \leq u_{1\max}$; with $\|e\| \leq e_{\max} \leq \sqrt{4\pi^2 + 2\bar{x}\bar{y}^2}$, for all $e \in \mathbb{E}$; then, the time derivative of V_s holds the following bound

$$\dot{V}_s \leq -\sqrt{2}\gamma V_s^{\frac{1}{2}},$$

and thus, due to the structure of the sliding variable, we have that $s(t) = 0$, for all $t \geq 0$. \square

Therefore, the robust controller u_1 will deal with the perturbations $F(\rho_2)d$ and part of the parametric uncertainty $\sum_{j=1}^4 \alpha_j(\rho_1) A_j e$, satisfying the input constraint $\|u_1\| \leq u_{1\max}$. Note that the term $F(\rho_2)d$ is matched with the control u_1 , and hence, it is completely compensated from the beginning. However, since the parametric uncertainty $\sum_{j=1}^4 \alpha_j(\rho_1) A_j e$ is not matched with the control u_1 , only its projection into the matched space of B^\top , i.e., the space directly affected by u_1 , could be compensated by u_1 .

Additionally, it is possible to fix $u_{1\max}$ according to the upper bound of the disturbances, e.g., $u_{1\max} = 1.1(F_{\max} + A_{\max} e_{\max})$; and then, to assign the rest of the control effort to u_0 , i.e., $u_{0\max} \leq \tau_{\max} - 1.1(F_{\max} + A_{\max} e_{\max})$.

Remark 1. It is well-known that the chattering is an undesirable effect in most of the ideal discontinuous sliding-mode controllers, as u_1 in (14), and it could be an issue for slow actuator dynamics. However, for this kind of mobile robot, the possible chattering effect is mitigated by the mobile robot itself since the control signals are commonly passed-through a PWM, which acts like a low-pass filter.

Remark 2. Alternatively, it is possible to substitute the proposed discontinuous controller u_1 in (14) by some continuous approximations avoiding high-frequency control signals. A well-known function that can be used is

$$u_1 = -\zeta(e) \frac{(GB)^\top s}{\|(GB)^\top s\| + \varepsilon},$$

where the tuning parameter $\varepsilon > 0$ is a small constant.

5.2 | Interval Predictor

Since the sliding-mode takes place, from (13) and recalling that $GB = I_2$, it follows that the equivalent control is

$$u_{1eq} = -G \sum_{j=1}^4 \alpha_j(\rho_1) A_j e - F(\rho_2) d(t). \quad (17)$$

Therefore, taking into account (10) and substituting (17) in (7), the dynamics of the system on the sliding surface is given by

$$\dot{e} = \left[A_0 + \sum_{j=1}^4 \alpha_j(\rho_1) \tilde{A}_j \right] e + B u_0, \quad (18)$$

where $\tilde{A}_j = (I_3 - BG)A_j$. Then, according to Lemma 1 and³⁵, it follows that

$$-\bar{A}e^- - \underline{A}e^+ \leq \sum_{j=1}^4 \alpha_j(\rho_1) \tilde{A}_j e \leq \bar{A}e^+ + \underline{A}e^-,$$

where $\bar{A} = \sum_{j=1}^4 \tilde{A}_j^+$, $\underline{A} = \sum_{j=1}^4 \tilde{A}_j^-$, and a couple of vectors $e_-, \bar{e} \in \mathbb{R}^3$ such that $e_- \leq e \leq \bar{e}$. Thus, it is possible to design the following interval predictor³⁶ for system (18)

$$\dot{\underline{e}} = A_0 \underline{e} - \bar{A} \bar{e}^- - \underline{A} \bar{e}^+ + B u_0, \quad (19a)$$

$$\dot{\bar{e}} = A_0 \bar{e} + \bar{A} \bar{e}^+ + \underline{A} \bar{e}^- + B u_0. \quad (19b)$$

Note that the previous dynamics may be represented in a compact way as follows

$$\dot{z} = \mathcal{A}_0 z + \mathcal{A}_1 z^+ + \mathcal{A}_2 z^- + B u_0, \quad (20)$$

where $z = (e_-, \bar{e}^-)^\top \in \mathbb{R}^6$, and the system matrices given as follows

$$\mathcal{A}_0 = \begin{pmatrix} A_0 & 0 \\ 0 & A_0 \end{pmatrix}, \quad \mathcal{A}_1 = \begin{pmatrix} 0 & -\bar{A} \\ 0 & \underline{A} \end{pmatrix}, \quad \mathcal{A}_2 = \begin{pmatrix} -\bar{A} & 0 \\ \underline{A} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} B \\ B \end{pmatrix}.$$

Then, in order to stabilize the tracking error dynamics (7), we need to design a state-feedback u_0 to take the trajectories of the system (20) to zero (see, e.g.,³⁷).

5.3 | State-Feedback Control Design

The control signal u_0 is proposed as

$$u_0(t) = \begin{cases} \mathcal{U}_0(t), & z(t_i) \notin \mathbb{E}_f, \\ \bar{u}_0(t), & z(t_i) \in \mathbb{E}_f, \end{cases} \quad (21)$$

where \mathcal{U}_0 is the control signal provided by the MPC scheme, for all $t \in [t_i, t_i + h)$, with $h \in (0, T)$ and T as the application time and the prediction interval for the MPC, respectively; and \bar{u}_0 is the state-feedback controller. The switching set \mathbb{E}_f is defined further on.

The state-feedback controller \bar{u}_0 is designed, based on (20), as

$$\bar{u}_0 = K_0 z + K_1 z^+ + K_2 z^-, \quad (22)$$

where $K_0, K_1, K_2 \in \mathbb{R}^{2 \times 6}$ are the matrix gains to be designed. The following lemma provides a constructive way to design the state-feedback gains in order to ensure the convergence of the trajectories of the system (20) to zero.

Lemma 3. *Let the state-feedback control law (22) be applied to the system (20), i.e., $u_0(t) = \bar{u}_0(t)$. Suppose that there exist two vectors $e_0, \bar{e}_0 \in \mathbb{R}^3$, such that $e_0 \leq e(0) \leq \bar{e}_0$, that there also exist diagonal matrices $0 < X_1 \in \mathbb{R}^{6 \times 6}$, $0 \leq R_1, R_2 \in \mathbb{R}^{6 \times 6}$, some*

diagonal matrices $Q_l, R_0 \in \mathbb{R}^{6 \times 6}$ and some matrices $Y_l \in \mathbb{R}^{2 \times 6}$, for $l = \overline{0, 2}$, such that the following LMIs

$$\Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \star & \Omega_{22} & \Omega_{23} \\ \star & \star & \Omega_{33} \end{pmatrix} \leq 0, \quad (23)$$

