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# Subdivisions in dicritical digraphs with large order or digirth\*

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## Abstract

Aboulker et al. proved that a digraph with large enough dichromatic number contains any fixed digraph as a subdivision. The dichromatic number of a digraph is the smallest order of a partition of its vertex set into acyclic induced subdigraphs. A digraph is dicritical if the removal of any arc or vertex decreases its dichromatic number. In this paper we give sufficient conditions on a dicritical digraph of large order or large directed girth to contain a given digraph as a subdivision. In particular, we prove that (i) for every integers  $k, \ell$ , large enough dicritical digraphs with dichromatic number  $k$  contain an orientation of a cycle with at least  $\ell$  vertices; (ii) there are functions  $f, g$  such that for every subdivision  $F^*$  of a digraph  $F$ , digraphs with directed girth at least  $f(F^*)$  and dichromatic number at least  $g(F)$  contain a subdivision of  $F^*$ , and if  $F$  is a tree, then  $g(F) = |V(F)|$ ; (iii) there is a function  $f$  such that for every subdivision  $F^*$  of  $TT_3$  (the transitive tournament on three vertices), digraphs with directed girth at least  $f(F^*)$  and minimum out-degree at least 2 contain  $F^*$  as a subdivision.

**Keywords:** Digraphs, dichromatic number, dicritical digraphs, subdivisions, digirth.

## 1 Introduction

Let  $D$  be a digraph. A  $k$ -colouring of  $D$  is a function  $\phi : V(D) \rightarrow [k]$ . It is a  $k$ -dicolouring if no directed cycle  $C$  in  $D$  is monochromatic for  $\phi$  (that is  $|\phi(V(C))| > 1$ ). Equivalently, it is a  $k$ -dicolouring if every colour class induces an acyclic subdigraph. The smallest integer  $k$  such that  $D$  has a  $k$ -dicolouring is the *dichromatic number* of  $D$  and is denoted by  $\bar{\chi}(D)$ . A digraph  $D$  is  $k$ -dicritical if  $\bar{\chi}(D) = k$  and every proper subdigraph  $H$  of  $D$  satisfies  $\bar{\chi}(H) < k$ . The following is a central result when dealing with substructures in digraphs of large dichromatic number.

**Theorem 1** (Aboulker et al. [1]). *Let  $F$  be a digraph on  $n$  vertices,  $m$  arcs and  $c$  connected components. Every digraph  $D$  satisfying  $\bar{\chi}(D) \geq 4^{m-n+c}(n-1) + 1$  contains a subdivision of  $F$ .*

For every digraph  $F$ , we denote by  $\text{mader}_{\bar{\chi}}(F)$  the least integer  $c_F$  for which every digraph  $D$  either contains a subdivision of  $F$  or has dichromatic number at most  $c_F - 1$ . Note that  $\text{mader}_{\bar{\chi}}(F)$  is well-defined by Theorem 1. The result above was generalized in a recent work of Steiner [21] (see also [15]) who extended it to subdivisions with modular constraints.

Since every digraph is a subdigraph of  $\overleftrightarrow{K}_n$ , the complete digraph on  $n$  vertices, it is natural to look for the value of  $\text{mader}_{\bar{\chi}}(\overleftrightarrow{K}_n)$ . The result above implies that  $\text{mader}_{\bar{\chi}}(\overleftrightarrow{K}_n) \leq 4^{n^2-2n+1}$ . A more precise computation using the tools developed in [1] yields the following best known upper bound, whose proof is presented in Section 2.

**Proposition 2.**  $\text{mader}_{\bar{\chi}}(\overleftrightarrow{K}_n) \leq 4^{\frac{2}{3}n^2+2n-\frac{8}{3}}$ .

For every digraph  $F$  on  $n$  vertices, we have  $\text{mader}_{\bar{\chi}}(F) \geq n$ . This is because  $\overleftrightarrow{K}_{n-1}$ , the complete digraph on  $n-1$  vertices, has dichromatic number  $n-1$  and does not contain any subdivision of  $F$ .

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For some digraphs  $F$ , it then appears that the value of  $\text{mader}_{\bar{\chi}}(F)$  does not capture the structure of  $F$  but only its order. For instance,  $\text{mader}_{\bar{\chi}}(\overrightarrow{C_n}) \geq n$ , but every 2-dicritical digraph on at least  $n$  vertices actually contains a subdivision of  $\overrightarrow{C_n}$ . In order to have a better understanding of digraphs forced to contain subdivisions of  $F$ , one may then ask for the minimum  $k$  such that there is a finite number of  $k$ -dicritical digraphs which do not contain any subdivision of  $F$ . The following question then naturally arises.

**Question 3.** Let  $F^*$  be a subdivision of a digraph  $F$ . Is it true that the set of  $(\text{mader}_{\bar{\chi}}(F))$ -dicritical digraphs that do not contain any subdivision of  $F^*$  is finite?

Unfortunately, the answer to this question is negative. To see that, consider for every positive integers  $k$  and  $\ell$  the digraph  $C(k, \ell)$ , which is the union of two internally disjoint directed paths from a vertex  $x$  to a vertex  $y$  of lengths respectively  $k$  and  $\ell$ . Observe that  $\text{mader}_{\bar{\chi}}(C(1, 2)) = 3$  because  $\overleftarrow{K_2}$  does not contain any subdivision of  $C(1, 2)$  and every 3-dicritical digraph is 2-arc-strong. The following answers Question 3 by the negative for  $F = C(1, 2)$  and  $F^* = C(3, 3)$ .

**Proposition 4.** For every integer  $n \geq 3$ , there exists a 3-dicritical digraph  $D_n$  on  $n$  vertices that does not contain any subdivision of  $C(3, 3)$ .

*Proof.* Let  $D_n$  be the digraph obtained from a directed cycle on  $n - 1$  vertices  $\overrightarrow{C_{n-1}}$  by adding a new vertex  $x$  and all possible digons between  $x$  and  $V(\overrightarrow{C_{n-1}})$  (see Figure 1 for an illustration). This digraph is 3-dicritical (it follows from Lemma 17 and the fact that directed cycles are 2-dicritical) and does not contain any subdivision of  $C(3, 3)$ .  $\square$

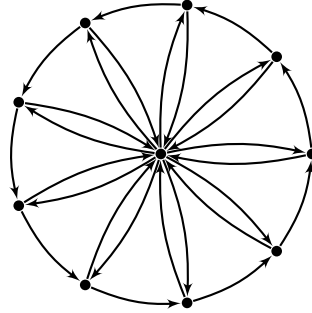


Figure 1: The digraph  $D_{10}$ .

In Section 3, we show that the answer to Question 3 is actually negative not only for  $C(1, 2)$  but for every digraph  $F$  on at least three vertices with at least one arc. Let  $F$  be such a digraph and  $F^*$  be a subdivision of  $F$  in which an arc has been subdivided at least  $3 \cdot \text{mader}_{\bar{\chi}}(F) + 1$  times. Since  $\text{mader}_{\bar{\chi}}(F) \geq |V(F)| \geq 3$ , the following result implies that the set of  $(\text{mader}_{\bar{\chi}}(F))$ -dicritical digraphs that do not contain any subdivision of  $F^*$  is infinite.

**Theorem 5.** For every integer  $k \geq 3$ , there are infinitely many  $k$ -dicritical digraphs without any directed path on  $3k + 1$  vertices.

Theorem 5 establishes a distinction between the directed and undirected cases. In the undirected case, for every fixed  $k \geq 3$ , there exists a non-decreasing function  $f_k : \mathbb{N} \rightarrow \mathbb{N}$  such that every  $k$ -critical graph on at least  $f_k(\ell)$  vertices contains a path on  $\ell$  vertices. This was first proved by Kelly and Kelly [13] in 1954, answering a question of Dirac. The bound on  $f_k$  was then improved by Alon, Krivelevich and Seymour [2] and finally settled by Shapira and Thomas [20], who proved that the largest cycle in a  $k$ -critical graph on  $n$  vertices has length at least  $c_k \cdot \log(n)$ , where  $c_k$  is a constant depending only on  $k$ . This bound is best possible up to the multiplicative constant  $c_k$ , as shown by a construction of Gallai [9, 10] (see [20]).

On the positive side, we adapt the proof of Alon et al. [2] and show that, in  $k$ -dicritical digraphs, the length of the longest oriented cycle (*i.e.* the longest cycle in the underlying graph) grows with the

number of vertices, and so does the length of its longest oriented path (*i.e.* the longest path in the underlying graph).

**Theorem 6.** *For every fixed integers  $k \geq 2$  and  $\ell \geq 3$ , there are finitely many  $k$ -dicritical digraphs with no oriented cycle on at least  $\ell$  vertices.*

Since Question 3 turns out to be almost always false, we propose as an alternative to restrict to digraphs with large digirth. The *girth* of a graph  $G$ , denoted by  $\text{girth}(G)$ , is the length of a smallest cycle in  $G$ , with the convention  $\text{girth}(G) = +\infty$  if  $G$  is a forest. The girth of a digraph  $D$ , denoted by  $\text{girth}(D)$ , is  $\text{girth}(\text{UG}(D))$ . The *digirth* of a digraph is the length of its shortest directed cycle. By convention we have  $\text{digirth}(D) = +\infty$  if  $D$  is acyclic. A celebrated result of Erdős (see [3]) states that there exist graphs of arbitrarily large chromatic number and arbitrarily large girth. It has been generalised by Bokal et al. [6] who showed the existence of digraphs of arbitrarily large girth and dichromatic number. For every integer  $g$ , we denote by  $\text{mader}_{\vec{\chi}}^{(g)}(F)$  the least integer  $k$  such that every digraph  $D$  satisfying  $\vec{\chi}(D) \geq k$  and  $\text{digirth}(D) \geq g$  contains a subdivision of  $F$ . Note that  $\text{mader}_{\vec{\chi}}(F) = \text{mader}_{\vec{\chi}}^{(2)}(F)$  and that  $\text{mader}_{\vec{\chi}}^{(g)}(F)$  is non-increasing in  $g$ . We conjecture that the analogue of Question 3 for digraphs of large digirth is actually true.

**Conjecture 7.** For every digraph  $F$  and every subdivision  $F^*$  of  $F$ , there exists  $g$  such that  $\text{mader}_{\vec{\chi}}^{(g)}(F^*) \leq \text{mader}_{\vec{\chi}}(F)$ .

This conjecture seems to be hard to prove since the exact value of  $\text{mader}_{\vec{\chi}}(F)$  is known for very few classes of digraphs. In order to provide some support to this conjecture, in Section 4 we show that the value of  $\text{mader}_{\vec{\chi}}^{(g)}(F^*)$  depends only on  $F$  when  $g$  is large enough. Our proof is strongly based on the key-lemma of [1].

**Theorem 8.** *Let  $k \geq 1$  be an integer. For every non-empty digraph  $F$ , if  $F^*$  is obtained from  $F$  by subdividing every arc at most  $k - 1$  times, then  $\text{mader}_{\vec{\chi}}^{(k)}(F^*) \leq \frac{1}{3} (4^{m(F)+1} n(F) - 1)$ .*

In Section 5, we prove that Conjecture 7 holds for every digraph  $F$  whose underlying undirected graph  $\text{UG}(F)$  is a forest.

**Theorem 9.** *Let  $k \geq 1$  be an integer and let  $T$  be a bidirected tree. If  $T^*$  is obtained from  $T$  by subdividing every arc at most  $k - 1$  times, then  $\text{mader}_{\vec{\chi}}^{(2k)}(T^*) \leq \text{mader}_{\vec{\chi}}(T) = n(T)$ .*

In the case of oriented trees, we improve Theorem 9 as follows.

