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► **To cite this version:**

Théophile Chaumont-Frelet, Euan A. Spence. The geometric error is less than the pollution error when solving the high-frequency Helmholtz equation with high-order FEM on curved domains. 2024. hal-04483175

HAL Id: hal-04483175

<https://inria.hal.science/hal-04483175>

Preprint submitted on 29 Feb 2024

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The geometric error is less than the pollution error when solving the high-frequency Helmholtz equation with high-order FEM on curved domains

T. Chaumont-Frelet*, E. A. Spence†

February 29, 2024

Abstract

We consider the h -version of the finite-element method, where accuracy is increased by decreasing the meshwidth h while keeping the polynomial degree p constant, applied to the Helmholtz equation. Although the question “how quickly must h decrease as the wavenumber k increases to maintain accuracy?” has been studied intensively since the 1990s, none of the existing rigorous wavenumber-explicit analyses take into account the approximation of the geometry. In this paper we prove that for nontrapping problems solved using straight elements the geometric error is order kh , which is then less than the pollution error $k(kh)^{2p}$ when k is large; this fact is then illustrated in numerical experiments. More generally, we prove that, even for problems with strong trapping, using degree four (in 2-d) or degree five (in 3-d) polynomials and isoparametric elements ensures that the geometric error is smaller than the pollution error for most large wavenumbers.

1 Introduction: informal statement of the main result

The results of this paper apply to general Helmholtz problems (interior problems, exterior problems, wave-guide problems), but for concreteness here we consider the Helmholtz equation

$$-k^2\mu u - \nabla \cdot (A\nabla u) = f, \quad (1.1)$$

with $k \gg 1$, posed in a bounded domain Ω corresponding to the exterior of an impenetrable obstacle where the Sommerfeld radiation condition has been approximated by a radial perfectly-matched layer (PML). For simplicity, we assume that Ω has characteristic length scale ~ 1 . The solution u satisfies either a zero Dirichlet or zero Neumann boundary condition on the part of $\partial\Omega$ corresponding to the impenetrable obstacle, and a zero Dirichlet boundary condition on the part of $\partial\Omega$ corresponding to the PML truncation boundary. The piecewise-smooth coefficients μ and A describe the PML near the truncation boundary, and variation of μ and A in the rest of the domain corresponds to penetrable obstacles.

We assume that the solution operator to (1.1) is bounded by $k^{\alpha-1}$ for some $\alpha > 0$, i.e., given $k_0 > 0$ there exists $C > 0$ such that given $f \in L^2(\Omega)$, the solution $u \in H^1(\Omega)$ to (1.1) satisfies

$$\|u\|_{H_k^1(\Omega)} \leq Ck^{\alpha-1} \|f\|_{L^2(\Omega)} \quad \text{for all } k \geq k_0, \quad (1.2)$$

where

$$\|u\|_{H_k^1(\Omega)}^2 := \|\nabla u\|_{L^2(\Omega)}^2 + k^2 \|u\|_{L^2(\Omega)}^2. \quad (1.3)$$

Recall that the bound (1.2) holds with $\alpha = 1$ when the problem is *nontrapping* (i.e., all geometric-optic rays starting in a neighbourhood of the scatterer escape to infinity in a uniform time); indeed in this case the solution of the exterior Helmholtz problem satisfies (1.2) with $\alpha = 1$ by [Mor75, Vai75], and the solution of the associated radial PML problem inherits this bound by [GLS23, Lemma 3.3].

*Inria Univ. Lille and Laboratoire Paul Painlevé, 59655 Villeneuve-d’Ascq, France, theophile.chaumont@inria.fr

†Department of Mathematical Sciences, University of Bath, Bath, BA2 7AY, UK, E.A.Spence@bath.ac.uk

Theorem 1.1 (Informal statement of the main result) *Suppose that the Helmholtz problem above is solved using the h -version of the finite-element method with shape-regular meshes with meshwidth h , fixed (but arbitrary) polynomial degree p , and fixed order of the geometric approximation q (so that $q = 1$ for straight elements and $q = p$ for isoparametric elements).*

Suppose further that the domain Ω and coefficients A and μ satisfy the natural regularity assumptions for using degree p polynomials, and that the data f in (1.1) comes from an incident plane wave or point source (or, more generally, a particular Helmholtz solution that is smooth in a neighbourhood of the scatterer). If

$$\left(kh + k^\alpha(kh)^p\right)(kh)^p + k^\alpha h^q \text{ is sufficiently small,} \quad (1.4)$$

then the Galerkin solution u_h exists, is unique, and satisfies

$$\frac{\|u - u_h\|_{H_k^1(\Omega)}}{\|u\|_{H_k^1(\Omega)}} \leq C \left[\left(1 + k^\alpha(kh)^p\right)(kh)^p + k^\alpha h^q \right]. \quad (1.5)$$

We make seven remarks about the interpretation and context of Theorem 1.1.

1. Near $\partial\Omega$, the error $\|u - u_h\|_{H_k^1(\Omega)}$ in (1.5) is measured on the subset of Ω where both u and u_h are well-defined. When A and μ are discontinuous, defining precisely where the error $\|u - u_h\|_{H_k^1(\Omega)}$ is measured is more involved; see Theorem 4.11 below.
2. In the limit $h \rightarrow 0$ with k fixed, the bound (1.5) shows that the Galerkin error is $O(h^{\min\{p,q\}})$; this is expected, at least in the isoparametric case when $p = q$, by [CR72a, Len86].
3. When $q = \infty$ (i.e., there is no geometric error), the condition (1.4) becomes

$$k^\alpha(kh)^{2p} \text{ is sufficiently small,}$$

and then (1.5) implies that choosing $k^\alpha(kh)^{2p}$ to be a sufficiently-small constant maintains accuracy of the Galerkin solution as $k \rightarrow \infty$. This result was

- proved for constant-coefficient problems in 1-d by [IB97] (see [IB97, Page 350, penultimate displayed equation], [Ihl98, Equation 4.7.41]),
- obtained in the context of dispersion analysis in [Ain04],
- proved for 2- and 3-d constant-coefficient Helmholtz problems with the radiation condition approximated by an impedance boundary condition in [DW15] (following [FW09, FW11, Wu14, ZW13]) and for variable-coefficient problems in [Pem20, Theorem 2.39], and then
- proved for general 2- and 3-d Helmholtz problems with truncation either by a PML or the exact Dirichlet-to-Neumann map in [GS23].

We note that bounds under the condition “ $k^\alpha(kh)^p$ sufficiently small” were proved for several specific Helmholtz problems in [MS10, MS11], and then for general Helmholtz problems in [CFN20].

4. When the problem is nontrapping, $\alpha = 1$, and $kh^q \ll 1$ when $k(kh)^{2p}$ is constant, regardless of q . Therefore, Theorem 1.1 implies that, regardless of q , choosing $k^\alpha(kh)^{2p}$ to be a sufficiently-small constant maintains accuracy of the Galerkin solution as $k \rightarrow \infty$. That is, even with straight elements ($q = 1$), in the limit $k \rightarrow \infty$ with h chosen to control the pollution error, the geometric error is smaller than the pollution error; this feature is illustrated in the numerical experiments in §2.
5. When the problem is trapping, $\alpha > 1$, and it is not immediate that $k^\alpha h^q \ll 1$ when $k^\alpha(kh)^{2p}$ is constant. For Helmholtz problems with strong trapping, the solution operator grows exponentially through an increasing sequence of wavenumbers, with this sequence becoming increasingly dense as k increases [PV99, CP02, BCWG+11]. However, if a set of wavenumbers of arbitrarily-small measure is excluded then (1.2) holds for any $\alpha > 5d/2 + 2$ by [LSW21] and

[GLS23, Lemma 3.3] ([LSW21] proves this result for the original exterior Helmholtz problem, and the PML problem inherits this bound by [GLS23, Lemma 3.3]). With isoparametric elements $q = p$, and then $h^q k \ll 1$ when $k^\alpha (kh)^{2p}$ is constant if $p \geq 4$ in 2-d and $p \geq 5$ in 3-d. That is, for moderate polynomial degree, isoparametric elements, and most (large) wavenumbers, the geometric error is smaller than the pollution error, even for problems with strong trapping.

6. To prove Theorem 1.1, we extend the duality argument recently introduced for general Helmholtz problems in [GS23] to incorporate a Strang-lemma-type argument to allow variational crimes. We then use the results of [Len86] to apply this abstract setting to the case of geometric error induced by polynomial element maps.
7. Finally, we highlight that Theorem 1.1 is conceptually similar to the results of [CFN18, CFN23]. Indeed, [CFN18, CFN23] show that, when solving 2-d or 3-d Helmholtz problems on domains with corners/conical points using a uniform mesh, the error from the corner singularity is smaller than the pollution error (for k sufficiently large). In other words, although *both* the geometric error *and* the error incurred by corner singularities are important in the limit $h \rightarrow 0$ with k fixed, in the limit $k \rightarrow \infty$ with h chosen to control the pollution error, both of these errors are smaller than the pollution error.

Outline of the paper. §2 gives numerical experiments illustrating Theorem 1.1 for two 2-d nontrapping problems ($\alpha = 0$) and straight elements. §3 contains an abstract analogue of Theorem 1.1 proved under the assumptions that the sesquilinear form is continuous and satisfies a Gårding-type inequality. §4 shows that a wide variety of Helmholtz problems fix into the abstract setting of §3 and proves Theorem 1.1.

2 Numerical experiments with straight elements in 2-d

In this section we demonstrate numerically for two 2-d scattering problems the consequence of Theorem 1.1 discussed in Point 4 above; i.e., that, for nontrapping problems solved using straight elements, in the limit $k \rightarrow \infty$ with h chosen to control the pollution error, the geometric error is smaller than the pollution error.

The two scattering problems considered.

Definition 2.1 (Sound-soft scattering by a 2-d ball) Given $u_{\text{inc}}(x) := e^{ikx_1}$, let u_{tot} satisfy

$$-k^2 u_{\text{tot}} - \Delta u_{\text{tot}} = 0 \quad \text{in } \mathbb{R}^d \setminus B_1(0), \quad u_{\text{tot}} = 0 \quad \text{on } \partial B_1(0),$$

where $u_{\text{sca}} := u_{\text{tot}} - u_{\text{inc}}$ satisfies the Sommerfeld radiation condition

$$\left(\frac{\partial}{\partial r} - ik \right) u_{\text{sca}} = o(r^{-(d-1)/2}) \quad \text{as } r := |x| \rightarrow \infty, \quad \text{uniformly in } x/r. \quad (2.1)$$

Definition 2.2 (Scattering by a penetrable 2-d ball) Given $u_{\text{inc}}(x) := e^{ikx_1}$ and

$$A := \begin{cases} 2 & \text{in } B_1(0), \\ 1 & \text{in } \mathbb{R}^d \setminus \overline{B_1(0)}, \end{cases} \quad \text{and} \quad \mu := \begin{cases} 1/2 & \text{in } B_1(0), \\ 1 & \text{in } \mathbb{R}^d \setminus \overline{B_1(0)}, \end{cases} \quad (2.2)$$

let u_{tot} satisfy

$$-k^2 \mu u_{\text{tot}} - \nabla \cdot (A \nabla u_{\text{tot}}) = 0 \quad \text{in } \mathbb{R}^d, \quad (2.3)$$

and

$$2\partial_r u_{\text{tot}}^+ = \partial_r u_{\text{tot}}^- \quad \text{and} \quad u_{\text{tot}}^+ = u_{\text{tot}}^- \quad \text{on } \partial B_1(0), \quad (2.4)$$

where the subscript $+$ denotes the limit taken with $r > 1$ and the subscript $-$ denotes the limit taken with $r < 1$, and where $u_{\text{sca}} := u_{\text{tot}} - u_{\text{inc}}$ satisfies the Sommerfeld radiation condition (2.1).

(Note that the transmission conditions in (2.4) follow from the variational formulation of the PDE (2.3).)

The choice of the coefficients A and μ in (2.2) implies that this problem is nontrapping, in the sense that outgoing solutions to (1.1), with A and μ given by (2.2), satisfy the bound (1.2) with $\alpha = 1$; see [MS19].

The reason for choosing these two scattering problems is that, in both cases, u_{tot} can be expressed explicitly as a Fourier series in the angular variable, with the Fourier coefficients expressed in terms of Hankel and Bessel functions.

Although it is possible to directly compute u_{tot} with the FEM, it is more convenient to compute

$$u := \chi u^I + u^S$$

for $\chi \in C_{\text{comp}}^\infty(\mathbb{R}^d)$ and $\chi \equiv 1$ in a neighbourhood of $B_1(0)$; this is because u then satisfies (i) the same boundary/transmission conditions as u_{tot} on $\partial B_1(0)$ and (ii) the Sommerfeld radiation condition (satisfied by u_{sca}) at infinity. Once u is computed, u_{tot} and u_{sca} can easily be recovered since χ and u_{inc} are known.

We then use a PML to approximate the radiation condition satisfied by u . Following [CM98], we let

$$\mu_{\text{PML}}(x) := \mathfrak{B}(r_{\text{P}})\beta(r_{\text{P}}) \quad (2.5)$$

and

$$A_{\text{PML}}(x) = \frac{\mathfrak{B}(r_{\text{P}})}{\beta(r_{\text{P}})} \begin{pmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{pmatrix} + \frac{\beta(r_{\text{P}})}{\mathfrak{B}(r_{\text{P}})} \begin{pmatrix} \sin^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \cos^2 \theta \end{pmatrix} \quad (2.6)$$

where $r_{\text{P}} = |x| - 4$, θ is such that $x = |x|(\cos \theta, \sin \theta)$, and

$$\beta(r) = \alpha r^\ell \quad \mathfrak{B}(r) = \frac{\alpha}{\ell + 1} r^{\ell+1}$$

with $\alpha = 10$ and $\ell = 2$.

We therefore approximate with the FEM the solutions of the following two problems.

Definition 2.3 (PML approximation to sound-soft scattering by a 2-d ball) Let $\Omega := B_5(0) \setminus \overline{B_1(0)}$. Let

$$A := \begin{cases} 1 & \text{in } B_4(0) \setminus \overline{B_1(0)}, \\ A_{\text{PML}} & \text{in } B_5(0) \setminus \overline{B_1(0)}, \end{cases} \quad \text{and} \quad \mu := \begin{cases} 1 & \text{in } B_4(0) \setminus \overline{B_1(0)}, \\ \mu_{\text{PML}} & \text{in } B_5(0) \setminus \overline{B_1(0)}, \end{cases}$$

where A_{PML} and μ_{PML} are given by (2.6) and (2.5) respectively. Given $\chi \in C_{\text{comp}}^\infty(\mathbb{R}^d)$ satisfying

$$\chi \equiv 1 \text{ on } B_1(0) \quad \text{and} \quad \chi \equiv 0 \text{ on } B_5(0) \setminus \overline{B_4(0)}, \quad (2.7)$$

let $u \in H_0^1(\Omega)$ satisfy

$$-k^2 \mu u - \nabla \cdot (A \nabla u) = f \quad (2.8)$$

with

$$f := -(\Delta \chi) u_{\text{inc}} - 2 \nabla \chi \cdot \nabla u_{\text{inc}} = (-\Delta \chi - 2ik \partial_1 \chi) e^{ikx_1}. \quad (2.9)$$

Definition 2.4 (PML approximation to scattering by a penetrable 2-d ball) Let $\Omega := B_5(0)$. Let

$$A := \begin{cases} 2 & \text{in } B_1(0), \\ 1 & \text{in } B_4(0) \setminus \overline{B_1(0)}, \\ A_{\text{PML}} & \text{in } B_5(0) \setminus \overline{B_1(0)} \end{cases} \quad \text{and} \quad \mu := \begin{cases} 1/2 & \text{in } B_1(0), \\ 1 & \text{in } B_4(0) \setminus \overline{B_1(0)}, \\ \mu_{\text{PML}} & \text{in } B_5(0) \setminus \overline{B_1(0)} \end{cases}$$

where A_{PML} and μ_{PML} are given by (2.6) and (2.5) respectively. Given $\chi \in C_{\text{comp}}^\infty(\mathbb{R}^d)$ satisfying (2.7), let $u \in H_0^1(\Omega)$ satisfy (2.8), with f given by (2.9), and the transmission conditions

$$2\partial_r u^+ = \partial_r u^- \quad \text{and} \quad u^+ = u^- \quad \text{on } \partial B_1(0).$$

For simplicity, in the numerical experiments below, we actually use the cutoff

$$\chi(x) := \frac{1}{2} \left(1 - \operatorname{erf} \left(\frac{|x| - 3}{\sigma} \right) \right),$$

with $\sigma = 0.2$ and

$$\operatorname{erf}(t) := \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2} ds.$$

This χ does not satisfy (2.7); however, $|1 - \chi(x)| < 10^{-12}$ if $|x| \leq 2$ and $|\chi(x)| < 10^{-12}$ if $|x| \geq 4$, so that the error incurred by this inconsistency is so small that it does not affect our examples.

