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Critical points of the distance function to a generic submanifold

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Preamble

In general, the critical points of the distance function d_M to a compact submanifold $M \subset \mathbb{R}^D$ can be poorly behaved. In this article, we show that this is generically not the case by listing regularity conditions on the critical and μ -critical points of a submanifold and by proving that they are generically satisfied and stable with respect to small C^2 perturbations. More specifically, for any compact abstract manifold M , the set of embeddings $i : M \rightarrow \mathbb{R}^D$ such that the submanifold $i(M)$ satisfies those conditions is open and dense in the Whitney C^2 -topology. When those regularity conditions are fulfilled, we prove that the critical points of the distance function to an ε -dense subset of the submanifold (e.g. obtained via some sampling process) are well-behaved. We also provide many examples that showcase how the absence of these conditions can result in pathological cases.

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*The authors contributed equally to this work.

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1 Introduction

Questions regarding the distance function $d_M : x \mapsto \inf\{d(x, y) : y \in M\}$ to a submanifold $M \subset \mathbb{R}^D$ stand at the intersection of a variety of domains; they feature preeminently in computational geometry [CCSL06], statistics on manifolds [NSW08, AKC⁺19, ABL23] and topological data analysis [EH22], and their study frequently involves tools from classical differential geometry as well [Yom81, Mat83]. In this article, we combine notions and methods from computational geometry and transversality theorems *à la* Thom to understand the behaviour of the critical points of the distance function to a generic submanifold of \mathbb{R}^D , and we show that they behave very nicely. We also explore some consequences in terms of sampling of the submanifold.

In 2004, A. Lieutier [Lie04] proposed a generalized notion of gradient for the distance function d_S to a compact set $S \subset \mathbb{R}^D$. Let $x \in \mathbb{R}^D$ and $\pi_S(x)$ be the set of all projections of x on S , that is

$$\forall y \in S, \quad y \in \pi_S(x) \iff d_S(x) = d(x, y). \quad (1)$$

The generalized gradient of d_S at $x \notin S$ is given by

$$\nabla d_S(x) = \frac{x - m(\pi_S(x))}{d_S(x)}, \quad (2)$$

where $m(\sigma)$ is the center of the smallest enclosing ball of a non-empty bounded set σ . This is illustrated in Figure 1. The gradient is extended by 0 on S , so that ∇d_S is defined on \mathbb{R}^D , and with this definition the norm $\|\nabla d_S\|$ of the generalized gradient is lower semicontinuous (see [Lie04]). The *medial axis* $\text{Med}(S)$ is the set of points $z \in \mathbb{R}^D$ such that $\pi_S(z)$ is of cardinality

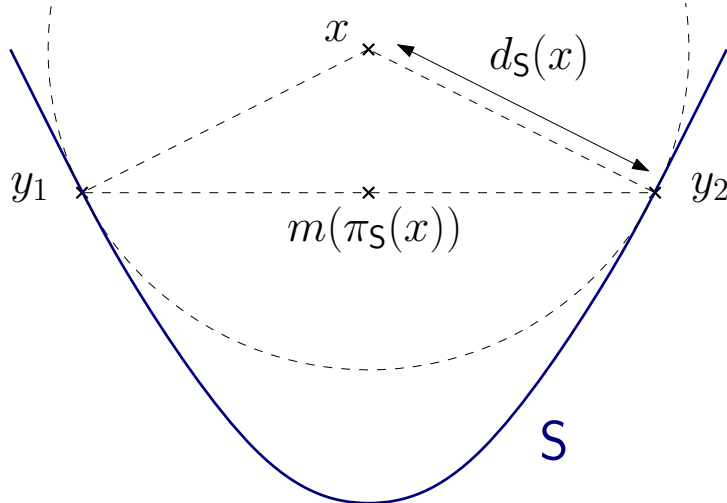


Figure 1: A point $x \in \text{Med}(\mathcal{S})$ with set of projections $\pi_{\mathcal{S}}(x) = \{y_1, y_2\}$.

at least 2, i.e. the points that do not have a unique projection on \mathcal{S} . Equivalently, $z \in \text{Med}(\mathcal{S})$ if and only if $\|\nabla d_{\mathcal{S}}(z)\| < 1$ and $z \notin \mathcal{S}$.

A *critical point* for the distance function is a point $z \in \mathbb{R}^D \setminus \mathcal{S}$ with $\nabla d_{\mathcal{S}}(z) = 0$, in which case $d_{\mathcal{S}}(z)$ is called a *singular value*; we let $Z(\mathcal{S})$ be the set of critical points for the distance function to \mathcal{S} , which we simply call critical points of \mathcal{S} . These will be our main object of study. This definition is generalized as follows: for $\mu \in [0, 1]$, one defines the set of μ -critical points as

$$Z_{\mu}(\mathcal{S}) := \{z \in \mathbb{R}^D \setminus \mathcal{S} : \|\nabla d_{\mathcal{S}}(z)\| \leq \mu\}. \quad (3)$$

The generalized gradient and the associated notions of criticality have many uses in computational geometry and data analysis. Among other applications, they lead to Morse-theoretic statements - it is for example shown in [CCSL09] (see also [SYM23]) that two offsets $\mathcal{S}^a = \{x \in \mathbb{R}^D : d_{\mathcal{S}}(x) \leq a\}$ and \mathcal{S}^b are isotopic whenever the segment $[a, b]$ does not contain any critical values of the distance function. In topological data analysis, the family of offsets $(\mathcal{S}^t)_{t \geq 0}$ allows one to compute the Čech persistence diagram of \mathcal{S} [CDSO14], which is routinely used in computations to analyze the topology of the set \mathcal{S} , with numerous applications in machine learning (see e.g. [TMO22] or the review [OPT⁺17] and references therein). The set of critical values of the distance function corresponds exactly to the coordinates of the points in the Čech persistence diagram, and understanding their behavior in different contexts is a central research topic. Among other uses, the generalized gradient is also employed to build homotopies between an open bounded set and its medial axis [Lie04], to give guarantees on the homology groups of an approximation of a given shape [CL05], or to control the convexity defect function of a set [ALS11].

Despite its importance, the set of critical points $Z(\mathcal{S})$ is still poorly understood. This is partly because without further assumptions, the set $Z(\mathcal{S})$ can be almost arbitrarily wild: consider any compact set $A \subset \mathbb{R}^D$, and let $\mathcal{S} := A \times \{-1\} \cup A \times \{1\} \subset \mathbb{R}^{D+1}$. Then $Z(\mathcal{S}) \cap (\mathbb{R}^D \times [-1/2, 1/2]) = (A \cup Z(A)) \times \{0\}$, which is extremely irregular for well-chosen¹ sets A . Another difficulty is the lack of stability of $Z(\mathcal{S})$ with respect to small perturbations of \mathcal{S} , even under strong regularity assumptions on \mathcal{S} . Indeed, the map $\mathcal{S} \mapsto Z(\mathcal{S})$ is not continuous for the Hausdorff distance d_H ,

¹One can for example let A be a Cantor set.

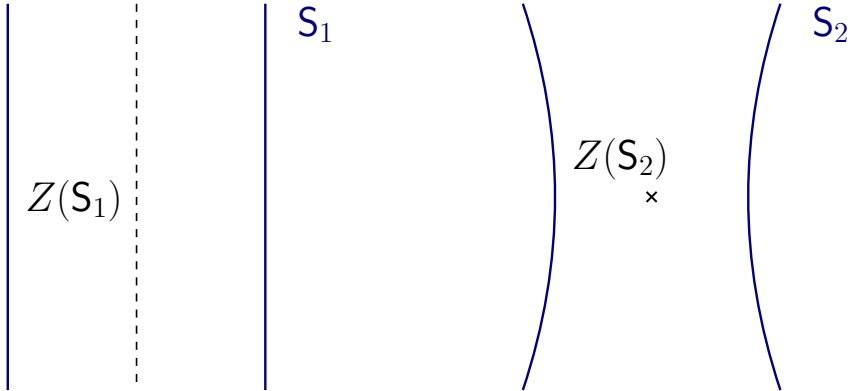


Figure 2: The Hausdorff distance between S_1 and S_2 can be made arbitrarily small while keeping the Hausdorff distance between $Z(S_1)$ and $Z(S_2)$ constant.

as is illustrated in Figure 2. This lack of stability can be particularly disastrous in practical situations, where the shape of interest S (e.g. a submanifold or a polyhedron) is often only accessible through a finite (possibly noisy) sample A . The set $Z(A)$ can then be very different from $Z(S)$, which can lead, in turn, to a poor estimation of the features of interest of S .

The μ -critical points are more stable: as shown by F. Chazal, D. Cohen-Steiner and A. Lieutier in [CCSL06], if S and A are two compact sets with $d_H(S, A) \leq \varepsilon$ and if $z \in \mathbb{R}^D$, then

$$z \in Z_\mu(A) \implies z \text{ is at distance at most } 2\sqrt{\varepsilon d_S(z)} \text{ from } Z_{\mu'}(S) \text{ for } \mu' = \mu + 2\sqrt{\varepsilon/d_S(z)}. \quad (4)$$

Nonetheless, this remains unsatisfactory when one is interested in actual critical points only.

The main goal of this article is to show that unlike what can be expected in general, the set of critical points of a *generic* compact C^2 submanifold $M \subset \mathbb{R}^D$ is well-behaved, and remains stable with respect to taking a dense enough sample of M . Furthermore, both these properties are stable with respect to small C^2 perturbations of M .

To formalize these statements, let us first clarify what we mean by “well-behaved”. Here are four conditions that we want our submanifolds to satisfy:

(P1) For every $z_0 \in Z(M)$, the projections $\pi_M(z_0)$ are the vertices of a non-degenerate simplex of \mathbb{R}^D . In particular, z_0 has at most $D + 1$ projections. Moreover, the point z_0 belongs to the relative interior of the convex hull of $\pi_M(z_0)$.

(P2) The set $Z(M)$ is finite.

(P3) For every $z_0 \in Z(M)$ and every $x_0 \in \pi_M(z_0)$, the sphere $S(z_0, d_M(z_0))$ is non-osculating M at x_0 , in the sense that there exist $\delta > 0$ and $\alpha > 0$ such that for all $y \in M \cap B(x_0, \delta)$,

$$\|y - z_0\|^2 \geq \|x_0 - z_0\|^2 + \alpha \|y - x_0\|^2. \quad (5)$$

(P4) There exist constants $C > 0$ and $\mu_0 \in (0, 1)$ such that for every $\mu \in [0, \mu_0)$, the set $Z_\mu(M)$ is included in a tubular neighborhood of size $C\mu$ of $Z(M)$, that is every point of $Z_\mu(M)$ is at distance less than $C\mu$ from $Z(M)$.

Properties (P1) and (P2) are quite self-explanatory; they control the discreteness of the set of the critical points and of their projections, and the genericity of their relative positions.

Property (P3) accepts many equivalent formulations, as shown in Section 3, and is essentially a property of the curvature of M at the projection x_0 . It is crucial to the stability of the projections of the critical points, as will become apparent later. Finally, Property (P4) controls the position of the μ -critical points relative to the critical points. It is equivalent to a local condition on the critical points and their projections, as shown in Section 6; somewhat surprisingly, it also plays a role in the stability of the critical points themselves, as can be seen in Section 7.

We also need definitions that appropriately capture the idea of genericity for a manifold. Consider an abstract C^r manifold M for some $r \in \mathbb{N} \cup \{\infty\}$, and let $\text{Emb}^k(M, \mathbb{R}^D)$ be the set of C^k embeddings from M to \mathbb{R}^D . For $0 \leq l \leq k$, we can endow $\text{Emb}^k(M, \mathbb{R}^D)$ with the Whitney C^l -topology, see Section 2. Intuitively, two embeddings $i, i' \rightarrow \mathbb{R}^D$ are close for the Whitney C^l -topology if their first l differentials (including their “0”th differentials, i.e. their values) are uniformly close over M when expressed in the coordinates of a fixed, finite atlas of M . In what follows, we are interested in showing that certain properties are such that for any C^k compact manifold M , the set of embeddings i such that $i(M)$ verifies the property is dense and open in $\text{Emb}^k(M, \mathbb{R}^D)$ for the Whitney C^l -topology (for some $l \leq k \in \mathbb{N}$). Openness captures the idea that the property should be stable with respect to small perturbations; density captures the idea that most submanifolds should satisfy the condition, as any submanifold can be made to satisfy it with an arbitrarily small modification. The conjunction of openness and density further reinforces this idea; by contrast, density alone can result in counter-intuitive situations, as \mathbb{Q} is dense in \mathbb{R} , yet few would claim that most real numbers are rational.²

We can now formulate our first main result, i.e. that generic C^2 submanifolds have well-behaved critical points and μ -critical points in the sense defined above.

Theorem 1.1 (The Genericity Theorem). *Let M be a compact C^k manifold for some $k \in \mathbb{N} \cup \{\infty\}$ with $k \geq 2$. Then the set of C^k embeddings $i : M \rightarrow \mathbb{R}^D$ such that $i(M)$ satisfies Conditions (P1-4) is dense for the Whitney C^k -topology and open for the Whitney C^2 -topology in $\text{Emb}^k(M, \mathbb{R}^D)$.*

Remark 1.2. *As the Whitney C^{l_1} -topology is finer than the Whitney C^{l_2} -topology for any $l_1 \geq l_2$, the statement implies that for any $2 \leq l \leq k$, the set of appropriate embeddings is dense and open in $\text{Emb}^k(M, \mathbb{R}^D)$ for the Whitney C^l -topology.*

Remark 1.3. *The density part of the statement easily extends to C^1 submanifolds: if M is a compact C^1 manifold, then the set of C^1 embeddings $i : M \rightarrow \mathbb{R}^D$ such that $i(M)$ satisfies Conditions (P1-4) is dense in $\text{Emb}^1(M, \mathbb{R}^D)$ for the Whitney C^1 -topology. We justify this remark in the second part of the proof of Theorem 1.1, at the end of Section 8.*

While openness is proved through arguments similar to those used in [CCSL06] or [Lie04], we use a more abstract method for the density part of Theorem 1.1. Indeed, while it is relatively easy to perturb any local non-generic situation into a generic one, ensuring that the submanifold $i(M)$ satisfies (P1-4) globally is more challenging. Inspired by Y. Yomdin’s work in [Yom81], we proceed as follows: we define various subsets W of the space of multijets such that embeddings whose associated multijet is transverse to W satisfy good global geometric properties, and we apply variants of Thom’s transversality theorem to show that the set of transverse embeddings is dense in the appropriate topology (all these notions are properly introduced in Section 8).

²This nuance can be particularly important for practical applications, as by nature computers can only manipulate rational numbers: one could imagine some algorithm that is guaranteed to perform well on a dense set of irrational inputs, yet fails for any rational input.

Various examples suggest that the openness part of the statement cannot be easily extended to C^1 submanifolds; in fact, it is even surprising that the theorem holds for C^2 manifolds, as most straightforward arguments seem to force us to resort to the Whitney C^3 -topology (see e.g. Remark 8.26). Showing that the property is true for the C^2 -topology greatly increases the difficulty of the proof. On the other hand, this rather convoluted proof yields as a byproduct a stronger stability result than the openness part of the Genericity Theorem 1.1, which we call the C^2 Stability Theorem. Let M be a submanifold that satisfies Properties (P1-4); then the theorem states that not only the properties themselves, but the numbers and positions of the critical points of M and of their projections are stable with respect to small C^2 perturbations, in the sense detailed below.

Theorem 1.4 (The C^2 Stability Theorem). *Let $i \in \text{Emb}^2(M, \mathbb{R}^D)$ be an embedding such that $M = i(M)$ satisfies (P1-4). Then, for all $\varepsilon > 0$ small enough, there exists a neighborhood \mathcal{U} of i in $\text{Emb}^2(M, \mathbb{R}^D)$ equipped with the Whitney C^2 -topology such that for every $i' \in \mathcal{U}$, with $M' = i'(M)$:*

1. M' satisfies (P1-4).
2. There is a bijection $\Psi : Z(M) \rightarrow Z(M')$.
3. Let $z_0 \in Z(M)$; then $\Psi(z_0) \in B(z_0, \varepsilon)$.
4. Let $z_0 \in Z(M)$ with $\pi_M(z_0) = \{x_1, \dots, x_s\}$. Then $\pi_{M'}(\Psi(z_0))$ is included in the union of balls of radius ε centered at the points x_j , with exactly one point in each ball.

Remark 1.5. *We only ask that ε be small enough to ensure that the balls of point 4. do not intersect.*

The C^2 Stability Theorem shows that Properties (P1-4), which we know to be generic in the sense of Theorem 1.1, are desirable not only for themselves, but also for the stability of the critical points and of their projections with respect to C^2 perturbations that they guarantee - various examples throughout this paper illustrate how removing some of the properties can lead to somewhat pathological instabilities. The following theorem states that Properties (P1-4) also induce another type of stability, this time with respect to taking an ε -dense subset A of the submanifold M - a typical example would be for A to be some point cloud sampled on M .

Theorem 1.6 (Stability Theorem for Subsets). *Let $M \subset \mathbb{R}^D$ be a compact C^2 submanifold. There exist positive constants C_1, C_2, C_3, C_4, C_5 and ε_0 such that the following holds. Let $A \subset M$ be a set with $d_H(A, M) \leq \varepsilon \leq \varepsilon_0$.*

1. *For every critical point $z \in Z(A)$, exactly one of these two possibilities is true: either z is very close to M , that is $d_M(z) \leq C_1\varepsilon^2$, or it is close to a μ -critical point of M , that is $d_{Z_\mu(M)}(z) \leq C_2\varepsilon$ for $\mu = C_3\varepsilon$.*
2. *If M satisfies (P4), then, in the second case, the critical point $z \in Z(A)$ is close to $Z(M)$, that is $d_{Z(M)}(z) \leq C_4\varepsilon$.*
3. *If M additionally satisfies conditions (P1), (P2) and (P3) then, in the second case, the set of projections $\pi_A(z)$ of the critical point $z \in Z(A)$ is included in a neighborhood of size $C_5\varepsilon$ of $\pi_M(z')$ for some $z' \in Z(M)$ such that $d(z, z') \leq C_4\varepsilon$.*

This is to be compared to the previously available guarantees for general compact sets with positive reach (see definition in Section 2): let $S \subset \mathbb{R}^D$ be a compact set with positive reach $\tau(S)$, and let $A \subset \mathbb{R}^D$ be such that $d_H(A, S) \leq \varepsilon$. Then Chazal, Cohen-Steiner and Lieutier [CCSL06] show that all critical points of A are either at distance 4ε from S , or are at distance at least $\tau(S) - 3\varepsilon$ from S . Hence, assuming that $\varepsilon < \tau(S)/7$, there are no critical values of d_A in the interval $(4\varepsilon, \tau(S)/2)$. In that case, Equation (4) implies that the critical points of A that are “far from S ” are only $O(\sqrt{\varepsilon})$ -close to μ -critical points for $\mu = O(\sqrt{\varepsilon})$.

Hence the added hypotheses on M yield several improvements with respect to the general case. Firstly,³ the critical points that are close to M are at distance $O(\varepsilon^2)$ from M instead of $O(\varepsilon)$. Secondly, the stability with respect to μ -critical points is also improved, going from $O(\sqrt{\varepsilon})$ to $O(\varepsilon)$ for both the distance to $Z_\mu(M)$ and the value of μ . Note that these two improvements are solely due to the set M being a submanifold and to A being a subset of M , and hold without any of the conditions (P1-4). Thirdly and finally, Properties (P1-4) allow us to get the $O(\varepsilon)$ -stability of both critical points and their projections.

At last, we get as a corollary to Theorem 1.6 a bound on the number of critical points that a finite approximation (i.e. a point cloud) A of M can have. We say that a finite subset $A \subset M$ is a (δ, ε) -sampling of M if $d_H(A, M) \leq \varepsilon$ and $\min_{x \neq y \in A} \|x - y\| \geq \delta$. For $\delta, \varepsilon > 0$, the δ -packing number $N(\delta, \varepsilon)$ of M is the maximal cardinality of a $(\delta, +\infty)$ -sampling of a ball $B(x, \varepsilon) \cap M$ centered at a point $x \in M$.

Corollary 1.7. *Let $M \subset \mathbb{R}^D$ be a compact C^2 submanifold of dimension m satisfying (P1-4). Consider the constants C_2 and C_5 from Theorem 1.6. There exists $\varepsilon_1 > 0$ such that if A is a (δ, ε) -sampling of M for some $0 < \delta \leq \varepsilon \leq \varepsilon_1$, and if $z \in Z(M)$ and s is the cardinality of the set of projections $\pi_M(z)$, then the number of critical points in $Z(A)$ at distance less than $C_2\varepsilon$ from z is bounded by $2^{N(\delta, C_5\varepsilon)^s}$. Furthermore*

$$N(\delta, C_5\varepsilon) \leq (C_m C_5 \varepsilon / \delta)^m$$

for some $C_m > 0$ that depends on m .

The practical applicability of Corollary 1.7 relies on the existence of (δ, ε) -samplings with δ of the same order as ε . The farthest point sampling algorithm [AL18] shows that such sets always exist: starting from a finite subset $A_0 \subset M$ with $d_H(A_0, M) \leq \varepsilon$, it outputs a subset $A_1 \subset A_0$ that is an $(\varepsilon, 2\varepsilon)$ -sampling of M . For such a subset A_1 , Corollary 1.7 ensures that the number of critical points at distance at least $O(\varepsilon^2)$ from M is at most $K2^{(2C_m C_5)^m}$, where K is the cardinality of $Z(M)$. In particular, this quantity stays bounded as ε goes to 0.