**Theorem 10.** *Let  $k \geq 1$  be an integer. Let  $D$  be a digraph,  $T$  be an oriented tree and  $T^*$  obtained from  $T$  by subdividing every arc at most  $k - 1$  times. If  $\vec{\chi}(D) \geq \text{mader}_{\vec{\chi}}(T)$  and  $\text{digirth}(D) \geq k$ , then  $D$  contains  $T^*$  as a subdigraph.*

Observe that every digraph  $D$  contains a subdigraph  $H$  such that  $\delta^+(H) \geq \vec{\chi}(D) - 1$  (by taking  $H$  a  $\vec{\chi}(D)$ -dicritical subdigraph of  $D$ ). Hence, for every integer  $k$ , if a digraph  $F$  is such that every digraph  $D$  with  $\delta^+(D) \geq k - 1$  contains a subdivision of  $F$ , then  $\text{mader}_{\vec{\chi}}(F) \leq k$ . In Section 6, we look for similar results using  $\delta^+$  instead of  $\vec{\chi}$ . Conjecture 7 for  $F = C(1, 2)$  appears to be a consequence of the following theorem (recall that  $\text{mader}_{\vec{\chi}}(C(1, 2)) = 3$ ).

**Theorem 11.** *For every integer  $k \geq 2$ , every digraph  $D$  with  $\delta^+(D) \geq 2$  and  $\text{digirth}(D) \geq 8k - 6$  contains a subdivision of  $C(k, k)$ .*

An *oriented graph* is a digraph with digirth at least 3. When  $k = 2$ , we improve Theorem 11 as follows.

**Theorem 12.** *Every oriented graph  $D$  with  $\delta^+(D) \geq 2$  contains a subdivision of  $C(2, 2)$ .*

We also consider out-stars. For integers  $k, \ell$ , let  $S_k^{+(\ell)}$  be the digraph consisting in  $k$  directed paths of length  $\ell$  sharing their origin (and no other common vertices). The *centre* of  $S_k^{+(\ell)}$  is its unique source.

**Theorem 13.** *For every positive integers  $k, \ell$ , every digraph with  $\delta^+(D) \geq k$  and  $\text{digirth}(D) \geq \frac{k^\ell - 1}{k - 1} + 1$  contains a copy of  $S_k^{+(\ell)}$  with centre  $u$  for every chosen vertex  $u$ .*

When  $k = 2$ , Theorem 13 can be improved by reducing the bound on  $\text{digirth}(D)$  down to  $2\ell$ .

**Theorem 14.** *Let  $D$  be a digraph with  $\delta^+(D) \geq 2$ . If  $\text{digirth}(D) \geq 2\ell$ , then  $D$  contains a copy of  $S_2^{+(\ell)}$ .*

We conclude in Section 7 by some open problems and further research directions.

## Notations

We refer the reader to [4] for notation and terminology not explicitly defined in this paper.

Given a digraph  $D$  (resp. a graph  $G$ ), we denote by  $V(D)$  (resp.  $V(G)$ ) its set of vertices and  $A(D)$  (resp.  $E(G)$ ) its set of arcs (resp. set of edges). For every vertex  $v \in V(D)$ ,  $N^+(v), N^-(v)$  denote respectively the in-neighbourhood and the out-neighbourhood of  $v$  in  $D$ ,  $d^+(v) = |N^+(v)|, d^-(v) = |N^-(v)|$ , and  $d(v) = d^+(v) + d^-(v)$ . Moreover, for every set  $A$  of vertices, we write  $N^+(A) = \bigcup_{a \in A} N^+(a)$ ,  $N^-(A) = \bigcup_{a \in A} N^-(a)$  and  $N(A) = N^+(A) \cup N^-(A)$ . We also define  $\delta^+(D) = \min_{v \in V(D)} d^+(v)$ ,  $\delta^-(D) = \min_{v \in V(D)} d^-(v)$ , and  $\delta(D) = \min_{v \in V(D)} d(v)$ . Similarly  $\Delta^+(D) = \max_{v \in V(D)} d^+(v)$ ,  $\Delta^-(D) = \max_{v \in V(D)} d^-(v)$ , and  $\Delta(D) = \max_{v \in V(D)} d(v)$ .

A *digon* in  $D$  is a pair of opposite arcs between two vertices. Such a pair of arcs  $\{uv, vu\}$  is denoted by  $[u, v]$ . The *underlying graph* of a digraph  $D$ , denoted by  $UG(D)$ , is the undirected graph on the same vertex set which contains an edge linking two vertices if  $D$  contains at least one arc linking these two vertices. An *oriented graph* is a digraph with  $\text{digirth}$  at least 3. A *bidirected digraph* is any digraph  $D$  such that for every  $x, y \in V(D)$ ,  $xy \in A(D) \Leftrightarrow yx \in A(D)$ , and we write  $D = \overleftrightarrow{G}$  where  $G = UG(D)$ . An *oriented cycle* (resp. *oriented path*) is an oriented graph whose underlying graph is a cycle (resp. path). A *directed cycle* is an oriented cycle in which every vertex has in- and out-degree 1. A *directed path* is obtained from a directed cycle by removing a vertex. The directed cycle on  $n$  vertices and the directed path on  $n$  vertices are respectively denoted by  $\overrightarrow{C}_n$  and  $\overrightarrow{P}_n$ . The *lengths* of  $\overrightarrow{P}_n$  and  $\overrightarrow{C}_n$  are respectively  $n - 1$  and  $n$ . An *antidirected path* is an orientation of a path where every vertex  $v$  satisfies  $\min(d^+(v), d^-(v)) = 0$ .

If  $u$  is a vertex of a digraph  $D$  with  $d^-(u) = 1$ , we define the *predecessor* of  $u$  in  $D$ , denoted by  $\text{pred}_D(u)$ , as the unique in-neighbour of  $u$  in  $D$ . Similarly, if  $d^+(u) = 1$ , we define the *successor* of  $u$  in  $D$ , denoted by  $\text{succ}_D(u)$ , as the unique out-neighbour of  $u$  in  $D$ .

Given two vertices  $a, b$  in a directed cycle  $C$  (with possibly  $a = b$ ), we denote by  $C[a, b]$  the directed path from  $a$  to  $b$  along  $C$  (which is the single-vertex path when  $a = b$ ). Moreover, we define  $C[a, b[ = C[a, b] - b$ ,  $C]a, b[ = C[a, b] - a$ , and  $C]a, b[ = C[a, b] - \{a, b\}$ . Note that these subpaths may be empty. Given a directed path  $P$  and two vertices  $a, b$  in  $V(P)$ , we use similar notations  $P[a, b], P[a, b[, P]a, b]$  and  $P]a, b[$ . We denote by  $\text{init}(P)$  the first vertex of  $P$  (i.e. the unique vertex with in-degree 0) and  $\text{term}(P)$  its last one. Given two directed paths  $P, Q$  such that  $V(P) \cap V(Q) = \{x\}$  where  $x = \text{term}(P) = \text{init}(Q)$ , the *concatenation* of  $P$  and  $Q$ , denoted by  $P \cdot Q$ , is the digraph  $(V(P) \cup V(Q), A(P) \cup A(Q))$ . The vertices in  $V(P) \setminus \{\text{init}(P), \text{term}(P)\}$  are called the *internal vertices* of  $P$ . If  $U$  and  $V$  are two sets of vertices in  $D$ , then a  $(U, V)$ -*path* in  $D$  is a directed path  $P$  in  $D$  with  $\text{init}(P) \in U$  and  $\text{term}(P) \in V$ , and we also say that  $P$  is a directed path from  $U$  to  $V$ . If  $U = \{u\}$  (resp.  $V = \{v\}$ ) then we simply write  $u$  for  $U$  (resp.  $v$  for  $V$ ) in these notations. The *distance* from  $u$  to  $v$ , denoted by  $\text{dist}(u, v)$ , is the length of a shortest  $(u, v)$ -path, with the convention  $\text{dist}(u, v) = +\infty$  if no such path exists.

A digraph is *connected* if its underlying graph is connected. It is *strongly connected* if for every ordered pair  $(u, v)$  of its vertices, there exists a directed path from  $u$  to  $v$ .

## 2 An improved bound on $\text{mader}_{\overleftrightarrow{\chi}}(\overleftrightarrow{K}_n)$

This section is devoted to the proof of Theorem 2. We need the two following Lemmas.

**Lemma 15** (Aboulker et al. [1, Lemma 31]). *For every integer  $k$  and every digraph  $D$  with  $\overleftrightarrow{\chi}(D) \geq 4k - 3$ , there is a subdigraph  $H$  of  $D$  with  $\overleftrightarrow{\chi}(H) \geq k$  such that for every pair  $u, v$  of distinct vertices in  $H$ , there is a directed path from  $u$  to  $v$  in  $D$  whose internal vertices are in  $V(D) \setminus V(H)$ .*

*In particular, for every digraph  $F$  and every arc  $e$  in  $F$ ,*

$$\text{mader}_{\overleftrightarrow{\chi}}(F) \leq 4 \cdot \text{mader}_{\overleftrightarrow{\chi}}(F \setminus e) - 3.$$

We skip the proof of the following easy lemma.

**Lemma 16.** *If  $F_1 + F_2$  denotes the disjoint union of two digraphs  $F_1$  and  $F_2$ , then  $\text{mader}_{\overleftarrow{\chi}}(F_1 + F_2) \leq \text{mader}_{\overleftarrow{\chi}}(F_1) + \text{mader}_{\overleftarrow{\chi}}(F_2)$ .*

We are now ready to prove Theorem 2, let us first restate it.

**Proposition 2.**  $\text{mader}_{\overleftarrow{\chi}}(\overleftarrow{K}_n) \leq 4^{\frac{2}{3}n^2 + 2n - \frac{8}{3}}$ .

*Proof.* Let  $f(n) = \text{mader}_{\overleftarrow{\chi}}(\overleftarrow{K}_n)$  for every  $n \geq 1$ . Clearly  $f(1) = 1$ . Let  $g(x) = 4^{\frac{2}{3}x^2 + 2x - \frac{8}{3}}$  for every positive real  $x$ . Observe that  $g$  is non-decreasing. We will show by induction on  $n$  that  $f(n) \leq g(n)$  for every positive integer  $n$ . For  $n = 1$ ,  $f(1) = 1 = g(1)$ . Now suppose  $n \geq 2$ .

If  $n$  is even, by Lemmas 15 and 16 we deduce the following inequalities.

$$\begin{aligned} f(n) &\leq 4^{\frac{n^2}{2}} \cdot \text{mader}_{\overleftarrow{\chi}}\left(\overleftarrow{K}_{\frac{n}{2}} + \overleftarrow{K}_{\frac{n}{2}}\right) \\ &\leq 4^{\frac{n^2}{2}} \cdot 2 \cdot f\left(\frac{n}{2}\right) \\ &\leq 4^{\frac{n^2}{2}} \cdot 2 \cdot g\left(\frac{n}{2}\right) \\ &\leq 4^{\frac{n^2}{2} + n} \cdot g\left(\frac{n}{2}\right). \end{aligned}$$

If  $n$  is odd, then  $f(n) \leq 4^{2(n-1)} \text{mader}_{\overleftarrow{\chi}}\left(\overleftarrow{K}_1 + \overleftarrow{K}_{n-1}\right) \leq 4^{2(n-1)}(1 + f(n-1)) \leq 4^{2n-1}f(n-1)$ . Hence:

$$\begin{aligned} f(n) &\leq 4^{2n-1}f(n-1) \\ &\leq 4^{2n-1} \cdot 4^{\frac{(n-1)^2}{2}} \cdot \text{mader}_{\overleftarrow{\chi}}\left(\overleftarrow{K}_{\frac{n-1}{2}} + \overleftarrow{K}_{\frac{n-1}{2}}\right) \\ &\leq 4^{2n-1} \cdot 4^{\frac{(n-1)^2}{2}} \cdot 2 \cdot f\left(\frac{n-1}{2}\right) \\ &\leq 4^{2n-1} \cdot 4^{\frac{(n-1)^2}{2}} \cdot 2 \cdot g\left(\frac{n-1}{2}\right) \\ &\leq 4^{\frac{n^2}{2} + n - \frac{1}{2}} \cdot 2 \cdot g\left(\frac{n}{2}\right) \\ &= 4^{\frac{n^2}{2} + n} \cdot g\left(\frac{n}{2}\right). \end{aligned}$$

In both cases we have:

$$f(n) \leq 4^{\frac{n^2}{2} + n} \cdot g\left(\frac{n}{2}\right) = 4^{n^2(\frac{1}{2} + \frac{1}{6}) + n(1+1) - \frac{8}{3}} = g(n).$$

□

### 3 Paths and cycles in large dicritical digraphs

This section is devoted to the proofs of Theorems 5 and 6. We first prove the following useful observation.