The finite-element solution and error measurement For simplicity, we only consider the error in the region where $\chi \equiv 1$, so that $u \equiv u_{\text{tot}}$, i.e.,

$$\Omega_{\text{tot}} := \begin{cases} B_2(0) \setminus \overline{B_1(0)} & \text{for the sound-soft ball,} \\ B_2(0) & \text{for the penetrable ball.} \end{cases}$$

Hence, our error measure is

$$\|u_{\text{tot}} - u_h\|_{H^1(\Omega_{\text{tot}})};$$

crucially, Ω_{tot} includes the interface $\partial B_1(0)$ where the geometric approximation takes place.

As a proxy for $u = u_{\text{tot}}$ we use the Fourier-series solution; this incurs an error due to the PML approximating the radiation condition, but this error is very small, especially for large wavenumbers; see [GLS23].

We consider finite element discretizations of degree $p = 2, 3$ or 4 and straight meshes. Since the analytical expression of $u_{\text{in}} := u_{\text{tot}}|_D$ and $u_{\text{out}} := u_{\text{tot}}|_{\mathbb{R}^d \setminus D}$ are known here, considering any triangle $K \in \mathcal{T}_h$ with barycenter x_K , we compute the error by $\|u_{\text{in}} - u_h\|_{H^1(K)}$ if $x_K \in D$ and $\|u_{\text{out}} - u_h\|_{H^1(K)}$ otherwise.

We use `gmsh` [GR09] to generate the meshes. `gmsh` produces nodally conforming meshes (meaning that interfaces are not cross by any edge) of specified maximal size h_{target} . Once such a mesh has been produced, we compute the actual mesh size h by measuring the length of all mesh edges. This value of h is the one plotted in the left-hand sides of the figures below.

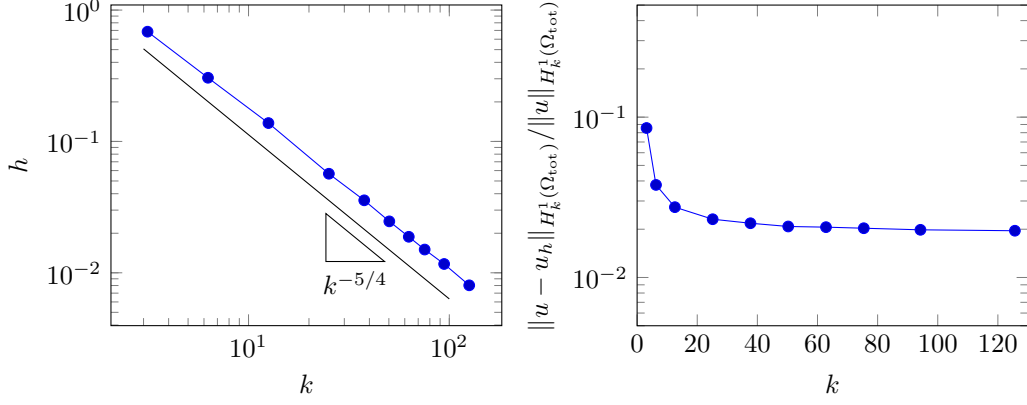
We consider an increasing sequence of wavenumbers ranging from $0.5 \cdot 2\pi$ to $20 \cdot 2\pi$, so that the number of wavelengths in the total field region where the error is measured ranges from 2 to 80. Fixing a polynomial degree p we then solve the problem for all wavenumber with increasingly refined meshes. Specifically, we ask `gmsh` to mesh the domain at size h_{target} , where $k^{2p+1} h_{\text{target}}^{2p} = C$, and C is chosen so that the relative error is about 1-2% for large wavenumbers.

Discussion of the numerical results Figures 1a, 1b and 1c present the results for $p = 2, 3$ and 4, respectively, for the PML approximation to the sound-soft scattering problem.

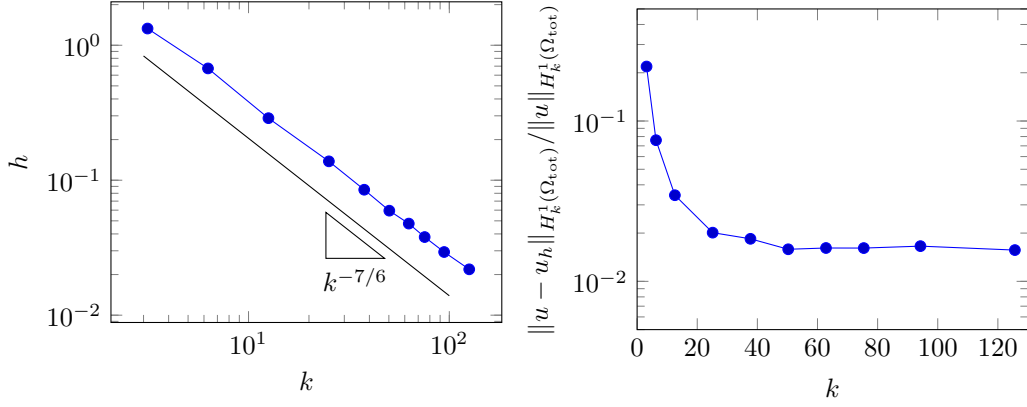
Figures 2a, 2b and 2c present the results for $p = 2, 3$ and 4, respectively, for the PML approximation to the penetrable-obstacle scattering problem.

As described above, the left-hand sides of the figures plot h as a function of k , illustrating that `gmsh` actually produces a mesh with the desired maximal meshwidth. The right-hand sides of the figures plot the relative Galerkin error in Ω_{tot} .

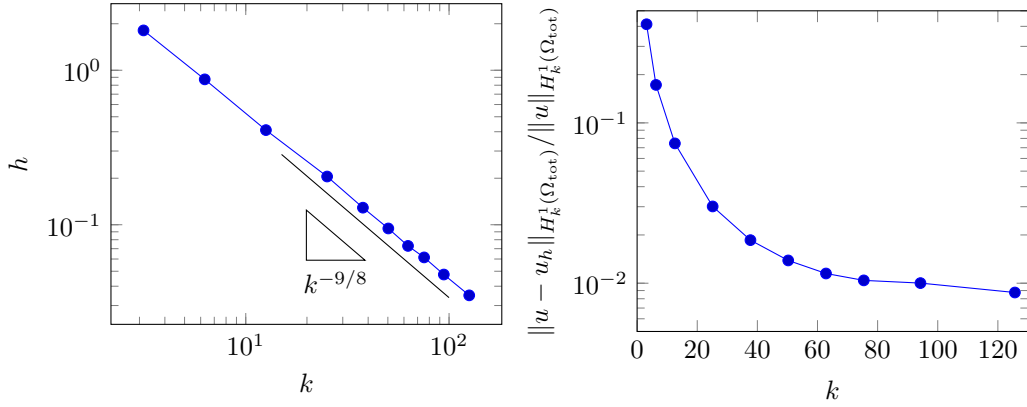
As anticipated from Theorem 1.1, the relative error remains uniformly bounded for all wavenumbers in all cases. We also see that, especially for higher polynomial degree, the error is larger for the first few wavenumbers and then stabilizes to a roughly constant value. This is because for the lowest wavenumbers, the geometric approximation error is not negligible, and in fact bigger than the pollution error: indeed, for k fixed, the pollution error on the right-hand side of (1.5) is $O(h^p)$ whereas the geometric error is $O(h)$. When the wavenumber increases, the pollution error (which remains constant thanks to our choice of h), becomes dominant, as predicted by Theorem 1.1.



(a) Numerical example: sound-soft scattering with $p = 2$

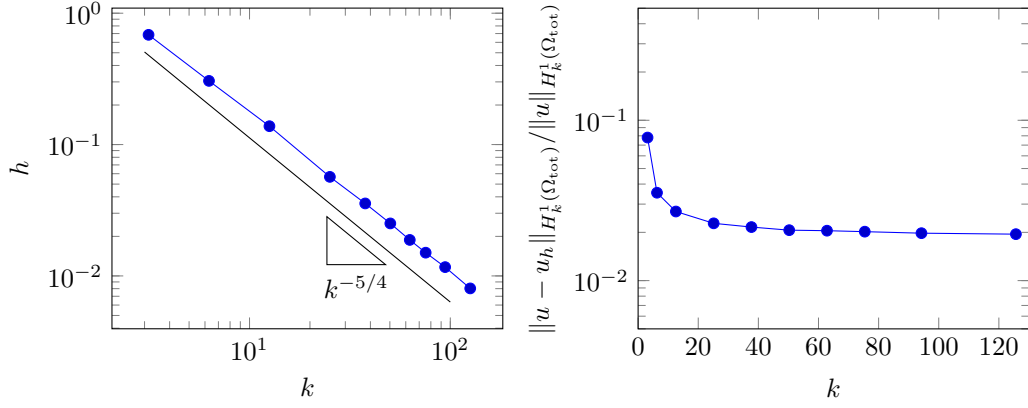


(b) Numerical example: sound-soft scattering with $p = 3$

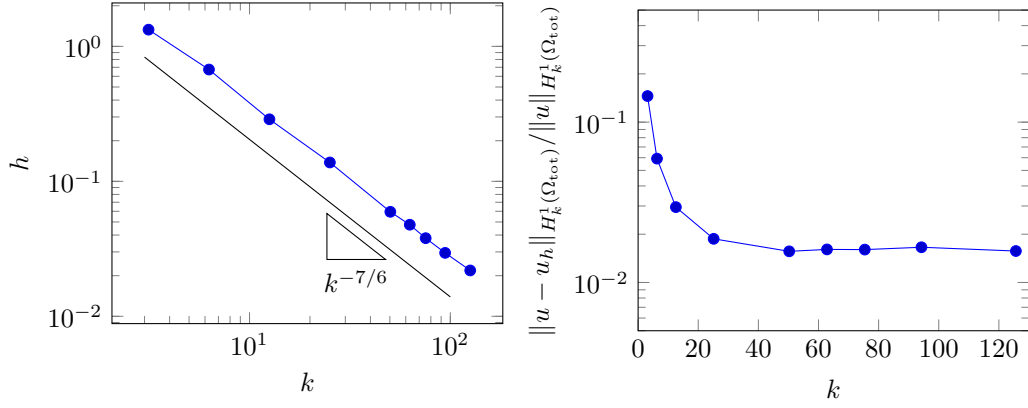


(c) Numerical example: sound-soft scattering with $p = 4$

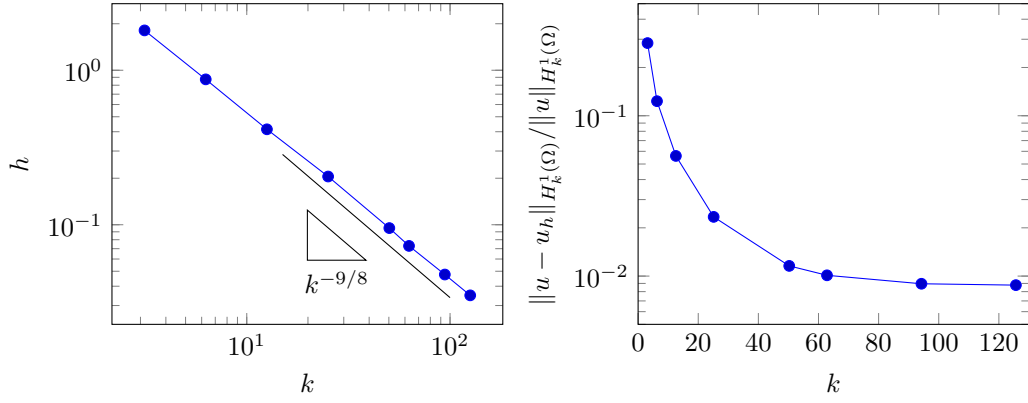
Figure 1: Numerical example: sound-soft scattering



(a) Numerical example: penetrable obstacle with $p = 2$



(b) Numerical example: penetrable obstacle with $p = 3$



(c) Numerical example: penetrable obstacle with $p = 4$

Figure 2: Numerical example: penetrable obstacle

3 Strang-type lemma for abstract Helmholtz problems

3.1 Assumptions on the sesquilinear form and finite-dimensional subspace

Continuity and Gårding-type inequality. Let $\mathcal{H} \subset H^1(\Omega)$ be a Hilbert space with the norm $\|\cdot\|_{H_k^1(\Omega)}$ (1.3). (In practice, \mathcal{H} will be $H^1(\Omega)$ with zero Dirichlet boundary conditions on part of $\partial\Omega$; see (4.1) below).

Let $b : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ be a sesquilinear form that is continuous, i.e.,

$$\sup_{\substack{v \in \mathcal{H} \\ \|v\|_{H_k^1(\Omega)}=1}} \sup_{\substack{w \in \mathcal{H} \\ \|w\|_{H_k^1(\Omega)}=1}} |b(v, w)| =: M < \infty \quad (3.1)$$

and satisfies the Gårding-type inequality

$$C_{\text{coer}} \|v\|_{H_k^1(\Omega)}^2 \leq \operatorname{Re} b(v, v) + 2k^2 \|\mathcal{K}v\|_{L^2(\Omega)}^2 \quad \text{for all } v \in \mathcal{H}, \quad (3.2)$$

for $C_{\text{coer}} > 0$ and for some self-adjoint operator $\mathcal{K} : L^2(\Omega) \rightarrow L^2(\Omega)$.

Remark 3.1 (The relationship between (3.2) and the standard Gårding inequality) *If b satisfies (3.2) then b also satisfies a “standard” Gårding inequality, since*

$$C_{\text{coer}} \|v\|_{H_k^1(\Omega)}^2 \leq \operatorname{Re} b(v, v) + 2k^2 \|\mathcal{K}\|_{L^2(\Omega) \rightarrow L^2(\Omega)}^2 \|v\|_{L^2(\Omega)}^2 \quad (3.3)$$

for all $v \in \mathcal{H}$. We show in Lemma 4.7 below (following [GS23]) that if b satisfies a standard Gårding inequality along with elliptic regularity, then b satisfies (3.2) with \mathcal{K} a smoothing operator; the smoothing property of \mathcal{K} is then key to obtaining the high-order convergence in the main result (Theorem 1.1).