1.1 Related work

Given a compact set A , both the medial axis $\text{Med}(A)$ and the skeleton $\text{Sk}(A)$ (the set of centers of maximal balls not intersecting A) are well studied in the literature [Gib00, GK04, Lie04, CS04]. The two concepts are tightly linked, as $\text{Med}(A) \subset \text{Sk}(A) \subset \overline{\text{Med}(A)}$, the latter inclusion being an equality in all “tame” cases. We refer to [ABE09] for a review of the (in)stability properties of the medial axis. Regarding specifically the issue of the stability of the critical and μ -critical points with respect to sampling, our main inspiration was [CCSL06], which considers the case of general compact sets.

³This first point was well-known folklore, frequently mentioned by F. Chazal among others.

The genericity of transversality is another key theme of this article; the first transversality theorems for jets date back to Thom’s [Tho56], later extended to multijets by J. Mather in [Mat70]. Early work on the application of those theorems to the distance function in Euclidean space include [Loo74] and [Wal77], which eventually led to the study of the properties of the skeleton $\text{Sk}(M)$ of a generic manifold M by Yomdin [Yom81], who proves among other results that the set of embeddings i of a manifold M such that for all $z \in \text{Sk}(i(M))$, the projection set $\pi_{i(M)}(z)$ is a non-degenerate simplex, is dense (and in fact residual), then by Mather [Mat83]. Later, J. Damon and E. Gasparovic gave in a series of papers (see [Dam97, DG17] and references therein) a thorough analysis of the geometry of the skeleton set for generic sets in different contexts. Very recently, A. Song, K.M. Yim and A. Monod [SYM23] studied the set of critical points of a generic submanifold, though only for surfaces S embedded into \mathbb{R}^3 and only in terms of density (without openness and stability) of regularity conditions weaker than (P1-4).

1.2 Outline

We recall a few definitions and prove an elementary lemma in Section 2, before studying in detail the non-oscularity condition (P3) in Section 3; in particular, we rephrase it as a curvature condition. In Section 4, we prove some properties of the projection on a submanifold M wherever it is defined and in particular under some non-oscularity assumptions; we are also interested in its stability with respect to perturbations of M . Later in Section 5, we use some of our findings to prove the Stability Theorem for Subsets Theorem 1.6. In Section 6, we introduce a condition that we call the Big Simplex Property (BSP) that is equivalent to (P4), but has the advantage of being phrased in terms of the local behaviour of M around the projections of its critical points. This helps us prove the C^2 Stability Theorem 1.4 in Section 7, which implies in particular the openness of the conjunction of Conditions (P1-4). Finally, the last section is dedicated to proving the density part of the statement of the Genericity Theorem 1.1; it starts with some exposition, where we introduce variants of Thom’s transversality theorem and some necessary notions regarding multijets, before moving on to the proof proper.

Throughout the paper, we always assume that the dimension m of the manifold M is greater than or equal to 1, and that the dimension D of the ambient Euclidean space \mathbb{R}^D is greater than or equal to 2, as otherwise all statements in the Introduction are trivially true.

2 Notations, definitions and preliminaries

For $s \in \mathbb{N}$, we let $[s]$ be the set $\{1, \dots, s\}$. We let $B(x, r)$ be the open ball centered at x of radius r and $\overline{B}(x, r)$ be its closure; we write $S(x, r)$ for the sphere centered at x of radius r . For a set $S \subset \mathbb{R}^D$, the *reach* of S is defined as

$$\tau(S) := \sup\{r \in \mathbb{R} : \forall x \in \mathbb{R}^D \text{ s.t. } d_S(x) < r, \exists! y \in S \text{ s.t. } d(x, y) = d_S(x)\}.$$

Note that if S has positive reach, then it is necessarily closed. Sets with positive reach are quite regular in many ways; they were extensively studied by H. Federer in [Fed59].

Throughout this article, we consider a fixed compact C^k abstract m -dimensional manifold M for some $k \in \{2, 3, \dots, \infty\}$ and $m \geq 1$,⁴ together with a fixed finite atlas $\mathcal{A} = (\phi_j, U_j)_{j \in J}$. For $1 \leq l \leq k$, we consider the set $C^l(M, \mathbb{R}^D)$ of functions of class C^l from M to \mathbb{R}^D , and for

⁴The different theorems stated in the introduction can easily be shown to be true for $m = 0$.

$1 \leq p \leq l$, if $f \in C^l(M, \mathbb{R}^D)$, we let

$$\|f\|_{C^p} := \max_{0 \leq r \leq p} \max_{j \in J} \sup_{u \in U_j} \|d^r(f \circ \phi_j)_u\|_{\text{op}}$$

be the C^p -norm of f , where $d^r g_u$ is the differential of order r of a function g at the point u . We can always assume this quantity to be finite, up to restricting each chart to a slightly smaller subset of its domain. This norm defines a topology on $C^l(M, \mathbb{R}^D)$, called the *Whitney C^p -topology*. This topology is independent of the choice of the atlas, and the Whitney C^{p_2} -topology is finer than the Whitney C^{p_1} -topology on $C^l(M, \mathbb{R}^D)$ for any $p_1 \leq p_2 \leq l$. If $k = \infty$, the union of all the Whitney C^p -topologies on $C^\infty(M, \mathbb{R}^D)$ for $p \geq 1$ defines the Whitney C^∞ -topology (i.e. the set of open sets of the Whitney C^∞ -topology is the union over $p \geq 1$ of the sets of open sets of the Whitney C^p -topologies). For $1 \leq l \leq k$, we let $\text{Emb}^l(M, \mathbb{R}^D)$ be the subset of $C^l(M, \mathbb{R}^D)$ consisting of all C^l embeddings (immersions that are homeomorphic to their image). It is an open set for the Whitney C^1 -topology (and hence for all Whitney C^p -topologies for $1 \leq p \leq l$), as shown in [Mic80, Prop 5.3].

Another, more intrinsic definition (based on jets) of the Whitney C^p -topology is given in Section 8.1; in particular, it extends to the case where the domain of the mappings is another C^k manifold N , rather than the Euclidean space \mathbb{R}^D . See also [GG74] for a more complete presentation.

If $i \in \text{Emb}^l(M, \mathbb{R}^D)$ for $l \geq 2$, then $M = i(M)$ is a compact C^2 submanifold of \mathbb{R}^D . For $x \in M$, we let $T_x M$ be the tangent space of M at x , and $T_x M^\perp$ the normal space, that are endowed with the norm induced by the ambient Euclidean space. When there is no possible ambiguity, we let $\pi_x : \mathbb{R}^D \rightarrow T_x M$ be the orthogonal projection on $T_x M$ and $\pi_x^\perp : \mathbb{R}^D \rightarrow T_x M^\perp$ be the orthogonal projection on $T_x M^\perp$. For $\eta \in T_x M^\perp$, we let $W_{x,\eta} : T_x M \rightarrow T_x M$ be the shape operator in the normal direction η . This is a self-adjoint operator such that $\langle W_{x,\eta}(v), v \rangle$ is the normal curvature in the normal direction $\eta \in T_x M^\perp$ along the tangent vector $v \in T_x M$, see [Lee18, Chapter 8].

Finally, we will make frequent use of the following elementary lemma.

Lemma 2.1. *Let S be any compact set. Let $t_0 > 0$ and let $z_0 \in Z(S)$. Then, there exists $r_0 > 0$ such that all the points in $B(z_0, d_S(z_0) + r_0) \cap S$ are at distance less than t_0 from the projections $\pi_S(z_0)$.*

Proof. Assume that the statement is false. Then, there exist sequences $r_n \rightarrow 0$ and $x_n \in S$, such that $\|x_n - z_0\| < d_S(z_0) + r_n$, but x_n is at distance at least t_0 from $\pi_S(z_0)$. By compactness, we can assume without loss of generality that $x_n \rightarrow x \in S$. But by continuity, we have $\|x - z_0\| \leq d_S(z_0)$, so that $x \in \pi_S(z_0)$, contradicting the condition that all the points x_n are at distance at least t_0 from $\pi_S(z_0)$. \square

3 Osculation

We fix an embedding $i \in \text{Emb}^2(M, \mathbb{R}^D)$, and consider the compact C^2 submanifold $M = i(M)$. In this section, we give several equivalent formulations of the condition of non-osculation from Property (P3). Some of the material presented here is already well-known; as the proofs are short, we nonetheless include them for the sake of completeness.

If $z_0 \in \mathbb{R}^D \setminus M$ and $x_0 \in \pi_M(z_0)$, then the sphere of radius $d_M(z_0)$ centered at z_0 is tangent to the manifold M at the point x_0 . As the inside of this sphere does not intersect M , the curvature

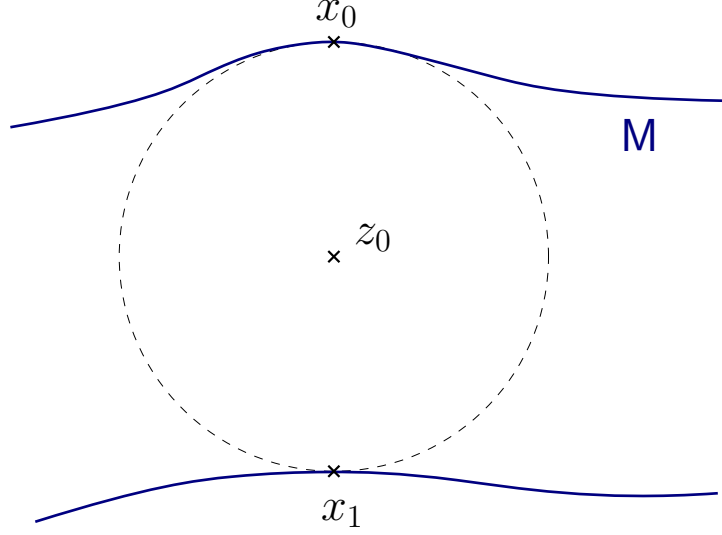


Figure 3: The sphere centered at z_0 of radius $d_M(z_0)$ is osculating M at x_0 , but not at x_1 .

at x_0 in the normal direction $z_0 - x_0$ has to be smaller than or equal to the one of the sphere along any tangent vector of $T_{x_0}M$. Two different situations can arise: either the curvature of the manifold is strictly smaller than the one of the sphere (as will be shown later, this is generically the case whenever z_0 is a critical point of M), or the curvature of the manifold in the normal direction $\eta := z_0 - x_0$ and in at least one of the directions of $T_{x_0}M$ is equal to the sphere's, that is there exists $v \in T_xM$ with $\langle W_{x_0,\eta}(v), v \rangle = \|v\|^2$. In that case, we say that the sphere of radius $d_M(z_0)$ centered at z_0 is *osculating* the manifold M at x_0 . Both situations are illustrated in Figure 3.

In the following lemma, we show that this condition is equivalent to a purely metric one, and that due to the C^2 regularity of M , this metric condition is equivalent to a local version of itself. Finally, though the non-osculation of $S(z_0, d_M(z_0))$ at x_0 only depends on M , and not on the embedding i such that $M = i(M)$, it will be convenient later on to have a formulation of the condition in terms of i , which is also provided by the lemma.

Lemma 3.1 (Osculation Characterization Lemma). *Let $z_0 \in \mathbb{R}^D \setminus M$, $x_0 \in \pi_M(z_0)$ and define $\eta := z_0 - x_0$. Then the sphere of radius $d_M(z_0)$ centered at z_0 not osculating the submanifold M at x_0 is equivalent to the existence of $\alpha > 0$ such that one of the following equivalent conditions holds:*

- (i) *The largest eigenvalue $\lambda_{\max}(W_{x_0,\eta})$ of the shape operator $W_{x_0,\eta} : T_{x_0}M \rightarrow T_{x_0}M$ at x_0 in the direction η satisfies $\lambda_{\max}(W_{x_0,\eta}) < 1 - \alpha$.*
- (ii) *Let $m_0 \in M$ be such that $i(m_0) = x_0$, with (ϕ, U) a chart of M around m_0 with $\phi(u_0) = m_0$. Then, the quadratic form*

$$\begin{aligned} \mathbb{R}^m &\longrightarrow \mathbb{R} \\ v &\longmapsto (1 - \alpha) \|d(i \circ \phi)_{u_0}(v)\|^2 - \langle \eta, d^2(i \circ \phi)_{u_0}(v, v) \rangle \end{aligned}$$

is positive definite.

(iii) There exists $\alpha' > \alpha$ and $\delta > 0$ such that, if $M_0 := M \cap \overline{B}(x_0, \delta)$, then for any $y \in M_0$,

$$\|y - z_0\|^2 \geq \|x_0 - z_0\|^2 + \alpha' \|y - x_0\|^2. \quad (6)$$

(iv) There exists $\alpha' > \alpha$ and $\delta, \delta' > 0$ such that, if $M_0 := M \cap \overline{B}(x_0, \delta)$, then for any $z \in B(z_0, \delta')$, $x \in \pi_{M_0}(z)$ and $y \in M_0$, it holds that

$$\|y - z\|^2 \geq \|x - z\|^2 + \alpha' \|y - x\|^2. \quad (7)$$

When the sphere of radius $d_M(z_0)$ centered at z_0 does not osculate M at x_0 , we call a number $\alpha > 0$ that satisfies the conditions of Lemma 3.1 a *degree of non-osculation at x_0 of the sphere $S(z_0, d_M(z_0))$* .

Proof. The equivalence between the sphere being non-osculating and the existence of some $\alpha > 0$ that satisfies condition (i) is clear. The equivalence between (i) and (ii) follows from the fact that, if $w = d(i \circ \phi)_{u_0}(v)$, then

$$\langle W_{x_0, \eta}(w), w \rangle = \langle \eta, d^2(i \circ \phi)_{u_0}(v, v) \rangle, \quad (8)$$

as $\eta \in T_{x_0} M^\top$ (see [Lee18, Chapter 8]).

Let us now show the equivalence between (ii) and (iii). Observe that for any $\alpha' > 0$ and any $y \in \mathbb{R}^D$,

$$\|y - z_0\|^2 - \alpha' \|y - x_0\|^2 - \|\eta\|^2 = (1 - \alpha') \|y - x_0\|^2 - 2\langle \eta, y - x_0 \rangle.$$

Let i, ϕ and u_0 be as in the statement of condition (ii); then condition (iii) is equivalent to the existence of some $\alpha' > \alpha$ such that

$$(1 - \alpha') \|i \circ \phi(u) - x_0\|^2 - 2\langle \eta, i \circ \phi(u) - x_0 \rangle > 0$$

for any $u \neq u_0$ close enough to u_0 . But, as $i \circ \phi$ is C^2 and $\text{Im}(d(i \circ \phi)_{u_0}) = T_{i \circ \phi(u_0)} M$ is orthogonal to η , it holds that

$$\begin{aligned} & (1 - \alpha') \|i \circ \phi(u) - x_0\|^2 - 2\langle \eta, i \circ \phi(u) - x_0 \rangle \\ &= (1 - \alpha') \|d(i \circ \phi)_{u_0}(u - u_0)\|^2 - \langle \eta, d^2(i \circ \phi)_{u_0}(u - u_0, u - u_0) \rangle + o(\|u - u_0\|^2). \end{aligned}$$

Hence, the existence of some $\alpha' > \alpha$ such that $(1 - \alpha') \|d(i \circ \phi)_{u_0}(v)\|^2 - \langle \eta, d^2(i \circ \phi)_{u_0}(v, v) \rangle + o(\|v\|^2)$ is strictly positive for $v \neq 0$ small enough is equivalent to the form

$$v \mapsto (1 - \alpha) \|d(i \circ \phi)_{u_0}(v)\|^2 - \langle \eta, d^2(i \circ \phi)_{u_0}(v, v) \rangle$$

being positive definite (due to $d(i \circ \phi)_{u_0}$ being an isomorphism). This shows that conditions (ii) and (iii) are equivalent.

Condition (iv) obviously implies condition (iii). It remains to show that (iii) implies (iv). We postpone the proof to Section 4, in Corollary 4.7. \square

In the next section, we show, among other things, that the non-oscularity with respect to M of the sphere $S(z_0, d_M(z_0))$ at $x_0 \in \pi_M(z_0)$ gives nice regularity and stability properties to the projection map around z_0 on a neighborhood of x_0 .

4 Regularity and stability of projections

Let as earlier $i \in \text{Emb}(M, \mathbb{R}^D)$ be a C^2 embedding of the abstract manifold M , and $\mathbf{M} := i(M)$ be the resulting submanifold. In this section, we prove various statements regarding the projections on \mathbf{M} or on some subsets of \mathbf{M} , particularly under some assumptions of non-osculation. These results will be instrumental later in the article.

4.1 Lipschitz continuity and local unicity of the projection

The following lemma states that if the sphere $S(z_0, d_{\mathbf{M}}(z_0))$ is not osculating \mathbf{M} at $x_0 \in \pi_{\mathbf{M}}(z_0)$, then one can define a Lipschitz projection function from a neighborhood of z_0 to a neighborhood of x_0 in \mathbf{M} .

Lemma 4.1. *Let $z_0 \in \mathbb{R}^D \setminus \mathbf{M}$ and $x_0 \in \pi_{\mathbf{M}}(z_0)$. Assume that the sphere $S(z_0, d_{\mathbf{M}}(z_0))$ does not osculate \mathbf{M} at x_0 , with $\alpha > 0$ a degree of non-osculation. Let $\delta, \delta' > 0$ be the constants given in Lemma 3.1.(iv). If $\mathbf{M}_0 = \mathbf{M} \cap \overline{B}(x_0, \delta)$ and $z \in B(z_0, \delta')$, then $\pi_{\mathbf{M}_0}(z)$ contains exactly one element x , which satisfies*

$$\|x - x_0\| \leq \frac{1}{\alpha} \|z - z_0\|. \quad (9)$$

Proof. Let $\delta, \delta' > 0$ and $\alpha' > \alpha$ be the constants given in Lemma 3.1.(iv). Let $x \in \pi_{\mathbf{M}_0}(z)$. Lemma 3.1.(iv) implies that every point $y \in \mathbf{M}_0$ different from x satisfies $\|y - z\| > \|x - z\|$. This proves that $\pi_{\mathbf{M}_0}(z)$ contains exactly one element x . Let us prove the second claim. Assume that $x \neq x_0$, for otherwise the claim is trivial. According to (7) with $y = x_0$,

$$\begin{aligned} \|x_0 - z\|^2 &\geq \|x - z\|^2 + \alpha' \|x_0 - x\|^2 \\ &\geq \|x - x_0\|^2 + \|x_0 - z\|^2 + 2\langle x - x_0, x_0 - z \rangle + \alpha' \|x_0 - x\|^2 \\ &\geq \|x - x_0\|^2 + \|x_0 - z\|^2 + 2\langle x - x_0, x_0 - z_0 \rangle + 2\langle x - x_0, z_0 - z \rangle + \alpha' \|x_0 - x\|^2, \end{aligned}$$

that is, using Cauchy-Schwartz inequality

$$(1 + \alpha') \|x - x_0\|^2 \leq -2\langle x - x_0, x_0 - z_0 \rangle + 2\|x - x_0\| \|z_0 - z\|. \quad (10)$$

Also, we may use (7) with $y = x$ and $x = x_0$ to obtain

$$\begin{aligned} \alpha' \|x_0 - x\|^2 + \|x_0 - z_0\|^2 &\leq \|x - z_0\|^2 \\ \alpha' \|x_0 - x\|^2 + \|x_0 - z_0\|^2 &\leq \|x - x_0\|^2 + \|x_0 - z_0\|^2 + 2\langle x - x_0, x_0 - z_0 \rangle. \end{aligned}$$

that is

$$-2\langle x - x_0, x_0 - z_0 \rangle < (1 - \alpha') \|x_0 - x\|^2. \quad (11)$$

Putting together (10) and (11) yields

$$2\alpha' \|x - x_0\|^2 \leq 2\|x - x_0\| \|z_0 - z\|.$$

Dividing by $2\alpha' \|x - x_0\|$ and using that $\alpha' > \alpha$ give the result. \square

4.2 Differential of the projection

The following proposition, which can be found e.g. in [LS21], describes the differential of the projection on M wherever it is well-defined.

Proposition 4.2. *Let $\overline{\text{Med}}(M)$ be the closure of the medial axis of M , and let $z \in \mathbb{R}^D \setminus (M \cup \overline{\text{Med}}(M))$. Then the projection on M uniquely defines a C^1 map p on a neighborhood of z , and the differential of p in z is given by*

$$dp_z = \left(\text{id}_{T_{p(z)}M} - W_{p(z), z-p(z)} \right)^{-1} \circ \pi_{p(z)} : \mathbb{R}^D \rightarrow T_{p(z)}M,$$

where we recall that $\pi_{p(z)} : \mathbb{R}^D \rightarrow T_{p(z)}M$ is the orthogonal projection onto $T_{p(z)}M$.

The following lemma serves two purposes: it shows that under simple conditions, the projection at z_0 on the image of the germ of a function $f : M \rightarrow \mathbb{R}^D$ (rather than on the image of some embedding $i : M \rightarrow \mathbb{R}^D$) and the differential of that projection are well-defined, and in that context it expresses the formula for the differential of the projection given in Proposition 4.2 in terms of coordinates of charts of M .⁵ This will be needed when working with jets in Section 8; sadly, it involves some rather tedious verifications.

Lemma 4.3. *Let $m_0 \in M$ and $x_0, z_0 \in \mathbb{R}^D$ and let $f \in C^2(M, \mathbb{R}^D)$ be some function (not necessarily an embedding) that maps m_0 to x_0 . Furthermore, suppose that df_{m_0} has full rank and that $\text{Im}(df_{m_0}) \perp z_0 - x_0$. Let $\phi : (V, u_0) \rightarrow (M, m_0)$ be any chart of M around m_0 . If the quadratic form*

$$\begin{aligned} \mathbb{R}^m &\longrightarrow \mathbb{R} \\ v &\longmapsto \|d(f \circ \phi)_{u_0}(v)\|^2 - \langle z_0 - x_0, d^2(f \circ \phi)_{u_0}(v, v) \rangle \end{aligned}$$

is positive definite, then there exist $\delta > 0$ and a neighborhood $U \subset M$ of m_0 such that

- $f|_U : U \rightarrow \mathbb{R}^D$ is an embedding,
- each $z \in B(z_0, \delta)$ admits a unique projection on $\overline{f(U)}$ and that unique projection belongs to $f(U)$, and
- this unique projection defines a C^1 map $p : B(z_0, \delta) \rightarrow f(U)$.

