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Subdivisions in dicritical digraphs with large order or digirth*

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Abstract

Aboulker et al. proved that a digraph with large enough dichromatic number contains any fixed digraph as a subdivision. The dichromatic number of a digraph is the smallest order of a partition of its vertex set into acyclic induced subdigraphs. A digraph is dicritical if the removal of any arc or vertex decreases its dichromatic number. In this paper we give sufficient conditions on a dicritical digraph of large order or large directed girth to contain a given digraph as a subdivision. In particular, we prove that (i) for every integers k, ℓ , large enough dicritical digraphs with dichromatic number k contain an orientation of a cycle with at least ℓ vertices; (ii) there are functions f, g such that for every subdivision F^* of a digraph F , digraphs with directed girth at least $f(F^*)$ and dichromatic number at least $g(F)$ contain a subdivision of F^* , and if F is a tree, then $g(F) = |V(F)|$; (iii) there is a function f such that for every subdivision F^* of TT_3 (the transitive tournament on three vertices), digraphs with directed girth at least $f(F^*)$ and minimum out-degree at least 2 contain F^* as a subdivision.

Keywords: Digraphs, dichromatic number, dicritical digraphs, subdivisions, digirth.

1 Introduction

Let D be a digraph. A k -colouring of D is a function $\phi : V(D) \rightarrow [k]$. It is a k -dicolouring if no directed cycle C in D is monochromatic for ϕ (that is $|\phi(V(C))| > 1$). Equivalently, it is a k -dicolouring if every colour class induces an acyclic subdigraph. The smallest integer k such that D has a k -dicolouring is the *dichromatic number* of D and is denoted by $\bar{\chi}(D)$. A digraph D is k -dicritical if $\bar{\chi}(D) = k$ and every proper subdigraph H of D satisfies $\bar{\chi}(H) < k$. The following is a central result when dealing with substructures in digraphs of large dichromatic number.

Theorem 1 (Aboulker et al. [1]). *Let F be a digraph on n vertices, m arcs and c connected components. Every digraph D satisfying $\bar{\chi}(D) \geq 4^{m-n+c}(n-1) + 1$ contains a subdivision of F .*

For every digraph F , we denote by $\text{mader}_{\bar{\chi}}(F)$ the least integer c_F for which every digraph D either contains a subdivision of F or has dichromatic number at most $c_F - 1$. Note that $\text{mader}_{\bar{\chi}}(F)$ is well-defined by Theorem 1. The result above was generalized in a recent work of Steiner [21] (see also [15]) who extended it to subdivisions with modular constraints.

Since every digraph is a subdigraph of \overleftrightarrow{K}_n , the complete digraph on n vertices, it is natural to look for the value of $\text{mader}_{\bar{\chi}}(\overleftrightarrow{K}_n)$. The result above implies that $\text{mader}_{\bar{\chi}}(\overleftrightarrow{K}_n) \leq 4^{n^2-2n+1}$. A more precise computation using the tools developed in [1] yields the following best known upper bound, whose proof is presented in Section 2.

Proposition 2. $\text{mader}_{\bar{\chi}}(\overleftrightarrow{K}_n) \leq 4^{\frac{2}{3}n^2+2n-\frac{8}{3}}$.

For every digraph F on n vertices, we have $\text{mader}_{\bar{\chi}}(F) \geq n$. This is because $\overleftrightarrow{K}_{n-1}$, the complete digraph on $n-1$ vertices, has dichromatic number $n-1$ and does not contain any subdivision of F .

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For some digraphs F , it then appears that the value of $\text{mader}_{\bar{\chi}}(F)$ does not capture the structure of F but only its order. For instance, $\text{mader}_{\bar{\chi}}(\overrightarrow{C_n}) \geq n$, but every 2-dicritical digraph on at least n vertices actually contains a subdivision of $\overrightarrow{C_n}$. In order to have a better understanding of digraphs forced to contain subdivisions of F , one may then ask for the minimum k such that there is a finite number of k -dicritical digraphs which do not contain any subdivision of F . The following question then naturally arises.

Question 3. Let F^* be a subdivision of a digraph F . Is it true that the set of $(\text{mader}_{\bar{\chi}}(F))$ -dicritical digraphs that do not contain any subdivision of F^* is finite?

Unfortunately, the answer to this question is negative. To see that, consider for every positive integers k and ℓ the digraph $C(k, \ell)$, which is the union of two internally disjoint directed paths from a vertex x to a vertex y of lengths respectively k and ℓ . Observe that $\text{mader}_{\bar{\chi}}(C(1, 2)) = 3$ because $\overleftarrow{K_2}$ does not contain any subdivision of $C(1, 2)$ and every 3-dicritical digraph is 2-arc-strong. The following answers Question 3 by the negative for $F = C(1, 2)$ and $F^* = C(3, 3)$.

Proposition 4. For every integer $n \geq 3$, there exists a 3-dicritical digraph D_n on n vertices that does not contain any subdivision of $C(3, 3)$.

Proof. Let D_n be the digraph obtained from a directed cycle on $n - 1$ vertices $\overrightarrow{C_{n-1}}$ by adding a new vertex x and all possible digons between x and $V(\overrightarrow{C_{n-1}})$ (see Figure 1 for an illustration). This digraph is 3-dicritical (it follows from Lemma 17 and the fact that directed cycles are 2-dicritical) and does not contain any subdivision of $C(3, 3)$. \square

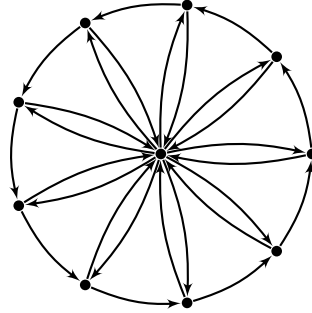


Figure 1: The digraph D_{10} .

In Section 3, we show that the answer to Question 3 is actually negative not only for $C(1, 2)$ but for every digraph F on at least three vertices with at least one arc. Let F be such a digraph and F^* be a subdivision of F in which an arc has been subdivided at least $3 \cdot \text{mader}_{\bar{\chi}}(F) + 1$ times. Since $\text{mader}_{\bar{\chi}}(F) \geq |V(F)| \geq 3$, the following result implies that the set of $(\text{mader}_{\bar{\chi}}(F))$ -dicritical digraphs that do not contain any subdivision of F^* is infinite.

Theorem 5. For every integer $k \geq 3$, there are infinitely many k -dicritical digraphs without any directed path on $3k + 1$ vertices.

Theorem 5 establishes a distinction between the directed and undirected cases. In the undirected case, for every fixed $k \geq 3$, there exists a non-decreasing function $f_k : \mathbb{N} \rightarrow \mathbb{N}$ such that every k -critical graph on at least $f_k(\ell)$ vertices contains a path on ℓ vertices. This was first proved by Kelly and Kelly [13] in 1954, answering a question of Dirac. The bound on f_k was then improved by Alon, Krivelevich and Seymour [2] and finally settled by Shapira and Thomas [20], who proved that the largest cycle in a k -critical graph on n vertices has length at least $c_k \cdot \log(n)$, where c_k is a constant depending only on k . This bound is best possible up to the multiplicative constant c_k , as shown by a construction of Gallai [9, 10] (see [20]).

On the positive side, we adapt the proof of Alon et al. [2] and show that, in k -dicritical digraphs, the length of the longest oriented cycle (*i.e.* the longest cycle in the underlying graph) grows with the

number of vertices, and so does the length of its longest oriented path (*i.e.* the longest path in the underlying graph).

Theorem 6. *For every fixed integers $k \geq 2$ and $\ell \geq 3$, there are finitely many k -dicritical digraphs with no oriented cycle on at least ℓ vertices.*

Since Question 3 turns out to be almost always false, we propose as an alternative to restrict to digraphs with large digirth. The *girth* of a graph G , denoted by $\text{girth}(G)$, is the length of a smallest cycle in G , with the convention $\text{girth}(G) = +\infty$ if G is a forest. The girth of a digraph D , denoted by $\text{girth}(D)$, is $\text{girth}(\text{UG}(D))$. The *digirth* of a digraph is the length of its shortest directed cycle. By convention we have $\text{digirth}(D) = +\infty$ if D is acyclic. A celebrated result of Erdős (see [3]) states that there exist graphs of arbitrarily large chromatic number and arbitrarily large girth. It has been generalised by Bokal et al. [6] who showed the existence of digraphs of arbitrarily large girth and dichromatic number. For every integer g , we denote by $\text{mader}_{\vec{\chi}}^{(g)}(F)$ the least integer k such that every digraph D satisfying $\vec{\chi}(D) \geq k$ and $\text{digirth}(D) \geq g$ contains a subdivision of F . Note that $\text{mader}_{\vec{\chi}}(F) = \text{mader}_{\vec{\chi}}^{(2)}(F)$ and that $\text{mader}_{\vec{\chi}}^{(g)}(F)$ is non-increasing in g . We conjecture that the analogue of Question 3 for digraphs of large digirth is actually true.

Conjecture 7. For every digraph F and every subdivision F^* of F , there exists g such that $\text{mader}_{\vec{\chi}}^{(g)}(F^*) \leq \text{mader}_{\vec{\chi}}(F)$.

This conjecture seems to be hard to prove since the exact value of $\text{mader}_{\vec{\chi}}(F)$ is known for very few classes of digraphs. In order to provide some support to this conjecture, in Section 4 we show that the value of $\text{mader}_{\vec{\chi}}^{(g)}(F^*)$ depends only on F when g is large enough. Our proof is strongly based on the key-lemma of [1].

Theorem 8. *Let $k \geq 1$ be an integer. For every non-empty digraph F , if F^* is obtained from F by subdividing every arc at most $k - 1$ times, then $\text{mader}_{\vec{\chi}}^{(k)}(F^*) \leq \frac{1}{3} (4^{m(F)+1} n(F) - 1)$.*

In Section 5, we prove that Conjecture 7 holds for every digraph F whose underlying undirected graph $\text{UG}(F)$ is a forest.

Theorem 9. *Let $k \geq 1$ be an integer and let T be a bidirected tree. If T^* is obtained from T by subdividing every arc at most $k - 1$ times, then $\text{mader}_{\vec{\chi}}^{(2k)}(T^*) \leq \text{mader}_{\vec{\chi}}(T) = n(T)$.*

In the case of oriented trees, we improve Theorem 9 as follows.