$$Q_0 + \min\{Q_1, Q_2\} + 2\min\{R_1, R_2\} > 0, \quad (24)$$

$$\begin{aligned} \Omega_{11} &= \text{He}(\mathcal{A}_0 X_0 + B Y_0) + Q_0, \\ \Omega_{12} &= \mathcal{A}_1 X_1 + B Y_1 + X_0 \mathcal{A}_0^\top + Y_0^\top B^\top + R_1, \\ \Omega_{13} &= \mathcal{A}_2 X_2 + B Y_2 - X_0 \mathcal{A}_0^\top - Y_0^\top B^\top - R_2, \\ \Omega_{22} &= \text{He}(\mathcal{A}_1 X_1 + B Y_1) + Q_1, \\ \Omega_{23} &= \mathcal{A}_2 X_2 + B Y_2 - X_1 \mathcal{A}_1^\top - Y_1^\top B^\top + R_0, \\ \Omega_{33} &= Q_2 - \text{He}(\mathcal{A}_2 X_2 + B Y_2), \end{aligned}$$

are feasible. If the state–feedback gains are designed as $K_l = Y_l X_l^{-1}$, for $l = \overline{0, 2}$; then, the trajectories of the system (20) exponentially converge to zero.

Proof. The nominal closed–loop system, taking into account (22), is given as

$$\dot{z} = \bar{\mathcal{A}}_0 z + \bar{\mathcal{A}}_1 z^+ + \bar{\mathcal{A}}_2 z^-, \quad (25)$$

where $\bar{\mathcal{A}}_0 = \mathcal{A}_0 + B K_0$, $\bar{\mathcal{A}}_1 = \mathcal{A}_1 + B K_1$ and $\bar{\mathcal{A}}_2 = \mathcal{A}_2 + B K_2$. Let us take a candidate Lyapunov function as in³⁶

$$V_z = z^\top P_0 z + z^\top P_1 z^+ - z^\top P_2 z^-, \quad (26)$$

with some positive diagonal matrices P_0, P_1 and P_2 , which imply that $P_0 + \min\{P_1, P_2\} > 0$, and then, ensure $V_z > 0$. Note that such a candidate Lyapunov function satisfies the following bounds

$$z^\top (P_0 + \min\{P_1, P_2\}) z \leq V_z \leq z^\top (P_0 + P_1 + P_2) z. \quad (27)$$

Thus, the time derivative of V_z , along the trajectories of the system (25) satisfies

$$\begin{aligned} \dot{V}_z &= z^\top \text{He}(P_0 \bar{\mathcal{A}}_0) z + 2z^\top P_0 (\bar{\mathcal{A}}_1 z^+ + \bar{\mathcal{A}}_2 z^-) \\ &\quad + z^{+\top} \text{He}(P_1 \bar{\mathcal{A}}_1) z^+ + 2z^{+\top} P_1 (\bar{\mathcal{A}}_0 z + \bar{\mathcal{A}}_2 z^-) \\ &\quad - z^{-\top} \text{He}(P_2 \bar{\mathcal{A}}_2) z^- - 2z^{-\top} P_2 (\bar{\mathcal{A}}_0 z + \bar{\mathcal{A}}_1 z^+), \end{aligned}$$

and defining an extended vector $\eta = (z^\top, z^{+\top}, z^{-\top})^\top$, it follows that \dot{V}_z could be rewritten as

$$\dot{V}_z = \eta^\top \bar{\Omega} \eta - z^\top \bar{Q}_0 z - z^{+\top} \bar{Q}_1 z^+ - z^{-\top} \bar{Q}_2 z^- - 2z^{+\top} \bar{R}_0 z^- - 2z^{+\top} \bar{R}_1 z - 2(-z^-)^\top \bar{R}_2 z,$$

for some diagonal matrices $\bar{Q}_l, \bar{R}_l \in \mathbb{R}^{6 \times 6}$, for $l = \overline{0, 2}$, such that $\Phi = \bar{Q}_0 + \min\{\bar{Q}_1, \bar{Q}_2\} + 2\min\{\bar{R}_1, \bar{R}_2\} > 0$, and with

$$\begin{aligned} \bar{\Omega} &= \begin{pmatrix} \bar{\Omega}_{11} & \bar{\Omega}_{12} & \bar{\Omega}_{13} \\ \star & \bar{\Omega}_{22} & \bar{\Omega}_{23} \\ \star & \star & \bar{\Omega}_{33} \end{pmatrix}, \\ \bar{\Omega}_{11} &= \text{He}(P_0 \bar{\mathcal{A}}_0) + \bar{Q}_0, \quad \bar{\Omega}_{12} = P_0 \bar{\mathcal{A}}_1 + \bar{\mathcal{A}}_0^\top P_1 + \bar{R}_1, \\ \bar{\Omega}_{13} &= P_0 \bar{\mathcal{A}}_2 - \bar{\mathcal{A}}_0^\top P_2 - \bar{R}_2, \quad \bar{\Omega}_{22} = \text{He}(P_1 \bar{\mathcal{A}}_1) + \bar{Q}_1, \\ \bar{\Omega}_{23} &= P_1 \bar{\mathcal{A}}_2 - \bar{\mathcal{A}}_1^\top P_2 + \bar{R}_0, \quad \bar{\Omega}_{33} = \bar{Q}_2 - \text{He}(P_2 \bar{\mathcal{A}}_2). \end{aligned}$$

It is worth noting that the constraints

$$z^{+\top} \bar{R}_0 z^- = 0, \quad z^{+\top} \bar{R}_1 z \geq 0, \quad (-z^-)^\top \bar{R}_2 z \geq 0,$$

hold for any diagonal matrix \bar{R}_0 , and positive diagonal matrices $\bar{R}_1, \bar{R}_2 \geq 0$. Therefore, taking into account that $\bar{\Omega} \leq 0$; then, the time derivative of V_z satisfies

$$\begin{aligned} \dot{V}_z &\leq -z^\top \bar{Q}_0 z - z^{+\top} \bar{Q}_1 z^+ - z^{-\top} \bar{Q}_2 z^- - 2z^{+\top} \bar{R}_1 z - 2(-z^-)^\top \bar{R}_2 z \\ &\leq -z^\top \Phi z, \end{aligned}$$

with the diagonal matrix $\Phi > 0$. Therefore, considering (27), we obtain that

$$\dot{V}_z \leq -\beta V_z,$$

where $\beta = \min_{\forall i=1,6} \lambda_i [\Phi(P_0 + P_1^+ + P_2^+)^{-1}]$; and hence, the trajectories of the system (25) exponentially converge to zero.

