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Schrödinger eigenfunctions sharing the same modulus and applications to the control of quantum systems

Ugo Boscain, Kévin Le Balc'h, Mario Sigalotti

March 12, 2024

Abstract

In this paper we investigate when linearly independent eigenfunctions of the Schrödinger operator may have the same modulus. General properties are established and the one-dimensional case is treated in full generality. The study is motivated by its application to the bilinear control of the Schrödinger equation. By assuming that the potentials of interaction satisfy a saturation property and by adapting a strategy recently proposed by Duca and Nersesyan, we discuss when the system can be steered arbitrarily fast between energy levels. Extensions of the previous results to quantum graphs are finally presented.

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1 Introduction

Consider a controlled (bilinear) quantum system

$$i\dot{\psi}(t) = \left(H_0 + \sum_{j=1}^m u_j(t)H_j \right) \psi(t), \quad t \in [0, +\infty). \quad (1.1)$$

In (1.1), $\psi(t)$ represents the state of the system, evolving in a Hilbert space \mathcal{H} , and $u(t) = (u_1(t), \dots, u_m(t)) \in \mathbb{R}^m$ is the control that can take arbitrarily large values.

One of the fundamental questions in quantum control is the following: is it possible to induce a transition from an energy level to another one in arbitrarily small time? Since in (1.1) quantum decoherence is neglected, the model may only be applicable for small times. This is why results of controllability in arbitrarily small time are particularly important for applications.

In the case in which the Hilbert space \mathcal{H} where ψ is evolving is finite-dimensional, this problem is well understood. Actually small-time controllability is possible if and only if the evaluation at each point of the unit sphere of the Lie algebra $\text{Lie}\{iH_1, \dots, iH_m\}$ generated by the (right-invariant) vector fields of the controlled part is of maximal rank, see for instance [ABGS17, Theorem 2]. As a consequence, small-time controllability is never possible when the operators H_1, \dots, H_m commute pairwise and in particular when $m = 1$.

The case of an infinite-dimensional Hilbert space could be very different due to the possible unboundedness of H_0 . This happens, for instance, for the controlled PDE

$$i\partial_t \psi = -\Delta_g \psi + V(x)\psi + \sum_{j=1}^m u_j(t)Q_j(x)\psi, \quad t \in [0, +\infty), \quad (1.2)$$

on a Riemannian manifold (M, g) . Here Δ_g is the Laplace–Beltrami operator, V is the *internal potential*, Q_1, \dots, Q_m are *potentials of interaction*, and H_0 is the unbounded operator $-\Delta_g + V$. Notice that in this case we are precisely in the situation in which the multiplication operators $H_1 = Q_1, \dots, H_m = Q_m$ commute pairwise.

In the last years, this subject attracted the attention of several researchers. A first example of system of the form (1.1) evolving in an infinite-dimensional space with $m = 1$ and in which transitions among energy levels in arbitrarily small time are possible has been presented in [BCC12]. Although very interesting, such an example is academic ($i\partial_t \psi = -\left(\frac{d^2}{dx^2}\right)^\alpha \psi + u(t) \cos(x)\psi$, on the circle \mathbb{S}^1 , $\alpha > 5/2$) and for almost ten years it seemed a safe bet that for systems in the form (1.2) such phenomenon never occurs. Several results in this direction have been obtained (see, for instance, [BCT14, BCT18] and [BBS21]).

A surprising result going in the opposite direction was provided more recently by Duca and Nersesyan [DN24], who proved that on the d -dimensional torus, for systems of the type (1.2) with $V = 0$, certain transitions among energy levels are possible in arbitrarily small time. In particular it was proven that it is possible to reach states of arbitrarily large energy in arbitrarily small time. This result has been obtained under an assumption on the potentials of interaction called saturation hypothesis (see Section 2.3 for more details). Such type of hypothesis was inspired by a similar one introduced by Agrachev and Sarychev for the control of the Navier–Stokes equations in [AS05] and [AS06].

The result from [DN24] was extended and proven in a simpler form in [CP23]. In particular in that paper examples of systems evolving on manifolds with a topology different from that of the torus were studied.

The techniques developed in [DN24] and [CP23] actually permit to prove that (under the saturation hypothesis) it is possible to steer in arbitrarily small time a state $\psi(x)$ arbitrarily close to a state of the form $e^{i\theta(x)}\psi(x)$ for some real-valued function $\theta(x)$.

Then the possibility of making an arbitrarily fast transition among different energy levels is reduced to the problem of *the existence of eigenfunctions corresponding to different energy levels that share the same modulus*. This is the property that we study in this paper.

To clarify the situation, let us discuss briefly the case of the circle $\mathbb{R}/2\pi\mathbb{Z}$. The eigenvalues of $H_0 = -\frac{d^2}{dx^2}$ are $\lambda_k = k^2$, $k = 0, 1, 2, \dots$. The fundamental state ($k = 0$) is non-degenerate and corresponds to the real eigenfunction $\phi_0 \equiv \frac{1}{\sqrt{2\pi}}$. The other eigenvalues ($k \geq 1$) are double and correspond to the real eigenfunctions $\phi_k^c(x) = \frac{1}{\sqrt{\pi}} \cos(kx)$ and $\phi_k^s(x) = \frac{1}{\sqrt{\pi}} \sin(kx)$. Although these eigenfunctions do not differ by a factor of phase only, a suitable linear combination in each eigenspace permits to construct eigenfunctions all with modulus $1/\sqrt{2\pi}$:

$$\phi_k^\pm(x) = \frac{1}{\sqrt{2}}(\phi_k^c(x) \pm i\phi_k^s(x)) = \frac{1}{\sqrt{2\pi}}e^{\pm ikx}, \quad k = 1, 2, \dots \quad (1.3)$$

With the technique developed by Duca and Nersesyan, it is possible to induce a transfer of population in arbitrarily small time among the eigenfunctions (1.3). This can be done approximately and suitable potentials of interaction are necessary.

The purpose of this paper is to study whether the presence of linearly independent eigenfunctions sharing the same modulus is a specific feature of this problem or can occur in more general situations.

The structure of the paper is the following.

In Section 2 we give some basic definitions and examples concerning Schrödinger operators having eigenfunctions sharing the same modulus. We then introduce basic concepts about small-time approximate controllability, we introduce the saturation hypothesis and the result by Duca and Nersesyan [DN24]. We then state in Theorem 2.19 a new result concerning small-time approximate controllability between states sharing the same modulus adapted to manifolds with boundary, in the spirit of the ones given by Duca and Nersesyan for the torus [DN24] and Chambrion and Pozzoli for manifolds without boundary [CP23]. The proof that we give in Appendix A generalizes immediately to the case of the graphs studied in Section 4.

In Section 3 we study conditions for Schrödinger operators to admit eigenfunctions sharing the same modulus in the general case and in the one-dimensional case (Section 3.2). Our main result in the one-dimensional case is Theorem 3.6 that we reformulate here for convenience.

Theorem 1.1. *Let M be a smooth and connected one-dimensional manifold (possibly with boundary) and a coordinate x on M . In $L^2(M, dx)$, consider the Schrödinger operator $H_0 = -\partial_x^2 + V$ (with $V \in L_{\text{loc}}^\infty(M, dx)$ bounded from below and Neumann or Dirichlet boundary conditions if M has boundary) that we assume to be self-adjoint and with compact resolvent. If H_0 admits two \mathbb{C} -linearly independent eigenfunctions ϕ_k and ϕ_ℓ sharing the same modulus, then necessarily M is a closed curve and ϕ_k, ϕ_ℓ are nowhere vanishing on M . If, moreover, ϕ_k, ϕ_ℓ correspond to distinct eigenvalues, then V is constant.*

One can read this result in the following way: the requirement of the existence of eigenfunctions sharing the same modulus for all eigenvalues gives a topological constraint (the manifold should be the circle) and a condition on the potential V that must be constant. However if we are interested in transitions in between the same eigenspace, then non-constant potentials are admitted. See Remark 3.8 for an explicit example.

In the two-dimensional case, we give some new results concerning the sphere \mathbb{S}^2 , showing that it is possible to induce small-time transitions among states (which are not eigenstates) having different expectation values of the energy (see Proposition 2.7). This fact is new since for the sphere \mathbb{S}^2 only the possibility of making small-time transitions between states of the type Y_m^l to Y_{-m}^l was known (see [CP23]). For the disk \mathbb{D} , we also prove that two eigenfunctions of

different energy levels cannot share the same modulus while it is possible to find two independent eigenfunctions of the same energy level that share the same modulus, see Proposition 2.8. This result is new and leads to the possibility of making small-time transitions among states having the same energy as for the sphere \mathbb{S}^2 .

Beside the two-dimensional torus, the sphere \mathbb{S}^2 and the disk \mathbb{D} , the study of other systems in dimension 2 is complicated by the lack of explicit expressions for the eigenfunctions.

We then study (Section 4) problems that in a sense are intermediate between dimension 1 and 2, namely problems on graphs. Actually graphs are obtained by gluing together one-dimensional manifolds, but their topology is much richer than the topology of one-dimensional manifolds and for certain properties they are more similar to higher dimensional manifolds. See, for instance, the survey [Kuc02] studying spectral properties in thin structures collapsing to a graph.

On graphs we exhibit some new phenomena. We show the existence of graphs for which

- there exists an infinite sequence of eigenfunctions corresponding to arbitrarily high energy levels that share the same modulus, but this is not the case for all energy levels. See Section 4.4;
- all energy levels but the ground state are degenerate, but there exists no pair of eigenfunctions sharing the same modulus and having different energy levels. See Section 4.5.

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2 Basic definitions

2.1 The Schrödinger operator on a Riemannian manifold

Let M be a smooth and connected manifold of dimension d , possibly with boundary, equipped with a Riemannian metric g . We assume that M , endowed with the Riemannian distance, is complete as a metric space. Let $\Delta_g = \operatorname{div}_{\omega_g} \circ \nabla_g$ be the Laplace–Beltrami operator on (M, g) , where $\operatorname{div}_{\omega_g}$ is the divergence operator with respect to the Riemannian volume and ∇_g is the Riemannian gradient. When $\partial M \neq \emptyset$, we split $\partial M = \cup_{j \in I} \mathcal{C}_j$ where $\{\mathcal{C}_j \mid j \in I\}$ is the set of the connected components of ∂M .

We define the Hilbert space $\mathcal{H} = L^2(M; \mathbb{C})$, where integration is considered with respect to the Riemannian volume. For simplicity, in the following we write $L^2(M)$ for $L^2(M; \mathbb{C})$ and, similarly, $H^k(M)$ for $H^k(M; \mathbb{C})$, $k \geq 1$. Let $V \in L_{\text{loc}}^\infty(M; \mathbb{R})$ be bounded from below, $H_0 = -\Delta_g + V$, and define

$$\operatorname{Dom}(H_0) := \{\psi \in H^2(M) \mid -\Delta_g \psi + V\psi \in L^2(M) \text{ and } P_j(\psi) = 0 \text{ on } \mathcal{C}_j \ \forall j \in I\}, \quad (2.1)$$

where, for every $j \in I$, $P_j(\psi)$ is either the trace of ψ on \mathcal{C}_j or the trace of $\partial_\nu \psi$ on \mathcal{C}_j with ν the outer unit normal vector to M . In the sequel, we split $I = I_D \cup I_N$, where I_D (respectively, I_N) corresponds to the set of indices such that $P_j(\psi)$ is the trace of ψ on \mathcal{C}_j (respectively, the trace of $\partial_\nu \psi$ on \mathcal{C}_j).

Assumption 2.1. The unbounded Schrödinger operator $(H_0, \operatorname{Dom}(H_0))$ is self-adjoint on $L^2(M)$ with compact resolvent.

As a consequence of Assumption 2.1, from the spectral theorem (see, e.g., [CR21, Theorem 6.2]), there exists an orthonormal basis of $L^2(M; \mathbb{C})$ composed of eigenfunctions $(\phi_k)_{k \geq 1}$ associated with the sequence of real eigenvalues $(\lambda_k)_{k \geq 1}$, that is for every $k \geq 1$, $\phi_k \in E_{\lambda_k} = \operatorname{Ker}(H_0 - \lambda_k I)$. Let us first give some examples, that are mainly taken from [Cha84], on which we will focus in the following.

1. Let $M = \mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$ be the d -dimensional torus, Δ_g be the Euclidean Laplace operator, $V \in L^\infty(\mathbb{T}^d; \mathbb{R})$. The unbounded operator $(-\Delta_g + V, H^2(\mathbb{T}^d))$ satisfies Assumption 2.1.
2. Let $M = \mathbb{S}^2$ be the two-dimensional sphere equipped with the standard Riemannian metric and Δ_g be the corresponding Laplace–Beltrami operator. Let $V \in L^\infty(\mathbb{S}^2; \mathbb{R})$. The unbounded operator $(-\Delta_g + V, H^2(\mathbb{S}^2))$ satisfies Assumption 2.1.
3. Let Ω be a smooth bounded connected domain of \mathbb{R}^d , Δ_g be the Euclidean Laplace operator, ν be the outer unit normal vector to $\partial\Omega$ and $V \in L^\infty(\Omega; \mathbb{R})$. The unbounded operators $(-\Delta_g + V, H^2(\Omega) \cap H_0^1(\Omega))$ and $(-\Delta_g + V, \{\psi \in H^2(\Omega) \mid \partial_\nu \psi = 0 \text{ on } \partial\Omega\})$ satisfy Assumption 2.1.
4. Let $M = \mathbb{R}^d$, Δ_g be the Euclidean Laplace operator, $V \in L_{\text{loc}}^\infty(\mathbb{R}^d; \mathbb{R})$ be such that $\lim_{\|x\| \rightarrow +\infty} V(x) = +\infty$. Then the unbounded operator $(-\Delta_g + V, \{\psi \in H^2(\mathbb{R}^d) \mid V\psi \in L^2(\mathbb{R}^d)\})$ satisfies Assumption 2.1.