In that case, let as above $\phi : (V, u_0) \rightarrow (M, m_0)$ be any chart of M around m_0 . For any $v \in \text{Im}(df_{m_0})$, let $\Lambda(v) \in \text{Im}(df_{m_0})$ be the only vector in $\text{Im}(df_{m_0})$ such that

$$\langle z_0 - x_0, d^2(f \circ \phi)_{u_0}(d(f \circ \phi)_{u_0}^{-1}(v), d(f \circ \phi)_{u_0}^{-1}(w)) \rangle = \langle \Lambda(v), w \rangle$$

for any $w \in \text{Im}(df_{m_0})$. This defines a linear endomorphism $\Lambda : \text{Im}(df_{m_0}) \rightarrow \text{Im}(df_{m_0})$ that is independent from the choice of the chart ϕ , and

$$\begin{aligned} Q : \text{Im}(df_{m_0}) &\longrightarrow \text{Im}(df_{m_0}) \\ v &\longmapsto v - \Lambda(v) \end{aligned}$$

is a linear isomorphism.

The differential at z_0 of the map p is then equal to

$$dp_{z_0} = Q^{-1} \circ \pi_{\text{Im}(df_{m_0})} : \mathbb{R}^D \rightarrow \text{Im}(df_{m_0}),$$

where $\pi_{\text{Im}(df_{m_0})} : \mathbb{R}^D \rightarrow \text{Im}(df_{m_0})$ is the orthogonal projection.

⁵Such an expression must be available somewhere in the literature, but we could not find it.

Remark 4.4. As shown in the proof below, the linear map Λ is simply the shape operator at x_0 in normal direction $z_0 - x_0$ of the submanifold $f(U)$.

Proof. The fact that $f|_U : U \rightarrow \mathbb{R}^D$ is an embedding for a small enough neighborhood U of m_0 is simply a consequence of the full rank of df_{m_0} . Let us choose such a neighborhood.

If the form $v \mapsto \|d(f \circ \phi)_{u_0}(v)\|^2 - \langle z_0 - x_0, d^2(f \circ \phi)_{u_0}(v, v) \rangle$ is positive definite for a given chart $\phi : (V, u_0) \rightarrow (M, m_0)$ of M around m_0 (we can assume that $\phi(V) \subset U$), it is so for any such chart (as $\text{Im}(df_{m_0}) \perp z_0 - x_0$). In that case, consider the function $g : V \rightarrow \mathbb{R}, u \mapsto \|f \circ \phi(u) - z_0\|^2$. Its first differential at u_0 is 0, as $\text{Im}(df_{m_0}) \perp z_0 - x_0$, and its second differential is

$$(v_1, v_2) \mapsto 2(\langle d(f \circ \phi)_{u_0}(v_1), d(f \circ \phi)_{u_0}(v_2) \rangle - \langle z_0 - x_0, d^2(f \circ \phi)_{u_0}(v_1, v_2) \rangle).$$

By hypothesis, the associated quadratic form is positive definite, hence u_0 is a strict local minimum of g , and x_0 is a strict local minimum of the distance function to z_0 restricted to $f(U)$. By making the neighborhood U of m_0 in M smaller, we can thus assume that x_0 is the unique projection of z_0 on $f(U)$.

Furthermore, we have shown that the sphere $S(z_0, d_{\overline{f(U)}}(z_0))$ does not osculate $\overline{f(U)}$ at x_0 (our definition of osculation naturally extends to this case), using characterization (ii) from the Osculation Lemma 3.1. If $\overline{f(U)}$ were a submanifold (without boundary), we could now use the Lipschitz continuity of the projection from Lemma 4.1 to conclude that there exists $\delta > 0$ such that each $z \in B(z_0, \delta)$ admits a unique projection $p(z)$ on $\overline{f(U)}$ that belongs to $f(U)$ (possibly by making U yet smaller). This would also mean that $B(z_0, \delta)$ would be included in the interior of the complement of the medial axis of $\overline{f(U)}$, hence the projection map $p : B(z_0, \delta) \rightarrow f(U)$ would be C^1 with its differential given by Proposition 4.2. One can see that the argument still works in the same way despite the boundary. Let δ (and U) satisfy the conditions above.

The only part of the proof left is to adapt the formula for the differential of the projection from Proposition 4.2 to express dp_{z_0} in the coordinates of $f \circ \phi|_V$. Let us show that Λ is indeed the shape operator at x_0 in normal direction $z_0 - x_0$ of $f(U)$, as claimed in the remark above. In [DCF92, p. 126], a bilinear map $B : \text{Im}(df_{m_0}) \times \text{Im}(df_{m_0}) \rightarrow \text{Im}(df_{m_0})^\perp$ is defined as follows: for $w_1, w_2 \in \text{Im}(df_{m_0}) = T_{x_0}f(U)$, let $W_1, W_2 : f(U) \supset \tilde{U} \rightarrow Tf(U)$ be vector fields on the submanifold $f(U)$ defined around x_0 such that $W_1(x_0) = w_1$ and $W_2(x_0) = w_2$, and let \overline{W}_1 and \overline{W}_2 be local extensions of W_1 and W_2 to \mathbb{R}^D . Then

$$B(w_1, w_2) = \pi_{T_{x_0}f(U)^\perp}(d(\overline{W}_1)_{x_0}(\overline{W}_2(x_0))),$$

where $\pi_{T_{x_0}f(U)^\perp}$ is the orthogonal projection on $T_{x_0}f(U)^\perp$. The same reference shows that $B(w_1, w_2)$ does not depend on the choice of $W_1, W_2, \overline{W}_1, \overline{W}_2$ and is symmetric.

Let us express B in terms of the embedding $f|_U$. We can assume without loss of generality that $T_{x_0}f(U) = \mathbb{R}^m \times \{0\} \subset \mathbb{R}^D$ and $x_0 = 0$. Then the projection $\pi : \mathbb{R}^m \times \mathbb{R}^{D-m} \rightarrow \mathbb{R}^m$ restricted to $f(U)$ is a diffeomorphism around 0, hence $\pi \circ f : U \rightarrow \mathbb{R}^m$ is locally the inverse of a chart $\psi : (\tilde{V}, 0) \rightarrow (M, m_0)$ of M around m_0 . Around $0 \in \mathbb{R}^D$, the submanifold $f(U)$ is parameterized as the graph of the map $f \circ \psi : \mathbb{R}^m \rightarrow \mathbb{R}^D$.

Now let $w_1, w_2 \in T_{x_0}f(U)$. We define a vector field W_i for $i = 1, 2$ as follows:

$$\begin{aligned} \overline{W}_i : \mathbb{R}^D = \mathbb{R}^m \times \mathbb{R}^{D-m} &\longrightarrow \mathbb{R}^D \\ (x, y) &\longmapsto d(f \circ \psi)_x((d(f \circ \psi)_0)^{-1}(w_i)). \end{aligned}$$

Then the vector fields \overline{W}_i are such that their restrictions to $f(U)$ take values in the tangent bundle of $f(U)$, and such that $\overline{W}_i(0) = w_i$. By definition,

$$\begin{aligned} B(w_1, w_2) &= \pi_{T_0 f(U)^\perp}(d(\overline{W}_1)_0(\overline{W}_2(0))) = \pi_{T_0 f(U)^\perp}(d^2(f \circ \psi)_0((d(f \circ \psi)_0)^{-1}(w_1), w_2)) \\ &= \pi_{T_0 f(U)^\perp}(d^2(f \circ \psi)_0((d(f \circ \psi)_0)^{-1}(w_1), (d(f \circ \psi)_0)^{-1}(w_2))) \end{aligned}$$

where the last equality is true because $d(f \circ \psi)_0$ is the inclusion $\mathbb{R}^m \rightarrow \mathbb{R}^m \subset \mathbb{R}^D$. In Definition 2.2 of [DCF92, Chapter 6], the shape operator at $x_0 = 0$ in normal direction $z_0 - x_0$ of the submanifold $f(U)$ is defined as the linear operator $W_{0, z_0 - x_0} : T_0 f(U) \rightarrow T_0 f(U)$ such that

$$\langle W_{0, z_0 - x_0}(w_1), w_2 \rangle = \langle z_0 - x_0, B(w_1, w_2) \rangle$$

for any $w_1, w_2 \in T_0 f(U)$, i.e. such that

$$\langle W_{0, z_0 - x_0}(w_1), w_2 \rangle = \langle z_0 - x_0, d^2(f \circ \psi)_0((d(f \circ \psi)_0)^{-1}(w_1), (d(f \circ \psi)_0)^{-1}(w_2)) \rangle.$$

This is precisely the definition that we gave of the operator Λ , hence $\Lambda = W_{0, z_0 - x_0}$ is the shape operator in normal direction $z_0 - x_0$ at x_0 . The fact that Λ does not depend on the choice of the chart ψ can then be deduced either from the fact that the shape operator does not depend on the choice of the vector fields \overline{W}_i (as shown in [DCF92]), or by direct computation using the expression $\langle z_0 - x_0, d^2(f \circ \psi)_0((d(f \circ \psi)_0)^{-1}(w_1), (d(f \circ \psi)_0)^{-1}(w_2)) \rangle$.

Proposition 4.2 then states the invertibility of $Q = \text{id} - \Lambda$ and the fact that $dp_{z_0} = Q^{-1} \circ \pi_{\text{Im}(df_{m_0})}$, which concludes the proof. \square

4.3 C^2 stability of the projection

In this subsection, we show that a non-osculatory projection on $i(M)$ behaves nicely locally, and that this is stable with respect to small C^2 perturbations of the embedding i . In particular, we finish the proof of Lemma 3.1. We first prove an almost trivial lemma.

Lemma 4.5. *Let $V \subset \mathbb{R}^m$ be an open set, let Λ be a topological space and let $F : V \times \Lambda \rightarrow \mathbb{R}$ be a continuous function. For $\lambda \in \Lambda$, write $F_\lambda : v \mapsto F(v, \lambda)$ and assume that the second differential*

$$V \times \Lambda \rightarrow \text{Bil}(\mathbb{R}^m, \mathbb{R}^m, \mathbb{R}), (v, \lambda) \mapsto d^2(F_\lambda)_v$$

of F with respect to $v \in V$ exists and is continuous with respect to $(v, \lambda) \in V \times \Lambda$. Assume further that the map $\Lambda \rightarrow C^0(V, \mathbb{R}), \lambda \mapsto F_\lambda$ is continuous with respect to the $\|\cdot\|_\infty$ norm, that $d^2(F_{\lambda_0})_{v_0}$ is positive definite for some $v_0 \in V$, $\lambda_0 \in \Lambda$ and that v_0 is a global minimum for F_{λ_0} . Then there exist open neighborhoods $U_{\mathbb{R}^m} \subset V$ of v_0 and $U_\Lambda \subset \Lambda$ of λ_0 such that $\overline{U_{\mathbb{R}^m}} \subset V$, that for any $\lambda \in U_\Lambda$ the function F_λ admits exactly one minimum v_λ on $\overline{U_{\mathbb{R}^m}}$ with $v_\lambda \in U_{\mathbb{R}^m}$ and that $d^2(F_\lambda)_v$ is positive definite for any $v \in U_{\mathbb{R}^m}, \lambda \in U_\Lambda$.

Proof. By assumption the second differential $d(v, \lambda) \mapsto d^2(F_\lambda)_v$ is continuous in $(v, \lambda) \in V \times \Lambda$, and being definite positive is an open property, hence there exists an open neighborhood of (v_0, λ_0) on which $d^2(F_\lambda)_v$ is positive definite. By restricting it, we can assume that it is of the shape $U_{\mathbb{R}^m} \times U_\Lambda \subset V \times \Lambda$ and that $U_{\mathbb{R}^m}$ is a convex subset of \mathbb{R}^m . For any fixed $\lambda \in U_\Lambda$, the second differential $d^2(F_\lambda)_v$ in v of the function $F_\lambda : U_{\mathbb{R}^m} \rightarrow \mathbb{R}^m$ is positive definite, hence F_λ is strictly convex and admits at most one minimum v_λ on the convex set $U_{\mathbb{R}^m}$.

As v_0 is a minimum for F_{λ_0} and $d^2(F_{\lambda_0})_{v_0}$ is positive definite, there exist open neighborhoods U_1, U_2 of v_0 and $\beta > 0$ such that $U_1 \subset U_2 \subset \overline{U_2} \subset U_{\mathbb{R}^m}$, that U_2 is convex and that $F_{\lambda_0}(v_0) <$

$\min\{F_{\lambda_0}(x) : x \in \overline{U_2} \setminus U_1\} - \beta$. By choosing U_Λ small enough that $\|F_\lambda - F_{\lambda_0}\|_\infty < \beta/4$ for any $\lambda \in U_\Lambda$, we have that $F_\lambda(v_0) < \min\{F_\lambda(x) : x \in \overline{U_2} \setminus U_1\} - \beta/2$ for such a $\lambda \in U_\Lambda$. Thus F_λ admits a minimum v_λ on $\overline{U_2}$ and v_λ must belong to $U_1 \subset U_2$. As U_2 is convex, the argument used above applies again and this minimum must be unique. By redefining $U_{\mathbb{R}^m}$ as its subset U_2 , we conclude that F_λ does admit exactly one minimum v_λ on $\overline{U_{\mathbb{R}^m}}$ with $v_\lambda \in U_{\mathbb{R}^m}$. \square

Now we specialize the lemma above to the special case of interest to us.

Proposition 4.6 (*C^2 stability of the projection*). *Let $i_0 \in \text{Emb}^2(M, \mathbb{R}^D)$ and $z_0 \in \mathbb{R}^D$. Let $x_0 = i_0(m_0)$ be a projection of z_0 on $i_0(M)$ such that $S(z_0, d_{i_0(M)}(z_0))$ is non-osculating $i_0(M)$ at x_0 . Then there exist $\alpha > 0$ and neighborhoods $m_0 \in U_M \subset M$, $z_0 \in U_{\mathbb{R}^D} \subset \mathbb{R}^D$ and $i_0 \in U_{\text{Emb}} \subset \text{Emb}^2(M, \mathbb{R}^D)$ such that for any $i \in U_{\text{Emb}}$, any $z \in U_{\mathbb{R}^D}$ has a unique projection $p_i(z)$ on $\overline{i(U_M)}$, that $p_i(z)$ belongs to $i(U_M)$, and that the sphere $S(z, d_{i(U_M)}(z))$ does not osculate $i(U_M)$ at $p_i(z)$ with α a degree of non-osculation.*

In particular, there exist $\delta, \delta' > 0$ such that for any $i \in U_{\text{Emb}}$ and $z \in U_{\mathbb{R}^D}$, if $x = p_i(z)$ and $M = i(M)$, then the function $p_i : B(z, \delta') \rightarrow M \cap \overline{B}(x, \delta) = M_0$ is the orthogonal projection on M_0 , which is C^1 and α^{-1} -Lipschitz continuous.

Proof. Let $\phi : V \rightarrow M$ be a chart of M with $\phi(u_0) = m_0$, and consider the function

$$F : V \times \mathbb{R}^D \times \text{Emb}^2(M, \mathbb{R}^D), (u, z, i) \mapsto \|i \circ \phi(u) - z\|^2 - \alpha \|i \circ \phi(u) - i(m_0)\|^2.$$

Its second differential with respect to $u \in V$ is the map

$$V \times \mathbb{R}^D \times \text{Emb}^2(M, \mathbb{R}^D) \rightarrow \text{Bil}(\mathbb{R}^m, \mathbb{R}^m, \mathbb{R})$$

that sends (u, z, i) to the bilinear map

$$d^2(F_{(z,i)})_u : (v_1, v_2) \mapsto 2(\langle d(i \circ \phi)_u(v_1), d(i \circ \phi)_u(v_2) \rangle + \langle i \circ \phi(u) - z, d^2(i \circ \phi)_u(v_1, v_2) \rangle).$$

Both F and $d^2(F_{(z,i)})_u$ are continuous with respect to (u, z, i) (with $\text{Emb}^2(M, \mathbb{R}^D)$ equipped with the Whitney C^2 -topology). Using Characterization (ii) from the Osculation Lemma 3.1, we see that if $i \circ \phi(u) \in i(M)$ is a projection of z on $i(M)$, the sphere $S(z, d_{i(M)}(z))$ not osculating $i(M)$ at $i \circ \phi(u)$ is equivalent to $d^2(F_{(z,i)})_u$ being positive definite.

Hence we can apply Lemma 4.5 to F and (u_0, z_0, i_0) (with $\Lambda = \mathbb{R}^D \times \text{Emb}^2(M, \mathbb{R}^D)$) to get that there exist neighborhoods $u_0 \in U_V \subset V$, $z_0 \in U_{\mathbb{R}^D} \subset \mathbb{R}^D$ and $i_0 \in U_{\text{Emb}} \subset \text{Emb}^2(M, \mathbb{R}^D)$ such that for any $(z, i) \in U_{\mathbb{R}^D} \times U_{\text{Emb}}$, the function $F_{(z,i)} : u \rightarrow \|i \circ \phi(u) - z\|^2$ admits exactly one minimum on $\overline{U_V}$ and that this minimum belongs to U_V . Define $U_M := \phi(U_V)$. For a given $i \in U_{\text{Emb}}$, this property is equivalent to the projection $U_{\mathbb{R}^D}$ on $\overline{i(U_M)}$ being uniquely defined and its image being contained in $i(U_M)$. Let us call p_i this projection. Lemma 4.5 also guarantees that if $i \in U_{\text{Emb}}$, $z \in U_{\mathbb{R}^D}$ and if we write $x := p_i(z)$ and $u := (i \circ \phi)^{-1}(x) \in U_V$, then

$$d^2(F_{(z,i)})_u : (v_1, v_2) \mapsto 2(\langle d(i \circ \phi)_u(v_1), d(i \circ \phi)_u(v_2) \rangle + \langle i \circ \phi(u) - z, d^2(i \circ \phi)_u(v_1, v_2) \rangle)$$

is positive definite. By possibly further restricting U_{Emb} , $U_{\mathbb{R}^D}$ and U_V , we can assume that there exists $c > 0$ such that the smallest eigenvalue of $d^2(F_{(z,i)})_u$ is greater than c for any such i, z, u and that the operator norm of $d(i \circ \phi)$ is bounded on U_{Emb} . Hence there exists $\alpha > 0$ such that the form $v \mapsto (1 - \alpha)\|d(i \circ \phi)_u(v)\|^2 - \langle i \circ \phi(u) - z, d^2(i \circ \phi)_u(v, v) \rangle$ is definite positive for any such i, z, u . Thus Characterization (ii) from the Osculation Lemma 3.1 guarantees that the

sphere $S(z, d_{i(U_M)}(z))$ does not osculate $i(U_M)$ at $p_i(z)$, with α a degree of non-osculation. This proves the first part of the proposition.

For each $i \in U_{\text{Emb}}$, the map p_i is the orthogonal projection from $U_{\mathbb{R}^D}$ to $\overline{i(U_M)}$. The non-osculation of each sphere $S(z, d_{i(U_M)}(z))$ at $p_i(z)$ for all $z \in U_{\mathbb{R}^D}$ entails that p_i is α^{-1} -Lipschitz continuous, as the proof of Lemma 4.1 shows. Besides, p_i is C^1 as the orthogonal projection on the manifold $i(U_M)$. Consider $\delta' > 0$ and $U \subset U_{\mathbb{R}^D}$ such that $B(z, \delta') \subset U_{\mathbb{R}^D}$ for all $z \in U$. Let $\delta = \delta'/\alpha$. Then, for all $z \in U$, the function p_i is defined on $B(z, \delta')$ and takes its values in $B(p_i(z), \delta) \cap i(U_M) \subset \overline{B}(p_i(z), \delta) \cap i(M)$. We replace $U_{\mathbb{R}^D}$ by U to conclude. \square

As a corollary, we get the final part of the proof of the Osculation Characterization Lemma.

Corollary 4.7. *Characterization (iii) from Lemma 3.1 implies Characterization (iv).*

Proof. Let z_0, x_0 and the embedding i from the statement of Lemma 3.1 and assume that they satisfy Characterization (iii). We already know that Characterization (iii) is equivalent to Characterization (ii), which is the one used in the first part of Proposition 4.6 above (hence there is no circularity in our argument). Thus the proposition can be applied to z_0, x_0 and i to get the existence of a neighborhood $U_{\mathbb{R}^D}$ of z_0 and a neighborhood M_0 of x_0 in $i(M)$ such that any $z \in U_{\mathbb{R}^D}$ and x its unique local projection on M_0 satisfy (iii). This, in turn, proves that z_0, x_0 and i verify Condition (iv). \square

4.4 Multiple local projections

This subsection is dedicated to an easy but handy lemma that helps us manage multiple local projections on a neighborhood of a critical point.

Assume that the C^2 submanifold M satisfies (P1), (P2) and (P3), and let $z_0 \in Z(M)$ be a critical point with projections $\pi_M(z_0) = \{x_1, \dots, x_s\}$, for some $s \in \{2, \dots, D+1\}$. As condition (P3) is satisfied, the sphere $S(z_0, d_M(z_0))$ does not osculate M at any of the x_i . Using Lemma 4.1, we can pick radii δ and δ' so that there exist local projections $p_j : B(z_0, \delta') \rightarrow M_j = \overline{B}(x_j, \delta) \cap M$ for all $j \in [s]$. Moreover, by making δ' smaller, we can ensure using the Lipschitz continuity of the projections that the image of p_i is included in the relative interior of M_i . Hence, these functions are equal to the projection on a C^2 manifold, and they are in particular of regularity C^1 , with known expressions for their gradients, as stated in Proposition 4.2. By (P2), the number of critical points is finite, so that the constants δ and δ' can be chosen independently of $z_0 \in Z(M)$. We summarize these properties in the next lemma.

