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COMPLEX CONJUGATION AND SIMPLICIAL ALGEBRAIC HYPERSURFACES

CHARLES ARNAL

ABSTRACT. We call a real algebraic hypersurface in $(\mathbb{C}^*)^n$ *simplicial* if it is given by a real Laurent polynomial in n -variables that has exactly $n + 1$ monomials with non-zero coefficients and such that the convex hull in \mathbb{R}^n of the $n + 1$ points of \mathbb{Z}^n corresponding to the exponents is a non-degenerate n -dimensional simplex. Such hypersurfaces are natural building blocks from which more complicated objects can be constructed, for example using O. Viro's Patchworking method. Drawing inspiration from related work by G. Kerr and I. Zharkov [KZ16], we describe the action of the complex conjugation on the homology of the coamoebas of simplicial real algebraic hypersurfaces, hoping it might prove useful in a variety of problems related to topology of real algebraic varieties. In particular, assuming a reasonable conjecture inspired by [KZ16], we identify the conditions under which such a hypersurface is Galois maximal.

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1. INTRODUCTION

Any non-constant real Laurent polynomial P in n variables defines a real algebraic hypersurface X_P in the complex torus $(\mathbb{C}^*)^n$. Given such a polynomial $P(z) = \sum_{\lambda=(\lambda_1, \dots, \lambda_n) \in \Lambda} c_\lambda z_1^{\lambda_1}, \dots, z_n^{\lambda_n}$, where Λ is a finite subset of \mathbb{Z}^n and $c_\lambda \in \mathbb{R}^*$ for all $\lambda \in \Lambda$, we call the convex hull in \mathbb{R}^n of Λ the *Newton polytope* of P .

If P has exactly $n + 1$ monomials with non-zero coefficients and if the Newton polytope Δ_P of P is a non-degenerate n -dimensional simplex, we call P a *simplicial* real polynomial and X_P a *simplicial* real algebraic hypersurface. In particular, the monomials of P and the vertices of Δ_P are in bijection.

Let X be a simplicial real algebraic hypersurface. We denote the real (respectively, complex) points of X in $(\mathbb{R}^*)^n$ (respectively, $(\mathbb{C}^*)^n$) by $\mathbb{R}X$ (respectively, $\mathbb{C}X$). The complex conjugation c on the complex torus naturally acts on $\mathbb{C}X$. Our goal here is to better understand this action and the induced action c_* on the homology $H_*(\mathbb{C}X)$ of $\mathbb{C}X$ with coefficients in $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$ (unless otherwise specified, we always consider homology with coefficients in \mathbb{Z}_2).

Simplicial real algebraic hypersurfaces play a key role in Viro's Patchworking method, in particular in the combinatorial case, where varieties with prescribed topology are obtained by gluing together simplicial real algebraic hypersurfaces. Viro's method is one of the main tools in the study of topology of real algebraic varieties, and is deeply connected to tropical geometry. It was used by Viro to disprove the *Ragsdale conjecture* regarding real algebraic curves in [Vir80], and later in [Ite93] by I. Itenberg to show that the conjecture was also asymptotically wrong. A complete exposition of the method can be found in [Vir06].

Hence, in addition to improving our understanding of simplicial real algebraic hypersurfaces, which are among the most natural and simple examples of real algebraic varieties, our results might also lead to a better comprehension of the varieties obtained using the combinatorial case of Viro's Patchworking.

In particular, A. Renaudineau and K. Shaw proved in [RS18] a bound conjectured by Itenberg, which states that in the case of a real algebraic projective hypersurface $X \subset \mathbb{R}\mathbb{P}^n$ obtained via *primitive* Patchworking,

$$\dim_{\mathbb{Z}_2}(H_q(\mathbb{R}X)) \leq h^{q, n-1-q}(\mathbb{C}X) + 1 \quad (1.1)$$

if $q \in \{0, \dots, n\}$ and $q \neq \frac{n-1}{2}$, and

$$\dim_{\mathbb{Z}_2}(H_{\frac{n-1}{2}}(\mathbb{R}X)) \leq h^{\frac{n-1}{2}, \frac{n-1}{2}}(\mathbb{C}X) \quad (1.2)$$

if $n - 1$ is even, where $h^{q,p}(\mathbb{C}X)$ is the (q, p) -th Hodge number of $\mathbb{C}X$. This can be compared to the much looser *Smith-Thom inequality*, valid for all real algebraic varieties, which states that

$$\dim_{\mathbb{Z}_2}(H_*(\mathbb{R}X)) \leq \dim_{\mathbb{Z}_2}(H_*(\mathbb{C}X)). \quad (1.3)$$

Taking the sum of inequalities (1.1) and (1.2) over all $q \in \{0, \dots, n\}$ yields the Smith-Thom inequality. A real algebraic variety for which (1.3) is an

equality is called *Smith-Thom maximal*, or simply *maximal*. They are objects of particular interest.

A primitive Patchwork is a Patchwork obtained by gluing together simplicial real algebraic hypersurfaces whose Newton polytopes are of Euclidean volume $\frac{1}{n!}$ (more details can be found in [RS18]). Such hypersurfaces are the simplest possible objects from which to construct more complicated hypersurfaces; in particular, they are maximal, which is a key ingredient in the proof of inequalities (1.1) and (1.2).

The *Borel-Swan inequality*, which implies the Smith-Thom inequality, states that

$$\dim_{\mathbb{Z}_2}(H_*(\mathbb{R}X)) \leq \dim_{\mathbb{Z}_2} \left(\frac{\text{Ker}(1 + c_*)}{\text{Im}(1 + c_*)} \right),$$

where $1 + c_* : H_*(\mathbb{C}X) \rightarrow H_*(\mathbb{C}X)$ (from now on, for any \mathbb{Z}_2 -vector space V , we write indifferently $\dim_{\mathbb{Z}_2}(V)$ or $\dim(V)$ to denote the dimension over \mathbb{Z}_2 of V). When this inequality is an equality, X is called *Galois maximal* (see [DIK00] for additional details). The notion has been considered of interest in itself; see for example Krasnov [Kra84], where the Galois maximality of various families of varieties is proved.

Galois maximality is equivalent to a condition of degeneration on the second page of a certain associated spectral sequence, called *Kalinin spectral sequence*, which was introduced by Kalinin in [Kal92] - further explanations can also be found in [Deg92]. Smith-Thom maximality is equivalent to degeneration on the first page of this spectral sequence, which plays an important role in the arguments used in [RS18].

Therefore, one possible way to relax the primitivity condition in order to generalize the results from [RS18] could be to consider Patchworks obtained from gluing together simplicial real algebraic hypersurfaces that are merely Galois maximal, rather than primitive (and thus Smith-Thom maximal). Note that we already know that inequalities (1.1) and (1.2) do not hold as they are under such hypotheses - additional work is needed.

Hence our interest in the action of the conjugation on the homology of simplicial real algebraic hypersurfaces, and in particular in the rank of the map $1 + c_*$, in order to determine the conditions under which such a hypersurface is Galois maximal.