Theorem 10. *Let $k \geq 1$ be an integer. Let D be a digraph, T be an oriented tree and T^* obtained from T by subdividing every arc at most $k - 1$ times. If $\vec{\chi}(D) \geq \text{mader}_{\vec{\chi}}(T)$ and $\text{digirth}(D) \geq k$, then D contains T^* as a subdigraph.*

Observe that every digraph D contains a subdigraph H such that $\delta^+(H) \geq \vec{\chi}(D) - 1$ (by taking H a $\vec{\chi}(D)$ -dicritical subdigraph of D). Hence, for every integer k , if a digraph F is such that every digraph D with $\delta^+(D) \geq k - 1$ contains a subdivision of F , then $\text{mader}_{\vec{\chi}}(F) \leq k$. In Section 6, we look for similar results using δ^+ instead of $\vec{\chi}$. Conjecture 7 for $F = C(1, 2)$ appears to be a consequence of the following theorem (recall that $\text{mader}_{\vec{\chi}}(C(1, 2)) = 3$).

Theorem 11. *For every integer $k \geq 2$, every digraph D with $\delta^+(D) \geq 2$ and $\text{digirth}(D) \geq 8k - 6$ contains a subdivision of $C(k, k)$.*

An *oriented graph* is a digraph with digirth at least 3. When $k = 2$, we improve Theorem 11 as follows.

Theorem 12. *Every oriented graph D with $\delta^+(D) \geq 2$ contains a subdivision of $C(2, 2)$.*

We also consider out-stars. For integers k, ℓ , let $S_k^{+(\ell)}$ be the digraph consisting in k directed paths of length ℓ sharing their origin (and no other common vertices). The *centre* of $S_k^{+(\ell)}$ is its unique source.

Theorem 13. *For every positive integers k, ℓ , every digraph with $\delta^+(D) \geq k$ and $\text{digirth}(D) \geq \frac{k^\ell - 1}{k - 1} + 1$ contains a copy of $S_k^{+(\ell)}$ with centre u for every chosen vertex u .*

When $k = 2$, Theorem 13 can be improved by reducing the bound on $\text{digirth}(D)$ down to 2ℓ .

Theorem 14. *Let D be a digraph with $\delta^+(D) \geq 2$. If $\text{digirth}(D) \geq 2\ell$, then D contains a copy of $S_2^{+(\ell)}$.*

We conclude in Section 7 by some open problems and further research directions.

Notations

We refer the reader to [4] for notation and terminology not explicitly defined in this paper.

Given a digraph D (resp. a graph G), we denote by $V(D)$ (resp. $V(G)$) its set of vertices and $A(D)$ (resp. $E(G)$) its set of arcs (resp. set of edges). For every vertex $v \in V(D)$, $N^+(v), N^-(v)$ denote respectively the in-neighbourhood and the out-neighbourhood of v in D , $d^+(v) = |N^+(v)|, d^-(v) = |N^-(v)|$, and $d(v) = d^+(v) + d^-(v)$. Moreover, for every set A of vertices, we write $N^+(A) = \bigcup_{a \in A} N^+(a)$, $N^-(A) = \bigcup_{a \in A} N^-(a)$ and $N(A) = N^+(A) \cup N^-(A)$. We also define $\delta^+(D) = \min_{v \in V(D)} d^+(v)$, $\delta^-(D) = \min_{v \in V(D)} d^-(v)$, and $\delta(D) = \min_{v \in V(D)} d(v)$. Similarly $\Delta^+(D) = \max_{v \in V(D)} d^+(v)$, $\Delta^-(D) = \max_{v \in V(D)} d^-(v)$, and $\Delta(D) = \max_{v \in V(D)} d(v)$.

A *digon* in D is a pair of opposite arcs between two vertices. Such a pair of arcs $\{uv, vu\}$ is denoted by $[u, v]$. The *underlying graph* of a digraph D , denoted by $UG(D)$, is the undirected graph on the same vertex set which contains an edge linking two vertices if D contains at least one arc linking these two vertices. An *oriented graph* is a digraph with digirth at least 3. A *bidirected digraph* is any digraph D such that for every $x, y \in V(D)$, $xy \in A(D) \Leftrightarrow yx \in A(D)$, and we write $D = \overleftrightarrow{G}$ where $G = UG(D)$. An *oriented cycle* (resp. *oriented path*) is an oriented graph whose underlying graph is a cycle (resp. path). A *directed cycle* is an oriented cycle in which every vertex has in- and out-degree 1. A *directed path* is obtained from a directed cycle by removing a vertex. The directed cycle on n vertices and the directed path on n vertices are respectively denoted by \overrightarrow{C}_n and \overrightarrow{P}_n . The *lengths* of \overrightarrow{P}_n and \overrightarrow{C}_n are respectively $n - 1$ and n . An *antidirected path* is an orientation of a path where every vertex v satisfies $\min(d^+(v), d^-(v)) = 0$.

If u is a vertex of a digraph D with $d^-(u) = 1$, we define the *predecessor* of u in D , denoted by $\text{pred}_D(u)$, as the unique in-neighbour of u in D . Similarly, if $d^+(u) = 1$, we define the *successor* of u in D , denoted by $\text{succ}_D(u)$, as the unique out-neighbour of u in D .

Given two vertices a, b in a directed cycle C (with possibly $a = b$), we denote by $C[a, b]$ the directed path from a to b along C (which is the single-vertex path when $a = b$). Moreover, we define $C[a, b[= C[a, b] - b$, $C]a, b[= C[a, b] - a$, and $C]a, b[= C[a, b] - \{a, b\}$. Note that these subpaths may be empty. Given a directed path P and two vertices a, b in $V(P)$, we use similar notations $P[a, b], P[a, b[, P]a, b]$ and $P]a, b[$. We denote by $\text{init}(P)$ the first vertex of P (i.e. the unique vertex with in-degree 0) and $\text{term}(P)$ its last one. Given two directed paths P, Q such that $V(P) \cap V(Q) = \{x\}$ where $x = \text{term}(P) = \text{init}(Q)$, the *concatenation* of P and Q , denoted by $P \cdot Q$, is the digraph $(V(P) \cup V(Q), A(P) \cup A(Q))$. The vertices in $V(P) \setminus \{\text{init}(P), \text{term}(P)\}$ are called the *internal vertices* of P . If U and V are two sets of vertices in D , then a (U, V) -*path* in D is a directed path P in D with $\text{init}(P) \in U$ and $\text{term}(P) \in V$, and we also say that P is a directed path from U to V . If $U = \{u\}$ (resp. $V = \{v\}$) then we simply write u for U (resp. v for V) in these notations. The *distance* from u to v , denoted by $\text{dist}(u, v)$, is the length of a shortest (u, v) -path, with the convention $\text{dist}(u, v) = +\infty$ if no such path exists.

A digraph is *connected* if its underlying graph is connected. It is *strongly connected* if for every ordered pair (u, v) of its vertices, there exists a directed path from u to v .

2 An improved bound on $\text{mader}_{\overleftrightarrow{\chi}}(\overleftrightarrow{K}_n)$

This section is devoted to the proof of Theorem 2. We need the two following Lemmas.

Lemma 15 (Aboulker et al. [1, Lemma 31]). *For every integer k and every digraph D with $\overleftrightarrow{\chi}(D) \geq 4k - 3$, there is a subdigraph H of D with $\overleftrightarrow{\chi}(H) \geq k$ such that for every pair u, v of distinct vertices in H , there is a directed path from u to v in D whose internal vertices are in $V(D) \setminus V(H)$.*

In particular, for every digraph F and every arc e in F ,

$$\text{mader}_{\overleftrightarrow{\chi}}(F) \leq 4 \cdot \text{mader}_{\overleftrightarrow{\chi}}(F \setminus e) - 3.$$

We skip the proof of the following easy lemma.

Lemma 16. *If $F_1 + F_2$ denotes the disjoint union of two digraphs F_1 and F_2 , then $\text{mader}_{\overleftarrow{\chi}}(F_1 + F_2) \leq \text{mader}_{\overleftarrow{\chi}}(F_1) + \text{mader}_{\overleftarrow{\chi}}(F_2)$.*

We are now ready to prove Theorem 2, let us first restate it.

Proposition 2. $\text{mader}_{\overleftarrow{\chi}}(\overleftarrow{K}_n) \leq 4^{\frac{2}{3}n^2 + 2n - \frac{8}{3}}$.

Proof. Let $f(n) = \text{mader}_{\overleftarrow{\chi}}(\overleftarrow{K}_n)$ for every $n \geq 1$. Clearly $f(1) = 1$. Let $g(x) = 4^{\frac{2}{3}x^2 + 2x - \frac{8}{3}}$ for every positive real x . Observe that g is non-decreasing. We will show by induction on n that $f(n) \leq g(n)$ for every positive integer n . For $n = 1$, $f(1) = 1 = g(1)$. Now suppose $n \geq 2$.

If n is even, by Lemmas 15 and 16 we deduce the following inequalities.

$$\begin{aligned} f(n) &\leq 4^{\frac{n^2}{2}} \cdot \text{mader}_{\overleftarrow{\chi}}(\overleftarrow{K}_{\frac{n}{2}} + \overleftarrow{K}_{\frac{n}{2}}) \\ &\leq 4^{\frac{n^2}{2}} \cdot 2 \cdot f\left(\frac{n}{2}\right) \\ &\leq 4^{\frac{n^2}{2}} \cdot 2 \cdot g\left(\frac{n}{2}\right) \\ &\leq 4^{\frac{n^2}{2} + n} \cdot g\left(\frac{n}{2}\right). \end{aligned}$$

If n is odd, then $f(n) \leq 4^{2(n-1)} \text{mader}_{\overleftarrow{\chi}}(\overleftarrow{K}_1 + \overleftarrow{K}_{n-1}) \leq 4^{2(n-1)}(1 + f(n-1)) \leq 4^{2n-1}f(n-1)$. Hence:

$$\begin{aligned} f(n) &\leq 4^{2n-1}f(n-1) \\ &\leq 4^{2n-1} \cdot 4^{\frac{(n-1)^2}{2}} \cdot \text{mader}_{\overleftarrow{\chi}}(\overleftarrow{K}_{\frac{n-1}{2}} + \overleftarrow{K}_{\frac{n-1}{2}}) \\ &\leq 4^{2n-1} \cdot 4^{\frac{(n-1)^2}{2}} \cdot 2 \cdot f\left(\frac{n-1}{2}\right) \\ &\leq 4^{2n-1} \cdot 4^{\frac{(n-1)^2}{2}} \cdot 2 \cdot g\left(\frac{n-1}{2}\right) \\ &\leq 4^{\frac{n^2}{2} + n - \frac{1}{2}} \cdot 2 \cdot g\left(\frac{n}{2}\right) \\ &= 4^{\frac{n^2}{2} + n} \cdot g\left(\frac{n}{2}\right). \end{aligned}$$

In both cases we have:

$$f(n) \leq 4^{\frac{n^2}{2} + n} \cdot g\left(\frac{n}{2}\right) = 4^{n^2(\frac{1}{2} + \frac{1}{6}) + n(1+1) - \frac{8}{3}} = g(n).$$

□

3 Paths and cycles in large dicritical digraphs

This section is devoted to the proofs of Theorems 5 and 6. We first prove the following useful observation.