The approximate sesquilinear form b_h and approximate right-hand side ψ_h . Given a finite-dimensional subspace $V_h \subset \mathcal{H}$, we consider a sesquilinear form $b_h : V_h \times V_h \rightarrow \mathbb{C}$ that is “close” to b , with this “closeness” characterised by the quantity

$$\gamma_{\text{q}} := \frac{1}{M} \sup_{\substack{v_h \in V_h \\ \|v_h\|_{H_k^1(\Omega)}=1}} \sup_{\substack{w_h \in V_h \\ \|w_h\|_{H_k^1(\Omega)}=1}} |b(v_h, w_h) - b_h(v_h, w_h)|. \quad (3.4)$$

Similarly, given $\psi \in (H_k^1(\Omega))^*$, let $\psi_h : V_h \rightarrow \mathbb{C}$ be “close” to ψ in the sense that

$$\tilde{\gamma}_{\text{q}} := \frac{1}{M \|\psi\|_{(H_k^1(\Omega))^*}} \sup_{\substack{v_h \in V_h \\ \|v_h\|_{H_k^1(\Omega)}=1}} |\langle \psi - \psi_h, v_h \rangle|. \quad (3.5)$$

Uniqueness of the solution of the variational problem. We assume that, given $\psi \in (H_k^1(\Omega))^*$, the solution of the variational problem: find $v \in \mathcal{H}$ such that

$$b(v, w) = \langle \psi, w \rangle \quad \text{for all } w \in \mathcal{H} \quad (3.6)$$

if it exists, is unique. The Gårding-type inequality (3.2) implies that b satisfies the standard Gårding inequality (3.3). Uniqueness of the solution of (3.14) is then equivalent to existence by, e.g., [McL00, Theorem 2.32], so that (3.6) has a unique solution $v \in \mathcal{H}$.

Notation for the adjoint solution operator and adjoint approximability. Define the adjoint solution operator $\mathcal{S}^* : L^2(\Omega) \rightarrow \mathcal{H}$ as the solution of the variational problem

$$b(v, \mathcal{S}^* \phi) = (v, \phi)_{L^2(\Omega)} \quad \text{for all } v \in \mathcal{H}, \phi \in L^2(\Omega) \quad (3.7)$$

(the fact that the solution to (3.6) is unique implies that \mathcal{S}^* is well-defined by, e.g., [McL00, Theorem 2.27]). We then define

$$\gamma_s^* := k \|\mathcal{S}^*\|_{L^2(\Omega) \rightarrow H_k^1(\Omega)};$$

the reason for including the factor of k is that γ_s^* is then dimensionless.

Denote the orthogonal projector in the $\|\cdot\|_{H_k^1(\Omega)}$ norm by

$$\pi_h w := \arg \min_{w_h \in V_h} \|w - w_h\|_{H_k^1(\Omega)} \quad \text{for all } w \in \mathcal{H}. \quad (3.8)$$

The following quantity measures how well solutions of the adjoint problem are approximated in V_h :

$$\gamma_a^* := k \|(I - \pi_h)\mathcal{S}^*\|_{L^2(\Omega) \rightarrow H_k^1(\Omega)}; \quad (3.9)$$

similar to with γ_s^* , the factor of k in (3.9) means that γ_a^* is dimensionless.

The modified sesquilinear form and its solution operator. Define

$$b^+(v, w) := b(v, w) + 2k^2(\mathcal{K}^2 v, w)_{L^2(\Omega)}, \quad (3.10)$$

so that b^+ is coercive on \mathcal{H} by (3.2) (with coercivity constant C_{coer}). Let

$$\sup_{\substack{v \in \mathcal{H} \\ \|v\|_{H_k^1(\Omega)}=1}} \sup_{\substack{w \in \mathcal{H} \\ \|w\|_{H_k^1(\Omega)}=1}} |b^+(v, w)| =: M^+ \leq M + 2\|\mathcal{K}^2\|_{L^2(\Omega) \rightarrow L^2(\Omega)}. \quad (3.11)$$

Let $\mathcal{S}^+ : L^2(\Omega) \rightarrow \mathcal{H}$ be defined as the solution of the variational problem

$$b^+(\mathcal{S}^+ \phi, w) = (\phi, w)_{L^2(\Omega)} \quad \text{for all } w \in \mathcal{H}, \phi \in L^2(\Omega). \quad (3.12)$$

It is convenient to define the following approximation factor

$$\gamma_{\mathcal{K}} := k \|(I - \pi_h)\mathcal{S}^+\mathcal{K}\|_{L^2(\Omega) \rightarrow H_k^1(\Omega)}; \quad (3.13)$$

i.e., $\gamma_{\mathcal{K}}$ measures how well solutions of the variational problem involving b^+ and with a \mathcal{K} on the right-hand side are approximated in V_h .

3.2 The main abstract result

The following result is an abstract version of the elliptic-projection argument from [GS23], extended to allow a variational crime.

Theorem 3.2 (Abstract elliptic-projection-type argument with variational crime) *Under the assumptions in §3.1, let $u \in \mathcal{H}$ be the unique solution of the variational problem*

$$b(u, w) = \langle \psi, w \rangle \quad \text{for all } w \in \mathcal{H}. \quad (3.14)$$

Let

$$A := 1 - 2 \frac{(M^+)^2}{C_{\text{coer}}} \|\mathcal{K}\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \gamma_a^* \gamma_{\mathcal{K}}$$

and

$$B := A(C_{\text{coer}} - M\gamma_q) - \frac{\sqrt{2}MM^+}{C_{\text{coer}}} \|\mathcal{K}\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \gamma_s^* \gamma_q.$$

If $B > 0$, then there exists a unique $u_h \in V_h$ such that

$$b_h(u_h, w_h) = \langle \psi_h, w_h \rangle \quad \text{for all } w_h \in V_h, \quad (3.15)$$

and the error $u - u_h$ satisfies the bound

$$\begin{aligned} B \|u - u_h\|_{H_k^1(\Omega)} &\leq \left[AM + \frac{\sqrt{2}MM^+}{C_{\text{coer}}} \gamma_a^* \|\mathcal{K}\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \right] \|(I - \pi_h)u\|_{H_k^1(\Omega)} \\ &\quad + \left[AM + \frac{\sqrt{2}MM^+}{C_{\text{coer}}} \gamma_s^* \|\mathcal{K}\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \right] \left(\gamma_q \|u\|_{H_k^1(\Omega)} + \tilde{\gamma}_q \|\psi\|_{(H_k^1(\Omega))^*} \right). \end{aligned} \quad (3.16)$$

We make two immediate remarks.

1. The bound (3.16) has a similar structure to the Strang lemma for coercive problems, namely the Galerkin error is bounded by the sum of (i) the best approximation error, (ii) the error in the sesquilinear form, and (iii) the error in the right-hand side; see [Str72], [Cia91, Theorem 26.1].
2. If $\gamma_q, \tilde{\gamma}_q = 0$ and the approximation factors $\gamma_a^*, \gamma_{\mathcal{K}} \rightarrow 0$, then $A \rightarrow 1$, $B \rightarrow C_{\text{coer}}$, and the bound (3.16) approaches

$$\|u - u_h\|_{H_k^1(\Omega)} \leq \frac{M}{C_{\text{coer}}} \|(I - \pi_h)u\|_{H_k^1(\Omega)};$$

observe that this is the bound given by C ea’s lemma for a coercive problem with continuity constant M and coercivity constant C_{coer} .

Interpretation and context of Theorem 3.2. Assuming that $\|\mathcal{K}\|_{L^2(\Omega) \rightarrow L^2(\Omega)}$, M, M^+ , and C_{coer} are independent of k (as we see below they are for Helmholtz problems), we see that Theorem 3.2 shows that if

$$\gamma_a^* \gamma_{\mathcal{K}} \quad \text{and} \quad \gamma_s^* \gamma_q \quad \text{are sufficiently small} \quad (3.17)$$

then

$$\|u - u_h\|_{H_k^1(\Omega)} \leq C(1 + \gamma_a^*) \|(I - \pi_h)u\|_{H_k^1(\Omega)} + C(1 + \gamma_s^*) \left(\gamma_q \|u\|_{H_k^1(\Omega)} + \tilde{\gamma}_q \|\psi\|_{(H_k^1(\Omega))^*} \right). \quad (3.18)$$

Continuity of b (3.1) implies that $\|\psi\|_{(H_k^1(\Omega))^*} \leq M \|u\|_{H_k^1(\Omega)}$, and thus (3.18) implies that

$$\|u - u_h\|_{H_k^1(\Omega)} \leq C(1 + \gamma_a^*) \|(I - \pi_h)u\|_{H_k^1(\Omega)} + C(1 + \gamma_s^*) (\gamma_q + \tilde{\gamma}_q) \|u\|_{H_k^1(\Omega)}. \quad (3.19)$$

Section 4 shows how for the particular Helmholtz problem considered in Section 1, (3.17) and (3.19) become (1.4) and (1.5), respectively; indeed, this is via the bounds $\gamma_a^* \leq C(kh + k^\alpha(kh)^p)$, $\gamma_{\mathcal{K}} \leq C(kh)^p$, and $\gamma_q, \tilde{\gamma}_q \leq Ch^q$ (for a domain Ω with characteristic length scale ~ 1).

We now discuss here the relation between (3.17), (3.19) and other results in the literature. Since these other results do not take into account errors in the sesquilinear form and data (via the quantities γ_q and $\tilde{\gamma}_q$), in this discussion we assume that $\gamma_q = \tilde{\gamma}_q = 0$.

Recall that the classic duality argument for sesquilinear forms satisfying a G arding inequality proves that, if γ_a^* is sufficiently small

$$\|u - u_h\|_{H_k^1(\Omega)} \leq C \|(I - \pi_h)u\|_{H_k^1(\Omega)};$$

this argument has its origins in the work of [Sch74], with then the notion of adjoint approximability encapsulated by γ_a^* introduced in [Sau06].

The advantage of (3.17), (3.19) over this classic argument is that the condition “ $\gamma_{\mathcal{K}} \gamma_a^*$ sufficiently small” can be less restrictive than “ γ_a^* sufficiently small”. Indeed, the “elliptic projection” argument introduced in [FW09, FW11] is essentially (3.17), (3.19) with the operator \mathcal{K} chosen to be a sufficiently-large multiple of the identity. In this case, we see below that

$$\gamma_{\mathcal{K}} := k \|(I - \pi_h) \mathcal{S}^+ \mathcal{K}\|_{L^2(\Omega) \rightarrow H_k^1(\Omega)} \leq Ck \|(I - \pi_h) \mathcal{S}^+\|_{L^2(\Omega) \rightarrow H_k^1(\Omega)} \leq C'kh,$$

and so the condition “ $\gamma_{\mathcal{K}} \gamma_a^*$ sufficiently small” is “ $kh \gamma_a^*$ is sufficiently small”, i.e., a weaker condition than “ γ_a^* is sufficiently small” (since $kh \ll 1$).

The generalisation of the elliptic-projection argument in [GS23] showed that, under natural elliptic-regularity assumptions, there exists a *smoothing* operator \mathcal{K} such that the G arding-type inequality (3.2) is satisfied. With this smoothing property, the condition “ $\gamma_a^* k \|(I - \pi_h) \mathcal{S}^+ \mathcal{K}\|_{L^2(\Omega) \rightarrow H_k^1(\Omega)}$ sufficiently small” is weaker than “ $\gamma_a^* k \|(I - \pi_h) \mathcal{S}^+\|_{L^2(\Omega) \rightarrow H_k^1(\Omega)}$ sufficiently small”; we recap the construction of \mathcal{K} in Lemma 4.7 below.

3.3 Proof of Theorem 3.2

Theorem 3.2 comes from combining Lemmas 3.3 and 3.4 below. These lemmas involve the following projection operator; let $\pi_h^+ : \mathcal{H} \rightarrow V_h$ be defined by

$$b^+(v_h, \pi_h^+ w) = b^+(v_h, w) \quad \text{for all } v_h \in V_h, w \in \mathcal{H};$$

i.e.,

$$b^+(v_h, (I - \pi_h^+)w) = 0 \quad \text{for all } v_h \in V_h, w \in \mathcal{H}. \quad (3.20)$$

We immediately note that, since b^+ is continuous and coercive, Céa's lemma implies that

$$\|(I - \pi_h^+)v\|_{H_k^1(\Omega)} \leq \frac{M^+}{C_{\text{coer}}} \|(I - \pi_h)v\|_{H_k^1(\Omega)} \quad \text{for all } v \in H_k^1(\Omega), \quad (3.21)$$

and

$$\|\pi_h^+ v\|_{H_k^1(\Omega)} \leq \frac{M^+}{C_{\text{coer}}} \|v\|_{H_k^1(\Omega)} \quad \text{for all } v \in \mathcal{H}, \quad (3.22)$$

since

$$C_{\text{coer}} \|v\|_{H_k^1(\Omega)}^2 \leq b^+(\pi_h^+ v, \pi_h^+ v) = b^+(\pi_h^+ v, v) \leq M^+ \|\pi_h^+ v\|_{H_k^1(\Omega)} \|v\|_{H_k^1(\Omega)}.$$

In both the statement and proof of the next lemma, we drop the (Ω) in norms, i.e., we write, e.g., $\|\mathcal{K}\|_{L^2 \rightarrow L^2}$ instead of $\|\mathcal{K}\|_{L^2(\Omega) \rightarrow L^2(\Omega)}$. to keep expressions compact.

Lemma 3.3 *Let*

$$\tilde{\gamma}_a^* := k^2 \|\mathcal{K}(I - \pi_h^+) \mathcal{S}^*\|_{L^2 \rightarrow L^2}. \quad (3.23)$$

If the solution u_h to (3.15) exists, then

$$\begin{aligned} & \left[\left(1 - 2\tilde{\gamma}_a^* \|\mathcal{K}\|_{L^2 \rightarrow L^2}\right) \left(C_{\text{coer}} - M\gamma_q\right) - \frac{\sqrt{2}MM^+}{C_{\text{coer}}} \|\mathcal{K}\|_{L^2 \rightarrow L^2} \gamma_s^* \gamma_q \right] \|u - u_h\|_{H_k^1(\Omega)} \\ & \leq \left[\left(1 - 2\tilde{\gamma}_a^* \|\mathcal{K}\|_{L^2 \rightarrow L^2}\right) M + \frac{\sqrt{2}(M^+)^2}{C_{\text{coer}}} \gamma_a^* \|\mathcal{K}\|_{L^2 \rightarrow L^2} \right] \|(I - \pi_h)u\|_{H_k^1(\Omega)} \\ & \quad + \left[\left(1 - 2\tilde{\gamma}_a^* \|\mathcal{K}\|_{L^2 \rightarrow L^2}\right) M + \frac{\sqrt{2}MM^+}{C_{\text{coer}}} \|\mathcal{K}\|_{L^2 \rightarrow L^2} \gamma_s^* \right] \left(\gamma_q \|u\|_{H_k^1(\Omega)} + \tilde{\gamma}_q \|\psi\|_{(H_k^1(\Omega))^*} \right). \end{aligned} \quad (3.24)$$

Note that (3.24) has the same structure as (3.16), except that the instance of γ_a^* in A has been replaced by a multiple of $\tilde{\gamma}_a^*$. The following lemma is then used to relate $\tilde{\gamma}_a^*$ to γ_a^* .

Lemma 3.4 *For all $v \in \mathcal{H}$,*

$$k \|\mathcal{K}(I - \pi_h^+)v\|_{L^2(\Omega)} \leq M^+ \gamma_{\mathcal{K}} \|(I - \pi_h^+)v\|_{H_k^1(\Omega)}.$$

Proof of Theorem 3.2 using Lemmas 3.3 and 3.4. Once the bound (3.16) is established under the assumption that u_h exists, applying this bound with $\psi = 0$ and using uniqueness of the continuous problem (3.14) implies that u_h , if it exists, is unique. Existence of u_h then follows since u_h is the solution of a finite-dimensional linear system, for which uniqueness is equivalent to existence.