**Lemma 17.** *For every integer  $k$ , if  $D$  is a  $k$ -dicritical digraph and if  $D'$  is obtained from  $D$  by adding a vertex  $u$  with  $N^+(u) = N^-(u) = V(D)$ , then  $D'$  is  $(k+1)$ -dicritical.*

*Proof.* First we show that  $\chi(D') \geq k+1$ . Indeed, if  $\phi : V(D') \rightarrow [k]$  is a  $k$ -dicolouring of  $D'$ , then  $\phi(v) \neq \phi(u)$  for every  $v \in V(D)$ , and so  $\phi$  induces a  $(k-1)$ -dicolouring of  $D$ , a contradiction.

It remains to show that for every arc  $vw$  in  $D'$ ,  $D' \setminus vw$  is  $(k-1)$ -dicolourable. If  $vw \in A(D)$ , then since  $D$  is  $k$ -dicritical,  $D \setminus vw$  admits a  $(k-1)$ -dicolouring  $\phi : V(D) \rightarrow [k-1]$ . Then extending  $\phi$  to  $V(D')$  by  $\phi(u) = k$  yields a  $k$ -dicolouring of  $D'$ . If  $u \in \{v, w\}$ , then consider a  $k$ -dicolouring  $\phi : V(D) \setminus \{v, w\} \rightarrow [k-1]$  of  $D' - \{v, w\}$ , which exists since  $D' - \{v, w\}$  is a proper subdigraph of  $D$ . Now set  $\phi(v) = \phi(w) = k$ . Colour  $k$  induces an acyclic digraph of  $D' \setminus vw$ , and this yields a  $k$ -dicolouring of  $D' \setminus vw$ . □

**Theorem 5.** *For every integer  $k \geq 3$ , there are infinitely many  $k$ -dicritical digraphs without any directed path on  $3k + 1$  vertices.*

*Proof.* Let  $n$  be an odd integer. Let  $D_{k,n}$  be the digraph constructed as follows. Start with the antidiirected path  $P = p_1 \dots p_n$  on  $n$  vertices in which  $d^+(p_1) = 1$ . Add the digon  $[p_1, p_n]$ , and two vertices  $x_1, x_2$  with a digon  $[x_1, x_2]$ . For every arc  $uv$  of  $P$ , add the arcs  $vx_i, x_iu$  for every  $i \in [2]$ . Finally, add  $k - 3$  vertices  $x_3, \dots, x_{k-1}$  inducing a copy of  $\overleftarrow{K}_{k-3}$  and add the digon  $[x_{i+2}, u]$  for every  $i \in [k - 3]$  and every  $u \in V(P) \cup \{x_1, x_2\}$ . See Figure 2 for an illustration.

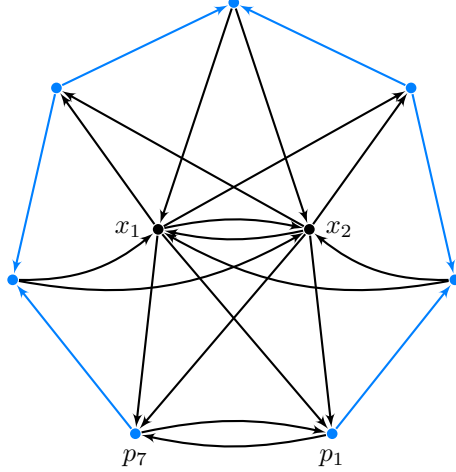


Figure 2: The digraph  $D_{3,7}$ . The antidiirected path  $P$  is in blue.

Let us show that  $D_{k,n}$  is  $k$ -dicritical. Since  $D_{k,n}$  is obtained from  $D_{3,n}$  by adding  $k - 3$  vertices linked to all other vertices by a digon, by Lemma 17 it is enough to show that  $D_{3,n}$  is 3-dicritical.

First we show that  $\bar{\chi}(D_{3,n}) > 2$ . Suppose for contradiction that there is a 2-dicolouring  $\phi : V(D_{3,n}) \rightarrow [2]$  of  $D_{3,n}$ . Without loss of generality,  $\phi(x_1) = 1$  and  $\phi(x_2) = 2$ . For every arc  $uv$  of  $P$ , for every  $i \in [2]$ ,  $uvx_iu$  is a directed triangle. This implies that  $\phi(u) \neq \phi(v)$ . Since  $P$  has an odd number of vertices,  $\phi(p_1) = \phi(p_n)$ , which is a contradiction as  $[p_1, p_n]$  is a digon in  $D_{3,n}$ .

Let  $uv$  be an arc in  $D_{3,n}$ . We show that  $\bar{\chi}(D_{3,n} \setminus uv) \leq 2$ . If  $\{u, v\} = \{x_1, x_2\}$ , set  $\phi(x_1) = \phi(x_2) = \phi(p_1) = 1$ ,  $\phi(p_2) = \dots = \phi(p_n) = 2$ . If  $\{u, v\} = \{p_1, p_n\}$ , set  $\phi(x_1) = 1$ ,  $\phi(x_2) = 2$ ,  $\phi(p_i) = 1$  if  $i$  is even,  $\phi(p_i) = 2$  if  $i$  is odd. If  $uv \in A(P)$ , set  $\phi(x_1) = 1$ ,  $\phi(x_2) = 2$ , and colour  $V(P)$  such that the only monochromatic pair of adjacent vertices in  $P$  is  $\{u, v\}$ . If  $u \in V(P)$  and  $v = x_i$ , set  $\phi(x_1) = 1$ ,  $\phi(x_2) = 2$ ,  $\phi(u) = i$ , and colour  $V(P - u)$  such that two adjacent vertices in  $V(P - u)$  receive distinct colours. The other cases are symmetric. In each case, one can check that this gives a proper 2-dicolouring of  $D_{3,n} \setminus uv$ .

It remains to prove that  $D_{k,n}$  does not contain a directed path on  $3k + 1$  vertices. Let  $Q$  be a directed path in  $D_{k,n}$ . Let  $y_1, \dots, y_\ell$  be the vertices of  $V(Q) \cap \{x_1, \dots, x_{k-1}\}$  in order of appearance along  $Q$ . Let  $Q_i$  be the subpath  $Q[y_i, y_{i+1}[$  of  $Q$  for every  $i \in [\ell - 1]$  and let  $Q_0 = Q[\text{init}(Q), y_1[$  and  $Q_\ell = Q[y_\ell, \text{term}(Q)]$ . Note that some  $Q_i$ s may be empty.

Then observe that at most one of the  $Q_i$ s intersects both  $p_1$  and  $p_n$ . Except this one, which has at most three vertices, all the  $Q_j$ s have at most two vertices (because  $P$  is anti-directed). We conclude that the number of vertices in  $P$  is at most  $3\ell + 3 \leq 3k$ .  $\square$

**Theorem 6.** *For every fixed integers  $k \geq 2$  and  $\ell \geq 3$ , there are finitely many  $k$ -dicritical digraphs with no oriented cycle on at least  $\ell$  vertices.*

*Proof.* Let  $k \geq 2$  be a fixed integer. We will show the existence of a function  $f_k : \mathbb{N} \rightarrow \mathbb{N}$  such that every  $k$ -dicritical digraph on at least  $f_k(\ell)$  vertices contains an oriented path on  $\ell$  vertices. We will then use a result of Dirac to show that every  $k$ -dicritical digraph on at least  $f_k(\frac{1}{4}\ell^2)$  vertices contains an oriented cycle on  $\ell$  vertices, implying the result.

Given a digraph  $H$ ,  $cc(H)$  is the number of connected components of  $H$  (i.e. the number of connected components of  $UG(H)$ ). Our proof is strongly based on the following claim.

**Claim 6.1.** Let  $D = (V, A)$  be a  $k$ -dicritical digraph and  $S \subseteq V$ , then  $\text{cc}(D - S) \leq (k - 1)^{|S|} \cdot 3^{\binom{|S|}{2}}$ .

*Proof of claim.* Assume this is not the case, *i.e.* there exists a  $k$ -dicritical digraph  $D$  and a subset of its vertices  $S$  such that  $\text{cc}(D - S) > (k - 1)^{|S|} \cdot 3^{\binom{|S|}{2}}$ . We denote by  $H_1, \dots, H_r$  the connected components of  $D - S$ .

For every  $i \in [r]$ , let  $\alpha_i$  be a  $(k - 1)$ -dicolouring of  $D - V(H_i)$ , the existence of which is guaranteed by the dicriticality of  $D$ . Let  $s = |S|$  and  $v_1, \dots, v_s$  be any fixed ordering of  $S$ . For every  $i \in [r]$ , we let  $\sigma_i^1$  be the ordered set  $(\alpha_i(v_1), \dots, \alpha_i(v_s))$ . We also define  $\sigma_i^2$  as the set of all ordered pairs  $(u, v) \in S^2$  such that  $D - V(H_i)$ , coloured with  $\alpha_i$ , contains a monochromatic directed path from  $u$  to  $v$ . We finally define the  $i^{\text{th}}$  configuration  $\sigma_i$  as the ordered pair  $(\sigma_i^1, \sigma_i^2)$ .

For every pair of vertices  $u, v$  in  $S$  and every  $i \in [r]$ , note that at most one of the ordered pairs  $(u, v), (v, u)$  actually belongs to  $\sigma_i^2$ , for otherwise  $D - V(H_i)$ , coloured with  $\alpha_i$ , contains a monochromatic directed cycle. Hence, the number of distinct configurations is at most  $(k - 1)^s \cdot 3^{\binom{s}{2}}$ . By the pigeonhole principle, since  $r > (k - 1)^s \cdot 3^{\binom{s}{2}}$ , there exist two distinct integers  $i, j \in [r]$  such that  $\sigma_i = \sigma_j$ . Let  $\alpha$  be the colouring of  $D$  defined as follows:

$$\alpha(v) = \begin{cases} \alpha_i(v) & \text{if } v \in (V(D) \setminus V(H_i)) \\ \alpha_j(v) & \text{otherwise.} \end{cases}$$

We claim that  $\alpha$  is a  $(k - 1)$ -dicolouring of  $D$ . Assume for a contradiction that it is not, so  $D$ , coloured with  $\alpha$ , contains a monochromatic directed cycle. Among all such cycles  $C$ , we choose one for which the size of  $V(C) \cap V(H_i)$  is minimised. If  $V(C) \cap V(H_i) = \emptyset$ , then  $C$  is a monochromatic directed cycle of  $D - H_i$ , a contradiction to the choice of  $\alpha_i$ . Analogously, we have  $V(C) \setminus V(H_i) \neq \emptyset$  by choice of  $\alpha_j$ .

Assume first that  $|V(C) \setminus V(H_i)| = 1$ , implying that  $C$  contains exactly one vertex  $s$  in  $S$  and  $(V(C) \setminus \{s\}) \subseteq V(H_i)$ . Since  $\sigma_i^1 = \sigma_j^1$ , we have  $\alpha_i(s) = \alpha_j(s)$ , which implies that  $C$  is a monochromatic directed cycle of  $D - H_j$  coloured with  $\alpha_j$ , a contradiction to the choice of  $\alpha_j$ .