Lemma 17. *For every integer k , if D is a k -dicritical digraph and if D' is obtained from D by adding a vertex u with $N^+(u) = N^-(u) = V(D)$, then D' is $(k+1)$ -dicritical.*

Proof. First we show that $\chi(D') \geq k+1$. Indeed, if $\phi : V(D') \rightarrow [k]$ is a k -dicolouring of D' , then $\phi(v) \neq \phi(u)$ for every $v \in V(D)$, and so ϕ induces a $(k-1)$ -dicolouring of D , a contradiction.

It remains to show that for every arc vw in D' , $D' \setminus vw$ is $(k-1)$ -dicolourable. If $vw \in A(D)$, then since D is k -dicritical, $D \setminus vw$ admits a $(k-1)$ -dicolouring $\phi : V(D) \rightarrow [k-1]$. Then extending ϕ to $V(D')$ by $\phi(u) = k$ yields a k -dicolouring of D' . If $u \in \{v, w\}$, then consider a k -dicolouring $\phi : V(D) \setminus \{v, w\} \rightarrow [k-1]$ of $D' - \{v, w\}$, which exists since $D' - \{v, w\}$ is a proper subdigraph of D . Now set $\phi(v) = \phi(w) = k$. Colour k induces an acyclic digraph of $D' \setminus vw$, and this yields a k -dicolouring of $D' \setminus vw$. □

Theorem 5. *For every integer $k \geq 3$, there are infinitely many k -dicritical digraphs without any directed path on $3k + 1$ vertices.*

Proof. Let n be an odd integer. Let $D_{k,n}$ be the digraph constructed as follows. Start with the antidiirected path $P = p_1 \dots p_n$ on n vertices in which $d^+(p_1) = 1$. Add the digon $[p_1, p_n]$, and two vertices x_1, x_2 with a digon $[x_1, x_2]$. For every arc uv of P , add the arcs vx_i, x_iu for every $i \in [2]$. Finally, add $k - 3$ vertices x_3, \dots, x_{k-1} inducing a copy of \overleftarrow{K}_{k-3} and add the digon $[x_{i+2}, u]$ for every $i \in [k - 3]$ and every $u \in V(P) \cup \{x_1, x_2\}$. See Figure 2 for an illustration.

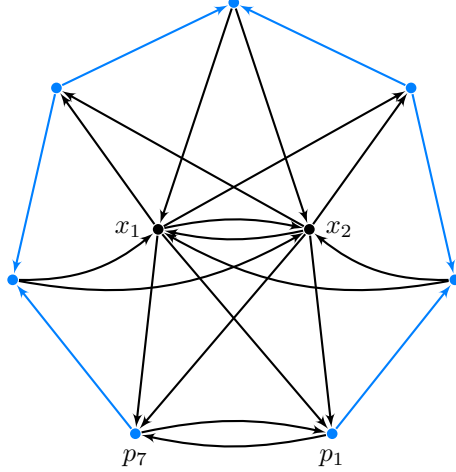


Figure 2: The digraph $D_{3,7}$. The antidiirected path P is in blue.

Let us show that $D_{k,n}$ is k -dicritical. Since $D_{k,n}$ is obtained from $D_{3,n}$ by adding $k - 3$ vertices linked to all other vertices by a digon, by Lemma 17 it is enough to show that $D_{3,n}$ is 3-dicritical.

First we show that $\bar{\chi}(D_{3,n}) > 2$. Suppose for contradiction that there is a 2-dicolouring $\phi : V(D_{3,n}) \rightarrow [2]$ of $D_{3,n}$. Without loss of generality, $\phi(x_1) = 1$ and $\phi(x_2) = 2$. For every arc uv of P , for every $i \in [2]$, uvx_iu is a directed triangle. This implies that $\phi(u) \neq \phi(v)$. Since P has an odd number of vertices, $\phi(p_1) = \phi(p_n)$, which is a contradiction as $[p_1, p_n]$ is a digon in $D_{3,n}$.

Let uv be an arc in $D_{3,n}$. We show that $\bar{\chi}(D_{3,n} \setminus uv) \leq 2$. If $\{u, v\} = \{x_1, x_2\}$, set $\phi(x_1) = \phi(x_2) = \phi(p_1) = 1$, $\phi(p_2) = \dots = \phi(p_n) = 2$. If $\{u, v\} = \{p_1, p_n\}$, set $\phi(x_1) = 1$, $\phi(x_2) = 2$, $\phi(p_i) = 1$ if i is even, $\phi(p_i) = 2$ if i is odd. If $uv \in A(P)$, set $\phi(x_1) = 1$, $\phi(x_2) = 2$, and colour $V(P)$ such that the only monochromatic pair of adjacent vertices in P is $\{u, v\}$. If $u \in V(P)$ and $v = x_i$, set $\phi(x_1) = 1$, $\phi(x_2) = 2$, $\phi(u) = i$, and colour $V(P - u)$ such that two adjacent vertices in $V(P - u)$ receive distinct colours. The other cases are symmetric. In each case, one can check that this gives a proper 2-dicolouring of $D_{3,n} \setminus uv$.

It remains to prove that $D_{k,n}$ does not contain a directed path on $3k + 1$ vertices. Let Q be a directed path in $D_{k,n}$. Let y_1, \dots, y_ℓ be the vertices of $V(Q) \cap \{x_1, \dots, x_{k-1}\}$ in order of appearance along Q . Let Q_i be the subpath $Q[y_i, y_{i+1}[$ of Q for every $i \in [\ell - 1]$ and let $Q_0 = Q[\text{init}(Q), y_1[$ and $Q_\ell = Q[y_\ell, \text{term}(Q)]$. Note that some Q_i s may be empty.

Then observe that at most one of the Q_i s intersects both p_1 and p_n . Except this one, which has at most three vertices, all the Q_j s have at most two vertices (because P is anti-directed). We conclude that the number of vertices in P is at most $3\ell + 3 \leq 3k$. \square

Theorem 6. *For every fixed integers $k \geq 2$ and $\ell \geq 3$, there are finitely many k -dicritical digraphs with no oriented cycle on at least ℓ vertices.*

Proof. Let $k \geq 2$ be a fixed integer. We will show the existence of a function $f_k : \mathbb{N} \rightarrow \mathbb{N}$ such that every k -dicritical digraph on at least $f_k(\ell)$ vertices contains an oriented path on ℓ vertices. We will then use a result of Dirac to show that every k -dicritical digraph on at least $f_k(\frac{1}{4}\ell^2)$ vertices contains an oriented cycle on ℓ vertices, implying the result.

Given a digraph H , $\text{cc}(H)$ is the number of connected components of H (i.e. the number of connected components of $\text{UG}(H)$). Our proof is strongly based on the following claim.

Claim 6.1. Let $D = (V, A)$ be a k -dicritical digraph and $S \subseteq V$, then $\text{cc}(D - S) \leq (k - 1)^{|S|} \cdot 3^{\binom{|S|}{2}}$.

Proof of claim. Assume this is not the case, *i.e.* there exists a k -dicritical digraph D and a subset of its vertices S such that $\text{cc}(D - S) > (k - 1)^{|S|} \cdot 3^{\binom{|S|}{2}}$. We denote by H_1, \dots, H_r the connected components of $D - S$.

For every $i \in [r]$, let α_i be a $(k - 1)$ -dicolouring of $D - V(H_i)$, the existence of which is guaranteed by the dicriticality of D . Let $s = |S|$ and v_1, \dots, v_s be any fixed ordering of S . For every $i \in [r]$, we let σ_i^1 be the ordered set $(\alpha_i(v_1), \dots, \alpha_i(v_s))$. We also define σ_i^2 as the set of all ordered pairs $(u, v) \in S^2$ such that $D - V(H_i)$, coloured with α_i , contains a monochromatic directed path from u to v . We finally define the i^{th} configuration σ_i as the ordered pair (σ_i^1, σ_i^2) .

For every pair of vertices u, v in S and every $i \in [r]$, note that at most one of the ordered pairs $(u, v), (v, u)$ actually belongs to σ_i^2 , for otherwise $D - V(H_i)$, coloured with α_i , contains a monochromatic directed cycle. Hence, the number of distinct configurations is at most $(k - 1)^s \cdot 3^{\binom{s}{2}}$. By the pigeonhole principle, since $r > (k - 1)^s \cdot 3^{\binom{s}{2}}$, there exist two distinct integers $i, j \in [r]$ such that $\sigma_i = \sigma_j$. Let α be the colouring of D defined as follows:

$$\alpha(v) = \begin{cases} \alpha_i(v) & \text{if } v \in (V(D) \setminus V(H_i)) \\ \alpha_j(v) & \text{otherwise.} \end{cases}$$

We claim that α is a $(k - 1)$ -dicolouring of D . Assume for a contradiction that it is not, so D , coloured with α , contains a monochromatic directed cycle. Among all such cycles C , we choose one for which the size of $V(C) \cap V(H_i)$ is minimised. If $V(C) \cap V(H_i) = \emptyset$, then C is a monochromatic directed cycle of $D - H_i$, a contradiction to the choice of α_i . Analogously, we have $V(C) \setminus V(H_i) \neq \emptyset$ by choice of α_j .