In [KZ16], G. Kerr and I. Zharkov have in particular showed that the complex part $\mathbb{C}X$ of a simplicial real algebraic hypersurface is homeomorphic to the associated *phase tropical variety* $\mathcal{T}X$. Moreover, it is easy to see that the phase tropical variety $\mathcal{T}X$ retracts by deformation to the *coamoeba* \mathcal{C}_X of X . Hence we get a homotopy equivalence $\mathbb{C}X \rightarrow \mathcal{C}_X$. Private conversations with Zharkov have led us to believe that this homotopy equivalence satisfies the condition of the following conjecture (which we hope to prove soon):

Conjecture 1.1. *Let X be a simplicial real algebraic hypersurface. There is a homotopy equivalence $\phi : \mathbb{C}X \rightarrow \mathcal{C}_X$ such that the following diagram commutes for all $i \geq 0$:*

$$\begin{array}{ccc}
H_i(\mathbb{C}X) & \xrightarrow{\phi_*} & H_i(\mathcal{C}_X) \\
\downarrow c_* & & \downarrow c_* \\
H_i(\mathbb{C}X) & \xrightarrow{\phi_*} & H_i(\mathcal{C}_X)
\end{array}$$

where c_* is induced by the complex conjugation on either $\mathbb{C}X$ or \mathcal{C}_X .

This would immediately imply

$$\begin{aligned}
\dim(\text{Im}(1 + c_* : H_i(\mathbb{C}X) \rightarrow H_i(\mathbb{C}X))) &= \\
\dim(\text{Im}(1 + c_* : H_i(\mathcal{C}_X) \rightarrow H_i(\mathcal{C}_X))), &
\end{aligned}$$

where 1 is the identity.

Our main focus in the rest of this text is to prove the following result, though the details of the proof might be of interest in themselves when considering other related questions. Consider a simplicial real algebraic hypersurface X , the Newton polytope Δ of the simplicial real Laurent polynomial P that defines X , and pick a vertex O of Δ . The edges of Δ containing O define n integer vectors (choosing O as their initial point). Define $A \in M_{n \times n}(\mathbb{Z}_2)$ as the matrix whose rows are these n vectors modulo 2. Then:

Theorem 1.2. *If $\mathbb{R}X$ intersects non-trivially each quadrant of $(\mathbb{R}^*)^n$, then*

$$\dim_{\mathbb{Z}_2}(H_*(\mathbb{R}X)) = \dim_{\mathbb{Z}_2} \left(\frac{\text{Ker}(1 + c_*)}{\text{Im}(1 + c_*)} \right),$$

where $1 + c_* : H_*(\mathcal{C}_X) \rightarrow H_*(\mathcal{C}_X)$.

Otherwise, we have

$$\dim_{\mathbb{Z}_2} \left(\frac{\text{Ker}(1 + c_*)}{\text{Im}(1 + c_*)} \right) - \dim_{\mathbb{Z}_2}(H_*(\mathbb{R}X)) = 2(2^{n - \text{rank}_{\mathbb{Z}_2}(A)} - 1 - (n - \text{rank}_{\mathbb{Z}_2}(A))).$$

The condition on the intersection of $\mathbb{R}X$ with the quadrants of $(\mathbb{R}^*)^n$ is equivalent to a condition on the matrix A and the signs of the monomials of P , as proved in Lemma 5.1.

If Conjecture 1.1 holds, then Theorem 1.2 immediately implies

Theorem 1.3. *If $\mathbb{R}X$ intersects non-trivially each quadrant of $(\mathbb{R}^*)^n$, then X is always Galois maximal.*

Otherwise, we have

$$\dim_{\mathbb{Z}_2} \left(\frac{\text{Ker}(1 + c_*)}{\text{Im}(1 + c_*)} \right) - \dim_{\mathbb{Z}_2}(H_*(\mathbb{R}X)) = 2(2^{n - \text{rank}_{\mathbb{Z}_2}(A)} - 1 - (n - \text{rank}_{\mathbb{Z}_2}(A))),$$

where $1 + c_* : H_*(\mathbb{C}X) \rightarrow H_*(\mathbb{C}X)$, and X is Galois maximal if and only if $n - \text{rank}_{\mathbb{Z}_2}(A) \in \{0, 1\}$.

The rest of this article is organized as follows: in Section 2, we go over some definitions and notations. In Section 3, the coamoeba \mathcal{C}_X associated to X is introduced and described in a way suited to computations. As the coamoeba \mathcal{C}_X is homotopically equivalent to $\mathbb{C}X$, we can study the action of the conjugation on \mathcal{C}_X and deduce from it its action on $\mathbb{C}X$ (assuming Conjecture 1.1). In Section 4, the action of the conjugation on the homology of \mathcal{C}_X is described. In particular, the rank of $1 + c_*$ is computed. Finally, Theorem 1.2 is proved in Section 5.

2. DEFINITIONS AND NOTATIONS

We denote the n -dimensional torus $(\mathbb{R}/2\pi\mathbb{Z})^n$ (seen as the product of n unit circles) by T^n , and use either additive or multiplicative notations, based on context, to describe its natural group law. In particular, we frequently apply matrices with integer coefficients to elements of T^n .

Given a vector of signs $(\varepsilon_1, \dots, \varepsilon_n) \in \{1, -1\}^n$, we define $\delta(\varepsilon) = (\delta_1, \dots, \delta_n) \in \mathbb{Z}_2^n$ by the relation $\varepsilon_i = (-1)^{\delta_i}$.

Given a finite set S , let $|S|$ denote the cardinality of S .

We define $Arg : (\mathbb{C}^*)^n \rightarrow T^n$, $(z_1, \dots, z_n) \mapsto (arg(z_1), \dots, arg(z_n))$.

Throughout this text, we use the following conventions: for any $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, any matrix $G \in M_{n \times n}(\mathbb{Z})$ and any vector $v = (v_1, \dots, v_n) \in \mathbb{Z}^n$, we define $z^v := z_1^{v_1} z_2^{v_2} \dots z_n^{v_n} \in \mathbb{C}$ and $z^G := (z^{G^1}, \dots, z^{G^n}) \in \mathbb{C}^n$, where G^i is the i -th **row** of G . Choosing rows instead of columns has the advantage of allowing us to write $Arg(z^G) = G \cdot Arg(z)$, and the disadvantage that for another matrix $H \in M_{n \times n}(\mathbb{Z})$, we have $(z^G)^H = z^{H \cdot G}$.

Consider as above a simplicial polynomial $P(z) = \sum_{\alpha \in \Delta_P} c_\alpha z^\alpha$, for some coefficients $c_\alpha \in \mathbb{R}^*$, and the associated simplicial real algebraic hypersurface X_P . Up to multiplication by a non-trivial Laurent monomial (which doesn't change X_P), the polynomial P can be chosen so that Δ_P has $0 \in \mathbb{Z}^n$ as one of its vertices. We assume this to be the case from now on. Moreover, let us denote the non-null vertices of Δ_P by $\alpha_P^1, \dots, \alpha_P^n \in \mathbb{Z}^n$, and define $A_P \in M_{n \times n}(\mathbb{Z})$ as the matrix whose i -th **row** is α_P^i for $i = 1, \dots, n$. If we also define $\alpha_P^0 := 0 \in \mathbb{Z}^n$, we can write indifferently $P(z) = \sum_{\alpha \in \Delta_P} c_\alpha z^\alpha = \sum_{i=0, \dots, n} c_{\alpha_P^i} z^{\alpha_P^i} = \sum_{i=1, \dots, n} c_{\alpha_P^i} z^{A_P^i} + c_0$.