Remark 2.2. Assumption 2.1 does not cover all interesting situations. Consider for instance the Schrödinger operator

$$-\frac{d^2}{dx^2} + \frac{c}{x^2(1-x)^2}, \quad x \in (0, 1).$$

If $c \geq 3/4$ this operator is essentially self-adjoint in $L^2((0, 1))$ (this is a consequence of [RS78], Theorem X.10) and hence this situation is covered by Assumption 2.1 (even if it does not satisfy the standing assumptions, since the metric space $(0, 1)$ endowed with the Euclidean distance is not complete). If $c < 3/4$ this operator is not essentially self-adjoint. Self-adjoint extensions exist, but they do not correspond to boundary conditions of Dirichlet or Neumann type. In other words the latter case is not covered by Assumption 2.1.

2.2 Eigenfunctions sharing the same modulus

The following definition introduces rigorously the key notion of eigenfunctions of the operator $H_0 = -\Delta_g + V$ sharing the same modulus.

Definition 2.3. For $k, \ell \geq 1$, two \mathbb{C} -linearly independent eigenfunctions $\phi_k \in E_{\lambda_k}$ and $\phi_\ell \in E_{\lambda_\ell}$ are said to *share the same modulus* if

$$|\phi_k(x)| = |\phi_\ell(x)|, \quad \text{for every } x \in M. \quad (2.2)$$

We are interested in different occurrences of eigenfunctions sharing the same modulus:

- For $k \geq 1$, H_0 may admit *eigenfunctions sharing the same modulus inside the energy level* λ_k , that is, there may exist two \mathbb{C} -linearly independent eigenfunctions in E_{λ_k} that share the same modulus;
- For $k, \ell \geq 1$, H_0 may admit *two eigenfunctions* $\phi_k \in E_{\lambda_k}$ and $\phi_\ell \in E_{\lambda_\ell}$ *sharing the same modulus and corresponding to different energy levels* λ_k and λ_ℓ ;
- H_0 may admit *eigenfunctions sharing the same modulus corresponding to all energy levels*, that is, there may exist a subsequence $(\phi_{k_j})_{j \geq 1}$ of an orthonormal basis of eigenfunctions $(\phi_k)_{k \geq 1}$ such that the functions ϕ_{k_j} all share the same modulus and such that $\{\lambda_k \mid k \geq 1\} = \{\lambda_{k_j} \mid j \geq 1\}$.

Let us illustrate these notions on some particular examples of Riemannian manifold M and operator $H_0 = -\Delta_g + V$.

1. Let $M = \mathbb{T}^d$, $V = 0$, and $H_0 = -\Delta$. The eigenvalues are given by

$$\lambda_k = n_1^2 + \cdots + n_d^2, \quad (n_1, \dots, n_d) \in \mathbb{N}^d. \quad (2.3)$$

With each λ_k one can associate the eigenstates

$$\Phi_k^\pm(x) = e^{\pm i \sum_{j=1}^d n_j x_j}, \quad x = (x_1, \dots, x_d) \in \mathbb{T}^d. \quad (2.4)$$

Since all eigenstates Φ_k^\pm share the same modulus (constantly equal to 1) and Φ_k^+, Φ_k^- are linearly independent if $\lambda_k > 0$, the following proposition holds true.

Proposition 2.4. *The operator H_0 admits eigenfunctions sharing the same modulus inside each energy level $\lambda_k > 0$ and corresponding to all energy levels.*

2. Let $M = \mathbb{S}^2$, $V = 0$, and $H_0 = -\Delta_g$. Denote by Y_m^l , $l \in \mathbb{N}$, $m \in \{-l, \dots, l\}$, the spherical harmonics, i.e., the eigenfunctions of $-\Delta_{\mathbb{S}^2}$. Then the eigenvalue associated with Y_m^l is $l(l+1)$ and

$$Y_l^m(\alpha, \beta) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos(\alpha)) e^{im\beta}, \quad (2.5)$$

where P_l^m is the corresponding Legendre polynomial and (α, β) are the spherical coordinates on \mathbb{S}^2 .

Proposition 2.5. *For every $l \in \mathbb{N}$, $m \in \{-l, \dots, l\}$, Y_l^m and Y_l^{-m} share the same modulus. As a consequence, for each $l \geq 1$, H_0 admits eigenfunctions sharing the same modulus inside the energy level $l(l+1)$.*

Proof. Recall that each P_l^m satisfies

$$P_l^{-m} = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m,$$

so that

$$Y_l^{-m}(\alpha, \beta) = (-1)^m e^{-2im\beta} Y_l^m(\alpha, \beta). \quad (2.6)$$

Therefore, Y_m^l and Y_{-m}^l share the same modulus. \square

Remark 2.6. A natural open question is the following one: for $k, l \geq 0$, $k \neq l$, do there exist $\phi_k \in E_{k(k+1)}$ and $\phi_l \in E_{l(l+1)}$ sharing the same modulus?

The next result tells us that appropriate linear combinations of spherical harmonics can also share the same modulus.

Proposition 2.7. *For every $l \geq 4$, $m \in \{-l-4, \dots, l-4\}$ there exist some constants $\alpha_l^m, \alpha_l^{m-2}, \alpha_l^{m+2}, \alpha_{l-2}^{m+2} \in \mathbb{R} \setminus \{0\}$ such that $\alpha_l^m Y_l^{-l} + \alpha_l^{m-2} Y_l^{m-2}$ and $\alpha_l^{m+2} Y_l^{m+2} + \alpha_{l-2}^{m+2} Y_{l-2}^{m+2}$ share the same modulus.*

Proof. The result follows from (2.5) and the recurrence relation

$$(l-m-1)(l-m)P_l^m(x) = -P_l^{m+2}(x) + P_{l-2}^{m+2}(x) + (l+m)(l+m-1)P_l^{m-2}(x).$$

This concludes the proof. \square

Note that Y_l^{m+2} and Y_{l-2}^{m+2} correspond to different energy levels, so the result tells us that an appropriate superposition of spherical harmonics of different energy levels shares the same modulus as an appropriate superposition of spherical harmonics from a single energy level.

3. Let us consider the unit disk $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. Denote by Δ_D the Dirichlet–Laplace operator on \mathbb{D} . From [Hen06, Proposition 1.2.14], the eigenvalues and eigenfunctions of $-\Delta_D$ are given by, for every $k \geq 1$,

$$\lambda_{0,k} = j_{0,k}^2, \quad u_{0,k}(r, \theta) = \sqrt{\frac{1}{\pi}} \frac{1}{|J'_0(j_{0,k})|} J_0(j_{0,k}r),$$

and, for every $n, k \geq 1$,

$$\begin{aligned} \lambda_{n,k} &= j_{n,k}^2 \quad \text{of multiplicity 2,} \\ u_{n,k}(r, \theta) &= \sqrt{\frac{2}{\pi}} \frac{1}{|J'_n(j_{n,k})|} J_n(j_{n,k}r) \cos(n\theta), \\ u_{n,k}(r, \theta) &= \sqrt{\frac{2}{\pi}} \frac{1}{|J'_n(j_{n,k})|} J_n(j_{n,k}r) \sin(n\theta), \end{aligned}$$

where $j_{n,k}$ is the k -th zero of the Bessel function J_n and (r, θ) are the polar coordinates on \mathbb{D} .

Proposition 2.8. *Let $n, m \geq 0$ be such that $n \neq m$ and $k, l \geq 1$. Assume that $\phi_{n,k}$ and $\phi_{m,l}$ are eigenfunctions corresponding to $j_{n,k}^2$ and $j_{m,l}^2$, respectively. Then $\phi_{n,k}$ and $\phi_{m,l}$ do not share the same modulus.*

On the other hand, for $n \geq 1$ and $k \geq 1$, there exist two \mathbb{C} -linearly independent eigenfunctions given by

$$\psi_{n,k}^+ = \sqrt{\frac{2}{\pi}} \frac{1}{|J'_n(j_{n,k})|} J_n(j_{n,k}r) e^{in\theta}, \quad \psi_{n,k}^- = \sqrt{\frac{2}{\pi}} \frac{1}{|J'_n(j_{n,k})|} J_n(j_{n,k}r) e^{-in\theta}, \quad (2.7)$$

corresponding to the eigenvalue $j_{n,k}^2$ that share the same modulus.

Proof. The first part is a standard application of a celebrated result by Siegel [Sie14] stating that J_n and J_m for $n \neq m$ have no common zeros, together with the simple observation that two eigenfunctions sharing the same modulus also share the same nodal set.

The second part consists in remarking that the eigenfunctions given by (2.7) are \mathbb{C} -linearly independent. \square

4. Let $M = \mathbb{R}^d$, $V(x) = \|x\|^2$, and $H_0 = -\Delta + V = -\Delta + \|x\|^2$ be the harmonic oscillator Hamiltonian.

The eigenfunctions of the one-dimensional harmonic oscillator $H_0 = -\partial_x^2 + x^2$ are the Hermite eigenfunctions

$$\Phi_k(x) = \frac{1}{\sqrt{2^k k! \sqrt{\pi}}} \left(x - \frac{d}{dx}\right)^k e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}, \quad k \in \mathbb{N}. \quad (2.8)$$

The eigenvalue corresponding to Φ_k is $2k+1$ and Φ_k can be written as $\Phi_k(x) = H_k(x)e^{-\frac{x^2}{2}}$, where H_k is a polynomial function of degree k . In the multi-dimensional case, for $d \geq 2$, $\alpha = (\alpha_j)_{1 \leq j \leq d} \in \mathbb{N}^d$, the Hermite eigenfunctions are given by

$$\Phi_\alpha(x) = \prod_{j=1}^d \Phi_{\alpha_j}(x_j) \quad x = (x_j)_{1 \leq j \leq d} \in \mathbb{R}^d, \quad \alpha \in \mathbb{N}^d, \quad (2.9)$$

and Φ_α corresponds to the eigenvalue $2|\alpha| + d$.

Proposition 2.9. *Let $d = 1$. For every $k_1, k_2 \in \mathbb{N}$, $k_1 \neq k_2$, H_0 does not admit two eigenfunctions corresponding to the energy levels $2k_1 + 1$ and $2k_2 + 1$ that share the same modulus.*

Proof. By contradiction, if Φ_k and Φ_ℓ share the same modulus, then $|H_k| = |H_\ell|$, so that H_k and H_ℓ have the same degree, which is not possible when $k \neq \ell$. \square

Note that Proposition 2.9 can also be seen as a consequence of Lemma 3.1 or Theorem 3.6, see below.

Remark 2.10. A natural open question is: does the result of Proposition 2.9 extend to the case $d \geq 2$? The difficulty is that in the contradiction argument, we will have

$$\left| \sum_{\alpha \in \mathbb{N}^d, |\alpha|=k_1} \prod_{j=1}^d H_{\alpha_j}(x_j) \right| = \left| \sum_{\beta \in \mathbb{N}^d, |\beta|=k_2} \prod_{j=1}^d H_{\beta_j}(x_j) \right| \quad x = (x_j)_{1 \leq j \leq d} \in \mathbb{R}^d. \quad (2.10)$$

An argument based on the degree does not work anymore because a priori one can have cancellations in the previous equality such that the polynomial in the left hand side has the same degree than the polynomial of the right hand side.

Eigenfunctions sharing the same modulus play a special role in the bilinear control of quantum systems, as explained in the next section.

2.3 Bilinear quantum control systems

Let $Q = (Q_1, \dots, Q_m) \in L_{\text{loc}}^\infty(M; \mathbb{R})^m$ be the potentials of interactions and let us consider the bilinear controlled Schrödinger equation

$$\begin{cases} i\partial_t \psi = -\Delta_g \psi + V\psi + \langle u(t), Q \rangle_{\mathbb{R}^m} \psi & \text{in } (0, +\infty) \times M, \\ \psi = 0 & \text{on } (0, +\infty) \times \cup_{j \in I_D} \mathcal{C}_j, \\ \partial_\nu \psi = 0 & \text{on } (0, +\infty) \times \cup_{j \in I_N} \mathcal{C}_j, \\ \psi(0, \cdot) = \psi_0 & \text{in } M. \end{cases} \quad (2.11)$$

In (2.11), at time $t \in [0, +\infty)$, $\psi(t) \in L^2(M)$ is the state and $u(t) \in \mathbb{R}^m$ is the control.

In all the sequel, we also make the following hypothesis.

Assumption 2.11. For every $j \in \{1, \dots, m\}$, $Q_j \in L^\infty(M; \mathbb{R})$.

By combining Assumptions 2.1 and 2.11, we have the following well-posedness result according to [BMS82]. For every $T > 0$, $\psi_0 \in L^2(M)$ and $u \in L^2(0, T; \mathbb{R}^d)$, there exists a unique mild solution $\psi \in C([0, T]; L^2(M))$ of (2.11), i.e.,

$$\psi(t) = e^{-itH_0} \psi_0 + \int_0^t e^{-i(t-s)H_0} \langle u(s), Q(x) \rangle \psi(s) ds, \quad \forall t \in [0, T].$$

If we assume furthermore that $\psi_0 \in \mathcal{S} = \{\psi \in L^2(M) \mid \|\psi\|_{L^2(M)} = 1\}$, then, for every $t \in [0, T]$, $\psi(t) \in \mathcal{S}$. In the sequel, for $\psi_0 \in L^2(M)$ and $u \in L^2(0, T; \mathbb{R}^d)$, the associated solution $\psi \in C([0, T]; L^2(M))$ of (2.11) will be denoted by $\psi(\cdot; \psi_0, u)$.