It is therefore sufficient to prove (3.16) under the assumption that u_h exists. By Lemma 3.4 and the definitions of $\tilde{\gamma}_a^*$ (3.23), $\gamma_{\mathcal{K}}$ (3.13), and γ_a^* (3.9),

$$\tilde{\gamma}_a^* \leq \frac{(M^+)^2}{C_{\text{coer}}} \gamma_{\mathcal{K}} \gamma_a^*.$$

and then using this in (3.24) gives the result (3.16). \blacksquare

We now prove Lemma 3.4 using a duality argument involving the sesquilinear form b^+ .

Proof of Lemma 3.4. By the definition of \mathcal{S}^+ (3.12), the Galerkin orthogonality (3.20), and continuity of b^+ (3.11)

$$\begin{aligned} \|\mathcal{K}(I - \pi_h^+)v\|_{L^2(\Omega)}^2 &= (\mathcal{K}^2(I - \pi_h^+)v, (I - \pi_h^+)v)_{L^2(\Omega)} \\ &= b^+(\mathcal{S}^+\mathcal{K}^2(I - \pi_h^+)v, (I - \pi_h^+)v) \\ &= b^+((I - \pi_h)\mathcal{S}^+\mathcal{K}^2(I - \pi_h^+)v, (I - \pi_h^+)v) \\ &\leq M^+ \|(I - \pi_h)\mathcal{S}^+\mathcal{K}\|_{L^2(\Omega) \rightarrow H_k^1(\Omega)} \|\mathcal{K}(I - \pi_h^+)v\|_{L^2(\Omega)} \|(I - \pi_h^+)v\|_{H_k^1(\Omega)}, \end{aligned}$$

and the result follows. \blacksquare

To complete the proof of Theorem 3.2, we therefore need to prove Lemma 3.3. To do this, we start with a simple result controlling the variational crime error.

Lemma 3.5 (“Galerkin orthogonality” in the sesquilinear form $b(\cdot, \cdot)$) *If $u \in \mathcal{H}$ and $u_h \in V_h$ satisfy (3.14) and (3.15), then, for all $w_h \in V_h$,*

$$|b(u - u_h, w_h)| \leq M \left(\gamma_q \|u - u_h\|_{H_k^1(\Omega)} + \gamma_q \|u\|_{H_k^1(\Omega)} + \tilde{\gamma}_q \|\psi\|_{(H_k^1(\Omega))^*} \right) \|w_h\|_{H_k^1(\Omega)}. \quad (3.25)$$

Proof. By (3.14) and (3.15),

$$\begin{aligned} b(u - u_h, w_h) &= b(u, w_h) - b(u_h, w_h) = \langle \psi, w_h \rangle - b_h(u_h, w_h) + (b_h(u_h, w_h) - b(u_h, w_h)) \\ &= \langle \psi - \psi_h, w_h \rangle + (b_h(u_h, w_h) - b(u_h, w_h)). \end{aligned}$$

Then, by (3.5) and (3.4),

$$|b(u - u_h, w_h)| \leq M \left(\tilde{\gamma}_q \|\psi\|_{(H_k^1(\Omega))^*} + \gamma_q \|u_h\|_{H_k^1(\Omega)} \right) \|w_h\|_{H_k^1(\Omega)}.$$

The result (3.25) then follows from the triangle inequality. \blacksquare

Lemma 3.6 (Useful quadratic inequality) *Let $0 < a \leq 1$, $b, c \geq 0$ and $x \geq 0$. If*

$$ax^2 \leq bx + c^2, \quad (3.26)$$

then

$$ax \leq b + c. \quad (3.27)$$

Proof. Multiplying the inequality (3.26) by $a > 0$, we find that

$$\left(ax - \frac{b}{2} \right)^2 - \frac{b^2}{4} \leq ac^2, \quad \text{and thus} \quad ax \leq \frac{b}{2} + \sqrt{\frac{b^2}{4} + ac^2}.$$

The result (3.27) follows since

$$\sqrt{\frac{b^2}{4} + ac^2} \leq \sqrt{\frac{b^2}{4}} + \sqrt{ac^2} \leq \frac{b}{2} + c,$$

since $a \leq 1$ by assumption. \blacksquare

Proof of Lemma 3.3. Let $e := u - u_h$. By (3.2), (3.25), and (3.1),

$$\begin{aligned} C_{\text{coer}} \|e\|_{H_k^1}^2 &\leq \text{Re } b(e, e) + 2k^2 \|\mathcal{K}e\|_{L^2}^2 \\ &= \text{Re } b(e, (I - \pi_h)u) + 2k^2 \|\mathcal{K}e\|_{L^2}^2 - \text{Re } b(e, u_h - \pi_h u) \\ &\leq M \|e\|_{H_k^1} \|(I - \pi_h)u\|_{H_k^1} + 2k^2 \|\mathcal{K}e\|_{L^2}^2 \\ &\quad + M \left(\gamma_q \|e\|_{H_k^1} + \gamma_q \|u\|_{H_k^1} + \tilde{\gamma}_q \|\psi\|_{(H_k^1)^*} \right) \|e\|_{H_k^1}, \end{aligned}$$

where to obtain the last inequality we have used (i) that $u_h - \pi_h u = \pi_h e$ and (ii) $\|\pi_h e\|_{H_k^1} \leq \|e\|_{H_k^1}$ since π_h is a projection (3.8). Rearranging the last displayed equation, we obtain that

$$(C_{\text{coer}} - M\gamma_q) \|e\|_{H_k^1}^2 \leq M \|e\|_{H_k^1} \left(\|(I - \pi_h)u\|_{H_k^1} + \gamma_q \|u\|_{H_k^1} + \tilde{\gamma}_q \|\psi\|_{(H_k^1)^*} \right) + 2k^2 \|\mathcal{K}e\|_{L^2}^2.$$

Then, by Lemma 3.6,

$$(C_{\text{coer}} - M\gamma_q) \|e\|_{H_k^1} \leq M \left(\|(I - \pi_h)u\|_{H_k^1} + \gamma_q \|u\|_{H_k^1} + \tilde{\gamma}_q \|\psi\|_{(H_k^1)^*} \right) + \sqrt{2}k \|\mathcal{K}e\|_{L^2}. \quad (3.28)$$

We now use a duality argument to bound $\|\mathcal{K}e\|_{L^2}$. By the self-adjointness of \mathcal{K} , the definition of \mathcal{S}^* (3.7), (3.25), and (3.22),

$$\begin{aligned} \|\mathcal{K}e\|_{L^2}^2 &= b(e, \mathcal{S}^* \mathcal{K}^2 e) \\ &= b(e, (I - \pi_h^+) \mathcal{S}^* \mathcal{K}^2 e) + b(e, \pi_h^+ \mathcal{S}^* \mathcal{K}^2 e) \\ &\leq b(e, (I - \pi_h^+) \mathcal{S}^* \mathcal{K}^2 e) \\ &\quad + M \left(\gamma_q \|e\|_{H_k^1} + \gamma_q \|u\|_{H_k^1} + \tilde{\gamma}_q \|\psi\|_{(H_k^1)^*} \right) \frac{M^+}{C_{\text{coer}}} \|\mathcal{S}^*\|_{L^2 \rightarrow H_k^1} \|\mathcal{K}\|_{L^2 \rightarrow L^2} \|\mathcal{K}e\|_{L^2}. \end{aligned} \quad (3.29)$$

Now, by the definition of b^+ (3.10), (3.20), the self-adjointness of \mathcal{K} , continuity of b^+ (3.11), (3.21), and the definitions of γ_a^* (3.9) and $\tilde{\gamma}_a^*$ (3.23),

$$\begin{aligned} b(e, (I - \pi_h^+) \mathcal{S}^* \mathcal{K}^2 e) &= b^+((I - \pi_h)u, (I - \pi_h^+) \mathcal{S}^* \mathcal{K}^2 e) - 2k^2 (\mathcal{K}^2 e, (I - \pi_h^+) \mathcal{S}^* \mathcal{K}^2 e)_{L^2} \\ &= b^+((I - \pi_h)u, (I - \pi_h^+) \mathcal{S}^* \mathcal{K}^2 e) - 2k^2 (\mathcal{K}e, \mathcal{K}(I - \pi_h^+) \mathcal{S}^* \mathcal{K}^2 e)_{L^2} \\ &\leq M^+ \|(I - \pi_h)u\|_{H_k^1} \frac{M^+}{C_{\text{coer}}} k^{-1} \gamma_a^* \|\mathcal{K}\|_{L^2 \rightarrow L^2} \|\mathcal{K}e\|_{L^2} + 2\tilde{\gamma}_a^* \|\mathcal{K}\|_{L^2 \rightarrow L^2} \|\mathcal{K}e\|_{L^2}^2. \end{aligned} \quad (3.30)$$

Combining (3.29) and (3.30), multiplying by k , and using the definition of γ_s^* , we obtain that

$$\begin{aligned} \left(1 - 2\tilde{\gamma}_a^* \|\mathcal{K}\|_{L^2 \rightarrow L^2}\right) k \|\mathcal{K}e\|_{L^2} &\leq \frac{(M^+)^2}{C_{\text{coer}}} \gamma_a^* \|\mathcal{K}\|_{L^2 \rightarrow L^2} \|(I - \pi_h)u\|_{H_k^1} \\ &\quad + \frac{MM^+}{C_{\text{coer}}} \|\mathcal{K}\|_{L^2 \rightarrow L^2} \gamma_s^* \left(\gamma_q \|e\|_{H_k^1} + \gamma_q \|u\|_{H_k^1} + \tilde{\gamma}_q \|\psi\|_{(H_k^1)^*} \right). \end{aligned} \quad (3.31)$$

Multiplying (3.28) by $(1 - 2\tilde{\gamma}_a^* \|\mathcal{K}\|_{L^2 \rightarrow L^2})$ and inputting (3.31), we obtain

$$\begin{aligned} &\left(1 - 2\tilde{\gamma}_a^* \|\mathcal{K}\|_{L^2 \rightarrow L^2}\right) (C_{\text{coer}} - M\gamma_q) \|e\|_{H_k^1} \\ &\leq \left(1 - 2\tilde{\gamma}_a^* \|\mathcal{K}\|_{L^2 \rightarrow L^2}\right) M \left(\|(I - \pi_h)u\|_{H_k^1} + \gamma_q \|u\|_{H_k^1} + \tilde{\gamma}_q \|\psi\|_{(H_k^1)^*} \right) \\ &\quad + \frac{\sqrt{2}(M^+)^2}{C_{\text{coer}}} \gamma_a^* \|\mathcal{K}\|_{L^2 \rightarrow L^2} \|(I - \pi_h)u\|_{H_k^1} \\ &\quad + \frac{\sqrt{2}MM^+}{C_{\text{coer}}} \|\mathcal{K}\|_{L^2 \rightarrow L^2} \gamma_s^* \left(\gamma_q \|e\|_{H_k^1} + \gamma_q \|u\|_{H_k^1} + \tilde{\gamma}_q \|\psi\|_{(H_k^1)^*} \right), \end{aligned}$$

which then rearranges to the result (3.24). \blacksquare

4 Applying the abstract theory to Helmholtz problems solved with curved finite elements

4.1 The Helmholtz problem with piecewise-smooth coefficients

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with characteristic length scale L . The boundary $\partial\Omega$ of Ω is partitioned into two disjoint relatively open Lipschitz subsets Γ_D and Γ_N (where the subscripts ‘‘D’’ and ‘‘N’’ stand for ‘‘Dirichlet’’ and ‘‘Neumann’’). Let \mathcal{H} be the Hilbert space

$$H_{\Gamma_D}^1(\Omega) := \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}. \quad (4.1)$$

The domain Ω is partitioned into disjoint open Lipschitz subsets $Q \in \mathcal{Q}$, and we assume that the restrictions of the PDE coefficients to each Q are smooth. Specifically, for each $Q \in \mathcal{Q}$, we consider two functions $\mu_Q : \mathbb{R}^d \rightarrow \mathbb{C}$ and $A_Q : \mathbb{R}^d \rightarrow \mathbb{C}^{d \times d}$ and the sesquilinear form then reads

$$\begin{aligned} b(v, w) &:= \sum_{Q \in \mathcal{Q}} \left\{ -k^2 (\mu_Q v, w)_{L^2(Q)} + (A_Q \nabla v, \nabla w)_{L^2(Q)} \right\} \\ &= -k^2 (\mu v, w)_{L^2(\Omega)} + (A \nabla v, \nabla w)_{L^2(\Omega)} \quad \text{for all } v, w \in H_{\Gamma_D}^1(\Omega), \end{aligned} \quad (4.2)$$

where $\mu|_Q := \mu_Q$, $A|_Q := A_Q$. To define the sesquilinear form b , the values of μ_Q and A_Q are not needed outside Q . However, assuming that they are also defined in a neighborhood of Q is useful when defining the discrete bilinear form (involving geometric error). We note that in the case of piecewise constant coefficients or coefficients given by analytical formula (e.g., in a PML) such extensions are automatically defined.

Assumption 4.1 (Strong ellipticity of the Helmholtz operator) *There exists $c > 0$ such that*

$$\operatorname{Re} \sum_{i=1}^d \sum_{j=1}^d A_{ij}(x) \xi_j \bar{\xi}_i \geq c |\xi|^2 \quad \text{for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^d.$$

Assumption 4.1 implies that the Helmholtz operator is strongly elliptic in the sense of [McL00, Equation 4.7] by, e.g., [McL00, Page 122].

Assumption 4.2 (Regularity assumption on A, μ , and ∂Q) *For some $\ell \in \mathbb{Z}^+$, $A_Q, \mu_Q \in C^{\ell-1,1}$ and $\partial Q \in C^{\ell,1}$ for all $Q \in \mathcal{Q}$.*

Assumption 4.2 implies that the Helmholtz operator satisfies the standard elliptic-regularity shift (see, e.g., [McL00, Theorem 4.20]). Thanks to [CFN20] and [GS23] this shift property can be used to bound the quantities γ_a^* and $\gamma_{\mathcal{K}}$, respectively, appearing in Theorem 3.2; see §4.6 below.

Similarly, we assume that

$$\langle \psi, v \rangle = (g, v)_{L^2(\Omega)} \quad (4.3)$$

with $g = g_Q$ on Q and $g_Q : \mathbb{R}^d \rightarrow \mathbb{C}$ with $g_Q \in H^1(\mathbb{R}^d)$.

Given the partition \mathcal{Q} of Ω , we use the notation that $v \in H^j(\mathcal{Q})$ iff $v|_Q \in H^j(Q)$ for all $Q \in \mathcal{Q}$, and we let

$$\|v\|_{H^m(\mathcal{Q})}^2 := \sum_{Q \in \mathcal{Q}} \|v\|_{H^m(Q)}^2 \quad (4.4)$$

where

$$\|u\|_{H^m(D)}^2 := \sum_{|\alpha| \leq m} L^{2(m-|\alpha|)} \|\partial^\alpha u\|_{L^2(D)}^2 \quad \text{for } m \in \mathbb{Z}^+,$$

where L is the characteristic length scale of Ω ; these factors of L are included to make each term in the norm have the same dimension.

4.2 Curved finite-element setting

Here, we introduce a finite element space on curved simplicial elements. For such spaces, there are the following two key considerations.

- (a) On the one hand, the mesh elements should be sufficiently curved so as to correctly approximate the geometry of the boundary value problem (i.e. Ω , Γ_D , Γ_N and \mathcal{Q}).
- (b) On the other hand, the elements should not be too distorted so as to preserve the approximation properties of the finite element space.

These two aspects have been thoroughly investigated in the finite-element literature [Cia02, CR72b, Len86], and this section summarises the key assumptions needed in our analysis.

