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ASYMPTOTICALLY CONSTANT-FREE AND POLYNOMIAL-DEGREE-ROBUST A POSTERIORI ERROR ESTIMATES FOR TIME-HARMONIC MAXWELL'S EQUATIONS

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ABSTRACT. We propose a novel a posteriori error estimator for the Nédélec finite element discretization of time-harmonic Maxwell's equations. After the approximation of the electric field is computed, we propose a fully localized algorithm to reconstruct approximations to the electric displacement and the magnetic field, with such approximations respectively fulfilling suitable divergence and curl constraints. These reconstructed fields are in turn used to construct an a posteriori error estimator which is shown to be reliable and efficient. Specifically, the estimator controls the error from above up to a constant that tends to one as the mesh is refined and/or the polynomial degree is increased, and from below up to constant independent of p . Both bounds are also fully-robust in the low-frequency regime. The properties of the proposed estimator are illustrated on a set of numerical examples.

KEYWORDS: a posteriori error estimates, Maxwell's equations, time-harmonic wave propagation, finite element methods

1. INTRODUCTION

We consider time-harmonic Maxwell's equations in an heterogeneous domain Ω : Given $\mathbf{J} : \Omega \rightarrow \mathbb{C}^3$, the solution $\mathbf{E} : \Omega \rightarrow \mathbb{C}^3$ satisfies

$$(1.1) \quad \begin{cases} -\omega^2 \boldsymbol{\varepsilon} \mathbf{E} + \nabla \times (\boldsymbol{\mu}^{-1} \nabla \times \mathbf{E}) &= i\omega \mathbf{J} & \text{in } \Omega, \\ \mathbf{E} \times \mathbf{n} &= \mathbf{0} & \text{on } \partial\Omega, \end{cases}$$

where $\omega > 0$ is the frequency, and $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are symmetric tensor-valued functions describing the electric permittivity and magnetic permeability of the materials contained inside Ω . Boundary value problem (1.1) is instrumental in a number of applications involving electrodynamics [25], where \mathbf{J} represents an applied current density, and \mathbf{E} is the (unknown) electric field.

Apart from simple geometrical configurations, the analytical solution to (1.1) is unavailable, which motivates the use of numerical schemes to compute approximate solutions. Here, we consider conforming Nédélec finite element discretizations [13, 29, 30, 31], and focus on two key aspects: reliable error estimation and adaptive mesh refinements using equilibrated error estimators.

The use of residual-based a posteriori error estimators has been considered in the literature for both time-harmonic Maxwell's equations [9] and the simpler model of the Helmholtz problem which is limited to specifically polarized electromagnetic fields [17, 34]. A shortcoming of residual-based estimators, however, is that they only control the error up to a generic constant depending on the flatness of mesh elements, even for asymptotically fine meshes. Besides, the

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efficiency constant typically depends on the polynomial degree p , meaning in particular that the upper bound is usually not very tight for high-order elements.

Equilibrated error estimators are slightly more complicated to compute than residual-based ones, but they alleviate the two aforementioned limitations [2, 20, 32]. Unfortunately, although equilibrated estimators have been developed and analyzed for the Helmholtz problem [7, 14], there is currently no construction for time-harmonic Maxwell's equations available in the literature. More precisely, equilibrated estimators have been proposed for the magnetostatic problem (which similar to (1.1), but with $\omega = 0$) recently [3, 12, 22, 23], and the purpose of this work is to bridge these results to the time-harmonic case.

As compared to the magnetostatic problem covered in [3, 12, 22, 23], the time-harmonic case considered here presents two key difficulties. (i) It is not coercive, and understanding the dependency of the reliability and efficiency constants in terms of the frequency is crucial. This difficulty is already present for Helmholtz problems, and it is addressed using duality techniques [7, 17, 34]. (ii) Duality techniques are significantly more complicated to perform for Maxwell's equations than for Helmholtz problems [6, 18, 28, 36]. Intuitively, the lack of compactness in the embedding $\mathbf{H}_0(\mathbf{curl}, \Omega) \subset \mathbf{L}^2(\Omega)$ requires to precisely take into account the divergence constraint

$$(1.2) \quad -\omega^2 \nabla \cdot (\varepsilon \mathbf{E}) = i\omega \nabla \cdot \mathbf{J}$$

hidden in (1.1). It is not an easy task, since Nédélec finite elements are $\mathbf{H}_0(\mathbf{curl}, \Omega)$ -conforming, but not $\mathbf{H}(\text{div}, \Omega)$ -conforming.

In the context of residual-based error estimators, these difficulties have recently been addressed in [9]. The key idea is to not only measure the residual appearing in (1.1), but also include a contribution that accounts for the violation of (1.2). With such an estimator, the duality techniques developed in [6, 18, 28, 36] in the context of a priori error analysis can be bridged to produce a posteriori error estimates.

In this work, we propose an equilibrated error estimator following the ideas of [9]. We therefore reconstruct two equilibrated fields respectively meant to measure how (1.1) and (1.2) are violated by the discrete solution \mathbf{E}_h . Specifically, these two fields physically correspond to an electric displacement \mathbf{D}_h and a magnetic field \mathbf{H}_h and satisfy the constraints

$$-\omega^2 \nabla \cdot \mathbf{D}_h = i\omega \nabla \cdot \mathbf{J}, \quad i\omega \nabla \times \mathbf{H}_h = i\omega \mathbf{J} + \omega^2 \mathbf{D}_h.$$

The construction of the divergence-constrained electric displacement \mathbf{D}_h closely follows the equilibration techniques introduced for the Poisson problem [2, 15, 19], whereas we adapt the construction in [12] for the magnetic field. In both cases, these reconstructions only hinge on small uncoupled finite element problems set on vertex patches which may be solved in parallel.

With the equilibrated fields \mathbf{D}_h and \mathbf{H}_h constructed, the estimator is defined as

$$\eta_K^2 := \omega^2 \|\mathbf{E}_h - \varepsilon^{-1} \mathbf{D}_h\|_{\varepsilon, K}^2 + \|\nabla \times \mathbf{E}_h - i\omega \mu \mathbf{H}_h\|_{\mu^{-1}, K}^2$$

for every mesh element K . Our key result is that

$$(1.3) \quad \|\mathbf{E} - \mathbf{E}_h\|_{\omega, \Omega}^2 \leq (1 + \theta_{\text{rel}}^2) \sum_K \eta_K^2, \quad \eta_K \lesssim (1 + \theta_{\text{eff}}) \|\mathbf{E} - \mathbf{E}_h\|_{\omega, \tilde{\omega}_K},$$

where

$$\|\mathbf{E} - \mathbf{E}_h\|_{\omega, D}^2 := \omega^2 \|\mathbf{E} - \mathbf{E}_h\|_{\varepsilon, D}^2 + \|\nabla \times (\mathbf{E} - \mathbf{E}_h)\|_{\mu^{-1}, D}^2, \quad D \subset \Omega,$$

is the natural energy norm and the coefficient-weighted $L^2(\Omega)$ -norms are rigorously introduced in Section 2.3 below. In (1.3), θ_{rel} tends to zero when the mesh is refined and/or the polynomial degree is increased, θ_{eff} is inversely proportional to the number of elements per wavelength, $\tilde{\omega}_K$ is a small region around the element K , and the symbol \lesssim means that the inequality holds up to constant that only depends on the shape-regularity parameter of the mesh. Crucially, our upper bound is asymptotically constant-free, and the lower bound does not depend on the polynomial degree p . Besides, the dependency of θ_{rel} on the mesh size, the polynomial degree, and the frequency is exactly the same as in the simpler case of Helmholtz problems. Hence, our results are in a certain sense optimal. We also emphasize that we do not assume that Ω is simply-connected, and that the estimates in (1.3) hold true irrespectively of the topology of the domain. Finally, our bounds are fully-robust, irrespectively of the mesh size and polynomial degree, in the low-frequency regime where $\omega \rightarrow 0$.

We also present a set of numerical examples illustrating how the constants in the lower bound and upper bound depend on the key parameters of the model and discretization. In particular, they show that the estimator indeed provides a very sharp constant-free upper bound for sufficiently fine meshes.

The remainder of this work is organized as follows. In Section 2, we recap notation and collect key results from the literature. Our main findings concerning the construction of the estimator and the corresponding error bounds are stated in Section 3, and later proved in Sections 4 and 5. Finally, numerical examples are presented in Section 6.

2. SETTINGS

This section introduces key notation and preliminary results.

2.1. Domain and coefficients. In this work, $\Omega \subset \mathbb{R}^3$ is a Lipschitz polyhedral domain.

In order to consider piecewise constant coefficients, we assume that Ω is subdivided into a ‘‘physical partition’’ \mathcal{P} consisting of a finite number of (open) polyhedral Lipschitz subdomains P such that $P \cap Q = \emptyset$ for all $P, Q \in \mathcal{P}$ and $\bar{\Omega} = \cup_{P \in \mathcal{P}} \bar{P}$.

We consider two real symmetric tensor-valued functions $\varepsilon, \chi : \Omega \rightarrow \mathbb{S}$. ε and χ are piecewise constant onto the partition \mathcal{P} which means that for all $P \in \mathcal{P}$, there exists two symmetric tensors $\varepsilon_P, \chi_P \in \mathbb{S}$ such that $\varepsilon(\mathbf{x}) = \varepsilon_P$ and $\chi(\mathbf{x}) = \chi_P$ for all $\mathbf{x} \in P$. We further assume that ε and χ are uniformly elliptic, which means that

$$\min_{P \in \mathcal{P}} \min_{\xi \in \mathbb{R}^3} \varepsilon_P \xi \cdot \xi > 0 \quad \min_{P \in \mathcal{P}} \min_{\xi \in \mathbb{R}^3} \chi_P \xi \cdot \xi > 0.$$

If $\mathbf{x} \in \Omega$, the notations $\varepsilon_{\min}(\mathbf{x})$ and $\varepsilon_{\max}(\mathbf{x})$ stand for the minimal and maximal eigenvalues of $\varepsilon(\mathbf{x})$, and if $D \subset \Omega$ is a measurable set, we introduce

$$\varepsilon_{\min, D} := \text{ess inf}_D \varepsilon_{\min}, \quad \varepsilon_{\max, D} := \text{ess sup}_D \varepsilon_{\min}.$$

We also employ similar notations for χ and $\mu := \chi^{-1}$.

2.2. Contrast and wavespeed. If $D \subset \Omega$ is a measurable set, we introduce the coefficient contrasts

$$\mathcal{C}_{\varepsilon, D} := \frac{\varepsilon_{\max, D}}{\varepsilon_{\min, D}}, \quad \mathcal{C}_{\mu, D} := \frac{\mu_{\max, D}}{\mu_{\min, D}}.$$

We also define the minimal wavespeed over D as

$$c_{\min, D} := \frac{1}{\sqrt{\varepsilon_{\max, D} \mu_{\max, D}}}.$$

2.3. Functional spaces. If $D \subset \Omega$ is an open subset with Lipschitz boundary, $L^2(D)$ stands for the Lebesgue space of real-valued square integrable functions defined on D , and $\mathbf{L}^2(D) := [L^2(D)]^d$ contains vector-valued functions [1]. The notations $(\cdot, \cdot)_D$ and $\|\cdot\|_D$ are respectively used for the inner-products and norms of both $L^2(D)$ and $\mathbf{L}^2(D)$. Besides, if $\boldsymbol{\rho} : D \rightarrow \mathbb{S}$ is a measurable symmetric matrix function whose eigenvalues are uniformly bounded away from 0 and ∞ , then the application

$$\|\mathbf{v}\|_{\boldsymbol{\rho}, D}^2 := \int_D \boldsymbol{\rho} \mathbf{v} \cdot \mathbf{v}, \quad \mathbf{v} \in \mathbf{L}^2(D)$$

is a norm on $\mathbf{L}^2(D)$ equivalent to its natural norm.

The symbols ∇ , $\nabla \times$ and $\nabla \cdot$ stand for the weak gradient, curl and divergence operators defined in the sense of distributions, and $H^1(D) := \{v \in L^2(D); \nabla v \in \mathbf{L}^2(D)\}$, $\mathbf{H}(\mathbf{curl}, D) := \{\mathbf{v} \in \mathbf{L}^2(D); \nabla \times \mathbf{v} \in \mathbf{L}^2(D)\}$ and $\mathbf{H}(\text{div}, D) := \{\mathbf{v} \in \mathbf{L}^2(D); \nabla \cdot \mathbf{v} \in L^2(D)\}$ are the associated Sobolev spaces. We refer the reader to [1, 24] for an in-depth presentation. The space

$$\mathbf{H}(\text{div}, \boldsymbol{\varepsilon}, D) := \{\mathbf{v} \in \mathbf{L}^2(D) \mid \nabla \cdot (\boldsymbol{\varepsilon} \mathbf{v}) \in L^2(D)\}$$

will also be useful, and $\mathbf{H}(\text{div}^0, D)$ and $\mathbf{H}(\text{div}^0, \boldsymbol{\varepsilon}, D)$ respectively collect the elements \mathbf{v} of $\mathbf{H}(\text{div}, D)$ and $\mathbf{H}(\text{div}, \boldsymbol{\varepsilon}, D)$ such that $\nabla \cdot \mathbf{v} = 0$ and $\nabla \cdot (\boldsymbol{\varepsilon} \mathbf{v}) = 0$.

Finally, $\mathbf{H}_0(\mathbf{curl}, \Omega)$ denotes the closure of smooth compactly supported functions into $\mathbf{H}(\mathbf{curl}, \Omega)$, i.e. elements of $\mathbf{H}(\mathbf{curl}, \Omega)$ with vanishing tangential trace.

2.4. Energy norm. If $D \subset \Omega$ is an open set with Lipschitz boundary, the application

$$\|\mathbf{v}\|_{\omega, D}^2 := \omega^2 \|\mathbf{v}\|_{\boldsymbol{\varepsilon}, D}^2 + \|\nabla \times \mathbf{v}\|_{\boldsymbol{\chi}, D}^2, \quad \mathbf{v} \in \mathbf{H}(\mathbf{curl}, D),$$

is a norm on $\mathbf{H}(\mathbf{curl}, D)$ which is natural to analyze the problem.

2.5. Well-posedness. Assuming that $\mathbf{J} \in \mathbf{L}^2(\Omega)$, we recast Maxwell's equations (1.1) into a weak form where we look for $\mathbf{E} \in \mathbf{H}_0(\mathbf{curl}, \Omega)$ such that

$$(2.1) \quad b(\mathbf{E}, \mathbf{v}) = i\omega(\mathbf{J}_h, \mathbf{v})_\Omega \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}, \Omega)$$

with

$$b(\mathbf{E}, \mathbf{v}) := -\omega^2(\boldsymbol{\varepsilon} \mathbf{E}, \mathbf{v})_\Omega + (\boldsymbol{\chi} \nabla \times \mathbf{E}, \nabla \times \mathbf{v})_\Omega.$$

We will work under the assumption that b is inf-sup stable, which means that there exists a constant $\gamma_{\text{st}} > 0$ such that

$$(2.2) \quad \inf_{\substack{\mathbf{e} \in \mathbf{H}_0(\mathbf{curl}, \Omega) \\ \|\mathbf{e}\|_{\omega, \Omega} = 1}} \sup_{\substack{\mathbf{v} \in \mathbf{H}_0(\mathbf{curl}, \Omega) \\ \|\mathbf{v}\|_{\omega, \Omega} = 1}} b(\mathbf{e}, \mathbf{v}) \geq \frac{1}{\gamma_{\text{st}}}.$$

Notice that (2.2) guaranties the well-posedness of (2.1). The dependence of γ_{st} with respect to ω is studied, e.g., in [10].

2.6. Computational mesh. We consider a mesh \mathcal{T}_h that partitions Ω into a collection of (open) tetrahedral elements K . We assume that the mesh is conforming in the sense that the intersection $\overline{K}_+ \cap \overline{K}_-$ of distinct elements $K_\pm \in \mathcal{T}_h$ is either empty, or a full face, edge or vertex of both K_+ and K_- . These assumption are standard [13, 29] and do not prevent strong mesh grading.

For each $K \in \mathcal{T}_h$, h_K and ρ_K respectively denote the diameter of the smallest ball containing K and of the largest ball contained in \overline{K} , and $\mathcal{V}(K)$ is the set of vertices of K . $\kappa_K := h_K/\rho_K$

then stands for the shape-regularity parameter of K , and if $\mathcal{T} \subset \mathcal{T}_h$, we let $\kappa_{\mathcal{T}} := \max_{K \in \mathcal{T}} \kappa_K$. We also introduce $h := \max_{K \in \mathcal{T}_h} h_K$.