For $\psi_0 \in L^2(M)$, let us denote

$$\mathcal{R}(\psi_0) := \{\psi(t; \psi_0, u) \mid t \geq 0, u \in L^2(0, t; \mathbb{R}^d)\}.$$

From a controllability point of view, we have the following well-known negative result from [Tur00], based on [BMS82] (see also [BCC20] for extensions to the case of L^1 and impulsive controls): For every $\psi_0 \in \text{Dom}(H_0) \cap \mathcal{S}$, the complement of $\mathcal{R}(\psi_0)$ in $\text{Dom}(H_0) \cap \mathcal{S}$ is dense in $\text{Dom}(H_0) \cap \mathcal{S}$, so in particular the interior of $\mathcal{R}(\psi_0)$ in $\text{Dom}(H_0) \cap \mathcal{S}$ for the topology of $\text{Dom}(H_0)$ is empty. With respect to this result of negative nature for the exact controllability, one may wonder if instead the approximate controllability holds. It is known that (2.11) is approximately controllable in large time, generically with respect to the parameters of the system, see [MS10]. Here, we will mainly focus on the small-time approximate controllability.

Definition 2.12. An element $\psi_1 \in \mathcal{S}$ belongs to the small-time approximately reachable set from $\psi_0 \in \mathcal{S}$, denoted by $\overline{\mathcal{R}_0(\psi_0)}$, if for every $\varepsilon > 0$, for every $\tau > 0$, there exists $T \in (0, \tau]$ and $u \in L^2(0, T; \mathbb{R}^m)$ such that

$$\|\psi(T; \psi_0, u) - \psi_1\|_{L^2(M)} < \varepsilon.$$

The characterization of small-time approximately reachable sets for (2.11) is an open problem in general. For instance, the authors in [BCT18] exhibits a minimal time concerning the approximate controllability between particular initial data and final target for (2.11) when $M = \mathbb{R}^d$. Results in the same spirit were also obtained in [BBS21] by using a WKB method. Nevertheless, there are examples of bilinear systems for which $\overline{\mathcal{R}_0(\psi_0)} = \mathcal{S}$ for all $\psi_0 \in \mathcal{S}$, see [BCC12].

In this article, we mainly focus on a weaker notion, that we call small-time isomodulus approximate controllability.

Definition 2.13. We say that (2.11) is *small-time isomodulus approximately controllable* from $\psi_0 \in \mathcal{S}$ if

$$\{e^{i\theta} \psi_0 \mid \theta \in L^2(M; \mathbb{T})\} \subset \overline{\mathcal{R}_0(\psi_0)}.$$

The motivation for studying such a notion comes from the recent papers [DN24] and [CP23]. Roughly speaking, in these papers, the authors prove that the small-time isomodulus approximate controllability holds assuming that the potentials of interaction Q_1, \dots, Q_m are smooth up to the boundary of M and satisfy a suitable saturation property.

Let us investigate for particular examples of Riemannian manifolds M , operators $H_0 = -\Delta_g + V$, and potentials of interaction Q_1, \dots, Q_m when small-time isomodulus approximate controllability holds.

1. In the case where M is a torus the following result holds.

Theorem 2.14. [DN24, Theorem A] Let $M = \mathbb{T}^d$, let \mathcal{K} be the set defined by

$$\mathcal{K} = \{(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1, 0), (1, 1, \dots, 1)\} \subset \mathbb{R}^d,$$

and assume that Q_1, \dots, Q_m are such that

$$x \mapsto 1, x \mapsto \sin\langle x, k \rangle, x \mapsto \cos\langle x, k \rangle \in \text{span}\{Q_1, \dots, Q_m\}, \quad \forall k \in \mathcal{K}.$$

Then (2.11) is *small-time isomodulus approximately controllable*.

Note that [DN24, Theorem A] actually claims the small-time isomodulus approximate controllability for $\psi_0 \in H^s(M) \cap \mathcal{S}$ for $s > d/2$ but this is due to the eventual presence of the semi-linearity in their equation.

A striking consequence of such a result is that the small-time approximate controllability among particular eigenstates holds.

Corollary 2.15. [DN24, Theorem B] *Take $V = 0$. Then, we have*

$$\Phi_k^\pm \in \overline{\mathcal{R}_0(\Phi_\ell^\pm)},$$

where $\Phi_k^\pm, \Phi_\ell^\pm$ are any eigenfunctions defined as in (2.4).

In particular, the system can be steered arbitrarily fast from one energy level to any other one. This is due to the fact that for every k, ℓ , Φ_k^\pm and Φ_ℓ^\pm share the same modulus.

2. For the two-dimensional sphere the following result holds.

Theorem 2.16. [CP23, Theorem 3] *Let $M = \mathbb{S}^2$, $V = 0$, $Q_1(x, y, z) = x$, $Q_2(x, y, z) = y$, $Q_3(x, y, z) = z$. Then (2.11) is small-time isomodulus approximately controllable.*

As an application, we have the following result, that claims the small-time approximate controllability between particular spherical harmonics.

Corollary 2.17. [CP23, Theorem 3] *We have*

$$Y_m^l \in \overline{\mathcal{R}_0(Y_{-m}^l)}, \quad \forall l \in \mathbb{N}, \forall m \in \{-l, \dots, l\}, \quad (2.12)$$

where Y_m^l are the eigenfunctions defined in (2.5).

Note that [CP23, Theorem 3] actually claims (2.12) for $l \in \mathbb{N}$ and $m = \pm l$ but the proof is still valid for any $m \in \{-l, \dots, l\}$ because of Proposition 2.5. The following result is new.

Corollary 2.18. *For every $l \geq 4$, $m \in \{-l - 4, \dots, l - 4\}$, there exist some constants $\alpha_l^m, \alpha_l^{m-2}, \alpha_l^{m+2}, \alpha_{l-2}^{m+2} \in \mathbb{R} \setminus \{0\}$ such that*

$$\alpha_l^m Y_l^{-l} + \alpha_l^{m-2} Y_l^{m-2} \in \overline{\mathcal{R}_0(\alpha_l^{m+2} Y_l^{m+2} + \alpha_{l-2}^{m+2} Y_{l-2}^{m+2})}.$$

Proof. This is a direct application of Corollary 2.15 and Proposition 2.7. \square

One of the contributions of this article is the generalization of [DN24, Theorem A] and [CP23, Theorem 3] to Riemannian manifolds with boundaries, assuming that the potentials of interaction satisfy a saturation property that we describe now.

We assume that $Q_1, \dots, Q_m \in C^\infty(M; \mathbb{R})$ and let us define the sequence of subspaces

$$\mathcal{H}_0 = \{\varphi \in \text{span}\{Q_1, \dots, Q_m\} \mid \varphi \text{Dom}(H_0) \subset \text{Dom}(H_0)\} \subset L^2(M; \mathbb{R}), \quad (2.13)$$

and iteratively

$$\mathcal{H}_{N+1} = \{\varphi \in \mathcal{H}_N + \text{span}\{g(\nabla_g \psi, \nabla_g \psi) \mid \psi \in \mathcal{H}_N\} \mid \varphi \text{Dom}(H_0) \subset \text{Dom}(H_0)\}, \quad N \geq 0. \quad (2.14)$$

Finally, we define

$$\mathcal{H}_\infty = \bigcup_{N \geq 0} \mathcal{H}_N. \quad (2.15)$$

Theorem 2.19. *Assume that \mathcal{H}_∞ is dense in $L^2(M; \mathbb{R})$. Then, (2.11) is small-time isomodulus approximately controllable.*

It is worth mentioning that the proof of [DN24, Theorem A] directly gives Theorem 2.19 when $M = \mathbb{T}^d$ and the proof of [CP23, Theorem 3] directly gives Theorem 2.19 when M is a manifold without boundary. Even if the treatment of the boundary is not the most difficult issue, we give a full proof of Theorem 2.19 in Appendix A in an abstract setting for the sake of completeness and for the application to other quantum control systems, in particular on quantum graphs.

As an application of Theorem 2.19, one can then consider the case of the Dirichlet–Laplace operator on the disk.

Corollary 2.20. *Assume that $Q_1, \dots, Q_m \in C^\infty(\mathbb{D}; \mathbb{R})$ and \mathcal{H}_∞ is dense in $L^2(\mathbb{D}; \mathbb{R})$. Then,*

$$\psi_{n,k}^+ \in \overline{\mathcal{R}_0(\psi_{n,k}^-)}, \quad \forall n, k \geq 1,$$

where the eigenfunctions $\psi_{n,k}^\pm$ are defined in (2.7).

Here, the C^∞ -regularity up to the boundary of the potentials leads to the fact that one can remove the conditions of the stabilization of $\text{Dom}(H_0)$ by the functions φ in the definitions (2.13) and (2.14) because it is automatically satisfied. This is specific to the homogeneous Dirichlet boundary conditions. It is not the case for homogeneous Neumann boundary conditions.

3 Eigenfunctions sharing the same modulus

3.1 General properties

The goal of this part is to study the implications of having two linearly independent eigenfunctions sharing the same modulus.

Let H_0 be as in Assumption 2.1. Notice that we could relax the assumptions on H_0 in this section, not requiring its entire spectrum to be discrete, but just focusing on two of its eigenvalues λ_k and λ_ℓ with corresponding eigenfunctions $\phi_k, \phi_\ell : M \rightarrow \mathbb{C}$. By elliptic regularity and Sobolev embeddings, see for instance [GT01, Theorem 7.26, Theorem 9.11], for every $0 < \alpha < 1$ one has that ϕ_k and ϕ_ℓ are $C_{\text{loc}}^{1,\alpha}(M; \mathbb{C})$. The one-dimensional case is specific because, by standard ODE arguments, one can prove that ϕ_k and ϕ_ℓ are $C_{\text{loc}}^{1,1}$.

Let us assume that ϕ_k and ϕ_ℓ share the same modulus $\rho := |\phi_k| = |\phi_\ell| \in C(M, [0, +\infty))$. Set $M_\rho = \{x \in M \mid \rho(x) \neq 0\}$ and let $\theta_k, \theta_\ell : M_\rho \rightarrow \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ be such that

$$\phi_k(x) = \rho(x)e^{i\theta_k(x)}, \quad \phi_\ell(x) = \rho(x)e^{i\theta_\ell(x)}, \quad x \in M_\rho. \quad (3.1)$$

Notice that both ϕ_k and ϕ_ℓ vanish on $M \setminus M_\rho = \{x \in M \mid \rho(x) = 0\}$ and that, by unique continuation, see [Aro57], M_ρ is dense in M . Moreover, for every $0 < \alpha < 1$ one has that $\rho|_{M_\rho}$ and $\theta_k : M_\rho \rightarrow \mathbb{T}$ are $C_{\text{loc}}^{1,\alpha}$, and even $C_{\text{loc}}^{1,1}$ in the one-dimensional case.

Let us observe the following.

Lemma 3.1. *If λ_k and λ_ℓ are simple and distinct eigenvalues of the Schrödinger operator H_0 with corresponding eigenfunctions ϕ_k and ϕ_ℓ , then ϕ_k and ϕ_ℓ cannot share the same modulus.*

Proof. Assume by contradiction that ϕ_k and ϕ_ℓ share the same modulus and let ρ, ϕ_k, ϕ_ℓ be as in (3.1).

Notice that, since V is real-valued, $\text{Re}(\phi_k)$ and $\text{Im}(\phi_k)$ satisfy

$$-\Delta \text{Re}(\phi_k) + V \text{Re}(\phi_k) = \lambda_k \text{Re}(\phi_k), \quad -\Delta \text{Im}(\phi_k) + V \text{Im}(\phi_k) = \lambda_k \text{Im}(\phi_k).$$

By using that λ_k is a simple eigenvalue, we then deduce that $\operatorname{Re}(\phi_k)$ and $\operatorname{Im}(\phi_k)$ are colinear, that is, up to exchanging the roles of $\operatorname{Re}(\phi_k)$ and $\operatorname{Im}(\phi_k)$, there exists $\mu_k \in \mathbb{R}$ such that $\operatorname{Re}(\phi_k) = \mu_k \operatorname{Im}(\phi_k)$, i.e., $\rho \cos(\theta_k) = \mu_k \rho \sin(\theta_k)$. We therefore deduce that

$$\cos(\theta_k) = \mu_k \sin(\theta_k) \quad \text{on } M_\rho.$$

By using that M_ρ is open and θ_k is continuous on M_ρ , we then deduce that θ_k is constant on each connected component M_ρ . As a consequence, we obtain that

$$-\Delta\rho + V\rho = \lambda_k\rho \quad \text{on } M_\rho.$$

By using the same argument for λ_ℓ , we obtain

$$-\Delta\rho + V\rho = \lambda_\ell\rho \quad \text{on } M_\rho.$$

Hence, since M_ρ is nonempty, $\lambda_k = \lambda_\ell$, contradicting the hypotheses. \square

Remark 3.2. One may wonder if the conclusion of Lemma 3.1 still holds true assuming that only λ_k is simple. This turns to be false by considering the example of $M = \mathbb{T}$ and $H_0 = -\Delta$. Indeed, 0 is a simple eigenvalue with corresponding eigenfunction constantly equal to 1, while 1 is an eigenvalue of multiplicity 2 with corresponding eigenfunctions e^{ix} and e^{-ix} , and the three eigenfunctions obviously share the same modulus.