Lemma 4.8. *Let $M \subset \mathbb{R}^D$ be a C^2 compact submanifold satisfying (P1), (P2) and (P3). There exist $\delta', \delta > 0$ such that for any $z_0 \in Z(M)$ with $\pi_M(z_0) = \{x_1, \dots, x_s\}$, there are C^1 functions $p_i : B(z_0, \delta') \rightarrow M_i = \overline{B}(x_i, \delta) \cap M$ for $i = 1, \dots, s$ such that for any $z \in B(z_0, \delta')$, we have $\pi_{M_i}(z) = \{p_i(z)\} \subset B(x_i, \delta) \cap M$ and $\pi_M(z) \subset \{p_1(z), \dots, p_s(z)\}$. Furthermore, if $p_i(z) = x$ and $\eta = z - x$, then $(dp_i)_z = (\text{id} - W_{x,\eta})^{-1} \circ \pi_x$.*

Proof. The only point that remains to be proven is the inclusion $\pi_M(z) \subset \{p_1(z), \dots, p_s(z)\}$. Let $z \in B(z_0, \delta')$ and $y \in \pi_M(z)$. Then,

$$\|y - z_0\| \leq \|y - z\| + \|z - z_0\| \leq \|x_0 - z\| + \|z - z_0\| \leq \|x_0 - z_0\| + 2\|z - z_0\|.$$

Hence, $y \in B(z_0, d_M(z_0) + 2\delta')$. By Lemma 2.1, if δ' is small enough, this implies that y is in one of the sets M_1, \dots, M_s , implying that $y \in \pi_{M_i}(z)$ for some $i = 1, \dots, s$. \square

5 Sampling theory for generic manifolds

In this section, we prove the Stability Theorem for Subsets (Theorem 1.6), which can be seen as a stronger analogue to the results for compact sets from [CCSL06] in the special case of well-behaved manifolds sampled without noise. We restate the theorem for the reader's convenience.

Theorem 1.6 (Stability Theorem for Subsets). *Let $M \subset \mathbb{R}^D$ be a compact C^2 submanifold. There exist positive constants C_1, C_2, C_3, C_4, C_5 and ε_0 such that the following holds. Let $A \subset M$ be a set with $d_H(A, M) \leq \varepsilon \leq \varepsilon_0$.*

1. *For every critical point $z \in Z(A)$, exactly one of these two possibilities is true: either z is very close to M , that is $d_M(z) \leq C_1\varepsilon^2$, or it is close to a μ -critical point of M , that is $d_{Z_\mu(M)}(z) \leq C_2\varepsilon$ for $\mu = C_3\varepsilon$.*
2. *If M satisfies (P4), then, in the second case, the critical point $z \in Z(A)$ is close to $Z(M)$, that is $d_{Z(M)}(z) \leq C_4\varepsilon$.*
3. *If M additionally satisfies conditions (P1), (P2) and (P3) then, in the second case, the set of projections $\pi_A(z)$ of the critical point $z \in Z(A)$ is included in a neighborhood of size $C_5\varepsilon$ of $\pi_M(z')$ for some $z' \in Z(M)$ such that $d(z, z') \leq C_4\varepsilon$.*

Proof of Theorem 1.6. Let $A \subset M$ be a subset with $d_H(A, M) \leq \varepsilon$, and let $\varepsilon < \tau(M)/8$.

1. According to the critical values separation theorem [CCSL06, Theorem 4.4], the distance function d_A has no critical values in the interval $(4\varepsilon, \tau(M) - 3\varepsilon)$. This proves that all critical points in $Z(A)$ are either at distance less than 4ε from A , or at distance at least $\tau(M) - 3\varepsilon$ from A . Let us focus on the first case. Let $z \in Z(A)$ be a point with $d_A(z) \leq 4\varepsilon$. By definition of critical points, z is the center of the smallest enclosing ball of $\pi_A(z) \subset A$, and this ball is at most of radius 4ε . According to [ALS13, Lemma 12], this implies that

$$d_M(z) \leq \tau(M) \left(1 - \sqrt{1 - \frac{16\varepsilon^2}{\tau(M)^2}} \right) \leq \frac{8\varepsilon^2}{\tau(M)} \left(1 + \frac{16\varepsilon^2}{\tau(M)^2} \right) \leq \frac{10\varepsilon^2}{\tau(M)}, \quad (12)$$

where we use the condition $\varepsilon < \tau(M)/8$ and the inequality $\sqrt{1 - u^2} \geq 1 - u/2 - u^2/2$ for $u \in [0, 1]$. This yields the first alternative in Theorem 1.6.

We now consider the second case, where $d_A(z) > \tau(M) - 3\varepsilon$. In that case, the inequality $\varepsilon \leq \tau(M)/8$ implies that $d_M(z) \geq d_A(z) - \varepsilon \geq \tau(M) - 4\varepsilon > \tau(M)/2$. We require a lemma, which is a variant of Lemma 3.3 from [CCSL06] suited to manifolds.

Lemma 5.1. *Let $M \subset \mathbb{R}^D$ be a compact C^2 submanifold. Let $A \subset M$ be a subset with $d_H(A, M) \leq \varepsilon$, and let $z \in Z(A)$ with $0 < r < d_M(z) < R$. Then, there exists a μ -critical point $z' \in Z_\mu(M)$, with $\|z - z'\| \leq \varepsilon$ and*

$$\mu \leq \frac{\varepsilon}{r} \left(1 + \frac{R}{2\tau(M)} \right). \quad (13)$$

Proof. We adapt the proof of Lemma 3.3 in [CCSL06]. We follow the gradient flow of ∇d_M starting at $z \in Z(A)$ along a trajectory parameterized by arc length. If the gradient flow

reaches a critical point of M before time ε , the result holds. Otherwise, let w be the point reached by the gradient flow after time ε . We have

$$d_M(w) - d_M(z) = \int_0^\varepsilon \|\nabla d_M(C(t))\| dt, \quad (14)$$

where $t \mapsto C(t)$ is the flow curve. Hence, there exists some point z' along this curve with

$$\|\nabla d_M(z')\| \leq \frac{d_M(w) - d_M(z)}{\varepsilon}.$$

Furthermore, it holds that $\|z - w\| \leq \varepsilon$. Let us show that z' is a μ -critical point of M for $\mu = \frac{\varepsilon}{r} \left(1 + \frac{R}{2\tau(M)}\right)$. As $z \in Z(A)$, we have according to [CCSL06, Lemma 3.2]

$$d_M(w)^2 \leq d_A(w)^2 \leq d_A(z)^2 + \|z - w\|^2 \leq d_A(z)^2 + \varepsilon^2. \quad (15)$$

Let $x \in \pi_M(z)$ and $y \in \pi_A(x)$, whence $\|x - y\| \leq \varepsilon$. We have

$$\begin{aligned} d_A(z)^2 &\leq \|z - y\|^2 = \|z - x\|^2 + \|x - y\|^2 + 2\langle z - x, x - y \rangle \\ &\leq d_M(z)^2 + \varepsilon^2 + 2\langle z - x, \pi_x^\perp(x - y) \rangle, \end{aligned} \quad (16)$$

where we use that $z - x$ is orthogonal to $T_x M$. According to [Fed59], $\|\pi_x^\perp(x - y)\| \leq \frac{\|x - y\|^2}{2\tau(M)}$. Hence, (15) and (16) yield that

$$\begin{aligned} \|\nabla d_M(z')\| &\leq \frac{d_M(w) - d_M(z)}{\varepsilon} = \frac{d_M(w)^2 - d_M(z)^2}{\varepsilon(d_M(w) + d_M(z))} \\ &\leq \frac{2\varepsilon^2 + \varepsilon^2 \frac{R}{\tau(M)}}{2\varepsilon r} = \frac{\varepsilon}{r} \left(1 + \frac{R}{2\tau(M)}\right). \end{aligned}$$

This concludes the proof. \square

Remark that we always have the trivial bound $d_M(z) \leq \text{diam}(M)$ for $z \in Z(A)$. Hence, the lemma implies that for $z \in Z(A)$ with $d_M(z) > \tau(M)/2$, there exists a μ -critical point $z' \in Z_\mu(M)$ with $\|z - z'\| \leq \varepsilon$ and $\mu = C_2\varepsilon$ for some constant C_2 depending on M , which completes the proof of the first point.

2. This is a straightforward consequence of the first item and (P4) - we only need to make $\varepsilon_0 > 0$ small enough that $C_2\varepsilon_0 < \mu_0$ (where μ_0 is from the statement of (P4)).

3. Let δ and δ' be radii such that the statement of Lemma 3.1.(iv) holds uniformly for all the critical points $z_0 \in Z(M)$ and $x_0 \in \pi_M(z_0)$ (these are in finite number, as M satisfies (P1) and (P2)). Let $z \in Z(A)$ be a critical point close to some critical point $z_0 \in Z(M)$, with $\|z - z_0\| \leq C_4\varepsilon$ (where C_4 is the constant from Theorem 1.6.2). Let $x \in \pi_A(z)$. Note that $\|x - z_0\| \leq \|x - z\| + \|z - z_0\| \leq \|x_0 - z\| + \|z - z_0\| \leq d_M(z_0) + 2\|z - z_0\|$. Hence, according to Lemma 2.1, for $\|z - z_0\|$ small enough, the point x is at distance less than δ from some vertex $x_0 \in \pi_M(z_0)$ (and this uniformly for each critical point z_0). According to Lemma 4.1, there exists a unique projection x_1 of z on $M_0 = M \cap \overline{B}(x_0, \delta)$, and $\|x_1 - x_0\| \leq \|z - z_0\|/\alpha$. In particular, for $\|z - z_0\|$ small enough, $\|x_1 - x_0\| < \delta$. Hence, x_1 belongs to the relative interior of M_0 and the vector $z - x_1$ is orthogonal to $T_{x_1} M$.

As $d_H(\mathbf{A}, \mathbf{M}) \leq \varepsilon$, there exists $y \in \mathbf{A}$ with $\|x_1 - y\| \leq \varepsilon$. We write thanks to [Fed59, Theorem 4.18]

$$\begin{aligned} \|z - x\|^2 &= d_{\mathbf{A}}(z)^2 \leq \|z - y\|^2 = \|z - x_1\|^2 + \|x_1 - y\|^2 + 2\langle z - x_1, \pi_{x_1}^\perp(x_1 - y) \rangle \\ &\leq d_{\mathbf{M}_0}(z)^2 + \varepsilon^2 \left(1 + \frac{d_{\mathbf{M}_0}(z)}{\tau(\mathbf{M})} \right). \end{aligned}$$

Note that $1 + d_{\mathbf{M}_0}(z)/\tau(\mathbf{M})$ can be roughly upper bounded by

$$1 + \frac{\text{diam}(\mathbf{M})}{\tau(\mathbf{M})} =: C'.$$

Hence, $x \in B(z, \sqrt{d_{\mathbf{M}_0}(z)^2 + C'\varepsilon^2})$. As $x \in \mathbf{M}_0$, this implies by (P3) that

$$\|x_1 - z\|^2 + C'\varepsilon^2 \geq \|x - z\|^2 \geq \|x_1 - z\|^2 + \alpha\|x - x_1\|^2,$$

so that $\|x - x_1\| \leq \sqrt{C'/\alpha\varepsilon}$. As we have $\|x_1 - x_0\| \leq \frac{1}{\alpha}\|z - z_0\|$, we obtain

$$\|x - x_0\| \leq \|x - x_1\| + \|x_1 - x_0\| \leq \sqrt{C'/\alpha\varepsilon} + \frac{1}{\alpha}C_4\varepsilon. \quad \square$$

Remark 5.2. *The reader will have noticed that the proof does not make use of all of Condition (P1), but rather only of the property that each critical point has only finitely many projections.*

Finally, we prove Corollary 1.7.

Proof of Corollary 1.7. Consider the constants $C_2, C_5, \varepsilon_0 > 0$ from Theorem 1.6, and let $\varepsilon_1 = \min(\tau(\mathbf{M})/8, \varepsilon_0/C_5)$. Let \mathbf{A} be a (δ, ε) -sampling of \mathbf{M} for some $0 < \delta \leq \varepsilon \leq \varepsilon_1$. Let $z \in Z(\mathbf{M})$ with $\pi_{\mathbf{M}}(z) = \{x_1, \dots, x_s\}$. According to Theorem 1.6, if $z' \in Z(\mathbf{A})$ is at distance less than $C_2\varepsilon$ from z , then $\pi_{\mathbf{A}}(z') \subset \bigcup_{i=1}^s B(x_i, C_5\varepsilon)$. Hence, the map

$$\begin{aligned} (\mathbf{A} \cap B(x_1, C_5\varepsilon)) \times \dots \times (\mathbf{A} \cap B(x_s, C_5\varepsilon)) &\longrightarrow \mathbb{R}^D \\ (y_1, \dots, y_s) &\longmapsto m(\{y_1, \dots, y_s\}) \end{aligned}$$

contains all the points of $Z(\mathbf{A})$ at distance less than $C_2\varepsilon$ from z in its image. As the cardinality of $\mathbf{A} \cap B(x_i, C_5\varepsilon)$ is at most $N(\delta, C_5\varepsilon)$, the total number of such points is $2^{N(\delta, C_5\varepsilon)s}$.

Let us now prove the second bound. Let $x \in \mathbf{M}$ and let $\{y_1, \dots, y_K\}$ be a maximal $(\delta, +\infty)$ -sampling of $B(x, \varepsilon) \cap \mathbf{M}$. Then, the balls $B(y_i, \delta/2)$ are pairwise disjoint for $i = 1, \dots, K$. According to [AL18, Proposition 8.7], as $\delta/2 \leq 2\varepsilon \leq 2\varepsilon_1 \leq \tau/4$, it holds that

$$\text{Vol}(B(y_i, \delta/2) \cap \mathbf{M}) \geq c_m \delta^m \quad \text{and} \quad \text{Vol}(B(x, 2\varepsilon) \cap \mathbf{M}) \leq C_m \varepsilon^m$$

for two positive constants c_m, C_m . Hence, as $\bigsqcup_{i=1}^K B(y_i, \delta/2) \cap \mathbf{M} \subset B(x, 2\varepsilon) \cap \mathbf{M}$, it holds that

$$c_m K \delta^m \leq C_m \varepsilon^m$$

that is $N(\delta, \varepsilon) \leq \frac{C_m}{c_m} \left(\frac{\varepsilon}{\delta}\right)^m$. □

6 The μ -critical points of M and the Big Simplex Property

Let as above M be a C^k compact manifold of dimension $m \geq 1$ for some $k \geq 2$, and let $i \in \text{Emb}^2(M, \mathbb{R}^D)$ and $M = i(M)$. In this section, we concern ourselves with the position of the μ -critical points of M . In particular, whenever M satisfies (P1), (P2) and (P3), we define a purely local condition on the critical points of M and their projections, called the Big Simplex Property, that turns out to be equivalent to the apparently more global condition (P4), yet is easier to manipulate in terms of the embedding i . This will help us show that (P4) is verified for a dense subset of the space $\text{Emb}^2(M, \mathbb{R}^D)$ of embeddings later in Section 8, using the language of jets.

Let us start with some preliminary observations. Consider the compact C^2 submanifold $M \subset \mathbb{R}^D$ and let us define the function

$$\eta_M : [0, 1] \longrightarrow [0, +\infty]$$

$$\mu \longmapsto \sup\{r : \exists x \in \mathbb{R}^D \text{ s.t. } \|\nabla d_M(x)\| \leq \mu \text{ and } d(Z(M), x) = r\},$$

the supremum of the distances to M at which a μ -critical point can be found. If M is of diameter $\text{diam}(M) > 0$ and $x \in \mathbb{R}^D$ is at distance $L + \text{diam}(M)$ from $Z(M) \subset \text{Conv}(M)$, then it is at distance at least L from M and the norm of the generalized gradient $\nabla d_M(x)$ is lower-bounded by $\frac{\sqrt{L^2 - \frac{1}{2}\text{diam}(M)^2}}{L}$. Hence, η_M is finite for $\mu \in [0, 1)$, and for such a μ the supremum of the definition is a maximum due to the lower semicontinuity of the norm of the gradient. On the other hand, we always have $\eta_M(1) = +\infty$. We are more interested in $\eta_M(\mu)$ for small values of μ , and we see that $\lim_{\mu \searrow 0} \eta_M(\mu) = 0$. Indeed, assume that it is not the case: there exists $\varepsilon > 0$ and a sequence $(x_n)_n \subset \mathbb{R}^D$ of points such that $\lim_{n \rightarrow \infty} \|\nabla d_M(x_n)\| = 0$ and such that $d(Z(M), x_n) \geq \varepsilon$ for all n . But thanks to the same argument as above, the sequence $(x_n)_n$ is bounded, hence a subsequence that converges to some point $x \notin Z(M)$ can be extracted, which yields a contradiction with the lower semicontinuity of the norm of the generalized gradient.

Property (P4) states that there exist $C = C(M)$ and $\mu_0 \in (0, 1)$ such that for any $0 < \mu \in [0, \mu_0)$ we have $\eta_M(\mu) \leq C\mu$, and will be our main object of interest in this section. Note that Property (P4) is equivalent to the apparently stronger Property (P4')

(P4') For any $\mu_0 \in (0, 1)$, there exists a constant $C > 0$ such that for every $\mu \in [0, \mu_0)$, the set $Z_\mu(M)$ is included in a tubular neighborhood of size $C\mu$ of $Z(M)$, that is every point of $Z_\mu(M)$ is at distance less than $C\mu$ from $Z(M)$.

Indeed, let $\mu_0 > 0$ and $C > 0$ be given to us by (P4), and let $\mu'_0 > 0$ be as in the statement of (P4'). We have shown above that $\eta_M(\mu'_0) < \infty$. Let z be a μ -critical point of M ; if $\mu < \mu_0$, then $d(z, Z(M)) < C\mu$ using (P4), and if $\mu \in [\mu_0, \mu'_0)$, then $d(z, Z(M)) \leq \eta_M(\mu'_0) = \mu_0 \frac{\eta_M(\mu'_0)}{\mu_0} \leq \mu \frac{\eta_M(\mu'_0)}{\mu_0}$, hence in all cases $d(z, Z(M)) \leq \mu \left(C + \frac{\eta_M(\mu'_0)}{\mu_0} \right)$ and M satisfies (P4').

6.1 Position of the μ -critical points

As a first step to reformulating Condition (P4), we need to understand the behavior of the medial axis $\text{Med}(M)$ around $z_0 \in Z(M)$. Assume that M satisfies properties (P1), (P2) and (P3), and let $z_0 \in Z(M)$ with projections $\{x_1, \dots, x_s\} = \pi_M(z_0)$, for some $s \in \{2, \dots, D+1\}$. Let

$$E = \text{Vec}(\{x_{j_1} - x_{j_2} : j_1, j_2 \in [s]\}).$$

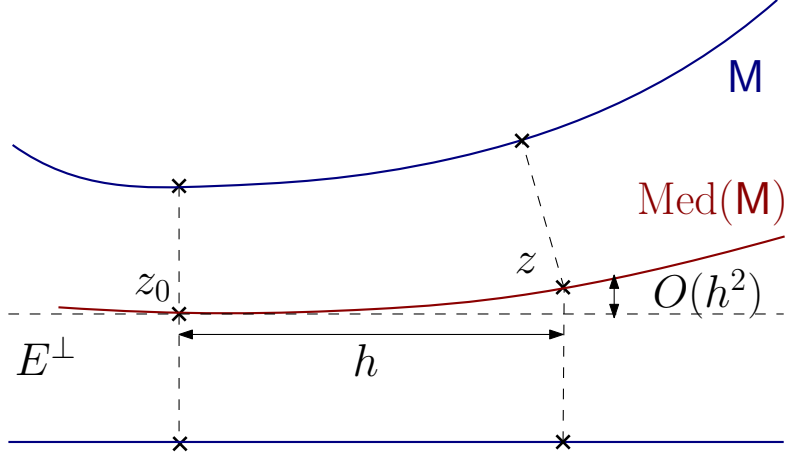


Figure 4: Using the notations of Subsection 6.1, for $\mu_0 > 0$ small enough, the tangent space of $Z_{\mu_0}(\mathbf{M})$ at the critical point $z_0 \in Z(\mathbf{M})$ is equal to E^\perp , and the points of $Z_{\mu_0}(\mathbf{M})$ that are at a small distance h from z_0 must be at distance $O(h^2)$ from the affine space $z_0 + E^\perp$.

As \mathbf{M} satisfies (P1), the dimension of E is $s - 1$. Let us also write $\pi_E : \mathbb{R}^D \rightarrow E$ and $\pi_E^\perp : \mathbb{R}^D \rightarrow E^\perp$ the perpendicular projections. We will show that for μ_0 small enough, the tangent space of the set $Z_{\mu_0}(\mathbf{M}) = \{z \in \mathbb{R}^D : \|\nabla d_{\mathbf{M}}(z)\| \leq \mu_0\}$ at z_0 (in the sense of Federer [Fed59]) is equal to the orthogonal E^\perp of the vector space E . Namely, we show that every point z of the medial axis close enough to z_0 are such that $\|\pi_E(z - z_0)\|$ is of order $\|z - z_0\|^2$, while for every direction $h \in E^\perp$, there is a point of $Z_{\mu_0}(\mathbf{M})$ at distance of order $\|h\|^2$ from $z_0 + h$. This is illustrated in Figure 4.