Finally, in order to obtain a constructive way to design the gains K_0 , K_1 and K_2 , by means of LMIs, we can pre- and post-multiply $\bar{\Omega}$ by the matrix $\text{diag}(P_0^{-1}, P_1^{-1}, P_2^{-1})$, in order to obtain the LMI (23), with $X_l = P_l^{-1}$, $Q_l = X_l \bar{Q}_l X_l$, $Y_l = K_l X_l$, for $l = 0, 2$, and $R_0 = X_0 \bar{R}_0 X_0$, $R_1 = X_0 \bar{R}_1 X_1$, and $R_2 = X_0 \bar{R}_2 X_2$. Note that, due to the diagonal structure, the constraint $\bar{Q}_0 + \min\{\bar{Q}_1, \bar{Q}_2\} + 2\min\{\bar{R}_1, \bar{R}_2\} > 0$ is equivalent to (24), *i.e.*, $Q_0 + \min\{Q_1, Q_2\} + 2\min\{R_1, R_2\} > 0$. Therefore, $\Omega \leq 0$ is equivalent to $\bar{\Omega} \leq 0$. This concludes the proof. \square

It is worth saying that the diagonal structure required for P_0 is feasible since the existence of a diagonal matrix P_0 , as a solution of the Lyapunov equation $\text{He}(P_0 \bar{A}_0) < 0$, is equivalent to the stability of a Metzler matrix \bar{A}_0 .

Therefore, the control (22) provides exponential convergence to zero for the interval predictor (20). Moreover, due to the fact that \mathcal{A}_0 is a Metzler matrix and $e_{\underline{0}} \leq e(0) \leq e_{\bar{0}}$, for two vectors $e_{\underline{0}}, e_{\bar{0}} \in \mathbb{R}^3$, the inclusion property $e(t) \leq e(t) \leq \bar{e}(t)$ is satisfied (see, for more details, *e.g.*,³⁸ and³⁷); hence, the trajectories of system (5) will also converge to zero and the considered problem will be properly solved provided that $x(t) \in \mathbb{X}$, $y(t) \in \mathbb{Y}$, $v(t) \in \mathbb{V}$ and $\omega(t) \in \mathbb{W}$, for all $t \geq 0$, hold.

Now, we are able to define the set \mathbb{E}_f as follows

$$\mathbb{E}_f = \{z \in \mathbb{R}^6 : V_z(z) \leq \beta^{-1} \varepsilon\}, \quad (28)$$

for a positive constant ε . It is clear that for any $\varepsilon > 0$, \mathbb{E}_f is an invariant set for the system (20); and moreover, all the trajectories, inside of \mathbb{E}_f , converge to zero. Therefore, it is always possible to select ε such that $\mathbb{E}_f \subset \mathbb{E} \times \mathbb{E}$, *i.e.*, such that $x(t) \in \mathbb{X}$ and $y(t) \in \mathbb{Y}$ hold, for all $t \geq t_f \geq 0$.

5.4 | Model Predictive Control Design

Before proceeding with the description of the MPC scheme, we need to introduce the following assumption.

Assumption 1. *There exist $\zeta(e)$, K_0 , K_1 and K_2 , satisfying the conditions of Lemmas 2 and 3, such that*

$$-\zeta(e(t)) \frac{(GB)^T s(t)}{\|(GB)^T s(t)\|} + K_0 z(t) + K_1 z^+(t) + K_2 z^-(t) \in \mathbb{U},$$

for any $z \in \mathbb{E}_f$ and all $t \geq t_f \geq 0$.

The previous assumption implies that there always exists a controller (11), given by (14) and (22), such that system (7) is stabilized, and inside \mathbb{E}_f , the state and input constraints hold.

Let us now describe the application of the MPC, which will deal with the state and input constraints for all $z(t_i) \notin \mathbb{E}_f$. Define T and $h \in (0, T)$ as the prediction interval and the application time for the MPC, respectively. Therefore, the optimal control problem for the MPC algorithm is given as follows (see, *e.g.*,¹⁵,¹⁶,¹⁷, and³⁶):

Problem 1. *For given matrices $0 \leq W_1 \in \mathbb{R}^{6 \times 6}$, $l = 0, 1$, a matrix $0 \leq W_2 \in \mathbb{R}^{2 \times 2}$, and $t_i = ih$, with $i \in \mathbb{N}_+$, to find the control signals*

$$\mathcal{U}_0 = \underset{u_0: [t_i, t_i+T] \rightarrow \mathbb{R}^2}{\text{argmin}} z^T(t_i + T) W_0 z(t_i + T) + \int_{t_i}^{t_i+T} [z^T(\sigma) W_1 z(\sigma) + u_0^T(\sigma) W_2 u_0(\sigma)] d\sigma, \quad (29)$$

such that the following constraints hold:

- $z : [t_i, t_i + T] \rightarrow \mathbb{R}^6$ is a solution of (25);
- $z(\sigma) \in \mathbb{E} \times \mathbb{E}$ and $u_0(\sigma) \in \mathbb{U}$ for $\sigma \in [t_i, t_i + T]$;
- $z(t_i + T) \in \mathbb{E}_f$.

Thus, if the above-mentioned optimal control problem is feasible, the trajectories of the system (20) will convergence to the terminal set \mathbb{E}_f , when $u_0(t) = \mathcal{U}_0(t)$; and then, inside \mathbb{E}_f , when $u_0(t) = \bar{u}_0(t)$, the trajectories will converge to zero satisfying the state and input constraints given in (8) and (9).

Finally, the statements of Lemmas 2 and 3, and the solution of the Problem 1 will provide the main result of this paper, which is described by the following Theorem.

Theorem 1. *Let Assumption 1, the conditions of Lemmas 2 and 3 be satisfied and Problem 1 be feasible. If the control law (11), given by (14) and (21), is applied to the system (5) and designed according to Lemmas 2 and 3, and the solution of Problem 1; then, the tracking error dynamics (5) is UES and satisfies the state and input constraints given in (8) and (9).*

Proof. Recall that the controller consists of two parts: u_1 , based on ISMC, that is able to perform the rejection of the uncertain matched terms affecting the system; while u_0 is a nominal stabilizing controller, which is composed of an interval predictor–based state–feedback controller and the MPC law, which will deal with the state and input constraints.

1. First, note that the interval predictor (20) generates an estimation $z \in \mathbb{R}^6$ such that the relation $\underline{e}(t) \leq e(t) \leq \bar{e}(t)$ is satisfied. Then, consider the control u_0 and note that, for some $t_f > 0$, if the initial conditions $z_0 \in \mathbb{E}_f \subset \mathbb{E} \times \mathbb{E}$; then, the control (21) equals to (22). According to (28) and Assumption 1, $z(t) \in \mathbb{E}_f$ and $u(t) \in \mathbb{U}$, for all $t \geq t_f$, and the trajectories of system converge to zero due to the results of Lemma 3.
2. Now, let $z_0 \notin \mathbb{E}_f$ and assume that there is a solution of the Problem 1. Then, applying the control (29), for $t \in [0, T)$, one has that $z(t) \in \mathbb{E} \times \mathbb{E}$, and $u(t) \in \mathbb{U}$, on this time interval. Note that \mathbb{E} is a neighborhood of the origin, and the cost function given in (29), with positive definite matrices W_0 , W_1 , and W_2 , is minimized inside $\mathbb{E} \times \mathbb{E}$. Using these arguments, since $z(t_i + T) \in \mathbb{E} \times \mathbb{E}$ provided that the control \mathcal{U} in (29) is applied, and $[e^\top(t_i), \bar{e}^\top(t_i)] \subset [e^\top(t_{i-1} + T), \bar{e}^\top(t_{i-1} + T)]$, for all $i \geq 1$, there is a finite time instant $t_f \geq T$ such that $(e^\top(t_f), \bar{e}^\top(t_f))^\top \in \mathbb{E}_f$; and then, the control law (21) switches from the control signal \mathcal{U}_0 computed by the MPC to the state–feedback control law u_0 , which provides exponential stability of $e = 0$.
3. As a result of the previous statements, the system constraints are satisfied. This concludes the proof. □