We consider a non-conforming mesh \mathcal{T}_h that does not exactly cover Ω nor the partition \mathcal{Q} . The (closed) elements $K \in \mathcal{T}_h$ are obtained by mapping a single (closed) reference simplex \hat{K} through

bilipschitz maps $\mathcal{F}_K : \widehat{K} \rightarrow K$. The maps \mathcal{F}_K could be affine (leading to straight elements) but we more generally allow polynomial mappings of degree $q \geq 1$ leading to curved elements. We classically assume that the mesh is “matching”, meaning that $\overset{\circ}{K} \cap \overset{\circ}{K}' = \emptyset$ for all $K, K' \in \mathcal{T}_h$, and $K \cap K'$ is either empty, or corresponds to a single vertex, a single edge or a single face of the reference simplex \widehat{K} mapped through \mathcal{F}_K (or equivalently, $\mathcal{F}_{K'}$).

Let

$$\Omega^h := \text{Int} \left(\bigcup_{K \in \mathcal{T}_h} K \right);$$

i.e., Ω^h is the domain covered by \mathcal{T}_h . In general, $\Omega^h \neq \Omega$, but we expect that \mathcal{T}_h is “close” to matching the boundary and interfaces in the PDE problem. To encode this, we assume that for each $K \in \mathcal{T}_h$, there exists a bilipschitz mapping $\Psi_K^h : K \rightarrow \mathbb{R}^d$, such that if $\tilde{K} := \Psi_K^h(K)$, then

- (i) there exists $Q \in \mathcal{Q}$ such that $\tilde{K} \subset \overline{Q}$,
- (ii) $\cup_{K \in \mathcal{T}_h} \tilde{K} = \overline{\Omega}$,

(iii) letting $\Psi^h|_K := \Psi_K^h$ for all $K \in \mathcal{T}_h$ defines a bilipschitz mapping $\Psi^h : \overline{\Omega^h} \rightarrow \overline{\Omega}$.

Let $\Phi_K^h := (\Psi_K^h)^{-1}$, and $\Phi^h := (\Psi^h)^{-1}$. Note then that $\Phi^h|_{\tilde{K}} = \Phi_K^h$ for all $K \in \mathcal{T}_h$. We let

$$\Gamma_D^h := \Phi^h(\Gamma_D), \quad \Gamma_N^h := \Phi^h(\Gamma_N), \quad Q^h := \Phi^h(Q) \text{ for } Q \in \mathcal{Q};$$

and $\mathcal{Q}^h := \{Q^h\}_{Q \in \mathcal{Q}}$. Notice that if $\Psi^h = I$, then $\Omega^h = \Omega$, $\Gamma_D^h = \Gamma_D$, $\Gamma_N^h = \Gamma_N$ and $Q^h = Q$ for all $Q \in \mathcal{T}_h$, so that the mesh is actually conforming. We therefore expect each map Ψ_K^h to be close to the identity in a suitable sense (made precise below). Without loss of generality we assume that $\det \Phi^h, \det \Psi^h > 0$.

Finally, we assume that both Γ_D^h and Γ_N^h are unions of full faces of elements $K \in \mathcal{T}_h$. In other words, every mesh face either sits in Γ_D^h or in Γ_N^h .

Fixing a polynomial degree $p \geq 1$, we associate with \mathcal{T}_h the following finite element space:

$$V_h := \left\{ v_h \in H_{\Gamma_D^h}^1(\Omega^h) \mid v_h \circ \mathcal{F}_K^{-1} \in \mathcal{P}_p(\widehat{K}) \right\}.$$

Note that elements of V_h need not be piecewise polynomials, since the map \mathcal{F}_K is not polynomial in general. Since piecewise polynomial spaces enjoy excellent approximation properties, we therefore want \mathcal{F}_K to be close to being affine, meaning that the mesh elements are not too distorted.

We can make the requirements needed for the maps \mathcal{F}_K and Ψ_K^h precise as follows.

Assumption 4.3 (Curved finite element space) *The elements $K \in \mathcal{T}_h$ are not too distorted, in the sense that the maps \mathcal{F}_K satisfy*

$$\|\nabla^s \mathcal{F}_K\|_{L^\infty(\widehat{K})} \leq CL \left(\frac{h_K}{L} \right)^s, \quad \text{and} \quad \|\nabla^s (\mathcal{F}_K^{-1})\|_{L^\infty(K)} \leq Ch_K^s, \quad (4.5)$$

for $1 \leq s \leq p+1$, where h_K is the diameter of K .

The elements $K \in \mathcal{T}_h$ approximate the geometry well, in the sense that

$$\|\nabla^s (\Psi_K^h - I)\|_{L^\infty(K)} \leq C \left(\frac{h_K}{L} \right)^q h_K^{1-s}, \quad \|\det(\nabla \Psi_K^h) - 1\|_{L^\infty(K)} \leq C \left(\frac{h_K}{L} \right)^q, \quad (4.6)$$

and

$$\|\nabla^s (\Phi_K^h - I)\|_{L^\infty(\tilde{K})} \leq C \left(\frac{h_K}{L} \right)^q h_K^{1-s}, \quad \|\det(\nabla \Phi_K^h) - 1\|_{L^\infty(\tilde{K})} \leq C \left(\frac{h_K}{L} \right)^q, \quad (4.7)$$

for $0 \leq s \leq q+1$.

The requirements in Assumption 4.3 are easily met in practice. Specifically, given a straight simplicial partition \mathcal{T}_h^\dagger that nodally conforms to \mathcal{Q} and Ω , suitable maps \mathcal{F}_K are constructed in [Len86, §3.2–3.3]. Note that constructing \mathcal{T}_h^\dagger is in turn easily done with standard meshing software like `gmsh` [GR09].

Under Assumption 4.3, there exists an interpolation operator $\mathcal{I}_h : H^2(\mathcal{Q}^h) \cap H_{\Gamma_D}^1(\Omega^h) \rightarrow V_h$ such that

$$h^{-1} \|(I - \mathcal{I}_h)v\|_{L^2(\Omega^h)} + |(I - \mathcal{I}_h)v|_{H^1(\Omega^h)} \leq Ch^p \|v\|_{H^{p+1}(\mathcal{Q}^h)} \quad (4.8)$$

for all $v \in H^{p+1}(\mathcal{Q}^h) \cap H_{\Gamma_D}^1(\Omega^h)$; see [CR72b, Len86]. (Note that the piecewise Sobolev spaces on \mathcal{Q}^h are defined exactly as the ones on \mathcal{Q} , with semi-norm given by the analogue of (4.4).)

We finally introduce a discrete sesquilinear form:

$$b_h(v_h, w_h) := \sum_{Q \in \mathcal{Q}} \left\{ -k^2 (\mu_Q v_h, w_h)_{L^2(Q^h)} + (A_Q \nabla v_h, \nabla w_h)_{L^2(Q^h)} \right\} \quad \text{for all } v_h, w_h \in V_h,$$

and a discrete right-hand side

$$\langle \psi_h, w_h \rangle = \sum_{Q \in \mathcal{Q}} (g_Q, w_h)_{L^2(Q^h)}. \quad (4.9)$$

(Note that since each \mathcal{Q}^h is exactly covered by \mathcal{T}_h , the coefficients of the matrix associated with b_h and the vector associated with ψ_h can be computed efficiently.) The discrete problem then consists of finding $u_h \in V_h$ such that

$$b_h(u_h, w_h) = \langle \psi_h, w_h \rangle \quad \text{for all } w_h \in V_h. \quad (4.10)$$

Remark 4.4 (Quadrature error) *In practice, if μ_Q and A_Q are not piecewise constant, the integrals in the definition of b_h are further approximated by a quadrature rule. Following [Len86, §6] this can be done by further considering approximations μ_Q^h and A_Q^h for which the quadrature can then be performed exactly; see also [Cia02, Section 4.4] for an alternative approach. For simplicity though, we focus here on the geometric error and assume that the matrix coefficient associated with b_h are computed exactly.*

4.3 A mapped finite element space

The setting introduced above for curved finite elements does not immediately fit our abstract framework. Indeed, b_h contains exact integrals on a wrong domain Ω^h , rather than inexact integrals on the right domain Ω . Following [Len86], this is easily remedied by introducing an abstract mapped finite element space. Notice that this space is never used in actual computations; rather, it is a convenient tool for the analysis.

For $v_h \in V_h$, let $\tilde{v}_h := v_h \circ \Psi_h^{-1}$; recall from Section 4.2 that $\Psi^h : \overline{\Omega^h} \rightarrow \overline{\Omega}$, so that $\tilde{v}_h : \Omega \rightarrow \mathbb{C}$ and $\tilde{v}_h \in H_{\Gamma_D}^1(\Omega)$. The mapped finite element space $\tilde{V}_h := \{\tilde{v}_h; v_h \in V_h\}$ is therefore conforming, i.e., $\tilde{V}_h \subset H_{\Gamma_D}^1(\Omega)$. Furthermore, instead of considering the solution u_h of the variational problem (4.10), we can (abstractly) solve a variational problem set in Ω with the mapped finite element space \tilde{V}_h . Specifically, we can equivalently consider the problem of finding $\tilde{u}_h \in \tilde{V}_h$ such that

$$\tilde{b}_h(\tilde{u}_h, \tilde{w}_h) = \langle \tilde{\psi}_h, \tilde{w}_h \rangle \quad \text{for all } \tilde{w}_h \in \tilde{V}_h, \quad (4.11)$$

where

$$\begin{aligned} \tilde{b}_h(\tilde{u}_h, \tilde{w}_h) &:= \sum_{Q \in \mathcal{Q}} \left\{ -k^2 (\mu_Q^h \tilde{u}_h, \tilde{w}_h)_{L^2(Q)} + (A_Q^h \nabla \tilde{u}_h, \nabla \tilde{w}_h)_{L^2(Q)} \right\} \\ &= -k^2 (\mu_h \tilde{u}_h, \tilde{w}_h)_{L^2(\Omega)} + (A^h \nabla \tilde{u}_h, \nabla \tilde{w}_h)_{L^2(\Omega)} \end{aligned} \quad (4.12)$$

with

$$\mu_Q^h := |\det \nabla \Phi^h| (\mu_Q \circ \Phi^h), \quad A_Q^h := |\det \nabla \Phi^h| \nabla \Psi^h (A_Q \circ \Phi^h) (\nabla \Psi^h)^T \quad (4.13)$$

and $\mu_h|_Q := \mu_Q^h$ and $A^h|_Q := A_Q^h$ for all $Q \in \mathcal{Q}$. Similarly,

$$\langle \tilde{\psi}_h, \tilde{w}_h \rangle := \sum_{Q \in \mathcal{Q}} (g_Q^h, \tilde{w}_h) \quad \text{with} \quad g_Q^h := |\det \nabla \Phi^h| (g_Q \circ \Phi^h). \quad (4.14)$$

These definitions are made so that, by the chain rule,

$$b_h(v_h, w_h) = \tilde{b}_h(\tilde{v}_h, \tilde{w}_h) \quad \text{for all } v_h, w_h \in V_h,$$

and thus $u_h = \tilde{u}_h \circ \Psi^h$ by (4.10) and (4.11).

It is clear from Assumption 4.3 and (4.8) that if we set $\tilde{\mathcal{I}}_h = \Psi^h \circ \mathcal{I}_h \circ (\Psi^h)^{-1}$, we obtain a suitable interpolation operator onto \tilde{V}_h . Specifically, [Len86, Lemma 7] shows that¹

$$h^{-1} \|v - \tilde{\mathcal{I}}_h v\|_{L^2(\Omega)} + \|v - \tilde{\mathcal{I}}_h v\|_{H^1(\Omega)} \leq Ch^p \|v\|_{H^{p+1}(\mathcal{Q})} \quad \text{for all } v \in H^{p+1}(\mathcal{Q}) \cap H_{\Gamma_D}^1(\Omega). \quad (4.15)$$

4.4 Bounds on γ_q and $\tilde{\gamma}_q$

With our framework established, we are ready to provide bounds on γ_q and $\tilde{\gamma}_q$. The proof for γ_q closely follows [Len86, Lemma 8] whereas the auxiliary result established in Appendix B is used for $\tilde{\gamma}_q$.

Theorem 4.5 (Bounds on γ_q and $\tilde{\gamma}_q$) *If V_h satisfies Assumption 4.3, then*

$$\gamma_q \leq C \left(\frac{h}{L} \right)^q W(\mu, A), \quad \tilde{\gamma}_q \|\psi\|_{(H_k^1)^*} \leq \frac{C}{M} \left(\frac{h}{L} \right)^q \frac{1}{k} \left(\sum_{Q \in \mathcal{Q}} \|g_Q\|_{L^2(\mathbb{R}^d)}^2 + h^2 \|\nabla g_Q\|_{L^2(\mathbb{R}^d)}^2 \right)^{1/2} \quad (4.16)$$

with

$$W(\mu, A) := \frac{1}{M} \max_{Q \in \mathcal{Q}} \left(\|\mu_Q\|_{L^\infty(\mathbb{R}^d)} + h \|\nabla \mu_Q\|_{L^\infty(\mathbb{R}^d)} + \|A_Q\|_{L^\infty(\mathbb{R}^d)} + h \|\nabla A_Q\|_{L^\infty(\mathbb{R}^d)} \right).$$

Proof. We first prove the bound on γ_q , and observe that we only need to show that

$$\|\mu - \mu^h\|_{L^\infty(\Omega)} + \|A - A^h\|_{L^\infty(\Omega)} \leq C \left(\frac{h}{L} \right)^q W(\mu, A), \quad (4.17)$$

since then, by the definition of \tilde{b}_h in (4.12),

$$|b(\tilde{v}_h, \tilde{w}_h) - \tilde{b}_h(\tilde{v}_h, \tilde{w}_h)| \leq C \left(\frac{h}{L} \right)^q W(\mu, A) \|\tilde{v}_h\|_{H_k^1(\Omega)} \|\tilde{w}_h\|_{H_k^1(\Omega)} \quad (4.18)$$

for all $\tilde{v}_h, \tilde{w}_h \in \tilde{V}_h$. The bound on γ_q then follows immediately from its definition (3.4) and (4.18).