Let $p_j : B(z_0, \delta') \rightarrow B(x_j, \delta) \cap \mathbf{M}$ for $j \in [s]$ be the local projections from Lemma 4.8 (for some $\delta', \delta > 0$), which we know to be Lipschitz continuous. We define

$$I : B(z_0, \delta') \longrightarrow \mathcal{P}([s]) \quad (17)$$

$$z \longmapsto \{j \in [s] : p_j(z) \in \pi_{\mathbf{M}}(z)\}.$$

In particular, $\pi_{\mathbf{M}}(z) = \{p_j(z) : j \in I(z)\}$. We also introduce the notation $m_I(z) = m(\{p_j(z) : j \in I\})$ for $I \subset [s]$ and $z \in B(z_0, \delta')$, where we recall that $m(\sigma)$ denotes the center of the smallest enclosing ball of the set σ .

For $I \subset [s]$, let

$$g_I : B(z_0, \delta') \longrightarrow \mathbb{R} \quad (18)$$

$$z \longmapsto \frac{\|z - m_I(z)\|}{d_{\mathbf{M}}(z)}.$$

Note that $\|\nabla d_{\mathbf{M}}(z)\| = g_{I(z)}(z)$ for any $z \in B(z_0, \delta')$.

The following Lemma states that $g_{[s]}$ and g_I (for any $I \subset [s]$) are Lipschitz continuous and Hölder continuous respectively around z_0 ; it also yields some control over their Lipschitz and Hölder constants, which will be needed later in Section 7 where we let the embedding $i : M \rightarrow \mathbb{R}^D$ vary.

Lemma 6.1. *There exist constants $\delta'', \ell_1, \ell_2 > 0$ that depend only, and continuously, on the Lipschitz constants of the projections $p_j : B(z_0, \delta') \rightarrow B(x_j, \delta) \cap \mathbf{M}$ and on the distance between z_0 and the boundary of the simplex with vertices $\{x_1, \dots, x_s\}$ such that the function*

$g_{[s]} : B(z_0, \min(\delta', \delta'')) \rightarrow \mathbb{R}$ is ℓ_1 -Lipschitz continuous and that for any $I \subset [s]$, the function $g_I : B(z_0, \min(\delta', \delta'')) \rightarrow \mathbb{R}$ is $(1/2)$ -Hölder continuous with Hölder constant ℓ_2 .

Proof. The distance $d_M(z_0)$ is lower bounded by the distance between z_0 and the boundary of the simplex with vertices $\{x_1, \dots, x_s\}$ and the function $z \mapsto d_M(z)$ is also lower bounded and Lipschitz continuous on a ball of radius $d_M(z_0)/2$ around z_0 . We know from Lemma 4.1 that the projections $p_j : B(z_0, \delta') \rightarrow B(x_j, \delta) \cap M$ are Lipschitz continuous. As the function $A \mapsto m(A)$ is locally $1/2$ -Hölder continuous with respect to the Hausdorff distance [ALS13, Lemma 16], the statement regarding Hölder continuity follows.

By hypothesis, z_0 belongs to the interior of the simplex spanned by $\{x_1, \dots, x_s\}$. Hence, $m_{[s]}(z)$ also belongs to the interior of the simplex spanned by $\{p_1(z), \dots, p_s(z)\}$ for z close enough to z_0 (depending on the Lipschitz constants of the p_j s and how far z_0 is from the boundary of the simplex spanned by $\{x_1, \dots, x_s\}$). When this is the case, the function $m_{[s]}(z)$ is actually equal to the center of the circumsphere of the points $p_1(z), \dots, p_s(z)$. When restricted to the simplices whose circumsphere has its center at distance at least $c > 0$ from their boundary, the coordinates of the center of the circumsphere is a Lipschitz function of the coordinates of the vertices for a Lipschitz constant that depends on c . Thus $m_{[s]}(z)$ is Lipschitz continuous with respect to the coordinates of the points $p_1(z), \dots, p_s(z)$, hence with respect to those of z , when z is close enough to z_0 , and so is the function $g_{[s]}$ as a composition of Lipschitz maps, with the definition of “close” and the Lipschitz constant depending on the Lipschitz constants of the p_j s and how far z_0 is from the boundary of the simplex spanned by $\{x_1, \dots, x_s\}$. \square

As another step towards understanding the local behavior of the μ -critical points around z_0 , the next lemma states that μ -critical points (for μ small enough) that are close to z_0 must belong to $z_0 + E^\perp$ as a first approximation.

Lemma 6.2. *There exist $\mu_0, \delta_0, C > 0$ such that for every μ_0 -critical point $z \in B(z_0, \delta_0)$, we have $I(z) = [s]$ and*

$$\|\pi_E(z - z_0)\| \leq C\|z - z_0\|^2.$$

Proof. Let $\delta, \delta' > 0$ and $p_j : B(z_0, \delta') \rightarrow B(x_j, \delta) \cap M$ for $j = 1, \dots, s$ be as in Lemma 4.8, and such that the functions g_I are Hölder continuous (Lemma 6.1). Note that $g_I(z_0) > 0$ for all $I \subsetneq [s]$. Hence, it holds that if $I \subsetneq [s]$, then

$$g_I(z) > \frac{1}{2} \min_{I \subsetneq [s]} g_I(z_0) =: \mu_0$$

for $\|z - z_0\| \leq \delta_1$ for some $\delta' > \delta_1 > 0$. If $z \in B(z_0, \delta_1)$ is μ_0 -critical, then

$$\|\nabla d_M(z)\| = \frac{\|z - m_{I(z)}(z)\|}{d_M(z)} = g_{I(z)}(z) \leq \mu_0,$$

hence $I(z) = [s]$. This proves the first point.

Let us now prove the second statement. Let $z \in B(z_0, \delta_1)$ be a μ_0 -critical point and $h = z - z_0$. For any $j \in [s]$,

$$\begin{aligned} \|z - p_j(z)\|^2 &= \|(z_0 - x_j) - (p_j(z) - h - x_j)\|^2 \\ &= \|z_0 - x_j\|^2 + 2\langle z_0 - x_j, h + x_j - p_j(z) \rangle + \|h + x_j - p_j(z)\|^2. \end{aligned}$$

As $z_0 \in \text{Conv}(\{x_j : j \in [s]\})$, we have $z_0 - x_j \in E$. This implies that $\langle z_0 - x_j, h \rangle = \langle z_0 - x_j, \pi_E(h) \rangle$. Also, as p_j is Lipschitz continuous, it holds that $\|p_j(z) - x_j\| \leq Lh$ for some constant $L > 0$. Furthermore, as $z_0 - x_j \in T_{x_j}M^\perp$ and $p_j(z) \in B(x_j, \delta) \cap M$, according to [Fed59, Theorem 4.8.7],

$$|\langle z_0 - x_j, x_j - p_j(z) \rangle| = |\langle z_0 - x_j, \pi_{x_j}^\perp(x_j - p_j(z)) \rangle| \leq \frac{d_{M_j}(z) \|x_j - p_j(z)\|^2}{2\tau(M)} \leq C_1 \|h\|^2$$

for some constant C_1 depending on $\tau(M)$, $d_M(z_0)$ and L . In total, we have shown that

$$\|z - p_j(z)\|^2 = \|z_0 - x_j\|^2 + 2\langle z_0 - x_j, \pi_E(h) \rangle + R_j(h), \quad (19)$$

where $|R_j(h)| \leq C_2 \|h\|^2$ for some positive constant C_2 . Now, by definition $\|z - p_{j_1}(z)\|^2 = \|z - p_{j_2}(z)\|^2$ for any $1 \leq j_1, j_2 \leq s$ (and s is at least 2). Hence,

$$\|z_0 - x_{j_1}\|^2 + 2\langle z_0 - x_{j_1}, \pi_E(h) \rangle + R_{j_1}(h) = \|z_0 - x_{j_2}\|^2 + 2\langle z_0 - x_{j_2}, \pi_E(h) \rangle + R_{j_2}(h),$$

so

$$|\langle x_{j_1} - x_{j_2}, \pi_E(h) \rangle| \leq |R_{j_1}(h) + R_{j_2}(h)|.$$

As E is spanned by the vectors $x_{j_1} - x_{j_2}$, this implies that $\|\pi_E(h)\| \leq C_4 h^2$ for some C_4 that depends on the geometry of the nondegenerate simplex spanned by the points x_j . \square

The geometry of the points close to the critical point z_0 that belong to the most central stratum of the medial axis, i.e. those such that $I(z) = [s]$, can be more precisely described:

Lemma 6.3. *There exists $\delta > 0$ such that the local projections $p_j : B(z_0, \delta) \rightarrow B(x_j, \delta') \cap M$ for $j \in [s]$ from Lemma 4.8 are well-defined and that the core medial axis of M around z_0*

$$M_c(z_0, M) := \{z \in B(z_0, \delta) : I(z) = [s]\}$$

is a $(D - s + 1)$ -dimensional C^2 submanifold with tangent space $T_{z_0}M_c(z_0, M) = E^\perp$ in z_0 .

Proof. Let $\delta, \delta' > 0$ and $p_j : B(z_0, \delta) \rightarrow B(x_j, \delta') \cap M = M_j$ for $j = 1, \dots, s$ be as in Lemma 4.8. When restricted to $B(z_0, \delta)$, the function $z \mapsto d(z, M_j)^2$ is equal to $\phi_j : B(z_0, \delta) \rightarrow \mathbb{R}, z \mapsto \|z - p_j(z)\|^2$. We know that the map p_j is C^1 , hence so is ϕ_j . Its differential is

$$(d\phi_j)_z : h \mapsto 2\langle z - p_j(z), h - d(p_j)_z(h) \rangle = 2\langle z - p_j(z), h \rangle$$

for $h \in \mathbb{R}^D$, as $\text{Im}(d(p_j)_z(h)) \subset T_{p_j(z)}M$ and thus $\text{Im}(d(p_j)_z(h)) \perp z - p_j(z)$. This shows that ϕ_j is in fact C^2 .

By definition, the core medial axis $M_c(z_0, M)$ is defined by the $s - 1$ equations

$$\phi_1(z) - \phi_j(z) = 0 \quad j = 2, \dots, s$$

on $B(z_0, \delta)$. The differential of the function $z \mapsto \phi_1(z) - \phi_j(z)$ is $h \mapsto 2\langle p_j(z) - p_1(z), h \rangle$, and the $s - 1$ vectors $p_j(z) - p_1(z)$ form a linearly independent family for $z_0 = z$, as M satisfies (P1). By continuity, we can make $\delta > 0$ smaller to ensure that this remains true for all $z \in B(z_0, \delta)$. Then $M_c(z_0, M)$ is the zero set of the submersion $F : B(z_0, \delta) \rightarrow \mathbb{R}^{s-1}, z \mapsto (\phi_1(z) - \phi_j(z))_{j \in [s]}$, and the kernel of the differential $d_{z_0}F$ is the perpendicular of $\text{Vec}(\{p_j(z_0) - p_1(z_0) : j = 2, \dots, s\}) = E$, i.e. E^\perp . This is enough to conclude. \square

We now show that μ_0 -critical points can be found by moving from the critical point z_0 in any orthogonal directions $h \in E^\perp$, up to a quadratic error of order $\|h\|^2$; this is essentially a consequence of the previous lemma.

Lemma 6.4 (Existence of μ -critical points). *For any $\mu_0 \in (0, 1)$, there exist $\delta_1, C > 0$ such that for any $h \in E^\perp$ with $\|h\| < \delta_1$, there exists a μ_0 -critical point z of \mathbf{M} with $\pi_{E^\perp}(z - z_0) = h$, $\|\pi_E(z - z_0)\| \leq C\|h\|^2$ and such that $I(z) = [s]$.*

Proof. This is a direct consequence of Lemma 6.3. Indeed, as $T_{z_0}M_c(z_0, M) = E^\perp$, the orthogonal projection $\psi : z \in M_c(z_0, M) \rightarrow \pi_{E^\perp}(z - z_0) \in E^\perp$ is a local diffeomorphism at z_0 . Its inverse ψ^{-1} is defined on a small ball around 0 in E^\perp . If h is in this ball, then $z = \psi^{-1}(h)$ is such that $\pi_{E^\perp}(z - z_0) = h$. Furthermore,

$$\|\pi_E(z - z_0)\| = \|z - z_0 - \pi_{E^\perp}(z - z_0)\| = \|\psi^{-1}(h) - \psi^{-1}(0) - h\| \leq C\|h\|^2$$

for some constant C as ψ^{-1} is of class C^2 and $d\psi_0^{-1}$ is equal to the identity on E^\perp . The point z is μ -critical for $\mu = \|\nabla d_M(z)\|$. As $I(z) = [s]$, we have $\|\nabla d_M(z)\| = g_{[s]}(z)$, which is a Lipschitz-continuous function on a neighborhood of z_0 according to Lemma 6.1. As $\|\nabla d_M(z_0)\| = 0$, μ will be smaller than μ_0 for any $\|h\| < \delta_1$ if δ_1 is small enough. \square

6.2 The Big Simplex Property

In the following proposition, we translate the global condition (P4) into a more practical and purely local (around a critical point $z_0 \in Z(\mathbf{M})$) condition, the *Big Simplex Property*, that can be expressed in terms of the volumes of the simplices formed by the points z close to z_0 and their projections $p_j(z)$ for $j \in [s]$, under the assumption that $z - z_0$ is perpendicular to $E = \text{Vec}(\{x_{j_1} - x_{j_2} : 1 \leq j_1, j_2 \leq s\})$. The proof makes direct use of the two previous lemmas shown in Section 6.1.

Proposition 6.5 (Big Simplex Property). *Let $\mathbf{M} \subset \mathbb{R}^D$ be a compact C^2 submanifold that satisfies conditions (P1), (P2), (P3). Then the following “Big Simplex Property”:*

(BSP) *For any critical point $z_0 \in Z(\mathbf{M})$, there are constants $\delta', \delta, L > 0$ such that if we write $\pi_M(z_0) = \{x_1, \dots, x_s\}$, the local projections $p_j : B(z_0, \delta') \rightarrow B(x_j, \delta) \cap \mathbf{M}$ from Lemma 4.8 are well-defined and the following holds. For $h \in E^\perp$ with $\|h\| \leq \delta'$, we let $\Delta(h)$ be the s -simplex with vertices $z = z_0 + h$ and $p_j(z)$ for $1 \leq j \leq s$. Then, the s -volume of $\Delta(h)$ satisfies*

$$\text{Vol}_s(\Delta(h)) \geq L\|h\|. \quad (20)$$

is equivalent to Property (P4) introduced earlier:

(P4) *There exist constants $C > 0$ and $\mu_0 \in (0, 1)$ such that for every $\mu \in [0, \mu_0)$, the set $Z_\mu(\mathbf{M})$ is included in a tubular neighborhood of size $C\mu$ of $Z(\mathbf{M})$, that is every point of $Z_\mu(\mathbf{M})$ is at distance less than $C\mu$ from $Z(\mathbf{M})$.*

Before proving the equivalence between (BSP) and (P4), we provide an example of a simple manifold \mathbf{M} not satisfying either conditions (albeit satisfying (P1), (P2) and (P3)). See also Figure 5 for an illustration.

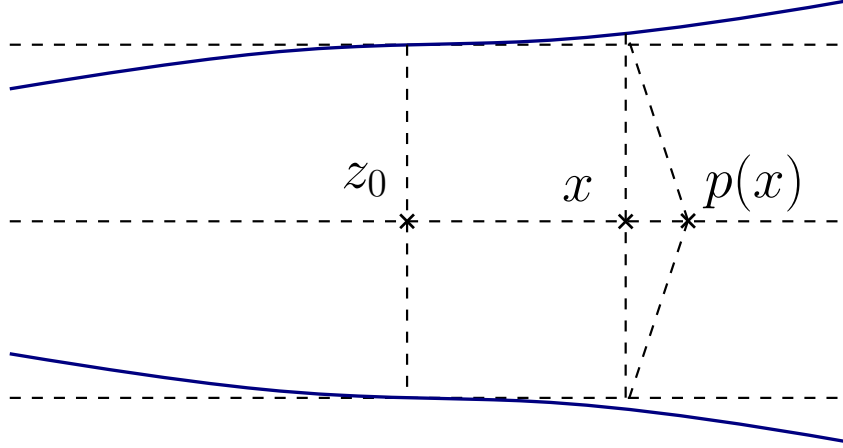


Figure 5: The point $p(x)$ satisfies $m(\pi_{\mathbf{M}}(p(x))) = x$, with $\|\nabla d_{\mathbf{M}}(p(x))\| = 3x^2 + o(x^2) \ll \|x\|$ for small values of x . Hence, $p(x)$ is a μ -critical point with $\mu = 3x^2 + o(x^2)$ but at distance from z_0 greater than $x \gg \mu$ (when x is small).

Example 6.6. Consider the C^2 submanifold $\mathbf{M} \subset \mathbb{R}^2$ that is equal on $[-1, 1] \times [-2, 2]$ to the union of the graph of the functions $x \mapsto 1 + x^3$ and $x \mapsto -1 - x^3$, see Figure 5. The distance function to \mathbf{M} admits a single critical point $z_0 = (0, 0)$ whose projections define a non-degenerate simplex, and the sphere $S(z_0, 1)$ is non-osculating \mathbf{M} at each projection. On the other hand, for each $x > 0$ the projections of the point $p(x) := (x + 3x^2 + 3x^5, 0)$ are $(x, 1 + x^3)$ and $(x, -1 - x^3)$, hence the norm of the generalized gradient of $d_{\mathbf{M}}$ at $p(x)$ is

$$\|\nabla d_{\mathbf{M}}(p(x))\| = \frac{d(p(x), (x, 0))}{d(p(x), (x, 1 + x^3))} = \frac{3x^2 + O(x^5)}{1 + O(x^3)} = 3x^2 + o(x^2),$$

while the distance from $p(x)$ to z_0 is $x + o(x)$. This shows that \mathbf{M} does not satisfy condition (P4). It is easy to modify this example so that \mathbf{M} be compact, yet the behaviour of the μ -critical points around z_0 remain the same.

Let us now prove Proposition 6.5.

Proof. Let $z_0 \in Z(\mathbf{M})$ be a critical point with projections $\pi_{\mathbf{M}}(z_0) = \{x_1, \dots, x_s\}$, and let $\delta', \delta, \mu_0 > 0$ be small enough that $p_j : B(z_0, \delta') \rightarrow B(x_j, \delta) \cap \mathbf{M}$ is as in Lemma 4.8 for $j = 1, \dots, s$ and that the conclusions of Lemma 6.2 stand, i.e. if $z \in B(z_0, \delta')$ is μ_0 -critical, then $I(z) = [s]$ and

$$\|\pi_E(z - z_0)\| \leq C\|z - z_0\|^2$$

for some $C > 0$ that depends on z_0 . Further assume that the conclusion of Lemma 6.1 hold, that is the function $g_{[s]}$ is ℓ -Lipschitz continuous. Let $z \in B(z_0, \delta')$ be μ_0 -critical. Then

$$\|\|\nabla d_{\mathbf{M}}(z)\| - g_{[s]}(z_0 + h)\| = |g_{[s]}(z) - g_{[s]}(z_0 + h)| \leq \ell C_0 \|z - z_0\|^2. \quad (21)$$

Now for $h \in E^\perp$ with $\|h\| \leq \delta'$, let $\tilde{\Delta}(h)$ be the $(s - 1)$ -simplex whose vertices are the points $p_j(z_0 + h)$ for $j \in [s]$, and let $\Delta(h)$ be the s -simplex obtained by adding the vertex $z_0 + h$ to

those of $\tilde{\Delta}(h)$. Let $\tilde{E}(h)$ be the affine space spanned by $\tilde{\Delta}(h)$. The s -volume of $\Delta(h)$ can be computed as

$$\text{Vol}_s(\Delta(h)) = \frac{1}{s} \text{Vol}_{s-1}(\tilde{\Delta}(h)) d(z_0 + h, \tilde{E}(h)).$$

Note that according to (P1), when $h = 0$, the projection of $z_0 + h$ on $\tilde{E}(h)$ belongs to the interior of the simplex $\tilde{\Delta}(h)$. By continuity of the projections p_j , this property is still valid when h is small enough. Hence, for h small enough,

$$\text{Vol}_s(\Delta(h)) = \frac{1}{s} \text{Vol}_{s-1}(\tilde{\Delta}(h)) d(z_0 + h, \tilde{\Delta}(h)). \quad (22)$$

(BSP) \implies (P4): Assume that (BSP) is satisfied. Then, $d(z_0 + h, \tilde{\Delta}(h)) \leq \|z_0 + h - m_{[s]}(z_0 + h)\| = d_M(z_0 + h) g_{[s]}(z_0 + h)$. Furthermore, the volume of $\text{Vol}_{s-1}(\tilde{\Delta}(h))$ can be crudely bounded by $(\text{diam}(M))^{s-1}$. Hence, if z is a μ_0 -critical point close enough to z_0 that $h = \pi_{E^\perp}(z - z_0)$ satisfies the condition (BSP), we obtain that

$$L\|h\| \leq \text{Vol}_s(\Delta(h)) \leq \frac{(\text{diam}(M))^{s-1}}{s} d_M(z_0 + h) g_{[s]}(z_0 + h) \leq \frac{(\text{diam}(M))^s}{s} g_{[s]}(z_0 + h).$$

Hence, (21) yields

$$\|\nabla d_M(z)\| \geq \frac{s}{(\text{diam}(M))^s} L\|h\| - \ell C_0 \|z - z_0\|^2.$$

Lemma 6.2 also implies that $\|h\| \geq \|z - z_0\|/2$ if $\|z - z_0\|$ is small enough. Hence, we obtain for $z \in B(z_0, \delta')$ and δ' small enough

$$\|\nabla d_M(z)\| \geq \|z - z_0\| \left(\frac{sL}{2(\text{diam}(M))^s} - \delta' \ell C_0 \right) = \frac{\|z - z_0\|}{C_1},$$

where $C_1 > 0$ for δ' small enough. In particular, if z is a μ -critical point for $z \in B(z_0, \delta')$ with $\mu < \mu_0$, then $\|z - z_0\| \leq C_1 \mu$.