Remark 3. *The nominal control u_0 is responsible for the performance of the tracking error since the ISMC only deals with the matched perturbations. In this sense, if we want to modify the performance; then, we would have to modify the performance index (29) to additionally include the optimization of the desired characteristics, e.g., the convergence time. This is a different, interesting, and not straightforward, optimization control problem for the MPC framework. Moreover, note that it is not possible to arbitrarily optimize the transients when we have energy, velocity, or input, constraints.*

6 | SIMULATION RESULTS

The simulations have been done in MATLAB with the Euler discretization method, sampling–time equal to 0.001[s] and the solutions for the corresponding LMIs have been found by means of SDPT3 solver among YALMIP in MATLAB while the MPC has been implemented using the `nLmpc` toolbox in MATLAB.

It is possible to fix the matrix A_0 as

$$A_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0.3914\bar{v}_d & 0 & -1 \end{pmatrix},$$

and then, the matrices A_i , with $i = \overline{1, 4}$, are computed by a convex polytopic approach, which guarantees the fulfillment of (10), i.e., such that $\sum_{j=1}^4 \alpha_j(\rho_1)A_j = A(\rho_1) - A_0$, obtaining

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 12.80 \\ 0.6086\bar{v}_d & -12.80 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -12.80 \\ 0.6086\bar{v}_d & 12.80 & 1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 12.80 \\ -0.6086\bar{v}_d & -12.80 & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 12.80 \\ -0.6086\bar{v}_d & -12.80 & 1 \end{pmatrix}.$$

6.1 | Tracking a Lemniscate Curve

The initial conditions for the kinematics are $x_0 = 1.5$ [m], $y_0 = -1.5$ [m], and $\theta_0 = 2$ [rad], and the state and input constraints sets are given as $\mathbb{X} = [-1.5, 1.5]$, $\mathbb{Y} = [-1.5, 1.5]$, $\mathbb{V} = [-5, 5]$ and $\mathbb{W} = [-8, 8]$. The desired lemniscate trajectory is given by

$$\omega_d(t) = \frac{\dot{x}_d \ddot{y}_d - \dot{y}_d \ddot{x}_d}{\dot{x}_d^2 + \dot{y}_d^2}, \quad v_d(t) = \sqrt{\dot{x}_d^2 + \dot{y}_d^2}, \quad x_d(t) = c(0.2094t), \quad y_d(t) = s(0.4188t), \quad \theta_d(t) = \int_0^t \omega_d(\tau) d\tau,$$

and thus, $\bar{\omega}_d = 1.221$ [rad/s] and $\bar{v}_d = 0.4682$ [m/s]. For robustness purposes, the external perturbations are taken as $d_1(t) = 0.1s(3t) + 0.5$ and $d_2(t) = 0.1c(t) + 0.5$, and hence, $d_{\max} = 0.6$.

For the interval predictor, the initial conditions are $\underline{z}_0 = (-0.6292, 1.3720, -0.3696)^\top$ and $\bar{z}_0 = (-0.2292, 1.7720, 0.0304)^\top$. The parameters for the ISMC are fixed, following the statements of Lemma 2, as $F_{\max} = 4.8$, $A_{\max} = 0.0512$, and $\gamma = 1$. The gains of the state–feedback controller are assessed with the LMIs of Lemma 3, *i.e.*,

$$\begin{aligned} K_0 &= \begin{pmatrix} -0.0212 & 0 & -0.0257 & -0.0212 & 0 & -0.0257 \\ 0 & -0.0134 & 0 & 0 & -0.0134 & 0 \end{pmatrix}, \\ K_1 &= \begin{pmatrix} -0.1392 & -0.0001 & -0.0530 & -0.2866 & -0.0620 & -0.1003 \\ -0.0001 & -0.1367 & -0.0001 & -0.0002 & -111.7967 & -0.0002 \end{pmatrix}, \\ K_2 &= \begin{pmatrix} 0.2863 & -0.0868 & 0.1000 & 0.1391 & -0.0001 & 0.0528 \\ -0.0002 & 111.7143 & -0.0002 & -0.0001 & 0.1365 & -0.0001 \end{pmatrix}. \end{aligned}$$

The prediction interval is selected as $T = 0.005$ [s], with weight matrices $W_0 = I$, $W_1 = 100I$ and $W_2 = 0.1I$, and we compute the corresponding switching set \mathbb{E}_f according to (28), with $\beta = 5.6484$ and $\varepsilon = 0.0001$.

The results are shown in Figs. 2, 3 and 4. The results depicted by Figs. 2 a) and 3 show that the proposed control strategy is able to ensure the trajectory tracking task without transgressing the state and input constraints, respectively. In Fig. 2 b), we can see the system trajectories without the ISMC approach. This result illustrates a comparison between the controller proposed in³⁶, *i.e.*, an interval predictor–based state–feedback controller with the MPC, and our proposed approach. It is clear that the perturbations, without being compensated, cause the system trajectories to transgress the state constraints. Additionally, Fig. 4 illustrates the behavior of the tracking errors, which exponentially converge to zero.

In order to illustrate, through simulations, that the computational burden is relaxed in our control algorithm, with respect to a classic MPC, it is sufficient to determine the time required to simulate the proposed switched scheme and the scheme when the MPC is active all the time. Considering that the simulations have been done in MATLAB with the Euler discretization method, sampling–time equal to 0.001[s], simulation time equal to 50[s], while the MPC has been implemented using the `nmpc` toolbox in MATLAB, the proposed switching scheme takes around 45 seconds to perform the simulation, while the scheme when the MPC is active all the time takes more than 120 seconds to perform the same simulation for the same initial conditions.