We now show how (4.17) follows from Assumption 4.3. For the coefficient A , if $Q \in \mathcal{Q}$ and $K \in \mathcal{T}_h$ with $\tilde{K} \subset \bar{Q}$, by (4.13), we have

$$\begin{aligned} A_Q - A_Q^h &= A_Q - (\det \nabla \Phi^h) \nabla \Psi^h (A_Q \circ \Phi^h) (\nabla \Psi^h)^T, \\ &= (1 - \det \nabla \Phi^h) A_Q - \det \nabla \Phi^h \{ A_Q - \nabla \Psi^h (A_Q \circ \Phi^h) (\nabla \Psi^h)^T \} \end{aligned}$$

so that

$$\begin{aligned} \|A_Q - A_Q^h\|_{L^\infty(\tilde{K})} &\leq \|1 - \det \nabla \Phi^h\|_{L^\infty(\tilde{K})} \|A_Q\|_{L^\infty(\mathbb{R}^d)} \\ &\quad + \|\det \nabla \Phi^h\|_{L^\infty(\tilde{K})} \|A_Q - \nabla \Psi^h (A_Q \circ \Phi^h) (\nabla \Psi^h)^T\|_{L^\infty(\tilde{K})} \\ &\leq C \left(\left(\frac{h}{L} \right)^q \|A_Q\|_{L^\infty(\tilde{K})} + \|A_Q - \nabla \Psi^h (A_Q \circ \Phi^h) (\nabla \Psi^h)^T\|_{L^\infty(\tilde{K})} \right) \quad (4.19) \end{aligned}$$

¹Strictly speaking, the result is only establish for the H^1 semi-norm, but the proof for the L^2 norm is analogous.

by (4.7). We then work on the second term on the right-hand side of (4.19). Specifically,

$$\begin{aligned} A_Q - \nabla \Psi^h(A_Q \circ \Phi^h)(\nabla \Psi^h)^T &= (\mathbf{I}_d - \nabla \Psi^h)A_Q + \nabla \Psi^h[A_Q - (A_Q \circ \Phi^h)(\nabla \Psi^h)^T] \\ &= (\mathbf{I}_d - \nabla \Psi^h)A_Q + \nabla \Psi^h[A_Q(\mathbf{I}_d - (\nabla \Psi^h)^T)] \\ &\quad + \nabla \Psi^h(A_Q - A_Q \circ \Phi^h)(\nabla \Psi^h)^T, \end{aligned}$$

and therefore

$$\begin{aligned} \|A_Q - \nabla \Psi^h(A_Q \circ \Phi^h)(\nabla \Psi^h)^T\|_{L^\infty(\tilde{K})} &\leq \|\mathbf{I}_d - \nabla \Psi^h\|_{L^\infty(\tilde{K})} \|A_Q\|_{L^\infty(\mathbb{R}^d)} \\ &\quad + C \|\nabla \Psi^h\|_{L^\infty(\tilde{K})} \|A_Q\|_{L^\infty(\mathbb{R}^d)} \|\mathbf{I}_d - (\nabla \Psi^h)^T\|_{L^\infty(\tilde{K})} \\ &\quad + C \|\nabla \Psi^h\|_{L^\infty(\tilde{K})}^2 \|A_Q - A_Q \circ \Phi^h\|_{L^\infty(\tilde{K})} \\ &\leq C \left(\left(\frac{h}{L} \right)^q \|A_Q\|_{L^\infty(\mathbb{R}^d)} + \|A_Q - A_Q \circ \Phi^h\|_{L^\infty(\tilde{K})} \right) \end{aligned} \quad (4.20)$$

by (4.6). We estimate the remaining term $\|A_Q - A_Q \circ \Phi^h\|_{L^\infty(\tilde{K})}$ by using Taylor's theorem and (4.7) to find that

$$\|A_Q - A_Q \circ \Phi^h\|_{L^\infty(\tilde{K})} \leq C \|I - \Phi^h\|_{L^\infty(\tilde{K})} \|\nabla A_Q\|_{L^\infty(\mathbb{R}^d)} \leq C \left(\frac{h}{L} \right)^q h \|\nabla A_Q\|_{L^\infty(\mathbb{R}^d)}. \quad (4.21)$$

Combining (4.19), (4.20), (4.21), and observing that these bounds are valid for all $K \in \mathcal{T}_h$, we obtain that

$$\|A - A^h\|_{L^\infty(\Omega)} \leq C \left(\frac{h}{L} \right)^q \max_{Q \in \mathcal{Q}} \left(\|A_Q\|_{L^\infty(\mathbb{R}^d)} + h \|\nabla A_Q\|_{L^\infty(\mathbb{R}^d)} \right) \quad (4.22)$$

The corresponding result for μ , namely,

$$\|\mu - \mu^h\|_{L^\infty(\Omega)} \leq C \left(\frac{h}{L} \right)^q \max_{Q \in \mathcal{Q}} \left(\|\mu_Q\|_{L^\infty(\mathbb{R}^d)} + h \|\nabla \mu_Q\|_{L^\infty(\mathbb{R}^d)} \right) \quad (4.23)$$

is established similarly, with details in [Len86, Lemma 8]. The bound on γ_q then follows from (4.22) and (4.23).

To prove the bound on $\tilde{\gamma}_q$, we first observe that, by (4.14),

$$\begin{aligned} \langle \tilde{\psi}_h, \tilde{v}_h \rangle &= \sum_{Q \in \mathcal{Q}} \int_Q \det(\nabla \Phi^h)(g_Q \circ \Phi^h) \tilde{v}_h \\ &= \sum_{Q \in \mathcal{Q}} \int_Q (\det(\nabla \Phi^h) - 1)(g_Q \circ \Phi^h) \tilde{v}_h + \sum_{Q \in \mathcal{Q}} \int_Q (g_Q \circ (\Phi^h - I)) \tilde{v}_h + \sum_{Q \in \mathcal{Q}} \int_Q g_Q \tilde{v}_h, \end{aligned}$$

so that, by (4.9),

$$\langle \tilde{\psi}_h - \psi_h, \tilde{v}_h \rangle = \sum_{Q \in \mathcal{Q}} \int_Q (\det(\nabla \Phi^h) - 1) g_Q \circ \Phi^h \tilde{v}_h + \sum_{Q \in \mathcal{Q}} \int_Q (g_Q \circ (\Phi^h - I)) \tilde{v}_h. \quad (4.24)$$

For a fixed $Q \in \mathcal{Q}$,

$$\begin{aligned} \left| \int_Q (\det \nabla \Phi^h - 1) g_Q \circ \Phi^h \tilde{v}_h \right| &= \left| \sum_{\substack{K \in \mathcal{T}_h \\ \tilde{K} \subset Q}} \int_{\tilde{K}} (\det \nabla \Phi_K^h - 1) (g_Q \circ \Phi_K^h) \tilde{v}_h \right| \\ &\leq \sum_{\substack{K \in \mathcal{T}_h \\ \tilde{K} \subset Q}} \|\det(\nabla \Phi_K^h) - 1\|_{L^\infty(\tilde{K})} \|g_Q \circ \Phi_K^h\|_{L^2(\tilde{K})} \|\tilde{v}_h\|_{L^2(\tilde{K})} \\ &\leq C \left(\frac{h}{L} \right)^q \|g_Q\|_{L^2(\mathbb{R}^d)} \|\tilde{v}_h\|_{L^2(Q)} \end{aligned} \quad (4.25)$$

by the second bound in (4.7). Similarly, by Lemma B.1 and the first bound in (4.7) with $s = 0$ and $s = 1$,

$$\left| \int_Q (g_Q \circ (\Phi^h - I)) \tilde{v}_h \right| \leq C \left(\frac{h}{L} \right)^q h \|\nabla g_Q\|_{L^2(\mathbb{R}^d)} \|\tilde{v}_h\|_{L^2(Q)}. \quad (4.26)$$

Using (4.25) and (4.26) in (4.24), we obtain

$$\begin{aligned} |\langle \tilde{\psi}_h - \psi_h, \tilde{v}_h \rangle| &\leq C \left(\frac{h}{L} \right)^q \left(\sum_{Q \in \mathcal{Q}} \|g_Q\|_{L^2(\mathbb{R}^d)}^2 + h^2 \|\nabla g_Q\|_{L^2(\mathbb{R}^d)}^2 \right)^{1/2} \|\tilde{v}_h\|_{L^2(\Omega)} \\ &\leq \frac{C}{k} \left(\frac{h}{L} \right)^q \left(\sum_{Q \in \mathcal{Q}} \|g_Q\|_{L^2(\mathbb{R}^d)}^2 + h^2 \|\nabla g_Q\|_{L^2(\mathbb{R}^d)}^2 \right)^{1/2} \|\tilde{v}_h\|_{H_k^1(\Omega)}. \end{aligned}$$

The second bound in (4.16) then follows from the definition of $\tilde{\gamma}_q$ (3.5). \blacksquare

4.5 Bound on γ_a^*

Theorem 4.6 (Bound on γ_a^*) *Suppose that A satisfies Assumption 4.1, that A , μ , and $\partial\Omega$ satisfy the regularity requirements in Assumption 4.2 for some $\ell \in \mathbb{Z}^+$, and that V_h satisfies Assumption 4.3 for some $p \leq \ell$. Suppose that $kh \leq C_0$ for some $C_0 > 0$. Then given $k_0 > 0$ there exists $C > 0$ such that, for $k \geq k_0$, with γ_a^* defined by (3.9) with the finite-element space equal to \tilde{V}_h ,*

$$\gamma_a^* \leq C(kh + \gamma_s^*(kh)^p).$$

Proof. The bound on γ_a^* follows from the general result of [CFN20, Lemma 2.13] (see also [GS23, Theorem 1.7]). Indeed, Assumptions 4.1 and 4.2 imply that the elliptic regularity assumptions of [CFN20, Definition 2.1] are satisfied, and (4.15) provides the polynomial approximation result of [CFN20, Equation 2.30]. \blacksquare

4.6 Bound on $\gamma_{\mathcal{K}}$

The next result uses the following k -weighted norms. For a Lipschitz domain D , let

$$\|u\|_{H_k^m(D)}^2 := \sum_{|\alpha| \leq m} k^{2(1-|\alpha|)} \|\partial^\alpha u\|_{L^2(D)}^2 \quad \text{for } m \in \mathbb{Z}^+ \quad (4.27)$$

(observe that this definition is equivalent to (1.3) when $m = 1$) and let

$$\|u\|_{H_k^m(\mathcal{Q})}^2 := \sum_{Q \in \mathcal{Q}} \|u\|_{H_k^m(Q)}^2 \quad \text{for } m \in \mathbb{Z}^+. \quad (4.28)$$

Lemma 4.7 (Existence of a smoothing \mathcal{K} satisfying the Gårding-type inequality (3.2)) *Suppose that A satisfies Assumption 4.1 and A , μ , and $\partial\Omega$ satisfy the regularity requirements in Assumption 4.2, for some $\ell \in \mathbb{Z}^+$. Let b be the sesquilinear form (4.2).*

Then there exists a self-adjoint $\mathcal{K} : L^2(\Omega) \rightarrow L^2(\Omega)$ such that given $k_0 > 0$ there exists $C > 0$ such that the Gårding-type inequality (3.2) is satisfied and $\mathcal{K} : L^2(\Omega) \rightarrow H^n(\mathcal{Q}) \cap H_{\Gamma_D}^1(\Omega)$ with

$$\|\mathcal{K}\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \leq C \quad \text{and} \quad \|\mathcal{K}\|_{L^2(\Omega) \rightarrow H_k^n(\mathcal{Q})} \leq Ck \quad \text{for all } k \geq k_0, \quad 1 \leq n \leq \ell + 1. \quad (4.29)$$

Furthermore $S^+ : H^{\ell-1}(\mathcal{Q}) \cap H_{\Gamma_D}^1(\Omega) \rightarrow H^{\ell+1}(\mathcal{Q})$ with

$$\|S^+\|_{H_k^{\ell-1}(\mathcal{Q}) \rightarrow H_k^{\ell+1}(\mathcal{Q})} \leq Ck^{-2}. \quad (4.30)$$

We now proceed with the operator \mathcal{K} introduced in Lemma 4.7.

Theorem 4.8 (Bound on $\gamma_{\mathcal{K}}$) Suppose that A satisfies Assumption 4.1, that A , μ , and $\partial\Omega$ satisfy the regularity requirements in Assumption 4.2 for some $\ell \in \mathbb{Z}^+$, and that V_h satisfies Assumption 4.3 for some $p \leq \ell$. Suppose that $kh \leq C_0$ for some $C_0 > 0$. Then given $k_0 > 0$ there exists $C > 0$ such that, for $k \geq k_0$, with $\gamma_{\mathcal{K}}$ defined by (3.13) with the finite-element space equal to \tilde{V}_h and the operator \mathcal{K} given by Lemma 4.7

$$\gamma_{\mathcal{K}} \leq C(kh)^p. \quad (4.31)$$

Proof of Theorem 4.8 using Lemma 4.7. By (4.15) and the definition of the weighted norm (4.28),

$$\begin{aligned} \|(I - \pi_h^+)v\|_{H_k^1(\Omega)} &\leq C(1 + kh)(kh)^\ell k^{-\ell} \|v\|_{H^{\ell+1}(\mathcal{Q})} \\ &\leq C(1 + kh)(kh)^\ell \|v\|_{H_k^{\ell+1}(\mathcal{Q})} \quad \text{for all } v \in H^{\ell+1}(\mathcal{Q}) \cap H_{\Gamma_D}^1(\Omega). \end{aligned}$$

Combining this with (4.30) and (4.29), we obtain that

$$\begin{aligned} \gamma_{\mathcal{K}} &\leq k \|(I - \pi_h^+)\mathcal{S}^+\mathcal{K}\|_{L^2(\Omega) \rightarrow H_k^1(\Omega)} \\ &\leq k \|I - \pi_h^+\|_{H^{\ell+1}(\mathcal{Q}) \rightarrow H_k^1(\Omega)} \|\mathcal{S}^+\|_{H_k^{\ell-1}(\mathcal{Q}) \rightarrow H_k^{\ell+1}(\mathcal{Q})} \|\mathcal{K}\|_{L^2(\Omega) \rightarrow H_k^{\ell-1}(\mathcal{Q})} \\ &\leq Ck(1 + kh)(kh)^\ell k^{-2}k; \end{aligned}$$

the result (4.31) then follows. \blacksquare

Proof of Lemma 4.7. This result was first proved in [GS23, Lemma 2.1]. For completeness, we give a (slightly different) proof here. Since A is a symmetric, positive definite matrix-valued function, $\text{Re } A$ is a real, symmetric, positive definite matrix-valued function. Combining this with the uniform positivity of $\text{Re } \mu$ and the spectral theorem (see, e.g., [McL00, Theorem 2.37]), we see that there exist $\{\lambda_j^2\}_{j=1}^\infty$ and $\{u_j\}_{j=1}^\infty$ with $\lambda_1^2 \leq \lambda_2^2 \leq \dots$ with $0 \leq \lambda_j^2 \rightarrow \infty$ as $j \rightarrow \infty$ and $u_j \in H_{\Gamma_D}^1(\Omega)$ with

$$(u_j, u_\ell)_{\text{Re } \mu} := ((\text{Re } \mu)u_j, u_\ell)_{L^2(\Omega)} = \delta_{j\ell}$$

such that

$$((\text{Re } A)\nabla u_j, \nabla w)_{L^2(\Omega)} = \lambda_j^2 ((\text{Re } \mu)u_j, w)_{L^2(\Omega)} \quad \text{for all } w \in H_{\Gamma_D}^1(\Omega) \quad (4.32)$$

i.e.,

$$-\nabla \cdot ((\text{Re } A)\nabla u_j) = \lambda_j^2 (\text{Re } \mu)u_j, \quad (4.33)$$

and

$$v = \sum_{j=1}^{\infty} (v, u_j)_{\text{Re } \mu} u_j \quad \text{for all } v \in L^2(\Omega). \quad (4.34)$$

We record immediately the consequences of (4.34) and (4.32) that

$$((\text{Re } \mu)v, v)_{L^2(\Omega)} = \sum_{j=1}^{\infty} |(v, u_j)_{\text{Re } \mu}|^2 \quad \text{and} \quad ((\text{Re } A)\nabla v, \nabla v)_{L^2(\Omega)} = \sum_{j=1}^{\infty} \lambda_j^2 |(v, u_j)_{\text{Re } \mu}|^2, \quad (4.35)$$

respectively. Let

$$\tilde{\mathcal{K}}v := \sum_{\substack{j=1 \\ \lambda_j^2 \leq 2k^2}}^{\infty} (v, u_j)_{\text{Re } \mu} u_j \quad \text{for all } v \in L^2(\Omega). \quad (4.36)$$

and let

$$\mathcal{K} := \left(\sup_{x \in \Omega} (\text{Re } \mu(x))^{1/2} \right) \tilde{\mathcal{K}}. \quad (4.37)$$

Since $\tilde{\mathcal{K}}^2 = \tilde{\mathcal{K}}$, $\tilde{\mathcal{K}}$ has norm one in the L^2 norm weighted with $\text{Re } \mu$, and thus the $L^2(\Omega) \rightarrow L^2(\Omega)$ bound in (4.29) holds.