For any $G \in M_{n \times n}(\mathbb{Z})$, we define the algebraic morphism

$$\begin{aligned} \Phi_G : (\mathbb{C}^*)^n &\rightarrow (\mathbb{C}^*)^n \\ z &\mapsto z^G. \end{aligned}$$

If G is invertible, Φ_G is an algebraic isomorphism that sends $X_{\tilde{P}} = \{\tilde{P}(z) = \sum_{i=1, \dots, n} c_{\alpha_P^i} z^{(A_P \cdot G)^i} + c_0 = 0\}$ to $X_P = \{P(z) = \sum_{i=1, \dots, n} c_{\alpha_P^i} z^{A_P^i} + c_0 = 0\}$ and respects the real structure. This means that up to such isomorphisms, we can consider the matrix A_P defining the simplex Δ_P up to right-multiplication by invertible matrices with integer coefficients.

Remark 2.1. *The complex part $\mathbb{C}X_P \subset (\mathbb{C}^*)^n$ of a simplicial real algebraic hypersurface X_P is smooth.*

Let X_P be given as above by $P(z) = \sum_{i=1, \dots, n} c_i z^{\alpha_i} + c_0$, for some coefficients $c_i \in \mathbb{R}^*$ and a simplex Δ_P . Let A be the $n \times n$ matrix whose rows correspond to the vertices α_i of Δ_P and the monomials of P (for $i = 1, \dots, n$). We also introduce B , the cofactor matrix of A , and $P_L(z) = \sum_{i=1, \dots, n} c_i z_i + c_0$ (the L stands for "linear").

Then $P(z) = P_L \circ \Phi_A(z)$, thus $\Phi_A : (\mathbb{C}^*)^n \rightarrow (\mathbb{C}^*)^n$ maps $\mathbb{C}X_P$ to the hyperplane $\mathbb{C}X_{P_L} := \{P_L(z) = 0\}$. But $\Phi_A \circ \Phi_B = \Phi_{\det(A) \cdot Id} : (z_1, \dots, z_n) \mapsto (z_1^{\det(A)}, \dots, z_n^{\det(A)})$ is a local diffeomorphism, which implies that so is Φ_A by a dimensional argument. Hence the smoothness of $\mathbb{C}X_P$ can be deduced from the smoothness of $\mathbb{C}X_{P_L}$.

3. COAMOEBAS

As above and for the remainder of this article, let $P(z) = \sum_{i=1,\dots,n} \varepsilon_i c_i z^{\alpha^i} + c_0$, where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{1, -1\}^n$ and $c_i \in \mathbb{R}_{>0}$ for all i , be a simplicial real polynomial (we can suppose without loss of generality that the constant term is positive). Let $X := X_P \subset (\mathbb{C}^*)^n$ be the associated simplicial real algebraic hypersurface, and let Δ_P be the Newton polytope of P , and A be the $n \times n$ matrix whose rows correspond to the vertices of Δ_P and the monomials of P , *i.e.* $A^i = \alpha^i$.

3.1. Definition and description of \mathcal{C}_X . The *coamoeba* $\mathcal{C}_X \subset T^n$ of X is the closure in T^n of the image $\text{Arg}(\mathbb{C}X)$. The conjugation c acts as $-\text{id}$ on T^n ; if we fix a representation of T^n as $[0, 2\pi]^n$ with its boundary quotiented as needed, which we do from now on, c is the central symmetry.

We will use the convenient description of \mathcal{C}_X given by Kerr and Zharkov in the linear case, *i.e.* when the coamoeba is given by an affine hyperplane, as an intermediate step to get to the general case. Let us introduce the simplicial polynomials $P_L^+(z) = \sum_{i=1,\dots,n} c_i z_i + c_0$ and $P_L(z) = \sum_{i=1,\dots,n} \varepsilon_i c_i z_i + c_0$, where the coefficients c_i and ε_i are the same as in P . Name $\mathcal{C}_{X_{P_L^+}}$ and $\mathcal{C}_{X_{P_L}}$ the associated coamoebas.

Consider $T := T^1 = \mathbb{R}/2\pi\mathbb{Z}$ and identify it with the unit circle via the map $E : [\theta] \mapsto \exp(i\theta)$. We say that $(\theta_1, \dots, \theta_n) \in T^n$ are in an *allowed configuration* if there is no open half-circle in T containing all the θ_i as well as the point $1 = E[0]$ (which corresponds to the constant term).

A *zonotope* is the Minkowski sum in \mathbb{R}^n of a finite collection of segments. Now consider such a zonotope \tilde{Z} ; if the quotient map $\mathbb{R}^n \rightarrow T^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ restricted to the interior of \tilde{Z} is an embedding, we call in the rest of this article, by extension, the image Z of the interior of \tilde{Z} an (open) zonotope of T^n .

Lemma 3.1. *The points of $\mathcal{C}_{X_{P_L^+}}$ are exactly the n -tuples $(\theta_1, \dots, \theta_n) \in T^n$ in allowed configurations. There is an n -dimensional zonotope \tilde{Z} in \mathbb{R}^n , generated by $n + 1$ segments, such that the quotient map $\mathbb{R}^n \rightarrow T^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ restricted to the interior of \tilde{Z} is an embedding, and such that the open zonotope Z of T^n which is the image of the interior of \tilde{Z} is the complement of the points in allowed configurations (see Figure 1).*

The proof is almost trivial - see [KZ16] for related details. Note that Z is contractible in T^n , hence $(T^n, \mathcal{C}_{X_{P_L^+}})$ is of the same homotopy type as $(T^n, T^n \setminus \{\star\})$.

We now consider the slightly more general case of $\mathcal{C}_{X_{P_L}}$. Let $\delta = (\delta_1, \dots, \delta_n) := \delta(\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{Z}_2^n$ be as in Section 2.

Lemma 3.2. *The coamoeba $\mathcal{C}_{X_{P_L}}$ is a translate of $\mathcal{C}_{X_{P_L^+}}$ by $\pi \cdot \delta \in T^n$, *i.e.* $\mathcal{C}_{X_{P_L}} = \mathcal{C}_{X_{P_L^+}} + \pi \cdot \delta \subset T^n$.*

The zonotope of forbidden configurations from Lemma 3.1 has simply been translated; the proof is once again almost trivial.

Let us now consider the general case of \mathcal{C}_X .

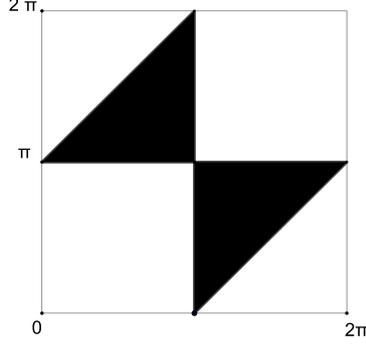


FIGURE 1. In black, the coamoeba $\mathcal{C}_{X_{P_L}^+}$. In white, the open zonotope Z

Lemma 3.3. *Using the same notations as above, the coamoeba \mathcal{C}_X is the preimage of the coamoeba $\mathcal{C}_{X_{P_L}}$ by the map $A : T^n \rightarrow T^n$, i.e. $\mathcal{C}_X = A^{-1}(\mathcal{C}_{X_{P_L}})$.*

Proof. First, let us show that $\text{Arg}(\mathbb{C}X) = A^{-1}(\text{Arg}(\mathbb{C}X_{P_L}))$.