6.2 | Tracking a Circumference Curve

The initial conditions for the kinematics are $x_0 = 1.4$ [m], $y_0 = 1.4$ [m], and $\theta_0 = 0$ [rad], and the state and input constraints sets are given as $\mathbb{X} = [-1.5, 1.5]$, $\mathbb{Y} = [-1.5, 1.5]$, $\mathbb{V} = [-5, 5]$ and $\mathbb{W} = [-8, 8]$. The desired circumference trajectory is given by

$$\omega_d(t) = \frac{\dot{x}_d \ddot{y}_d - \dot{y}_d \ddot{x}_d}{\dot{x}_d^2 + \dot{y}_d^2}, \quad v_d(t) = \sqrt{\dot{x}_d^2 + \dot{y}_d^2}, \quad x_d(t) = 1.5c(0.4t), \quad y_d(t) = 1.5s(0.4t), \quad \theta_d(t) = \int_0^t \omega_d(\tau) d\tau,$$

and thus, $\bar{\omega}_d = 0.4$ [rad/s] and $\bar{v}_d = 0.6$ [m/s]. For robustness purposes, the external perturbations are taken as $d_1(t) = 0.1s(3t) + 0.5$ and $d_2(t) = 0.1c(t) + 0.5$, and hence, $d_{\max} = 0.6$.

For the interval predictor, the initial conditions are $\underline{z}_0 = (1.3708, -0.1, -1.6)^\top$ and $\bar{z}_0 = (1.7708, 0.3, -1.2)^\top$. The parameters for the ISMC are fixed, following the statements of Lemma 2, as $F_{\max} = 4.8$, $A_{\max} = 0.0512$, and $\gamma = 1$. The gains of the

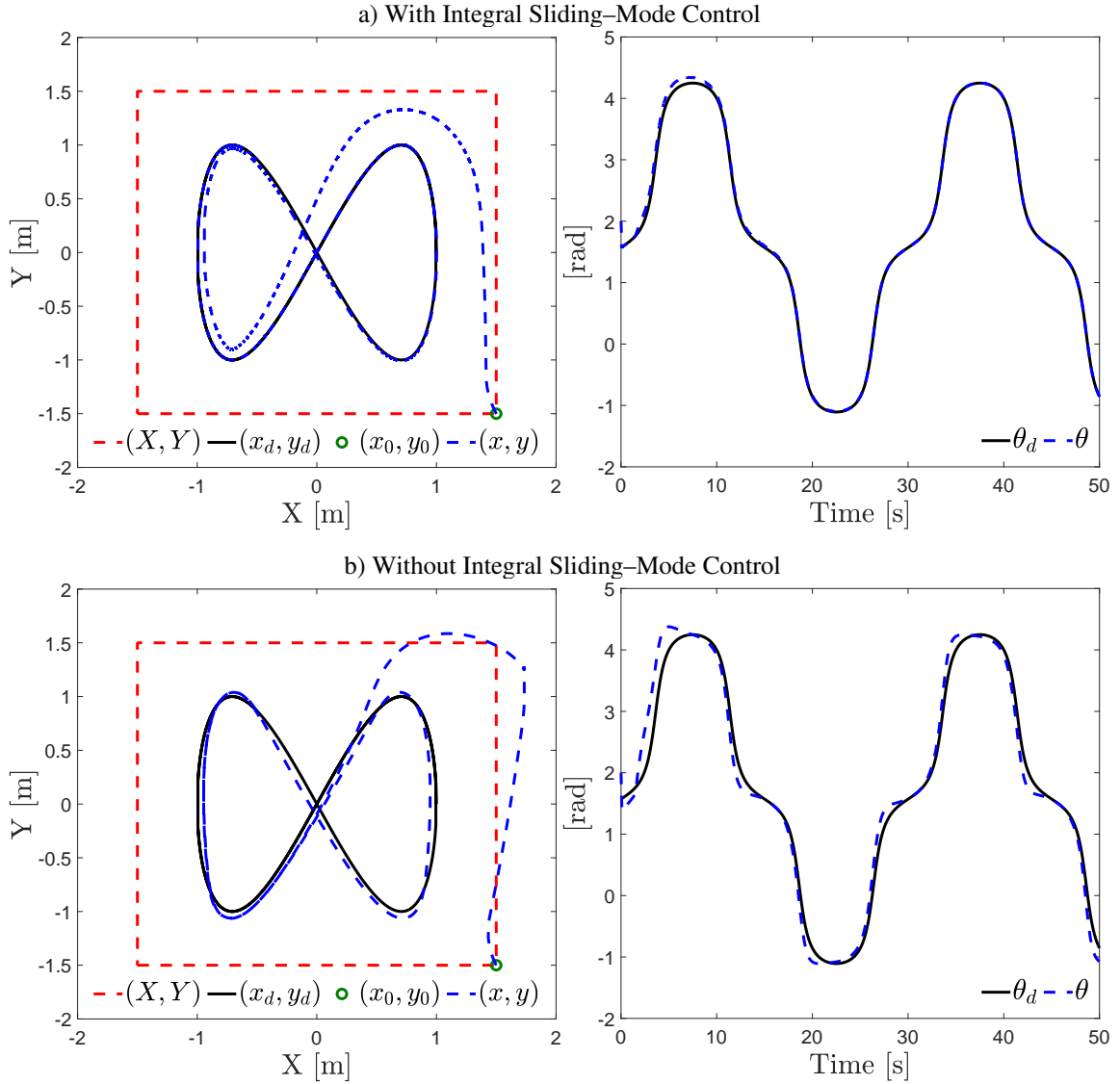


FIGURE 2 System Trajectories – Lemniscate Curve

state-feedback controller are assessed with the LMIs of Lemma 3, *i.e.*,

$$\begin{aligned}
 K_0 &= \begin{pmatrix} -0.0270 & 0 & -0.0323 & -0.0270 & 0 & -0.0323 \\ 0 & -0.0134 & 0 & 0 & -0.0134 & 0 \end{pmatrix}, \\
 K_1 &= \begin{pmatrix} -0.1412 & -0.0001 & -0.0667 & -0.3749 & -0.0641 & -0.1266 \\ -0.0001 & -0.1366 & -0.0001 & -0.0002 & -115.5969 & -0.0002 \end{pmatrix}, \\
 K_2 &= \begin{pmatrix} 0.3744 & -0.0897 & 0.1263 & 0.1410 & -0.0001 & 0.0666 \\ -0.0003 & 115.5138 & -0.0002 & -0.0001 & 0.1365 & -0.0001 \end{pmatrix}.
 \end{aligned}$$

The prediction interval is selected as $T = 0.005$ [s], with weight matrices $W_0 = I$, $W_1 = 10000I$ and $W_2 = 0.1I$, and we compute the corresponding switching set \mathbb{E}_f according to (28), with $\beta = 5.4637$ and $\varepsilon = 0.0011$.

The results are given in Figs. 5, 6 and 7. The results shown by Figs. 5 a) and 6 demonstrate that the proposed control strategy is able to ensure the trajectory tracking task without transgressing the state and input constraints, respectively. The system trajectories without the ISMC approach are shown in Fig. 5 b). Once again, we can see that the perturbations, without being compensated, cause the system trajectories to transgress the state constraints. Finally, Fig. 7 depicts the behavior of the tracking errors that also converge to zero exponentially.