We have shown that (BSP) implies (P4) "locally". To conclude "globally", we proceed as follows : the constants μ_0, δ' and C_1 depend on the chosen critical point z_0 . As M satisfies property (P2), it admits only finitely many critical points. Let $\bar{\mu}_0, \bar{\delta}'$ and \bar{C}_1 be the minima over the critical points of the associated constants μ_0, δ' and C_1 respectively. Then any $\bar{\mu}_0$ -critical $z \in \mathbb{R}^D$ that is at distance less than $\bar{\delta}'$ from a critical point z_0 of M verifies

$$\bar{C}_1 \|\nabla d_M(z)\| \geq \|z - z_0\| \geq d(z, Z(M)). \quad (23)$$

Now remember that we have shown at the start of this section that $\eta_M(r) = \sup\{r : \exists x \in \mathbb{R}^D \text{ s.t. } \|\nabla d_M(x)\| \leq \mu \text{ and } d(Z(M), x) = r\}$ goes to 0 as $\mu \rightarrow 0$. This means that there exists $0 < \mu_1 < \bar{\mu}_0$ such that any μ_1 -critical point z must be at distance less than $\bar{\delta}'$ from $Z(M)$, where (23) applies. This allows us to conclude.

(P4) \implies (BSP): Assume now that (P4) is satisfied with constants C and μ_0 , and consider again $z_0 \in Z(M)$ with the associated spaces E and E^\perp . Let $h \in E^\perp$ be small enough that Lemma 6.4 applies: there exists a μ_0 critical point z of M with $\pi_{E^\perp}(z - z_0) = h$, $I(z) = [s]$ and $\|\pi_E(z - z_0)\| \leq C_2 \|h\|^2$ for some constant $C_2 > 0$ that is independent from the choice of h . Condition (P4) ensures that

$$C \|\nabla d_M(z)\| \geq \|z - z_0\| \geq \|h\|. \quad (24)$$

Let $\tilde{\Delta}(z - z_0)$ be the simplex with vertices $p_j(z)$ for $j \in [s]$ (assuming that h is small enough for the projections $p_j(z)$ to be well-defined). By continuity, as $\tilde{\Delta}(0)$ is a nondegenerate simplex (since \mathbf{M} satisfies (P1)), the volume $\text{Vol}_{s-1}(\tilde{\Delta}(h))$ is lower bounded for h small enough. As the projections $\pi_{\mathbf{M}}(z)$ coincide with the vertices of $\tilde{\Delta}(z - z_0)$, we also have that $d(z, \tilde{\Delta}(z - z_0)) = \|z - m_{[s]}(z)\|$ (as noted in [CCSL06, Section 2.1]). Thus, starting from Equation (22) and for h small enough,

$$\begin{aligned} \text{Vol}_s(\Delta(h)) &= \frac{1}{s} \text{Vol}_{s-1}(\tilde{\Delta}(h)) d(z_0 + h, \tilde{\Delta}(h)) \\ &\geq C_3(d(z, \tilde{\Delta}(z - z_0)) - \|z_0 + h - z\| - d_H(\tilde{\Delta}(h), \tilde{\Delta}(z - z_0))) \\ &\geq C_3(\|z - m_{[s]}(z)\| - C_2\|h\|^2 - \max_{j \in [s]} \|p_j(z) - p_j(z_0 + h)\|) \\ &\geq C_3(d_{\mathbf{M}}(z)\|\nabla d_{\mathbf{M}}(z)\| - C_2\|h\|^2 - KC_2\|h\|^2) \\ &\geq C_3(C_4\|h\| - C_2\|h\|^2 - KC_2\|h\|^2) \end{aligned}$$

where we use the K -Lipschitz continuity of the p_j s on a neighborhood of z_0 for some constant $K > 0$, Inequality (24), and the fact that $d_{\mathbf{M}}(z)$ is lower bounded on a neighborhood of z_0 . Hence, for $\|h\|$ smaller than some constant δ'' , this quantity is larger than $\frac{C_3 C_4}{2} \|h\| = L \|h\|$. This proves that (BSP) holds at any given point $z_0 \in \mathbf{M}$, concluding the proof. \square

We choose to translate (P4) into the Big Simplex Property because the volume of the simplex $\Delta(h)$ (in the notations of the proof of Proposition 6.5 above) can be easily expressed in terms of the first derivative of the projections p_i , which themselves depend on the shape operator of \mathbf{M} around the points $x_i \in \pi_{\mathbf{M}}(z_0)$; this is formally stated in the Lemma 6.7 further below. This will be used for both showing that the condition is open in Section 7 (along with Conditions (P1), (P2) and (P3)) and that it is dense in Section 8.

6.3 Computing the volume of the simplex

We compute an approximation of the volume of the simplex $\Delta(h)$ introduced in Subsection 6.2 expressed using the differentials of the local projections $p_i : B(z_0, \delta') \rightarrow B(x_i, \delta) \cap \mathbf{M}$, where \mathbf{M} is as above a C^2 compact manifold satisfying (P1), (P2) and (P3). This allows us to characterize (BSP) (and hence (P4)) purely locally, in terms of the critical points z_0 , their projections x_i and the curvature of \mathbf{M} at the points x_i which determines the differentials of the maps p_i .

Lemma 6.7. *Let $z_0 \in Z(\mathbf{M})$ be a critical point of \mathbf{M} with projections $\pi_{\mathbf{M}}(z_0) = \{x_1, \dots, x_s\}$. Let $\delta', \delta > 0$ be small enough that the projections $p_i : B(z_0, \delta') \rightarrow B(x_i, \delta) \cap \mathbf{M}$ from Lemma 4.8 are well-defined. For $h \in E^\perp$ small enough, let as before $\Delta(h)$ be the s -simplex with vertices $z_0 + h$ and $p_j(z_0 + h)$ for $j \in [s]$. Let $X = [x_1 - z_0, \dots, x_s - z_0] \in \mathbb{R}^{D \times s}$, and let $R_i(h)$ be the matrix obtaining by replacing the i th column of X by $r_i(h) = p_i(z_0 + h) - x_i - h$. Define likewise $A_i(h)$, where we replace the i th column of X by $(dp_i)_{z_0}(h) - h$. Then, for any $h \in E^\perp$ with $\|h\| < \delta'$ it holds that*

$$\begin{aligned} \text{Vol}_s(\Delta(h))^2 &= \frac{1}{(s!)^2} \sum_{i,j=1}^s \det(R_i(h)^\top R_j(h)) + O(\|h\|^3) \\ &= \frac{1}{(s!)^2} B(h) + o(\|h\|^2), \end{aligned}$$

where B is the quadratic form given by

$$B(h) = \sum_{i,j=1}^s \det(A_i(h)^\top A_j(h)). \quad (25)$$

Furthermore, the constant in the term $O(h^3)$ depends only on s , $\text{diam}(M)$, and the Lipschitz constants of the projections p_j , $j \in [s]$.

Controlling the constant in the term $O(h^3)$ will be needed in Section 7, once we let the embedding of the manifold into \mathbb{R}^D vary.

Proof. Let $h \in E^\perp$ be such that $\|h\| < \delta'$ and let

$$v_j = p_j(z_0 + h) - (z_0 + h) = (x_j - z_0) + r_j(h) = (x_j - z_0) + (dp_j)_{z_0}(h) - h + o(\|h\|)$$

for $j \in [s]$. Let $L(h)$ be the $l \times l$ matrix with $L_{i,j}(h) = \langle v_i, v_j \rangle$. Then,

$$\text{Vol}_s(\Delta(h)) = \frac{1}{s!} \det(L(h))^{\frac{1}{2}}.$$

Define the $D \times s$ matrices $V = [v_1, \dots, v_s]$, $X = [x_1 - z_0, \dots, x_s - z_0]$. For $\tau \subset [s]$, let $R_\tau(h)$ be the matrix obtained from X by replacing the i th column of X by $r_i(h)$ for all $i \in \tau$. We write $R_i(h) = R_{\{i\}}(h)$ and $R_{i,j}(h) = R_{\{i,j\}}(h)$. By multilinearity of the determinant, as the columns of the matrix $V^\top V$ are of the form $V^\top(x_i - z_0) + V^\top r_i(h)$, it holds that

$$\begin{aligned} \det(L(h)) &= \sum_{\tau \subset [s]} \det(V^\top R_\tau(h)) \\ &= \det(V^\top X) + \sum_{j=1}^s \det(V^\top R_j(h)) + \sum_{i \neq j} \det(V^\top R_{i,j}(h)) + O(\|h\|^3). \end{aligned}$$

Similarly, we write for any matrix M of size $D \times l$,

$$\det(V^\top M) = \det(X^\top M) + \sum_{i=1}^s \det(R_i(h)^\top M) + \sum_{i \neq j} \det(R_{i,j}(h)^\top M) + O(\|h\|^3).$$

Hence,

$$\begin{aligned} \det(L(h)) &= \det(X^\top X) + 2 \sum_{i=1}^s \det(R_i(h)^\top X) + 2 \sum_{i \neq j} \det(R_{i,j}(h)^\top X) \\ &\quad + \sum_{i,j=1}^l \det(R_i(h)^\top R_j(h)) + O(\|h\|^3). \end{aligned}$$

Any matrix of the form $X^\top M$ is non-invertible as X is not of full rank. Hence,

$$\det(L(h)) = \sum_{i,j=1}^l \det(R_i(h)^\top R_j(h)) + O(\|h\|^3).$$

Note that the $O(\|h\|^3)$ can actually be made explicit, and is given as a sum of terms of the form $\det(R_\tau(h)^\top R_{\tau'}(h))$ where $|\tau| + |\tau'| \geq 3$. As the p_j s are ℓ -Lipschitz continuous for some $\ell > 0$, we have $\|r_j(h)\| \leq (\ell + 1)\|h\|$ for $j \in [s]$. As $\|x_j - z_0\| \leq \text{diam}(\mathbf{M})$, we can use the multilinearity of the determinant to show that $|\det(R_\tau(h)^\top R_{\tau'}(h))|$ is smaller than

$$C \text{diam}(\mathbf{M})^{s-|\tau|} ((\ell + 1)\|h\|)^{|\tau|} \text{diam}(\mathbf{M})^{s-|\tau'|} ((\ell + 1)\|h\|)^{|\tau'|},$$

where C is the maximal determinant of a $s \times s$ matrix of dot products of vectors of \mathbb{R}^D of norm smaller than 1. This is of order $\|h\|^3$, with a constant depending only on $\text{diam}(\mathbf{M})$, s and ℓ .

It remains to prove the second equality in Lemma 6.7, which follows from the equality $R_i(h) = A_i(h) + o(\|h\|)$ and that the i th columns of $A_i(h)$ and $R_i(h)$ are of order $O(\|h\|)$. \square

We get the following characterization of (BSP) as a simple corollary of Lemma 6.7:

Corollary 6.8. *Let \mathbf{M} be a C^2 compact manifold satisfying (P1), (P2) and (P3). Then \mathbf{M} satisfies (P4) if and only if for any $z_0 \in Z(\mathbf{M})$ the associated quadratic form $B : E^\perp \rightarrow \mathbb{R}$ defined in Lemma 6.7 is non-degenerate.*

Proof. Property (BSP) is satisfied if and only if for any $z_0 \in Z(\mathbf{M})$, there exist $L, \delta' > 0$ such that for any $h \in E^\perp$ with $\|h\| \leq \delta'$, the s -volume of the associated s -simplex $\Delta(h)$ satisfies

$$\text{Vol}_s(\Delta(h)) \geq L\|h\|.$$

As we have shown in Lemma 6.7 that $\text{Vol}_s(\Delta(h)) = \frac{1}{s!} \sqrt{B(h) + o(\|h\|^2)}$, this is true if and only if $B : E^\perp \rightarrow \mathbb{R}$ is non-degenerate for any such z_0 . As (BSP) is equivalent to (P4) according to Proposition 6.5, the conclusion holds. \square

7 The C^2 stability theorem

Let as usual M be a C^k compact manifold for some $k \geq 2$. In this section, we show that the set of all embeddings $i \in \text{Emb}(M, \mathbb{R}^D)$ such that $i(M)$ satisfies Conditions (P1), (P2), (P3) and (P4) is open for the Whitney C^2 -topology, or in other words that the conjunction of the four conditions is stable with respect to small C^2 perturbations. This proves the ‘‘open’’ part of the Genericity Theorem 1.1. We also show that for an embedding i that satisfies (P1-4), small C^2 perturbations of i leave the critical points of $i(M)$ and their projections stable. The C^2 Stability Theorem below, whose proof makes many of the more technical results of the previous sections necessary, is the combination of these statements.

Theorem 1.4 (The C^2 Stability Theorem). *Let $i \in \text{Emb}^2(M, \mathbb{R}^D)$ be an embedding such that $\mathbf{M} = i(M)$ satisfies (P1-4). Then, for all $\varepsilon > 0$ small enough, there exists a neighborhood \mathcal{U} of i in $\text{Emb}^2(M, \mathbb{R}^D)$ equipped with the Whitney C^2 -topology such that for every $i' \in \mathcal{U}$, with $\mathbf{M}' = i'(M)$:*

1. \mathbf{M}' satisfies (P1-4).
2. There is a bijection $\Psi : Z(\mathbf{M}) \rightarrow Z(\mathbf{M}')$.
3. Let $z_0 \in Z(\mathbf{M})$; then $\Psi(z_0) \in B(z_0, \varepsilon)$.
4. Let $z_0 \in Z(\mathbf{M})$ with $\pi_{\mathbf{M}}(z_0) = \{x_1, \dots, x_s\}$. Then $\pi_{\mathbf{M}'}(\Psi(z_0))$ is included in the union of balls of radius ε centered at the points x_j , with exactly one point in each ball.

Before proving Theorem 1.4, we give three (counter)examples showcasing degenerate behaviors when (P1-4) is not satisfied.

Example 7.1. *It is interesting to note that while the conjunction of (P1), (P2), (P3) and (P4) defines an open condition on the set of embeddings $\text{Emb}^2(M, \mathbb{R}^D)$, this is e.g. not true of (P1), (P2) and (P4) without (P3), as showcased by the following example: consider the curve M illustrated on the left of Figure 6. The right part is parameterized around $x = (0, 1)$ as $\theta \mapsto (1 + \theta^4)(\cos(\theta), \sin(\theta))$ for $\theta \in] -\frac{\pi}{4}, \frac{\pi}{4}[$, and there is a critical point at $z_0 = (0, 0)$ with $\pi_M(z_0) = \{x, x_1, x_2\}$. The curve M satisfies (P1), (P2) and (P4), but not (P3) due to its curvature around x . Consider now any C^∞ function $f : \mathbb{R} \rightarrow [0, +\infty)$ such that $f(\theta) = \theta^2$ for any $\theta \in] -\frac{\pi}{8}, \frac{\pi}{8}[$, $f(\theta) \leq \theta^2$ for any $\theta \in \mathbb{R}$ and $f(\theta) = 0$ for any θ with $|\theta| \geq \frac{\pi}{4}$. For $a > 0$, consider the modified curve M_a shown on the right of Figure 6 and obtained by leaving M untouched, except around x where it is parameterized by*

$$\theta \mapsto (1 + \theta^4 - af(\theta))(\cos(\theta), \sin(\theta))$$

for $\theta \in] -\frac{\pi}{4}, \frac{\pi}{4}[$. Let $i_0 : M \rightarrow \mathbb{R}^2$ be some embedding whose image is M ; then for any $k \geq 0$, there is a constant $C_k > 0$ and an embedding $i_a : M \rightarrow \mathbb{R}^2$ whose image is M_a and at distance less than $C_k a$ in norm $\| - \|_{C^k}$ from i_0 . Yet for any $a > 0$ small enough, there is a critical point z_a of M_a close to z_0 with four projections (which is more than $D + 1 = 3$), two of them resulting from a “splitting” of x , hence M_a does not satisfy (P1).

Example 7.2. *Consider (some compact version of) the curve M shown on the left of Figure 7. While Conditions (P2), (P3), (P4) and the first part of Condition (P1) are satisfied, the second part of Condition (P1) is not, as the critical point z_0 belongs to the boundary of the simplex whose vertices are its projections $\pi_M(z_0) = \{x_1, x_2, x_3\}$. A small perturbation results in the curve M' shown on the right of the figure, where z_0 has split into two critical points (one with three projections and one with only two), illustrating how (P1) is necessary to the stability of $Z(M)$.*

Example 7.3. *Critical points may not be stable when Condition (P4) is not satisfied. Consider the curve $M \subset \mathbb{R}^2$ given in Example 6.6, that is equal on $\Omega = [-1, 1] \times [-2, 2]$ to the union of the graph of the functions $f_0 : x \mapsto 1 + x^3$ and $-f_0 : x \mapsto -1 - x^3$. For $a > 0$, consider the curve M_a given by the union of the graph of $f_a : x \mapsto 1 + ax + x^3$ and $-f_a$ on Ω (and modified outside Ω so that M_a is close to M in the C^2 -Whitney topology). The point $(0, 0)$ is a critical point of M , but the curve M_a has no critical points in $B((0, 0), 1)$. Indeed, if z was such a point, it would have a strictly greater x coordinate than all of its projections, due to the orthogonality of the projection and to the strict monotonicity of f_a ; hence z would not belong to the convex hull of its projections, leading to a contradiction. Hence, the critical point $(0, 0) \in Z(M)$ disappears in the perturbed submanifold M_a , and the existence of a bijection Ψ between critical points as in Theorem 1.4 does not hold on a neighborhood of M .*

The remainder of this section is dedicated to proving Theorem 1.4. We decompose the proof in a sequence of claims.

Let $\varepsilon > 0$, and let us assume that ε is small enough that the balls of radius ε around the critical points $z_0 \in Z(M)$ and around their projections are all pairwise disjoint.

Claim 1. *For any $\delta > 0$, we can make a neighborhood $\mathcal{U} \subset \text{Emb}^2(M, \mathbb{R}^D)$ of i small enough that if $i' \in \mathcal{U}$, the critical points of M' are δ -close to those of M .*

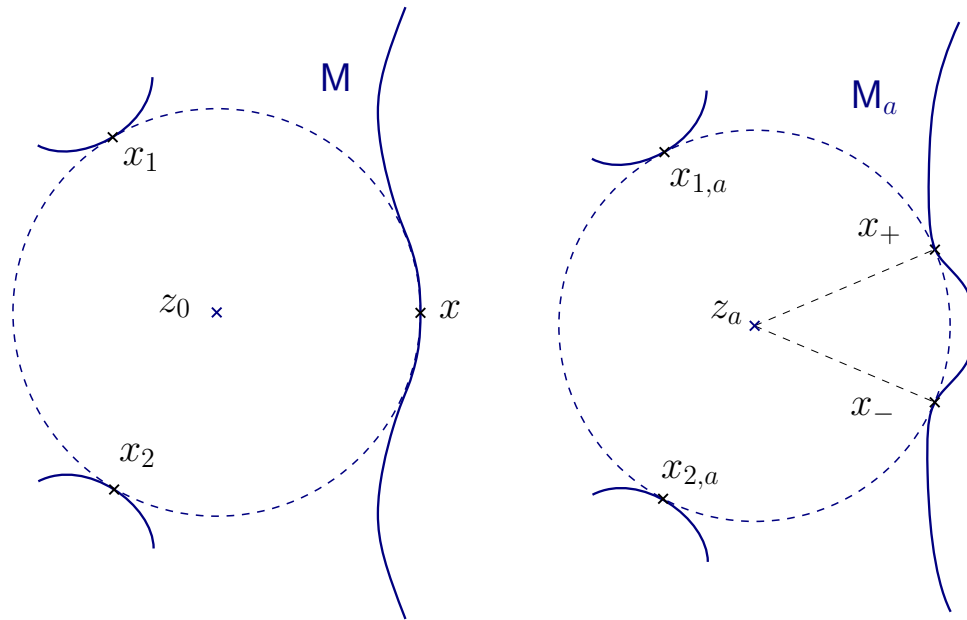


Figure 6: Illustration of Example 7.1: the curve M on the left satisfies Conditions (P1), (P2) and (P4), but not (P3). An arbitrarily small modification results in the curve M_a on the right, which does not fulfill Condition (P1).

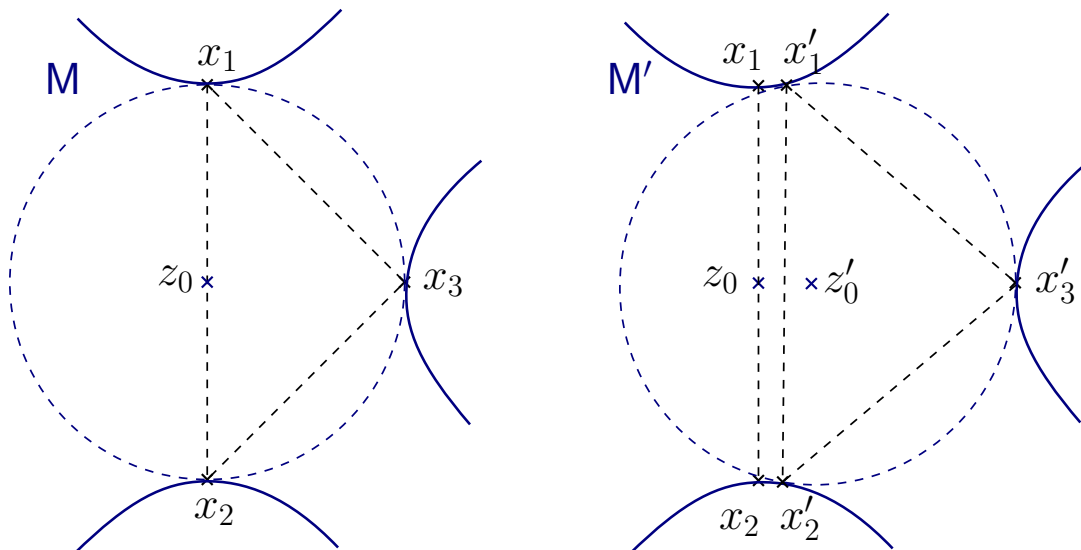


Figure 7: Illustration of Example 7.2: the critical point $z_0 \in Z(M)$ splits into two critical points of the perturbed curve M' .