Assume first that $|V(C) \setminus V(H_i)| = 1$, implying that C contains exactly one vertex s in S and $(V(C) \setminus \{s\}) \subseteq V(H_i)$. Since $\sigma_i^1 = \sigma_j^1$, we have $\alpha_i(s) = \alpha_j(s)$, which implies that C is a monochromatic directed cycle of $D - H_j$ coloured with α_j , a contradiction to the choice of α_j .

Henceforth we can assume that C contains a directed path P on at least three vertices, with initial vertex u and terminal vertex v , such that $V(P) \cap S = \{u, v\}$ and $V(P) \subseteq (V(H_i) \cup \{u, v\})$. The existence of P ensures that (u, v) belongs to σ_j^2 . Hence, since $\sigma_i = \sigma_j$, there exists a monochromatic directed path P' in $D - V(H_i)$ coloured with α_i , from u to v , and with the same color as P . Hence, replacing P by P' in C , we obtain a closed walk which, coloured with α , contains a monochromatic directed cycle C' such that $|V(C') \cap V(H_i)| < |V(C) \cap V(H_i)|$, a contradiction to the choice of C . \diamond

We are now ready to prove the existence of f_k . Let D be a k -dicritical digraph whose underlying graph G does not contain any path on ℓ vertices. Let v be any vertex of D and T be a spanning DFS-tree of G rooted in v (recall that D is connected since it is dicritical). Let h be the depth of T (*i.e.* the maximum number of vertices in a branch of T), then h is at most ℓ since G does not contain any path of length ℓ .

For every vertex x , let S_x be the ancestors of x (including x itself) in T and $d_T(x)$ be the number of children of x in T . Since T is a DFS-tree, note that for every neighbour y of x , x and y must belong to the same branch. Hence, $d_T(x) \leq \text{cc}(D - S_x)$. Since $|S_x| \leq h \leq \ell$, we deduce from Claim 6.1 that $d_T(x) \leq (k - 1)^{\ell} \cdot 3^{\binom{\ell}{2}}$. Since T is spanning, we obtain that $|V(D)| \leq \left((k - 1)^{\ell} \cdot 3^{\binom{\ell}{2}} \right)^{\ell - 1} = f_k(\ell) - 1$.

Dirac proved that every 2-connected graph that contains a path of length t actually contains a cycle of length at least $2\sqrt{t}$ (see [17]). It is straightforward to show that every k -dicritical digraph is 2-connected. Hence, if D is a k -dicritical digraph on at least $f_k(\frac{1}{4}\ell^2)$ vertices, then D contains an oriented cycle of length at least ℓ , implying the result. \square

4 Subdivisions in digraphs with large digirth

This section is devoted to the proof of Theorem 8.

Theorem 8. Let $k \geq 1$ be an integer. For every non-empty digraph F , if F^* is obtained from F by subdividing every arc at most $k - 1$ times, then $\text{mader}_{\chi}^{(k)}(F^*) \leq \frac{1}{3} (4^{m(F)+1} n(F) - 1)$.

Proof. We proceed by induction on $m(F)$, the result being trivial when $m(F) = 0$. Let F be any digraph with $m > 0$ arcs, and let F^* be a digraph obtained from F by subdividing every arc at most $k - 1$ times.

Let $uv \in A(F)$ be any arc, and $P = x_1, \dots, x_r$ its corresponding directed path in F^* (where $u = x_1$ and $v = x_r$). Then we only have to prove that $\text{mader}_{\vec{\chi}}^{(k)}(F^* \setminus x_1x_2) \leq \frac{4^{m(F)}n(F)-1}{3} + 1$. If this is true, then by Lemma 15, we get that $\text{mader}_{\vec{\chi}}^{(k)}(F^*) \leq 4 \left(\frac{4^{m(F)}n(F)-1}{3} + 1 \right) - 3$ which shows the result.

Let D be any digraph with dichromatic number at least $\frac{4^{m(F)}n(F)-1}{3} + 1$ and digirth at least k and let $B \subseteq V(D)$ be a maximal acyclic set in D . Then $\vec{\chi}(D - B) \geq \frac{4^{m(F)}n(F)-1}{3}$, so by induction $D - B$ must contain a subdivision of $F \setminus uv$ where each arc has been subdivided at least $k - 1$ times. This is also a subdivision of $F^* - \{x_2, \dots, x_{r-1}\}$. Let y be the vertex in $D - B$ corresponding to x_r . By maximality of B , there must be a directed cycle C in D such that $V(C) \cap V(D - B) = \{y\}$. Note that C has length at least k . Thus, ignoring the leaving arc of y in C , we have found a subdivision of $F^* \setminus x_1x_2$ in D , showing the result. \square

5 Subdivisions of trees in digraphs with large digirth

This section is devoted to the proofs of Theorems 9 and 10.

Theorem 9. *Let $k \geq 1$ be an integer and let T be a bidirected tree. If T^* is obtained from T by subdividing every arc at most $k - 1$ times, then $\text{mader}_{\vec{\chi}}^{(2k)}(T^*) \leq \text{mader}_{\vec{\chi}}(T) = n(T)$.*

Proof. We proceed by induction on $n(T)$. Suppose $n(T) \geq 2$, the result being trivial when $n(T) = 1$.

Let f be a leaf of T with neighbour p , and we denote by $(T - f)^*$ the bidirected tree $T - f$ with every arc subdivided exactly $k - 1$ times. By induction hypothesis $\text{mader}_{\vec{\chi}}^{(2k)}((T - f)^*) \leq \text{mader}_{\vec{\chi}}(T - f) = n(T) - 1$. Let D be a digraph with $\text{digirth}(D) \geq 2k$ and $\vec{\chi}(D) \geq n(T)$, and consider a maximal acyclic set A in D . Then $\vec{\chi}(D - A) \geq n(T) - 1$ and so by induction hypothesis, $D - A$ contains a subdivision of $(T - f)^*$. Let $y \in V(D)$ be the vertex corresponding to $p \in V(T)$ in the subdivision of $(T - f)^*$ contained in D . By maximality of A , $A + y$ contains a directed cycle C with $V(C) \setminus A = \{y\}$. As $\text{digirth}(D) \geq 2k$, C has length at least $2k$. Then the subdivision of $(T - f)^*$ in $D - A$ together with C gives the desired subdivision of T^* . \square

Theorem 10. *Let $k \geq 1$ be an integer. Let D be a digraph, T be an oriented tree and T^* obtained from T by subdividing every arc at most $k - 1$ times. If $\vec{\chi}(D) \geq \text{mader}_{\vec{\chi}}(T)$ and $\text{digirth}(D) \geq k$, then D contains T^* as a subdigraph.*

Proof. We proceed by induction on $n(T)$. Suppose $n(T) \geq 2$, the result being trivial when $n(T) = 1$.

For every arc e of T , we denote by $s(e)$ the number of subdivisions of e in T^* . Let f be a leaf of T with neighbour p , and we denote by $(T - f)^*$ the oriented tree $T - f$ where every arc e is subdivided $s(e) \leq k - 1$ times.

Let D be a digraph with $\text{digirth}(D) \geq k$ and $\vec{\chi}(D) \geq n(T)$, and consider a maximal acyclic set A in D . We have $\vec{\chi}(D - A) \geq n(T) - 1$ and, by the induction hypothesis, $D - A$ contains a copy of $(T - f)^*$. Let $y \in V(D) \setminus A$ be the vertex corresponding to $p \in V(T)$ in the copy of $(T - f)^*$ contained in D . By maximality of A , $A + y$ contains a directed cycle C with $C \setminus A = \{y\}$. As $\text{digirth}(D) \geq k$, C has length at least k .

If the arc between p and f goes from p to f then we define P as the directed path on $s(pf)$ vertices starting from y along C . Otherwise, it goes from f to p and then we define P as the directed path on $s(fp)$ vertices, along C , ending on y . In both cases, the copy of $(T - f)^*$ in $D - A$ together with P gives the desired copy of T^* . \square

6 Subdivisions in digraphs of large out-degree and large digirth

6.1 Subdivisions of $C(k, k)$

This section is devoted to the proof of Theorem 11.

Theorem 11. For every integer $k \geq 2$, every digraph D with $\delta^+(D) \geq 2$ and $\text{digirth}(D) \geq 8k - 6$ contains a subdivision of $C(k, k)$.

Proof. We will prove the following stronger statement: for every digraph D with $\text{digirth}(D) \geq 8k - 6$ and $v_0 \in V(D)$, if $d^+(v_0) \geq 1$ and $d^+(v) \geq 2$ for every $v \in V(D) \setminus \{v_0\}$, then D contains a subdivision of $C(k, k)$. We now consider a counterexample to this statement with minimum number of vertices, and minimum number of arcs if equality holds.

Claim 11.1. D is strongly connected.

Proof of claim. Let C be a terminal strongly connected component of D , that is a strongly connected component such that there is no arc going out of C . Then C is also a counterexample, so by minimality of D we have $D = C$, and D is strongly connected. \diamond

Claim 11.2. $d^+(v) = 2$ for every vertex $v \neq v_0$ and $d^+(v_0) = 1$.

Proof of claim. If $v \neq v_0$ is a vertex with at least 3 out-neighbours w_1, w_2, w_3 , then $D \setminus vw_3$ is a smaller counterexample. Similarly, if $d^+(v_0) > 1$, then v_0 has at least two distinct out-neighbours w_1, w_2 , and $D \setminus vw_2$ is a smaller counterexample. \diamond

Given two vertices u, v of D , a (u, v) -vertex-cut is a vertex $x \in V(D) \setminus \{u, v\}$ which intersects every (u, v) -path of D .

Claim 11.3. Let u, v be two vertices in D . If $\text{dist}(u, v) \leq 7k - 6$, then there exists a (v, u) -vertex-cut.