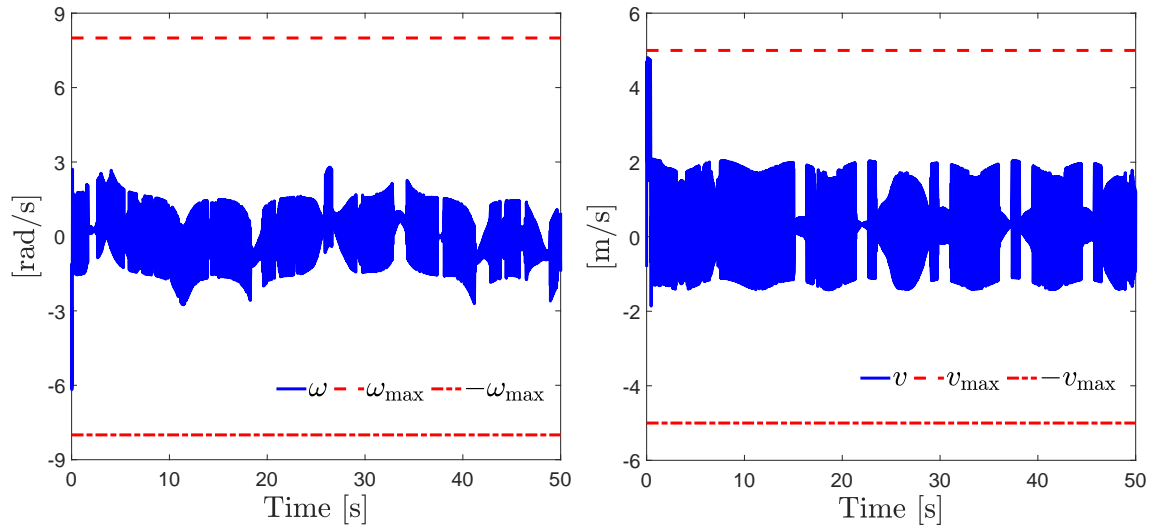


FIGURE 3 Control Signals – Lemniscate Curve

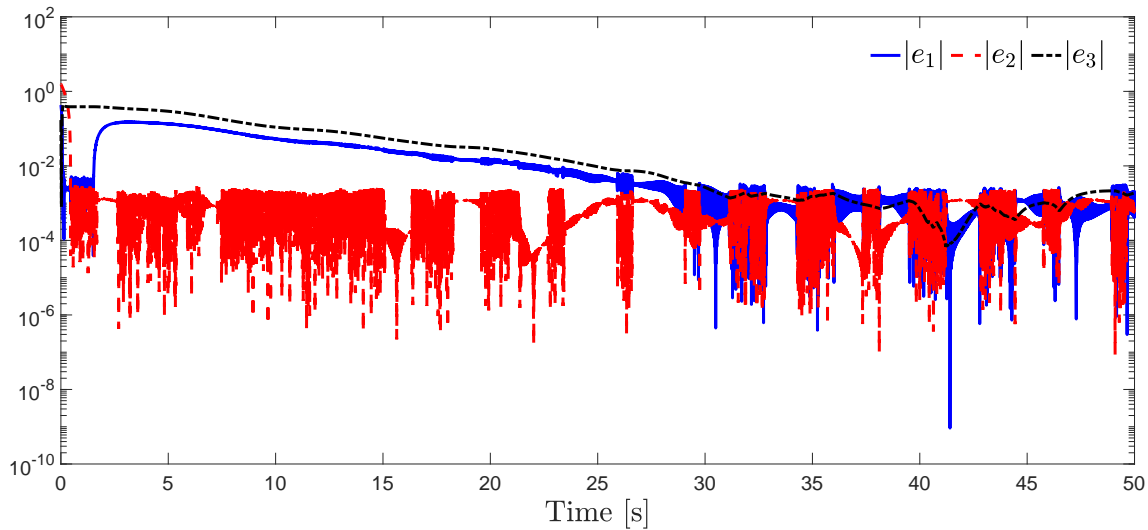
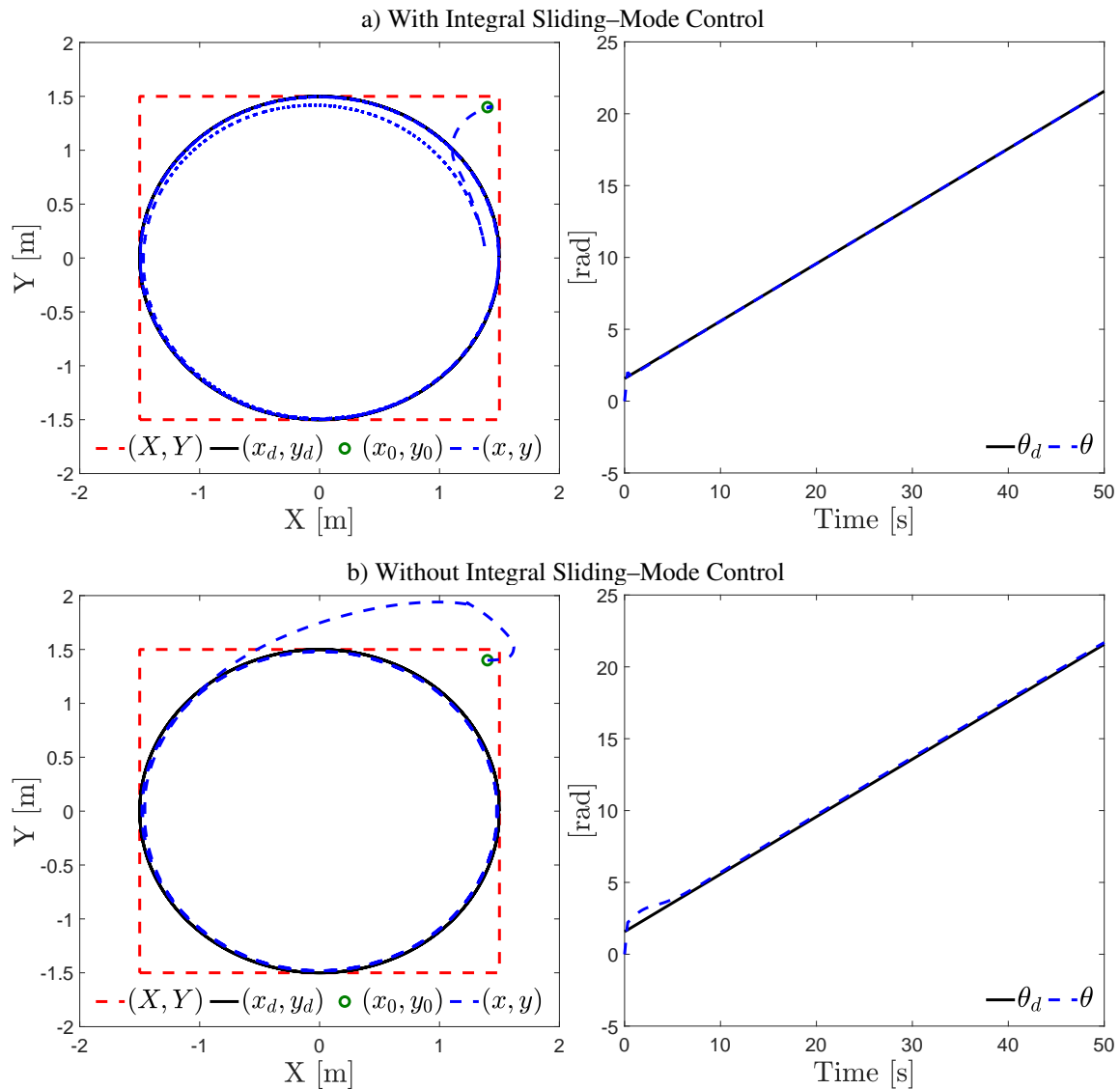