Corollary 3.3. *Let M be a compact connected C^∞ manifold M without boundary of dimension larger than or equal to 2. Then, generically with respect to the Riemannian metric, no pair of \mathbb{C} -linearly independent eigenfunctions of the Laplace–Beltrami operator $-\Delta_g$ share the same modulus.*

Proof. This is straightforward application of Lemma 3.1 and the well-known result by Uhlenbeck [Uhl76, Tan79]¹ ensuring that, given a compact connected C^∞ manifold M of dimension larger than or equal to 2, the set

$$\{g \in \mathcal{M} \mid \text{all eigenvalues of } -\Delta_g \text{ have multiplicity one}\},$$

is residual in the topological space \mathcal{M} of all smooth Riemannian metrics on M endowed with the C^∞ topology. \square

Remark 3.4. Results similar to Corollary 3.3 can be obtained when considering the genericity with respect to the potential V (with no restriction on the dimension of M). Indeed, the spectrum of the Schrödinger operator is known to be generically simple with respect to V . This general statement requires precise functional analysis setups to be rigorously formulated. For the case of compact connected manifolds without boundary and C^∞ potentials V see [Alb75]. The case of bounded Euclidean domains with Dirichlet boundary conditions and $V \in L^\infty$ or of full Euclidean spaces with $V \in L_{\text{loc}}^\infty$ such that $\lim_{\|x\| \rightarrow \infty} V(x) = +\infty$ can be found in [MS10, Proposition 3.2].

Let us collect in the following result some differential identities satisfied by the modulus and phases of two eigenfunctions sharing the same modulus.

¹The version of Uhlenbeck’s result given by Tanikawa in [Tan79] provides the formulation in the C^∞ topology and also covers the case where M has nonempty boundary and the Schrödinger operator has Dirichlet boundary conditions. One could therefore extend the statement of Corollary 3.3 accordingly.

Lemma 3.5. *Let $\phi_k, \phi_\ell : M \rightarrow \mathbb{C}$ be two eigenfunctions of the Schrödinger operator A sharing the same modulus with corresponding eigenvalues λ_k, λ_ℓ . Let $\rho, \theta_k, \theta_\ell$ be as in (3.1). The following elliptic equations are fulfilled*

$$-\Delta\rho + |\nabla\theta_j|^2\rho + V\rho = \lambda_j\rho \quad \text{on } M_\rho, \quad j = k, l, \quad (3.2)$$

$$-\rho\Delta\theta_j - 2\nabla\rho \cdot \nabla\theta_j = 0 \quad \text{on } M_\rho, \quad j = k, l. \quad (3.3)$$

Moreover,

$$|\nabla\theta_k|^2 - |\nabla\theta_\ell|^2 = \lambda_k - \lambda_\ell \quad \text{on } M_\rho. \quad (3.4)$$

Proof. First, we have

$$\Delta\phi_k = \Delta\rho e^{i\theta_k} + 2i\nabla\rho \cdot \nabla\theta_k e^{i\theta_k} + \rho(i\Delta\theta_k e^{i\theta_k} - |\nabla\theta_k|^2 e^{i\theta_k}) \quad \text{on } M_\rho. \quad (3.5)$$

Plugging (3.5) in the equality $-\Delta\phi_k + V\phi_k = \lambda_k\phi_k$, simplifying by $e^{i\theta_k}$, and taking the real and imaginary parts of the obtained equality, we get (3.2) and (3.3). Equation (3.4) is obtained by taking the difference of the two equations in (3.2) corresponding to $j = k, l$ and by simplifying by $\rho \neq 0$ on M_ρ . \square

3.2 The one-dimensional case

A noticeable feature of the one-dimensional case is that, locally around each point of M , the Riemannian manifold is isometric to an interval in \mathbb{R} endowed with the Euclidean metric, as it follows by taking an arclength coordinate (see, for instance, [Mil65, Appendix Classifying 1-dimensional manifolds]). Hence, since M is complete, four situations may occur:

- M is isometric to the line \mathbb{R} ,
- M is isometric to the half-line $[0, +\infty)$,
- M is isometric to a compact interval $[0, L]$ for some $L > 0$,
- M is a closed curve isometric to the quotient $\mathbb{R}/L\mathbb{Z}$ for some $L > 0$.

Moreover, in the arclength-coordinate the metric is constant and the Laplace–Beltrami operator coincides with the second derivative.

The main result of this section is Theorem 3.6 below, stating, in particular, that a one-dimensional manifold M admits a pair of eigenfunctions sharing the same modulus and corresponding to different eigenvalues only if $M = \mathbb{T}$ and V is constant, that is, in the one-dimensional occurrence of the system studied in [DN24].

Theorem 3.6. *If M is one-dimensional and the Schrödinger operator H_0 admits two \mathbb{C} -linearly independent eigenfunctions ϕ_k and ϕ_ℓ sharing the same modulus, then necessarily M is a closed curve and ϕ_k, ϕ_ℓ are nowhere vanishing on M . If, moreover, the two eigenfunctions correspond to distinct eigenvalues, then V is constant.*

The proof of the previous result is based on the following lemma.

Lemma 3.7. *Let (a, b) be an interval in \mathbb{R} and $\lambda_k, \lambda_\ell \in \mathbb{R}$. Assume that $\rho \in C^1(a, b; (0, +\infty))$ and $\theta_k, \theta_\ell \in C^{1,1}(a, b; \mathbb{T})$ are such that*

$$\rho \neq 0, \quad -\rho\theta_k'' - 2\rho'\theta_k' = 0, \quad -\rho\theta_\ell'' - 2\rho'\theta_\ell' = 0, \quad |\theta_k'|^2 - |\theta_\ell'|^2 = \lambda_k - \lambda_\ell \quad \text{in } (a, b).$$

Then there exist $C_k, C_\ell \in \mathbb{R}$ such that $\theta_k' = C_k\rho^{-2}$ and $\theta_\ell' = C_\ell\rho^{-2}$ in (a, b) . Moreover, if $\lambda_k = \lambda_\ell$, then there exists $c \in \mathbb{T}$ such that either $\theta_k - \theta_\ell = c$ or $\theta_k + \theta_\ell = c$ in (a, b) . If, instead, $\lambda_k \neq \lambda_\ell$ then $C_k^2 \neq C_\ell^2$ and there exists $\rho_ \in (0, +\infty)$ such that*

$$\rho(x) = \rho_*, \quad x \in (a, b). \quad (3.6)$$

Proof. Setting $\gamma_k = \theta'_k$, we have that

$$\gamma'_k = -2(\rho'/\rho)\gamma_k \quad \text{almost everywhere in } (a, b).$$

Integrating, we deduce that there exists $C_k \in \mathbb{R}$ such that

$$\theta'_k(x) = \gamma_k(x) = C_k \rho(x)^{-2}, \quad x \in (a, b). \quad (3.7)$$

Analogously, there exists $C_\ell \in \mathbb{R}$ such that $\theta'_\ell(x) = C_\ell \rho(x)^{-2}$, $x \in (a, b)$. Hence, using the equality $|\theta'_k|^2 - |\theta'_\ell|^2 = \lambda_k - \lambda_\ell$, we deduce that

$$(C_k^2 - C_\ell^2)\rho(x)^{-4} = \lambda_k - \lambda_\ell, \quad x \in (a, b). \quad (3.8)$$

If $\lambda_k = \lambda_\ell$ then necessarily $C_k^2 = C_\ell^2$, meaning that either $\theta'_k = \theta'_\ell$ or $\theta'_k = -\theta'_\ell$ in (a, b) . The latter means that either $\theta_k - \theta_\ell$ or $\theta_k + \theta_\ell$ is constant.

Let now consider the case $\lambda_k \neq \lambda_\ell$. We deduce from (3.8) that $C_k^2 \neq C_\ell^2$ and that ρ is constant on (a, b) , that is, there exists $\rho_* > 0$ such that $\rho(x) = \rho_*$, $x \in (a, b)$. \square

Proof of Theorem 3.6. Let ϕ_k and ϕ_ℓ be two eigenfunctions of H_0 sharing the same modulus and corresponding to the eigenvalues λ_k and λ_ℓ . In particular, ϕ_k and ϕ_ℓ are assumed to be \mathbb{C} -linearly independent. Let $\rho, \theta_k, \theta_\ell$ be as in (3.1). Up to taking an arclength coordinate, equations (3.3) and (3.4) read $-\rho\theta''_k - 2\theta'_k\rho' = 0 = -\rho\theta''_\ell - 2\theta'_\ell\rho'$ and $|\theta'_k|^2 - |\theta'_\ell|^2 = \lambda_k - \lambda_\ell$ on M_ρ . These are the equations for $\theta_k, \theta_\ell, \rho$ appearing in the statement of Lemma 3.7.

Let us first exclude the case in which H_0 has boundary conditions of Neumann type at a point of ∂M . Without loss of generality such point is 0 and $M = [0, L]$ for some $L > 0$ or $M = [0, +\infty)$. Since for every $\lambda \in \mathbb{R}$ the space of real-valued solutions φ of $-\varphi''(x) = (\lambda - V(x))\varphi(x)$ with $\varphi'(0) = 0$ is one-dimensional, then λ_k and λ_ℓ are both simple and the conclusion follows from Lemma 3.1.

Let us consider now the case $\lambda_k \neq \lambda_\ell$. We obtain from Lemma 3.7 that ρ is locally constant on M_ρ . Since, moreover, M is connected and ρ is continuous, then ρ coincides with some constant $\rho_* > 0$ on M . This excludes the case in which H_0 has boundary conditions of Dirichlet type at a point of ∂M and the case in which M is isometric to \mathbb{R} or to $[0, +\infty)$, since otherwise ϕ_k and ϕ_ℓ would not be in L^2 . In particular M is a closed curve. We also obtain from (3.2) and (3.6) that V is constant (globally on $M = M_\rho$).

Let us now turn to the case where $\lambda_k = \lambda_\ell$. We claim that ρ does not vanish on M . Assume by contradiction that this is not the case and identify isometrically a connected component of M_ρ with a possibly unbounded interval (a, b) or $(a, b]$, with $a \in M \setminus M_\rho$. Since $\phi'_k e^{-i\theta_k} = \rho' + i\theta'_k \rho$ on (a, b) and both ρ' and $\theta'_k \rho$ are real-valued, we deduce that ρ' and $\theta'_k \rho$ are bounded in a right-neighborhood of a . By Lemma 3.7, there exists $C_k \in \mathbb{R}$ such that $\theta'_k \rho = C_k \rho^{-1}$ on (a, b) . Since $\rho(x) \rightarrow 0$ as $x \downarrow a$, it follows from the boundedness of $\theta'_k \rho$ that $C_k = 0$. Similarly, $C_\ell = 0$, yielding that θ_k and θ_ℓ are constant on a connected component of M_ρ . Hence ϕ_k and ϕ_ℓ are \mathbb{C} -linearly dependent on a nonempty open set of M , yielding by the unique continuation principle that they are \mathbb{C} -linearly dependent on M . This concludes the contradiction argument proving that ρ does not vanish on M (excluding, in particular, the case in which H_0 has boundary conditions of Dirichlet type at a point of ∂M).

We are then left to exclude the case where $M = M_\rho$ is isometric to \mathbb{R} . It follows from Lemma 3.7 that either $\theta_k - \theta_\ell$ or $\theta_k + \theta_\ell$ is constant on M . By \mathbb{C} -linear independence of ϕ_k and ϕ_ℓ we can exclude the case where $\theta_k - \theta_\ell$ is constant and we can assume that $\theta'_k = C_k \rho^{-2}$ on M with $C_k \neq 0$.

Since ϕ_k and ϕ_ℓ are in $H^1(M)$, then

$$\lim_{|x| \rightarrow \infty} \rho(x) = 0. \quad (3.9)$$

It follows from (3.2) that ρ satisfies

$$-\rho'' + C_k^2 \rho^{-3} + V\rho = \lambda_k \rho \quad \text{on } M = \mathbb{R}.$$

Since V is bounded from below, we deduce that there exists $\rho_* > 0$ such that $\rho''(x) > 1$ at every x such that $\rho(x) < \rho_*$. This is in contradiction with (3.9) and the proof is concluded. \square

Remark 3.8. The theorem allows to conclude that V is constant only in the case where $\lambda_k \neq \lambda_\ell$. Let us see through an example that this is not true in general if $\lambda_k = \lambda_\ell$. We proceed in a constructive way, starting from any non-constant $\rho \in C^{1,1}(\mathbb{T}; (0, +\infty))$ and selecting V and θ , both non-constant, such that $\rho e^{i\theta}$ is an eigenfunction of $-\Delta + V$ corresponding to the eigenvalue 0. In particular, $\rho e^{i\theta}$ and $\rho e^{-i\theta}$ are \mathbb{C} -linearly independent eigenfunctions of $-\Delta + V$ on \mathbb{T} .

Consider a positive integer j to be fixed later and set

$$C = \frac{2\pi}{\int_0^{2\pi} \rho(x)^{-2} dx},$$

$$\theta(x) = Cj \int_0^x \rho(y)^{-2} dy, \quad x \in [0, 2\pi].$$

It holds by construction that $\theta(0) = 0$ and $\theta(2\pi) = 2j\pi$. Hence θ , seen as a function from \mathbb{T} to itself, is well defined and non-constant. Notice that θ has been defined in such a way that $\theta' = Cj\rho^{-2}$. Setting then

$$V = \frac{\rho''}{\rho} - (\theta')^2 = \frac{\rho''}{\rho} - \frac{C^2 j^2}{\rho^4} \quad \text{on } \mathbb{T},$$

we have that ρ solves

$$-\rho'' + (\theta')^2 \rho + V\rho = 0, \quad \text{on } \mathbb{T}.$$

Moreover, j can be chosen so that V is non-constant.