2.7. Hidden constants. In order to lighten the notation in the remainder of this work, we will write that $A \lesssim_{\mathcal{T}} B$ if $A, B > 0$ are two constants such that $A \leq C(\kappa_{\mathcal{T}})B$ where the hidden constant $C(\kappa_{\mathcal{T}})$ only depends on $\kappa_{\mathcal{T}}$, $\mathcal{T} \subset \mathcal{T}_h$ being a submesh.

2.8. Finite element spaces. If $q \geq 0$ and $K \in \mathcal{T}_h$, we denote by $\mathcal{P}_q(K)$ the space of polynomials of degree less than or equal to q , and $\mathbf{P}_q(K) := [\mathcal{P}_q(K)]^3$ contains vector-valued polynomial functions. We will also employ the spaces $\mathbf{N}_q(K) := \mathbf{x} \times \mathcal{P}_q(K) + \mathcal{P}_q(K)$ and $\mathbf{RT}_q(K) := \mathbf{x}\mathcal{P}_q(K) + \mathcal{P}_q(K)$ of Nédélec and Raviart–Thomas polynomials [30, 33].

If $\mathcal{T} \subset \mathcal{T}_h$ is a submesh, then $\mathcal{P}_q(\mathcal{T})$, $\mathbf{N}_q(\mathcal{T})$ and $\mathbf{RT}_q(\mathcal{T})$ stand for the spaces of functions whose restriction to each $K \in \mathcal{T}$ respectively belong to $\mathcal{P}_q(K)$, $\mathbf{N}_q(K)$ and $\mathbf{RT}_q(K)$. Notice that these spaces do not exhibit any built-in compatibility conditions.

2.9. Discrete solution. In the remainder of this work, we consider a fixed conforming finite element space $\mathbf{W}_h \subset \mathbf{H}_0(\mathbf{curl}, \Omega)$. Specifically, we assume that (i) $\mathcal{P}_1(\mathcal{T}_h) \cap \mathbf{H}_0(\mathbf{curl}, \Omega) \subset \mathbf{W}_h$ and (ii) for each $K \in \mathcal{T}_h$, there exists a value $q_K \geq 1$ such that $\mathbf{v}_h|_K \in \mathbf{N}_{q_K}(K)$ for all $\mathbf{v}_h \in \mathbf{W}_h$ and we will denote by p_K the smallest integer q_K for which the inclusion holds. The notation $p := \min_{K \in \mathcal{T}_h} p_K$ will be useful.

We further assume that we are given an element $\mathbf{E}_h \in \mathbf{W}_h$ satisfying

$$(2.3) \quad b(\mathbf{E}_h, \mathbf{v}_h) = i\omega(\mathbf{J}, \mathbf{v}_h)_{\Omega}$$

for all $\mathbf{v}_h \in \mathbf{W}_h$. Notice that if \mathbf{W}_h is sufficiently rich (in particular, if $(\omega h)/(pc_{\min, \Omega})$ is small enough), the linear system in (2.3) in fact uniquely defines \mathbf{E}_h . Precise statements may be found in [6, 18, 27, 28, 36], where the dependency on the frequency is also analyzed.

2.10. Right-hand side. We will assume for the remainder of this work that $\mathbf{J} \in \mathbf{RT}_p(\mathcal{T}_h) \cap \mathbf{H}(\mathbf{div}, \Omega)$. The requirement that $\mathbf{J} \in \mathbf{H}(\mathbf{div}, \Omega)$ is crucial, and is already demanded in earlier works [9]. On the other hand, we only require that $\mathbf{J} \in \mathbf{RT}_p(\mathcal{T}_h)$ for the sake of simplicity. The general case may be treated, as usual, by including a “data oscillation” term in the estimator, as detailed in [12]. To emphasize that we restrict our attention to a piecewise polynomial right-hand sides, we employ the notation \mathbf{J}_h instead of \mathbf{J} in the remainder of this work.

2.11. Cohomology. We introduce

$$\mathbf{\Lambda}_{\varepsilon}(\Omega) := \{\varepsilon^{-1} \nabla \times \phi \mid \phi \in \mathbf{H}_0(\mathbf{curl}, \Omega)\}$$

and denote by $\mathbf{\Lambda}_{\varepsilon}^{\perp}(\Omega)$ its orthogonal complement in the $(\varepsilon \cdot, \cdot)_{\Omega}$ inner product, i.e., $\mathbf{v} \in \mathbf{\Lambda}_{\varepsilon}^{\perp}(\Omega)$ if and only if $\mathbf{v} \in \mathbf{L}^2(\Omega)$ with

$$(\mathbf{v}, \nabla \times \phi)_{\Omega} = 0 \quad \forall \phi \in \mathbf{H}_0(\mathbf{curl}, \Omega).$$

When Ω is simply-connected, we have $\mathbf{\Lambda}_{\varepsilon}(\Omega) = \mathbf{H}(\mathbf{div}^0, \varepsilon, \Omega)$ and $\mathbf{\Lambda}_{\varepsilon}^{\perp}(\Omega) = \nabla H_0^1(\Omega)$. In the general case, the inclusions $\mathbf{\Lambda}_{\varepsilon}(\Omega) \subset \mathbf{H}(\mathbf{div}^0, \varepsilon, \Omega)$ and $\nabla H_0^1(\Omega) \subset \mathbf{\Lambda}_{\varepsilon}^{\perp}(\Omega)$ may be strict [21].

2.12. Approximation factor. The real number

$$(2.4) \quad \gamma_{\text{ba}} := \sup_{\substack{\boldsymbol{\theta} \in \boldsymbol{\Lambda}_\varepsilon(\Omega) \\ \|\boldsymbol{\theta}\|_{\varepsilon, \Omega} = 1}} \inf_{\mathbf{v}_h \in \mathbf{W}_h} \|\boldsymbol{\xi}_\boldsymbol{\theta} - \mathbf{v}_h\|_{\omega, \Omega},$$

where $\boldsymbol{\xi}_\boldsymbol{\theta}$ is the unique element of $\mathbf{H}_0(\mathbf{curl}, \Omega)$ such that

$$b(\boldsymbol{\xi}_\boldsymbol{\theta}, \mathbf{v}) = \omega(\boldsymbol{\varepsilon}\boldsymbol{\theta}, \mathbf{v})_\Omega \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}, \Omega),$$

is called the approximation factor. It is instrumental in the a priori and a posteriori analysis of time-harmonic wave propagation problems [6, 7, 9, 17, 34, 26, 27]. In fact, the supremum and infimum in (2.4) can be replaced by a maximum and a minimum, and γ_{ba} is the best constant such that

$$(2.5) \quad \min_{\mathbf{v}_h \in \mathbf{W}_h} \|\boldsymbol{\xi}_\boldsymbol{\theta} - \mathbf{v}_h\|_{\omega, \Omega} \leq \gamma_{\text{ba}} \|\boldsymbol{\theta}\|_{\varepsilon, \Omega}$$

for all $\boldsymbol{\theta} \in \boldsymbol{\Lambda}_\varepsilon(\Omega)$. Taking $\mathbf{v}_h := \mathbf{0}$ in (2.4), we see that

$$(2.6) \quad \gamma_{\text{ba}} \leq \gamma_{\text{st}}$$

since

$$\|\boldsymbol{\xi}_\boldsymbol{\theta}\|_{\omega, \Omega} \leq \gamma_{\text{st}} \sup_{\substack{\mathbf{v} \in \mathbf{H}_0(\mathbf{curl}, \Omega) \\ \|\mathbf{v}\|_{\omega, \Omega} = 1}} b(\boldsymbol{\xi}_\boldsymbol{\theta}, \mathbf{v}) = \gamma_{\text{st}} \sup_{\substack{\mathbf{v} \in \mathbf{H}_0(\mathbf{curl}, \Omega) \\ \|\mathbf{v}\|_{\omega, \Omega} = 1}} \omega(\boldsymbol{\varepsilon}\boldsymbol{\theta}, \mathbf{v}) \leq \gamma_{\text{st}} \|\boldsymbol{\theta}\|_{\varepsilon, \Omega}$$

for all $\boldsymbol{\theta} \in \boldsymbol{\Lambda}_\varepsilon(\Omega)$.

Because $\boldsymbol{\theta} \in \boldsymbol{\Lambda}_\varepsilon(\Omega) \subset \mathbf{H}(\text{div}^0, \boldsymbol{\varepsilon}, \Omega)$, the associated solution $\boldsymbol{\xi}_\boldsymbol{\theta}$ typically exhibits some smoothness, and there is a rate $s > 0$ such that $\gamma_{\text{ba}} \sim \{(\omega h)/(c_{\min, \Omega} p)\}^s$. The value of s and the dependency of the hidden constant in \sim on the frequency are analyzed, e.g., in [6, 9, 27]. Furthermore, because we restrict the right-hand sides in (2.4) to $\boldsymbol{\Lambda}_\varepsilon(\Omega)$ (as opposed to $\mathbf{H}_0(\text{div}^0, \boldsymbol{\varepsilon}, \Omega)$), γ_{ba} remains uniformly bounded as $\omega \rightarrow 0$.

2.13. Vertex patches and hat functions. We denote by \mathcal{V}_h the set of the vertices of \mathcal{T}_h . If $\mathbf{a} \in \mathcal{V}_h$, $\mathcal{T}_h^\mathbf{a} \subset \mathcal{T}_h$ is the submesh consisting of those elements $K \in \mathcal{T}_h$ such that $\mathbf{a} \in \overline{K}$, and $\omega^\mathbf{a} := \text{int}\left(\bigcup_{K \in \mathcal{T}_h^\mathbf{a}} \overline{K}\right)$ is the associated domain, which is always topologically trivial. We also set $p^\mathbf{a} := \max_{K \in \mathcal{T}_h^\mathbf{a}} p_K$.

We use the notation $\psi^\mathbf{a}$ for the ‘‘hat function’’ associated with \mathbf{a} , i.e., the only element of $\mathcal{P}_1(\mathcal{T}_h) \cap H^1(\Omega)$ such that $\psi^\mathbf{a}(\mathbf{b}) = \delta_{\mathbf{a}, \mathbf{b}}$ for all $\mathbf{b} \in \mathcal{V}_h$. We will frequently use the facts that $0 \leq \psi^\mathbf{a} \leq 1$ in Ω ,

$$(2.7) \quad \sum_{\mathbf{a} \in \mathcal{V}_h} \psi^\mathbf{a} = 1 \text{ in } \Omega$$

and

$$(2.8) \quad \|\nabla \psi^\mathbf{a}\|_{\mathbf{L}^\infty(\omega^\mathbf{a})} \lesssim_{\mathcal{T}_h^\mathbf{a}} h_{\omega^\mathbf{a}}^{-1},$$

where $\|\nabla \psi^\mathbf{a}\|_{\mathbf{L}^\infty(\omega^\mathbf{a})}$ denotes the essential supremum of $|\nabla \psi^\mathbf{a}|$ over $\omega^\mathbf{a}$.

For $\mathbf{a} \in \mathcal{V}_h$, the extended vertex patch $\tilde{\mathcal{T}}_h^\mathbf{a}$ collects all the elements $K \in \mathcal{T}_h$ that share at least one vertex with at least one element of $\mathcal{T}_h^\mathbf{a}$. In effect, $\tilde{\mathcal{T}}_h^\mathbf{a}$ as one additional layer of elements around \mathbf{a} as compared to $\mathcal{T}_h^\mathbf{a}$. $\tilde{\omega}^\mathbf{a}$ is the open domain covered by the elements of $\tilde{\mathcal{T}}_h^\mathbf{a}$.

For $K \in \mathcal{T}_h$, we also employ the notation ω_K for the domain covered by all the elements sharing at least one vertex with K , and $\tilde{\omega}_K$ for the one covered by all elements sharing at

least one vertex with an element contained in ω_K . The corresponding sets of elements are respectively denoted by $\mathcal{T}_{h,K}$ and $\tilde{\mathcal{T}}_{h,K}$.

2.14. Local functional spaces and inequalities. For a vertex $\mathbf{a} \in \mathcal{V}_h$, we define local spaces as follow. First, if $\mathbf{a} \in \partial\Omega$, we denote by $\Gamma_{\mathbf{a}} \subset \partial\Omega$ the portion of the boundary covered by faces sharing the vertex \mathbf{a} , $\Gamma_{\mathbf{a}}^c := \partial\omega^{\mathbf{a}} \setminus \Gamma_{\mathbf{a}}$ and $L_{\star}^2(\omega^{\mathbf{a}}) := L^2(\omega^{\mathbf{a}})$. If on the other hand $\mathbf{a} \in \Omega$, then $\Gamma_{\mathbf{a}} := \emptyset$, $\Gamma_{\mathbf{a}}^c := \partial\Omega$ and $L_{\star}^2(\omega^{\mathbf{a}}) := \{v \in L^2(\omega^{\mathbf{a}}) \mid (v, 1)_{\omega^{\mathbf{a}}} = 0\}$.

For all vertices $\mathbf{a} \in \mathcal{V}_h$, we then introduce the spaces $H_{\dagger}^1(\omega^{\mathbf{a}}) := \{v \in H^1(\omega^{\mathbf{a}}) \mid v = 0 \text{ on } \Gamma_{\mathbf{a}}\}$, $\mathbf{H}_0(\text{div}, \omega^{\mathbf{a}}) := \{\mathbf{v} \in \mathbf{H}(\text{div}, \omega^{\mathbf{a}}) \mid \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma_{\mathbf{a}}^c\}$, $\mathbf{H}_{\dagger}(\text{curl}, \omega^{\mathbf{a}}) := \{\mathbf{v} \in \mathbf{H}(\text{curl}, \omega^{\mathbf{a}}) \mid \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma_{\mathbf{a}}\}$, $\mathbf{H}_{\star}(\text{div}^0, \omega^{\mathbf{a}}) := \{\mathbf{v} \in \mathbf{H}(\text{div}^0, \omega^{\mathbf{a}}) \mid \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma_{\mathbf{a}}^c\}$, and $\mathbf{H}_0(\text{curl}, \omega^{\mathbf{a}}) := \{\mathbf{v} \in \mathbf{H}(\text{curl}, \omega^{\mathbf{a}}) \mid \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma_{\mathbf{a}}^c\}$, where \mathbf{n} denote the outward normal unit vector on $\partial\omega^{\mathbf{a}}$ and the normal and tangential traces are defined by integration by parts following [21].

For all $\mathbf{a} \in \mathcal{V}_h$, the following inequalities hold true

$$(2.9) \quad \|v\|_{\omega^{\mathbf{a}}} \lesssim_{\mathcal{T}_h^{\mathbf{a}}} h_{\omega^{\mathbf{a}}} \|\nabla v\|_{\omega^{\mathbf{a}}}, \quad \|\mathbf{w}\|_{\omega^{\mathbf{a}}} \lesssim_{\mathcal{T}_h^{\mathbf{a}}} h_{\omega^{\mathbf{a}}} \|\nabla \times \mathbf{w}\|_{\omega^{\mathbf{a}}}$$

for all $v \in H_{\dagger}^1(\omega^{\mathbf{a}}) \cap L_{\star}^2(\omega^{\mathbf{a}})$ and $\mathbf{w} \in \mathbf{H}_{\dagger}(\text{curl}, \omega^{\mathbf{a}}) \cap \mathbf{H}_{\star}(\text{div}^0, \omega^{\mathbf{a}})$, see [35] and [8, Theorem A.1].

Notice that in contrast to Ω , the cohomology of the vertex patches is always trivial, i.e., the image of the curl operator is the space of divergence-free functions. Specifically, we have

$$(2.10) \quad \nabla \times \mathbf{H}_0(\text{curl}, \omega) = \mathbf{H}_0(\text{div}, \omega^{\mathbf{a}})$$

for all $\mathbf{a} \in \mathcal{V}_h$. This is due to the fact that each $\omega^{\mathbf{a}}$ is homotopy equivalent to a ball, and that both the portion on the boundary on which the essential boundary conditions are imposed and its complement are simply-connected.

3. MAIN RESULTS

Here, we state the main theoretical contribution of this work.

3.1. Prager-Syngé type upper-bound. In this section, we consider two fields $\mathbf{D}_h \in \mathbf{H}(\text{div}, \Omega)$ and $\mathbf{H}_h \in \mathbf{H}(\text{curl}, \Omega)$ such that

$$(3.1) \quad i\omega \nabla \times \mathbf{H}_h = i\omega \mathbf{J}_h + \omega^2 \mathbf{D}_h.$$

These fields are meant to resemble the electric displacement $\mathbf{D}_{\star} := \varepsilon \mathbf{E}$ and magnetic field $\mathbf{H}_{\star} := (i\omega \mu)^{-1} \nabla \times \mathbf{E}_h$, and the subscript h indicates that they will eventually be constructed using discrete finite element spaces.