Henceforth we can assume that  $C$  contains a directed path  $P$  on at least three vertices, with initial vertex  $u$  and terminal vertex  $v$ , such that  $V(P) \cap S = \{u, v\}$  and  $V(P) \subseteq (V(H_i) \cup \{u, v\})$ . The existence of  $P$  ensures that  $(u, v)$  belongs to  $\sigma_j^2$ . Hence, since  $\sigma_i = \sigma_j$ , there exists a monochromatic directed path  $P'$  in  $D - V(H_i)$  coloured with  $\alpha_i$ , from  $u$  to  $v$ , and with the same color as  $P$ . Hence, replacing  $P$  by  $P'$  in  $C$ , we obtain a closed walk which, coloured with  $\alpha$ , contains a monochromatic directed cycle  $C'$  such that  $|V(C') \cap V(H_i)| < |V(C) \cap V(H_i)|$ , a contradiction to the choice of  $C$ .  $\diamond$

We are now ready to prove the existence of  $f_k$ . Let  $D$  be a  $k$ -dicritical digraph whose underlying graph  $G$  does not contain any path on  $\ell$  vertices. Let  $v$  be any vertex of  $D$  and  $T$  be a spanning DFS-tree of  $G$  rooted in  $v$  (recall that  $D$  is connected since it is dicritical). Let  $h$  be the depth of  $T$  (*i.e.* the maximum number of vertices in a branch of  $T$ ), then  $h$  is at most  $\ell$  since  $G$  does not contain any path of length  $\ell$ .

For every vertex  $x$ , let  $S_x$  be the ancestors of  $x$  (including  $x$  itself) in  $T$  and  $d_T(x)$  be the number of children of  $x$  in  $T$ . Since  $T$  is a DFS-tree, note that for every neighbour  $y$  of  $x$ ,  $x$  and  $y$  must belong to the same branch. Hence,  $d_T(x) \leq \text{cc}(D - S_x)$ . Since  $|S_x| \leq h \leq \ell$ , we deduce from Claim 6.1 that  $d_T(x) \leq (k - 1)^{\ell} \cdot 3^{\binom{\ell}{2}}$ . Since  $T$  is spanning, we obtain that  $|V(D)| \leq \left( (k - 1)^{\ell} \cdot 3^{\binom{\ell}{2}} \right)^{\ell - 1} = f_k(\ell) - 1$ .

Dirac proved that every 2-connected graph that contains a path of length  $t$  actually contains a cycle of length at least  $2\sqrt{t}$  (see [17]). It is straightforward to show that every  $k$ -dicritical digraph is 2-connected. Hence, if  $D$  is a  $k$ -dicritical digraph on at least  $f_k(\frac{1}{4}\ell^2)$  vertices, then  $D$  contains an oriented cycle of length at least  $\ell$ , implying the result.  $\square$

## 4 Subdivisions in digraphs with large digirth

This section is devoted to the proof of Theorem 8.

**Theorem 8.** *Let  $k \geq 1$  be an integer. For every non-empty digraph  $F$ , if  $F^*$  is obtained from  $F$  by subdividing every arc at most  $k - 1$  times, then  $\text{mader}_{\chi}^{(k)}(F^*) \leq \frac{1}{3} (4^{m(F)+1} n(F) - 1)$ .*



*Proof.* We proceed by induction on  $m(F)$ , the result being trivial when  $m(F) = 0$ . Let  $F$  be any digraph with  $m > 0$  arcs, and let  $F^*$  be a digraph obtained from  $F$  by subdividing every arc at most  $k - 1$  times.

Let  $uv \in A(F)$  be any arc, and  $P = x_1, \dots, x_r$  its corresponding directed path in  $F^*$  (where  $u = x_1$  and  $v = x_r$ ). Then we only have to prove that  $\text{mader}_{\vec{\chi}}^{(k)}(F^* \setminus x_1x_2) \leq \frac{4^{m(F)}n(F)-1}{3} + 1$ . If this is true, then by Lemma 15, we get that  $\text{mader}_{\vec{\chi}}^{(k)}(F^*) \leq 4 \left( \frac{4^{m(F)}n(F)-1}{3} + 1 \right) - 3$  which shows the result.

Let  $D$  be any digraph with dichromatic number at least  $\frac{4^{m(F)}n(F)-1}{3} + 1$  and digirth at least  $k$  and let  $B \subseteq V(D)$  be a maximal acyclic set in  $D$ . Then  $\vec{\chi}(D - B) \geq \frac{4^{m(F)}n(F)-1}{3}$ , so by induction  $D - B$  must contain a subdivision of  $F \setminus uv$  where each arc has been subdivided at least  $k - 1$  times. This is also a subdivision of  $F^* - \{x_2, \dots, x_{r-1}\}$ . Let  $y$  be the vertex in  $D - B$  corresponding to  $x_r$ . By maximality of  $B$ , there must be a directed cycle  $C$  in  $D$  such that  $V(C) \cap V(D - B) = \{y\}$ . Note that  $C$  has length at least  $k$ . Thus, ignoring the leaving arc of  $y$  in  $C$ , we have found a subdivision of  $F^* \setminus x_1x_2$  in  $D$ , showing the result.  $\square$

## 5 Subdivisions of trees in digraphs with large digirth

This section is devoted to the proofs of Theorems 9 and 10.

**Theorem 9.** *Let  $k \geq 1$  be an integer and let  $T$  be a bidirected tree. If  $T^*$  is obtained from  $T$  by subdividing every arc at most  $k - 1$  times, then  $\text{mader}_{\vec{\chi}}^{(2k)}(T^*) \leq \text{mader}_{\vec{\chi}}(T) = n(T)$ .*

*Proof.* We proceed by induction on  $n(T)$ . Suppose  $n(T) \geq 2$ , the result being trivial when  $n(T) = 1$ .

Let  $f$  be a leaf of  $T$  with neighbour  $p$ , and we denote by  $(T - f)^*$  the bidirected tree  $T - f$  with every arc subdivided exactly  $k - 1$  times. By induction hypothesis  $\text{mader}_{\vec{\chi}}^{(2k)}((T - f)^*) \leq \text{mader}_{\vec{\chi}}(T - f) = n(T) - 1$ . Let  $D$  be a digraph with  $\text{digirth}(D) \geq 2k$  and  $\vec{\chi}(D) \geq n(T)$ , and consider a maximal acyclic set  $A$  in  $D$ . Then  $\vec{\chi}(D - A) \geq n(T) - 1$  and so by induction hypothesis,  $D - A$  contains a subdivision of  $(T - f)^*$ . Let  $y \in V(D)$  be the vertex corresponding to  $p \in V(T)$  in the subdivision of  $(T - f)^*$  contained in  $D$ . By maximality of  $A$ ,  $A + y$  contains a directed cycle  $C$  with  $V(C) \setminus A = \{y\}$ . As  $\text{digirth}(D) \geq 2k$ ,  $C$  has length at least  $2k$ . Then the subdivision of  $(T - f)^*$  in  $D - A$  together with  $C$  gives the desired subdivision of  $T^*$ .  $\square$

**Theorem 10.** *Let  $k \geq 1$  be an integer. Let  $D$  be a digraph,  $T$  be an oriented tree and  $T^*$  obtained from  $T$  by subdividing every arc at most  $k - 1$  times. If  $\vec{\chi}(D) \geq \text{mader}_{\vec{\chi}}(T)$  and  $\text{digirth}(D) \geq k$ , then  $D$  contains  $T^*$  as a subdigraph.*

*Proof.* We proceed by induction on  $n(T)$ . Suppose  $n(T) \geq 2$ , the result being trivial when  $n(T) = 1$ .

For every arc  $e$  of  $T$ , we denote by  $s(e)$  the number of subdivisions of  $e$  in  $T^*$ . Let  $f$  be a leaf of  $T$  with neighbour  $p$ , and we denote by  $(T - f)^*$  the oriented tree  $T - f$  where every arc  $e$  is subdivided  $s(e) \leq k - 1$  times.

Let  $D$  be a digraph with  $\text{digirth}(D) \geq k$  and  $\vec{\chi}(D) \geq n(T)$ , and consider a maximal acyclic set  $A$  in  $D$ . We have  $\vec{\chi}(D - A) \geq n(T) - 1$  and, by the induction hypothesis,  $D - A$  contains a copy of  $(T - f)^*$ . Let  $y \in V(D) \setminus A$  be the vertex corresponding to  $p \in V(T)$  in the copy of  $(T - f)^*$  contained in  $D$ . By maximality of  $A$ ,  $A + y$  contains a directed cycle  $C$  with  $C \setminus A = \{y\}$ . As  $\text{digirth}(D) \geq k$ ,  $C$  has length at least  $k$ .

If the arc between  $p$  and  $f$  goes from  $p$  to  $f$  then we define  $P$  as the directed path on  $s(pf)$  vertices starting from  $y$  along  $C$ . Otherwise, it goes from  $f$  to  $p$  and then we define  $P$  as the directed path on  $s(fp)$  vertices, along  $C$ , ending on  $y$ . In both cases, the copy of  $(T - f)^*$  in  $D - A$  together with  $P$  gives the desired copy of  $T^*$ .  $\square$

## 6 Subdivisions in digraphs of large out-degree and large digirth

### 6.1 Subdivisions of $C(k, k)$

This section is devoted to the proof of Theorem 11.

**Theorem 11.** For every integer  $k \geq 2$ , every digraph  $D$  with  $\delta^+(D) \geq 2$  and  $\text{digirth}(D) \geq 8k - 6$  contains a subdivision of  $C(k, k)$ .

*Proof.* We will prove the following stronger statement: for every digraph  $D$  with  $\text{digirth}(D) \geq 8k - 6$  and  $v_0 \in V(D)$ , if  $d^+(v_0) \geq 1$  and  $d^+(v) \geq 2$  for every  $v \in V(D) \setminus \{v_0\}$ , then  $D$  contains a subdivision of  $C(k, k)$ . We now consider a counterexample to this statement with minimum number of vertices, and minimum number of arcs if equality holds.

**Claim 11.1.**  $D$  is strongly connected.

*Proof of claim.* Let  $C$  be a terminal strongly connected component of  $D$ , that is a strongly connected component such that there is no arc going out of  $C$ . Then  $C$  is also a counterexample, so by minimality of  $D$  we have  $D = C$ , and  $D$  is strongly connected.  $\diamond$

**Claim 11.2.**  $d^+(v) = 2$  for every vertex  $v \neq v_0$  and  $d^+(v_0) = 1$ .

*Proof of claim.* If  $v \neq v_0$  is a vertex with at least 3 out-neighbours  $w_1, w_2, w_3$ , then  $D \setminus vw_3$  is a smaller counterexample. Similarly, if  $d^+(v_0) > 1$ , then  $v_0$  has at least two distinct out-neighbours  $w_1, w_2$ , and  $D \setminus vw_2$  is a smaller counterexample.  $\diamond$

Given two vertices  $u, v$  of  $D$ , a  $(u, v)$ -vertex-cut is a vertex  $x \in V(D) \setminus \{u, v\}$  which intersects every  $(u, v)$ -path of  $D$ .

**Claim 11.3.** Let  $u, v$  be two vertices in  $D$ . If  $\text{dist}(u, v) \leq 7k - 6$ , then there exists a  $(v, u)$ -vertex-cut.

*Proof of claim.* Suppose the contrary for contradiction. Then by Menger's theorem, there exist two internally vertex-disjoint  $(v, u)$ -paths  $P_1$  and  $P_2$ . As  $\text{digirth}(D) \geq 8k - 6$ , both  $P_1$  and  $P_2$  have length at least  $k$ , and so  $P_1 \cup P_2$  is a subdivision of  $C(k, k)$  with source  $v$  and sink  $u$ .  $\diamond$

For every directed cycle  $C$  in  $D$ , let  $\rho(C)$  be the number of vertices in the largest connected component of  $D - V(C)$ . We say that  $C$  is isometric if for every  $u, v \in V(C)$ ,  $C$  contains a shortest  $(u, v)$ -path in  $D$ . Clearly  $D$  contains an isometric cycle (it is enough to take a minimum directed cycle), and we consider among them an isometric cycle  $C$  which maximises  $\rho(C)$ .