FIGURE 4 Tracking Errors – Lemniscate Curve

7 | CONCLUSIONS

This paper contributes to the design of a robust control strategy for the trajectory tracking problem in perturbed unicycle mobile robots, considering state and input constraints. We propose a robust controller, which is based on an ISMC approach together with an interval predictor-based state-feedback controller with an MPC scheme. The robust controller deals with some perturbations in the kinematic model, which represent slipping effects, and with state and input constraints that are related to restrictions on the workspace and saturated actuators, respectively; and it guarantees the exponential convergence to zero of the tracking error. Moreover, the synthesis of the proposed controller is constructive since is based on LMIs. Finally, the performance of the proposed approach is verified through some simulations, which show the tracking performance for a lemniscate and a circumference curve.



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CONFLICT OF INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

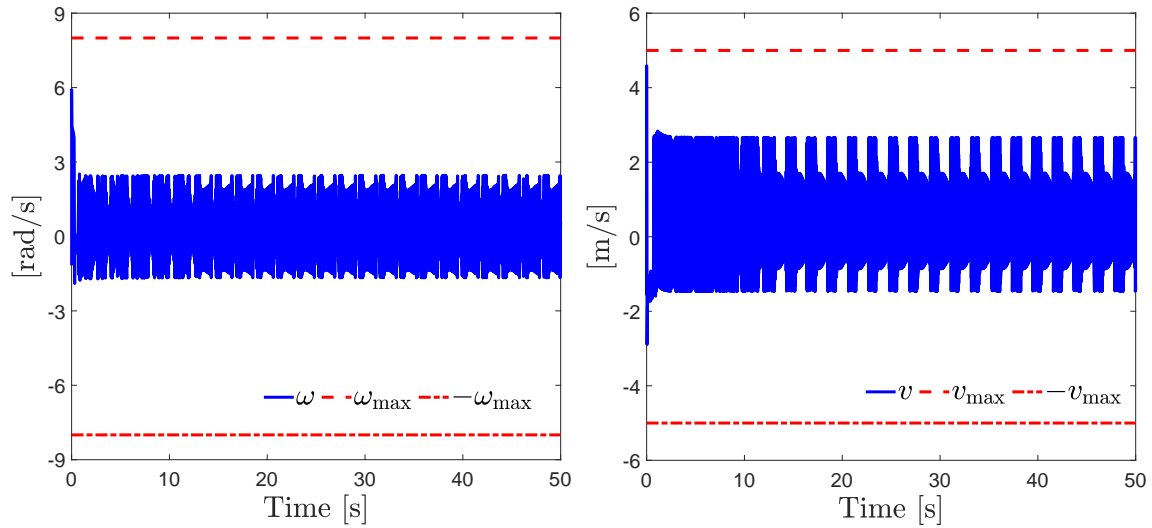


FIGURE 6 Control Signals – Circumference Curve

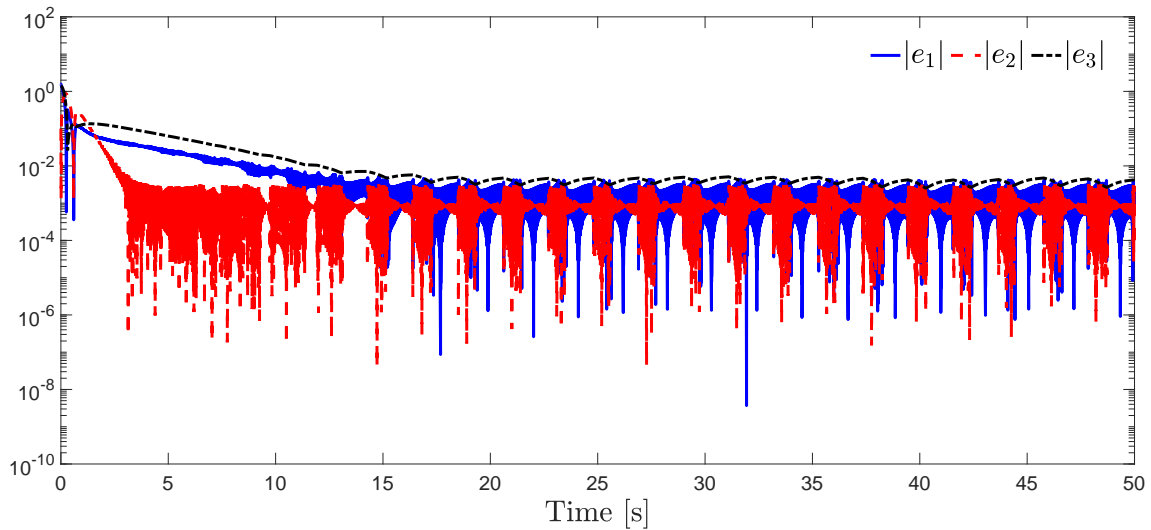


FIGURE 7 Tracking Errors – Circumference Curve

DATA AVAILABILITY STATEMENT

Data sharing not applicable—no new data generated, or the article describes entirely theoretical research.

References

1. Zhang H, Sun J, Wang Z. Game-Theoretical Persistent Distributed control of nonholonomic robots without global position measurements subject to unknown slippage constraints. *IEEE/CAA Journal of Automatica Sinica* 2022; 9(2): 354-364.
2. Khaledyan M, Liu T, Fernandez-Kim V, Queiroz dM. Flocking and Target Interception Control for Formations of Nonholonomic Kinematic Agents. *IEEE Transactions on Control Systems Technology* 2015; 28(4): 1603-1610.
3. Zhang M, Liu H. Game-Theoretical Persistent Tracking of a Moving Target Using a Unicycle-Type Mobile Vehicle. *IEEE Transactions on Industrial Electronics* 2014; 61(11): 6222-6233.