For all $K \in \mathcal{T}_h$, we define the local estimators

$$(3.2) \quad \eta_{\text{div},K} := \omega \|\mathbf{E}_h - \varepsilon^{-1} \mathbf{D}_h\|_{\varepsilon,K} \quad \eta_{\text{curl},K} := \|\nabla \times \mathbf{E}_h - i\omega \chi^{-1} \mathbf{H}_h\|_{\chi,K}$$

and $\eta_K^2 := \eta_{\text{div},K}^2 + \eta_{\text{curl},K}^2$. We also consider the global versions

$$(3.3a) \quad \eta_{\text{div}}^2 := \sum_{K \in \mathcal{T}_h} \eta_{\text{div},K}^2 = \omega^2 \|\mathbf{E}_h - \varepsilon^{-1} \mathbf{D}_h\|_{\varepsilon,\Omega}^2$$

and

$$(3.3b) \quad \eta_{\text{curl}}^2 := \sum_{K \in \mathcal{T}_h} \eta_{\text{curl},K}^2 = \|\nabla \times \mathbf{E}_h - i\omega \chi^{-1} \mathbf{H}_h\|_{\chi,\Omega}^2$$

as well as $\eta^2 := \eta_{\text{div}}^2 + \eta_{\text{curl}}^2$. Our first key result consists of two reliability estimates of Prager-Synge type.

Theorem 3.1 (Error estimates in energy norm). *The error estimates*

$$(3.4a) \quad \|\mathbf{E} - \mathbf{E}_h\|_{\omega, \Omega} \leq \gamma_{\text{st}} \eta$$

and

$$(3.4b) \quad \|\mathbf{E} - \mathbf{E}_h\|_{\omega, \Omega} \leq (1 + 2\gamma_{\text{ba}} + 3\gamma_{\text{ba}}^2)^{1/2} \eta.$$

hold true.

Estimate (3.4b) is asymptotically constant-free, since $\gamma_{\text{ba}} \rightarrow 0$ as $(\omega h)/(pc_{\min, \Omega}) \rightarrow 0$. Besides, we obtain a guaranteed bound whenever an estimate for γ_{st} or γ_{ba} is available.

3.2. Local reconstruction of auxiliary fields. We now propose a fully local construction of equilibrated fields satisfying (3.1). These fields are defined as the sum of local contributions solution to finite element problems set on vertex patches. The fact that all the local problems appearing in the definitions below are well-posed is established later in Section 5.

3.2.1. Electric displacement. We first introduce a local reconstruction designed to mimic the “true” electric displacement $\mathbf{D}_\star := \varepsilon \mathbf{E}$. Specifically, the key property we want to reproduce is that $-\omega^2 \nabla \cdot \mathbf{D}_\star = i\omega \nabla \cdot \mathbf{J}_h$. Following [15], we remark that $-\omega^2 \nabla \cdot (\psi^a \mathbf{D}_\star) = i\omega \psi^a \nabla \cdot \mathbf{J}_h - \omega^2 \nabla \psi^a \cdot (\varepsilon \mathbf{E})$, which lead to the definition

$$(3.5) \quad \mathbf{D}_h^a := \arg \min_{\substack{\mathbf{v}_h \in \mathbf{RT}_{p^a+2}(\mathcal{T}_h^a) \cap \mathbf{H}_0(\text{div}, \omega^a) \\ -\omega^2 \nabla \cdot \mathbf{v}_h = i\omega \psi^a \nabla \cdot \mathbf{J}_h - \omega^2 \nabla \psi^a \cdot (\varepsilon \mathbf{E}_h)}} \|\psi^a \mathbf{E}_h - \varepsilon^{-1} \mathbf{v}_h\|_{\varepsilon, \omega^a}.$$

We also set

$$(3.6) \quad \mathbf{D}_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \mathbf{D}_h^a.$$

3.2.2. Total current variation. Next, we are interested in the total current variation

$$\mathbf{G}_\star := \nabla \times (\chi \nabla \times \mathbf{E}) = i\omega \mathbf{J}_h + \omega^2 \varepsilon \mathbf{E} = i\omega \mathbf{J}_h + \omega^2 \mathbf{D}_\star$$

and its discrete counterpart

$$\mathbf{G}_h := i\omega \psi^a \mathbf{J}_h + \omega^2 \mathbf{D}_h.$$

Crucially, $\nabla \cdot \mathbf{G}_\star = \nabla \cdot \mathbf{G}_h = 0$, and we would like to obtain a divergence-free decomposition of \mathbf{G}_\star and \mathbf{G}_h into patch-wise contributions \mathbf{G}_\star^a and \mathbf{G}_h^a . We start by remarking that

$$\mathbf{G}_\star^a := \nabla \times (\psi^a \chi \nabla \times \mathbf{E}) = i\omega \psi^a \mathbf{J}_h + \omega^2 \psi^a \mathbf{D}_\star + \boldsymbol{\theta}_\star^a$$

with

$$\boldsymbol{\theta}_\star^a := \nabla \psi^a \times (\chi \nabla \times \mathbf{E}).$$

It is key to observe that at the continuous level,

$$\nabla \cdot \boldsymbol{\theta}_\star^a = -\nabla \psi^a \cdot \nabla \times (\chi \nabla \times \mathbf{E}),$$

so that $\boldsymbol{\theta}_\star^a \in \mathbf{H}_0(\text{div}, \omega^a)$ and $\nabla \cdot \mathbf{G}_\star^a = 0$. Unfortunately [12], the “naive” discrete counterpart to $\boldsymbol{\theta}_\star^a$, namely $\nabla \psi^a \times (\chi \nabla \times \mathbf{E}_h)$, does not belong to $\mathbf{H}(\text{div}, \omega^a)$, and in particular, does not lead to a divergence-free decomposition of \mathbf{G}_h .

Following [12], we thus construct a discrete counterpart to $\boldsymbol{\theta}_\star^a$ in two steps. In the first step, we remark that

$$\nabla \cdot (\boldsymbol{\theta}_\star^a) = -\nabla \psi^a \cdot \nabla \times (\chi \nabla \times \mathbf{E}) = -\nabla \psi^a \cdot (i\omega \mathbf{J}_h + \omega^2 \varepsilon \mathbf{E}).$$

This motivates the definitions

$$(3.7) \quad \tilde{\boldsymbol{\theta}}_h^{\mathbf{a}} := \arg \min_{\substack{\mathbf{v}_h \in \mathbf{RT}_{p^{\mathbf{a}}+1}(\mathcal{T}_h^{\mathbf{a}}) \cap \mathbf{H}_0(\text{div}, \omega^{\mathbf{a}}) \\ \nabla \cdot \mathbf{v}_h = -\nabla \psi^{\mathbf{a}} \cdot (i\omega \mathbf{J}_h + \omega^2 \boldsymbol{\varepsilon} \mathbf{E}_h) \\ (\mathbf{v}_h, \mathbf{r})_{\omega^{\mathbf{a}}} = (\nabla \psi^{\mathbf{a}} \times (\boldsymbol{\chi} \nabla \times \mathbf{E}_h), \mathbf{r})_{\omega^{\mathbf{a}}} \quad \forall \mathbf{r} \in \mathcal{P}_0(\mathcal{T}_h^{\mathbf{a}})}} \|\nabla \psi^{\mathbf{a}} \times (\boldsymbol{\chi} \nabla \times \mathbf{E}_h) - \mathbf{v}_h\|_{\boldsymbol{\mu}, \omega^{\mathbf{a}}},$$

and

$$\tilde{\boldsymbol{\theta}}_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \tilde{\boldsymbol{\theta}}_h^{\mathbf{a}}.$$

As compared to $\boldsymbol{\theta}_*^{\mathbf{a}}$, $\tilde{\boldsymbol{\theta}}_h^{\mathbf{a}}$ has the desired divergence, but does not sum up to zero. This is remedied in the second step. For all $\mathbf{a} \in \mathcal{V}_h$ and $K \in \mathcal{T}_h^{\mathbf{a}}$, we introduce

$$(3.8) \quad \hat{\boldsymbol{\theta}}_h^{\mathbf{a}}|_K := \arg \min_{\substack{\mathbf{v}_h \in \mathbf{RT}_{p^{\mathbf{a}}+2}(K) \\ \nabla \cdot \mathbf{v}_h = 0 \text{ in } K \\ \mathbf{v}_h \cdot \mathbf{n}_K = \psi^{\mathbf{a}} \tilde{\boldsymbol{\theta}}_h^{\mathbf{a}} \cdot \mathbf{n}_K \text{ on } \partial K}} \|\psi^{\mathbf{a}} \tilde{\boldsymbol{\theta}}_h^{\mathbf{a}} - \mathbf{v}_h\|_{\boldsymbol{\mu}, K}.$$

We can now define $\boldsymbol{\theta}_h^{\mathbf{a}} := \tilde{\boldsymbol{\theta}}_h^{\mathbf{a}} - \hat{\boldsymbol{\theta}}_h^{\mathbf{a}}$, and the localized total current variation is given by

$$(3.9) \quad \mathbf{G}_h^{\mathbf{a}} := i\omega \psi^{\mathbf{a}} \mathbf{J}_h + \omega^2 \mathbf{D}_h^{\mathbf{a}} + \boldsymbol{\theta}_h^{\mathbf{a}}.$$

Notice that the element-wise mean-value constraints in (3.7), which may appear peculiar at first glance, are actually crucial to ensure that the minimization problem of second step, (3.8), is well-posed. This is detailed in [12] and analyzed in Section 5.2 below.

3.2.3. Magnetic field. Having properly decomposed the total current density, we can now easily introduce a local magnetic field reconstruction. Specifically, for each vertex $\mathbf{a} \in \mathcal{V}_h$, the magnetic field is constructed as

$$(3.10) \quad \mathbf{H}_h^{\mathbf{a}} := \arg \min_{\substack{\mathbf{v}_h \in \mathbf{N}_{p^{\mathbf{a}}+2}(\mathcal{T}_h^{\mathbf{a}}) \cap \mathbf{H}_0(\text{curl}, \omega^{\mathbf{a}}) \\ i\omega \nabla \times \mathbf{v}_h = \mathbf{G}_h^{\mathbf{a}}}} \|\psi^{\mathbf{a}} \nabla \times \mathbf{E}_h - i\omega \boldsymbol{\chi}^{-1} \mathbf{v}_h\|_{\boldsymbol{\chi}, \omega^{\mathbf{a}}}.$$

and we set

$$(3.11) \quad \mathbf{H}_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \mathbf{H}_h^{\mathbf{a}}.$$

3.2.4. Properties of the local reconstructions. We now summarize the key properties of our local reconstructions, namely, that they are well-posed and lead to suitable equilibrated fields.

Theorem 3.2 (Equilibrated local reconstructions). *The minimization problems (3.5), (3.7), (3.8) and (3.10) admit unique minimizers. The fields \mathbf{D}_h and \mathbf{H}_h defined by (3.6) and (3.11) satisfy the assumptions of Section 3.1 given in (3.1).*

3.3. Efficiency. In this section, we specifically consider the estimator $\eta_{\text{div}, K}$ and $\eta_{\text{curl}, K}$ defined by (3.2) with the fields \mathbf{D}_h and \mathbf{H}_h obtained from the construction presented in Section 3.2. Our key result is that the resulting estimator is locally efficient, with a constant independent of the polynomial degree distribution $p_{\mathcal{T}_h}$.

Theorem 3.3 (Efficiency). *The estimate*

$$(3.12) \quad \eta_K \lesssim_{\tilde{\mathcal{T}}_h, K} \left(1 + \frac{\omega h_K}{c_{\min, \tilde{\omega}_K}}\right) \mathcal{C}_{\varepsilon, \tilde{\omega}_K} \mathcal{C}_{\boldsymbol{\mu}, \tilde{\omega}_K}^{5/2} \|\mathbf{E} - \mathbf{E}_h\|_{\omega, \tilde{\omega}_K}$$

holds true for all $K \in \mathcal{T}_h$,

This result is established in Section 5, together with “vertex patch versions” of estimates (3.12) where the (unsummed) local contributions \mathbf{D}_h^α and \mathbf{H}_h^α appear.

3.4. Remarks. We close this section with some remarks on the main results.

(i) For each vertex patch \mathbf{a} , the local contributions to our reconstructions use uniformly the maximum polynomial degree p^α on the patch \mathcal{T}_h^α . It is possible to employ slightly cheaper local problems where p_K is used in each element instead of p^α . In this case, we can still show the constant-free reliability of the estimator and its efficiency, but we are not able to show the p -robustness of the lower bound.

(ii) We end up requiring an increase of polynomial degree by 2 to construct \mathbf{H}_h , as opposed to the traditional increase by one degree for equilibrated estimators. This could be remedied by using Brezzi–Douglas–Marini elements [4] instead of Raviart–Thomas elements [33], and the second family of Nédélec elements [31] instead of the first one [30].

(iii) For $\eta_{\text{div},K}$, the sharper efficiency bound

$$\eta_{\text{div},K} \lesssim_{\mathcal{T}_h,K} \mathcal{C}_{\varepsilon,\omega_K} \omega \|\mathbf{E} - \mathbf{E}_h\|_{\varepsilon,\omega_K}$$

is available for all $K \in \mathcal{T}_h$, as the proof of Theorem 3.3 shows.

(iv) Our magnetic field reconstruction closely follows the approach introduced in [12] for the case $\omega = 0$. As opposed to [12], we do require that $\mathcal{P}_1(\mathcal{T}_h) \cap \mathbf{H}_0(\mathbf{curl}, \Omega) \subset \mathbf{W}_h$. This condition is satisfied for the second-order Nédélec elements of the first-family and the lowest-order Nédélec elements of the second family, but not for the lowest-order Nédélec element of the first family. This additional requirement is due to the presence of the zero-order term in the model problem.

(v) The dependence on the contrast in the magnetic permeability, $\mathcal{C}_{\mu,\tilde{\omega}_K}$ seems suboptimal. In fact, defining $\boldsymbol{\theta}$ using unweighted norms leads to a linear scaling in $\mathcal{C}_{\mu,\tilde{\omega}_K}$ instead of order 5/2. This is readily checked by going through the proof. Nevertheless, we have chosen to define $\boldsymbol{\theta}$ with the weights, since it is the physically relevant norm.

(vi) To be physically consistent, we have added factors $i\omega$ in front of \mathbf{J}_h in the right-hand of (1.1) and in front of \mathbf{H}_h for its definition. Mathematically, however, this factor can be removed. This may be advantageous since then, if \mathbf{J}_h is real-valued, the solution \mathbf{E}_h and the reconstructions \mathbf{D}_h and \mathbf{E}_h may be computed using only real floating point numbers.

4. RELIABILITY

We start by establishing the reliability properties of our estimator stated in Theorem 3.1. In this section, the estimators η_{div} and η_{curl} of (3.3) can be obtained from any reconstructions \mathbf{D}_h and \mathbf{H}_h of the electric displacement and the magnetic field satisfying (3.1). In particular, these fields do not need to be provided by the local construction presented in Section 3.2.

Following [7, 9] the first step is to obtain an upper-bound for the “residual term”, that is, given $\mathbf{v} \in \mathbf{H}_0(\mathbf{curl}, \Omega)$ the quantity $|b(\mathbf{E} - \mathbf{E}_h, \mathbf{v})|$.