Let  $ab$  be an arc along  $C$ . Let  $c_1, \dots, c_\ell$  be the  $(b, a)$ -vertex-cuts in  $D$ . Observe that at least one such vertex-cut exists by Claim 11.3. As  $C$  contains a  $(b, a)$ -path, all these vertices belong to  $V(C)$ , and we suppose that they appear in this order  $c_1, \dots, c_\ell$  along  $C$  starting at  $b$ . By convention we also define  $c_0 = b$  and  $c_{\ell+1} = a$ .

**Claim 11.4.**  $\text{dist}(c_i, c_{i+1}) \leq k - 1$  for every  $i = 0, \dots, \ell$ .

*Proof of claim.* Suppose for contradiction that  $\text{dist}(c_i, c_{i+1}) \geq k$ . Assume first that there exists a  $(c_i, c_{i+1})$ -vertex-cut  $x$ . We claim that  $x$  is also a  $(b, a)$ -vertex-cut. Consider a  $(b, a)$ -path  $Q$ . Then  $Q$  passes through  $c_i$  and  $c_{i+1}$  in this order, otherwise the concatenation of  $Q[b, c_{i+1}]$  and  $C[c_{i+1}, a]$  is a  $(b, a)$ -path avoiding  $c_i$ , a contradiction. By definition of  $x$ , it belongs to  $Q[c_i, c_{i+1}]$ . Hence  $x$  intersects every  $(b, a)$ -path, and so  $x \in \{c_1, \dots, c_\ell\}$ , a contradiction since  $c_1, \dots, c_\ell$  are in this order along  $C$ .

This shows, by Menger's theorem, that there are two internally vertex-disjoint  $(c_i, c_{i+1})$ -paths  $P_1, P_2$ . Then  $P_1$  and  $P_2$  have length at least  $k$ , and so  $P_1 \cup P_2$  is a subdivision of  $C(k, k)$  with source  $c_i$  and sink  $c_{i+1}$ , a contradiction.  $\diamond$

Let  $i_0$  be the least index  $i$  such that  $\text{dist}(b, c_i) \geq k$ , and let  $i_1$  be the largest index  $i$  such that  $\text{dist}(c_i, a) \geq k$ . By choice of  $i_0$ , we have  $\text{dist}(b, c_{i_0-1}) \leq k - 1$ . By Claim 11.4, we have  $\text{dist}(c_{i_0-1}, c_{i_0}) \leq k - 1$ , which implies  $\text{dist}(b, c_{i_0}) \leq \text{dist}(b, c_{i_0-1}) + \text{dist}(c_{i_0-1}, c_{i_0}) \leq 2k - 2$ . Similarly we have  $\text{dist}(c_{i_1}, a) \leq 2k - 2$ . Therefore, we have

$$\text{dist}(c_{i_0}, c_{i_1}) = |V(C)| - \text{dist}(c_{i_1}, a) - 1 - \text{dist}(b, c_{i_0}) \geq 4k - 3,$$

which implies  $i_1 - i_0 \geq 5$  by Claim 11.4. See Figure 3 for an illustration.

We now define  $d_i = c_{i+i_0}$  for  $i = 0, \dots, 5$  and we consider these indices modulo 6. For every  $i = 0, \dots, 5$ , let  $X_i$  be the set of vertices reachable from  $d_i$  in  $D - d_{i+1}$ .

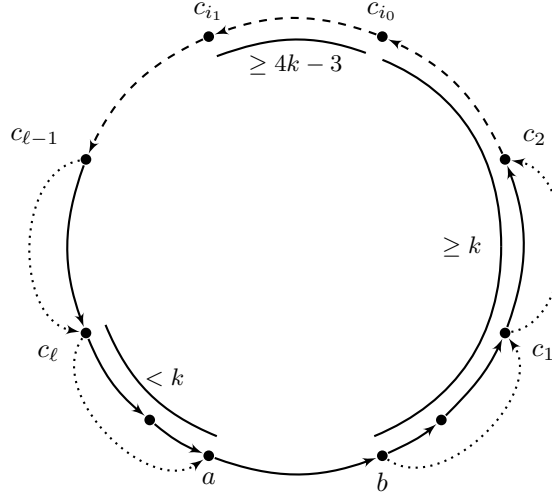


Figure 3: The structure of  $C$  in  $D$ . The solid and dashed arcs represent the arcs of  $C$ . A dotted arc from  $u$  to  $v$  illustrates the existence of two internally-disjoint directed path from  $u$  to  $v$  in  $D$ .

**Claim 11.5.** For every  $i = 0, \dots, 5$ ,  $X_i \cap C = C[d_i, d_{i+1}[$ .

*Proof of claim.* We first consider the case  $i \in \{0, \dots, 4\}$ . Assume for a contradiction that there exists a directed path  $P$  from  $V(C[d_i, d_{i+1}[$  to  $V(C) \setminus V(C[d_i, d_{i+1}[$  with internal vertices disjoint from  $C$ . Let  $u$  be  $\text{init}(P)$  and  $v$  be  $\text{term}(P)$ . If  $v \in V(C[d_{i+1}, a])$ , then  $C[b, u] \cup P \cup C[v, a]$  is a  $(b, a)$ -path disjoint which avoids  $d_{i+1}$ , a contradiction. Otherwise  $v \in C[b, d_i[$ , and then  $P$  has length at least  $k$  because, since  $C$  is isometric,

$$\text{dist}_P(u, v) \geq \text{dist}_C(u, v) \geq \text{dist}_C(d_5, b) \geq \text{dist}_C(c_{i_1}, a) \geq k.$$

But then  $P \cup C[u, v]$  is a subdivision of  $C(k, k)$  with source  $u$  and sink  $v$ , a contradiction to  $D$  being a counterexample.

Now suppose  $i = 5$ . Consider first a directed path  $P$  from  $u \in C[d_5, a]$  to  $v \in C[d_0, d_5[$  internally disjoint from  $C$ . Then  $P$  has length at least  $k$  because

$$\text{dist}_P(u, v) \geq \text{dist}_C(u, v) \geq \text{dist}_C(b, d_0) \geq k.$$

Thus  $P \cup C[u, v]$  is a subdivision of  $C(k, k)$  with source  $u$  and sink  $v$ , a contradiction. Consider finally a directed path  $P$  from  $u \in C[b, d_0[$  to  $v \in C[d_0, d_5[$ . Then  $d_0$  is not a  $(b, a)$ -cut in  $D$ , a contradiction.  $\diamond$

**Claim 11.6.** For every distinct  $i, j \in \{0, \dots, 5\}$ , we have

- (i)  $X_i \cap X_j = \emptyset$  if  $i \notin \{j-1, j+1\}$ , and
- (ii)  $X_i \cap X_{i+1} = \emptyset$  if  $v_0 \notin X_i \cup X_{i+1}$ .

*Proof of claim.* We first prove (i). Let us fix two distinct integers  $i, j \in \{0, \dots, 5\}$  such that  $i \notin \{j-1, j+1\}$ . Assume for a contradiction that  $X_i \cap X_j \neq \emptyset$ . Note that  $X_i \cap X_j \neq V(D)$  because  $d_{i+1} \notin X_i$ . Therefore, since  $D$  is strongly connected by Claim 12.2, there is an arc  $uv$  such that  $u \in X_i \cap X_j$  and  $v \in V(D) \setminus (X_i \cap X_j)$ . Assume first that  $v \in X_j \setminus X_i$ . Since  $v \notin X_i$ , we must have  $v = d_{i+1}$ . Hence  $d_{i+1} \in X_j$ , a contradiction to Claim 11.5. Symmetrically, if  $v \in X_i \setminus X_j$  then  $v = d_{j+1}$  by definition of  $X_j$ , implying that  $d_{j+1} \in X_i$ , a contradiction to Claim 11.5. Finally if  $v \notin (X_i \cup X_j)$ , by definition of  $X_i$  and  $X_j$ , we must have  $v = d_{i+1} = d_{j+1}$ , a contradiction. This proves (i).

We now prove (ii). Assume for a contradiction that  $v_0 \notin X_i \cup X_{i+1}$  and  $X_i \cap X_{i+1} \neq \emptyset$ . Recall that  $X_i \cap X_{i+1} \neq V(D)$  because  $d_{i+1} \notin X_i$ . Therefore, since  $D$  is strongly connected, there is an arc  $uv$  such that  $u \in X_i \cap X_{i+1}$  and  $v \in V(D) \setminus (X_i \cap X_{i+1})$ . First, if  $v \in V(D) \setminus (X_i \cup X_{i+1})$  then, by definition of  $X_i$ ,

$v$  must be  $d_{i+1}$ , and by definition of  $X_{i+1}$ ,  $v$  must be  $d_{i+2}$ , a contradiction. Next if  $v \in X_i \setminus X_{i+1}$ , then by definition of  $X_{i+1}$ ,  $v$  must be  $d_{i+2}$ , but  $d_{i+2} \notin X_i$  by Claim 11.5, a contradiction. Then we may assume that  $v \in X_{i+1} \setminus X_i$ , and by definition of  $X_i$ ,  $v$  must be  $d_{i+1}$ . As  $u \in X_{i+1}$ , there is a directed path  $P$  from  $V(C[d_{i+1}, d_{i+2}])$  to  $u$  in  $D - d_{i+2}$  internally disjoint from  $C$ . Let  $x$  be  $\text{init}(P)$ . If  $x \neq d_{i+1}$ , then the union of  $P \cup ud_{i+1}$  (which has length at least  $k$  because  $\text{digirth}(D) \geq 8k - 6 \geq 2k$  and  $\text{dist}(d_{i+1}, x) \leq k$  by Claim 11.4) and  $C[x, d_{i+1}]$  (which has length at least  $k$  because  $|V(C)| \geq 8k - 6 \geq 2k$  and  $\text{dist}(d_{i+1}, x) \leq k$  by Claim 11.4) is a subdivision of  $C(k, k)$  with source  $x$  and sink  $d_{i+1}$ , a contradiction.

So we assume that  $x = d_{i+1}$ , that is  $P \cup ud_{i+1}$  is a cycle  $C'$  with  $V(C') \cap V(C) = \{d_{i+1}\}$ , and  $u \in V(C') \cap X_i \cap X_{i+1}$ . Let  $w$  be the vertex in  $V(C') \cap X_i$  such that  $\text{dist}_{C'}(w, d_{i+1})$  is maximum (the existence of  $w$  is guaranteed because  $u \in V(C') \cap X_i$ ), and let  $Q$  be a  $(d_i, w)$ -path in  $D[X_i]$ . If  $\text{dist}_{C'}(w, d_{i+1}) \leq k - 1$ , then  $C'[d_{i+1}, w]$  has length at least  $k$ , and  $C[d_{i+1}, d_i] \cup Q$  has length at least  $k$ . Moreover, the directed paths  $C'[d_{i+1}, w]$  and  $C[d_{i+1}, d_i] \cup Q$  are internally vertex-disjoint by the choice of  $P$ ,  $w$  and  $Q$ . Hence their union is a subdivision of  $C(k, k)$  with source  $d_{i+1}$  and sink  $w$ , a contradiction. Henceforth we suppose that  $\text{dist}_{C'}(w, d_{i+1}) \geq k$ .