One can now check that $\phi = \rho e^{i\theta}$ solves $-\phi'' + V\phi = 0$. As a consequence, $\bar{\phi} = \rho e^{-i\theta}$ solves the same equation. This completes our construction.

For an explicit example, one can take $j = 1$ and

$$\rho(x) = \cos(x) + 2, \quad x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}.$$

The resulting V and the linearly independent eigenfunctions $\rho \cos(\theta)$ and $\rho \sin(\theta)$ of $-\Delta + V$ are illustrated in Figure 1.

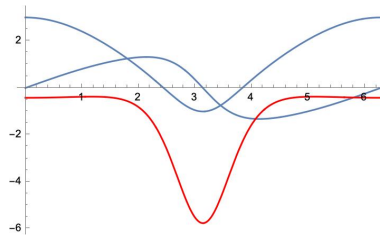


Figure 1: Plot of V (in red), $\rho \cos(\theta)$ and $\rho \sin(\theta)$ (in blue) from Remark 3.8.

As a consequence of Theorem 2.14 in the case $d = 1$, even in the case $V \neq 0$ it might be possible to steer system (2.11) in arbitrarily small time from one eigenfunction arbitrarily close to a linearly independent one (sharing the same eigenvalue). Notice that the Schrödinger equation on \mathbb{T} with non-constant potential V has a relevant role in solid state physics (see, e.g, [RS78, Section XIII.16]).

4 Quantum graphs

4.1 Setting

Let \mathcal{G} be a compact connected metric graph with N edges e_1, \dots, e_N of respective lengths L_1, \dots, L_N and M vertices v_1, \dots, v_M . For every vertex v , we denote

$$N(v) := \{l \in \{1, \dots, N\} \mid v \in e_l\}, \quad n(v) := |N(v)|. \quad (4.1)$$

We denote by \mathcal{V}_i the set of internal vertices of \mathcal{G} , that is, the set of those vertices v for which either $n(v) > 1$ or $n(v) = 1$ and the single edge to which v belongs is actually a loop. The remaining vertices are called external and we denote by \mathcal{V}_e the set comprising them.

Each edge e_j is parameterized by an arclength coordinate going from 0 to its length L_j . On \mathcal{G} we consider functions $f := (f^1, \dots, f^N)$, where $f^j : e_j \rightarrow \mathbb{C}$ for every $j = 1, \dots, N$. We denote

$$L^2(\mathcal{G}) = \bigoplus_{j=1}^N L^2(e_j).$$

The Hilbert space $L^2(\mathcal{G})$ is equipped with the norm $\|\cdot\|_{L^2}$ defined by

$$\|f\|_{L^2(\mathcal{G})}^2 = \sum_{j=1}^N \|f^j\|_{L^2(e_j)}^2.$$

We then denote by $H^1(\mathcal{G})$ the Sobolev space of all functions $f = (f^1, \dots, f^N)$ on \mathcal{G} that are continuous (that is, such that $f^j(v) = f^l(v)$ if $j, l \in N(v)$) and such that f^j belong to $H^1(e_j)$ for every $j = 1, \dots, N$, equipped with the norm defined by

$$\|f\|_{H^1(\mathcal{G})}^2 = \sum_{j=1}^N \|f^j\|_{H^1(e_j)}^2.$$

Finally, the Sobolev space $\tilde{H}^2(\mathcal{G})$ is defined as

$$\tilde{H}^2(\mathcal{G}) = \bigoplus_{j=1}^N H^2(e_j),$$

equipped with the natural norm.

Let $f \in \tilde{H}^2(\mathcal{G}) \cap H^1(\mathcal{G})$ and v be a vertex of \mathcal{G} connected once to an edge e_j with $j = 1, \dots, N$. When the coordinate parameter e_j at the vertex v is equal to 0 (respectively, L_j), we denote

$$\partial_x f^j(v) = \partial_x f^j(0), \quad (\text{respectively, } \partial_x f^j(v) = -\partial_x f^j(L_j)). \quad (4.2)$$

When e_j is a loop connected to v at both its extremities, we use the notation

$$\partial_x f^j(v) = \partial_x f^j(0) - \partial_x f^j(L_j). \quad (4.3)$$

In the following, we consider two type of boundary conditions. For $f \in \tilde{H}^2(\mathcal{G}) \cap H^1(\mathcal{G})$, we say that f satisfies

- a *Dirichlet boundary condition* at a vertex $v \in \mathcal{V}_e$ if $f(v) = 0$,
- a *Neumann–Kirchhoff boundary condition* at a vertex $v \in \mathcal{V}_e \cup \mathcal{V}_i$ if $\sum_{j \in N(v)} \partial_x f^j(v) = 0$.

We partition \mathcal{V}_e in a two subsets of vertices \mathcal{V}_D and \mathcal{V}_N at which we impose, respectively, a Dirichlet and a Neumann–Kirchoff boundary condition.

Let $V \in L^\infty(\mathcal{G}; \mathbb{R})$, $H_0 = -\partial_x^2 + V$ be the Hamiltonian, and let us define

$$\text{Dom}(H_0) = \{f \in \tilde{H}^2(\mathcal{G}) \cap H^1(\mathcal{G}) \mid \\ f(v) = 0 \ \forall v \in \mathcal{V}_D \text{ and } \sum_{j \in N(v)} \partial_x f^j(v) = 0 \ \forall v \in \mathcal{V}_N \cup \mathcal{V}_i\}.$$

Definition 4.1. A quantum graph is a compact connected metric graph \mathcal{G} , equipped with the unbounded operator $(H_0, \text{Dom}(H_0))$.

Given a quantum graph, the Hamiltonian H_0 is a self-adjoint operator that has compact resolvent, see [BK13, Theorems 1.4.4 and 3.1.1]. So, there exists an orthonormal basis of eigenfunctions $(\phi_k)_{k \geq 1}$ associated with the sequence of real eigenvalues $(\lambda_k)_{k \geq 1}$.

4.2 General results for quantum graphs

The definition of eigenfunctions sharing the same modulus is the same as in Definition 2.3.

Lemma 4.2. *If λ_k and λ_ℓ are simple and distinct eigenvalues of the Hamiltonian $H_0 = -\partial_x^2 + V$ with corresponding eigenfunctions ϕ_k and ϕ_ℓ , then ϕ_k and ϕ_ℓ cannot share the same modulus.*

Proof. The proof is analogue to the one of Lemma 3.1. □

Corollary 4.3. *Let \mathcal{G} be a compact graph with the Hamiltonian $H_0 = -\partial_x^2$, equipped with Neumann–Kirchoff conditions. Assume that \mathcal{G} has at least one vertex v with $N(v) > 1$. Then, generically with respect to the lengths of the edges of the graph, no pair of distinct eigenfunctions belonging to the same orthonormal basis of the Laplace–Beltrami operator share the same modulus.*

Proof. We know from Friedlander’s result (see [Fri05] or [BK13, Theorem 3.1.7]) that, given a compact graph such that $N(v) > 1$ for at least one vertex v and with Hamiltonian $H_0 = -\partial_x^2$, the set

$$\{(L_1, \dots, L_N) \in [0, +\infty)^N \mid \text{all eigenvalues of } -\partial_x^2 \text{ have multiplicity one}\},$$

is residual in the topological space $[0, +\infty)^N$. The conclusion then follows from Lemma 4.2. □

Remark 4.4. By using [BL17], one can generalize Corollary 4.3 to quantum graphs with so-called δ -type condition, including in particular mixed Dirichlet and Neumann–Kirchoff conditions at the vertices.

Corollary 4.5. *Let \mathcal{G} be a compact graph with Hamiltonian $H_0 = -\partial_x^2$ and Dirichlet boundary conditions at at least one Dirichlet vertex of \mathcal{G} . Then, there exists a subsequence of eigenfunctions among which no pair share the same modulus.*

Proof. We use [PT21, Theorem 2] to obtain that there exists an increasing sequence of eigenvalues $(\lambda_{k_l})_{l \geq 1}$ of multiplicity one. Therefore, by Lemma 4.2, the corresponding eigenfunctions cannot share the same modulus. □

Remark 4.6. By using [PT21, Theorem 4], one can generalize Corollary 4.5 to Hamiltonians of the type $H_0 = -\partial_x^2 + V$, where V is edgewise constant.

4.3 Bilinear control of the Schrödinger equation in quantum graphs

Let $Q = (Q_1, \dots, Q_m) \in L^\infty(\mathcal{G}; \mathbb{R})^m$ be the m -tuple potentials of interaction and let us consider the bilinear controlled Schrödinger equation

$$\begin{cases} i\partial_t \psi = -\partial_x^2 \psi + V(x)\psi + \langle u(t), Q(x) \rangle_{\mathbb{R}^d} \psi & (t, x) \in (0, T) \times \mathcal{G}, \\ \psi(t, v) = 0 & (t, v) \in (0, T) \times \mathcal{V}_D, \\ \sum_{j \in N(v)} \partial_x \psi^j(t, v) = 0 & (t, v) \in (0, T) \times (\mathcal{V}_N \cup \mathcal{V}_i), \\ \psi(0, \cdot) = \psi_0 & \text{in } \mathcal{G}. \end{cases} \quad (4.4)$$

As for Riemannian manifolds, see Section 2.3, the well-posedness of (2.11) is a consequence of [BMS82]. The notions of mild solutions, reachable sets, controllability are then easily adapted. In particular, we keep the same notations. The interior of $\mathcal{R}(\psi_0)$ for the topology of $\text{Dom}(H_0) \cap \mathcal{S}$ is empty. For positive results of exact or approximate controllability in subspaces of $L^2(\mathcal{G})$, we refer to [Duc20] and [Duc21].

Assume that $Q_1, \dots, Q_m \in C^0(\mathcal{G}; \mathbb{R})$ and the restriction of Q_1, \dots, Q_m to each edge of \mathcal{G} is C^∞ up to the boundary. Let us define as before the sequence of subspaces

$$\mathcal{H}_0 = \{\varphi \in \text{span}\{Q_1, \dots, Q_m\} \mid \varphi \text{Dom}(H_0) \subset \text{Dom}(H_0)\} \subset L^2(\mathcal{G}), \quad (4.5)$$

and by recurrence

$$\mathcal{H}_{N+1} = \{\varphi \in \mathcal{H}_N + \text{span}\{g(\nabla_g \psi, \nabla_g \psi) \mid \psi \in \mathcal{H}_N\} \mid \varphi \text{Dom}(H_0) \subset \text{Dom}(H_0)\}, \quad N \geq 0. \quad (4.6)$$

Finally, we define

$$\mathcal{H}_\infty = \bigcup_{N \geq 0} \mathcal{H}_N. \quad (4.7)$$

We have the following result, that is an adaptation of [DN24, Theorem A] in the context of quantum graphs.

Theorem 4.7. *Assume that \mathcal{H}_∞ is dense in $L^2(\mathcal{G}; \mathbb{R})$. Equation (4.4) is small-time isomodulus approximately controllable, i.e.,*

$$\{e^{i\phi} \psi_0 \mid \phi \in L^2(\mathcal{G}; \mathbb{R})\} \subset \overline{\mathcal{R}_0(\psi_0)}, \quad \text{for every } \psi_0 \in \mathcal{S}.$$

The proof of Theorem 4.7 is given in Appendix A.3.

In the next two parts, we consider two examples of quantum graphs in which Theorem 4.7 can be applied. In the first example we exhibit a quantum graph with topology different from that of the circle and admitting eigenfunctions that share the same modulus. Each of these eigenfunctions corresponds to an eigenvalue of multiplicity 3. Note also that in this case the Schrödinger operator admits eigenfunctions corresponding to simple eigenvalues that do not share the same modulus between themselves because of Lemma 4.2 and actually do not share the same modulus with all other eigenfunctions. This highlights the variety of situations that may occur. In the second example we exhibit a quantum graph whose corresponding eigenvalues are all nonsimple (apart from the ground state) and that does not admit eigenfunctions sharing the same modulus and corresponding to different eigenvalues.

For convenience, we use in the following two section the following notation: for every $\lambda \in \mathbb{R}$, $c_\lambda : [0, 2\pi] \rightarrow \mathbb{R}$, $s_\lambda : [0, 2\pi] \rightarrow \mathbb{C}$, $\eta_\lambda : [0, 2\pi] \rightarrow \mathbb{R}$ denote the functions defined by

$$c_\lambda(x) = \cos(\lambda x), \quad s_\lambda(x) = \sin(\lambda x), \quad \eta_\lambda(x) = e^{i\lambda x}.$$

According to this notation, $\eta_\lambda = c_\lambda + is_\lambda$.

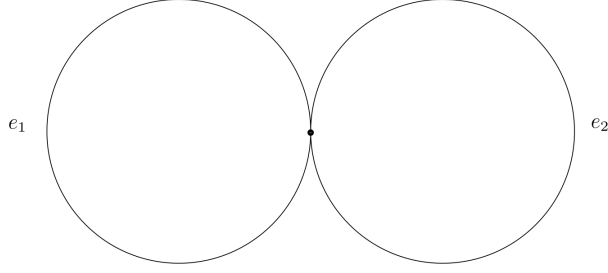


Figure 2: The eight graph

4.4 The example of the eight graph

We consider here the compact graph \mathcal{G} with $N = 2$ edges e_1, e_2 , both of length 2π , with $M = 1$ vertex. Namely, \mathcal{G} is composed of two loops having the same length $L = 2\pi$ that are linked to the same vertex, see Figure 2.

We consider the Hamiltonian $H_0 = -\partial_x^2$ on \mathcal{G} and Neumann–Kirchoff boundary conditions at the single vertex.

We have the following result.