Lemma 4.1 (Residual term). *For all $\mathbf{v} \in \mathbf{H}_0(\mathbf{curl}, \Omega)$, we have*

$$(4.1a) \quad |b(\mathbf{E} - \mathbf{E}_h, \mathbf{v})| \leq \omega \eta_{\text{div}} \|\mathbf{v}\|_{\varepsilon,\Omega} + \eta_{\text{curl}} \|\nabla \times \mathbf{v}\|_{\chi,\Omega},$$

and

$$(4.1b) \quad |b(\mathbf{E} - \mathbf{E}_h, \mathbf{v})| \leq \eta \|\mathbf{v}\|_{\omega,\Omega}.$$

Proof. Let $\mathbf{v} \in \mathbf{H}_0(\mathbf{curl}, \Omega)$. We have

$$\begin{aligned} b(\mathbf{E} - \mathbf{E}_h, \mathbf{v}) &= (i\omega \mathbf{J}_h + \omega^2 \boldsymbol{\varepsilon} \mathbf{E}_h, \mathbf{v}) - (\boldsymbol{\chi} \boldsymbol{\nabla} \times \mathbf{E}_h, \boldsymbol{\nabla} \times \mathbf{v}) \\ &= \omega^2 (\boldsymbol{\varepsilon} \mathbf{E}_h - \mathbf{D}_h, \mathbf{v}) + (i\omega \mathbf{J} + \omega^2 \mathbf{D}_h, \mathbf{v}) - (\boldsymbol{\chi} \boldsymbol{\nabla} \times \mathbf{E}_h, \boldsymbol{\nabla} \times \mathbf{v}) \\ &= \omega^2 (\boldsymbol{\varepsilon} \mathbf{E}_h - \mathbf{D}_h, \mathbf{v}) + i\omega (\boldsymbol{\nabla} \times \mathbf{H}_h, \mathbf{v}) - (\boldsymbol{\chi} \boldsymbol{\nabla} \times \mathbf{E}_h, \boldsymbol{\nabla} \times \mathbf{v}) \\ &= \omega^2 (\boldsymbol{\varepsilon} \mathbf{E}_h - \mathbf{D}_h, \mathbf{v}) + (i\omega \mathbf{H}_h - \boldsymbol{\chi} \boldsymbol{\nabla} \times \mathbf{E}_h, \boldsymbol{\nabla} \times \mathbf{v}), \end{aligned}$$

and (4.1a) follows from Cauchy-Schwartz inequalities. Then, (4.1b) follows from (4.1a) since

$$\omega \eta_{\text{div}} \|\mathbf{v}\|_{\boldsymbol{\varepsilon}, \Omega} + \eta_{\text{curl}} \|\boldsymbol{\nabla} \times \mathbf{v}\|_{\boldsymbol{\chi}, \Omega} \leq (\eta_{\text{div}}^2 + \eta_{\text{curl}}^2)^{1/2} (\omega^2 \|\mathbf{v}\|_{\boldsymbol{\varepsilon}, \Omega}^2 + \|\boldsymbol{\nabla} \times \mathbf{v}\|_{\boldsymbol{\chi}, \Omega}^2)^{1/2}.$$

□

The next step of the reliability proof consists in employing the estimates in (4.1), as done in [7, 9]. Specifically, we follow the approach given in [9] for residual-based estimator, but we take special care to obtain clean constants as in [7] for the Helmholtz equation.

Introducing \mathbf{g} as the only element of $\boldsymbol{\Lambda}_\varepsilon^\perp(\Omega)$ such that

$$(\boldsymbol{\varepsilon} \mathbf{g}, \mathbf{v}) = (\boldsymbol{\varepsilon}(\mathbf{E} - \mathbf{E}_h), \mathbf{v}) \quad \forall \mathbf{v} \in \boldsymbol{\Lambda}_\varepsilon^\perp(\Omega)$$

we arrive the Helmholtz-Hodge decomposition of the error

$$\mathbf{E} - \mathbf{E}_h = \mathbf{g} + \boldsymbol{\theta}$$

with $\boldsymbol{\theta} \in \mathbf{H}_0(\mathbf{curl}, \Omega) \cap \boldsymbol{\Lambda}_\varepsilon(\Omega)$. We have

$$(4.2) \quad \|\mathbf{E} - \mathbf{E}_h\|_{\boldsymbol{\varepsilon}, \Omega}^2 = \|\mathbf{g}\|_{\boldsymbol{\varepsilon}, \Omega}^2 + \|\boldsymbol{\theta}\|_{\boldsymbol{\varepsilon}, \Omega}^2.$$

The two terms in the right-hand side of (4.2) are then estimated separately. The letter \mathbf{g} is chosen to remind us that in the case of a simply-connected domain, we infact have $\mathbf{g} = \boldsymbol{\nabla} q$ for some $q \in H_0^1(\Omega)$.

Lemma 4.2 (Error estimates in $L^2(\Omega)$ norm). *We have*

$$(4.3a) \quad \omega \|\boldsymbol{\theta}\|_{\boldsymbol{\varepsilon}, \Omega} \leq \gamma_{\text{ba}} \eta$$

and

$$(4.3b) \quad \omega \|\mathbf{g}\|_{\boldsymbol{\varepsilon}, \Omega} \leq \eta_{\text{div}}.$$

Proof. We introduce $\boldsymbol{\xi}$ as the only element of $\mathbf{H}_0(\mathbf{curl}, \Omega)$ such that

$$b(\mathbf{w}, \boldsymbol{\xi}) = \omega(\boldsymbol{\varepsilon} \mathbf{w}, \boldsymbol{\theta}) \quad \forall \mathbf{w} \in \mathbf{H}_0(\mathbf{curl}, \Omega).$$

We then have

$$\omega \|\boldsymbol{\theta}\|_{\boldsymbol{\varepsilon}, \Omega}^2 = (\boldsymbol{\varepsilon}(\mathbf{E} - \mathbf{E}_h), \boldsymbol{\theta}) = b(\mathbf{E} - \mathbf{E}_h, \boldsymbol{\xi}) = b(\mathbf{E} - \mathbf{E}_h, \boldsymbol{\xi} - \mathbf{v}_h)$$

for all $\mathbf{v}_h \in \mathbf{W}_h$. Recalling that $\boldsymbol{\theta} \in \boldsymbol{\Lambda}_\varepsilon(\Omega)$, (4.1b) and (2.5) show that

$$\omega \|\boldsymbol{\theta}\|_{\boldsymbol{\varepsilon}, \Omega}^2 \leq \gamma_{\text{ba}} \eta \|\boldsymbol{\theta}\|_{\boldsymbol{\varepsilon}, \Omega},$$

and (4.3a) follows.

On the other hand, since $\boldsymbol{\nabla} \times \mathbf{g} = \mathbf{0}$, we have

$$\omega^2 \|\mathbf{g}\|_{\boldsymbol{\varepsilon}, \Omega}^2 = \omega^2 (\boldsymbol{\varepsilon}(\mathbf{E} - \mathbf{E}_h), \mathbf{g}) = -b(\mathbf{E} - \mathbf{E}_h, \mathbf{g}) \leq \omega \eta_{\text{div}} \|\mathbf{g}\|_{\boldsymbol{\varepsilon}, \Omega},$$

and (4.3b) follows. □

We are now fully equipped to establish Theorem 3.1. This is done by combining (4.1), (4.3) and the Gårding inequality satisfied by the bilinear form $b(\cdot, \cdot)$.

Proof of Theorem 3.1. Estimate (3.4a) simply follows from inf-sup condition (2.2) combined with (4.1b). For (3.4b), we have

$$\begin{aligned} \|\nabla \times \boldsymbol{\theta}\|_{\mathcal{X}, \Omega}^2 &= b(\boldsymbol{\theta}, \boldsymbol{\theta}) + \omega^2 \|\boldsymbol{\theta}\|_{\boldsymbol{\varepsilon}, \Omega}^2 \\ &= b(\mathbf{E} - \mathbf{E}_h, \boldsymbol{\theta}) + \omega^2 \|\boldsymbol{\theta}\|_{\boldsymbol{\varepsilon}, \Omega}^2 \\ &\leq \omega \eta_{\text{div}} \|\boldsymbol{\theta}\|_{\boldsymbol{\varepsilon}, \Omega} + \eta_{\text{curl}} \|\nabla \times \boldsymbol{\theta}\|_{\mathcal{X}, \Omega} + \omega^2 \|\boldsymbol{\theta}\|_{\boldsymbol{\varepsilon}, \Omega}^2 \\ &\leq \gamma_{\text{ba}} \eta_{\text{div}} \eta + \gamma_{\text{ba}}^2 \eta^2 + \eta_{\text{curl}} \|\nabla \times \boldsymbol{\theta}\|_{\mathcal{X}, \Omega}, \end{aligned}$$

and it follows from Young's inequality that

$$(4.4) \quad \|\nabla \times \boldsymbol{\theta}\|_{\mathcal{X}, \Omega}^2 \leq \eta_{\text{curl}}^2 + 2\gamma_{\text{ba}} \eta_{\text{div}} \eta + 2\gamma_{\text{ba}}^2 \eta^2.$$

Then, (3.4b) is a direct consequence of (4.3a), (4.3b) and (4.4) since

$$\|\mathbf{E} - \mathbf{E}_h\|_{\boldsymbol{\omega}, \Omega}^2 = \omega^2 \|\boldsymbol{\theta}\|_{\boldsymbol{\varepsilon}, \Omega}^2 + \omega^2 \|\boldsymbol{g}\|_{\boldsymbol{\varepsilon}, \Omega}^2 + \|\nabla \times \boldsymbol{\theta}\|_{\mathcal{X}, \Omega}^2$$

and $\eta^2 + 2\gamma_{\text{ba}} \eta_{\text{div}} \eta + 3\gamma_{\text{ba}}^2 \eta^2 \leq (1 + 2\gamma_{\text{ba}} + 3\gamma_{\text{ba}}^2) \eta^2$. \square

5. EFFICIENCY

In this section, we show that the localized reconstruction of the fields \mathbf{D}_h and \mathbf{H}_h introduced in Section 3.2 are well-defined. We then establish Theorem 3.3 which states that the corresponding estimators $\{\eta_K\}_{K \in \mathcal{T}_h}$ are locally efficient.

5.1. Electric displacement reconstruction. We first address the electric displacement \mathbf{D}_h . Recall that it is defined as the sum of patch-wise contributions $\mathbf{D}_h^{\mathbf{a}}$ that solve the local minimization problems in (3.5). Following [19], we start by analyzing a ‘‘continuous version’’ of (3.5), where the Raviart–Thomas space is replaced by the full $\mathbf{H}(\text{div}, \omega^{\mathbf{a}})$ space.

Lemma 5.1 (Idealized electric displacement reconstruction). *For all $\mathbf{a} \in \mathcal{V}_h$, there exists a unique minimizer*

$$(5.1) \quad \mathbf{D}^{\mathbf{a}} := \arg \min_{\substack{\mathbf{v} \in \mathbf{H}_0(\text{div}, \omega^{\mathbf{a}}) \\ -\omega^2 \nabla \cdot \mathbf{v} = i\omega \psi^{\mathbf{a}} \nabla \cdot \mathbf{J}_h - \omega^2 \nabla \psi^{\mathbf{a}} \cdot (\boldsymbol{\varepsilon} \mathbf{E}_h)}} \|\psi^{\mathbf{a}} \mathbf{E}_h - \boldsymbol{\varepsilon}^{-1} \mathbf{v}\|_{\boldsymbol{\varepsilon}, \omega^{\mathbf{a}}}.$$

In addition, we have

$$(5.2) \quad \|\psi^{\mathbf{a}} \mathbf{E}_h - \boldsymbol{\varepsilon}^{-1} \mathbf{D}^{\mathbf{a}}\|_{\boldsymbol{\varepsilon}, \omega^{\mathbf{a}}} \lesssim \sqrt{\mathcal{C}_{\boldsymbol{\varepsilon}, \omega^{\mathbf{a}}}} \|\mathbf{E} - \mathbf{E}_h\|_{\boldsymbol{\varepsilon}, \omega^{\mathbf{a}}}.$$

Proof. We first check that the minimization set of (5.1) is non-empty. If $\mathbf{a} \in \partial\Omega$, the constraint on the normal trace is not imposed on the whole boundary of the patch, so that it is always non-empty. For $\mathbf{a} \in \Omega$, the usual compatibility condition

$$\begin{aligned} (i\omega \psi^{\mathbf{a}} \nabla \cdot \mathbf{J}_h - \omega^2 \nabla \psi^{\mathbf{a}} \cdot (\boldsymbol{\varepsilon} \mathbf{E}_h), 1)_{\omega^{\mathbf{a}}} &= i\omega (\nabla \cdot \mathbf{J}_h, \psi^{\mathbf{a}})_{\omega^{\mathbf{a}}} - \omega^2 (\boldsymbol{\varepsilon} \mathbf{E}_h, \nabla \psi^{\mathbf{a}})_{\omega^{\mathbf{a}}} \\ &= -i\omega (\mathbf{J}_h, \nabla \psi^{\mathbf{a}})_{\omega^{\mathbf{a}}} + b(\mathbf{E}_h, \nabla \psi^{\mathbf{a}}) \\ &= 0 \end{aligned}$$

holds true, since $\nabla \psi^{\mathbf{a}} \in \mathbf{N}_0(\mathcal{T}_h) \cap \mathbf{H}_0(\text{curl}, \Omega) \subset \mathbf{W}_h$. Then, the existence and uniqueness of the minimizer $\mathbf{D}^{\mathbf{a}}$ follows from standard convexity argument.

The Euler-Lagrange equations associated with (5.1) show the existence (and uniqueness) of $q \in L^2_\star(\omega^\alpha)$ such that

$$\begin{cases} (\psi^\alpha \mathbf{E}_h - \varepsilon^{-1} \mathbf{D}^\alpha, \mathbf{v})_{\omega^\alpha} + (q, \nabla \cdot \mathbf{v})_{\omega^\alpha} = 0 \\ -\omega^2 (\nabla \cdot \mathbf{D}^\alpha, r)_{\omega^\alpha} = (i\omega \psi^\alpha \nabla \cdot \mathbf{J}_h - \omega^2 \nabla \psi^\alpha \cdot (\varepsilon \mathbf{E}_h), r)_{\omega^\alpha} \end{cases}$$

for all $\mathbf{v} \in \mathbf{H}_0(\text{div}, \omega^\alpha)$ and $r \in L^2_\star(\omega^\alpha)$, and we readily see that $q \in H^1_\dagger(\omega^\alpha) \cap L^2_\star(\omega^\alpha)$, with

$$(5.3) \quad -\omega^2 \varepsilon \nabla q = -\omega^2 \psi^\alpha \varepsilon \mathbf{E}_h + \omega^2 \mathbf{D}^\alpha.$$

As a result, we have

$$\|\psi^\alpha \mathbf{E}_h - \varepsilon^{-1} \mathbf{D}^\alpha\|_{\varepsilon, \omega^\alpha} = \|\nabla q\|_{\varepsilon, \omega^\alpha}.$$

On the other hand, taking the $L^2(\Omega)$ inner product of both sides of (5.3) with ∇q , we have

$$-\omega^2 \|\nabla q\|_{\varepsilon, \omega^\alpha}^2 = (i\omega \psi^\alpha \nabla \cdot \mathbf{J}_h - \omega^2 \nabla \psi^\alpha \cdot (\varepsilon \mathbf{E}_h), q)_{\omega^\alpha} + \omega^2 (\psi^\alpha \varepsilon \mathbf{E}_h, \nabla q)_{\omega^\alpha},$$

since $\omega^2 \nabla \cdot \mathbf{D}^\alpha = i\omega \psi^\alpha \nabla \cdot \mathbf{J}_h - \omega^2 \nabla \psi^\alpha \cdot (\varepsilon \mathbf{E}_h)$. We then remark that

$$\begin{aligned} \omega^2 (\psi^\alpha \varepsilon \mathbf{E}, \nabla q)_{\omega^\alpha} &= -\omega^2 (\nabla \cdot (\psi^\alpha \varepsilon \mathbf{E}), q)_{\omega^\alpha} = -\omega^2 (\psi^\alpha \nabla \cdot (\varepsilon \mathbf{E}), q)_{\omega^\alpha} - \omega^2 (\nabla \psi^\alpha \cdot (\varepsilon \mathbf{E}), q)_{\omega^\alpha} \\ &= i\omega (\psi^\alpha \nabla \cdot \mathbf{J}_h, q)_{\omega^\alpha} - \omega^2 (\nabla \psi^\alpha \cdot (\varepsilon \mathbf{E}), q)_{\omega^\alpha}, \end{aligned}$$

and therefore

$$-\omega^2 \|\nabla q\|_{\varepsilon, \omega^\alpha}^2 = \omega^2 (\nabla \psi^\alpha \cdot (\varepsilon \mathbf{E} - \mathbf{E}_h), q)_{\omega^\alpha} - \omega^2 (\psi^\alpha \varepsilon (\mathbf{E} - \mathbf{E}_h), \nabla q)_{\omega^\alpha}.$$

It follows that

$$\|\nabla q\|_{\varepsilon, \omega^\alpha}^2 \leq \|\mathbf{E} - \mathbf{E}_h\|_{\varepsilon, \omega^\alpha} \|q \nabla \psi^\alpha - \psi^\alpha \nabla q\|_{\varepsilon, \omega^\alpha}.$$

The conclusion then follow since on the one hand,

$$\begin{aligned} \|q \nabla \psi^\alpha - \psi^\alpha \nabla q\|_{\varepsilon, \omega^\alpha} &\leq \|q \nabla \psi^\alpha\|_{\varepsilon, \omega^\alpha} + \|\psi^\alpha \nabla q\|_{\varepsilon, \omega^\alpha} \\ &\leq \sqrt{\varepsilon_{\max, \omega^\alpha}} \|q \nabla \psi^\alpha\|_{\omega^\alpha} + \|\nabla q\|_{\varepsilon, \omega^\alpha} \\ &\leq \sqrt{\varepsilon_{\max, \omega^\alpha}} \|\nabla \psi^\alpha\|_{L^\infty(\omega^\alpha)} \|q\|_{\omega^\alpha} + \|\nabla q\|_{\varepsilon, \omega^\alpha}, \end{aligned}$$

and on the other hand

$$\|q\|_{\omega^\alpha} \lesssim_{\mathcal{T}_h^\alpha} h_{\omega^\alpha} \|\nabla q\|_{\omega^\alpha} \leq h_{\omega^\alpha} \frac{1}{\sqrt{\varepsilon_{\min, \omega^\alpha}}} \|\nabla q\|_{\varepsilon, \omega^\alpha}.$$

where we employed (2.9) and (2.8). \square

Theorem 5.2 (Discrete electric displacement reconstruction). *Problem (3.5) admits a unique minimizer. In addition, we have*

$$(5.4) \quad \|\psi^\alpha \mathbf{E}_h - \varepsilon^{-1} \mathbf{D}_h^\alpha\|_{\varepsilon, \omega^\alpha} \lesssim_{\mathcal{T}_h^\alpha} \mathcal{C}_{\varepsilon, \omega^\alpha} \|\mathbf{E} - \mathbf{E}_h\|_{\varepsilon, \omega^\alpha}.$$

Proof. Let us denote by $g^\alpha := i\omega \psi^\alpha \nabla \cdot \mathbf{J}_h - \omega^2 \nabla \psi^\alpha \cdot (\varepsilon \mathbf{E}_h)$ the divergence constraint in (3.5). We already showed in Lemma 5.1 that the divergence constraint satisfies the compatibility condition $(g^\alpha, 1)_{\omega^\alpha} = 0$ when $\mathbf{a} \in \Omega$. Since in addition $g^\alpha \in \mathcal{P}_{p+1}(\mathcal{T}_h^\alpha)$, the minimization set of (3.5) is non-empty, and the existence and uniqueness of \mathbf{D}_h^α follows from convexity considerations.