We now prove the following statement.

$$D - d_{i+1} \text{ does not contain any directed path } R \text{ from } V(C') \text{ to } V(C). \quad (1)$$

Assume for a contradiction that such a directed path  $R$  exists. We assume that  $R$  is internally disjoint from  $V(C') \cup V(C)$ , for otherwise we can extract a subpath of  $R$  with this extra property. Let  $y = \text{init}(R)$  and  $z = \text{term}(R)$ . Then by Claim 11.5,  $z$  belongs to  $V(C)[d_{i+1}, d_{i+2}]$ . Observe that  $y \in V(C')[d_{i+1}, w]$ , for otherwise  $y$  belongs to  $X_i$  and so does  $z$ , a contradiction to Claim 11.5. But then the union of  $R \cup C[z, d_{i+1}]$  and  $C'[y, d_{i+1}]$  is a subdivision of  $C(k, k)$  with source  $y$  and sink  $d_{i+1}$ , a contradiction to  $D$  being a counterexample. This shows (1).

Let  $U$  be the set of vertices reachable from  $d_i$  in  $D \setminus d_{i+1}t$  where  $t$  is the successor of  $d_{i+1}$  in  $C$ . We claim that  $U \subseteq X_i \cup X_{i+1}$ . Let  $u$  be any vertex in  $U$ . By definition, there is a directed path  $R'$  from  $d_i$  to  $u$  in  $D \setminus d_{i+1}t$ . If  $d_{i+1} \notin V(R')$ , then  $u \in X_i$ . Else if  $d_{i+1} \in V(R')$  and  $d_{i+2} \notin V(R')$ , then  $u \in X_{i+1}$ . Henceforth assume that both  $d_{i+1}$  and  $d_{i+2}$  belong to  $R'$ . Observe that  $d_{i+1}$  is before  $d_{i+2}$  along  $R'$ , otherwise  $d_{i+2} \in X_i$ , a contradiction to Claim 11.5. Since  $d_D^+(d_{i+1}) = 2$ , the successor of  $d_{i+1}$  in  $R'$  is also its successor in  $C'$ . Hence  $R'[d_{i+1}, d_{i+2}]$  contains a subpath  $R$  from  $V(C') \setminus \{d_{i+1}\}$  to  $V(C) \setminus \{d_{i+1}\}$  internally disjoint from  $V(C') \cup V(C)$ , a contradiction to (1).

This proves that  $U \subseteq X_i \cup X_{i+1}$  and in particular,  $v_0 \notin U$ . Set  $v'_0 = d_{i+1}$ ,  $D' = D[U]$ . Then  $D'$  equipped with  $v'_0$  is such that every vertex in  $U$  has out-degree 2 in  $D'$  except  $v'_0$  which has out-degree at least 1. By minimality of  $|V(D)|$ ,  $D'$  contains a subdivision of  $C(k, k)$  and so does  $D$ , a contradiction.  $\diamond$

By (i) of the previous claim, there is an index  $j \in \{0, \dots, 5\}$  such that  $v_0 \notin X_{j-1}$ . Since  $D$  is strongly connected,  $\bigcup_{i=0}^5 X_i = V(D)$ , and so there is an index  $i \in \{0, \dots, 5\}$  such that  $v_0 \in (X_i \setminus X_{i-1})$ . From now on, we fix such an index  $i \in \{0, \dots, 5\}$ , and we set

$$\begin{aligned} Y_0 &= X_{i-1} \cup X_i \cup X_{i+1} \cup X_{i+2}, \\ Y_1 &= X_{i+3}, \text{ and} \\ Y_2 &= X_{i+4}. \end{aligned}$$

Note that  $v_0 \notin X_{i-1} \cup X_{i+2}$ , and so, by Claim 11.6,  $Y_0, Y_1, Y_2$  are pairwise vertex-disjoint. Moreover,  $Y_0 \setminus V(C), Y_1 \setminus V(C), Y_2 \setminus V(C)$  are pairwise non adjacent by definition of the  $X_j$ s (*i.e.* there is no arc of  $D$  with head and tail in different parts of  $(Y_0 \setminus V(C), Y_1 \setminus V(C), Y_2 \setminus V(C))$ ). Consider a connected component  $A$  of  $D - V(C)$  of maximal size, that is with  $|A| = \rho(C)$ . Then  $A$  is included in one of  $Y_0, Y_1, Y_2$ . Let  $j \in \{1, 2\}$  be such that  $A \cap Y_j = \emptyset$ . Let  $q$  be the predecessor of  $d_{i+j+3}$  in  $C$ . Let  $S$  be the set of vertices reachable from  $q$  in  $D - d_{i+j+3}$ . Observe that  $S$  is a subset of  $X_{i+j+2}$ . We claim that  $D[S]$  is not acyclic. Indeed, for every vertex  $u \in S$ ,  $N_D^+(u) \subseteq N_{D[S]}^+(u) \cup \{d_{i+j+3}\}$ . Since  $v_0 \notin S$ , for every vertex  $u \in S$ ,  $d_{D[S]}^+(u) \geq d_D^+(u) - 1 = 1$ . Therefore  $D[S]$  has minimum out-degree at least 1. Let  $C'$  be an isometric cycle in  $D[S]$ .

Let us show that  $C'$  is an isometric cycle in  $D$ . Suppose on the contrary that there is a directed path  $P$  from  $x \in V(C')$  to  $y \in V(C')$  internally disjoint from  $V(C')$  of length smaller than  $\text{dist}_{C'}(x, y)$ . As  $P$  is not included in  $S$ ,  $P$  must contain  $d_{i+j+3}$ . Let  $d_k$  be the last vertex along  $P$  in  $\{d_j \mid j = 0, \dots, 5\}$ . We

have  $y \in X_k$  by definition of  $k$  and  $y \in X_{i+j+2}$  because  $y \in S \subseteq X_{i+j+2}$ . Therefore  $y \in X_k \cap X_{i+j+2}$ . Since  $v_0 \notin X_{i+j+1} \cup X_{i+j+2} \cup X_{i+j+3}$ , by Claim 11.6, we deduce that  $k = i + j + 2$ . Hence  $P$  contains a directed path from  $d_{i+j+3}$  to  $d_{i+j+2}$ . This implies:

$$\begin{aligned}
\text{dist}_{C'}(x, y) &\geq \text{length}(P) && \text{by definition of } P \\
&\geq \text{dist}_D(d_{i+j+3}, d_{i+j+2}) && \text{because } P \text{ contains } d_{i+j+3} \text{ and } d_{i+j+2} \\
&= \text{dist}_C(d_{i+j+3}, d_{i+j+2}) && \text{because } C \text{ is isometric} \\
&= |V(C)| - \text{dist}_C(d_{i+j+2}, d_{i+j+3}) \\
&\geq (8k - 6) - (k - 1) \geq k && \text{by Claim 11.4.}
\end{aligned}$$

Therefore both  $P$  and  $C'[x, y]$  have length at least  $k$ , implying that  $P \cup C'[x, y]$  is a subdivision of  $C(k, k)$  with source  $x$  and sink  $y$ , a contradiction to  $D$  being a counterexample. This proves that  $C'$  is isometric in  $D$ .

By definition,  $N(A) \subseteq V(C) \cup A$ . Since  $D$  is strongly connected,  $A$  has an in-neighbour in  $V(C)$ . Since  $A \cap Y_j = \emptyset$  by choice of  $j$ ,  $A$  has an in-neighbour in  $C[d_{i+j+3}, d_{i+j+2}]$ . Hence, the connected component in  $D - V(C')$  which contains  $A$  is strictly larger than  $A$ , which contradicts the maximality of  $\rho(C)$ , and concludes the proof of the theorem.  $\square$

## 6.2 Subdivisions of $C(2, 2)$ in oriented graphs

This section is devoted to the proof of Theorem 12.

**Theorem 12.** *Every oriented graph  $D$  with  $\delta^+(D) \geq 2$  contains a subdivision of  $C(2, 2)$ .*

*Proof.* Suppose for contradiction that there exists an oriented graph  $D$  with  $\delta^+(D) \geq 2$  that contains no subdivision of  $C(2, 2)$ . Assume that  $|V(D)|$  is minimum, and that among such minimum counterexamples,  $|A(D)|$  is minimum.

**Claim 12.1.** For every vertex  $v \in V(D)$ ,  $d^+(v) = 2$ .

*Proof of claim.* If  $v$  is a vertex with least 3 out-neighbours  $w_1, w_2, w_3$ , then  $D \setminus vw_3$  is a smaller counterexample.  $\diamond$

**Claim 12.2.**  $D$  is strongly connected. In particular,  $d^-(v) \geq 1$  for every vertex  $v$ .

*Proof of claim.* Let  $C$  be a terminal strongly connected component of  $D$ . Then  $C$  is also a counterexample, so by minimality of  $D$  we have  $D = C$ , and  $D$  is strongly connected.  $\diamond$

**Claim 12.3.** For every vertex  $v \in V(D)$ ,  $d^-(v) \geq 2$ .

*Proof of claim.* Suppose that  $v$  is a vertex which has at most one in-neighbour. By Claim 12.2, it must have a unique in-neighbour  $u$ , and let  $w_1, w_2$  be its two out-neighbours. If  $w_1$  is non adjacent to  $u$ , then consider  $D' = D - v \cup uw_1$ . By minimality of  $D$ ,  $D'$  contains a subdivision  $F$  of  $C(2, 2)$ , and as  $F \not\subseteq D$ , we have  $uw_1 \in A(F)$ . But then  $F \setminus uw_1 \cup uv \cup vw_1 \subseteq D$  is a subdivision of  $C(2, 2)$ . Hence there is an arc between  $u$  and  $w_1$ . Similarly, there is an arc between  $u$  and  $w_2$ .

If  $w_1u, w_2u \in A(D)$ , then the union of the directed paths  $vw_1u$  and  $vw_2u$  yields a copy of  $C(2, 2)$  in  $D$ . Moreover, if  $uw_1, uw_2 \in A(D)$ , then  $d^+(u) \geq 3$ , a contradiction to Claim 12.1. Hence, without loss of generality,  $w_1u, uw_2 \in A(D)$ .

As  $D$  is strongly connected, there is a directed path  $P$  from  $w_2$  to  $\{u, v, w_1\}$  with internal vertices disjoint from  $\{u, v, w_1, w_2\}$ . The terminal vertex of  $P$  is not  $v$ , as the only in-neighbour of  $v$  is  $u$ . So the terminal vertex of  $P$  is either  $u$  or  $w_1$ . If it is  $w_1$ , then the union of the directed paths  $uw_1$  and  $uP$  yields a subdivision of  $C(2, 2)$ . If  $u$  is the end-vertex of  $P$ , then the union of the directed paths  $vw_1u$  and

$vP$  yields a subdivision of  $C(2, 2)$ . In both cases, we find a subdivision of  $C(2, 2)$  in  $D$ , a contradiction.  $\diamond$

**Claim 12.4.**  $D$  is 2-diregular.

*Proof of claim.* By Claim 12.1, we know that, for each  $v \in V(D)$ ,  $d^+(v) = 2$ . It implies that  $|A(D)| = \sum_{v \in V(D)} d^+(v) = 2|V(D)|$ . Since  $|A(D)|$  is also equal to  $\sum_{v \in V(D)} d^-(v)$ , we get by Claim 12.3 that for every vertex  $v$  of  $D$ ,  $d^-(v) = d^+(v) = 2$ , which implies that  $D$  is 2-diregular.  $\diamond$

**Claim 12.5.** For every arc  $vw$ ,  $w$  has a neighbour in  $N^-(v)$ .

*Proof of claim.* Let  $u_1, u_2$  be the in-neighbours of  $v$ . If  $w$  has no neighbour in  $\{u_1, u_2\}$ , consider  $D' = D - v \cup u_1w \cup u_2w$ . By minimality of  $D$ , there exists a subdivision  $F$  of  $C(2, 2)$  in  $D'$ . If neither  $u_1w$  nor  $u_2w$  belongs to  $A(F)$ , then  $F \subseteq D$ , a contradiction. If both  $u_1w$  and  $u_2w$  belong to  $A(F)$ , then  $w$  is the sink of  $F$  and  $F \setminus w_1 \cup v$  is a subdivision of  $C(2, 2)$  in  $D$ , a contradiction. If exactly one of  $u_1w$  and  $u_2w$  belongs to  $A(F)$ , say  $u_1w$ , then  $F \setminus u_1w \cup \{u_1v, vw\}$  is a subdivision of  $C(2, 2)$  in  $D$ , a contradiction. Hence  $w_1$  has a neighbour in  $\{u_1, u_2\}$ .  $\diamond$

**Claim 12.6.** For every vertex  $v$  with in-neighbourhood  $u_1, u_2$  and out-neighbourhood  $w_1, w_2$ , either  $\{w_1u_1, w_2u_2\} \subseteq A(D)$  or  $\{w_1u_2, w_2u_1\} \subseteq A(D)$ . In particular, every vertex belongs to two different directed triangles.