Proof of claim. Suppose the contrary for contradiction. Then by Menger's theorem, there exist two internally vertex-disjoint (v, u) -paths P_1 and P_2 . As $\text{digirth}(D) \geq 8k - 6$, both P_1 and P_2 have length at least k , and so $P_1 \cup P_2$ is a subdivision of $C(k, k)$ with source v and sink u . \diamond

For every directed cycle C in D , let $\rho(C)$ be the number of vertices in the largest connected component of $D - V(C)$. We say that C is isometric if for every $u, v \in V(C)$, C contains a shortest (u, v) -path in D . Clearly D contains an isometric cycle (it is enough to take a minimum directed cycle), and we consider among them an isometric cycle C which maximises $\rho(C)$.

Let ab be an arc along C . Let c_1, \dots, c_ℓ be the (b, a) -vertex-cuts in D . Observe that at least one such vertex-cut exists by Claim 11.3. As C contains a (b, a) -path, all these vertices belong to $V(C)$, and we suppose that they appear in this order c_1, \dots, c_ℓ along C starting at b . By convention we also define $c_0 = b$ and $c_{\ell+1} = a$.

Claim 11.4. $\text{dist}(c_i, c_{i+1}) \leq k - 1$ for every $i = 0, \dots, \ell$.

Proof of claim. Suppose for contradiction that $\text{dist}(c_i, c_{i+1}) \geq k$. Assume first that there exists a (c_i, c_{i+1}) -vertex-cut x . We claim that x is also a (b, a) -vertex-cut. Consider a (b, a) -path Q . Then Q passes through c_i and c_{i+1} in this order, otherwise the concatenation of $Q[b, c_{i+1}]$ and $C[c_{i+1}, a]$ is a (b, a) -path avoiding c_i , a contradiction. By definition of x , it belongs to $Q[c_i, c_{i+1}]$. Hence x intersects every (b, a) -path, and so $x \in \{c_1, \dots, c_\ell\}$, a contradiction since c_1, \dots, c_ℓ are in this order along C .

This shows, by Menger's theorem, that there are two internally vertex-disjoint (c_i, c_{i+1}) -paths P_1, P_2 . Then P_1 and P_2 have length at least k , and so $P_1 \cup P_2$ is a subdivision of $C(k, k)$ with source c_i and sink c_{i+1} , a contradiction. \diamond

Let i_0 be the least index i such that $\text{dist}(b, c_i) \geq k$, and let i_1 be the largest index i such that $\text{dist}(c_i, a) \geq k$. By choice of i_0 , we have $\text{dist}(b, c_{i_0-1}) \leq k - 1$. By Claim 11.4, we have $\text{dist}(c_{i_0-1}, c_{i_0}) \leq k - 1$, which implies $\text{dist}(b, c_{i_0}) \leq \text{dist}(b, c_{i_0-1}) + \text{dist}(c_{i_0-1}, c_{i_0}) \leq 2k - 2$. Similarly we have $\text{dist}(c_{i_1}, a) \leq 2k - 2$. Therefore, we have

$$\text{dist}(c_{i_0}, c_{i_1}) = |V(C)| - \text{dist}(c_{i_1}, a) - 1 - \text{dist}(b, c_{i_0}) \geq 4k - 3,$$

which implies $i_1 - i_0 \geq 5$ by Claim 11.4. See Figure 3 for an illustration.

We now define $d_i = c_{i+i_0}$ for $i = 0, \dots, 5$ and we consider these indices modulo 6. For every $i = 0, \dots, 5$, let X_i be the set of vertices reachable from d_i in $D - d_{i+1}$.

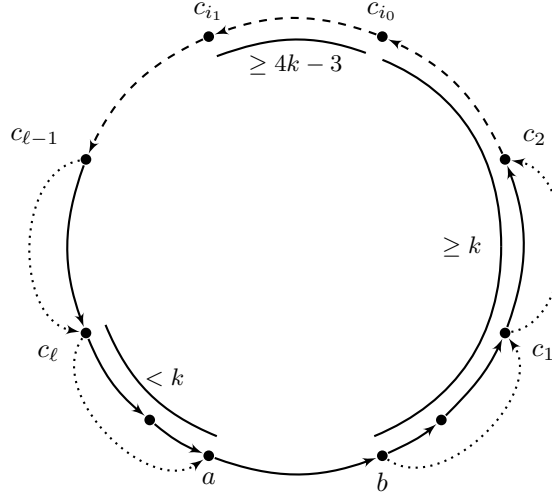


Figure 3: The structure of C in D . The solid and dashed arcs represent the arcs of C . A dotted arc from u to v illustrates the existence of two internally-disjoint directed path from u to v in D .

Claim 11.5. For every $i = 0, \dots, 5$, $X_i \cap C = C[d_i, d_{i+1}[$.

Proof of claim. We first consider the case $i \in \{0, \dots, 4\}$. Assume for a contradiction that there exists a directed path P from $V(C[d_i, d_{i+1}[$ to $V(C) \setminus V(C[d_i, d_{i+1}[$ with internal vertices disjoint from C . Let u be $\text{init}(P)$ and v be $\text{term}(P)$. If $v \in V(C[d_{i+1}, a])$, then $C[b, u] \cup P \cup C[v, a]$ is a (b, a) -path disjoint which avoids d_{i+1} , a contradiction. Otherwise $v \in C[b, d_i[$, and then P has length at least k because, since C is isometric,

$$\text{dist}_P(u, v) \geq \text{dist}_C(u, v) \geq \text{dist}_C(d_5, b) \geq \text{dist}_C(c_{i_1}, a) \geq k.$$

But then $P \cup C[u, v]$ is a subdivision of $C(k, k)$ with source u and sink v , a contradiction to D being a counterexample.

Now suppose $i = 5$. Consider first a directed path P from $u \in C[d_5, a]$ to $v \in C[d_0, d_5[$ internally disjoint from C . Then P has length at least k because

$$\text{dist}_P(u, v) \geq \text{dist}_C(u, v) \geq \text{dist}_C(b, d_0) \geq k.$$

Thus $P \cup C[u, v]$ is a subdivision of $C(k, k)$ with source u and sink v , a contradiction. Consider finally a directed path P from $u \in C[b, d_0[$ to $v \in C[d_0, d_5[$. Then d_0 is not a (b, a) -cut in D , a contradiction. \diamond

Claim 11.6. For every distinct $i, j \in \{0, \dots, 5\}$, we have

- (i) $X_i \cap X_j = \emptyset$ if $i \notin \{j-1, j+1\}$, and
- (ii) $X_i \cap X_{i+1} = \emptyset$ if $v_0 \notin X_i \cup X_{i+1}$.

Proof of claim. We first prove (i). Let us fix two distinct integers $i, j \in \{0, \dots, 5\}$ such that $i \notin \{j-1, j+1\}$. Assume for a contradiction that $X_i \cap X_j \neq \emptyset$. Note that $X_i \cap X_j \neq V(D)$ because $d_{i+1} \notin X_i$. Therefore, since D is strongly connected by Claim 12.2, there is an arc uv such that $u \in X_i \cap X_j$ and $v \in V(D) \setminus (X_i \cap X_j)$. Assume first that $v \in X_j \setminus X_i$. Since $v \notin X_i$, we must have $v = d_{i+1}$. Hence $d_{i+1} \in X_j$, a contradiction to Claim 11.5. Symmetrically, if $v \in X_i \setminus X_j$ then $v = d_{j+1}$ by definition of X_j , implying that $d_{j+1} \in X_i$, a contradiction to Claim 11.5. Finally if $v \notin (X_i \cup X_j)$, by definition of X_i and X_j , we must have $v = d_{i+1} = d_{j+1}$, a contradiction. This proves (i).

We now prove (ii). Assume for a contradiction that $v_0 \notin X_i \cup X_{i+1}$ and $X_i \cap X_{i+1} \neq \emptyset$. Recall that $X_i \cap X_{i+1} \neq V(D)$ because $d_{i+1} \notin X_i$. Therefore, since D is strongly connected, there is an arc uv such that $u \in X_i \cap X_{i+1}$ and $v \in V(D) \setminus (X_i \cap X_{i+1})$. First, if $v \in V(D) \setminus (X_i \cup X_{i+1})$ then, by definition of X_i ,

v must be d_{i+1} , and by definition of X_{i+1} , v must be d_{i+2} , a contradiction. Next if $v \in X_i \setminus X_{i+1}$, then by definition of X_{i+1} , v must be d_{i+2} , but $d_{i+2} \notin X_i$ by Claim 11.5, a contradiction. Then we may assume that $v \in X_{i+1} \setminus X_i$, and by definition of X_i , v must be d_{i+1} . As $u \in X_{i+1}$, there is a directed path P from $V(C[d_{i+1}, d_{i+2}])$ to u in $D - d_{i+2}$ internally disjoint from C . Let x be $\text{init}(P)$. If $x \neq d_{i+1}$, then the union of $P \cup ud_{i+1}$ (which has length at least k because $\text{digirth}(D) \geq 8k - 6 \geq 2k$ and $\text{dist}(d_{i+1}, x) \leq k$ by Claim 11.4) and $C[x, d_{i+1}]$ (which has length at least k because $|V(C)| \geq 8k - 6 \geq 2k$ and $\text{dist}(d_{i+1}, x) \leq k$ by Claim 11.4) is a subdivision of $C(k, k)$ with source x and sink d_{i+1} , a contradiction.

So we assume that $x = d_{i+1}$, that is $P \cup ud_{i+1}$ is a cycle C' with $V(C') \cap V(C) = \{d_{i+1}\}$, and $u \in V(C') \cap X_i \cap X_{i+1}$. Let w be the vertex in $V(C') \cap X_i$ such that $\text{dist}_{C'}(w, d_{i+1})$ is maximum (the existence of w is guaranteed because $u \in V(C') \cap X_i$), and let Q be a (d_i, w) -path in $D[X_i]$. If $\text{dist}_{C'}(w, d_{i+1}) \leq k - 1$, then $C'[d_{i+1}, w]$ has length at least k , and $C[d_{i+1}, d_i] \cup Q$ has length at least k . Moreover, the directed paths $C'[d_{i+1}, w]$ and $C[d_{i+1}, d_i] \cup Q$ are internally vertex-disjoint by the choice of P , w and Q . Hence their union is a subdivision of $C(k, k)$ with source d_{i+1} and sink w , a contradiction. Henceforth we suppose that $\text{dist}_{C'}(w, d_{i+1}) \geq k$.