Indeed, if $\theta = (\theta_1, \dots, \theta_n) \in \text{Arg}(\mathbb{C}X)$, by definition there exists $r = (r_1, \dots, r_n) \in (\mathbb{R}_{>0})^n$ such that $z := (r_1\theta_1, \dots, r_n\theta_n)$ belongs to $\mathbb{C}X$, i.e. $P(r_1\theta_1, \dots, r_n\theta_n) = \sum_{i=1, \dots, n} \varepsilon_i c_i r^{A^i} \theta^{A^i} + c_0 = P_L(r^{A^1} \theta^{A^1}, \dots, r^{A^n} \theta^{A^n}) = 0$. Hence $(r^{A^1} \theta^{A^1}, \dots, r^{A^n} \theta^{A^n}) \in \mathbb{C}X_{P_L}$ and $\theta^A = A \cdot \theta$ belongs to $\text{Arg}(\mathbb{C}X_{P_L})$.

Conversely, suppose that $A \cdot \theta \in \text{Arg}(\mathbb{C}X_{P_L})$. Then by definition there exists $s = (s_1, \dots, s_n) \in (\mathbb{R}_{>0})^n$ such that $P_L(s_1\theta^{A^1}, \dots, s_n\theta^{A^n}) = \sum_{i=1, \dots, n} \varepsilon_i c_i s_i \theta^{A^i} + c_0 = 0$. If there exists $r = (r_1, \dots, r_n) \in (\mathbb{R}_{>0})^n$ such that $r^A = s$, then $z := (r_1\theta_1, \dots, r_n\theta_n)$ is such that $P(z) = 0$ and we can conclude that $\theta \in \text{Arg}(\mathbb{C}X)$. Now consider the cofactor matrix B of A , and

$$\begin{aligned} (\mathbb{R}_{>0})^n &\xrightarrow{\Phi_B} (\mathbb{R}_{>0})^n \xrightarrow{\Phi_A} (\mathbb{R}_{>0})^n \\ r = (r_1, \dots, r_n) &\mapsto r^B \mapsto (r^B)^A = r^{\det(A)} = (r_1^{\det(A)}, \dots, r_n^{\det(A)}). \end{aligned}$$

This clearly shows that Φ_A is surjective from $(\mathbb{R}_{>0})^n$ onto itself; hence we have shown that $\text{Arg}(\mathbb{C}X) = A^{-1}(\text{Arg}(\mathbb{C}X_{P_L}))$.

As $A \cdot : T^n \rightarrow T^n$ is continuous, we immediately have that $\mathcal{C}_X = \overline{\text{Arg}(\mathbb{C}X)} \subset A^{-1}(\overline{\text{Arg}(\mathbb{C}X_{P_L})}) = A^{-1}(\mathcal{C}_{X_{P_L}})$. Following the same reasoning as in Remark 2.1, the map $A \cdot : T^n \rightarrow T^n$ is a local diffeomorphism, because $(AB) \cdot = (\det(A) \cdot \text{Id}) \cdot : T^n \rightarrow T^n$ is a local diffeomorphism, where B is the comatrix of A . Consider $\theta \in A^{-1}(\overline{\text{Arg}(\mathbb{C}X_{P_L})})$, an open neighborhood $U \subset T^n$ of θ such that $(A \cdot)|_U$ is an embedding, and some open neighborhood $V \subset T^n$ of θ . Then $A \cdot (U \cap V)$ is an open neighborhood of $A \cdot \theta \in \overline{\text{Arg}(\mathbb{C}X_{P_L})}$, hence there exists $\rho \in A \cdot (U \cap V) \cap \text{Arg}(\mathbb{C}X_{P_L})$. Then $((A \cdot)|_U)^{-1}(\rho) \in V \cap A^{-1}(\text{Arg}(\mathbb{C}X_{P_L})) = V \cap \text{Arg}(\mathbb{C}X)$, which shows that $\theta \in \overline{\text{Arg}(\mathbb{C}X)}$. \square

3.2. A more explicit description of \mathcal{C}_X . It is well-known (for example, using the theorem of structure of finitely generated abelian groups) that

there exists two (non-uniquely defined) invertible matrices $G, H \in M_{n \times n}(\mathbb{Z})$ such that

$$G \cdot A \cdot H = D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix},$$

where $d_1 | d_2 | \dots | d_n \in \mathbb{Z}$ and $d_1 d_2 \dots d_k$ is the greatest common divisor of the non-trivial k -minors of A (for $k = 1, \dots, n$).

Consider $\delta^G = G \cdot \delta \in \mathbb{Z}_2^n$, where $\delta = \delta(\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{Z}_2^n$ is as above.

We partition $\{1, \dots, n\}$ in the following way:

$$\begin{aligned} I^{0,0} &:= \{i \in \{1, \dots, n\} \mid \delta_i^G \equiv [0]_2, d_i \equiv [0]_2\}, \\ I^{1,0} &:= \{i \in \{1, \dots, n\} \mid \delta_i^G \equiv [1]_2, d_i \equiv [0]_2\}, \\ I^{0,1} &:= \{i \in \{1, \dots, n\} \mid \delta_i^G \equiv [0]_2, d_i \equiv [1]_2\}, \\ I^{1,1} &:= \{i \in \{1, \dots, n\} \mid \delta_i^G \equiv [1]_2, d_i \equiv [1]_2\}. \end{aligned}$$

For the remainder of this text, we fix two such matrices G and H . Moreover, as observed in section 2, we can consider A up to right-multiplication by invertible matrices with integer coefficients: hence we can, and do, assume that H is the identity matrix. Hence from now on we have

$$A = G^{-1} \cdot D.$$

Then \mathcal{C}_X lends itself to the following description: define the set of indices $\Omega := \{0, \dots, d_1 - 1\} \times \dots \times \{0, \dots, d_n - 1\}$ and let $\widetilde{\delta}^G \in \{0, 1\}^n$ be the unique lifting of δ^G . Let also $G \cdot \widetilde{\mathcal{C}}_{X_{P_L}}$ be the preimage of $G \cdot \mathcal{C}_{X_{P_L}}$ by the quotient map $[0, 2\pi]^n \rightarrow T^n$ (see the left part of Figure 2).

Cover $[0, 2\pi]^n$ with $d_1 \dots d_n$ hyperrectangular cells $C_\alpha := [\alpha_1 \frac{2\pi}{d_1}, (\alpha_1 + 1) \frac{2\pi}{d_1}] \times \dots \times [\alpha_n \frac{2\pi}{d_n}, (\alpha_n + 1) \frac{2\pi}{d_n}]$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \Omega$. Define $\widetilde{\mathcal{C}}_X \subset [0, 2\pi]^n$ as the set such that the pair $(C_\alpha, C_\alpha \cap \widetilde{\mathcal{C}}_X)$ is mapped to the pair $([0, 2\pi]^n, G \cdot \widetilde{\mathcal{C}}_{X_{P_L}})$ by $(z_1, \dots, z_n) \mapsto (d_1 z_1 - \alpha_1 2\pi, \dots, d_n z_n - \alpha_n 2\pi)$ (see the right part of Figure 2). Then

Proposition 3.4. *The pair (T^n, \mathcal{C}_X) is the image of $([0, 2\pi]^n, \widetilde{\mathcal{C}}_X)$ by the quotient map $[0, 2\pi]^n \rightarrow T^n$.*

The coamoeba \mathcal{C}_X is the complement in T^n of $d_1 \dots d_n$ open zonotopes indexed by the set of indices Ω , such that the center of zonotope Z_α , for $\alpha = (\alpha_1, \dots, \alpha_n) \in \Omega$, is in $\pi \cdot (\widetilde{\delta}_1^G/d_1, \dots, \widetilde{\delta}_n^G/d_n) + 2\pi \cdot (\alpha_1/d_1, \dots, \alpha_n/d_n) \in T^n$. Those zonotopes are translate of each other.