Proof. Indeed, let $\tau(\mathbf{M}) > 0$ be the reach of \mathbf{M} . If $\|i - i'\|_{C^2}$ is small enough, then $\tau(\mathbf{M}') > \tau(\mathbf{M})/2$ (see [Fed59, Theorem 4.19]) and Theorem 3.4 from [CCSL06] states that each critical point $z'_0 \in Z(\mathbf{M}')$ (which is necessarily at distance at least $\tau(\mathbf{M}')$ from \mathbf{M}') is at distance at most $C_1\sqrt{d_H(\mathbf{M}, \mathbf{M}')} \leq C_1\sqrt{\|i - i'\|_{C^2}}$ from a μ -critical point of \mathbf{M} , with $\mu = C_2\sqrt{d_H(\mathbf{M}, \mathbf{M}')} \leq C_2\sqrt{\|i - i'\|_{C^2}}$ for some constants $C_1, C_2 > 0$ that depend on \mathbf{M} . As observed at the start of Section 6, there exists $\mu_0 > 0$ such that any μ_0 -critical point of \mathbf{M} is at distance at most $\delta/2$ from $Z(\mathbf{M})$. Hence if $\|i - i'\|_{C^2}$ is small enough, any critical point of \mathbf{M}' is at distance at most $\delta/2$ of a μ_0 -critical point of \mathbf{M} , which is itself at distance at most $\delta/2$ from a critical point of \mathbf{M} , which proves our claim. Note that the proof does not make use of Properties (P1-4). \square

Now let \mathcal{U} be a neighborhood of i small enough that for $i' \in \mathcal{U}$ each point $z'_0 \in Z(\mathbf{M}')$ is ε -close to some $z_0 \in Z(\mathbf{M})$; then for $i' \in \mathcal{U}$ we can define a map $\phi : Z(\mathbf{M}') \rightarrow Z(\mathbf{M})$ which associates to each critical point z'_0 the (unique) critical point $z_0 \in Z(\mathbf{M})$ at distance less than ε . We make that assumption for the remainder of the proof. We consider a fixed $z_0 \in \mathbf{M}$, for which we write $\pi_{\mathbf{M}}(z_0) = \{x_1, \dots, x_s\}$ for some $2 \leq s \leq D + 1$, and a point $z'_0 \in Z(\mathbf{M}')$ such that $\phi(z'_0) = z_0$.

Claim 2. *Let $0 < \delta < \varepsilon$. We can make \mathcal{U} small enough that each projection of z'_0 must be δ -close to one of the x_j s.*

Proof. Let $i' \in \mathcal{U}$, $z'_0 \in Z(\mathbf{M}')$ and $x' \in \pi_{\mathbf{M}'}(z'_0)$. There exists a point $y \in \mathbf{M}$ at distance less than $d_H(\mathbf{M}, \mathbf{M}')$ from x' . This point is such that

$$\begin{aligned} \|\phi(z'_0) - y\| &\leq d_H(\mathbf{M}, \mathbf{M}') + \|\phi(z'_0) - x'\| \leq d_H(\mathbf{M}, \mathbf{M}') + \|\phi(z'_0) - z'_0\| + \|z'_0 - x'\| \\ &= d_H(\mathbf{M}, \mathbf{M}') + \|\phi(z'_0) - z'_0\| + d_{\mathbf{M}'}(z'_0) \leq 2d_H(\mathbf{M}, \mathbf{M}') + 2\|\phi(z'_0) - z'_0\| + d_{\mathbf{M}}(\phi(z'_0)). \end{aligned}$$

Hence, $y \in B(\phi(z'_0), 2d_H(\mathbf{M}, \mathbf{M}') + 2\|\phi(z'_0) - z'_0\| + d_{\mathbf{M}}(\phi(z'_0))) \cap \mathbf{M}$. The Hausdorff distance $d_H(\mathbf{M}, \mathbf{M}')$ and the distance between each $z'_0 \in Z(\mathbf{M}')$ and the corresponding $\phi(z'_0) \in Z(\mathbf{M})$ can be made arbitrarily small (as shown above) by making \mathcal{U} small enough; hence, using Lemma 2.1, we see that y can be made as close as we want to $\pi_{\mathbf{M}}(\phi(z'_0))$ (this is true even without \mathbf{M} satisfying Conditions (P1-4)). Hence, x' can also be made as close as we want to $\pi_{\mathbf{M}}(\phi(z'_0))$. By applying the same reasoning to each of the finitely many critical points of \mathbf{M} (here we use (P2), and implicitly (P1) to assert that $\pi_{\mathbf{M}}(\phi(z'_0))$ is a finite set), we prove the claim. \square

In the remainder of the proof, we assume that \mathcal{U} is small enough that for any $i' \in \mathcal{U}$, each projection of each critical point of \mathbf{M}' is ε -close to some projection of some critical point of \mathbf{M} .

Claim 3. *We can make \mathcal{U} small enough for there to be constants $\delta, \delta', \alpha > 0$ such that the following holds for any $i' \in \mathcal{U}$ and any $z'_0 \in Z(\mathbf{M}')$: the point z'_0 has at most a single projection x'_j that is ε -close to any given projection x_j of $\phi(z'_0)$. Furthermore, the sphere $S(z'_0, d_{\mathbf{M}'}(z'_0))$ does not osculate \mathbf{M}' at x'_j , and there exist local orthogonal projections $p'_j : B(z'_0, \delta') \rightarrow \mathbf{M}'_j = \overline{B}(x'_j, \delta) \cap \mathbf{M}'$ given by Lemma 4.8 that are α^{-1} -Lipschitz continuous. In particular, \mathbf{M}' satisfies (P3).*

Proof. This is a consequence of the C^2 stability of the projection proved in Proposition 4.6, and of the crucial hypothesis that \mathbf{M} satisfies (P3). Indeed, let $z_0 \in Z(\mathbf{M})$, $x_0 \in \pi_{\mathbf{M}}(z_0)$ and $m_0 \in \mathbf{M}$ with $i(m_0) = x_0$. As the sphere $S(z_0, d_{\mathbf{M}}(z_0))$ is non-osculating at the point $x_j \in \pi_{\mathbf{M}}(x_j)$, there exist $\alpha > 0$ and neighborhoods $U_M \subset \mathbf{M}$, $U_{\mathbb{R}^D} \subset \mathbb{R}^D$ and $U_{\text{Emb}} \subset \text{Emb}^2(\mathbf{M}, \mathbb{R}^D)$ of m_0, z_0 and i such that for any $i' \in U_{\text{Emb}}$, any $z'_0 \in U_{\mathbb{R}^D}$ has a unique projection $x'_j = p'_j(z'_0)$ on $i'(U_M)$, that

x'_j belongs to $i'(U_M)$, and that the sphere $S(z'_0, d_{i'(U_M)}(z'_0))$ does not osculate $i'(U_M)$ at x'_j with α a degree of non-osculation.

Furthermore, if $M' = i'(M)$, then there exist $\delta, \delta' > 0$ such that the function $p'_j : B(z'_0, \delta') \rightarrow M' \cap \overline{B}(x'_j, \delta) = M'_j$ is the orthogonal projection on M'_j , which is C^1 and α^{-1} -Lipschitz continuous. We can assume that $\delta, \delta' < \varepsilon$.

Let us make \mathcal{U} small enough that $\mathcal{U} \subset U_{\text{Emb}}$ and that if $i' \in \mathcal{U}$, any critical point $z'_0 \in Z(M')$ in $\phi^{-1}(z_0)$ belongs to $U_{\mathbb{R}^D}$ and any projection $x' \in \pi_{M'}(z'_0)$ of z'_0 close to x_j is at distance at most δ from x_j . Then for such an embedding i' , critical point z'_0 and projection x' , we have shown that $x' \in i'(U_M)$. As we know from the conclusions of Proposition 4.6 on the C^2 stability of the projection that z'_0 has a unique projection on M' in $i'(U_M)$, the projection x' is necessarily the only one to be δ -close to x_j (hence the only one to be ε -close to x_j). We also get that the sphere $S(z'_0, d_{i'(U_M)}(z'_0)) = S(z'_0, d_M(z'_0))$ does not osculate $i'(U_M)$ at x' . By making \mathcal{U} small enough for the same reasoning to work for each of the finitely many critical points of M and their finitely many projections (and by taking the minima of the constants δ, δ' and α corresponding to each critical point z_0 and each projection x_j), we prove the claim. \square

For the remainder of the proof, we assume that \mathcal{U} is small enough for the conclusions of the previous claim to hold.

Claim 4. *We can make \mathcal{U} small enough that there is exactly one projection $x'_j \in \pi_{M'}(z'_0)$ that is ε -close to each x_j .*

Proof. Let us consider again $z_0 \in Z(M)$ with $\pi_M(z_0) = \{x_1, \dots, x_s\}$. If $i' \in \mathcal{U}$ and $z'_0 \in \phi^{-1}(z_0)$, we already know that for each $x_j \in \pi_M(z_0)$ there can be at most one projection $x'_j \in \pi_{M'}(z'_0)$ that is ε -close to x_j , and that each projection of z'_0 must be ε -close to one of the x_j .

As M satisfies (P1), the critical point z_0 must belong to the interior of the simplex with vertices x_1, \dots, x_s . In particular, for any $J \subsetneq [s]$

$$\frac{\|z_0 - m(\{x_j : j \in J\})\|}{d_M(z_0)} > 0.$$

For $i' \in \mathcal{U}$ and $z'_0 \in \phi^{-1}(z_0)$, let $I(z'_0) \subset [s]$ be the set of indices j such that z'_0 has a (unique) projection x'_j that is ε -close to x_j . By making \mathcal{U} small enough, we can ensure (by continuity) that for any $i' \in \mathcal{U}$ and any $z'_0 \in \phi^{-1}(z_0)$ the following holds: the point z'_0 is close enough to z_0 and each of its projections x'_j is close enough to x_j that if we let $J \subset I(z'_0)$ with $J \subsetneq [s]$, then

$$\frac{\|z'_0 - m(\{x'_j : j \in J\})\|}{d_{M'}(z'_0)} > 0.$$

In particular, as

$$\nabla d_{M'}(z'_0) = \frac{z'_0 - m(\{x'_j : j \in I(z'_0)\})}{d_{M'}(z'_0)},$$

the point z'_0 being critical forces $I(z'_0) = [s]$. Hence the projections of z'_0 are in bijection with those of z_0 . By applying the same reasoning to each of the finitely $z_0 \in Z(M)$, we prove the claim. \square

Again, we assume \mathcal{U} to be small enough for its conclusions to hold for the remainder of the proof.

Claim 5. *We can make \mathcal{U} small enough that M' satisfies (P1).*

Proof. This is a simple consequence of the previous claims. As M satisfies (P1), we know that for each $z_0 \in Z(M)$, the projections $\pi_M(z_0)$ are the vertices of a non-degenerate simplex to the relative interior of which z_0 belongs. These properties are open in the coordinates of z_0 and its projections (under the constraint that z_0 necessarily belongs to the simplex, which is satisfied for any critical point). We have shown that by making \mathcal{U} small enough, we can make any critical point $z'_0 \in M'$ arbitrarily close to a corresponding critical point $\phi(z'_0) \in M$ and the projections of z'_0 arbitrarily close to the corresponding projections of $\phi(z'_0)$ (with which they are in bijection) if $i' \in \mathcal{U}$. As there are only finitely many critical points of M with finitely many projections, this is enough to conclude that any M' satisfies (P1) for $i' \in \mathcal{U}$ and \mathcal{U} small enough, which we assume to be the case for the remainder of the proof. \square

For $z_0 \in Z(M)$ with $\pi_M(z_0) = \{x_1, \dots, x_s\}$, let $E = \text{Vec}(\{x_j - x_k : j, k \in [s]\})$ and $E' = \text{Vec}(\{x'_j - x'_k : j, k \in [s]\})$. For $I \subset [s]$, recall the functions m_I and g_I from Section 6, and define the functions $m'_I(z) = m(\{p'_j(z) : j \in I\})$ and $g'_I(z) = \frac{\|z - m'_I(z)\|}{d_{M'}(z)}$ implicitly associated to some $i' \in \mathcal{U}$. They are defined for $z \in B(z'_0, \delta')$, where δ' is the constant of Claim 3. As the projections p'_j are Lipschitz continuous for a uniform constant α^{-1} over $i' \in \mathcal{U}$ and as the distance between z'_0 and the boundary of the simplex with vertices $\pi_{M'}(z'_0)$ can be uniformly controlled over $i' \in \mathcal{U}$ (as in the proof of the previous claim), Lemma 6.1 implies that the functions g'_I are (1/2)-Hölder continuous and $g'_{[s]}$ is Lipschitz continuous on a small ball $B(z'_0, \delta'')$, where δ'' , the Lipschitz constant of $g'_{[s]}$ and the Hölder constants of the g'_I are uniform over $i' \in \mathcal{U}$.

Remark also that the various constants in Lemma 6.2 depend only on the reach and the diameter of M , on the Lipschitz continuity of the p_j s, the Hölder constants of the g_I s and the Lipschitz constant of $g_{[s]}$, on the level of non-degeneracy of the simplex with vertices x_1, \dots, x_s (e.g. measured through its determinant) and on the distance between z_0 and the boundary of this simplex. As the local projections p'_j s are all α^{-1} -Lipschitz continuous and defined on $B(z'_0, \delta')$ for some α and δ' uniform over $i' \in \mathcal{U}$, by making the non-degeneracy of the simplices and the distances of the critical points to their boundaries uniform over $i' \in \mathcal{U}$ in the conclusion of Claim 5, and using the remark on the Hölder constant of the previous paragraph, the constants of Lemma 6.2 can be made uniform over $i' \in \mathcal{U}$ as well, thence the following claim holds.

Claim 6. *We can make \mathcal{U} small enough that there exist constants $\delta_1, \mu_0, C > 0$ such that for all $i' \in \mathcal{U}$ and $z'_0 \in Z(M')$, if $z \in B(z'_0, \delta_1)$ is μ_0 -critical (for M'), then $\|\pi_{E'}(z - z'_0)\| \leq C\|z - z'_0\|^2$.*

We now turn to proving that Condition (P4), which we have shown in Proposition 6.5 to be equivalent under our hypotheses to Condition (BSP), is open.

Claim 7. *We can make \mathcal{U} small enough that there exist constants $\delta, \delta', L > 0$ such that for all $i' \in \mathcal{U}$, the submanifold $M' = i'(M)$ satisfies (BSP) with constants δ, δ', L for any $z'_0 \in Z(M')$. In particular, M' satisfies (P4).*

The uniformity of δ, δ' and L over $i' \in \mathcal{U}$ is not needed for M' to satisfy (BSP) (hence (P4)), but it will prove useful for the following claims.

Proof. Using the same notations as above, for $i' \in \mathcal{U}$, let $h' \in E'^{\perp}$ be small enough that $p'_j(z'_0 + h')$ is well-defined for $j \in [s]$. Let $\Delta'(h')$ be the simplex with vertices $z'_0 + h'$ and $p'_j(z'_0 + h')$ for $j \in [s]$. Define $X' = [x'_1 - z'_0, \dots, x'_s - z'_0]$, $r'_j(h') = p'_j(z'_0 + h') - x'_j - h'$ and $R'_j(h')$ the matrix

obtained from X' by replacing the j th column of X' by $r'_j(h')$. According to Lemma 6.7, the volume of $\Delta'(h')$ satisfies

$$\text{Vol}_s(\Delta'(h'))^2 \geq \frac{1}{(s!)^2} \sum_{i,j=1}^s \det(R'_i(h')^\top R'_j(h')) - C_0 \|h'\|^3,$$

where C_0 depends on $\text{diam}(M')$ and the Lipschitz constant ℓ of the projections p'_j (and s , which is the same for any $i' \in \mathcal{U}$ and $z'_0 \in \phi^{-1}(z_0)$). As those quantities are uniformly bounded over \mathcal{U} , the constant C_0 can be chosen independent of $i' \in \mathcal{U}$.

Consider the matrices $\tilde{R}_j(X', h)$, obtained by replacing the j th column of X' by $r_j(h)$ (similarly defined as $r_j(h) = p_j(z_0 + h) - x_j - h$). Let $\delta_0 > 0$ be such that the p_j s are defined and ℓ -Lipschitz continuous on $B(z_0, 2\delta_0)$. For $\|h\| \leq \delta_0$, define

$$F(X', h) = \sum_{i,j=1}^s \frac{1}{\|h\|^2} \det(\tilde{R}_j(X', h)^\top \tilde{R}_i(X', h))$$

and $F(X') = \inf_{h \in E^\perp, \|h\| \leq \delta_0} F(X', h)$. As $\|r_j(h)\|/\|h\|$ is bounded by $\ell+1$ for $h \in E^\perp$ with $\|h\| \leq \delta_0$, the family of functions $X' \mapsto F(X', h)$ indexed by $h \in E^\perp$ with $\|h\| \leq \delta_0$ is equicontinuous. This implies in particular that the function $X' \mapsto F(X')$ is continuous. Let $X = [x_1 - z_0, \dots, x_s - z_0]$. As the submanifold M satisfies Condition (P4) (which is equivalent to Condition (BSP)), Lemma 6.7 ensures that $F(X) > C_1$ for δ_0 small enough and for some constant $C_1 > 0$. By continuity, as long as z'_0 and the projections x'_j are close enough to z_0 and to the projections x_j respectively, then

$$\sum_{i,j=1}^s \det(\tilde{R}_j(X', h)^\top \tilde{R}_i(X', h)) \geq C_1 \|h\|^2$$

for $h \in E^\perp$ with $\|h\| \leq \delta_0$.

Let $h' \in E^\perp$ and let $h = \pi_{E^\perp}(h')$. For any $\beta > 0$, we can ensure that $\|h' - h\| \leq \beta \|h'\|$ for any such h' by making the x'_j s close enough to the x_j s. Then

$$\begin{aligned} \|r_j(h) - r'_j(h')\| &\leq \beta \|h'\| + \|p'_j(z'_0 + h') - x'_j - (p_j(z_0 + h) - x_j)\| \\ &\leq \beta \|h'\| + \left\| \int_0^1 (d(p'_j)_{z'_0+th'}(h') - d(p_j)_{z_0+th}(h)) dt \right\| \end{aligned}$$

Note that $p'_j(z'_0 + th')$ and $p_j(z_0 + th)$ belong respectively to $i'(U_M)$ and $i(U_M)$ for some open set U_M of M that can be made arbitrarily small (by making \mathcal{U} smaller). We know from Proposition 4.2 that the differential $d(p_j)_z$ of the projection can be expressed in terms of the orthogonal projection onto $T_{p_j(z)}M$ and of the shape operator of M at $p_j(z)$. As a result, for any constant $\gamma > 0$, if $\|i' - i\|_{C^2}$, $\delta' > 0$ and the set U_M are small enough, then for any $z, z' \in B(z_0, \delta')$ the projections $p'_j(z')$ and $p_j(z)$ are the images by i and i' of points $m, m' \in U_M$ that are sufficiently close for us to have $\left\| d(p'_j)_{z'} - d(p_j)_z \right\|_{\text{op}} \leq \gamma$. Assuming that this is the case,

$$\begin{aligned} \|r_j(h) - r'_j(h')\| &\leq \beta \|h'\| + \left\| \int_0^1 d(p_j)_{z_0+th}(h - h') dt \right\| + \left\| \int_0^1 (d(p'_j)_{z'_0+th'} - d(p_j)_{z_0+th})(h') dt \right\| \\ &\leq \beta \|h'\| + \ell \beta \|h'\| + \gamma \|h'\| = (\beta(\ell + 1) + \gamma) \|h'\|. \end{aligned}$$

As the determinant is locally Lipschitz continuous, it holds that for some constant $C_2 > 0$ and under the same hypotheses

$$\begin{aligned} & |\det(R'_j(h')^\top R'_i(h')) - \det(\tilde{R}_j(X', h)^\top \tilde{R}_i(X', h))| \\ & \leq C_2(\|r'_j(h') - r_j(h)\| \|r_i(h)\| + \|r'_i(h') - r_i(h)\| \|r'_j(h')\|) \\ & \leq 2C_2\ell(\beta(\ell + 1) + \gamma)\|h'\|^2. \end{aligned}$$

Hence for any $h' \in E'^\perp$ with $\|h'\| \leq \delta_0$ and any $i' \in \mathcal{U}$,

$$\begin{aligned} \text{Vol}_s(\Delta'(h'))^2 & \geq \frac{C_1}{(s!)^2}\|h\|^2 - \frac{1}{(s!)^2} \sum_{i,j=1}^s 2C_2\ell(\beta(\ell + 1) + \gamma)\|h'\|^2 - C_0\|h'\|^3 \\ & \geq \frac{C_1}{2(s!)^2}\|h'\|^2 \end{aligned}$$

if we choose \mathcal{U} small enough so that β and γ are small enough. We can then make \mathcal{U} even smaller so that the above considerations hold simultaneously for all the finitely many critical points $z_0 \in Z(\mathbf{M})$. \square

Claim 8. *We can make \mathcal{U} small enough that the associated map $\phi : Z(\mathbf{M}') \rightarrow Z(\mathbf{M})$ is injective. In particular, \mathbf{M}' satisfies (P2).*