We now prove the following statement.

$$D - d_{i+1} \text{ does not contain any directed path } R \text{ from } V(C') \text{ to } V(C). \quad (1)$$

Assume for a contradiction that such a directed path R exists. We assume that R is internally disjoint from $V(C') \cup V(C)$, for otherwise we can extract a subpath of R with this extra property. Let $y = \text{init}(R)$ and $z = \text{term}(R)$. Then by Claim 11.5, z belongs to $V(C[d_{i+1}, d_{i+2}])$. Observe that $y \in V(C'[d_{i+1}, w])$, for otherwise y belongs to X_i and so does z , a contradiction to Claim 11.5. But then the union of $R \cup C[z, d_{i+1}]$ and $C'[y, d_{i+1}]$ is a subdivision of $C(k, k)$ with source y and sink d_{i+1} , a contradiction to D being a counterexample. This shows (1).

Let U be the set of vertices reachable from d_i in $D \setminus d_{i+1}t$ where t is the successor of d_{i+1} in C . We claim that $U \subseteq X_i \cup X_{i+1}$. Let u be any vertex in U . By definition, there is a directed path R' from d_i to u in $D \setminus d_{i+1}t$. If $d_{i+1} \notin V(R')$, then $u \in X_i$. Else if $d_{i+1} \in V(R')$ and $d_{i+2} \notin V(R')$, then $u \in X_{i+1}$. Henceforth assume that both d_{i+1} and d_{i+2} belong to R' . Observe that d_{i+1} is before d_{i+2} along R' , otherwise $d_{i+2} \in X_i$, a contradiction to Claim 11.5. Since $d_D^+(d_{i+1}) = 2$, the successor of d_{i+1} in R' is also its successor in C' . Hence $R'[d_{i+1}, d_{i+2}]$ contains a subpath R from $V(C') \setminus \{d_{i+1}\}$ to $V(C) \setminus \{d_{i+1}\}$ internally disjoint from $V(C') \cup V(C)$, a contradiction to (1).

This proves that $U \subseteq X_i \cup X_{i+1}$ and in particular, $v_0 \notin U$. Set $v'_0 = d_{i+1}$, $D' = D[U]$. Then D' equipped with v'_0 is such that every vertex in U has out-degree 2 in D' except v'_0 which has out-degree at least 1. By minimality of $|V(D)|$, D' contains a subdivision of $C(k, k)$ and so does D , a contradiction. \diamond

By (i) of the previous claim, there is an index $j \in \{0, \dots, 5\}$ such that $v_0 \notin X_{j-1}$. Since D is strongly connected, $\bigcup_{i=0}^5 X_i = V(D)$, and so there is an index $i \in \{0, \dots, 5\}$ such that $v_0 \in (X_i \setminus X_{i-1})$. From now on, we fix such an index $i \in \{0, \dots, 5\}$, and we set

$$\begin{aligned} Y_0 &= X_{i-1} \cup X_i \cup X_{i+1} \cup X_{i+2}, \\ Y_1 &= X_{i+3}, \text{ and} \\ Y_2 &= X_{i+4}. \end{aligned}$$

Note that $v_0 \notin X_{i-1} \cup X_{i+2}$, and so, by Claim 11.6, Y_0, Y_1, Y_2 are pairwise vertex-disjoint. Moreover, $Y_0 \setminus V(C), Y_1 \setminus V(C), Y_2 \setminus V(C)$ are pairwise non adjacent by definition of the X_j s (*i.e.* there is no arc of D with head and tail in different parts of $(Y_0 \setminus V(C), Y_1 \setminus V(C), Y_2 \setminus V(C))$). Consider a connected component A of $D - V(C)$ of maximal size, that is with $|A| = \rho(C)$. Then A is included in one of Y_0, Y_1, Y_2 . Let $j \in \{1, 2\}$ be such that $A \cap Y_j = \emptyset$. Let q be the predecessor of d_{i+j+3} in C . Let S be the set of vertices reachable from q in $D - d_{i+j+3}$. Observe that S is a subset of X_{i+j+2} . We claim that $D[S]$ is not acyclic. Indeed, for every vertex $u \in S$, $N_D^+(u) \subseteq N_{D[S]}^+(u) \cup \{d_{i+j+3}\}$. Since $v_0 \notin S$, for every vertex $u \in S$, $d_{D[S]}^+(u) \geq d_D^+(u) - 1 = 1$. Therefore $D[S]$ has minimum out-degree at least 1. Let C' be an isometric cycle in $D[S]$.

Let us show that C' is an isometric cycle in D . Suppose on the contrary that there is a directed path P from $x \in V(C')$ to $y \in V(C')$ internally disjoint from $V(C')$ of length smaller than $\text{dist}_{C'}(x, y)$. As P is not included in S , P must contain d_{i+j+3} . Let d_k be the last vertex along P in $\{d_j \mid j = 0, \dots, 5\}$. We

have $y \in X_k$ by definition of k and $y \in X_{i+j+2}$ because $y \in S \subseteq X_{i+j+2}$. Therefore $y \in X_k \cap X_{i+j+2}$. Since $v_0 \notin X_{i+j+1} \cup X_{i+j+2} \cup X_{i+j+3}$, by Claim 11.6, we deduce that $k = i + j + 2$. Hence P contains a directed path from d_{i+j+3} to d_{i+j+2} . This implies:

$$\begin{aligned}
\text{dist}_{C'}(x, y) &\geq \text{length}(P) && \text{by definition of } P \\
&\geq \text{dist}_D(d_{i+j+3}, d_{i+j+2}) && \text{because } P \text{ contains } d_{i+j+3} \text{ and } d_{i+j+2} \\
&= \text{dist}_C(d_{i+j+3}, d_{i+j+2}) && \text{because } C \text{ is isometric} \\
&= |V(C)| - \text{dist}_C(d_{i+j+2}, d_{i+j+3}) \\
&\geq (8k - 6) - (k - 1) \geq k && \text{by Claim 11.4.}
\end{aligned}$$

Therefore both P and $C'[x, y]$ have length at least k , implying that $P \cup C'[x, y]$ is a subdivision of $C(k, k)$ with source x and sink y , a contradiction to D being a counterexample. This proves that C' is isometric in D .

By definition, $N(A) \subseteq V(C) \cup A$. Since D is strongly connected, A has an in-neighbour in $V(C)$. Since $A \cap Y_j = \emptyset$ by choice of j , A has an in-neighbour in $C[d_{i+j+3}, d_{i+j+2}]$. Hence, the connected component in $D - V(C')$ which contains A is strictly larger than A , which contradicts the maximality of $\rho(C)$, and concludes the proof of the theorem. \square

6.2 Subdivisions of $C(2, 2)$ in oriented graphs

This section is devoted to the proof of Theorem 12.

Theorem 12. *Every oriented graph D with $\delta^+(D) \geq 2$ contains a subdivision of $C(2, 2)$.*

Proof. Suppose for contradiction that there exists an oriented graph D with $\delta^+(D) \geq 2$ that contains no subdivision of $C(2, 2)$. Assume that $|V(D)|$ is minimum, and that among such minimum counterexamples, $|A(D)|$ is minimum.

Claim 12.1. For every vertex $v \in V(D)$, $d^+(v) = 2$.

Proof of claim. If v is a vertex with least 3 out-neighbours w_1, w_2, w_3 , then $D \setminus vw_3$ is a smaller counterexample. \diamond

Claim 12.2. D is strongly connected. In particular, $d^-(v) \geq 1$ for every vertex v .

Proof of claim. Let C be a terminal strongly connected component of D . Then C is also a counterexample, so by minimality of D we have $D = C$, and D is strongly connected. \diamond

Claim 12.3. For every vertex $v \in V(D)$, $d^-(v) \geq 2$.

Proof of claim. Suppose that v is a vertex which has at most one in-neighbour. By Claim 12.2, it must have a unique in-neighbour u , and let w_1, w_2 be its two out-neighbours. If w_1 is non adjacent to u , then consider $D' = D - v \cup uw_1$. By minimality of D , D' contains a subdivision F of $C(2, 2)$, and as $F \not\subseteq D$, we have $uw_1 \in A(F)$. But then $F \setminus uw_1 \cup uv \cup vw_1 \subseteq D$ is a subdivision of $C(2, 2)$. Hence there is an arc between u and w_1 . Similarly, there is an arc between u and w_2 .

If $w_1u, w_2u \in A(D)$, then the union of the directed paths vw_1u and vw_2u yields a copy of $C(2, 2)$ in D . Moreover, if $uw_1, uw_2 \in A(D)$, then $d^+(u) \geq 3$, a contradiction to Claim 12.1. Hence, without loss of generality, $w_1u, uw_2 \in A(D)$.

As D is strongly connected, there is a directed path P from w_2 to $\{u, v, w_1\}$ with internal vertices disjoint from $\{u, v, w_1, w_2\}$. The terminal vertex of P is not v , as the only in-neighbour of v is u . So the terminal vertex of P is either u or w_1 . If it is w_1 , then the union of the directed paths uw_1 and uP yields a subdivision of $C(2, 2)$. If u is the end-vertex of P , then the union of the directed paths vw_1u and

vP yields a subdivision of $C(2, 2)$. In both cases, we find a subdivision of $C(2, 2)$ in D , a contradiction. \diamond

Claim 12.4. D is 2-diregular.

Proof of claim. By Claim 12.1, we know that, for each $v \in V(D)$, $d^+(v) = 2$. It implies that $|A(D)| = \sum_{v \in V(D)} d^+(v) = 2|V(D)|$. Since $|A(D)|$ is also equal to $\sum_{v \in V(D)} d^-(v)$, we get by Claim 12.3 that for every vertex v of D , $d^-(v) = d^+(v) = 2$, which implies that D is 2-diregular. \diamond

Claim 12.5. For every arc vw , w has a neighbour in $N^-(v)$.