Proof. From 3.3, we know that $\mathcal{C}_X = A^{-1}(\mathcal{C}_{X_{P_L}}) = D^{-1}(G \cdot \mathcal{C}_{X_{P_L}})$.

As explained in Lemmas 3.1 and 3.2, the coamoeba $\mathcal{C}_{X_{P_L}^+}$ is the complement in T^n of an open zonotope centered in 0, and $\mathcal{C}_{X_{P_L}}$ is the complement in T^n of the same zonotope translated and now centered in $(\delta_1 \pi, \dots, \delta_n \pi) = \pi \cdot \delta \in T^n$.

The linear isomorphism G is then applied to it, so that $G \cdot \mathcal{C}_{X_{P_L}}$ is the complement in T^n of an open zonotope (geometrically the starting zonotope deformed by G) centered in $\pi \cdot \delta^G \in T^n$. The last step is to take the preimage by the covering map $D \cdot : T^n \rightarrow T^n$.

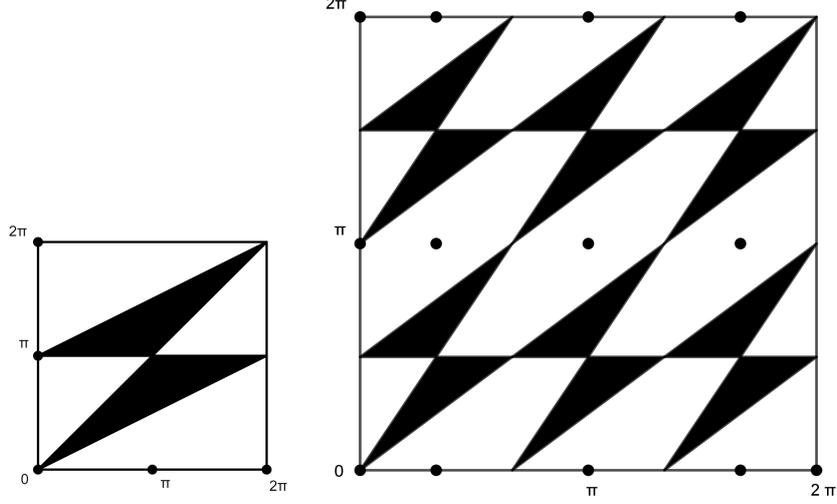


FIGURE 2. On the left, $G \cdot \widetilde{\mathcal{C}}_{X_{P_L}}$. On the right: the coamoeba \mathcal{C}_X in black and the zonotopes in white.

□

4. HOMOLOGICAL COMPUTATIONS

Let us now study the action of the conjugation c on \mathcal{C}_X , using the description from Proposition 3.4. As pointed out earlier, c simply acts as the central symmetry on T^n seen as an appropriate quotient of $[0, 2\pi]^n$. It maps \mathcal{C}_X to itself, and zonotopes to one another. Let us describe that action more precisely.

4.1. Action of the conjugation on the zonotopes. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \Omega = \{0, \dots, d_1 - 1\} \times \dots \times \{0, \dots, d_n - 1\}$. We define $c(\alpha) \in \Omega$ as the index such that $Z_{c(\alpha)} = c(Z_\alpha)$. Then:

If $i \in I^{0,0}$, then $c(\alpha)_i$ is the unique lifting to $\{0, \dots, d_i - 1\}$ of $[d_i - \alpha_i]_{d_i}$. In particular, notice that $c(\alpha)_i = \alpha_i$ if and only if $\alpha_i \in \{0, d_i/2\}$.

If $i \in I^{0,1}$, then $c(\alpha)_i$ is the unique lifting to $\{0, \dots, d_i - 1\}$ of $[d_i - \alpha_i]_{d_i}$. In particular, notice that $c(\alpha)_i = \alpha_i$ if and only if $\alpha_i = 0$.

If $i \in I^{1,0}$, then $c(\alpha)_i = (d_i - 1) - \alpha_i$. In particular, notice that $c(\alpha)_i$ is never equal to α_i .

If $i \in I^{1,1}$, then $c(\alpha)_i = (d_i - 1) - \alpha_i$. In particular, notice that $c(\alpha)_i = \alpha_i$ if and only if $\alpha_i = (d_i - 1)/2$.

Denote by $F := \{\alpha \in \Omega | c(\alpha) = \alpha\}$ the set of fixed points of c . If $|I^{1,0}| \neq 0$, then $F = \emptyset$. If $|I^{1,0}| = 0$, there are $2^{|I^{0,0}|}$ fixed points: the indices $\beta \in \Omega$ such that $\beta_i \in \{0, d_i/2\}$ if $i \in I^{0,0}$, $\beta_i = 0$ if $i \in I^{0,1}$ and $\beta_i = (d_i - 1)/2$ if $i \in I^{1,1}$.

4.2. Homology of \mathcal{C}_X . Denote by $\Gamma_1, \dots, \Gamma_n \in H_{n-1}(T^n)$ the n homology classes induced by the $(n-1)$ -dimensional tori $\{[\theta_i] = 0\} \subset T^n$. They form a basis of $H_{n-1}(T^n)$.

Using a Mayer-Vietoris long exact sequence, it is easy to see that

$$H_k(\mathcal{C}_X) \xrightarrow{in_*} H_k(T^n),$$

where in_* is induced by the inclusion, is an isomorphism for $k = 0, \dots, n-2$, and that $H_k(\mathcal{C}_X) = 0$ for $k \geq n$. We also get a short exact sequence

$$0 \longrightarrow H_n(T^n) \longrightarrow H_{n-1}\left(\bigsqcup_{\alpha \in \Omega} \partial Z_\alpha\right) \longrightarrow H_{n-1}(\mathcal{C}_X) \longrightarrow H_{n-1}(T^n) \longrightarrow 0,$$

where ∂Z_α is the border of zonotope Z_α , which is homeomorphic to a $(n-1)$ -sphere. From it, we deduce the following Lemma:

Lemma 4.1. *Given $B_1, \dots, B_n \in H_{n-1}(\mathcal{C}_X)$ such that $in_*(B_i) = \Gamma_i$ for all i , we have an isomorphism*

$$H_{n-1}(\mathcal{C}_X) \cong \left(\bigoplus_{i=1, \dots, n} \mathbb{Z}_2 \cdot B_i \right) \oplus \left(\frac{\bigoplus_{\alpha \in \Omega} \mathbb{Z}_2 \cdot [\partial Z_\alpha]}{\mathbb{Z}_2 \cdot \sum_{\alpha \in \Omega} [\partial Z_\alpha]} \right).$$

We already know that $c_*([\partial Z_\alpha]) = [\partial Z_{c(\alpha)}]$ (as we consider homology with coefficients in \mathbb{Z}_2 , we need not concern ourselves with orientation), where $c(\alpha)$ is as above - we only need to find suitable classes B_i and describe how c_* acts on them.

Let us define B_i , for $i \in \{1, \dots, n\}$. Consider the $(n-1)$ -dimensional torus $\{[\theta_i] = 0\} \subset T^n$, and the set I_i of intersections between $\{[\theta_i] = 0\}$ and the open zonotopes of the complement of \mathcal{C}_X in T^n . Notice that a given zonotope Z_α can intersect several times $\{[\theta_i] = 0\}$ - each of these intersections appears as a distinct element of I_i . Call $N_i(\alpha)$ the number of such intersections. We would like to define B_i as the class of $\{[\theta_i] = 0\}$, with a modification (since $\{[\theta_i] = 0\}$ is in general not included in \mathcal{C}_X).