The same reasoning also applies to the problems with unweighted norms, and we can introduce

$$\mathbf{v}^\star := \arg \min_{\substack{\mathbf{v} \in \mathbf{H}_0(\text{div}, \omega^\alpha) \\ -\omega^2 \nabla \cdot \mathbf{v} = i\omega \mathbf{J}_h - \omega^2 \psi^\alpha \cdot (\varepsilon \mathbf{E}_h)}} \|\psi^\alpha \varepsilon \mathbf{E}_h - \mathbf{v}\|_{\omega^\alpha}$$

and

$$\mathbf{v}_h^* := \arg \min_{\substack{\mathbf{v}_h \in \mathbf{RT}_{p+2}(\omega^a) \cap \mathbf{H}_0(\operatorname{div}, \omega^a) \\ -\omega^2 \nabla \cdot \mathbf{v}_h = i\omega \mathbf{J}_h - \omega^2 \psi^a \cdot (\boldsymbol{\varepsilon} \mathbf{E}_h)}} \|\psi^a \boldsymbol{\varepsilon} \mathbf{E}_h - \mathbf{v}_h\|_{\omega^a}.$$

Observing in addition that $\psi^a \boldsymbol{\varepsilon} \mathbf{E}_h \in \mathcal{P}_{p+2}(\mathcal{T}_h^a) \subset \mathbf{RT}_{p+2}(\mathcal{T}_h^a)$, [11, Proposition 3.1] (see also [20]) shows that

$$\|\psi^a \boldsymbol{\varepsilon} \mathbf{E}_h - \mathbf{v}_h^*\|_{\omega^a} \lesssim_{\mathcal{T}_h^a} \|\psi^a \boldsymbol{\varepsilon} \mathbf{E}_h - \mathbf{v}^*\|_{\omega^a}.$$

We can now conclude since on the one hand

$$\begin{aligned} \|\psi^a \boldsymbol{\varepsilon} \mathbf{E}_h - \varepsilon^{-1} \mathbf{D}_h^a\|_{\varepsilon, \omega^a}^2 &\leq \|\psi^a \boldsymbol{\varepsilon} \mathbf{E}_h - \varepsilon^{-1} \mathbf{v}_h^*\|_{\varepsilon, \omega^a}^2 \\ &= \|\psi^a \boldsymbol{\varepsilon} \mathbf{E}_h - \mathbf{v}_h^*\|_{\varepsilon^{-1}, \omega^a}^2 \leq \frac{1}{\varepsilon_{\min, \omega^a}} \|\psi^a \boldsymbol{\varepsilon} \mathbf{E}_h - \mathbf{v}_h^*\|_{\omega^a}^2 \end{aligned}$$

and on the other hand

$$\|\psi^a \boldsymbol{\varepsilon} \mathbf{E}_h - \mathbf{v}^*\|_{\omega^a} \leq \|\psi^a \boldsymbol{\varepsilon} \mathbf{E}_h - \mathbf{D}^a\|_{\omega^a} \leq \sqrt{\varepsilon_{\max, \omega^a}} \|\psi^a \boldsymbol{\varepsilon} \mathbf{E}_h - \varepsilon^{-1} \mathbf{D}^a\|_{\varepsilon, \omega^a},$$

where \mathbf{D}^a is the minimizer in the left-hand side of (5.1). We now conclude with (5.2). \square

5.2. Total current variation. We next analyze the total current variation \mathbf{G}_h^a from (3.9). The key point is the intermediate quantity $\boldsymbol{\theta}_h^a$. The proofs of this section heavily rely on [12, Appendix A]. For $\mathbf{a} \in \mathcal{V}_h$ we thus introduce

$$(5.5) \quad g^a := -\nabla \psi^a \cdot (i\omega \mathbf{J}_h + \omega^2 \boldsymbol{\varepsilon} \mathbf{E}_h) \quad \boldsymbol{\tau}_h^a := \nabla \psi^a \times (\boldsymbol{\chi} \nabla \times \mathbf{E}_h)$$

for the two constraints appearing in (3.7). We start by introducing a continuous version of $\tilde{\boldsymbol{\theta}}_h^a$.

Lemma 5.3 (Idealized intermediate problem). *There exists a unique minimizer*

$$(5.6) \quad \tilde{\boldsymbol{\theta}}^a := \arg \min_{\substack{\mathbf{v} \in \mathbf{H}_0(\operatorname{div}, \omega^a) \\ \nabla \cdot \mathbf{v} = g^a}} \|\mathbf{v} - \boldsymbol{\tau}_h^a\|_{\boldsymbol{\mu}, \omega^a},$$

and the following estimate holds true:

$$(5.7) \quad \|\tilde{\boldsymbol{\theta}}^a - \boldsymbol{\tau}_h^a\|_{\boldsymbol{\mu}, \omega^a} \lesssim_{\mathcal{T}_h^a} \sqrt{\mathcal{C}_{\boldsymbol{\mu}, \omega^a}} h_{\omega^a}^{-1} \left(1 + \frac{\omega h_{\omega^a}}{c_{\min, \omega^a}}\right) \|\mathbf{E} - \mathbf{E}_h\|_{\omega, \omega^a}.$$

Proof. The existence and uniqueness of the minimizer $\tilde{\boldsymbol{\theta}}^a$ in (5.6) follows from the usual compatibility condition. If $\mathbf{a} \in \partial\Omega$, there is nothing to check. On the other hand, if $\mathbf{a} \notin \partial\Omega$, then $\nabla \psi^a \in \mathbf{W}_h$, and therefore

$$(g^a, 1)_{\omega^a} = -i\omega (\mathbf{J}_h, \nabla \psi^a)_{\omega^a} + \omega^2 (\boldsymbol{\varepsilon} \mathbf{E}_h, \nabla \psi^a)_{\omega^a} = -i\omega (\mathbf{J}_h, \nabla \psi^a)_{\Omega} + b(\mathbf{E}_h, \nabla \psi^a) = 0.$$

The Euler-Lagrange equations for the right-hand side of (5.6) ensures the existence and uniqueness of $q \in L_*^2(\omega^a)$ such that

$$\begin{cases} (\boldsymbol{\mu}(\tilde{\boldsymbol{\theta}}^a - \boldsymbol{\tau}_h^a), \mathbf{w})_{\omega^a} + (q, \nabla \cdot \mathbf{w})_{\omega^a} = 0 \\ (\nabla \cdot \tilde{\boldsymbol{\theta}}^a, r)_{\omega^a} = (g^a, r)_{\omega^a} \end{cases}$$

for all $\mathbf{w} \in \mathbf{H}_0(\text{div}, \omega^{\mathbf{a}})$ and $r \in L^2(\omega^{\mathbf{a}})$. We easily deduce that $q \in H_{\dagger}^1(\omega^{\mathbf{a}}) \cap L_{\star}^2(\omega^{\mathbf{a}})$ with $\chi \nabla q = \tilde{\boldsymbol{\theta}}^{\mathbf{a}} - \boldsymbol{\tau}_h^{\mathbf{a}}$. Then, we have

$$\begin{aligned} \|\nabla q\|_{\chi, \omega^{\mathbf{a}}}^2 &= (\tilde{\boldsymbol{\theta}}^{\mathbf{a}} - \boldsymbol{\tau}_h^{\mathbf{a}}, \nabla q)_{\omega^{\mathbf{a}}} = -(\nabla \cdot \tilde{\boldsymbol{\theta}}^{\mathbf{a}}, q)_{\omega^{\mathbf{a}}} - (\boldsymbol{\tau}_h^{\mathbf{a}}, \nabla q)_{\omega^{\mathbf{a}}} = -(g^{\mathbf{a}}, q)_{\omega^{\mathbf{a}}} - (\boldsymbol{\tau}_h^{\mathbf{a}}, \nabla q)_{\omega^{\mathbf{a}}} \\ &= (i\omega \mathbf{J}_h + \omega^2 \boldsymbol{\varepsilon} \mathbf{E}_h) \cdot \nabla \psi^{\mathbf{a}}, q)_{\omega^{\mathbf{a}}} - ((\chi \nabla \times \mathbf{E}_h) \times \nabla \psi^{\mathbf{a}}, \nabla q)_{\omega^{\mathbf{a}}} \\ &= (i\omega \mathbf{J}_h + \omega^2 \boldsymbol{\varepsilon} \mathbf{E}_h, q \nabla \psi^{\mathbf{a}})_{\omega^{\mathbf{a}}} - ((\chi \nabla \times \mathbf{E}_h), \nabla \psi^{\mathbf{a}} \times \nabla q)_{\omega^{\mathbf{a}}} \\ &= i\omega (\mathbf{J}_h, q \nabla \psi^{\mathbf{a}}) + \omega^2 (\boldsymbol{\varepsilon} \mathbf{E}_h, q \nabla \psi^{\mathbf{a}})_{\omega^{\mathbf{a}}} - ((\chi \nabla \times \mathbf{E}_h), \nabla \times (q \nabla \psi^{\mathbf{a}}))_{\omega^{\mathbf{a}}} \\ &= i\omega (\mathbf{J}_h, q \nabla \psi^{\mathbf{a}}) - b(\mathbf{E} - \mathbf{E}_h, q \nabla \psi^{\mathbf{a}})_{\omega^{\mathbf{a}}} \\ &= b(\mathbf{E} - \mathbf{E}_h, q \nabla \psi^{\mathbf{a}})_{\omega^{\mathbf{a}}} \\ &\leq \|\mathbf{E} - \mathbf{E}_h\|_{\omega, \omega^{\mathbf{a}}} \|q \nabla \psi^{\mathbf{a}}\|_{\omega, \omega^{\mathbf{a}}}, \end{aligned}$$

and since $\|\nabla q\|_{\chi, \omega^{\mathbf{a}}} = \|\tilde{\boldsymbol{\theta}}^{\mathbf{a}} - \boldsymbol{\tau}_h^{\mathbf{a}}\|_{\mu, \omega^{\mathbf{a}}}$, (5.7) follows with

$$\begin{aligned} \|q \nabla \psi^{\mathbf{a}}\|_{\omega, \omega^{\mathbf{a}}} &\lesssim \mathcal{T}_h^{\mathbf{a}} \omega \|q \nabla \psi^{\mathbf{a}}\|_{\varepsilon, \omega^{\mathbf{a}}} + \|\nabla q \times \nabla \psi^{\mathbf{a}}\|_{\chi, \omega^{\mathbf{a}}} \\ &\lesssim \mathcal{T}_h^{\mathbf{a}} \|\nabla \psi^{\mathbf{a}}\|_{L^{\infty}(\omega^{\mathbf{a}})} (\omega \sqrt{\varepsilon_{\max, \omega^{\mathbf{a}}}} \|q\|_{\omega^{\mathbf{a}}} + \sqrt{\chi_{\max, \omega^{\mathbf{a}}} \|\nabla q\|_{\omega^{\mathbf{a}}}}) \\ &\lesssim \mathcal{T}_h^{\mathbf{a}} \sqrt{\chi_{\max, \omega^{\mathbf{a}}}} \|\nabla \psi^{\mathbf{a}}\|_{L^{\infty}(\omega^{\mathbf{a}})} \left(1 + \omega h_{\omega^{\mathbf{a}}} \frac{\sqrt{\varepsilon_{\max, \omega^{\mathbf{a}}}}}{\sqrt{\chi_{\max, \omega^{\mathbf{a}}}}}\right) \|\nabla q\|_{\omega^{\mathbf{a}}} \\ &\lesssim \mathcal{T}_h^{\mathbf{a}} \sqrt{\mathcal{C}_{\mu, \omega^{\mathbf{a}}}} \|\nabla \psi^{\mathbf{a}}\|_{L^{\infty}(\omega^{\mathbf{a}})} \left(1 + \omega h_{\omega^{\mathbf{a}}} \frac{\sqrt{\varepsilon_{\max, \omega^{\mathbf{a}}}}}{\sqrt{\chi_{\max, \omega^{\mathbf{a}}}}}\right) \|\nabla q\|_{\chi, \omega^{\mathbf{a}}}. \end{aligned}$$

□

Lemma 5.4 (Discrete intermediate problem). *Problem (3.7) admits a unique minimizer $\tilde{\boldsymbol{\theta}}_h^{\mathbf{a}}$. In addition, we have*

$$(5.8) \quad \|\nabla \psi^{\mathbf{a}} \times (\chi \nabla \times \mathbf{E}_h) - \tilde{\boldsymbol{\theta}}_h^{\mathbf{a}}\|_{\mu, \omega^{\mathbf{a}}} \lesssim \mathcal{T}_h^{\mathbf{a}} \mathcal{C}_{\mu, \omega^{\mathbf{a}}} h_{\omega^{\mathbf{a}}}^{-1} \left(1 + \frac{\omega h_{\omega^{\mathbf{a}}}}{c_{\min, \omega^{\mathbf{a}}}}\right) \|\mathbf{E} - \mathbf{E}_h\|_{\omega, \omega^{\mathbf{a}}}.$$

Proof. Recalling the definition of $g^{\mathbf{a}}$ and $\boldsymbol{\tau}_h^{\mathbf{a}}$ at (5.5), we see that Problem (3.7) takes the form

$$\min_{\substack{\mathbf{v}_h \in \mathbf{RT}_{p, \mathbf{a}+1}(\mathcal{T}_h^{\mathbf{a}}) \cap \mathbf{H}_0(\text{div}, \omega^{\mathbf{a}}) \\ \nabla \cdot \mathbf{v}_h = g^{\mathbf{a}} \\ (\mathbf{v}_h, \boldsymbol{\tau})_{\omega^{\mathbf{a}}} = (\boldsymbol{\tau}_h^{\mathbf{a}}, \boldsymbol{\tau})_{\omega^{\mathbf{a}}} \quad \forall \boldsymbol{\tau} \in \mathcal{P}_0(\mathcal{T}_h^{\mathbf{a}})}} \|\mathbf{v}_h - \boldsymbol{\tau}_h^{\mathbf{a}}\|_{\mu, \omega^{\mathbf{a}}},$$

treated in [12, Appendix A]. The analysis in [12] is performed without the weight μ in the norm, but the analysis holds true here, up to adding $\sqrt{\mathcal{C}_{\mu, \omega^{\mathbf{a}}}}$ as a multiplicative factor. As a result, we simply need to check that the conditions in [12, Assumption A.1] are satisfied. First (i), it is clear that $g^{\mathbf{a}} \in L^2(\omega^{\mathbf{a}})$. Second (ii), if $\mathbf{a} \notin \partial\Omega$, then $\nabla \psi^{\mathbf{a}} \in \mathbf{W}_h$, and we have

$$(g^{\mathbf{a}}, 1)_{\omega^{\mathbf{a}}} = -\omega^2 (\boldsymbol{\varepsilon} \mathbf{E}_h, \nabla \psi^{\mathbf{a}}) - i\omega (\mathbf{J}_h, \nabla \psi^{\mathbf{a}}) = b(\mathbf{E}_h, \nabla \psi^{\mathbf{a}}) - i\omega (\mathbf{J}_h, \nabla \psi^{\mathbf{a}}) = 0.$$