Proof of claim. Let u_1, u_2 be the in-neighbours of v . If w has no neighbour in $\{u_1, u_2\}$, consider $D' = D - v \cup u_1w \cup u_2w$. By minimality of D , there exists a subdivision F of $C(2, 2)$ in D' . If neither u_1w nor u_2w belongs to $A(F)$, then $F \subseteq D$, a contradiction. If both u_1w and u_2w belong to $A(F)$, then w is the sink of F and $F \setminus w_1 \cup v$ is a subdivision of $C(2, 2)$ in D , a contradiction. If exactly one of u_1w and u_2w belongs to $A(F)$, say u_1w , then $F \setminus u_1w \cup \{u_1v, vw\}$ is a subdivision of $C(2, 2)$ in D , a contradiction. Hence w_1 has a neighbour in $\{u_1, u_2\}$. \diamond

Claim 12.6. For every vertex v with in-neighbourhood u_1, u_2 and out-neighbourhood w_1, w_2 , either $\{w_1u_1, w_2u_2\} \subseteq A(D)$ or $\{w_1u_2, w_2u_1\} \subseteq A(D)$. In particular, every vertex belongs to two different directed triangles.

Proof of claim. By Claim 12.5, w_1 has a neighbour in $\{u_1, u_2\}$. Without loss of generality, suppose that it is u_1 . We now show that $w_1u_1 \in A(D)$, so assume for a contradiction that $u_1w_1 \in A(D)$. By Claim 12.5 u_1 has an in-neighbour x which is also a neighbour of w_1 .

If $x = u_2$, then $\{u_1, u_2, v, w_1\}$ contains a copy of $C(2, 2)$ with source u_2 and sink w_1 . If $x = w_2$, then either $w_2w_1 \in A(D)$ and w_1 has in-degree 3, a contradiction to Claim 12.4, or $w_1w_2 \in A(D)$ and $\{u_1, v, w_1, w_2\}$ contains a copy of $C(2, 2)$ with source u_1 and sink w_2 . Hence x, u_1, u_2, v, w_1, w_2 are distinct.

Moreover, $xw_1 \notin A(D)$ for otherwise w_1 has in-degree 3 contradicting Claim 12.4. Hence $w_1x \in A(D)$. Consider the out-neighbour y of w_1 distinct from x . By Claim 12.5, y has a neighbour in $N^-(w_1) = \{u_1, v\}$. If v is a neighbour of y , then $y \in \{u_2, w_2\}$. If $y = w_2$, then $\{u_1, v, w_1, w_2\}$ contains a copy of $C(2, 2)$ with source u_1 and sink w_2 . If $y = u_2$, then the union of the directed paths w_1u_2v and w_1xu_1v yields a subdivision of $C(2, 2)$ with source w_1 and sink v .

Hence y is not a neighbour of v , and so y is a neighbour of u_1 . If $u_1y \in A(D)$, then u_1 has out-degree at least 3, contradicting Claim 12.1. If $yu_1 \in A(D)$, then $\{w_1, x, u_1, y\}$ contains a copy of $C(2, 2)$ with source w_1 and sink u_1 . In both cases, we reach a contradiction.

This proves that $w_1u_1 \in A$. Similarly, w_2 has an out-neighbour in $N^-(v) = \{u_1, u_2\}$. If $w_2u_1 \in A(D)$, then $\{v, w_1, w_2, u_1\}$ contains a copy of $C(2, 2)$ with source v and sink u_1 , a contradiction. Hence $w_2u_1 \notin A(D)$, and so $w_2u_2 \in A(D)$ as claimed. \diamond

Claim 12.7. Let t_1 and t_2 be two different directed triangles of D . Then $|V(t_1) \cap V(t_2)| \leq 1$.

Proof of claim. Since D is an oriented graph, then it is clear that $|V(t_1) \cap V(t_2)| \leq 2$. Assume now that $V(t_1) = \{x, y, z\}$ and $V(t_2) = \{x, y, w\}$, where $z \neq w$. Assume without loss of generality that $xy \in A(t_1)$, then $xy \in A(t_2)$ because x and y must be adjacent in t_2 , and D does not contain any digon. Now $t_1 \cup t_2$ contains a copy of $C(2, 2)$ with source y and sink x , a contradiction. \diamond

Consider the undirected graph H whose vertices are the directed triangles in D , and such that two triangles t and t' are adjacent if and only if they share a common vertex.

By Claim 12.7 and Claim 12.4, H is a subcubic graph. Moreover, by Claim 12.6, H must be a cubic graph. In particular H is not a forest and so it contains an induced cycle $C = (t_1, \dots, t_k, t_1)$. Let $t_1 = (x, y, z, x)$ and suppose (by possibly relabelling t_1 and C) that $V(t_1) \cap V(t_k) = \{x\}$ and $V(t_1) \cap V(t_2) = \{y\}$. Let P be a directed path in D with vertices in $V(t_2) \cup \dots \cup V(t_k)$ from y to x . Observe that $z \notin V(P)$ because C is an induced cycle of H . Then $P \cup (y, z, x)$ is a subdivision of $C(2, 2)$ in D , a contradiction. This proves the theorem. \square

6.3 Subdivisions of out-stars

This section is devoted to the proofs of Theorems 13 and 14.

Theorem 13. *For every positive integers k, ℓ , every digraph with $\delta^+(D) \geq k$ and $\text{digirth}(D) \geq \frac{k^\ell - 1}{k - 1} + 1$ contains a copy of $S_k^{+(\ell)}$ with centre u for every chosen vertex u .*

Proof. Let D be such a digraph. By taking $|A(D)|$ minimal, we can suppose that $d^+(v) = k$ for every vertex $v \in V(D)$. Let W be the set of vertices at distance at least ℓ from u . If there are k vertex-disjoint (u, W) -paths then these directed paths have length at least ℓ and so they form a copy of $S_k^{+(\ell)}$. Otherwise, by Menger's Theorem, there is a set $S \subseteq V(D) \setminus \{u\}$ of $k - 1$ vertices such that there is no (u, W) -path in $D - S$. Let R be the set of vertices reachable from u in $D - S$. Then every vertex in R is at distance at most $\ell - 1$ from u , so $|R| \leq \frac{k^\ell - 1}{k - 1}$. As D has digirth at least $\frac{k^\ell - 1}{k - 1} + 1$, this implies that $D[R]$ is acyclic. Let $r \in R$ be a sink in $D[R]$. Then all the out-neighbours of r in D are in S , and so $d^+(r) \leq k - 1$, a contradiction. \square

Theorem 14. *Let D be a digraph with $\delta^+(D) \geq 2$. If $\text{digirth}(D) \geq 2\ell$, then D contains a copy of $S_2^{+(\ell)}$.*

Proof. Suppose for contradiction that there is such a digraph D containing no copy of $S_2^{+(\ell)}$. We assume $\ell \geq 2$, the result being trivial when $\ell = 1$. Without loss of generality, we may also assume that $d^+(v) = 2$ for every vertex v in D . By considering only one terminal strongly connected component of D , we can also assume that D is strong. Let u be a vertex in D , and let w be a vertex at distance exactly ℓ from u . Such a vertex exists because, as D is strong, u has an in-neighbour, which is at distance at least $2\ell - 1 \geq \ell + 1$ from u .

Let us fix P a shortest directed path from u to w . A P -tricot is a sequence of pairwise vertex-disjoint directed paths Q_1, \dots, Q_r (where r is the size of the tricot) such that for every $i \in [r]$:

- $V(Q_i) \cap V(P) = \{\text{term}(Q_i), \text{init}(Q_i)\}$,
- $\text{init}(Q_1) = u$,
- $\text{init}(Q_{i+1}) = \text{pred}_P(\text{term}(Q_i))$ if $i < r$,
- there is an arc from $\text{pred}_{Q_i}(\text{term}(Q_i))$ to $V(P) \setminus \{\text{init}(Q_i), \text{term}(Q_i)\}$.

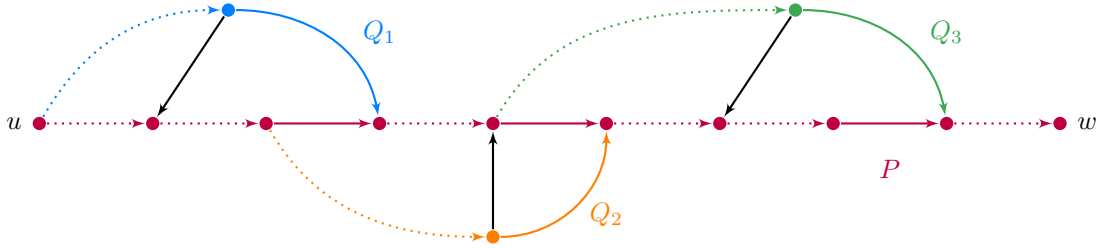


Figure 4: An example of a P -tricot of size 3. Dotted arcs represent directed paths.

Let us first prove that D admits a P -tricot. Let Q be a maximum directed path in D , starting on u , that is disjoint from $V(P) \setminus \{u\}$. Since Q is maximum, the two out-neighbours of $\text{term}(Q)$ belong to $V(P) \cup V(Q)$. We have that the length of Q is at most $\ell - 1$, for otherwise the union of P and Q contains a copy of $S_2^{+(\ell)}$, a contradiction to the choice of D . This implies that the out-neighbourhood of $\text{term}(Q)$ is in $V(P) \setminus V(Q)$, for otherwise $D[V(Q)]$ contains a directed cycle of length at most $\ell - 1$, a contradiction. Let x be the out-neighbour of Q which is the furthest from u and y be its other out-neighbour. Let Q' be the extension of Q with x , then (Q') is a P -tricot of size one since Q' starts at u , intersects P exactly on $\{u, x\}$ and y belongs to $P[u, x[$.

Among all P -tricot of D , we choose one with maximum size r and denote it by \mathcal{T} . Let P_1 be the directed path corresponding to the concatenation of Q_1 and all $P[\text{term}(Q_{i-2}), \text{init}(Q_i)] \cdot Q_i$ for odd $i \in$

$\{3, \dots, r\}$. Let P_2 be the directed path corresponding to the concatenation of all $P[\text{term}(Q_{i-2}), \text{init}(Q_i)] \cdot Q_i$ for even $i \in \{2, \dots, r\}$ (we identify $\text{term}(Q_0)$ with u).

Observe that P_1 and P_2 are two directed paths starting from u , and that they intersect exactly on $\{u\}$. Note that P_1 can be completed into a (u, w) -path $\tilde{P}_1 = P_1 \cdot P[\text{term}(P_1), w]$ disjoint from $P_2 - \text{term}(P_2)$. Since it is a (u, w) -path, \tilde{P}_1 has length at least ℓ . Therefore P_2 has length at most ℓ , for otherwise $\tilde{P}_1 \cup (P_2 - \text{term}(P_2))$ is a copy of $S_2^{+(\ell)}$ in D . Analogously, P_2 can be completed into a (u, w) -path, which implies that P_1 has length at most ℓ .