For each intersection $\gamma \in I_i$, let $\alpha(\gamma) \subset \Omega$ be the index of the zonotope corresponding to γ . The intersection γ of $\{[\theta_i] = 0\}$ and $\overline{Z_{\alpha(\gamma)}}$ cuts $\partial Z_{\alpha(\gamma)}$ in two (topological) half-spheres of dimension $n-1$. Call S_γ^+ the half-sphere that lies in the direction e_i of $\{[\theta_i] = 0\}$, and S_γ^- the one lying in direction $-e_i$.

We start our construction with $\{[\theta_i] = 0\}$. For each $\gamma \in I_i$, we remove the intersection $\{[\theta_i] = 0\} \cap Z_{\alpha(\gamma)}$, and we add S_γ^+ . Thus we obtain a $(n-1)$ -cycle - name it \widetilde{B}_i and name its class B_i . Observe that parts of the border of a given zonotope Z_α can appear several times in \widetilde{B}_i : in other words, there can be several S_γ^+ that are subsets of the same $\overline{Z_\alpha}$ (with non-trivial intersections). Notice also that S_γ^+ can be homotopically contracted in T^n to the intersection $\{[\theta_i] = 0\} \cap \overline{Z_{\alpha(\gamma)}}$. Thus B_i is a lift of Γ_i , as required. This construction is illustrated in Figure 3.

4.3. Image of $1+c_*$. We want to compute the image of $1+c_* : H_{n-1}(\mathcal{C}_X) \longrightarrow H_{n-1}(\mathcal{C}_X)$, and in particular its dimension. Using the description in Lemma 4.1, we see that

$$Im(1+c_*) = \sum_{i=1, \dots, n} \mathbb{Z}_2 \cdot ((1+c_*)(B_i)) + \sum_{\alpha \in \Omega} \mathbb{Z}_2 \cdot [\partial Z_\alpha + \partial Z_{c(\alpha)}] \quad (4.1)$$

Let us make some observations about the image of B_i under c_* . It is the class of the mirror image by central symmetry of \widetilde{B}_i . This means that

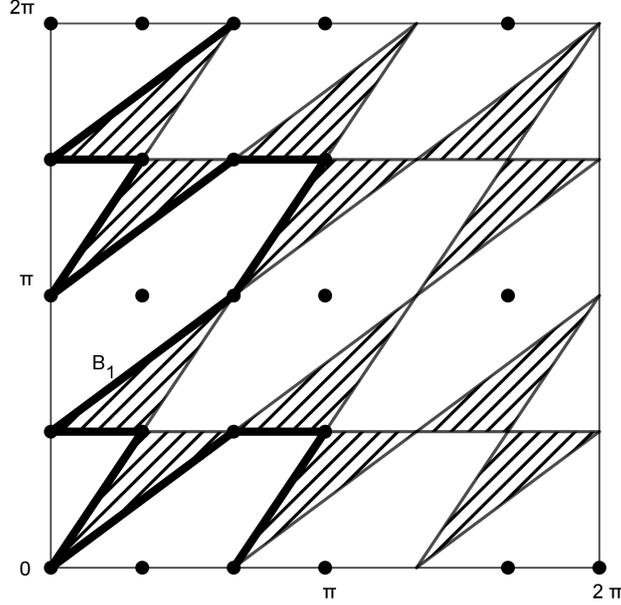


FIGURE 3. As a black broken line, the cycle \widetilde{B}_1 . In that example, I_1 is of cardinal 4.

it consists of $\{[\theta_i] = 0\}$, without the intersections $\{[\theta_i] = 0\} \cap Z_{\alpha(\gamma)}$ (for $\gamma \in I_i$), and with all the S_γ^- , i.e. the half-spheres lying in the direction $-e_i$. Thus $B_i + c_*(B_i) = \sum_{\gamma \in I_i} [\partial Z_{\alpha(\gamma)}]$ (since by definition $S_\gamma^+ + S_\gamma^- = \partial Z_\alpha$) - each ∂Z_α appears $N_i(\alpha)$ times.

It would be possible to compute $N_i(\alpha)$ exactly. However, we only want to compute the image of $1 + c_*$ - the following observations suffice for our purpose.

Since we are considering coefficients in \mathbb{Z}_2 , we only have to pay attention to the parity of $N_i(\alpha)$. Let us define $J_i := \{\alpha \in \Omega \mid N_i(\alpha) \equiv [1]_2\}$. The number of intersections $N_i(\alpha)$ depends only on the length of the zonotope Z_α in the direction e_i (which is the same for all α) and on the i -th coordinate of the center of Z_α , i.e. on α_i .

As we only have to consider the coordinate α_i and because of the symmetry of the situation, it is clear that if $\beta \in J_i$, then $c(\beta) \in J_i$. Consider the description (4.1) of $Im(1 + c_*)$; if $\beta \neq c(\beta)$, we have

$$\partial Z_\beta + \partial Z_{c(\beta)} \in \sum_{\alpha \in \Omega} \mathbb{Z}_2 \cdot [\partial Z_\alpha + \partial Z_{c(\alpha)}].$$

Thus for any $i \in \{1, \dots, n\}$ we get

$$\begin{aligned}
& \mathbb{Z}_2 \cdot ((1 + c_*)(B_i)) + \sum_{\alpha \in \Omega} \mathbb{Z}_2 \cdot [\partial Z_\alpha + \partial Z_{c(\alpha)}] = \\
& \mathbb{Z}_2 \cdot \sum_{\beta \in J_i} [\partial Z_\beta] + \sum_{\alpha \in \Omega} \mathbb{Z}_2 \cdot [\partial Z_\alpha + \partial Z_{c(\alpha)}] = \\
& \mathbb{Z}_2 \cdot \sum_{\beta \in J_i \cap F} [\partial Z_\beta] + \sum_{\alpha \in \Omega} \mathbb{Z}_2 \cdot [\partial Z_\alpha + \partial Z_{c(\alpha)}],
\end{aligned}$$

where as above F is the set of fixed elements $\beta \in \Omega$.

We need to determine for all $i \in \{1, \dots, n\}$ the set $J_i \cap F$. Doing so will allow us to prove the following Lemma.

Lemma 4.2. *If $|I^{1,0}| \neq 0$, the image of $1 + c_*$ is $\sum_{\alpha \in \Omega} \mathbb{Z}_2 \cdot [\partial Z_\alpha + \partial Z_{c(\alpha)}]$.
If $|I^{1,0}| = 0$,*

$$Im(1 + c_*) = \sum_{i \in I^{0,0}} \mathbb{Z}_2 \cdot ((1 + c_*)(B_i)) + \sum_{\alpha \in \Omega} \mathbb{Z}_2 \cdot [\partial Z_\alpha + \partial Z_{c(\alpha)}].$$

Proof. Indeed, as noted in Subsection 4.1, there are no fixed points if $|I^{1,0}| \neq 0$.

Let us now consider the case $|I^{1,0}| = 0$.

If $i \in I^{0,0}$, $\beta \in J_i$ for all β such that $\beta_i = 0$, and $\beta \notin J_i$ for all β such that $\beta_i = (d_i - 1)/2$. In particular, $J_i \cap F$ is the set of cardinality $2^{|I^{0,0}|-1}$ of all $\beta \in \Omega$ such that $\beta_i = 0$, $\beta_j \in \{0, d_j/2\}$ if $j \in I^{0,0} - \{i\}$, $\beta_j = 0$ if $j \in I^{0,1}$ and $\beta_j = (d_j - 1)/2$ if $j \in I^{1,1}$.