*Proof of claim.* By Claim 12.5,  $w_1$  has a neighbour in  $\{u_1, u_2\}$ . Without loss of generality, suppose that it is  $u_1$ . We now show that  $w_1u_1 \in A(D)$ , so assume for a contradiction that  $u_1w_1 \in A(D)$ . By Claim 12.5  $u_1$  has an in-neighbour  $x$  which is also a neighbour of  $w_1$ .

If  $x = u_2$ , then  $\{u_1, u_2, v, w_1\}$  contains a copy of  $C(2, 2)$  with source  $u_2$  and sink  $w_1$ . If  $x = w_2$ , then either  $w_2w_1 \in A(D)$  and  $w_1$  has in-degree 3, a contradiction to Claim 12.4, or  $w_1w_2 \in A(D)$  and  $\{u_1, v, w_1, w_2\}$  contains a copy of  $C(2, 2)$  with source  $u_1$  and sink  $w_2$ . Hence  $x, u_1, u_2, v, w_1, w_2$  are distinct.

Moreover,  $xw_1 \notin A(D)$  for otherwise  $w_1$  has in-degree 3 contradicting Claim 12.4. Hence  $w_1x \in A(D)$ . Consider the out-neighbour  $y$  of  $w_1$  distinct from  $x$ . By Claim 12.5,  $y$  has a neighbour in  $N^-(w_1) = \{u_1, v\}$ . If  $v$  is a neighbour of  $y$ , then  $y \in \{u_2, w_2\}$ . If  $y = w_2$ , then  $\{u_1, v, w_1, w_2\}$  contains a copy of  $C(2, 2)$  with source  $u_1$  and sink  $w_2$ . If  $y = u_2$ , then the union of the directed paths  $w_1u_2v$  and  $w_1xu_1v$  yields a subdivision of  $C(2, 2)$  with source  $w_1$  and sink  $v$ .

Hence  $y$  is not a neighbour of  $v$ , and so  $y$  is a neighbour of  $u_1$ . If  $u_1y \in A(D)$ , then  $u_1$  has out-degree at least 3, contradicting Claim 12.1. If  $yu_1 \in A(D)$ , then  $\{w_1, x, u_1, y\}$  contains a copy of  $C(2, 2)$  with source  $w_1$  and sink  $u_1$ . In both cases, we reach a contradiction.

This proves that  $w_1u_1 \in A$ . Similarly,  $w_2$  has an out-neighbour in  $N^-(v) = \{u_1, u_2\}$ . If  $w_2u_1 \in A(D)$ , then  $\{v, w_1, w_2, u_1\}$  contains a copy of  $C(2, 2)$  with source  $v$  and sink  $u_1$ , a contradiction. Hence  $w_2u_1 \notin A(D)$ , and so  $w_2u_2 \in A(D)$  as claimed.  $\diamond$

**Claim 12.7.** Let  $t_1$  and  $t_2$  be two different directed triangles of  $D$ . Then  $|V(t_1) \cap V(t_2)| \leq 1$ .

*Proof of claim.* Since  $D$  is an oriented graph, then it is clear that  $|V(t_1) \cap V(t_2)| \leq 2$ . Assume now that  $V(t_1) = \{x, y, z\}$  and  $V(t_2) = \{x, y, w\}$ , where  $z \neq w$ . Assume without loss of generality that  $xy \in A(t_1)$ , then  $xy \in A(t_2)$  because  $x$  and  $y$  must be adjacent in  $t_2$ , and  $D$  does not contain any digon. Now  $t_1 \cup t_2$  contains a copy of  $C(2, 2)$  with source  $y$  and sink  $x$ , a contradiction.  $\diamond$

Consider the undirected graph  $H$  whose vertices are the directed triangles in  $D$ , and such that two triangles  $t$  and  $t'$  are adjacent if and only if they share a common vertex.

By Claim 12.7 and Claim 12.4,  $H$  is a subcubic graph. Moreover, by Claim 12.6,  $H$  must be a cubic graph. In particular  $H$  is not a forest and so it contains an induced cycle  $C = (t_1, \dots, t_k, t_1)$ . Let  $t_1 = (x, y, z, x)$  and suppose (by possibly relabelling  $t_1$  and  $C$ ) that  $V(t_1) \cap V(t_k) = \{x\}$  and  $V(t_1) \cap V(t_2) = \{y\}$ . Let  $P$  be a directed path in  $D$  with vertices in  $V(t_2) \cup \dots \cup V(t_k)$  from  $y$  to  $x$ . Observe that  $z \notin V(P)$  because  $C$  is an induced cycle of  $H$ . Then  $P \cup (y, z, x)$  is a subdivision of  $C(2, 2)$  in  $D$ , a contradiction. This proves the theorem.  $\square$

### 6.3 Subdivisions of out-stars

This section is devoted to the proofs of Theorems 13 and 14.

**Theorem 13.** *For every positive integers  $k, \ell$ , every digraph with  $\delta^+(D) \geq k$  and  $\text{digirth}(D) \geq \frac{k^\ell - 1}{k - 1} + 1$  contains a copy of  $S_k^{+(\ell)}$  with centre  $u$  for every chosen vertex  $u$ .*

*Proof.* Let  $D$  be such a digraph. By taking  $|A(D)|$  minimal, we can suppose that  $d^+(v) = k$  for every vertex  $v \in V(D)$ . Let  $W$  be the set of vertices at distance at least  $\ell$  from  $u$ . If there are  $k$  vertex-disjoint  $(u, W)$ -paths then these directed paths have length at least  $\ell$  and so they form a copy of  $S_k^{+(\ell)}$ . Otherwise, by Menger's Theorem, there is a set  $S \subseteq V(D) \setminus \{u\}$  of  $k - 1$  vertices such that there is no  $(u, W)$ -path in  $D - S$ . Let  $R$  be the set of vertices reachable from  $u$  in  $D - S$ . Then every vertex in  $R$  is at distance at most  $\ell - 1$  from  $u$ , so  $|R| \leq \frac{k^\ell - 1}{k - 1}$ . As  $D$  has digirth at least  $\frac{k^\ell - 1}{k - 1} + 1$ , this implies that  $D[R]$  is acyclic. Let  $r \in R$  be a sink in  $D[R]$ . Then all the out-neighbours of  $r$  in  $D$  are in  $S$ , and so  $d^+(r) \leq k - 1$ , a contradiction.  $\square$

**Theorem 14.** *Let  $D$  be a digraph with  $\delta^+(D) \geq 2$ . If  $\text{digirth}(D) \geq 2\ell$ , then  $D$  contains a copy of  $S_2^{+(\ell)}$ .*

*Proof.* Suppose for contradiction that there is such a digraph  $D$  containing no copy of  $S_2^{+(\ell)}$ . We assume  $\ell \geq 2$ , the result being trivial when  $\ell = 1$ . Without loss of generality, we may also assume that  $d^+(v) = 2$  for every vertex  $v$  in  $D$ . By considering only one terminal strongly connected component of  $D$ , we can also assume that  $D$  is strong. Let  $u$  be a vertex in  $D$ , and let  $w$  be a vertex at distance exactly  $\ell$  from  $u$ . Such a vertex exists because, as  $D$  is strong,  $u$  has an in-neighbour, which is at distance at least  $2\ell - 1 \geq \ell + 1$  from  $u$ .

Let us fix  $P$  a shortest directed path from  $u$  to  $w$ . A  $P$ -tricot is a sequence of pairwise vertex-disjoint directed paths  $Q_1, \dots, Q_r$  (where  $r$  is the size of the tricot) such that for every  $i \in [r]$ :

- $V(Q_i) \cap V(P) = \{\text{term}(Q_i), \text{init}(Q_i)\}$ ,
- $\text{init}(Q_1) = u$ ,
- $\text{init}(Q_{i+1}) = \text{pred}_P(\text{term}(Q_i))$  if  $i < r$ ,
- there is an arc from  $\text{pred}_{Q_i}(\text{term}(Q_i))$  to  $V(P) \setminus \{\text{init}(Q_i), \text{term}(Q_i)\}$ .

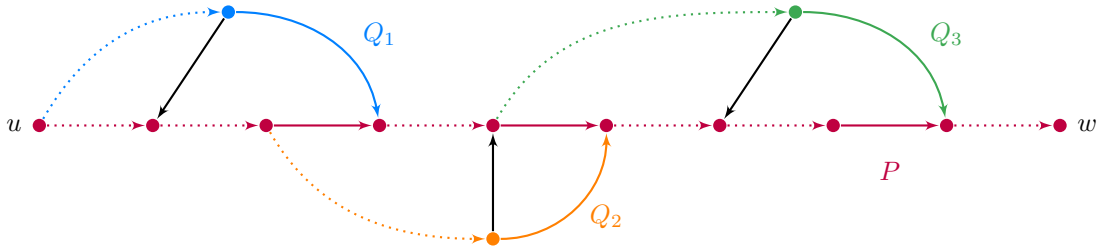


Figure 4: An example of a  $P$ -tricot of size 3. Dotted arcs represent directed paths.

Let us first prove that  $D$  admits a  $P$ -tricot. Let  $Q$  be a maximum directed path in  $D$ , starting on  $u$ , that is disjoint from  $V(P) \setminus \{u\}$ . Since  $Q$  is maximum, the two out-neighbours of  $\text{term}(Q)$  belong to  $V(P) \cup V(Q)$ . We have that the length of  $Q$  is at most  $\ell - 1$ , for otherwise the union of  $P$  and  $Q$  contains a copy of  $S_2^{+(\ell)}$ , a contradiction to the choice of  $D$ . This implies that the out-neighbourhood of  $\text{term}(Q)$  is in  $V(P) \setminus V(Q)$ , for otherwise  $D[V(Q)]$  contains a directed cycle of length at most  $\ell - 1$ , a contradiction. Let  $x$  be the out-neighbour of  $Q$  which is the furthest from  $u$  and  $y$  be its other out-neighbour. Let  $Q'$  be the extension of  $Q$  with  $x$ , then  $(Q')$  is a  $P$ -tricot of size one since  $Q'$  starts at  $u$ , intersects  $P$  exactly on  $\{u, x\}$  and  $y$  belongs to  $P[u, x[$ .

Among all  $P$ -tricot of  $D$ , we choose one with maximum size  $r$  and denote it by  $\mathcal{T}$ . Let  $P_1$  be the directed path corresponding to the concatenation of  $Q_1$  and all  $P[\text{term}(Q_{i-2}), \text{init}(Q_i)] \cdot Q_i$  for odd  $i \in$

$\{3, \dots, r\}$ . Let  $P_2$  be the directed path corresponding to the concatenation of all  $P[\text{term}(Q_{i-2}), \text{init}(Q_i)] \cdot Q_i$  for even  $i \in \{2, \dots, r\}$  (we identify  $\text{term}(Q_0)$  with  $u$ ).