Let $i \in \{1, 2\}$ be such that P_i does not contain Q_r and let $j \in \{1, 2\}$ be different from i . Let v be $\text{pred}_P(\text{term}(Q_r))$. We consider Q' a maximal directed path starting from v in $D - (V(P) \cup V(P_1) \cup V(P_2) \setminus \{v\})$. Let t be $\text{term}(Q')$, let P'_i be the concatenation $P_i \cdot P[\text{term}(P_i), v] \cdot Q'$ and P'_j be the concatenation $P_j \cdot P[\text{term}(Q_r), w]$. See Figure 5 for an illustration.

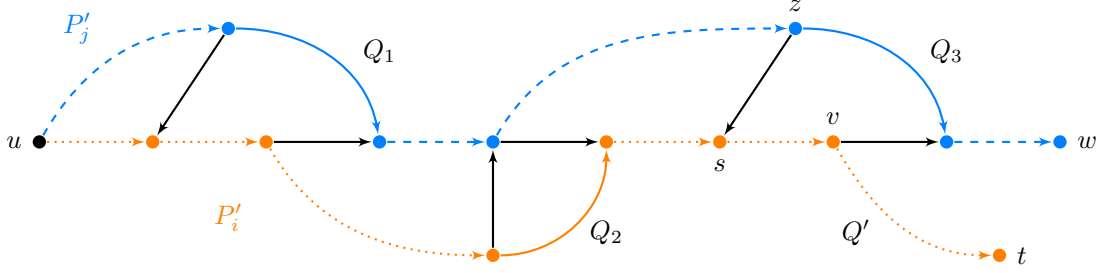


Figure 5: An illustration of the paths P'_i and P'_j .

Since P'_j is a (u, w) -path, P'_j has length at least ℓ . If also P'_i has length at least ℓ , then P'_i and P'_j are two directed paths of length at least ℓ , sharing their source u and no other vertices, a contradiction since D is a counterexample. We know that t has two out-neighbours x, y in D . Since Q' is a maximal directed path, we know that both x and y belong to $V(P'_i) \cup V(P'_j) \cup V(P)$.

- First if one of x, y , say x , belongs to $V(P'_i)$, $P'_i[x, t] \cup \text{term}(P'_i)x$ is a directed cycle of length at most

$$\text{length}(P'_i[x, t]) + 1 \leq \text{length}(P'_i) + 1 \leq \ell,$$

where in the last inequality we used that P'_i has length at most $\ell - 1$. This is a contradiction since $\text{digirth}(D) \geq 2\ell > \ell - 1$.

- Else if one of x, y , say x , belongs to $V(P[u, v])$, then $P[x, v] \cdot Q' \cup tx$ is a directed cycle of length at most:

$$\begin{aligned} & \text{length}(P[x, v]) + \text{length}(Q'[v, t]) + 1 \\ & \leq \text{length}(P[u, v]) + \text{length}(Q'[v, t]) + 1 && \text{because } u \text{ is before } x \text{ in } P \\ & \leq \text{length}(P'_i[u, v]) + \text{length}(Q'[v, t]) + 1 && \text{because } P \text{ is a shortest path} \\ & = \text{length}(P'_i) + 1 \leq \ell, \end{aligned}$$

a contradiction since $\text{digirth}(D) \geq 2\ell > \ell$.

- Else if one of x, y , say x , belongs to $V(P_j) \setminus V(P[u, v])$, let z be $\text{pred}_{P_j}(\text{term}(P_j))$, which is also $\text{pred}_{Q_r}(\text{term}(Q_r))$. By definition of \mathcal{T} , z has an out-neighbour s in $V(P[\text{init}(Q_r), \text{term}(Q_r)])$. Then

$P_j[x, z] \cdot zs \cdot P[s, v] \cdot Q' \cup tx$ is a directed cycle with length at most

$$\begin{aligned}
& \text{length}(P_j[x, z]) + \text{length}(P[s, v]) + \text{length}(Q') + 2 \\
& \leq \text{length}(P_j) + \text{length}(P[s, v]) + \text{length}(Q') && \text{because } x \neq u \text{ and } z \neq \text{term}(P_j) \\
& \leq \text{length}(P_j) + \text{length}(P'_i[u, v]) + \text{length}(Q') && \text{because } P \text{ is a shortest path} \\
& \leq \text{length}(P_j) + \text{length}(P'_i) \leq 2\ell - 1,
\end{aligned}$$

a contradiction since $\text{digirth}(D) \geq 2\ell$.

- Finally if both x and y belong to $P[\text{term}(Q_r), w]$, we can assume that x is before y on the path P . But then the P -tricot $(Q_1, \dots, Q_r, Q' \cdot ty)$ contradicts the maximality of \mathcal{T} .

□

7 Further research directions

In this work, for a fixed digraph F , we gave some sufficient conditions on a digraph D to ensure that D contains F as a subdivision. This is just the tip of the iceberg and many open questions arise. In particular, the exact value of $\text{mader}_{\overleftarrow{\chi}}(F)$ is known only for very few graphs F . The smallest digraph F for which it is unknown is \overleftrightarrow{K}_3 .

Conjecture 16 (Gishboliner, Steiner and Szabó [11]).

$$\text{mader}_{\overleftarrow{\chi}}(\overleftrightarrow{K}_3) = 4$$

In the first part of this paper, we looked for paths and cycles in large dicritical digraphs. In particular, we proved in Theorem 5 that for every integer $k \geq 3$, there are infinitely many k -dicritical digraphs without any directed path on $3k + 1$ vertices. Conversely Bermond et al. [5] proved that every connected digraph with $\delta^+(D) \geq k$ and $\delta^-(D) \geq \ell$ contains a directed path of order at least $\min\{n, k + \ell + 1\}$. As every vertex in a k -dicritical digraph has in- and out-degree at least $k - 1$, we obtain that there are finitely many k -dicritical digraphs with no directed path on $2k - 1$ vertices. The following problem then naturally comes.

Problem 17. For every integer $k \geq 3$, find the largest integer $f(k) \in [2k - 1, 3k]$ such that the set of k -dicritical $\overrightarrow{P}_{f(k)}$ -free digraphs is finite.

Given a digraph F , we say that F is δ^+ -maderian if there is an integer k such that every digraph D with $\delta^+(D) \geq k$ contains a subdivision of F . The smallest such integer k is then denoted by $\text{mader}_{\delta^+}(F)$. The problem of characterising δ^+ -maderian digraphs is widely open. In particular, Mader [18] conjectured that every acyclic digraph is δ^+ -maderian, but this remains unproven albeit many effort to prove or disprove it (see [16] for a partial answering to the conjecture).

In the remaining of the paper, we focused on digraphs of large digirth. Given a digraph F , and an integer g , we can define $\text{mader}_{\delta^+}^{(g)}(F)$ to be the smallest integer k , if it exists, such that every digraph D with $\delta^+(D) \geq k$ and $\text{digirth}(D) \geq g$ contains a subdivision of F .

It is interesting to note that there are digraphs which are not δ^+ -maderian even when restricted to digraphs of large digirth. Indeed, for every integers g, d there is a digraph D with $\text{digirth}(D) \geq g$ and $\delta^+(D) \geq d$ such that D does not contains any subdivision of \overleftrightarrow{K}_3 . Such a digraph D can be easily obtained from a construction by DeVos et al. [8] of digraphs with arbitrarily large out-degree in which every directed cycle has odd length, by removing a few arcs in order to increase the digirth. Since every subdivision of \overleftrightarrow{K}_3 has an even directed cycle, such a digraph does not contain \overleftrightarrow{K}_3 as a subdivision.

In Theorem 11, we proved that $\text{mader}_{\delta^+}^{(8k-4)}(C(k, k)) \leq 2$. On the other hand, the value $8k - 4$ can not be replaced by $k - 1$. To see this, consider the digraph with vertex set $\mathbb{Z}/(2k - 1)\mathbb{Z}$ and arc set $\{(i, i + 1), (i, i + 2) \mid i \in V(G)\}$. Since $|V(D)| < 2k$, D has no subdivision of $C(k, k)$. Since $\delta^+(D) = 2$ and $\text{digirth}(G) = k - 1$, we deduce that $\text{mader}_{\delta^+}^{(k-1)}(C(k, k)) > 2$. Thus the following problem arises.

Problem 18. Find the minimum $g \in [k, 8k - 4]$ such that $\text{mader}_{\delta^+}^{(g)}(C(k, k)) \leq 2$.

In this paper, we studied the value of $\min\{\text{mader}_{\vec{\chi}}^{(g)}(X) \mid g \geq 0\}$ given a digraph X . Actually, we believe that this value is upper bounded by a function of the maximum degree.

Conjecture 19. There is a function f such that for every digraph F with maximum degree Δ , there is an integer g such that $\text{mader}_{\vec{\chi}}^{(g)}(F) \leq f(\Delta)$.

This is motivated by the following result by Mader [19], which is somehow the undirected analog of Conjecture 19.

Theorem 20 (Mader [19]). *There is a function f such that for every graph F , for every graph G with $\delta(G) \geq \max\{\Delta(F), 3\}$, if $\text{girth}(G) \geq f(F)$ then G contains a subdivision of F .*

It was later proved by Kühn and Osthus [14] that one can take $f(H) = 166 \frac{\log |V(H)|}{\log \Delta(H)}$, which is optimal up to the constant factor.

Harutyunyan and Mohar [12] proved that there is a positive constant c such that for every large enough Δ, g , there is a digraph D with $\text{girth}(D) \geq g$, $\Delta(D) \leq \Delta$ and $\vec{\chi}(D) \geq c \cdot \frac{\Delta}{\log \Delta}$. This is a generalisation to the directed case of a classical result by Bollobás [7]. This implies that any function f satisfying Conjecture 19 is such that $f(\Delta) \geq c \cdot \frac{\Delta}{\log \Delta}$. We are inclined to believe that this is optimal and that there is such a function f in $O\left(\frac{\Delta}{\log \Delta}\right)$.

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