Finally (iii), fix $q_h \in \mathcal{P}_1(\mathcal{T}_h^{\mathbf{a}}) \cap H_{\dagger}^1(\omega^{\mathbf{a}}) \cap L_{\star}^2(\omega^{\mathbf{a}})$, and let $\mathbf{w}_h := q_h \nabla \psi^{\mathbf{a}}$. Since $\nabla \psi^{\mathbf{a}} \in \mathbf{N}_0(\mathcal{T}_h) \cap \mathbf{H}(\text{curl}, \omega^{\mathbf{a}})$, it is clear that $\mathbf{w}_h \in \mathcal{P}_1(\mathcal{T}_h^{\mathbf{a}}) \cap \mathbf{H}(\text{curl}, \omega^{\mathbf{a}})$. In addition, combining the boundary conditions satisfied by $\nabla \psi^{\mathbf{a}}$ and q_h , we see that $\mathbf{w}_h \in \mathcal{P}_1(\mathcal{T}_h^{\mathbf{a}}) \cap \mathbf{H}_0(\text{curl}, \omega^{\mathbf{a}}) \subset \mathbf{W}_h$. As a result, we can employ (2.3), and since $\nabla \times \mathbf{w}_h = \nabla q_h \times \nabla \psi^{\mathbf{a}}$, we have

$$\begin{aligned} (\boldsymbol{\tau}_h^{\mathbf{a}}, \nabla q_h)_{\omega^{\mathbf{a}}} &= (\chi \nabla \times \mathbf{E}_h \times \nabla \psi^{\mathbf{a}}, \nabla q_h) = (\chi \nabla \times \mathbf{E}_h, \nabla \times \mathbf{w}_h) \\ &= i\omega (\mathbf{J}_h, \mathbf{w}_h) + \omega^2 (\boldsymbol{\varepsilon} \mathbf{E}_h, \mathbf{w}_h) = -(g^{\mathbf{a}}, q_h)_{\omega^{\mathbf{a}}}. \end{aligned}$$

We can thus invoke [12, Theorem A.2], which ensures the existence and uniqueness of $\tilde{\boldsymbol{\theta}}_h^a$ and provides the estimate

$$\|\tilde{\boldsymbol{\theta}}_h^a - \boldsymbol{\tau}_h^a\|_{\boldsymbol{\mu}, \omega^a} \lesssim_{\mathcal{T}_h^a} \sqrt{\mathcal{C}_{\boldsymbol{\mu}, \omega^a}} \|\tilde{\boldsymbol{\theta}}^a - \boldsymbol{\tau}_h^a\|_{\boldsymbol{\mu}, \omega^a},$$

where, again, the presence of $\mathcal{C}_{\boldsymbol{\mu}, \omega^a}$ follows from the use of the weighted norm. Then, (5.8) follows from (5.7). \square

Lemma 5.5 (Key properties of $\hat{\boldsymbol{\theta}}_h^a$). *Problem (3.8) admits a unique minimizer $\hat{\boldsymbol{\theta}}_h^a|_K$. For each $\mathbf{a} \in \mathcal{V}_h$, the resulting function $\hat{\boldsymbol{\theta}}_h^a$ satisfies*

$$\hat{\boldsymbol{\theta}}_h^a \in \mathbf{H}_0(\operatorname{div}, \omega^a) \quad \nabla \cdot \hat{\boldsymbol{\theta}}_h^a = 0,$$

and we have

$$\tilde{\boldsymbol{\theta}}_h = \sum_{\mathbf{a} \in \mathcal{V}_h} \hat{\boldsymbol{\theta}}_h^a.$$

In addition, the estimate

$$(5.9) \quad \|\hat{\boldsymbol{\theta}}_h^a\|_{\boldsymbol{\mu}, K} \lesssim_{\mathcal{T}_h^a} \sqrt{\mathcal{C}_{\boldsymbol{\mu}, K}} \|\tilde{\boldsymbol{\theta}}_h\|_{\boldsymbol{\mu}, K}$$

holds true.

Proof. The above results are immediate consequences of [12, Theorem B.1]. As above, the results in [12] are given for an unweighted norm, which is corrected here by the presence of the multiplicative factor $\sqrt{\mathcal{C}_{\boldsymbol{\mu}, K}}$ in (5.9). We rapidly check the assumptions of theorem, namely that $\tilde{\boldsymbol{\theta}}_h \in \mathbf{H}(\operatorname{div}, \omega^a)$ and that for all elements $K \in \mathcal{T}_h$

$$\begin{aligned} \nabla \cdot \tilde{\boldsymbol{\theta}}_h|_K &= -\nabla \cdot \left(\sum_{\mathbf{a} \in \mathcal{V}(K)} [\nabla \psi^{\mathbf{a}} \cdot (i\omega \mathbf{J}_h + \omega^2 \boldsymbol{\varepsilon} \mathbf{E}_h)] \right) \\ &= -\nabla \cdot \left(\left(\sum_{\mathbf{a} \in \mathcal{V}(K)} \nabla \psi^{\mathbf{a}} \right) \cdot (i\omega \mathbf{J}_h + \omega^2 \boldsymbol{\varepsilon} \mathbf{E}_h) \right) \\ &= -\nabla \cdot ((\nabla \mathbf{1}) \cdot (i\omega \mathbf{J}_h + \omega^2 \boldsymbol{\varepsilon} \mathbf{E}_h)) = 0, \end{aligned}$$

and

$$(\tilde{\boldsymbol{\theta}}_h, \mathbf{r})_K = \sum_{\mathbf{a} \in \mathcal{V}_h} (\hat{\boldsymbol{\theta}}_h^a, \mathbf{r})_K = 0$$

for all $\mathbf{r} \in \mathcal{P}_0(K)$. \square

We are now ready to give an estimate to the mismatch between \mathbf{G}_h^a and its continuous counterpart

$$\mathbf{G}^a := i\omega \psi^{\mathbf{a}} \mathbf{J}_h + \omega^2 \psi^{\mathbf{a}} \boldsymbol{\varepsilon} \mathbf{E} + \nabla \psi^{\mathbf{a}} \times (\boldsymbol{\chi} \nabla \times \mathbf{E}).$$

Theorem 5.6 (Total current variation reconstruction). *We have $\mathbf{G}_h^a \in \mathbf{RT}_{p^a+2}(\mathcal{T}_h^a) \cap \mathbf{H}_0(\operatorname{div}, \omega^a)$ with*

$$(5.10) \quad \nabla \cdot \mathbf{G}_h^a = 0.$$

In addition, the following estimate holds true:

$$(5.11) \quad \|\mathbf{G}^a - \mathbf{G}_h^a\|_{\boldsymbol{\mu}, \omega^a} \lesssim_{\tilde{\mathcal{T}}_h^a} h_{\omega^a}^{-1} \mathcal{C}_{\boldsymbol{\mu}, \tilde{\omega}^a}^{3/2} \mathcal{C}_{\boldsymbol{\varepsilon}, \tilde{\omega}^a} \left(1 + \frac{\omega h_{\omega^a}}{c_{\min, \tilde{\omega}^a}} \right) \|\mathbf{E} - \mathbf{E}_h\|_{\omega, \tilde{\omega}^a}.$$

Proof. The first part of the statement immediately follows from the fact that $\psi^{\mathbf{a}} \mathbf{J}_h \in \mathbf{RT}_{p^{\mathbf{a}}+1}(\mathcal{T}_h^{\mathbf{a}}) \cap \mathbf{H}_0(\operatorname{div}, \omega^{\mathbf{a}})$, $\mathbf{D}_h^{\mathbf{a}} \in \mathbf{RT}_{p^{\mathbf{a}}+2}(\mathcal{T}_h^{\mathbf{a}}) \cap \mathbf{H}_0(\operatorname{div}, \omega^{\mathbf{a}})$ and $\boldsymbol{\theta}_h^{\mathbf{a}} \in \mathbf{RT}_{p^{\mathbf{a}}+2}(\mathcal{T}_h^{\mathbf{a}}) \cap \mathbf{H}_0(\operatorname{div}, \omega^{\mathbf{a}})$. Then, recalling the divergence constraints in (3.5) and (3.7), we have

$$\begin{aligned} \nabla \cdot \mathbf{G}_h^{\mathbf{a}} &= i\omega \nabla \cdot (\psi^{\mathbf{a}} \mathbf{J}_h) + \omega^2 \nabla \cdot (\mathbf{D}_h^{\mathbf{a}}) + \nabla \cdot \boldsymbol{\theta}_h^{\mathbf{a}} = \\ &= i\omega (\nabla \psi^{\mathbf{a}} \cdot \mathbf{J}_h + \psi^{\mathbf{a}} \nabla \cdot \mathbf{J}_h) - (i\omega \psi^{\mathbf{a}} \nabla \cdot \mathbf{J}_h - \omega^2 \nabla \psi^{\mathbf{a}} \cdot (\boldsymbol{\varepsilon} \mathbf{E}_h)) \\ &\quad - \nabla \psi^{\mathbf{a}} \cdot (i\omega \mathbf{J}_h + \omega^2 \boldsymbol{\varepsilon} \mathbf{E}_h) = 0. \end{aligned}$$

and (5.10) follows.

We now establish (5.11). We have

$$\mathbf{G}^{\mathbf{a}} - \mathbf{G}_h^{\mathbf{a}} = \omega^2 (\psi^{\mathbf{a}} \boldsymbol{\varepsilon} \mathbf{E} - \mathbf{D}_h^{\mathbf{a}}) + (\nabla \psi^{\mathbf{a}} \times (\boldsymbol{\chi} \nabla \times \mathbf{E}) - \tilde{\boldsymbol{\theta}}_h^{\mathbf{a}}) + \hat{\boldsymbol{\theta}}_h^{\mathbf{a}}$$

In addition, we have

$$\omega^2 (\psi^{\mathbf{a}} \boldsymbol{\varepsilon} \mathbf{E} - \mathbf{D}_h^{\mathbf{a}}) = \omega^2 \psi^{\mathbf{a}} (\boldsymbol{\varepsilon} \mathbf{E} - \boldsymbol{\varepsilon} \mathbf{E}_h) + \omega^2 (\psi^{\mathbf{a}} \boldsymbol{\varepsilon} \mathbf{E}_h - \mathbf{D}_h^{\mathbf{a}}),$$

and using (5.4), we have

$$\begin{aligned} (5.12) \quad \omega^2 \|\psi^{\mathbf{a}} \boldsymbol{\varepsilon} \mathbf{E} - \mathbf{D}_h^{\mathbf{a}}\|_{\boldsymbol{\mu}, \omega^{\mathbf{a}}} &\leq \sqrt{\frac{\varepsilon_{\max, \omega^{\mathbf{a}}}}{\chi_{\min, \omega^{\mathbf{a}}}}} \omega^2 (\|\mathbf{E} - \mathbf{E}_h\|_{\boldsymbol{\varepsilon}, \omega^{\mathbf{a}}} + \|\psi^{\mathbf{a}} \mathbf{E}_h - \varepsilon^{-1} \mathbf{D}_h^{\mathbf{a}}\|_{\boldsymbol{\varepsilon}, \omega^{\mathbf{a}}}) \\ &\lesssim_{\mathcal{T}_h^{\mathbf{a}}} \mathcal{C}_{\boldsymbol{\varepsilon}, \omega^{\mathbf{a}}} \sqrt{\frac{\varepsilon_{\max, \omega^{\mathbf{a}}}}{\chi_{\min, \omega^{\mathbf{a}}}}} \omega^2 \|\mathbf{E} - \mathbf{E}_h\|_{\boldsymbol{\varepsilon}, \omega^{\mathbf{a}}} \\ &= h_{\omega^{\mathbf{a}}}^{-1} \mathcal{C}_{\boldsymbol{\varepsilon}, \omega^{\mathbf{a}}} \frac{\omega h_{\omega^{\mathbf{a}}}}{c_{\min, \omega^{\mathbf{a}}}} \omega \|\mathbf{E} - \mathbf{E}_h\|_{\boldsymbol{\varepsilon}, \omega^{\mathbf{a}}}. \end{aligned}$$

Recalling (5.9), we have

$$\begin{aligned} \|\hat{\boldsymbol{\theta}}_h^{\mathbf{a}}\|_{\boldsymbol{\mu}, K} &\lesssim_{\mathcal{T}_h^{\mathbf{a}}} \sqrt{\mathcal{C}_{\boldsymbol{\mu}, K}} \|\tilde{\boldsymbol{\theta}}_h\|_{\boldsymbol{\mu}, K} = \sqrt{\mathcal{C}_{\boldsymbol{\mu}, K}} \left\| \sum_{\mathbf{b} \in \mathcal{V}_h} \left(\tilde{\boldsymbol{\theta}}_h^{\mathbf{b}} - \nabla \psi^{\mathbf{b}} \times (\boldsymbol{\chi} \nabla \times \mathbf{E}_h) \right) \right\|_{\boldsymbol{\mu}, K} \\ &\lesssim_{\tilde{\mathcal{T}}_h^{\mathbf{a}}} \sqrt{\mathcal{C}_{\boldsymbol{\mu}, K}} \sum_{\mathbf{b} \in \mathcal{V}_h} h_{\omega^{\mathbf{b}}}^{-1} \mathcal{C}_{\boldsymbol{\mu}, \omega^{\mathbf{b}}} \left(1 + \frac{\omega h_{\omega^{\mathbf{b}}}}{c_{\min, \omega^{\mathbf{b}}}} \right) \|\mathbf{E} - \mathbf{E}_h\|_{\boldsymbol{\omega}, \omega^{\mathbf{b}}}, \end{aligned}$$

and summing over $K \in \mathcal{T}_h^{\mathbf{a}}$, since $h_{\omega^{\mathbf{b}}} \lesssim_{\tilde{\mathcal{T}}_h^{\mathbf{a}}} h_{\omega^{\mathbf{a}}}$ for all vertices \mathbf{b} in the patch, we arrive at

$$(5.13) \quad \|\hat{\boldsymbol{\theta}}_h^{\mathbf{a}}\|_{\boldsymbol{\mu}, \omega^{\mathbf{a}}} \lesssim \mathcal{C}_{\boldsymbol{\mu}, \tilde{\omega}^{\mathbf{a}}}^{3/2} h_{\omega^{\mathbf{a}}}^{-1} \left(1 + \frac{\omega h_{\omega^{\mathbf{a}}}}{c_{\min, \tilde{\omega}^{\mathbf{a}}}} \right) \|\mathbf{E} - \mathbf{E}_h\|_{\boldsymbol{\omega}, \tilde{\omega}^{\mathbf{a}}}.$$

Then, (5.11) follows from (5.8), (5.12) and (5.13). \square

5.3. Magnetic fields reconstruction. We now analyze the magnetic field reconstruction $\mathbf{H}_h^{\mathbf{a}}$. Here too, we start by introducing a continuous version $\mathbf{H}^{\mathbf{a}}$ of $\mathbf{H}_h^{\mathbf{a}}$ defined with a minimization problem over $\mathbf{H}_0(\operatorname{curl}, \omega^{\mathbf{a}})$.

Lemma 5.7 (Idealized magnetic field reconstruction). *For all $\mathbf{a} \in \mathcal{V}_h$, there exists a unique minimizer*

$$(5.14) \quad \mathbf{H}^{\mathbf{a}} := \arg \min_{\substack{\mathbf{v} \in \mathbf{H}_0(\operatorname{curl}, \omega^{\mathbf{a}}) \\ i\omega \nabla \times \mathbf{v} = \mathbf{G}_h^{\mathbf{a}}}} \|\psi^{\mathbf{a}} \nabla \times \mathbf{E}_h - i\omega \boldsymbol{\chi}^{-1} \mathbf{v}\|_{\boldsymbol{\chi}, \omega^{\mathbf{a}}}$$

In addition, we have

$$(5.15) \quad \|\psi^{\mathbf{a}} \nabla \times \mathbf{E}_h - i\omega \boldsymbol{\chi}^{-1} \mathbf{H}^{\mathbf{a}}\|_{\boldsymbol{\chi}, \omega^{\mathbf{a}}} \lesssim_{\mathcal{T}_h^{\mathbf{a}}} \|\nabla \times (\mathbf{E} - \mathbf{E}_h)\|_{\boldsymbol{\chi}, \omega^{\mathbf{a}}} + \sqrt{\mathcal{C}_{\boldsymbol{\mu}, \omega^{\mathbf{a}}}} h_{\omega^{\mathbf{a}}} \|\mathbf{G}^{\mathbf{a}} - \mathbf{G}_h^{\mathbf{a}}\|_{\omega^{\mathbf{a}}}.$$

Proof. Recalling Theorem 5.6, $\mathbf{G}_h^a \in \mathbf{H}_0(\operatorname{div}, \omega^a)$ with $\nabla \cdot \mathbf{G}_h^a = 0$. Due to (2.10), this proves that the minimization set in (5.14) is non-empty and ensures the existence and uniqueness of \mathbf{H}^a .