Proposition 4.8. *The eigenvalues and eigenfunctions are given by*

$$\lambda_0 = 0 \text{ of multiplicity } 1, \phi_0 = (1, 1),$$

plus, for every $k \geq 0$,

$$\lambda_{k,o} = \left(\frac{2k+1}{2}\right)^2 \text{ of multiplicity } 1, \phi_{k,o} = \left(s_{\frac{2k+1}{2}}, -s_{\frac{2k+1}{2}}\right),$$

and finally, for $k \geq 1$,

$$\begin{aligned} \lambda_{k,e} &= k^2 \text{ of multiplicity } 3, \\ \phi_{k,e,1} &= (c_k, c_k), \phi_{k,e,2} = (s_k, 0), \phi_{k,e,3} = (0, s_k). \end{aligned}$$

Proof. We know that the eigenvalues are of the form λ^2 , $\lambda \in \mathbb{R}$, because the Hamiltonian operator is nonnegative. As a consequence, on each edge, the eigenfunction is a solution to

$$\partial_x^2 + \lambda^2 \psi = 0.$$

So, the eigenfunctions are of the form

$$\phi = (\phi^1, \phi^2) = (A_1 c_\lambda + B_1 s_\lambda, A_2 c_\lambda + B_2 s_\lambda).$$

We then write the boundary conditions, that are the continuity and the Neumann–Kirchoff conditions at the vertex, leading to

$$\phi^1(0) = \phi^2(0) \Rightarrow A = A_1 = A_2, \tag{4.8}$$

$$\phi^1(2\pi) = \phi^2(2\pi) \Rightarrow B_1 \sin(2\pi\lambda) = B_2 \sin(2\pi\lambda), \tag{4.9}$$

$$\phi^1(0) = \phi^1(2\pi) \Rightarrow A = A \cos(2\pi\lambda) + B_1 \sin(2\pi\lambda), \tag{4.10}$$

$$\begin{aligned} \partial_x \phi^1(0) - \partial_x \phi^1(2\pi) + \partial_x \phi^2(0) - \partial_x \phi^2(2\pi) &= 0 \\ \Rightarrow B_1 \lambda - (-A \lambda \sin(2\pi\lambda) + B_1 \lambda \cos(2\pi\lambda)) + B_2 \lambda - (-A \lambda \sin(2\pi\lambda) + B_2 \lambda \cos(2\pi\lambda)) &= 0. \end{aligned} \tag{4.11}$$

If $\lambda \notin \mathbb{N}^*/2$, then we obtain from (4.9) that $B = B_1 = B_2$. We therefore deduce from (4.11) that

$$2B(\lambda - \lambda \cos(2\pi\lambda)) + 2A\lambda \sin(2\pi\lambda) = 0.$$

Taking into account also (4.10), we find that (A, B) satisfies the linear system

$$\begin{pmatrix} 1 - \cos(2\pi\lambda) & -\sin(2\pi\lambda) \\ \sin(2\pi\lambda) & 1 - \cos(2\pi\lambda) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant of the matrix is different from 0, leading to $A = B = 0$.

So necessarily $\lambda \in \mathbb{N}^*/2$. If $\lambda = k/2$ with k odd, we find that $A = 0$ by (4.10) and $B_1 = -B_2$ by the Neumann–Kirchhoff condition (4.11). Finally, if $\lambda = k/2$ with k even, we find that all the conditions are satisfied if $A_1 = A_2$, that is, A, B_1, B_2 are free parameters. In the case $k = \lambda = 0$ the space parameterized by A, B_1, B_2 is one-dimensional, otherwise it has dimension 3.

To conclude, we have that $\lambda = k/2$, k odd, is a simple eigenvalue with corresponding eigenfunction $\phi_{k,o}$ and $\lambda = k/2$, k even, corresponds either to a simple eigenvalue for $k = 0$ or to an eigenvalue of multiplicity 3 for $k > 0$, associated with eigenfunctions of the type $\phi_{k,e,j}$, $j = 1, 2, 3$. \square

Proposition 4.9. *For every $k, l \geq 0$, $k \neq l$, there exists an eigenfunction corresponding to k^2 and an eigenfunction corresponding to l^2 that share the same modulus.*

For every $k \geq 1$, there exist two \mathbb{C} -linearly independent eigenfunctions corresponding to k^2 that share the same modulus.

For every $k, l \geq 0$, $k \neq l$, two eigenfunctions corresponding to $(\frac{2k+1}{2})^2$ and $(\frac{2l+1}{2})^2$ do not share the same modulus.

For every $k \geq 0$, $l \geq 1$, two eigenfunctions corresponding to $(\frac{2k+1}{2})^2$ and l^2 do not share the same modulus.

Proof. For the first point, let us remark that, for $k \geq 1$,

$$\phi_{k,e,1} + i\phi_{k,e,2} + i\phi_{k,e,3} = (\eta_k, \eta_k),$$

so that $|\phi_{k,e,1} + i\phi_{k,e,2} + i\phi_{k,e,3}| = (1, 1)$.

For the second point, let us notice that

$$-\phi_{k,e,1} + i\phi_{k,e,2} + i\phi_{k,e,3} = (\eta_{-k}, \eta_{-k}).$$

For the third point, we use the fact that the eigenvalues are simple and Lemma 4.2.

For the fourth point, let us argue by contradiction. Assume that there exist $k \geq 0$, $l \geq 1$, $C_1, C_2 \in \mathbb{C}$ such that

$$\left| \sin\left(\frac{2k+1}{2}x\right) \right| = |C_1 \cos(lx) + C_2 \sin(lx)|, \quad x \in [0, 2\pi].$$

Taking $x = 0$ leads to $C_1 = 0$ and then taking $x = \pi$ one gets a contradiction. \square

We define the potentials

$$Q_1 = (1, 1), \quad Q_2 = (c_1, c_1), \quad Q_3 = (s_1, 0), \quad Q_4 = (0, s_1), \quad Q_5 = \left(\frac{s_1}{2}, -\frac{s_1}{2}\right), \quad Q_6 = \left(\frac{c_1}{2}, \frac{c_1}{2}\right).$$

We have the following lemma.

Lemma 4.10. *The set \mathcal{H}_∞ is dense in $L^2(\mathcal{G}; \mathbb{R})$.*

Proof. The main ingredient of the proof consists in establishing that

$$\{\phi_0\} \cup \{\phi_{k,o} \mid k \geq 0\} \cup \{\phi_{k,e,j} \mid k \geq 1, j = 1, 2, 3\} \subset \mathcal{H}_\infty.$$

First note that these functions stabilize $\text{Dom}(H_0)$ and that $\phi_0 = Q_1$, $\phi_{0,o} = Q_5$, $\phi_{1,e,1} = Q_2$, $\phi_{1,e,2} = Q_3$, $\phi_{1,e,3} = Q_4$ belong to $\mathcal{H}_0 \subset \mathcal{H}_\infty$.

Let us then prove that $\phi_{k,e,j} \in \mathcal{H}_\infty$ for every $k \geq 1$ and $j = 1, 2, 3$. Notice that the property is true for $k = 1$. The argument is based on the trigonometric formulas

$$\begin{aligned} \cos((m+l)x) &= \cos(mx) \cos(lx) - \sin(mx) \sin(lx) \\ &= \left(\frac{1}{2} (\cos(mx) + \cos(lx))^2 - \frac{1}{2} \cos(mx)^2 - \frac{1}{2} \cos(lx)^2 \right) \\ &\quad - \left(\frac{1}{2} (\sin(mx) + \sin(lx))^2 - \frac{1}{2} \sin(mx)^2 - \frac{1}{2} \sin(lx)^2 \right), \end{aligned} \quad (4.12)$$

and

$$\begin{aligned} \sin((m+l)x) &= \sin(mx) \cos(lx) - \sin(lx) \cos(mx) \\ &= \left(\frac{1}{2} (\sin(mx) + \cos(lx))^2 - \frac{1}{2} \sin(mx)^2 - \frac{1}{2} \cos(lx)^2 \right) \\ &\quad - \left(\frac{1}{2} (\sin(lx) + \cos(mx))^2 - \frac{1}{2} \sin(lx)^2 - \frac{1}{2} \cos(mx)^2 \right), \end{aligned} \quad (4.13)$$

holding for every $l, m \in \mathbb{R}$. For $m, l \neq 0$ we can rewrite (4.12) as

$$2c_{m+l} = \left(\frac{s'_m}{m} + \frac{s'_l}{l} \right)^2 - \left(\frac{s'_m}{m} \right)^2 - \left(\frac{s'_l}{l} \right)^2 - \left(\frac{c'_m}{m} + \frac{c'_l}{l} \right)^2 + \left(\frac{c'_m}{m} \right)^2 + \left(\frac{c'_l}{l} \right)^2. \quad (4.14)$$

This proves that if (c_m, c_m) , (c_l, c_l) , (s_m, s_m) , (s_l, s_l) belong to \mathcal{H}_∞ , then the same is true for (c_{m+l}, c_{m+l}) . Taking $l = 1$ and by recurrence on $m \geq 1$ (integer) this proves that $\phi_{k,e,1}$ is in \mathcal{H}_∞ for every $k \geq 1$.

We proceed in the same way to prove that $(s_{m+l}, 0)$, $(0, s_{m+l}) \in \mathcal{H}_\infty$ provided that (c_m, c_m) , (c_l, c_l) , $(s_m, 0)$, $(s_l, 0)$, $(0, s_m)$, $(0, s_l)$ belong to \mathcal{H}_∞ . To see that, notice that, according to (4.13),

$$\begin{aligned} 2(s_{m+l}, 0) &= \left[-\frac{1}{m}(c'_m, c'_m) + \frac{1}{l}(s'_l, 0) \right]^2 - \left[\frac{1}{m}(c'_m, c'_m) \right]^2 - \left[\frac{1}{l}(s'_l, 0) \right]^2 \\ &\quad - \left[-\frac{1}{l}(c'_l, c'_l) + \frac{1}{m}(s'_m, 0) \right]^2 + \left[\frac{1}{l}(c'_l, c'_l) \right]^2 + \left[\frac{1}{m}(s'_m, 0) \right]^2. \end{aligned}$$

Hence $\phi_{k,e,j}$, $j = 2, 3$, is in \mathcal{H}_∞ for every $k \geq 1$.

The proof for the functions $\phi_{k,o}$ uses the fact that $(s_1, -s_1)$, (c_1, c_1) , $(s_{1/2}, -s_{1/2})$, $(c_{1/2}, c_{1/2})$ belong to \mathcal{H}_∞ and that

$$\pm 2s_{m+\frac{1}{2}} = \left(-\frac{c'_m}{m} \pm 2s'_{\frac{1}{2}} \right)^2 - \left(-\frac{c'_m}{m} \right)^2 - \left(\pm 2s'_{\frac{1}{2}} \right)^2 - \left(-2c'_{\frac{1}{2}} \pm \frac{s'_m}{m} \right)^2 + \left(-2c'_{\frac{1}{2}} \right)^2 + \left(\pm \frac{s'_m}{m} \right)^2,$$

which follows taking $m \neq 0$ and $l = 1/2$ in (4.13) and in its equivalent reformulation

$$-s_{m+l} = -s_m c_l + s_l c_m = \left(\frac{1}{2} (s_m - c_l)^2 - \frac{1}{2} s_m^2 - \frac{1}{2} c_l^2 \right) - \left(\frac{1}{2} (s_l - c_m)^2 - \frac{1}{2} s_l^2 - \frac{1}{2} c_m^2 \right).$$

The conclusion is then obtained by recurrence on $m \geq 1$ (integer). \square

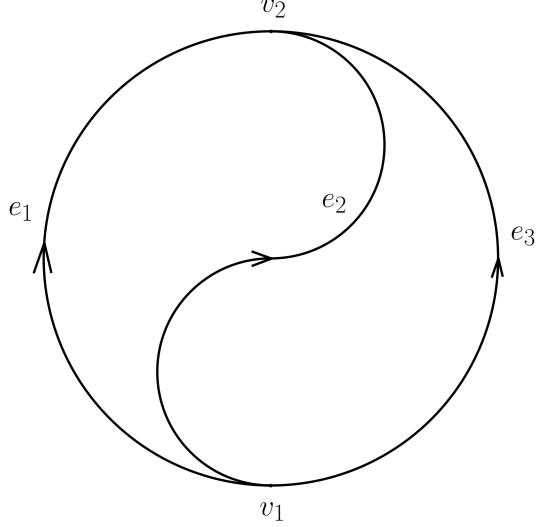


Figure 3: The graph with three branches.

Theorem 4.11. Equation (4.4) is small-time isomodulus approximately controllable. Moreover, for every $k, l \in \mathbb{Z}$, we have that

$$(\eta_l, \eta_l) \in \overline{\mathcal{R}_0(\eta_k, \eta_k)}.$$

Proof. This is a direct consequence of Theorem 4.7, Proposition 4.9, and Lemma 4.10. \square

4.5 The example of the graph with three branches

We consider now as \mathcal{G} the compact graph with $N = 3$ edges e_1, e_2, e_3 , each of them of length 2π , with $M = 2$ vertices v_1, v_2 and no loops. Namely, \mathcal{G} is composed of three edges having the same length $L = 2\pi$ that are linked to the same vertices. We orient the three edges from v_1 to v_2 . See Figure 3.

We consider the Hamiltonian $H_0 = -\partial_x^2$ on \mathcal{G} and Neumann–Kirchoff boundary conditions.

We have the following result.