4. Brockett R. Control theory and singular Riemannian geometry. *New directions in applied mathematics* 1982; 11–27.
5. Duleba I, Khefifi W, Karcz-Duleba I. Layer, Lie algebraic method of motion planning for nonholonomic systems. *Journal of the Franklin Institute* 2012; 349(1): 201–215.
6. Tayebi A, Rachid A. Adaptive controller for non-holonomic mobile robots with matched uncertainties. *Advanced Robotics* 2000; 14(2): 105–118.
7. Wang D, Low CB. Modeling and Analysis of Skidding and Slipping in Wheeled Mobile Robots: Control Design Perspective. *IEEE Transactions on Robotics* 2008; 24(3): 676–687.
8. Thomas M, Bandyopadhyay B, Vachhani L. Finite-time posture stabilization of the unicycle mobile robot using only position information: A discrete-time sliding mode approach. *International Journal of Robust and Nonlinear Control* 2019; 29(6): 1990–2006.
9. Mera M, Ríos H, Martínez E. A sliding-mode based controller for trajectory tracking of perturbed Unicycle Mobile Robots. *Control Engineering Practice* 2020; 102: 104548.
10. Rochel P, Ríos H, Mera M, Dzul A. Trajectory tracking for uncertain Unicycle Mobile Robots: A Super-Twisting approach. *Control Engineering Practice* 2022; 122: 105078.
11. Singhal K, Kumar V, Rana K. Robust trajectory tracking control of non-holonomic wheeled mobile robots using an adaptive fractional order parallel fuzzy PID controller. *Journal of the Franklin Institute* 2022; 359(9): 4160–4215.
12. Rocha E, Castañón F, Moreno J. Robust finite-time stabilisation of an arbitrary-order nonholonomic system in chained form. *Automatica* 2022; 135: 109956.
13. Ríos H, Mera M, Polyakov A. Perturbed Unicycle Mobile Robots: A Second–Order Sliding–Mode Trajectory Tracking Control. *IEEE Transactions on Industrial Electronics* 2024; 71(3): 2864–2872.
14. Zhou Y, Ríos H, Mera M, Polyakov A, Zheng G, Dzul A. Homogeneity–based Control Strategy for Trajectory Tracking in Perturbed Unicycle Mobile Robots. *IEEE Transactions on Control Systems Technology* 2023. DOI:10.1109/TCST.2023.3300273.
15. Mayne DQ, Ravokic SV, Findeisen R, Allgower F. Robust output feedback model predictive control of constrained linear systems: time-varying case. *Automatica* 2009; 45: 2082–2087.
16. Mayne DQ, Rawlings JB, Rao CV, Sokaert POM. Constrained model predictive control: stability and optimality. *Automatica* 2000; 36: 789–814.
17. Michalska H, Mayne DQ. Robust receding horizon control of constrained nonlinear systems. *IEEE Transactions on Automatic Control* 1993; 38(11): 1623–1633.
18. Xiao H, Li Z, Yang C, et al. Robust Stabilization of a Wheeled Mobile Robot Using Model Predictive Control Based on Neurodynamics Optimization. *IEEE Transactions on Industrial Electronics* 2017; 64(1): 505–516.
19. Sun Z, Dai L, Liu K, Xia Y, Johansson K. Robust MPC for tracking constrained unicycle robots with additive disturbances. *Automatica* 2018; 90: 172–184.
20. Cao Q, Sun Z, Xia Y, Dai L. Self-triggered MPC for trajectory tracking of unicycle-type robots with external disturbance. *Journal of the Franklin Institute* 2019; 356(11): 5593–5610.
21. Sun Z, Xia Y, Dai L, Campoy P. Tracking of Unicycle Robots Using Event-Based MPC With Adaptive Prediction Horizon. *IEEE/ASME Transactions on Mechatronics* 2020; 25(2): 739–749.
22. Galván-Guerra R, Incremona GP, Fridman L, Ferrara A. Robust multi–model predictive control via integral sliding modes. *IEEE Control Systems Letters* 2022; 6: 2623–2628.
23. Incremona GP, Ferrara A, Magni L. Asynchronous networked MPC with ISM for uncertain nonlinear systems. *IEEE Transactions on Automatic Control* 2017; 62(9): 4305–4317.

24. Rubagotti M, Incremona GP, Raimondo DM, Ferrara A. Constrained nonlinear discrete-time sliding mode control based on a receding horizon approach. *IEEE Transactions on Automatic Control* 2021; 66(8): 3802–3809.
25. Steinberger M, Castillo I, Horn M, Fridman L. Robust output tracking of constrained perturbed linear systems via model predictive sliding mode control. *International Journal of Robust and Nonlinear Control* 2020; 30(3): 1258-1274.
26. Ríos H, Mera M, Raïssi T, Efimov D. An Integral Sliding-Mode-based Robust Interval Predictive Control for Perturbed Unicycle Mobile Robots. *Proceedings of the 62nd IEEE Conference on Decision and Control* 2023.
27. Guerra M, Efimov D, Zheng G, Perruquetti W. Avoiding local minima in the potential field method using input-to-state stability. *Control Engineering Practice* 2016; 55: 174–184.
28. Filippov A. *Differential equations with discontinuous righthand sides*. Kluwer Academic Publishers . 1988.
29. Khalil H. *Nonlinear Systems*. New Jersey, U.S.A.: Prentice Hall . 2002.
30. Polyakov A. Nonlinear feedback design for fixed-time stabilization of linear control systems. *IEEE Transactions on Automatic Control* 2012; 57(8): 2106-2110.
31. Efimov D, Perruquetti W, Raïssi T, Zolghadri A. Interval observers for time-varying discrete-time systems. *IEEE Transactions on Automatic Control* 2013; 58(12): 3218-3224.
32. Utkin V, Shi J. Integral sliding mode in systems operating under uncertainty conditions. *Proceedings of the 35th IEEE Conference on Decision and Control* 1996: 4591-4596.
33. Castaños F, Fridman L. Analysis and Design of Integral Sliding Manifolds for Systems With Unmatched Perturbations. *IEEE Transactions on Automatic Control* 2006; 51(5): 853-858.
34. Utkin V, Guldner J, Shi J. *Sliding Modes in Electromechanical Systems*. London: Taylor and Francis . 1999.
35. dos Reis de Souza A, Efimov D, Raïssi T. Robust Output Feedback MPC for LPV Systems Using Interval Observers. *IEEE Transactions on Automatic Control* 2022; 67(6): 3188-3195.
36. Leurent E, Efimov D, Maillard OA. Robust-Adaptive Interval Predictive Control for Linear Uncertain Systems. *Proceedings of the 59th IEEE Conference on Decision and Control* 2020: 1429-1434.
37. Efimov D, Raïssi T, Zolghadri A. Control of nonlinear and LPV systems: Interval observer-based framework. *IEEE Transactions on Automatic Control* 2013; 58(3): 773-778.
38. Efimov D, Fridman L, Raïssi T, Zolghadri A, Seydou R. Interval Estimation for LPV Systems Applying High Order Sliding Mode Techniques. *Automatica* 2012; 48(9): 2365-2371.