We next prove that the $L^2(\Omega) \rightarrow H_k^n(\Omega)$ bound in (4.29) holds with $1 \leq n \leq \ell + 1$. Since A , μ , and $\partial\Omega$ satisfy Assumption 4.2, by elliptic regularity (see, e.g., [McL00, Theorem 4.20]),

$$\|u_j\|_{H^2(\mathcal{Q})} \leq C \left(\|\nabla \cdot ((\text{Re } A)\nabla u_j)\|_{L^2(\Omega)} + \|u_j\|_{H^1(\Omega)} \right) \leq C \left(\lambda_j^2 \|u_j\|_{L^2(\Omega)} + \|u_j\|_{H^1(\Omega)} \right).$$

By (4.32) with $w = u_j$, $\|u\|_{H^1(\Omega)} \leq C|\lambda_j| \|u\|_{L^2(\Omega)}$, and thus

$$\|u_j\|_{H^2(\mathcal{Q})} \leq C\lambda_j^2 \|u_j\|_{L^2(\Omega)}. \quad (4.38)$$

The bound (4.38) and interpolation (see, e.g. [McL00, Theorems B.2 and B.8]) imply that

$$\|u_j\|_{H^1(\mathcal{Q})} \leq C|\lambda_j| \|u_j\|_{L^2(\Omega)}. \quad (4.39)$$

By elliptic regularity, (4.33), and (4.39), for $0 \leq m \leq \ell + 1$,

$$\begin{aligned} \|u_j\|_{H^{m+2}(\mathcal{Q})} &\leq C \left(\|\nabla \cdot ((\operatorname{Re} A)\nabla u_j)\|_{H^m(\mathcal{Q})} + \|u\|_{H^1(\Omega)} \right) \\ &\leq C \left(\lambda_j^2 \|u_j\|_{H^m(\mathcal{Q})} + \|u\|_{H^1(\Omega)} \right) \leq C\lambda_j^2 \|u_j\|_{H^m(\mathcal{Q})} \end{aligned} \quad (4.40)$$

(where the upper limit of $m = \ell + 1$ is determined by the regularity of A, μ , and $\partial\Omega$ in Assumption 4.2). The combination of (4.38), (4.39), and (4.40) imply that

$$\|u_j\|_{H^n(\mathcal{Q})} \leq C|\lambda_j|^n \|u_j\|_{L^2(\Omega)} \quad \text{for all } 0 \leq n \leq \ell + 1. \quad (4.41)$$

Therefore, by the definition of $\tilde{\mathcal{K}}$ (4.36), (4.41), and (4.35),

$$\|\tilde{\mathcal{K}}v\|_{H^n(\mathcal{Q})}^2 \leq C \sum_{\substack{j=1 \\ \lambda_j^2 \leq 2k^2}} \lambda_j^{2n} |(v, u_j)_{\operatorname{Re} \mu}|^2 \leq Ck^{2n} \|v\|_{L^2(\Omega)}^2 \quad \text{for all } 0 \leq n \leq \ell + 1;$$

the bound (4.29) then follows $0 \leq n \leq \ell + 1$ by the definition of the weighted norm (4.28).

To prove the Gårding-type inequality (3.2), observe that, by (4.35) and the definition of $\tilde{\mathcal{K}}$ (4.36),

$$\begin{aligned} \operatorname{Re}(b(v, v)) + 2k^2 (\tilde{\mathcal{K}}v, \tilde{\mathcal{K}}v)_{\operatorname{Re} \mu} &= (\operatorname{Re} b)(v, v) + 2k^2 (\tilde{\mathcal{K}}v, v)_{\operatorname{Re} \mu}, \\ &= \sum_{j=1}^{\infty} (\lambda_j^2 - k^2) |(v, u_j)_{\operatorname{Re} \mu}|^2 + 2k^2 \sum_{\substack{j=1 \\ \lambda_j^2 \leq 2k^2}}^{\infty} |(v, u_j)_{\operatorname{Re} \mu}|^2, \\ &= \sum_{\lambda_j^2 \leq 2k^2} (\lambda_j^2 - k^2) |(v, u_j)_{\operatorname{Re} \mu}|^2 + \sum_{\lambda_j^2 > 2k^2} (\lambda_j^2 + k^2) |(v, u_j)_{\operatorname{Re} \mu}|^2. \end{aligned} \quad (4.42)$$

Now,

$$\text{if } \lambda_j^2 \geq 2k^2 \quad \text{then} \quad (\lambda_j^2 - k^2) \geq \frac{1}{2}\lambda_j^2 \geq \frac{1}{4}(\lambda_j^2 + k^2). \quad (4.43)$$

Therefore, combining (4.37), (4.42), and (4.43), we obtain that, for all $v \in H_{\Gamma_D}^1(\Omega)$,

$$\begin{aligned} \operatorname{Re}(b(v, v)) + 2k^2 \|\tilde{\mathcal{K}}v\|_{L^2(\Omega)}^2 &\geq \operatorname{Re}(b(v, v)) + 2k^2 \|\tilde{\mathcal{K}}v\|_{\operatorname{Re} \mu}^2 \\ &\geq \frac{1}{4} \sum_{j=1}^{\infty} (\lambda_j^2 + k^2) |(v, u_j)_{\operatorname{Re} \mu}|^2 \\ &\geq \frac{1}{4} \left(((\operatorname{Re} A)\nabla v, \nabla v)_{L^2(\Omega)} + k^2 ((\operatorname{Re} \mu)v, v)_{L^2(\Omega)} \right) \geq C_{\text{coer}} \|v\|_{H_k^1(\Omega)}^2, \end{aligned}$$

which is (3.2) with

$$C_{\text{coer}} := \frac{1}{4} \min \left\{ \inf_{x \in \Omega} ((\operatorname{Re} A)_*(x)), \inf_{x \in \Omega} (\operatorname{Re} \mu(x)) \right\}.$$

where, for $x \in \Omega$, $(\operatorname{Re} A)_*(x) := \inf_{\xi \in \mathbb{R}^d} (\operatorname{Re} A)(x)_{ij} \xi_i \xi_j$.

To complete the proof, we need to prove the bound (4.30). First observe that, by coercivity of b^+ , the Lax–Milgram lemma (see, e.g., [McL00, Lemma 2.32]) and the definitions of $\|\cdot\|_{(H_k^1(\Omega))^*}$ and $\|\cdot\|_{H_k^1(\Omega)}$,

$$\|\mathcal{S}^+\|_{(H_k^1(\Omega))^* \rightarrow H_k^1(\Omega)} \leq C, \quad \|\mathcal{S}^+\|_{L^2(\Omega) \rightarrow H_k^1(\Omega)} \leq \frac{C}{k}. \quad \text{and} \quad \|\mathcal{S}^+\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \leq \frac{C}{k^2}. \quad (4.44)$$

Given $\phi \in L^2(\Omega)$, by the definition of \mathcal{S}^+ (3.12), $v := \mathcal{S}^+\phi \in H_{\Gamma_D}^1(\Omega)$ is the solution of the equation

$$-k^2(\operatorname{Re} \mu)v - \nabla \cdot ((\operatorname{Re} A)\nabla v) + k^2\mathcal{K}v = \phi.$$

Since A , μ , and $\partial\Omega$ satisfy Assumption 4.2, by elliptic regularity (see, e.g., [McL00, Theorem 4.20]) the second bound in (4.44), and the third bound in (4.44), for $0 \leq m \leq \ell - 1$,

$$\begin{aligned} |v|_{H^{m+2}(\mathcal{Q})} &\leq C \left(\|k^2\mathcal{K}v + k^2(\operatorname{Re} \mu)v + \phi\|_{H^m(\mathcal{Q})} + \|v\|_{H^1(\Omega)} \right) \\ &\leq C \left(k^{2+m} \|v\|_{L^2(\Omega)} + k^2 \|v\|_{H^m(\mathcal{Q})} + \|\phi\|_{H^m(\mathcal{Q})} \right) \\ &\leq C \left(k^m \|\phi\|_{L^2(\Omega)} + k^2 \|v\|_{H^m(\mathcal{Q})} + \|\phi\|_{H^m(\mathcal{Q})} \right). \end{aligned}$$

After multiplying by k^{-m-1} and using the definitions of the weighted norms (4.27) and (4.28), we obtain that

$$k^{-m-1}|v|_{H^{m+2}(\mathcal{Q})} \leq C \left(\|v\|_{H_k^m(\mathcal{Q})} + k^{-2} \|\phi\|_{H_k^m(\mathcal{Q})} \right) \quad \text{for } 0 \leq m \leq \ell - 1. \quad (4.45)$$

By iteratively applying (4.45), and then using either the second and third bounds in (4.44) (depending on whether m is odd or even), we obtain that $\|v\|_{H_k^{n+2}(\mathcal{Q})} \leq k^{-2} \|\phi\|_{H_k^n(\mathcal{Q})}$ for all $0 \leq n \leq \ell - 1$; using this last bound with $n = m - 2$ in (4.45) and then setting $m = \ell - 1$, we obtain the result (4.30). \blacksquare

4.7 Transferring the error bound from the mapped finite-element space to the true finite-element space

With Theorems 4.5, 4.6, and 4.8 we have all the tools needed to apply the abstract framework of Section 3 to obtain an error estimate for the abstract mapped finite-element solution, i.e., a bound on $\|u - \tilde{u}_h\|_{H_k^1(\Omega)}$. We now provide an additional set of results to give an error estimate for the ‘‘concrete’’ finite element solution u_h . Since we can expect that $K = \tilde{K}$ for the elements $K \in \mathcal{T}_h$ not touching an interface, the error bound for \tilde{u}_h already gives an error bound for u_h on those elements, but we can be a bit more precise. To do so, we need the following additional assumption that essentially requires that for the elements $K \in \mathcal{T}_h$ not touching an interface, $\Psi_K^h = I$.

Assumption 4.9 (Ruling out pathological behaviour of Ψ^h inside Q) For all $Q \in \mathcal{Q}$,

$$Q \cap Q^h = \bigcup_{\substack{K \in \mathcal{T}_h \\ K \subset Q^h}} \tilde{K} \cap K.$$

Under Assumption 4.9, we can transform an error estimate on \tilde{u}_h into an error estimate on u_h .

Lemma 4.10 (Error transfer) Let $v \in H_{\Gamma_D}^1(\Omega)$ and $v_h \in V_h$. Then, for all $Q \in \mathcal{Q}$,

$$\|v - v_h\|_{H_k^1(Q \cap Q^h)} \leq C \left(\|v - \tilde{v}_h\|_{H_k^1(Q)} + \left(\frac{h}{L}\right)^q \|v\|_{H_k^1(Q)} \right). \quad (4.46)$$

Proof. If we can prove that

$$\|\tilde{v}_h - v_h\|_{H_k^1(Q \cap Q^h)} \leq C \left(\frac{h}{L}\right)^q \|\tilde{v}_h\|_{H_k^1(Q \cap Q^h)}, \quad (4.47)$$

then the result (4.46) follows by the triangle inequality:

$$\begin{aligned} \|v - v_h\|_{H_k^1(Q \cap Q^h)} &\leq \|v - \tilde{v}_h\|_{H_k^1(Q \cap Q^h)} + \|\tilde{v}_h - v_h\|_{H_k^1(Q \cap Q^h)} \\ &\leq \|v - \tilde{v}_h\|_{H_k^1(Q \cap Q^h)} + C \left(\frac{h}{L}\right)^q \|\tilde{v}_h\|_{H_k^1(Q \cap Q^h)} \\ &\leq \left(1 + C \left(\frac{h}{L}\right)^q\right) \|v - \tilde{v}_h\|_{H_k^1(Q \cap Q^h)} + C \left(\frac{h}{L}\right)^q \|v\|_{H_k^1(Q \cap Q^h)} \\ &\leq C \left(\|v - \tilde{v}_h\|_{H_k^1(Q \cap Q^h)} + \left(\frac{h}{L}\right)^q \|v\|_{H_k^1(Q \cap Q^h)} \right). \end{aligned}$$

We now prove (4.47); by Assumption 4.9,

$$\|\tilde{v}_h - v_h\|_{L^2(Q \cap Q^h)}^2 = \sum_{\substack{K \in \mathcal{T}_h \\ K \subset Q^h}} \|\tilde{v}_h - v_h\|_{L^2(K \cap \tilde{K})}^2.$$

We now focus on a single element K . Since $\tilde{v}_h = v_h \circ \Phi^h$ and $\Phi^h(x) = \Phi_K^h(x)$ when $x \in \tilde{K}$,

$$\|\tilde{v}_h - v_h\|_{L^2(K \cap \tilde{K})}^2 = \int_{K \cap \tilde{K}} |v_h(\Phi^h(x)) - v_h(x)|^2 dx = \int_{K \cap \tilde{K}} |v_h(\Phi_K^h(x)) - v_h(x)|^2 dx. \quad (4.48)$$

We now work on the integrand; specifically,

$$\begin{aligned} |v_h(\Phi_K^h(x)) - v_h(x)| &= |v_h(x + (I - \Phi_K^h)(x)) - v_h(x)| \\ &\leq |(I - \Phi_K^h)(x)| \|\nabla v_h\|_{L^\infty(K)} \leq \|I - \Phi_K^h\|_{L^\infty(K)} \|\nabla v_h\|_{L^\infty(K)}; \end{aligned}$$

using this in (4.48) gives

$$\|\tilde{v}_h - v_h\|_{L^2(K \cap \tilde{K})}^2 \leq \|I - \Phi_K^h\|_{L^\infty(K)}^2 |K| \|\nabla v_h\|_{L^\infty(K)}^2 \leq Ch_K^{-2} \|I - \Phi_K^h\|_{L^\infty(K)}^2 \|v_h\|_{L^2(K)}^2$$

due to standard equivalence of norms and scaling argument on polynomial spaces recapped in Lemma A.1 below. Summing over K and using the bound (4.7) on Φ in Assumption 4.3, we obtain that

$$\|\tilde{v}_h - v_h\|_{L^2(Q \cap Q^h)} \leq C \left(\frac{h}{L}\right)^q \|v_h\|_{L^2(Q^h)} \leq C \left(\frac{h}{L}\right)^q \|\tilde{v}_h\|_{L^2(Q)}. \quad (4.49)$$

We argue similarly for the gradient term; since $\tilde{v}_h = v_h \circ \Phi^h$ and $\Phi^h(x) = \Phi_K^h(x)$ when $x \in \tilde{K}$,

$$\|\nabla(\tilde{v}_h - v_h)\|_{L^2(Q \cap Q^h)}^2 = \sum_{\substack{K \in \mathcal{T}_h \\ K \subset Q^h}} \int_{K \cap \tilde{K}} |(\nabla \Phi_K^h)^T(x) (\nabla v_h)(\Phi_K^h(x)) - (\nabla v_h)(x)|^2 dx.$$

Then, with I_d the $d \times d$ identity matrix,

$$\begin{aligned} |(\nabla \Phi_K^h)^T(x) (\nabla v_h)(\Phi_K^h(x)) - (\nabla v_h)(x)| &\leq |(I_d - \nabla \Phi_K^h)^T(x) (\nabla v_h)(\Phi_K^h(x))| + |(\nabla v_h)(\Phi_K^h(x)) - (\nabla v_h)(x)| \\ &\leq \|I_d - \nabla \Phi_K^h\|_{L^\infty(\tilde{K})} \|\nabla v_h\|_{L^\infty(K)} + \|I - \Phi_K^h\|_{L^\infty(\tilde{K})} \|\nabla^2 v_h\|_{L^\infty(K)}, \end{aligned}$$

and it follows from the scaling arguments in Lemma A.1 that

$$\|\nabla(\tilde{v}_h - v_h)\|_{L^2(Q \cap Q^h)} \leq C \left(\|I_d - \nabla \Phi_K^h\|_{L^\infty(\tilde{K})} + h_K^{-1} \|I - \Phi_K^h\|_{L^\infty(\tilde{K})} \right) \|\nabla v_h\|_{L^2(K)}.$$

Summing over K and using the bound (4.7) on Φ in Assumption 4.3, we obtain that

$$\|\nabla(\tilde{v}_h - v_h)\|_{L^2(Q \cap Q^h)} \leq C \left(\frac{h}{L}\right)^q \|\nabla v_h\|_{L^2(Q^h)} \leq C \left(\frac{h}{L}\right)^q \|\nabla \tilde{v}_h\|_{L^2(Q)}. \quad (4.50)$$

Combining (4.49) and (4.50), we obtain (4.47) and the proof is complete. \blacksquare

4.8 Theorem 3.2 applied to Helmholtz problems solved with curved finite elements

Theorem 4.11 (Preasymptotic Helmholtz h -FEM error bound with curved finite elements.) *Suppose that A, μ , and g satisfy the assumptions in §4.1 (in particular, A satisfies Assumption 4.1 and A, μ , and $\partial\Omega$ satisfy the regularity requirements in Assumption 4.2). Suppose that V_h satisfies Assumptions 4.3 and 4.9. Suppose that $kh \leq C_0$ for some $C_0 > 0$.*

(i) *Given $k_0 > 0$ there exists $C_1, C_2 > 0$ such that, for all $k \geq k_0$, if*

$$\left(kh + \gamma_s^*(kh)^p\right)(kh)^p + (1 + \gamma_s^*) \left(\frac{h}{L}\right)^q \leq C_1 \quad (4.51)$$

then the Galerkin solution u_h exists, is unique, and satisfies

$$\begin{aligned} & \left(\sum_{Q \in \mathcal{Q}} \|u - u_h\|_{H_k^1(Q \cap Q^h)}^2 \right)^{1/2} \\ & \leq C_2 \left[\left(1 + \gamma_s^*(kh)^p\right) \|(I - \tilde{\pi}_h)u\|_{H_k^1(\Omega)} \right. \\ & \quad \left. + (1 + \gamma_s^*) \left(\frac{h}{L}\right)^q \left(\|u\|_{H_k^1(\Omega)} + \frac{1}{k^2} \left(\sum_{Q \in \mathcal{Q}} \|g_Q\|_{H_k^1(\mathbb{R}^d)}^2 \right)^{1/2} \right) \right], \end{aligned} \quad (4.52)$$

where $\tilde{\pi}_h$ is the orthogonal projection in the mapped finite-element space \tilde{V}_h (which, in particular, satisfies the interpolation bound (4.15)).