The Euler-Lagrange conditions associated with the minimization problem ensures that there exists a (unique) $\phi \in \mathbf{H}_*(\operatorname{div}^0, \omega^a)$ such that

$$\begin{cases} (\psi^a \nabla \times \mathbf{E}_h - i\omega \chi^{-1} \mathbf{H}^a, \mathbf{v})_{\omega^a} + (\phi, \nabla \times \mathbf{v})_{\omega^a} = 0 \\ i\omega (\nabla \times \mathbf{H}^a, \mathbf{w})_{\omega^a} = (\mathbf{G}_h^a, \mathbf{w})_{\omega^a} \end{cases}$$

for all $\mathbf{v} \in \mathbf{H}_0(\operatorname{curl}, \omega^a)$ and $\mathbf{w} \in \mathbf{H}_*(\operatorname{div}^0, \omega^a)$. Integrating by parts in the first equation reveals that $\phi \in \mathbf{H}_+(\operatorname{curl}, \omega^a) \cap \mathbf{H}_*(\operatorname{div}^0, \omega^a)$, with

$$\nabla \times \phi = \psi^a \nabla \times \mathbf{E}_h - i\omega \chi^{-1} \mathbf{H}^a.$$

Similarly, integration by parts in the second equation gives

$$-i\omega (\mathbf{H}^a, \nabla \times \mathbf{w})_{\omega^a} = -(\mathbf{G}_h^a, \mathbf{w})_{\omega^a},$$

and we deduce that

$$(\chi \nabla \times \phi, \nabla \times \mathbf{w})_{\omega^a} = (\psi^a \chi \nabla \times \mathbf{E}_h, \nabla \times \mathbf{w})_{\omega^a} - (\mathbf{G}_h^a, \mathbf{w})_{\omega^a}.$$

On the other hand, we have

$$(\psi^a \chi \nabla \times \mathbf{E}, \nabla \times \mathbf{w}) - (\mathbf{G}^a, \mathbf{w})_{\omega^a} = 0,$$

so that

$$(\chi \nabla \times \phi, \nabla \times \mathbf{w})_{\omega^a} = (\mathbf{G}^a - \mathbf{G}_h^a, \mathbf{w})_{\omega^a} - (\psi^a \chi \nabla \times (\mathbf{E} - \mathbf{E}_h), \nabla \times \mathbf{w})_{\omega^a}.$$

It follows that

$$\|\nabla \times \phi\|_{\chi, \omega^a}^2 \leq \|\mathbf{G}^a - \mathbf{G}_h^a\|_{\mu, \omega^a} \|\phi\|_{\chi, \omega^a} + \|\nabla \times (\mathbf{E} - \mathbf{E}_h)\|_{\chi, \omega^a} \|\psi^a \nabla \times \phi\|_{\chi, \omega^a},$$

and we conclude since recalling (2.9), we have $\|\psi^a \nabla \times \phi\|_{\chi, \omega^a} \leq \|\nabla \times \phi\|_{\chi, \omega^a}$ and

$$\frac{1}{\sqrt{\chi_{\max, \omega^a}}} \|\phi\|_{\chi, \omega^a} \leq \|\phi\|_{\omega^a} \lesssim \mathcal{T}_h^a h_{\omega^a} \|\nabla \times \phi\|_{\omega^a} \leq \frac{h_{\omega^a}}{\sqrt{\chi_{\min, \omega^a}}} \|\nabla \times \phi\|_{\chi, \omega^a}.$$

□

Theorem 5.8 (Discrete magnetic field reconstruction). *Problem (3.10) admits a unique minimizer \mathbf{H}_h^a . In addition, we have*

$$(5.16) \quad \|\psi^a \nabla \times \mathbf{E}_h - i\omega \chi \mathbf{H}_h^a\|_{\chi, \omega^a} \lesssim \tilde{\mathcal{T}}_h^a \left(1 + \frac{\omega h_{\omega^a}}{c_{\min, \tilde{\omega}^a}}\right) \mathcal{C}_{\varepsilon, \tilde{\omega}^a} \mathcal{C}_{\mu, \tilde{\omega}^a}^{5/2} \|\mathbf{E} - \mathbf{E}_h\|_{\omega, \tilde{\omega}^a}.$$

Proof. We have established in Theorem 5.6 that $\mathbf{G}_h^a \in \mathbf{RT}_{p^a+2}(\mathcal{T}_h^a) \cap \mathbf{H}_0(\operatorname{div}, \omega^a)$ with $\nabla \cdot \mathbf{G}_h^a = 0$. Then, the existence and uniqueness of a minimizer $\mathbf{H}_h^a \in \mathbf{N}_{p^a+2}(\mathcal{T}_h^a) \cap \mathbf{H}_0(\operatorname{curl}, \omega^a)$ to (3.10) follows from [11, Theorem 3.3]. The same reasoning also applies to the minimization in an unweighted norm, and since $\psi^a \chi^{-1} \nabla \times \mathbf{E}_h \in \mathbf{N}_{p^a+1}(\mathcal{T}_h^a)$, [11, Theorem 3.3] additionally ensures that

$$(5.17) \quad \min_{\substack{\mathbf{v}_h \in \mathbf{N}_{p^a+2}(\mathcal{T}_h^a) \cap \mathbf{H}_0(\operatorname{curl}, \omega^a) \\ \nabla \times \mathbf{v}_h = \mathbf{G}_h^a}} \|\psi^a \chi \nabla \times \mathbf{E}_h - i\omega \mathbf{v}_h\|_{\omega^a} \\ \lesssim \mathcal{T}_h^a \min_{\substack{\mathbf{v} \in \mathbf{H}_0(\operatorname{curl}, \omega^a) \\ \nabla \times \mathbf{v} = \mathbf{G}_h^a}} \|\psi^a \chi \nabla \times \mathbf{E}_h - i\omega \mathbf{v}\|_{\omega^a}.$$

For the sake of shortness, we introduce the notation \mathbf{v}_h^* and \mathbf{v}^* for the minimizers of both sides of (5.17). Then, using that $\|\mathbf{w}\|_{\chi,\omega^a} = \|\chi\mathbf{w}\|_{\chi^{-1},\omega^a}$ whenever $\mathbf{w} \in \mathbf{L}^2(\omega^a)$, we further infer that

$$\begin{aligned} \|\psi^a \nabla \times \mathbf{E}_h - i\omega \chi^{-1} \mathbf{H}_h^a\|_{\chi,\omega^a} &\leq \|\psi^a \chi \nabla \times \mathbf{E}_h - i\omega \mathbf{v}_h^*\|_{\chi^{-1},\omega^a} \leq \frac{1}{\sqrt{\chi_{\min,\omega^a}}} \|\psi^a \chi \nabla \times \mathbf{E}_h - i\omega \mathbf{v}_h^*\|_{\omega^a} \\ &\lesssim_{\mathcal{T}_h^a} \frac{1}{\sqrt{\chi_{\min,\omega^a}}} \|\psi^a \chi \nabla \times \mathbf{E}_h - i\omega \mathbf{v}^*\|_{\omega^a} \leq \frac{1}{\sqrt{\chi_{\min,\omega^a}}} \|\psi^a \chi \nabla \times \mathbf{E}_h - i\omega \mathbf{H}^a\|_{\omega^a} \\ &\leq \sqrt{\mathcal{C}_{\mu,\omega^a}} \|\psi^a \chi \nabla \times \mathbf{E}_h - i\omega \mathbf{H}^a\|_{\chi^{-1},\omega^a} = \sqrt{\mathcal{C}_{\mu,\omega^a}} \|\psi^a \nabla \times \mathbf{E}_h - i\omega \chi^{-1} \mathbf{H}^a\|_{\chi,\omega^a}. \end{aligned}$$

We finally observe that by combining (5.15) and (5.11), we obtain

$$\begin{aligned} &\sqrt{\mathcal{C}_{\mu,\omega^a}} \|\psi^a \nabla \times \mathbf{E}_h - i\omega \chi^{-1} \mathbf{H}^a\|_{\chi,\omega^a} \\ &\lesssim_{\tilde{\mathcal{T}}_h^a} \sqrt{\mathcal{C}_{\mu,\omega^a}} \left(\|\nabla \times (\mathbf{E} - \mathbf{E}_h)\|_{\chi,\omega^a} + \sqrt{\mathcal{C}_{\mu,\tilde{\omega}^a} h_{\omega^a}} \|\mathbf{G}^a - \mathbf{G}_h^a\|_{\mu,\omega^a} \right) \\ &\lesssim_{\tilde{\mathcal{T}}_h^a} \sqrt{\mathcal{C}_{\mu,\omega^a}} \left(\|\nabla \times (\mathbf{E} - \mathbf{E}_h)\|_{\chi,\omega^a} + \mathcal{C}_{\mu,\tilde{\omega}^a}^2 \mathcal{C}_{\varepsilon,\tilde{\omega}^a} \left(1 + \frac{\omega h_{\omega^a}}{c_{\min,\tilde{\omega}^a}} \right) \|\mathbf{E} - \mathbf{E}_h\|_{\omega,\tilde{\omega}^a} \right) \\ &\lesssim_{\tilde{\mathcal{T}}_h^a} \sqrt{\mathcal{C}_{\mu,\omega^a}} \left(1 + \mathcal{C}_{\mu,\tilde{\omega}^a}^2 \mathcal{C}_{\varepsilon,\tilde{\omega}^a} \left(1 + \frac{\omega h_{\omega^a}}{c_{\min,\tilde{\omega}^a}} \right) \right) \|\mathbf{E} - \mathbf{E}_h\|_{\omega,\tilde{\omega}^a}, \end{aligned}$$

and the result follows since $\mathcal{C}_{\mu,\omega^a} \leq \mathcal{C}_{\mu,\tilde{\omega}^a}$ and

$$1 \leq \mathcal{C}_{\mu,\tilde{\omega}^a}^2 \mathcal{C}_{\varepsilon,\tilde{\omega}^a} \left(1 + \frac{\omega h_{\omega^a}}{c_{\min,\tilde{\omega}^a}} \right).$$

□

Remark 5.9 (Image of the curl). *Theorem 5.8 shows that \mathbf{H}_h^a is well-defined for each \mathbf{a} , and therefore, after summation, that*

$$i\omega \nabla \times \mathbf{H}_h = i\omega \mathbf{J}_h + \omega^2 \mathbf{D}_h.$$

Hence, $i\omega \mathbf{J}_h + \omega^2 \mathbf{D}_h$ is in the image of the curl operator. This is, in general, a stronger property than the observation that

$$-\omega^2 \nabla \cdot \mathbf{D}_h = i\omega \nabla \cdot \mathbf{J}_h$$

which naturally follows from the local divergence constraints defining \mathbf{D}_h^a . This is to be linked with [5] where a similar observation is made for the magnetostatic problem, but with a completely different proof.

5.4. Efficiency. We can now conclude the efficiency proof by combining the results established in this section.

Proof of Theorem 3.3. Recalling the definition of $\eta_{\text{div},K}$ in (3.2), the fact that the hat functions ψ^a form a partition of unity (2.7), and that \mathbf{D}_h decomposes into local contributions \mathbf{D}_h^a according to (3.6), we have

$$\begin{aligned} \eta_{\text{div},K} &= \omega \|\mathbf{E}_h - \varepsilon^{-1} \mathbf{D}_h\|_{\varepsilon,K} = \omega \left\| \sum_{\mathbf{a} \in \mathcal{V}(K)} (\psi^a \mathbf{E}_h - \varepsilon^{-1} \mathbf{D}_h^a) \right\|_{\varepsilon,K} \\ &\leq \omega \sum_{\mathbf{a} \in \mathcal{V}(K)} \|\psi^a \mathbf{E}_h - \varepsilon^{-1} \mathbf{D}_h^a\|_{\varepsilon,K} \leq \omega \sum_{\mathbf{a} \in \mathcal{V}(K)} \|\psi^a \mathbf{E}_h - \varepsilon^{-1} \mathbf{D}_h^a\|_{\varepsilon,\omega^a}. \end{aligned}$$

Then, recalling (5.4), we conclude that

$$(5.18) \quad \eta_{\text{div},K} \lesssim_{\mathcal{T}_h,K} \omega \sum_{\mathbf{a} \in \mathcal{V}(K)} \mathcal{C}_{\varepsilon,\omega^{\mathbf{a}}} \|\mathbf{E} - \mathbf{E}_h\|_{\varepsilon,\omega^{\mathbf{a}}} \lesssim_{\mathcal{T}_h,K} \mathcal{C}_{\varepsilon,\omega_K} \omega \|\mathbf{E} - \mathbf{E}_h\|_{\varepsilon,\omega_K} \\ \lesssim_{\tilde{\mathcal{T}}_h,K} \left(1 + \frac{\omega h_K}{c_{\min,\tilde{\omega}_K}}\right) \mathcal{C}_{\varepsilon,\tilde{\omega}_K} \mathcal{C}_{\mu,\tilde{\omega}_K}^{5/2} \|\mathbf{E} - \mathbf{E}_h\|_{\omega,\tilde{\omega}_K},$$

Following a similar path for $\eta_{\text{curl},K}$, but using (5.16) instead of (5.4), we arrive at

$$(5.19) \quad \eta_{\text{curl},K} \lesssim_{\tilde{\mathcal{T}}_h,K} \left(1 + \frac{\omega h_K}{c_{\min,\tilde{\omega}_K}}\right) \mathcal{C}_{\varepsilon,\tilde{\omega}_K} \mathcal{C}_{\mu,\tilde{\omega}_K}^{5/2} \|\mathbf{E} - \mathbf{E}_h\|_{\omega,\tilde{\omega}_K},$$

since $h_{\omega^{\mathbf{a}}} \lesssim_{\mathcal{T}_h} h_K$ due to the shape-regularity assumption on the mesh. Then, (3.12) follows from (5.18) and (5.19). \square

6. NUMERICAL EXAMPLES

This section illustrates our key theoretical results with a set of numerical examples.

6.1. Continuous setting. Throughout this section $\Omega := (0,1)^3$ is the unit cube, and the coefficients $\boldsymbol{\chi} = \boldsymbol{\varepsilon} = \mathbf{I}$ are constant. We will select frequencies of the form $\omega = 2\pi(m/2 + \delta)$ with $m \in \mathbb{N}$ and $\delta > 0$. Notice that the case $\delta = 0$ corresponds to a resonance frequency.

Letting $\mathbf{e}_2 = (0,1,0)$, the right-hand side reads

$$\mathbf{J}(\mathbf{x}) := \sin(m\pi\mathbf{x}_3)\mathbf{e}_2,$$

and the associated solution is

$$\mathbf{E}(\mathbf{x}) := \frac{1}{k^2} \left((\cos(k\mathbf{x}_1) - 1) - (\cos(k) - 1) \frac{\sin(k\mathbf{x}_1)}{\sin(k)} \right) \sin(m\pi\mathbf{x}_3)\mathbf{e}_2$$

where $k := \sqrt{\omega^2 - (m\pi)^2}$.

6.2. Discrete setting. The domain Ω is partitioned into unstructured tetrahedral meshes \mathcal{T}_h . We employ `mmg3d` to generate such meshes [16]. The parameter h corresponds to the argument `-hmax` passed to `mmg3d` when generating each mesh. The mesh sizes go through zero through the sequence $h = 1, 0.5, 0.25, 0.125$. On these meshes, we consider Nédélec finite element of the first family, i.e. $\mathbf{W}_h = \mathbf{N}_p(\mathcal{T}_h) \cap \mathbf{H}_0(\mathbf{curl}, \Omega)$, with p ranging from 1 to 3.

6.3. Results. Our theoretical results indicate that we may expect the estimator to be unreliable on coarse meshes when m is large or δ is close to zero. This lack of reliability should be only pre-asymptotic, and disappear on fine meshes or when using higher polynomial degrees. We also expect higher polynomial degrees to be more robust.

To lighten notation in the graphs below, we set

$$\mathcal{E} := \|\mathbf{E} - \mathbf{E}_h\|_{\omega,\Omega} \quad \mathcal{I} := \frac{\eta}{\mathcal{E}}.$$

As discussed above, we expect that $\mathcal{I} \leq 1$ for coarse meshes, and $\mathcal{I} \rightarrow 1$ as $(\omega h)/(c_{\min,\Omega} p) \rightarrow 0$.