If $i \in I^{0,1}$, $\beta \in J_i$ for all β such that $\beta_i = 0$. In particular, $J_i \cap F = F$. Remember description (4.1); we have $\sum_{\alpha \in \Omega} [\partial Z_\alpha] = 0$ in $H_{n-1}(\mathcal{C}_X)$. Then,

$$\sum_{\alpha \in F} [\partial Z_\alpha] = \sum_{\alpha \in F} [\partial Z_\alpha] + \sum_{\alpha \in \Omega} [\partial Z_\alpha] \subset \sum_{\alpha \in \Omega} [\partial Z_\alpha + \partial Z_{c(\alpha)}].$$

This implies that $\mathbb{Z}_2 \cdot ((1 + c_*)(B_i)) \subset \sum_{\alpha \in \Omega} \mathbb{Z}_2 \cdot [\partial Z_\alpha + \partial Z_{c(\alpha)}]$.

If $i \in I^{1,1}$, $\beta \notin J_i$ for all β such that $\beta_i = (d_i - 1)/2$. In particular, $J_i \cap F = \emptyset$.

Using description (4.1), we can conclude. \square

4.4. Rank of $1 + c_*$. Let us compute the rank of $1 + c_*$. We get the following proposition.

Proposition 4.3. *If $|I^{1,0}| \neq 0$,*

$$\dim(Im(1 + c_*)) = \frac{d_1 \dots d_n}{2} - 1.$$

If $|I^{1,0}| = 0$,

$$\dim(Im(1 + c_*)) = |I^{0,0}| + \frac{d_1 \dots d_n - 2^{|I^{0,0}|}}{2}.$$

Proof. We will first compute the dimension of

$$\sum_{i=1, \dots, n} \mathbb{Z}_2 \cdot ((1 + c_*)(B_i)) + \sum_{\alpha \in \Omega} \mathbb{Z}_2 \cdot [\partial Z_\alpha + \partial Z_{c(\alpha)}]$$

in

$$\left(\bigoplus_{i=1, \dots, n} \mathbb{Z}_2 \cdot B_i \right) \oplus \left(\bigoplus_{\alpha \in \Omega} \mathbb{Z}_2 \cdot [\partial Z_\alpha] \right),$$

then quotient by $\mathbb{Z}_2 \cdot \sum_{\alpha \in \Omega} [\partial Z_\alpha]$ to get the dimension in $H_{n-1}(\mathcal{C}_X)$ (see Lemma 4.1).

Denote by $\tilde{\Omega} \subset \Omega$ a set defined as such: for each pair of elements $\{\alpha, c(\alpha)\}$, where $\alpha \in \Omega \setminus F$, choose exactly one element to be included in $\tilde{\Omega}$. Thus $\sum_{\alpha \in \tilde{\Omega}} \mathbb{Z}_2 \cdot [\partial Z_\alpha + \partial Z_{c(\alpha)}] = \sum_{\alpha \in \tilde{\Omega}} \mathbb{Z}_2 \cdot [\partial Z_\alpha + \partial Z_{c(\alpha)}]$. This sum is clearly a direct sum before quotienting.

If $|I^{1,0}| \neq 0$, then $\text{Im}(1 + c_*) = \sum_{\alpha \in \tilde{\Omega}} \mathbb{Z}_2 \cdot [\partial Z_\alpha + \partial Z_{c(\alpha)}] = \sum_{\alpha \in \tilde{\Omega}} \mathbb{Z}_2 \cdot [\partial Z_\alpha + \partial Z_{c(\alpha)}]$. When we quotient by $\mathbb{Z}_2 \cdot \sum_{\alpha \in \Omega} [\partial Z_\alpha]$, the dimension decreases by exactly one, as $\sum_{\alpha \in \Omega} [\partial Z_\alpha] \in \sum_{\alpha \in \tilde{\Omega}} \mathbb{Z}_2 \cdot [\partial Z_\alpha + \partial Z_{c(\alpha)}]$ (since there is no fixed point). Hence

$$\dim(\text{Im}(1 + c_*)) = |\tilde{\Omega}| - 1 = \frac{d_1 \dots d_n}{2} - 1.$$

If $|I^{1,0}| = 0$, we can see that

$$\sum_{i \in I^{0,0}} \mathbb{Z}_2 \cdot ((1 + c_*)(B_i)) + \sum_{\alpha \in \tilde{\Omega}} \mathbb{Z}_2 \cdot [\partial Z_\alpha + \partial Z_{c(\alpha)}]$$

is actually a free sum before quotienting. Indeed, suppose that for each $\alpha \in \tilde{\Omega}$ (respectively, $i \in I^{0,0}$) there is $\lambda_\alpha \in \mathbb{Z}_2$ (respectively, $\lambda_i \in \mathbb{Z}_2$), not all 0, such that

$$\sum_{i \in I^{0,0}} \lambda_i (1 + c_*(B_i)) + \sum_{\alpha \in \tilde{\Omega}} \lambda_\alpha [\partial Z_\alpha + \partial Z_{c(\alpha)}] = 0. \quad (4.2)$$

For $i \in I^{0,0}$, consider β defined by $\beta_i = 0$, $\beta_j = d_j/2$ if $j \in I^{0,0} - \{i\}$, $\beta_j = 0$ if $j \in I^{0,1}$ and $\beta_j = (d_j - 1)/2$ if $j \in I^{1,1}$. As explained in the description of $J_i \cap F$ in the proof of Lemma 4.2, the class $[\partial Z_\beta]$ only appears in $((1 + c_*)(B_i))$ among all the terms of equation (4.2). Thus λ_i is necessarily 0. We can then conclude from the independence of the family $\{[\partial Z_\alpha + \partial Z_{c(\alpha)}]\}_{\alpha \in \tilde{\Omega}}$ before the quotient that all λ_α are 0, which proves our point.

Consider $\beta \in \Omega$ defined by $\beta_j = d_j/2$ if $j \in I^{0,0}$, $\beta_j = 0$ if $j \in I^{0,1}$ and $\beta_j = (d_j - 1)/2$ if $j \in I^{1,1}$. Since $[\partial Z_\beta]$ appears in $\sum_{\alpha \in \tilde{\Omega}} [\partial Z_\alpha + \partial Z_{c(\alpha)}]$ but not in $\sum_{i \in I^{0,0}} \mathbb{Z}_2 \cdot ((1 + c_*)(B_i)) + \sum_{\alpha \in \tilde{\Omega}} \mathbb{Z}_2 \cdot [\partial Z_\alpha + \partial Z_{c(\alpha)}]$ (once again going back to the proof of Lemma 4.2), we see that $\sum_{\alpha \in \tilde{\Omega}} [\partial Z_\alpha + \partial Z_{c(\alpha)}] \notin \sum_{i \in I^{0,0}} \mathbb{Z}_2 \cdot ((1 + c_*)(B_i)) + \sum_{\alpha \in \tilde{\Omega}} \mathbb{Z}_2 \cdot [\partial Z_\alpha + \partial Z_{c(\alpha)}]$. This means that the dimension of $\sum_{i \in I^{0,0}} \mathbb{Z}_2 \cdot ((1 + c_*)(B_i)) + \sum_{\alpha \in \tilde{\Omega}} \mathbb{Z}_2 \cdot [\partial Z_\alpha + \partial Z_{c(\alpha)}]$ does not decrease when we quotient. Thus

$$\dim(\text{Im}(1 + c_*)) = |I^{0,0}| + |\tilde{\Omega}| = |I^{0,0}| + \frac{|\Omega| - |F|}{2} = |I^{0,0}| + \frac{d_1 \dots d_n - 2|I^{0,0}|}{2},$$

as stated. \square

5. GALOIS MAXIMALITY

We are still using the notations of the previous sections.