Proposition 4.12. The eigenvalues and eigenfunctions are given by

$$\lambda_0 = 0, \quad \phi_0 = 1,$$

and, for $k \geq 1$,

$$\lambda_k = \left(\frac{k}{2}\right)^2 \quad \text{of multiplicity three,}$$

$$\phi_{k,1} = \left(c_{\frac{k}{2}}, c_{\frac{k}{2}}, c_{\frac{k}{2}}\right), \quad \phi_{k,2} = \left(-s_{\frac{k}{2}}, s_{\frac{k}{2}}, 0\right), \quad \phi_{k,3} = \left(-s_{\frac{k}{2}}, 0, s_{\frac{k}{2}}\right).$$

Proof. We know that the eigenvalues are of the form λ^2 , $\lambda \in \mathbb{R}$, because the Hamiltonian operator is nonnegative. As a consequence, on each edge, the eigenfunction is a solution to

$$\partial_x^2 + \lambda^2 \psi = 0.$$

So, the eigenfunctions are of the form

$$\phi = (\phi^j)_{1 \leq j \leq 3} = (A_j c_\lambda + B_j s_\lambda)_{1 \leq j \leq 3}.$$

The case $\lambda = 0$ being easy, let us focus on the case $\lambda \neq 0$.

We then write the boundary conditions, that are the continuity and the Neumann–Kirchoff conditions at each vertex, leading to

$$\phi^1(0) = \phi^2(0) = \phi^3(0) \Rightarrow A_1 = A_2 = A_3 = A, \quad (4.15)$$

$$\phi^1(2\pi) = \phi^2(2\pi) = \phi^3(2\pi) \Rightarrow B_1 \sin(2\pi\lambda) = B_2 \sin(2\pi\lambda) = B_3 \sin(2\pi\lambda), \quad (4.16)$$

$$\partial_x \phi^1(0) + \partial_x \phi^2(0) + \partial_x \phi^3(0) = 0 \Rightarrow B_1 + B_2 + B_3 = 0, \quad (4.17)$$

$$\begin{aligned} \partial_x \phi^1(2\pi) + \partial_x \phi^2(2\pi) + \partial_x \phi^3(2\pi) &= 0 \\ \Rightarrow -3A\lambda \sin(2\pi\lambda) + (B_1 + B_2 + B_3)\lambda \cos(2\pi\lambda) &= 0 \Rightarrow A \sin(2\pi\lambda) = 0. \end{aligned} \quad (4.18)$$

If $\sin(2\pi\lambda) \neq 0$, i.e., $\lambda \notin \mathbb{N}^*/2$, then we obtain from (4.16) and (4.18) that $B_1 = B_2 = B_3 = B$ and $A = 0$. Then $3B = 0$ by (4.17), so that also $B = 0$. This case can then be excluded.

If $\sin(2\pi\lambda) = 0$ then, all the conditions are fulfilled provided that $A = A_1 = A_2 = A_3$ and $B_1 = -B_2 - B_3$. This corresponds to the eigenfunctions introduced in the statement. \square

We remark now that the eigenvalues are not simple so there is no a priori obstruction to the isomodulus approximate controllability between two eigenfunctions corresponding to different energy levels. Another obstruction actually holds, as presented below.

Proposition 4.13. *For every $k, l \geq 0$, $k \neq l$, two eigenfunctions corresponding to $(\frac{k}{2})^2$ and $(\frac{l}{2})^2$ do not share the same modulus.*

Proof. Let us proceed by contradiction, supposing that two distinct energy levels are connectable. Since k and l are different, one of the two, say k , is larger than zero. A general eigenfunction corresponding to the eigenvalue $(k/2)^2$ can be written as

$$\phi_k = A_1 \phi_{k,1} + A_2 \phi_{k,2} + A_3 \phi_{k,3},$$

with $A_1, A_2, A_3 \in \mathbb{C}$ and $|A_1|^2 + |A_2|^2 + |A_3|^2 \neq 0$. We know from Lemma 3.7 that the modulus of ϕ_k on each edge is a constant so we have

$$\left| A_1 \cos\left(\frac{k}{2}x\right) - A_2 \sin\left(\frac{k}{2}x\right) - A_3 \sin\left(\frac{k}{2}x\right) \right|^2 = C_1, \quad x \in (0, 2\pi), \quad (4.19)$$

$$\left| A_1 \cos\left(\frac{k}{2}x\right) + A_2 \sin\left(\frac{k}{2}x\right) \right|^2 = C_2, \quad x \in (0, 2\pi), \quad (4.20)$$

$$\left| A_1 \cos\left(\frac{k}{2}x\right) + A_3 \sin\left(\frac{k}{2}x\right) \right|^2 = C_3, \quad x \in (0, 2\pi), \quad (4.21)$$

for some $C_1, C_2, C_3 \geq 0$.

If $A_1 = 0$ then we deduce from (4.20) and (4.21) that also $A_2, A_3 = 0$. We can then assume without loss of generality that $A_1 = 1$. Taking $x = 0$, we get that $C_1 = C_2 = C_3 = 1$ so that we can rewrite (4.19)–(4.21) as

$$\left| \cos\left(\frac{k}{2}x\right) - A_2 \sin\left(\frac{k}{2}x\right) - A_3 \sin\left(\frac{k}{2}x\right) \right|^2 = 1,$$

$$\left| \cos\left(\frac{k}{2}x\right) + A_2 \sin\left(\frac{k}{2}x\right) \right|^2 = 1,$$

$$\left| \cos\left(\frac{k}{2}x\right) + A_3 \sin\left(\frac{k}{2}x\right) \right|^2 = 1.$$

Taking $x = \frac{\pi}{4k}$ (so that $\frac{k}{2}x = \frac{\pi}{2}$), we get that $|A_2 + A_3| = |A_2| = |A_3| = 1$. Take now $x = \frac{\pi}{8k}$ (so that $\frac{k}{2}x = \frac{\pi}{4}$) and deduce that $|1 - A_1 - A_2| = |1 + A_2| = |1 + A_3| = 1$. Since the only $z \in \mathbb{C}$ such that $|z| = 1$ and $|1 + z| = 2$ is $z = 1$, we deduce that $-A_2 - A_3 = A_2 = A_3 = 1$, leading to a contradiction. \square

A Small-time isomodulus approximate controllability

A.1 Abstract setting

Let \mathcal{H} be an infinite-dimensional Hilbert space, H_0 be an unbounded self-adjoint operator with domain $\text{Dom}(H_0) \subset \mathcal{H}$, H_1, \dots, H_m be bounded self-adjoint operators on \mathcal{H} . Let us consider the abstract bilinear control system

$$\begin{cases} i\partial_t \psi(t) = H_0 \psi(t) + \sum_{j=1}^m u_j(t) H_j \psi(t), & t \in (0, +\infty), \\ \psi(0) = \psi_0. \end{cases} \quad (\text{A.1})$$

We have the following well-posedness result according to [BMS82]. For every $T > 0$, $\psi_0 \in \mathcal{H}$, and $u = (u_1, \dots, u_m) \in L^2(0, T; \mathbb{R}^d)$, there exists a unique mild solution $\psi \in C([0, T]; \mathcal{H})$ of (A.1), i.e.,

$$\psi(t) = e^{-itH_0} \psi_0 + \int_0^t e^{-i(t-s)H_0} \left(\sum_{j=1}^m u_j(s) H_j \right) \psi(s) ds, \quad \forall t \in [0, T].$$

If we assume furthermore that $\psi_0 \in \mathcal{S} = \{\psi \in \mathcal{H} \mid \|\psi\|_{\mathcal{H}} = 1\}$, then $\psi(t) \in \mathcal{S}$ for every $t \in [0, T]$. In the sequel, for $\psi_0 \in \mathcal{H}$ and $u \in L^2(0, T; \mathbb{R}^d)$, the associated solution $\psi \in C([0, T]; \mathcal{H})$ of (A.1) will be denoted by $\psi(\cdot; \psi_0, u)$.

The following result is exactly [CP23, Theorem 8].

Theorem A.1. *Let S be a bounded self-adjoint operator satisfying*

$$[S, H_j] = 0, \quad j = 1, \dots, m, \quad (\text{A.2})$$

$$S \text{Dom}(H_0) \subset \text{Dom}(H_0), \quad \text{ad}_S^3(H_0) \text{Dom}(H_0) = 0. \quad (\text{A.3})$$

Then, for each $\psi_0 \in \mathcal{H}$ and $u = (u_1, \dots, u_m) \in \mathbb{R}^m$, the following limit holds in \mathcal{H}

$$\begin{aligned} & \lim_{\delta \rightarrow 0} e^{-i\delta^{-1/2}S} \exp \left(-i\delta \left(H_0 + \sum_{j=1}^m \frac{u_j}{\delta} H_j \right) \right) e^{i\delta^{-1/2}S} \psi_0 \\ &= \exp \left(\frac{i}{2} \text{ad}_S^2(H_0) - i \sum_{j=1}^m u_j H_j \right) \psi_0, \end{aligned} \quad (\text{A.4})$$

where $\text{ad}_A^0 B = B$, $\text{ad}_A B = [A, B] = AB - BA$, and $\text{ad}_A^{n+1} B = [A, \text{ad}_A^n B]$.

A.2 Applications to manifolds with boundary

We start from the setting of Section 2.3. Consider the following asymptotic result.

Theorem A.2. For every $\psi_0 \in L^2(M)$, $(u_1, \dots, u_m) \in \mathbb{R}^m$, and $\varphi \in C^\infty(M; \mathbb{R})$ such that $\varphi \text{Dom}(H_0) \subset \text{Dom}(H_0)$, the following limit holds in $L^2(M)$

$$\begin{aligned} \lim_{\delta \rightarrow 0} e^{-i\delta^{-1/2}\varphi} \exp \left(-i\delta \left(-\Delta_g + V + \sum_{j=1}^m \frac{u_j}{\delta} Q_j \right) \right) e^{i\delta^{-1/2}\varphi} \psi_0 \\ = \exp \left(-ig(\nabla_g \varphi, \nabla_g \varphi) - i \sum_{j=1}^m u_j Q_j \right) \psi_0. \end{aligned} \quad (\text{A.5})$$

This is an adapted version of [CP23, Theorem 2] to the case of a manifold with boundary. The result is a direct application of Theorem A.1. Indeed, let us take $\mathcal{H} = L^2(M)$, $H_0 = -\Delta_g + V$, $H_j = Q_j$ for every $j \in \{1, \dots, m\}$, and let S be the bounded multiplication operator by φ on \mathcal{H} . Then (A.2) and (A.3) are fulfilled. Moreover, we have

$$\frac{1}{2} \text{ad}_S^2(H_0) = -g(\nabla_g \varphi, \nabla_g \varphi). \quad (\text{A.6})$$

For $Q_1, \dots, Q_m \in C^\infty(M; \mathbb{R})$, we recall the notation (2.13), (2.14), and (2.15). We have the following result.

Theorem A.3. For every $\psi_0 \in L^2(M)$, we have

$$\{e^{i\phi} \psi_0 \mid \phi \in \mathcal{H}_\infty\} \subset \overline{\mathcal{R}_0(\psi_0)}.$$

The proof of the theorem, included for completeness, is an adaptation of the ones of [CP23, Theorem 3] and [DP23, Theorem 1], and works because, for every $N \geq 0$ and every $\varphi \in \mathcal{H}_N$, $\varphi \text{Dom}(H_0) \subset \text{Dom}(H_0)$, so the arguments of the original proof still make sense. Notice that this last property is included in the definitions (2.13), (2.14) while this inclusion was automatically satisfied in the specific setting chosen for [CP23, Theorem 3]. Finally, note that Theorem A.3 directly implies Theorem 2.19.

Proof of Theorem A.3. Let us introduce the following notation for the concatenation of two controls u and v : given $\tau \geq 0$ such that u is defined at least up to time τ , $u \diamond_\tau v$ denotes the function

$$(u \diamond_\tau v)(t) = \begin{cases} u(t) & \text{for } t \in [0, \tau), \\ v(t - \tau) & \text{for } t \geq \tau. \end{cases} \quad (\text{A.7})$$

It is sufficient to prove by recurrence on $N \geq 0$ that for every $\psi_0 \in L^2(M)$,

$$\{e^{i\phi} \psi_0 \mid \phi \in \mathcal{H}_N\} \subset \overline{\mathcal{R}_0(\psi_0)}. \quad (\text{A.8})$$

Initialization: The case $N = 0$ is a straightforward consequence of Theorem A.2 by taking $\varphi = 0$. Indeed, we have that for every $(u_1, \dots, u_m) \in \mathbb{R}^m$

$$\lim_{\delta \rightarrow 0} \exp \left(-i\delta \left(-\Delta_g + V + \sum_{j=1}^m \frac{u_j}{\delta} Q_j \right) \right) \psi_0 = \exp \left(-i \sum_{j=1}^m u_j Q_j \right) \psi_0 \quad \text{in } L^2(M). \quad (\text{A.9})$$

Recurrence step: Assume that for some $N \in \mathbb{N}$, we have (A.8) for every $\psi_0 \in L^2(M)$. Let us take $\varepsilon > 0$ and $\phi \in \mathcal{H}_{N+1}$. Then there exist $\phi_0, \dots, \phi_k \in \mathcal{H}_N$ and $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ such that

$$\phi = \phi_0 + \sum_{j=1}^k \alpha_j g(\nabla \phi_j, \nabla \phi_j).$$

Let us first consider the case $\alpha_1 \geq 0$. Note that $\phi_1 \in \mathcal{H}_N$ stabilizes $\text{Dom}(H_0)$. By applying Theorem A.2 with $u = 0$, we obtain that there exists $\gamma \in (0, T/3)$ such that

$$\left\| e^{-i\gamma^{-1/2}\phi_1} e^{-i\gamma(-\Delta_g+V)} e^{i\gamma^{-1/2}\phi_1} e^{i\phi_0} \psi_0 - e^{i\alpha_1 g(\nabla\phi_1, \nabla\phi_1)} e^{i\phi_0} \psi_0 \right\|_{L^2(M)} < \varepsilon.$$