(ii) Suppose, in addition, that $\text{supp } g \subset Q$ for some $Q \in \mathcal{Q}$ and that given $k_0 > 0$ there exists $C > 0$ such that for all $0 \leq j \leq p - 1$,

$$\|g\|_{H^j(\Omega)} \leq Ck^{j+1} \sup_{\substack{v \in \mathcal{H} \\ \|v\|_{H_k^1(\Omega)} = 1}} |\langle g, v \rangle| \quad \text{for all } k \geq k_0. \quad (4.53)$$

Then, given $k_0 > 0$, there exists $C_3 > 0$ such that, for all $k \geq k_0$, if (4.51) holds, then

$$\left(\sum_{Q \in \mathcal{Q}} \|u - u_h\|_{H_k^1(Q \cap Q^h)}^2 \right)^{1/2} \leq C_3 \left[\left(1 + \gamma_s^*(kh)^p\right) (kh)^p + \gamma_s^* \left(\frac{h}{L}\right)^q \right] \|u\|_{H_k^1(\Omega)}.$$

Proof. (i) Under the assumptions of the theorem, Theorems 4.5, 4.6, and 4.8 all hold. Applying Theorem 3.2 to the variational problem (4.11) (i.e., in the mapped space \tilde{V}_h) and inputting the bounds on $\gamma_q, \tilde{\gamma}_q, \gamma_a^*$, and $k\|(I - \pi_h)\mathcal{S}^+\mathcal{K}\|_{L^2(\Omega) \rightarrow H_k^1(\Omega)}$, we obtain, for some $C > 0$ depending on A and μ ,

$$\begin{aligned} & \|u - \tilde{u}_h\|_{H_k^1(\Omega)} \\ & \leq C(1 + \gamma_s^*(kh)^p) \|(I - \tilde{\pi}_h)u\|_{H_k^1(\Omega)} \\ & \quad + C(1 + \gamma_s^*) \left(\left(\frac{h}{L}\right)^q \|u\|_{H_k^1(\Omega)} + \left(\frac{h}{L}\right)^q \frac{1}{k} \left(\sum_{Q \in \mathcal{Q}} \|g_Q\|_{L^2(\mathbb{R}^d)}^2 + h^2 \|\nabla g_Q\|_{L^2(\mathbb{R}^d)}^2 \right)^{1/2} \right) \end{aligned} \quad (4.54)$$

(compare to (3.19)). The bound (4.46) and the fact that $(h/L)^q$ is bounded (by (4.51)) allows us to replace $\|u - \tilde{u}_h\|_{H_k^1(\Omega)}$ on the left-hand side of (4.54) with $\sum_{Q \in \mathcal{Q}} \|u - u_h\|_{H^1(Q \cup Q^h)}$. The result (4.52) then follows by using $h \leq C_0/k$ in the terms involving g_Q .

(ii) The result follows from Part (i) if we can show that

(a) given $k_0 > 0$ there exists $C > 0$ such that, for all $k \geq k_0$, if $kh \leq C_0$,

$$\|(I - \tilde{\pi}_h)u\|_{H_k^1(\Omega)} \leq C(kh)^p \|u\|_{H_k^1(\Omega)}, \quad (4.55)$$

(b) if g satisfies (4.53), then

$$\frac{1}{k^2} \left(\sum_{Q \in \mathcal{Q}} \|g_Q\|_{H_k^1(\mathbb{R}^d)}^2 \right)^{1/2} \leq C \|u\|_{H_k^1(\Omega)}. \quad (4.56)$$

For (a), by (4.15)

$$\|(I - \tilde{\pi}_h)u\|_{H_k^1(\Omega)} \leq C(1 + kh)h^p \|u\|_{H^{p+1}(\mathcal{Q})}. \quad (4.57)$$

If g satisfies (4.53), then by elliptic regularity (using the regularity assumptions on A , μ , and $\partial\Omega$ in Assumption 4.2),

$$\|u\|_{H^{p+1}(\mathcal{Q})} \leq Ck^p \|u\|_{H_k^1(\Omega)} \quad \text{for all } k \geq k_0; \quad (4.58)$$

(i.e., the data being k -oscillatory implies that the solution is k -oscillatory); the proof is very similar to the uses of elliptic regularity in the proof of Lemma 4.7; for the details see [GS23, end of the proof of Theorem 1.5]. Combining (4.57) and (4.58), we obtain that

$$\|(I - \tilde{\pi}_h)u\|_{H_k^1(\Omega)} \leq C(1 + kh)(kh)^p \|u\|_{H_k^1(\Omega)},$$

and (4.55) follows since $kh \leq C_0$.

For (b), by (4.53) applied with $j = 1$ and the fact that $\text{supp } g \subset Q$ for some $Q \in \mathcal{Q}$,

$$\frac{1}{k^2} \left(\sum_{Q \in \mathcal{Q}} \|g_Q\|_{H_k^1(\mathbb{R}^d)} \right)^{1/2} = k^{-2} \|g\|_{H_k^1(\Omega)} \leq C \|g\|_{(H_k^1(\Omega))^*}. \quad (4.59)$$

By continuity of b (3.1) and (4.3),

$$\|g\|_{(H_k^1(\Omega))^*} \leq M \|u\|_{H_k^1(\Omega)}, \quad (4.60)$$

and (4.56) follows from combining (4.59) and (4.60). \blacksquare

Remark 4.12 (Obtaining Theorem 1.1 from Part (ii) of Theorem 4.11) *For the radial PML originally introduced in [CM98], the coefficient A satisfies Assumption 4.1 by, e.g., [GLSW24, Lemma 2.3]. Theorem 1.1 therefore follows from Part (ii) of Theorem 4.11 if the data satisfies (4.53) and is such that $\text{supp } g \subset Q$ for some $Q \in \mathcal{Q}$.*

As described in Definitions 2.3 and 2.4, the solution of the plane-wave scattering problem can be approximated using a PML by solving for the unknown $u := \chi u^I + u^S$, with $\chi \in C_{\text{comp}}^\infty(\mathbb{R}^d)$ equal one on the scatterer, and corresponding data $g = f$ with f defined by (2.9). By construction, the PML must be away from the support of χ ; thus $\nabla \chi$, and hence also g , are supported where $A = I$ and $\mu = 1$, and thus on a single $Q \in \mathcal{Q}$.

A Scaling arguments and inverse estimates

In this section, we consider a mesh \mathcal{T}_h satisfying the requirements of Assumption 4.3. Recall that, in this case, the elements $K \in \mathcal{T}_h$ are obtained by mapping a single reference simplex \hat{K} through bilipschitz maps $\mathcal{F}_K : \hat{K} \rightarrow K$. To simplify the notation, we let $\mathcal{G}_K := \mathcal{F}_K^{-1}$. By (4.5),

$$\|\det \nabla \mathcal{F}_K\|_{L^\infty(K)} \leq C \|\nabla \mathcal{F}_K\|_{L^\infty(K)}^d \leq Ch_K^d, \quad (A.1a)$$

and

$$\|\det \nabla \mathcal{G}_K\|_{L^\infty(\hat{K})} \leq C \|\nabla \mathcal{G}_K\|_{L^\infty(\hat{K})}^d \leq Ch_K^{-d}. \quad (A.1b)$$

Lemma A.1 *For all $v_h \in V_h$ and $K \in \mathcal{T}_h$,*

$$|K| \|\nabla v_h\|_{L^\infty(K)}^2 \leq Ch_K^{-2} \|v_h\|_{L^2(K)}^2, \quad |K| \|\nabla^2 v_h\|_{L^\infty(K)}^2 \leq Ch_K^{-2} \|\nabla v_h\|_{L^2(K)}^2.$$

Proof. By (A.1),

$$|K| = \int_K 1 dx = \int_{\hat{K}} (\det \nabla \mathcal{F}_K)(\hat{x}) d\hat{x} \leq |\hat{K}| \|\det \nabla \mathcal{F}_K\|_{L^\infty(\hat{K})} \leq Ch_K^d. \quad (A.2)$$

Let $\hat{v} \in \mathcal{P}_p(\hat{K})$ and set $v := \hat{v} \circ \mathcal{F}_K^{-1} = \hat{v} \circ \mathcal{G}_K$. By the chain rule,

$$\partial_j v = \partial_j \mathcal{G}_K^\ell (\partial_\ell \hat{v}) \circ \mathcal{G}_K$$

and a second application gives

$$\partial_{j\ell}^2 v = \partial_{j\ell}^2 \mathcal{G}_K^\ell (\partial_\ell \hat{v}) \circ \mathcal{G}_K + \partial_j \mathcal{G}_K^\ell \partial_\ell \mathcal{G}_K^m (\partial_{\ell m}^2 \hat{v}) \circ \mathcal{G}_K.$$

It then follows from (4.5) that

$$\|\nabla v\|_{L^\infty(K)} \leq Ch_K^{-1} \|\nabla \hat{v}\|_{L^\infty(\hat{K})} \quad \text{and} \quad \|\nabla^2 v\|_{L^\infty(K)} \leq Ch_K^{-2} \left(\|\nabla \hat{v}\|_{L^\infty(\hat{K})} + \|\nabla^2 \hat{v}\|_{L^\infty(\hat{K})} \right).$$

Since $\mathcal{P}_p(\widehat{K})$ is a finite dimensional space, equivalence of semi-norms gives that

$$\|\nabla\widehat{v}\|_{L^\infty(\widehat{K})} \leq C\|\widehat{v}\|_{L^2(\widehat{K})} \quad \text{and} \quad \|\nabla\widehat{v}\|_{L^\infty(\widehat{K})} + \|\nabla^2\widehat{v}\|_{L^\infty(\widehat{K})} \leq C\|\nabla\widehat{v}\|_{L^2(\widehat{K})},$$

leading to

$$\|\nabla v\|_{L^\infty(K)} \leq Ch_K^{-1}\|\widehat{v}\|_{L^2(\widehat{K})} \quad \text{and} \quad \|\nabla^2 v\|_{L^\infty(K)} \leq Ch_K^{-2}\|\nabla\widehat{v}\|_{L^2(\widehat{K})}. \quad (\text{A.3})$$

Using the chain rule in the other direction gives us that

$$|\nabla\widehat{v}|^2 \leq Ch_K^2 |\nabla v \circ \mathcal{F}_K|^2,$$

and, by the change-of-variable formula, we have

$$\|\widehat{v}\|_{L^2(\widehat{K})}^2 \leq Ch^{-d}\|v\|_{L^2(K)}^2 \quad \text{and} \quad \|\nabla\widehat{v}\|_{L^2(\widehat{K})}^2 \leq Ch^{-d}h_K^2\|\nabla v\|_{L^2(K)}^2, \quad (\text{A.4})$$

where we again used (A.1); combining (A.3), (A.4), and (A.2) concludes the proof. \blacksquare

B Composition with a map close to the identity

The following result is used to estimate the geometric error on the right-hand side (in Theorem 4.5).

Lemma B.1 *Let $U \subset \mathbb{R}^d$ be a measurable set and $\phi : U \rightarrow \phi(U)$ a bilipschitz mapping such that $\|I - \phi\|_{L^\infty(U)} \leq \varepsilon$ and $\|I_d - \nabla\phi\|_{L^\infty(U)} \leq \varepsilon' < 1$. Let $U_\varepsilon := \{x \in \mathbb{R}^d : \text{dist}(x, U) < \varepsilon\}$ and $f \in H^1(U_\varepsilon)$. Then*

$$\|f - f \circ \phi\|_{L^2(U)} \leq \frac{\varepsilon}{1 - \varepsilon'} \|\nabla f\|_{L^2(U_\varepsilon)}.$$

Proof. Let $\phi_t(x) = x + t(\phi(x) - x)$ and $g_x(t) = f(\phi_t(x))$ $t \in [0, 1]$. Then

$$f(\phi(x)) - f(x) = \int_0^1 g'_x(t) dt = \int_0^1 (\phi(x) - x) \cdot (\nabla f)(\phi_t(x)) dt,$$

and

$$\begin{aligned} \int_U |f(\phi(x)) - f(x)|^2 dx &= \int_U \left| \int_0^1 (\phi(x) - x) \cdot (\nabla f)(\phi_t(x)) dt \right|^2 dx \\ &\leq \int_0^1 \int_U |(\phi(x) - x) \cdot (\nabla f)(\phi_t(x))|^2 dx dt, \end{aligned}$$

where we have used the Cauchy-Schwartz inequality and Fubini's theorem.

Since $\nabla\phi_t = I_d + t(\nabla\phi - I_d)$, we have $|\nabla\phi_t(x)| \geq 1 - t\varepsilon' \geq 1 - \varepsilon'$. It follows that ϕ_t^{-1} exists for all $t \in [0, 1]$ with

$$|\nabla(\phi_t^{-1})(y)| \leq \frac{1}{1 - \varepsilon'} \quad \text{for all } y \in \phi_t(U). \quad (\text{B.1})$$

Hence, for a fixed $t \in [0, 1]$, by (B.1) and the fact that $\phi_t(U) \subset U_\varepsilon$ for all $t \in [0, 1]$,

$$\begin{aligned} \int_U |(\phi(x) - x) \cdot (\nabla f)(\phi_t(x))|^2 dx &\leq \|\phi - I\|_{L^\infty(U)}^2 \int_U |(\nabla f)(\phi_t(x))|^2 dx \\ &= \|\phi - I\|_{L^\infty(U)}^2 \int_{\phi_t(U)} |\det \nabla(\phi_t^{-1})(y)|^2 |(\nabla f)(y)|^2 dy \\ &\leq \left(\frac{\varepsilon}{1 - \varepsilon'} \right)^2 \int_{U_\varepsilon} |(\nabla f)(y)|^2 dy, \end{aligned}$$

and the result follows. \blacksquare

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