Observe that  $P_1$  and  $P_2$  are two directed paths starting from  $u$ , and that they intersect exactly on  $\{u\}$ . Note that  $P_1$  can be completed into a  $(u, w)$ -path  $\tilde{P}_1 = P_1 \cdot P[\text{term}(P_1), w]$  disjoint from  $P_2 - \text{term}(P_2)$ . Since it is a  $(u, w)$ -path,  $\tilde{P}_1$  has length at least  $\ell$ . Therefore  $P_2$  has length at most  $\ell$ , for otherwise  $\tilde{P}_1 \cup (P_2 - \text{term}(P_2))$  is a copy of  $S_2^{+(\ell)}$  in  $D$ . Analogously,  $P_2$  can be completed into a  $(u, w)$ -path, which implies that  $P_1$  has length at most  $\ell$ .

Let  $i \in \{1, 2\}$  be such that  $P_i$  does not contain  $Q_r$  and let  $j \in \{1, 2\}$  be different from  $i$ . Let  $v$  be  $\text{pred}_P(\text{term}(Q_r))$ . We consider  $Q'$  a maximal directed path starting from  $v$  in  $D - (V(P) \cup V(P_1) \cup V(P_2) \setminus \{v\})$ . Let  $t$  be  $\text{term}(Q')$ , let  $P'_i$  be the concatenation  $P_i \cdot P[\text{term}(P_i), v] \cdot Q'$  and  $P'_j$  be the concatenation  $P_j \cdot P[\text{term}(Q_r), w]$ . See Figure 5 for an illustration.

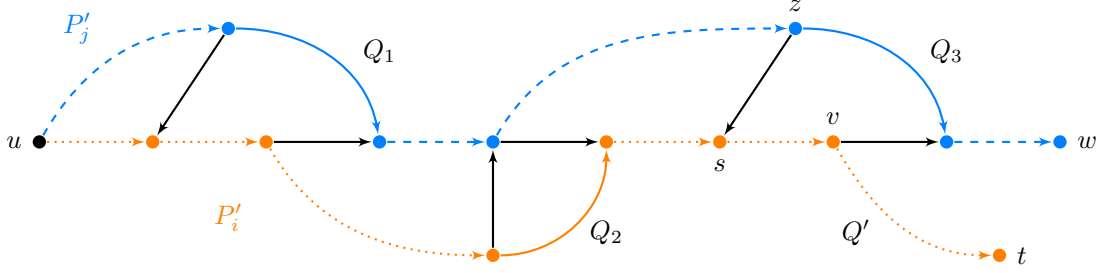


Figure 5: An illustration of the paths  $P'_i$  and  $P'_j$ .

Since  $P'_j$  is a  $(u, w)$ -path,  $P'_j$  has length at least  $\ell$ . If also  $P'_i$  has length at least  $\ell$ , then  $P'_i$  and  $P'_j$  are two directed paths of length at least  $\ell$ , sharing their source  $u$  and no other vertices, a contradiction since  $D$  is a counterexample. We know that  $t$  has two out-neighbours  $x, y$  in  $D$ . Since  $Q'$  is a maximal directed path, we know that both  $x$  and  $y$  belong to  $V(P'_i) \cup V(P'_j) \cup V(P)$ .

- First if one of  $x, y$ , say  $x$ , belongs to  $V(P'_i)$ ,  $P'_i[x, t] \cup \text{term}(P'_i)x$  is a directed cycle of length at most

$$\text{length}(P'_i[x, t]) + 1 \leq \text{length}(P'_i) + 1 \leq \ell,$$

where in the last inequality we used that  $P'_i$  has length at most  $\ell - 1$ . This is a contradiction since  $\text{digirth}(D) \geq 2\ell > \ell - 1$ .

- Else if one of  $x, y$ , say  $x$ , belongs to  $V(P[u, v])$ , then  $P[x, v] \cdot Q' \cup tx$  is a directed cycle of length at most:

$$\begin{aligned} & \text{length}(P[x, v]) + \text{length}(Q'[v, t]) + 1 \\ & \leq \text{length}(P[u, v]) + \text{length}(Q'[v, t]) + 1 && \text{because } u \text{ is before } x \text{ in } P \\ & \leq \text{length}(P'_i[u, v]) + \text{length}(Q'[v, t]) + 1 && \text{because } P \text{ is a shortest path} \\ & = \text{length}(P'_i) + 1 \leq \ell, \end{aligned}$$

a contradiction since  $\text{digirth}(D) \geq 2\ell > \ell$ .

- Else if one of  $x, y$ , say  $x$ , belongs to  $V(P_j) \setminus V(P[u, v])$ , let  $z$  be  $\text{pred}_{P_j}(\text{term}(P_j))$ , which is also  $\text{pred}_{Q_r}(\text{term}(Q_r))$ . By definition of  $\mathcal{T}$ ,  $z$  has an out-neighbour  $s$  in  $V(P[\text{init}(Q_r), \text{term}(Q_r)])$ . Then



$P_j[x, z] \cdot zs \cdot P[s, v] \cdot Q' \cup tx$  is a directed cycle with length at most

$$\begin{aligned}
& \text{length}(P_j[x, z]) + \text{length}(P[s, v]) + \text{length}(Q') + 2 \\
& \leq \text{length}(P_j) + \text{length}(P[s, v]) + \text{length}(Q') && \text{because } x \neq u \text{ and } z \neq \text{term}(P_j) \\
& \leq \text{length}(P_j) + \text{length}(P'_i[u, v]) + \text{length}(Q') && \text{because } P \text{ is a shortest path} \\
& \leq \text{length}(P_j) + \text{length}(P'_i) \leq 2\ell - 1,
\end{aligned}$$

a contradiction since  $\text{digirth}(D) \geq 2\ell$ .

- Finally if both  $x$  and  $y$  belong to  $P[\text{term}(Q_r), w]$ , we can assume that  $x$  is before  $y$  on the path  $P$ . But then the  $P$ -tricot  $(Q_1, \dots, Q_r, Q' \cdot ty)$  contradicts the maximality of  $\mathcal{T}$ .

□

## 7 Further research directions

In this work, for a fixed digraph  $F$ , we gave some sufficient conditions on a digraph  $D$  to ensure that  $D$  contains  $F$  as a subdivision. This is just the tip of the iceberg and many open questions arise. In particular, the exact value of  $\text{mader}_{\overleftarrow{\chi}}(F)$  is known only for very few graphs  $F$ . The smallest digraph  $F$  for which it is unknown is  $\overleftrightarrow{K}_3$ .

**Conjecture 16** (Gishboliner, Steiner and Szabó [11]).

$$\text{mader}_{\overleftarrow{\chi}}(\overleftrightarrow{K}_3) = 4$$

In the first part of this paper, we looked for paths and cycles in large dicritical digraphs. In particular, we proved in Theorem 5 that for every integer  $k \geq 3$ , there are infinitely many  $k$ -dicritical digraphs without any directed path on  $3k + 1$  vertices. Conversely Bermond et al. [5] proved that every connected digraph with  $\delta^+(D) \geq k$  and  $\delta^-(D) \geq \ell$  contains a directed path of order at least  $\min\{n, k + \ell + 1\}$ . As every vertex in a  $k$ -dicritical digraph has in- and out-degree at least  $k - 1$ , we obtain that there are finitely many  $k$ -dicritical digraphs with no directed path on  $2k - 1$  vertices. The following problem then naturally comes.

**Problem 17.** For every integer  $k \geq 3$ , find the largest integer  $f(k) \in [2k - 1, 3k]$  such that the set of  $k$ -dicritical  $\overrightarrow{P}_{f(k)}$ -free digraphs is finite.

Given a digraph  $F$ , we say that  $F$  is  $\delta^+$ -maderian if there is an integer  $k$  such that every digraph  $D$  with  $\delta^+(D) \geq k$  contains a subdivision of  $F$ . The smallest such integer  $k$  is then denoted by  $\text{mader}_{\delta^+}(F)$ . The problem of characterising  $\delta^+$ -maderian digraphs is widely open. In particular, Mader [18] conjectured that every acyclic digraph is  $\delta^+$ -maderian, but this remains unproven albeit many effort to prove or disprove it (see [16] for a partial answering to the conjecture).

In the remaining of the paper, we focused on digraphs of large digirth. Given a digraph  $F$ , and an integer  $g$ , we can define  $\text{mader}_{\delta^+}^{(g)}(F)$  to be the smallest integer  $k$ , if it exists, such that every digraph  $D$  with  $\delta^+(D) \geq k$  and  $\text{digirth}(D) \geq g$  contains a subdivision of  $F$ .

It is interesting to note that there are digraphs which are not  $\delta^+$ -maderian even when restricted to digraphs of large digirth. Indeed, for every integers  $g, d$  there is a digraph  $D$  with  $\text{digirth}(D) \geq g$  and  $\delta^+(D) \geq d$  such that  $D$  does not contains any subdivision of  $\overleftrightarrow{K}_3$ . Such a digraph  $D$  can be easily obtained from a construction by DeVos et al. [8] of digraphs with arbitrarily large out-degree in which every directed cycle has odd length, by removing a few arcs in order to increase the digirth. Since every subdivision of  $\overleftrightarrow{K}_3$  has an even directed cycle, such a digraph does not contain  $\overleftrightarrow{K}_3$  as a subdivision.

In Theorem 11, we proved that  $\text{mader}_{\delta^+}^{(8k-4)}(C(k, k)) \leq 2$ . On the other hand, the value  $8k - 4$  can not be replaced by  $k - 1$ . To see this, consider the digraph with vertex set  $\mathbb{Z}/(2k - 1)\mathbb{Z}$  and arc set  $\{(i, i + 1), (i, i + 2) \mid i \in V(G)\}$ . Since  $|V(D)| < 2k$ ,  $D$  has no subdivision of  $C(k, k)$ . Since  $\delta^+(D) = 2$  and  $\text{digirth}(G) = k - 1$ , we deduce that  $\text{mader}_{\delta^+}^{(k-1)}(C(k, k)) > 2$ . Thus the following problem arises.

**Problem 18.** Find the minimum  $g \in [k, 8k - 4]$  such that  $\text{mader}_{\delta^+}^{(g)}(C(k, k)) \leq 2$ .

In this paper, we studied the value of  $\min\{\text{mader}_{\vec{\chi}}^{(g)}(X) \mid g \geq 0\}$  given a digraph  $X$ . Actually, we believe that this value is upper bounded by a function of the maximum degree.

**Conjecture 19.** There is a function  $f$  such that for every digraph  $F$  with maximum degree  $\Delta$ , there is an integer  $g$  such that  $\text{mader}_{\vec{\chi}}^{(g)}(F) \leq f(\Delta)$ .

This is motivated by the following result by Mader [19], which is somehow the undirected analog of Conjecture 19.

**Theorem 20** (Mader [19]). *There is a function  $f$  such that for every graph  $F$ , for every graph  $G$  with  $\delta(G) \geq \max\{\Delta(F), 3\}$ , if  $\text{girth}(G) \geq f(F)$  then  $G$  contains a subdivision of  $F$ .*

It was later proved by Kühn and Osthus [14] that one can take  $f(H) = 166 \frac{\log |V(H)|}{\log \Delta(H)}$ , which is optimal up to the constant factor.

Harutyunyan and Mohar [12] proved that there is a positive constant  $c$  such that for every large enough  $\Delta, g$ , there is a digraph  $D$  with  $\text{girth}(D) \geq g$ ,  $\Delta(D) \leq \Delta$  and  $\vec{\chi}(D) \geq c \cdot \frac{\Delta}{\log \Delta}$ . This is a generalisation to the directed case of a classical result by Bollobás [7]. This implies that any function  $f$  satisfying Conjecture 19 is such that  $f(\Delta) \geq c \cdot \frac{\Delta}{\log \Delta}$ . We are inclined to believe that this is optimal and that there is such a function  $f$  in  $O\left(\frac{\Delta}{\log \Delta}\right)$ .

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