Then, by the inductive hypothesis, there exist $\tau_1 \in (0, T/3)$ and a control $u_1 \in L^2(0, \tau_1; \mathbb{R}^m)$ such that

$$\left\| \psi(\tau_1; \psi_0, u_1) - e^{i\gamma^{-1/2}\phi_1} e^{i\phi_0} \psi_0 \right\|_{L^2(M)} < \varepsilon. \quad (\text{A.10})$$

By using that the semigroup $(e^{-it(-\Delta_g+V)})_{t \geq 0}$ is unitary and (A.10), we obtain that for $\tau_2 = \gamma$ and $u_2 = 0$ in $(0, \tau_2)$,

$$\begin{aligned} & \left\| \psi(\tau_2 + \tau_1; \psi_0, u_1 \diamond_{\tau_1} u_2) - e^{-i\gamma(-\Delta_g+V)} e^{i\gamma^{-1/2}\phi_1} e^{i\phi_0} \psi_0 \right\|_{L^2(M)} \\ &= \left\| e^{-i\gamma(-\Delta_g+V)} \psi(\tau_1; \psi_0, u_1) - e^{-i\gamma(-\Delta_g+V)} e^{i\gamma^{-1/2}\phi_1} e^{i\phi_0} \psi_0 \right\|_{L^2(M)} < \varepsilon. \end{aligned} \quad (\text{A.11})$$

By using again the inductive hypothesis to the data $e^{-i\gamma(-\Delta_g+V)} e^{i\gamma^{-1/2}\phi_1} e^{i\phi_0} \psi_0$, we obtain that there exist a time $\tau_3 \in (0, T/3)$ and a control $u_3 \in L^2(0, \tau_3; \mathbb{R}^m)$ such that

$$\left\| \psi(\tau_3; e^{-i\gamma(-\Delta_g+V)} e^{i\gamma^{-1/2}\phi_1} e^{i\phi_0} \psi_0, u_3) - e^{-i\gamma^{-1/2}\phi_1} e^{-i\gamma(-\Delta_g+V)} e^{i\gamma^{-1/2}\phi_1} e^{i\phi_0} \psi_0 \right\|_{L^2(M)} < \varepsilon. \quad (\text{A.12})$$

We now set $\tau = \tau_1 + \tau_2 + \tau_3$ and $u = u_1 \diamond_{\tau_1} u_2 \diamond_{\tau_2} u_3 \in L^2(0, \tau; \mathbb{R}^m)$ and we have from (A.11) and (A.12),

$$\begin{aligned} & \left\| \psi(\tau; \psi_0, u) - e^{i\alpha_1 g(\nabla\phi_1, \nabla\phi_1)} e^{i\phi_0} \psi_0 \right\|_{L^2(M)} \\ & \leq \left\| \psi(\tau_3; \psi(\tau_2 + \tau_1; \psi_0, u_1 \diamond_{\tau_1} u_2), u_3) - \psi(\tau_3; e^{-i\gamma(-\Delta_g+V)} e^{i\gamma^{-1/2}\phi_1} e^{i\phi_0} \psi_0, u_3) \right\|_{L^2(M)} \\ & + \left\| \psi(\tau_3; e^{-i\gamma(-\Delta_g+V)} e^{i\gamma^{-1/2}\phi_1} e^{i\phi_0} \psi_0, u_3) - e^{-i\gamma^{-1/2}\phi_1} e^{-i\gamma(-\Delta_g+V)} e^{i\gamma^{-1/2}\phi_1} e^{i\phi_0} \psi_0 \right\|_{L^2(M)} \\ & + \left\| e^{-i\gamma^{-1/2}\phi_1} e^{-i\gamma(-\Delta_g+V)} e^{i\gamma^{-1/2}\phi_1} e^{i\phi_0} \psi_0 - e^{i\alpha_1 g(\nabla\phi_1, \nabla\phi_1)} e^{i\phi_0} \psi_0 \right\|_{L^2(M)} \\ & < 3\varepsilon. \end{aligned}$$

By arbitrariness of $\varepsilon > 0$, this proves that

$$e^{i\alpha_1 g(\nabla\phi_1, \nabla\phi_1)} e^{i\phi_0} \psi_0 \in \overline{\mathcal{R}_0(\psi_0)}. \quad (\text{A.13})$$

If $\alpha_1 < 0$, one only needs to replace δ by $-\delta$ in the limit of (A.5). We then iterate the argument by considering the limit (A.5) with $\varphi = -\alpha_2^{1/2} \phi_2$ if $\alpha_2 \geq 0$, starting from the initial data $e^{i\alpha_1 g(\nabla\phi_1, \nabla\phi_1)} e^{i\phi_0} \psi_0$. Again, note that $\phi_2 \in \mathcal{H}_N$ stabilizes $\text{Dom}(H_0)$, therefore this leads to the possibility of applying of Theorem A.2.

Therefore, thanks to (A.13), one obtains

$$e^{i(\phi_0 + \sum_{j=1}^k \alpha_j g(\nabla\phi_j, \nabla\phi_j))} \in \overline{\mathcal{R}_0(\psi_0)},$$

and the proof is complete. \square

A.3 Application to quantum graphs

We adopt here the setting of Section 4.3. Consider the following asymptotic result.

Theorem A.4. *For every $\psi_0 \in L^2(\mathcal{G})$, $(u_1, \dots, u_m) \in \mathbb{R}^m$, and $\varphi \in C^0(\mathcal{G}; \mathbb{R})$ such that the restriction of φ to each edge is C^∞ and $\varphi \text{Dom}(H_0) \subset \text{Dom}(H_0)$, the following limit holds in $L^2(\mathcal{G})$*

$$\begin{aligned} & \lim_{\delta \rightarrow 0} e^{-i\delta^{-1/2}\varphi} \exp \left(-i\delta \left(-\partial_x^2 + V + \sum_{j=1}^m \frac{u_j}{\delta} Q_j \right) \right) e^{i\delta^{-1/2}\varphi} \psi_0 \\ &= \exp \left(-i(\partial_x \varphi)^2 - i \sum_{j=1}^m u_j Q_j \right) \psi_0. \end{aligned} \quad (\text{A.14})$$

This is an adapted version of [CP23, Theorem 2] to the case of a quantum graph. The result is a direct application of Theorem A.1. Indeed, let us take $\mathcal{H} = L^2(\mathcal{G})$, $H_0 = -\Delta_g + V$, $H_j = Q_j$ for every $j \in \{1, \dots, m\}$, and let S be the bounded multiplication operator by φ on \mathcal{H} . Then (A.2) and (A.3) are fulfilled. Moreover, we have

$$\frac{1}{2} \text{ad}_S^2(H_0) = -g(\nabla_g \varphi, \nabla_g \varphi). \quad (\text{A.15})$$

Assume that $Q_1, \dots, Q_m \in C^0(\mathcal{G}; \mathbb{R})$ and the restrictions of Q_1, \dots, Q_m to each edge of \mathcal{G} are C^∞ up to the boundary. Recalling the notations (4.5), (4.6), and (4.7), we have the following result.

Theorem A.5. *For every $\psi_0 \in L^2(\mathcal{G})$, we have*

$$\{e^{i\phi} \psi_0 \mid \phi \in \mathcal{H}_\infty\} \subset \overline{\mathcal{R}_0(\psi_0)}.$$

The proof of the theorem is the same as the one of Theorem A.3. Finally, note that Theorem A.5 directly implies Theorem 4.7.

References

- [ABGS17] Andrei Agrachev, Ugo Boschain, Jean-Paul Gauthier, and Mario Sigalotti. A note on time-zero controllability and density of orbits for quantum systems. In *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*, pages 5535–5538. IEEE, 2017.
- [Alb75] Jeffrey H. Albert. Genericity of simple eigenvalues for elliptic PDE's. *Proc. Amer. Math. Soc.*, 48:413–418, 1975.
- [Aro57] Nachman Aronszajn. A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order. *J. Math. Pures Appl. (9)*, 36:235–249, 1957.
- [AS05] Andrey A. Agrachev and Andrey V. Sarychev. Navier-Stokes equations: controllability by means of low modes forcing. *J. Math. Fluid Mech.*, 7(1):108–152, 2005.
- [AS06] Andrey A. Agrachev and Andrey V. Sarychev. Controllability of 2D Euler and Navier-Stokes equations by degenerate forcing. *Comm. Math. Phys.*, 265(3):673–697, 2006.

- [BBS21] Ivan Beschastnyi, Ugo Boscain, and Mario Sigalotti. An obstruction to small-time controllability of the bilinear Schrödinger equation. *J. Math. Phys.*, 62(3):Paper No. 032103, 14, 2021.
- [BCC12] Nabile Boussaïd, Marco Caponigro, and Thomas Chambrion. Small time reachable set of bilinear quantum systems. In *Proceedings of the 51st IEEE Conference on Decision and Control (CDC)*, pages 1083–1087, 2012.
- [BCC20] Nabile Boussaïd, Marco Caponigro, and Thomas Chambrion. Regular propagators of bilinear quantum systems. *J. Funct. Anal.*, 278(6):108412, 66, 2020.
- [BCT14] Karine Beauchard, Jean-Michel Coron, and Holger Teismann. Minimal time for the bilinear control of Schrödinger equations. *Systems & Control Letters*, 71:1–6, 2014.
- [BCT18] Karine Beauchard, Jean-Michel Coron, and Holger Teismann. Minimal time for the approximate bilinear control of Schrödinger equations. *Math. Methods Appl. Sci.*, 41(5):1831–1844, 2018.
- [BK13] Gregory Berkolaiko and Peter Kuchment. *Introduction to quantum graphs*, volume 186 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2013.
- [BL17] Gregory Berkolaiko and Wen Liu. Simplicity of eigenvalues and non-vanishing of eigenfunctions of a quantum graph. *J. Math. Anal. Appl.*, 445(1):803–818, 2017.
- [BMS82] John M. Ball, Jerrold E. Marsden, and Marshall Slemrod. Controllability for distributed bilinear systems. *SIAM J. Control Optim.*, 20(4):575–597, 1982.
- [Cha84] Isaac Chavel. *Eigenvalues in Riemannian geometry*, volume 115 of *Pure and Applied Mathematics*. Academic Press, Inc., Orlando, FL, 1984. Including a chapter by Burton Randol, With an appendix by Jozef Dodziuk.
- [CP23] Thomas Chambrion and Eugenio Pozzoli. Small-time bilinear control of Schrödinger equations with application to rotating linear molecules. *Automatica J. IFAC*, 153:Paper No. 111028, 7, 2023.
- [CR21] Christophe Cheverry and Nicolas Raymond. *A guide to spectral theory—applications and exercises*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser/Springer, Cham, [2021] ©2021. With a foreword by Peter D. Hislop.
- [DN24] Alessandro Duca and Vahagn Nersesyan. Bilinear control and growth of Sobolev norms for the nonlinear Schrödinger equation. *J. Eur. Math. Soc.*, pages 1–20, 2024.
- [DP23] Alessandro Duca and Eugenio Pozzoli. Small-time controllability for the nonlinear Schrödinger equation on \mathbb{R}^n via bilinear electromagnetic fields. ArXiv:2307.15819, 2023.
- [Duc20] Alessandro Duca. Global exact controllability of bilinear quantum systems on compact graphs and energetic controllability. *SIAM J. Control Optim.*, 58(6):3092–3129, 2020.
- [Duc21] Alessandro Duca. Bilinear quantum systems on compact graphs: well-posedness and global exact controllability. *Automatica J. IFAC*, 123:Paper No. 109324, 13, 2021.

- [Fri05] Leonid Friedlander. Genericity of simple eigenvalues for a metric graph. *Israel J. Math.*, 146:149–156, 2005.
- [GT01] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
- [Hen06] Antoine Henrot. *Extremum problems for eigenvalues of elliptic operators*. Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006.
- [Kuc02] Peter Kuchment. Graph models for waves in thin structures. *Waves Random Media*, 12(4):R1–R24, 2002.
- [Mil65] John W. Milnor. *Topology from the differentiable viewpoint*. University Press of Virginia, Charlottesville, VA, 1965. Based on notes by David W. Weaver.
- [MS10] Paolo Mason and Mario Sigalotti. Generic controllability properties for the bilinear Schrödinger equation. *Comm. Partial Differential Equations*, 35(4):685–706, 2010.
- [PT21] Marvin Plümer and Matthias Täufer. On fully supported eigenfunctions of quantum graphs. *Lett. Math. Phys.*, 111(6):Paper No. 153, 23, 2021.
- [RS78] Michael Reed and Barry Simon. *Methods of modern mathematical physics*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.
- [Sie14] Carl L. Siegel. Über einige Anwendungen diophantischer Approximationen [reprint of Abhandlungen der Preußischen Akademie der Wissenschaften. Physikalisch-mathematische Klasse 1929, Nr. 1]. In *On some applications of Diophantine approximations*, volume 2 of *Quad./Monogr.*, pages 81–138. Ed. Norm., Pisa, 2014.
- [Tan79] Masao Tanikawa. The spectrum of the Laplacian and smooth deformation of the Riemannian metric. *Proc. Japan Acad. Ser. A Math. Sci.*, 55(4):125–127, 1979.
- [Tur00] Gabriel Turinici. On the controllability of bilinear quantum systems. In *Mathematical models and methods for ab initio quantum chemistry*, volume 74 of *Lecture Notes in Chem.*, pages 75–92. Springer, Berlin, 2000.
- [Uhl76] Karen Uhlenbeck. Generic properties of eigenfunctions. *Amer. J. Math.*, 98(4):1059–1078, 1976.