To prove Theorem 1.2, we need to compare $\dim_{\mathbb{Z}_2} \left(\frac{\text{Ker}(1+c_*)}{\text{Im}(1+c_*)} \right)$ (where $1+c_* : H_*(\mathcal{C}_X) \rightarrow H_*(\mathcal{C}_X)$) and $\dim_{\mathbb{Z}_2}(H_*(\mathbb{R}X))$. We consider $H_*(\mathbb{R}X)$ first, with the following lemma.

Lemma 5.1. *The real part $\mathbb{R}X \subset (\mathbb{R}^*)^n$ consists of 2^n contractible connected components if $\delta = \delta(\varepsilon) \in \mathbb{Z}_2^n$ is not in the image of $A : \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2^n$, and of $2^n - 2^{n-\text{rank}(A)}$ contractible connected components if it is. Thus $H_*(\mathbb{R}X)$ is of \mathbb{Z}_2 -dimension 2^n if $\delta \notin \text{Im}(A)$ and of \mathbb{Z}_2 -dimension $2^n - 2^{n-\text{rank}(A)}$ if $\delta \in \text{Im}(A)$.*

Proof. Consider the 2^n quadrants of $(\mathbb{R}^*)^n$, and let $\gamma = (\gamma_1, \dots, \gamma_n) \in \{1, -1\}^n$ index the quadrant $Q_\gamma := \{(x_1, \dots, x_n) \in (\mathbb{R}^*)^n \mid x_1\gamma_1 > 0, \dots, x_n\gamma_n > 0\}$. Consider as in Section 3 the real polynomial $P_L(z) = \sum_{i=1, \dots, n} \varepsilon_i c_i z_i + c_0$ associated to $P(z) = \sum_{i=1, \dots, n} \varepsilon_i c_i z^{A_i} + c_0$. Then the map

$$\begin{aligned} \Phi_{A^{-1}} : Q_{(1, \dots, 1)} &\rightarrow Q_{(1, \dots, 1)} \\ z &\mapsto z^{A^{-1}}, \end{aligned}$$

where we extend the notation to include rational exponents, is a well defined homeomorphism that maps $\mathbb{R}X_{P_L} \cap Q_{(1, \dots, 1)}$ to $\mathbb{R}X \cap Q_{(1, \dots, 1)}$. In particular, $\mathbb{R}X \cap Q_{(1, \dots, 1)}$ is empty if $\varepsilon_i = 1$ for all $i \in \{1, \dots, n\}$, and is non-empty and contractible otherwise. Now observe that $\mathbb{R}X \cap Q_\gamma$ is isomorphic to $\mathbb{R}X_{P_\gamma} \cap Q_{(1, \dots, 1)}$, where $P_\gamma(z) := \sum_{i=1, \dots, n} \varepsilon_i \gamma^{A_i} c_i z^{A_i} + c_0$. Since $\delta(\varepsilon_1 \gamma^{A_1}, \dots, \varepsilon_n \gamma^{A_n}) = \delta(\varepsilon) + A \cdot \delta(\gamma)$, we see that $\mathbb{R}X \cap Q_\gamma$ is empty if $\delta(\varepsilon) = A \cdot \delta(\gamma)$, and non-empty and contractible otherwise. \square

The next lemma links the conditions of Theorem 1.2, Lemma 4.2 and Proposition 4.3.

Lemma 5.2. *The real part $\mathbb{R}X$ intersects non-trivially each quadrant of $(\mathbb{R}^*)^n$ if and only if $\delta \notin \text{Im}(A)$ if and only if $|I^{1,0}| \neq 0$.*

Proof. The first equivalence comes from Lemma 5.1. Then

$$\begin{aligned} \delta \in \text{Im}(A) &\iff \exists x \in \mathbb{Z}_2^n \text{ s.t. } A \cdot x = G^{-1} \cdot D \cdot x = \delta \iff \\ &\exists x \in \mathbb{Z}_2^n \text{ s.t. } D \cdot x = G \cdot \delta = \delta^G \iff \delta^G \in \text{Im}(D) \iff \\ &(\delta^G)_i = 0 \forall i \text{ s.t. } d_i = [0]_2 \iff |I^{1,0}| = 0. \end{aligned}$$

\square

We are now ready to prove the main result.

Proof of Theorem 1.2. As mentioned at the beginning of Subsection 4.2, $1+c_* : H_k(\mathcal{C}_X) \rightarrow H_k(\mathcal{C}_X)$ is trivial for $k \neq n-1$. Moreover, we know that $H_k(\mathcal{C}_X)$ is isomorphic to $H_k(T^n)$ for $k \neq n-1$, and that $\dim(H_{n-1}(\mathcal{C}_X)) = n + d_1 \dots d_n - 1$ (as is shown by Lemma 4.1).

Therefore, from Proposition 4.3 and Lemmas 5.1 and 5.2, we see that if $\mathbb{R}X$ intersects non-trivially each quadrant of $(\mathbb{R}^*)^n$,

$$\begin{aligned} \dim\left(\frac{\text{Ker}(1+c_*)}{\text{Im}(1+c_*)}\right) &= \dim(H_*(\mathbb{C}X)) - 2\dim(\text{Im}(1+c_*)) = \\ 2^n - 1 + (d_1 \dots d_n - 1) - 2\left(\frac{d_1 \dots d_n}{2} - 1\right) &= 2^n = \dim(H_*(\mathbb{R}X)). \end{aligned}$$

Hence X is Galois maximal.

Otherwise, we have by definition of the sets $I^{\pm 1, \pm 1}$ in Section 3 that $\text{rank}_{\mathbb{Z}_2}(\Delta_P) = \text{rank}_{\mathbb{Z}_2}(A) = |I^{0,1}| + |I^{1,1}| = n - |I^{0,0}|$ (since $|I^{1,0}| = 0$) and

$$\begin{aligned} \dim\left(\frac{\text{Ker}(1+c_*)}{\text{Im}(1+c_*)}\right) &= \dim(H_*(\mathbb{C}X)) - 2\dim(\text{Im}(1+c_*)) = \\ 2^n - 1 + (d_1 \dots d_n - 1) - 2\left(|I^{0,0}| + \frac{d_1 \dots d_n - 2|I^{0,0}|}{2}\right) &= \\ 2^n - 2 - 2(n - \text{rank}_{\mathbb{Z}_2}(A)) + 2^{n - \text{rank}_{\mathbb{Z}_2}(A)}, \end{aligned}$$

hence

$$\begin{aligned} \dim\left(\frac{\text{Ker}(1+c_*)}{\text{Im}(1+c_*)}\right) - \dim(H_*(\mathbb{R}X)) &= \\ 2^n - 2 - 2(n - \text{rank}_{\mathbb{Z}_2}(A)) + 2^{n - \text{rank}_{\mathbb{Z}_2}(A)} - (2^n - 2^{n - \text{rank}(A)}) &= \\ 2(2^{n - \text{rank}_{\mathbb{Z}_2}(A)} - 1 - (n - \text{rank}_{\mathbb{Z}_2}(A))) & \quad . \end{aligned}$$

This ends the proof. \square

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