We first set $m = 3$ and let $\delta = 10^{-2}, 10^{-3}, 10^{-4}$ approach zero. Figures 6.1a, 6.1b and 6.1c represent the behaviour error \mathcal{E} and the estimator η as the mesh is refined for the different choices of p . The expected convergence rates are observed as well as the asymptotic guaranteed reliability of the estimator. This is seen in more details on Figure 6.2. In particular, we indeed see that, as predicted, the error underestimation is more pronounced for smaller δ , but reduced when using a higher value of p .

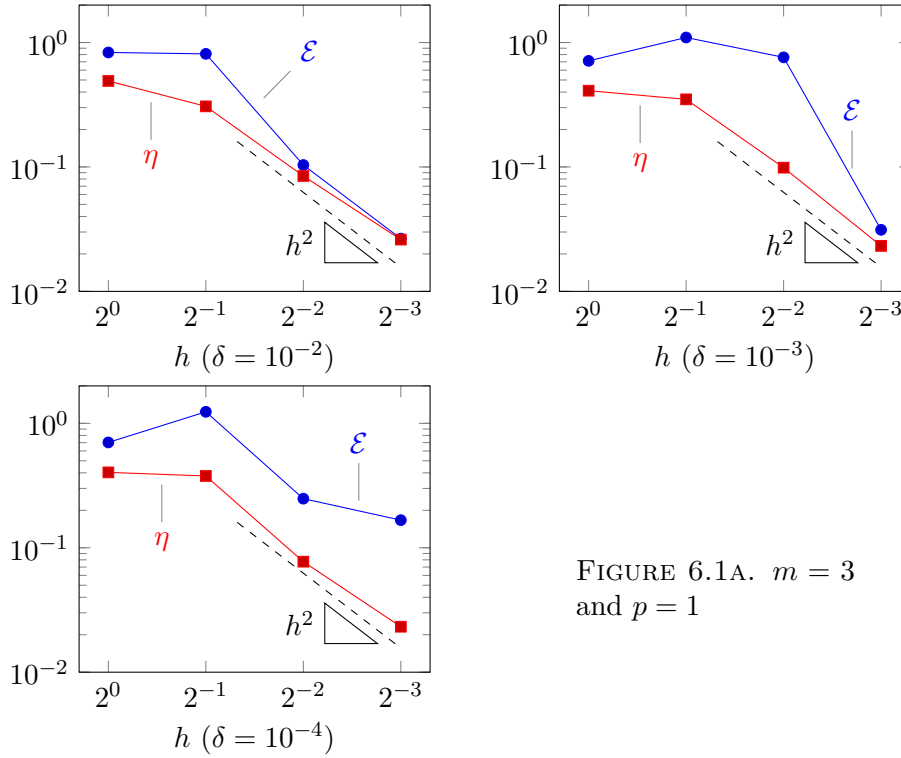


FIGURE 6.1A. $m = 3$
and $p = 1$

Figures 6.3a, 6.3b, 6.3c and 6.4 present the same results for the case $m = 5$. The comments made above for the case $m = 3$ still applies. We further see that the error estimation is more pronounced for $m = 5$ than $m = 3$, which is in line with the theoretical prediction.

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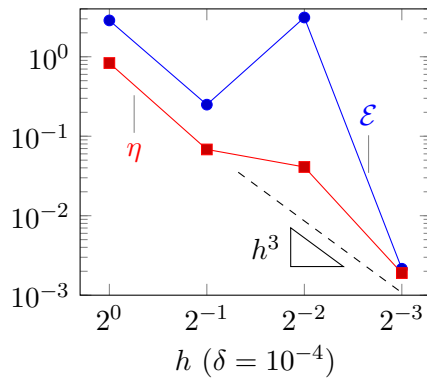
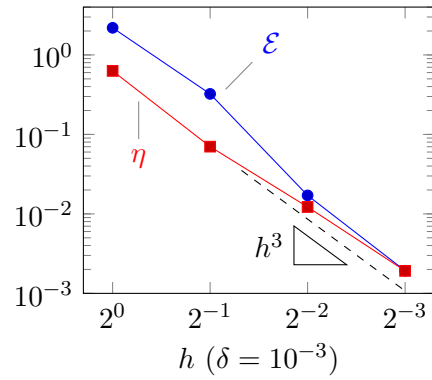
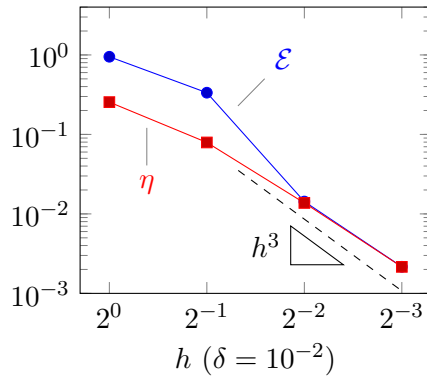


FIGURE 6.1B. $m = 3$
and $p = 2$

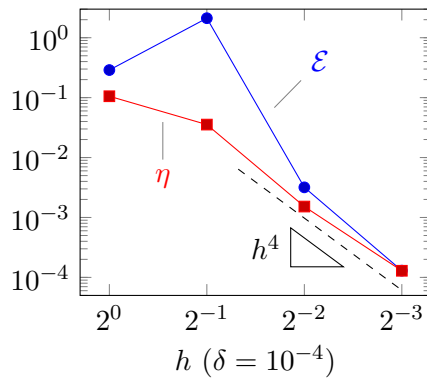
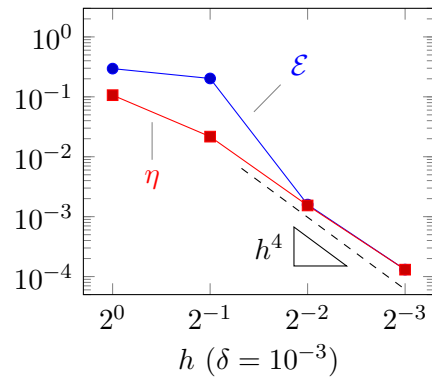
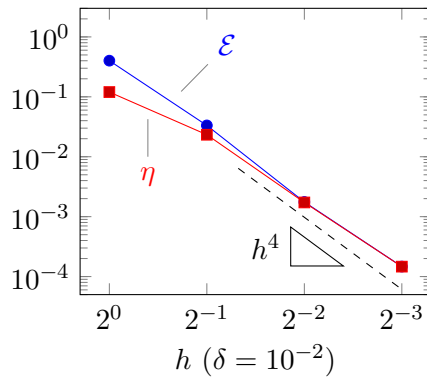


FIGURE 6.1C. $m = 3$
and $p = 3$

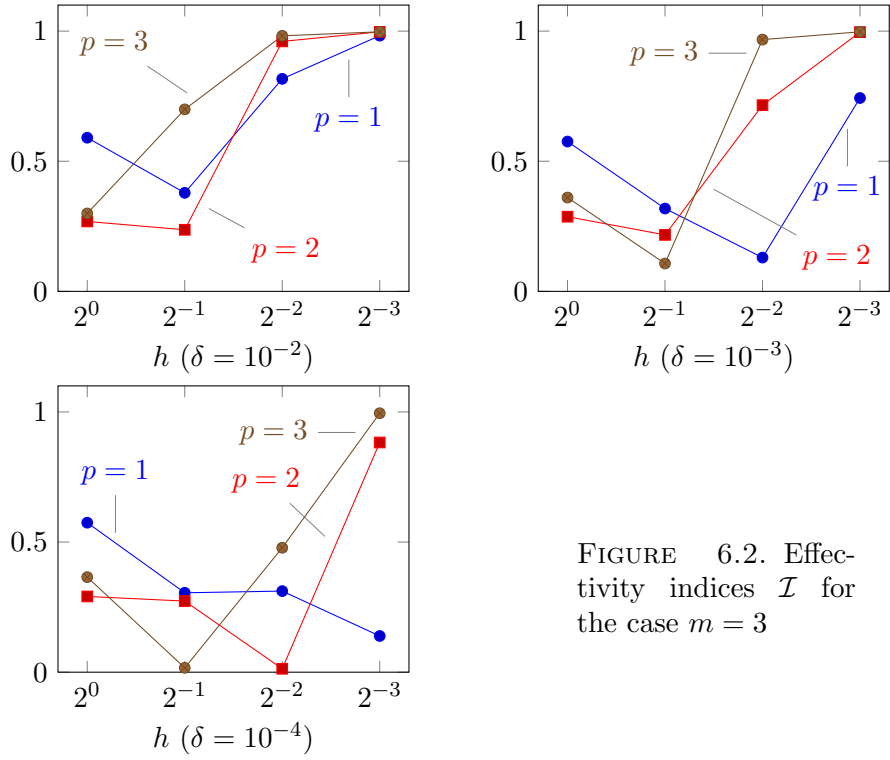


FIGURE 6.2. Effectivity indices \mathcal{I} for the case $m = 3$

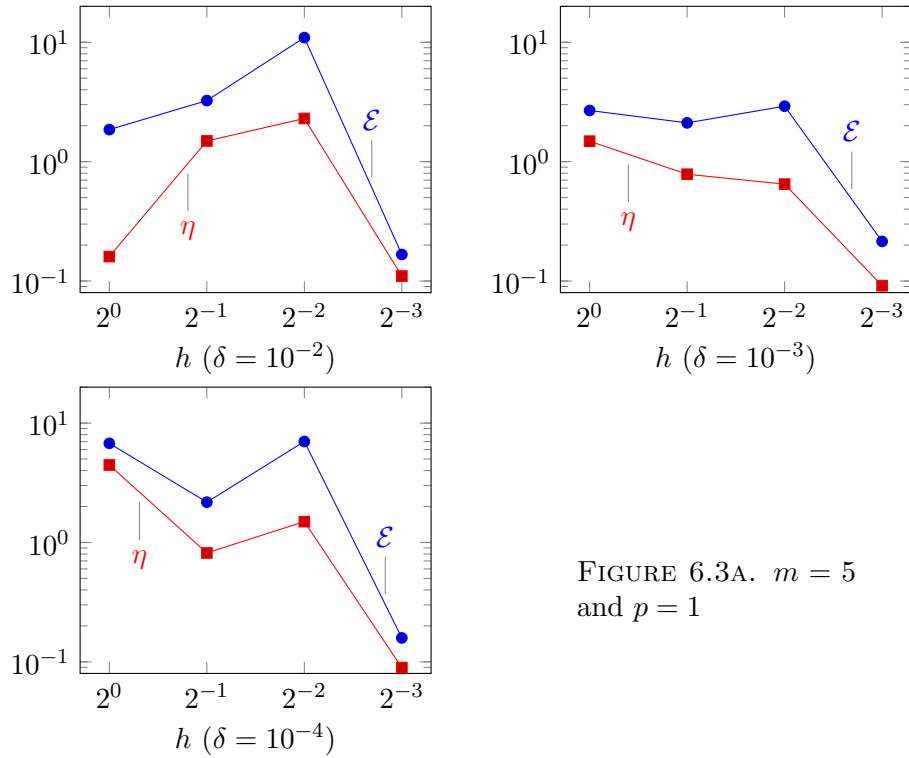


FIGURE 6.3A. $m = 5$ and $p = 1$

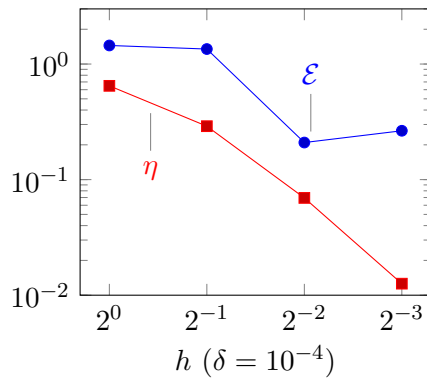
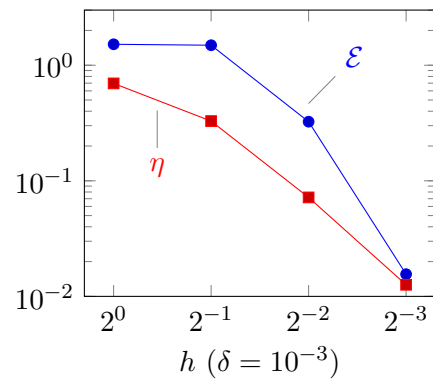
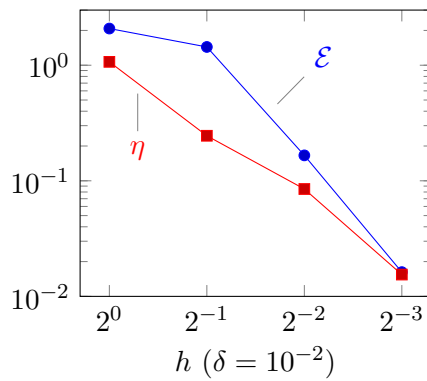


FIGURE 6.3B. $m = 5$
and $p = 2$

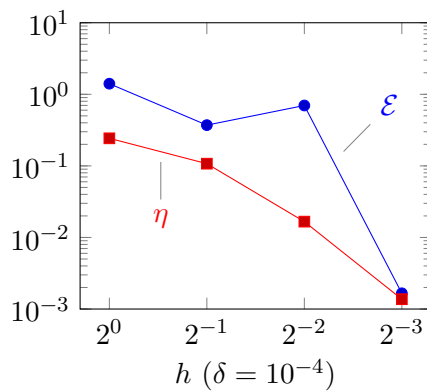
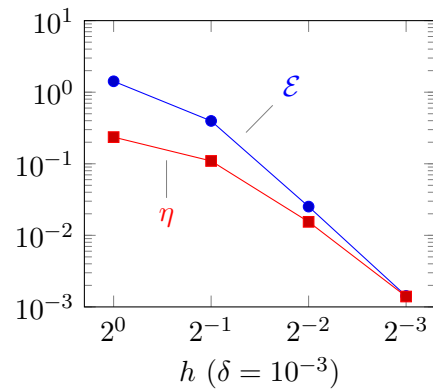
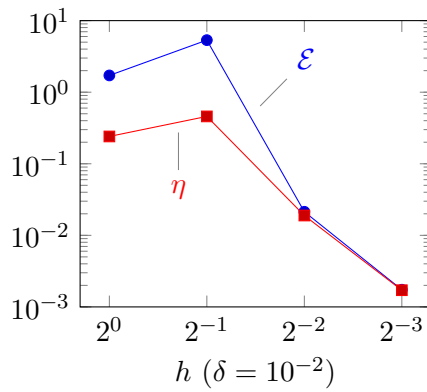


FIGURE 6.3C. $m = 5$
and $p = 3$

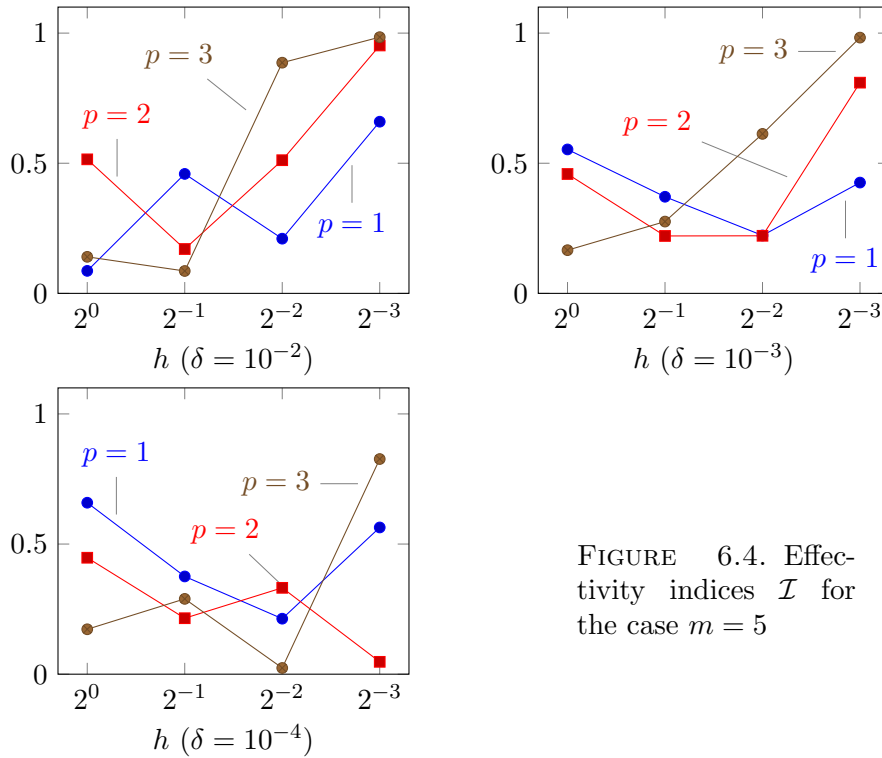


FIGURE 6.4. Effectivity indices \mathcal{I} for the case $m = 5$

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