Proof. Interestingly, the proof of this claim once again makes use of Condition (P4) (see also Example 7.2 regarding the importance of the second part of Condition (P1)). Let z and z' be critical points in $Z(\mathbf{M}')$ with $\phi(z) = \phi(z') = z_0$. Recall from Claim 1 that we can make $\|z - z_0\|$ and $\|z' - z_0\|$ arbitrarily small if needed. Write $\pi_{\mathbf{M}'}(z') = \{x'_1, \dots, x'_s\}$ and let $E' = \text{Vec}(\{x'_j - x'_k : j, k \in [s]\})$. According to Claim 6, whenever $\|z - z'\| \leq \|z - z_0\| + \|z' - z_0\| < \delta_1$, as z is a critical point of \mathbf{M}' , we have

$$\|\pi_{E'}(z - z')\| \leq C\|z - z'\|^2.$$

Let p'_j be the local projections defined on $B(z', \delta')$, for some δ' given by Claim 3. Note that $z \in B(z', \delta')$ when $\|z - z'\|$ is made small enough by taking \mathcal{U} small enough. In particular, we have $\pi_{\mathbf{M}'}(z) = \{p'_1(z), \dots, p'_s(z)\}$ thanks to Claims 3 and 4. Let $h = \pi_{E'^\perp}(z - z')$ and let $\tilde{\Delta}'(h)$ be the simplex with vertices $p'_1(z' + h), \dots, p'_s(z' + h)$. Let $\Delta'(h)$ be the simplex obtained by adding $z' + h$ to the vertices of $\tilde{\Delta}'(h)$. As \mathbf{M}' satisfies (BSP), it also holds that if $\|z - z'\|$ is small enough, then

$$\text{Vol}_s(\Delta'(h)) \geq L\|h\|,$$

where we use the uniformity with respect to $i' \in \mathcal{U}$ of the constants from Claim 7. Let $m'(h)$ be the center of the smallest enclosing ball of the vertices $p'_1(z' + h), \dots, p'_s(z' + h)$. Remark that, as $g'_{[s]}(z) = 0$ and $g'_{[s]}$ is ℓ -Lipschitz continuous for a constant ℓ that is uniform over all $i' \in \mathcal{U}$,

$$\begin{aligned} \text{Vol}_s(\Delta'(h)) & \leq \frac{1}{s} \text{Vol}_{s-1}(\tilde{\Delta}'(h)) \|z' + h - m'(h)\| \leq \frac{1}{s} (\text{diam}(\mathbf{M}'))^{s-1} d_{\mathbf{M}'}(z' + h) g'_{[s]}(z' + h) \\ & \leq \frac{2}{s} (\text{diam}(\mathbf{M}'))^s (g'_{[s]}(z' + h) - g'_{[s]}(z)) \\ & \leq \frac{2}{s} (\text{diam}(\mathbf{M}'))^s \ell \|\pi_{E'}(z - z')\| \\ & \leq \frac{2}{s} (\text{diam}(\mathbf{M}'))^s \ell C \|z - z'\|^2. \end{aligned}$$

As $\|h\| \geq \|z - z'\|/2$ as long as $\|z - z'\|$ is small enough, we obtain that

$$\frac{L}{2}\|z - z'\| \leq \frac{2}{s}(\text{diam}(\mathbf{M}'))^s \ell C \|z - z'\|^2.$$

When $\|z - z'\|$ is small enough, this is only possible when $z = z'$. We make \mathcal{U} small enough so that this argument applies around all critical points $z_0 \in Z(\mathbf{M})$ to conclude regarding the injectivity of ϕ . \square

The last point of the proof of Theorem 1.4 is that the map $\phi : Z(\mathbf{M}') \rightarrow Z(\mathbf{M})$ can also be made surjective. The map Ψ from the statement of the theorem is then defined as ϕ^{-1} .

Claim 9. *We can make \mathcal{U} small enough that for any $i' \in \mathcal{U}$, the associated map $\phi : Z(\mathbf{M}') \rightarrow Z(\mathbf{M})$ is surjective.*

Proof. Let $z_0 \in Z(\mathbf{M})$. We first show that z_0 is a topological critical point of the distance function $d_{\mathbf{M}}$ in the sense of topological Morse theory. To do so, we follow closely the proof outlined by Song & al. in [SYM23] (which is only stated for hypersurfaces). Their arguments rely on the Morse theory for Min-type functions developed by V. Gershkovich and H. Rubinstein [GR97]; we refer to [SYM23] for a succinct summary. We have already established that for $\delta > 0$ small enough, the distance function $d_{\mathbf{M}}$ restricted to $B(z_0, \delta)$ can be expressed as the minimum of the C^2 functions $d_{\mathbf{M}_j}$ for $j \in [s]$. Such a function is called of C^2 Min-type at the point z_0 , while the set of functions $\{d_{\mathbf{M}_j} : j \in [s]\}$ is called a representation of the function $d_{\mathbf{M}}$. This representation is efficient, in the sense that no representation using a smaller number of functions can be found, see [SYM23, Proof of Lemma 8]. The fact that z_0 belongs to the convex set formed by the points x_1, \dots, x_s implies that z_0 is a Min-type critical point of $d_{\mathbf{M}}$ [SYM23, Definition 5].

Furthermore, let us remark that the function f given by the restriction of $d_{\mathbf{M}}$ to the core medial axis $M_c(z_0, \mathbf{M})$ described in Lemma 6.3 is C^2 (it is equal to the C^2 function $d_{\mathbf{M}_1}$ on the core medial axis). Besides, f has a nondegenerate Hessian at z_0 . Indeed, for $z \in M_c(z_0, \mathbf{M})$ and any $j \in [s]$, the gradient $\nabla f(z)$ is given by the projection on $T_z M_c(z_0, \mathbf{M})$ of $z - p_j(z)$, that is $\nabla f(z) = \frac{\pi_{T_z M_c(z_0, \mathbf{M})}(z - p_j(z))}{d_{\mathbf{M}}(z)}$. As the center $m_z = m(\{p_j(z) : j \in [s]\})$ of the smallest enclosing ball of the projections $p_1(z), \dots, p_s(z)$ is written as a convex combination of the projections, we also have

$$\nabla f(z) = \frac{\pi_{T_z M_c(z_0, \mathbf{M})}(z - m_z)}{d_{\mathbf{M}}(z)}.$$

But m_z is the orthogonal projection of z onto the affine space spanned by the points $p_j(z)$. As $M_c(z_0, \mathbf{M})$ is defined as the set of points equidistant to all the sets M_j s, the vector space spanned by the vectors $\{p_j(z) - p_1(z) : j \in [s]\}$ is actually the orthogonal of the tangent space $T_z M_c(z_0, \mathbf{M})$. Hence, $z - m_z \in T_z M_c(z_0, \mathbf{M})$, and

$$\nabla f(z) = \frac{z - m_z}{d_{\mathbf{M}}(z)}.$$

According to (P4), which is the key ingredient here, when z is close enough to z_0 , the norm of $\nabla f(z)$ satisfies an inequality of the type $\|\nabla f(z)\| \geq c\|z - z_0\|$ (as $\|\nabla f(z)\|$ is also the norm of the generalized gradient of the unrestricted distance function $d_{\mathbf{M}}$, which means that z is $\|\nabla f(z)\|$ -critical). In particular, the Hessian of f at z_0 is non degenerate.

In the language of Min-type Morse theory, the previous properties (together with (P1)) imply that z_0 is a nondegenerate critical point of d_M (see [SYM23, Definition 6]), and the same holds for all critical points in $Z(M)$. Let $t > 0$ be a critical value for the distance function d_M , and let $z_1, \dots, z_l \in Z(M)$ be all the critical points with associated critical value t . Then the Handle Attachment Lemma for topological Morse functions [SYM23, Theorem 5] states that for $\eta > 0$ small enough, the set $d_M^{-1}(-\infty, t + \eta]$ has the homotopy type of $d_M^{-1}(-\infty, t - \eta]$ with cells (i.e. closed balls) $e_{1,d_1}, \dots, e_{l,d_l}$ attached along their boundary, where $d_j \in \{1, \dots, D\}$ is the dimension of e_{j,d_j} . Conversely, the Isotopy Lemma for topological Morse functions [SYM23, Theorem 4] ensures that there is no change in homotopy type between $d_M^{-1}(-\infty, a]$ and $d_M^{-1}(-\infty, b]$ if d_M has no critical value in the interval $[a, b]$.

Consider now the persistence diagram $\text{dgm}(M)$ of the sublevel filtration of d_M [EH22]. Corollary 3 from [SYM23] ensures that to each non-zero coordinate (either birth or death) of a point in the persistence diagram $\text{dgm}(M)$ corresponds exactly one critical point of d_M . Conversely, let us show that each critical point corresponds to a non-zero coordinate of some point in the persistence diagram. Indeed, let t be as above some critical value with associated critical points z_1, \dots, z_l and let $\eta > 0$ be small enough that $d_M^{-1}(-\infty, t + \eta]$ has the homotopy type of

$$d_M^{-1}(-\infty, t - \eta] \cup (e_{1,d_1} \cup \dots \cup e_{l,d_l}),$$

where the cells $e_{1,d_1}, \dots, e_{l,d_l}$ are attached along their boundaries (and the homotopy equivalence is compatible with the inclusion of $d_M^{-1}(-\infty, t - \eta]$). Let us consider the Mayer–Vietoris exact sequence with coefficients \mathbb{Z}_2 associated to the pair $(d_M^{-1}(-\infty, t - \eta], (e_{1,d_1} \cup \dots \cup e_{l,d_l}))$. As $d_M^{-1}(-\infty, t - \eta] \cap (e_{1,d_1} \cup \dots \cup e_{l,d_l})$ is homeomorphic to $\bigsqcup_{j=1}^l \partial e_{j,d_j} = \bigsqcup_{j=1}^l S^{d_j-1}$, where S^d is the d -dimensional sphere, the homology group $H_k(d_M^{-1}(-\infty, t - \eta] \cap (e_{1,d_1} \cup \dots \cup e_{l,d_l}))$ is isomorphic to $(\mathbb{Z}_2)^{n_{k+1}}$ for $k \geq 1$, where n_k is the number of k -dimensional cells among $e_{1,d_1}, \dots, e_{l,d_l}$. Furthermore, we have $H_k(e_{1,d_1} \cup \dots \cup e_{l,d_l}) = 0$ for $k \geq 1$.

Let us write $A_- := d_M^{-1}(-\infty, t - \eta]$ and $A_+ := d_M^{-1}(-\infty, t + \eta]$ for brevity. Then, around $k \geq 1$, the sequence is isomorphic to

$$\dots \longrightarrow H_{k+1}(A_-) \xrightarrow{i_{k+1}} H_{k+1}(A_+) \longrightarrow \mathbb{Z}_2^{n_{k+1}} \longrightarrow H_k(A_-) \xrightarrow{i_k} H_k(A_+) \longrightarrow \dots$$

The dimension $D_{k+1,b}$ of the cokernel of $H_{k+1}(d_M^{-1}(-\infty, t - \eta]) \xrightarrow{i_{k+1}} H_{k+1}(d_M^{-1}(-\infty, t + \eta])$ is precisely the number of births of intervals between $t - \eta$ and $t + \eta$ (hence precisely at t) in the persistence module of the filtration. Similarly, the dimension $D_{k,d}$ of the kernel of $H_k(d_M^{-1}(-\infty, t - \eta]) \xrightarrow{i_k} H_k(d_M^{-1}(-\infty, t + \eta])$ is the number of deaths of intervals at t in the persistence module of the filtration. Using the exactness of the sequence, one finds that $n_{k+1} = D_{k+1,b} + D_{k,d}$, meaning that each $k + 1$ -cell among $e_{1,d_1}, \dots, e_{l,d_l}$ corresponds exactly either to the birth of an interval for the homology of degree $k + 1$, or to the death of an interval for the homology of degree $k + 1$ (in particular, a k -cell and a $k + 1$ -cell cannot "compensate" each other). An almost identical reasoning applies for $k = 0$. Hence we have shown that the non-zero birth and death values of the intervals of the filtration are in bijection with the critical points of d_M .

Now let $\rho > 0$ and consider an embedding i' close enough to i that all the previous claims stand and such that $M' = i'(M)$ is ρ -close to M for the Hausdorff distance. This means that $\|d_{M'} - d_M\|_\infty \leq \rho$, and the bottleneck stability theorem [CSEH05] ensures that the two persistence diagrams $\text{dgm}(M)$ and $\text{dgm}(M')$ associated to the sublevel filtrations of $d_{M'}$ and d_M are ρ -close

(for the bottleneck distance). When ρ is small enough, this implies that the points of $\text{dgm}(M')$ have at least as many non-zero coordinates (counted with multiplicity if several points share a coordinate) as those of $\text{dgm}(M)$, hence that $d_{M'}$ has at least as many critical points as d_M . As $\phi : Z(M') \rightarrow Z(M)$ is injective, it must then be bijective, and in particular surjective. This proves the claim, and thence the theorem. \square

8 Density of Conditions (P1-4)

Let as usual M be a compact abstract manifold of regularity C^k with $k \geq 2$. In this section, we show that the set of C^k embeddings $i : M \rightarrow \mathbb{R}^D$ such that $i(M)$ satisfies geometric properties (P1-4) is dense in $\text{Emb}^k(M, \mathbb{R}^D)$ for the Whitney C^k -topology, which completes the proof of the Genericity Theorem 1.1.

It would not be too difficult to show that these geometric properties are “locally generic” using elementary methods. However, the difficulty resides in controlling $i(M)$ globally to make sure that situations contradicting one of the properties (P1-4) do not emerge anywhere.

To do so, we make repeated use of one of the many variants of R. Thom’s transversality theorem (see e.g. [GG74] for more on this theorem and its applications), following the examples of Yomdin [Yom81], Mather [Mat83] or Damon and Gasparovic [DG17] (among others) who applied similar methods.

Remember that if X, Y and $Z \subset Y$ are manifolds of regularity at least C^1 , a mapping $\phi : X \rightarrow Y$ is *transverse* to Z if for each $x \in \phi^{-1}(Z)$ we have $\text{Im}(d\phi_x) + T_{\phi(x)}Z = T_{\phi(x)}Y$. Informally, most versions of Thom’s theorem state that for any such X, Y, Z , the set of mappings $\phi : X \rightarrow Y$ such that ϕ is transverse to Z is dense (or sometimes residual, or even open and dense) in the set of such mappings; in other words, “most” mappings are transverse to Z . The exact statement depends on the exact set of mappings considered, the topology with which we equip it, and whether or not we consider *(multi)jets*, of which we now give a terse definition.

8.1 A brief introduction to jets and multijets

Jets and multijets are useful tools that help us study and control the values and derivatives of maps between manifolds; these, in turn, determine the geometric properties that are of interest to us when considering the special case of embeddings $i : M \rightarrow \mathbb{R}^D$. As we make extensive use of multijets in the next subsections, we choose to succinctly introduce them here for completeness; the reader already familiar with the definitions can safely skip this subsection. Our presentation mostly follows [GG74] (see also [KMS13] for more details).

Let X and Y be C^k manifolds (in the cases of interest to us, X will be our compact manifold M and Y will be the euclidean space \mathbb{R}^D), and let $x \in X$ and $y \in Y$. For $0 \leq r \leq k$, let us consider the set of mappings $C^r(X, Y)_{x,y} := \{f \in C^r(X, Y) : f(x) = y\}$ and let us define the following relation of equivalence on $C^r(X, Y)_{x,y}$:

Let U be a neighborhood of x in X and V be a neighborhood of y in Y , and let $\phi : U \rightarrow \mathbb{R}^{\dim X}, x \mapsto 0$ and $\psi : V \rightarrow \mathbb{R}^{\dim Y}, y \mapsto 0$ be charts of X and Y respectively. Then we say that $f, g \in C^r(X, Y)_{x,y}$ have r -th order contact at x , which we write as “ $f \sim_r g$ at x ”, if the differentials $d^l(\psi \circ f \circ \phi^{-1})_0$ and $d^l(\psi \circ g \circ \phi^{-1})_0$ in 0 of order $l = 0, \dots, r$ coincide.

It is easy to see that if the property is verified for a choice of charts ϕ, ψ around x and y , then it is verified for any such pair of charts; hence, the definition is coherent. Note that the equivalence

class of $f \in C^r(X, Y)_{x,y}$ depends only on the germ of f at x . The r -jets $J^r(X, Y)_{x,y}$ from x to y are defined as the set $C^r(X, Y)_{x,y}$ quotiented by the equivalence relation “ \sim_r at x ”, and the r -jets from X to Y are defined as

$$J^r(X, Y) := \bigcup_{x \in X, y \in Y} J^r(X, Y)_{x,y}.$$

We call x the *source* and y the *target* of an r -jet of $J^r(X, Y)_{x,y}$. Out of convenience, given an r -jet $a \in J^r(X, Y)$, we explicitly specify its source x and target y by writing it as

$$a = (x, y, [f]),$$

where $f \in C^r(X, Y)_{x,y}$ is any representative of the equivalence class a .

Let A_n^r be the vector space of polynomials in n variables of degree less than or equal to r with 0 as their constant term, and let $B_{n,m}^r := \bigoplus_{i=1}^m A_n^r$. It is a real vector space of dimension $m \dim_{\mathbb{R}} A_n^r = m \binom{n+r}{n} - 1$. Given an open set $U \subset \mathbb{R}^n$ and a C^r map $g : U \rightarrow \mathbb{R}$, define $T^r g : U \rightarrow A_n^r$ as the map that associates to $x \in U$ the terms of degree 1 to r of the Taylor series of g in x . Now let $U \subset \mathbb{R}^n$ be an open set, and consider a C^r map $f = (f_1, \dots, f_m) : U \rightarrow \mathbb{R}^m$. For any $x \in U$, the polynomials $T^r(f)(x) := (T^r f_1(x), \dots, T^r f_m(x)) \in B_{n,m}^r$ fully describe the first r derivatives $d^r f_x$ of f in x (excluding the “0-th derivative” $f(x)$). Conversely, for any $P \in B_{n,m}^r$ and any $x \in U$, there exists a C^r map $g : U \rightarrow \mathbb{R}^m$ such that $T^r(g)(x) = P$. In other words, there is a canonical bijection between $B_{n,m}^r$ and the possible values of the first r derivatives of a C^r map $\mathbb{R}^n \supset U \rightarrow \mathbb{R}^m$ at any point $x \in U$.

Given as before two C^k manifolds X and Y , this means that the space of r -jets $J^r(X, Y)$ (for $0 \leq r \leq k$) can naturally be seen as a C^{k-r} manifold of dimension $\dim X + \dim Y + \dim B_{\dim X, \dim Y}^r = \dim X + \dim Y + \binom{\dim X + r}{\dim X}$. Indeed, a pair of charts $\phi : U \rightarrow \mathbb{R}^{\dim X}$ and $\psi : V \rightarrow \mathbb{R}^{\dim Y}$ of X and Y induces a map

$$\begin{aligned} \xi_{\phi, \psi} : \{(x, y, [f]) \in J^r(X, Y) : x \in U, y \in V\} &\longrightarrow \phi(U) \times \psi(V) \times B_{\dim X, \dim Y}^r \\ a = (x, y, [f]) &\longmapsto (\phi(x), \psi(y), T^r(\psi \circ f \circ \phi^{-1})(\phi(x))). \end{aligned}$$

In other words, given the r -jet a , we simply consider the first r derivatives in the source x of a of one of its representatives f expressed in the system of coordinates induced by the charts ϕ and ψ . It is easy to show that the family of such maps $\{\xi_{\phi, \psi}\}$ to which an atlas $\{\phi\}$ of X and an atlas $\{\psi\}$ of Y give rise does yield an atlas of $J^r(X, Y)$. In what follows, we always consider $J^r(X, Y)$ with this manifold structure and with the associated topology. We will call charts of $J^r(X, Y)$ induced in such a way by charts of X and Y *adapted charts* (this is not standard terminology).

Any map $f \in C^k(X, Y)$ induces a canonically defined C^{k-r} map

$$\begin{aligned} j^r(f) : X &\rightarrow J^r(X, Y) \\ x &\longmapsto (x, f(x), [f]). \end{aligned}$$

called the r -jet of f , which essentially encodes the first r -derivatives of f in all $x \in X$. It can be shown that the sets

$$\{f \in C^k(X, Y) : j^r(f)(X) \subset U\},$$

where U is any open set of $J^r(X, Y)$, form a basis for the Whitney C^r -topology (again, see [GG74] for details) which we introduced in the special case where X is a compact manifold and $Y = \mathbb{R}^D$ in Section 2; in fact, this is how the topology is often defined.

The space of jets and the r -jet $j^r(f)$ associated to a mapping $f \in C^k(X, Y)$ can help us monitor the r first derivatives of f in each $x \in X$ separately, but the geometric properties that are of interest to us require that we control the image and derivatives of a mapping in several points simultaneously and with respect to each other; we want embeddings $i : M \rightarrow \mathbb{R}^D$ for which “there is no $m_1, \dots, m_s \in M$ such that $i(m_1), \dots, i(m_s)$ and the (higher) derivatives of i in m_1, \dots, m_s are in a certain configuration”. To that end, we need to consider *multijets*, defined as follows :

Consider the generalized diagonal

$$\Delta^{(s)} =: \{(x_1, \dots, x_s) \in X^s : \text{there exist } 1 \leq i < j \leq s \text{ s.t. } x_i = x_j\}$$

and define $X^{(s)} := X^s \setminus \Delta^{(s)}$, the product s times of X without repetitions. It is an open subset of X^s . Let $\alpha : J^r(X, Y) \rightarrow X$ be the source map, i.e. α maps $(x, y, [f])$ to its source x . One can consider the product map $\alpha^s = \alpha \times \dots \times \alpha : J^r(X, Y)^s \rightarrow X^s$. Then

$$J_s^r(X, Y) := (\alpha^s)^{-1} \left(X^{(s)} \right)$$

is the s -fold r -jet bundle from X to Y ; in other words, the space of s -tuples of r -jets with pairwise distinct sources. A *multijet bundle* is such a space for any $s \geq 2$. As α^s is continuous, the set of multijets $J_s^r(X, Y)$ is an open set of $J^r(X, Y)^s$ and inherits a C^{k-r} manifold structure as a result (where X and Y are as above C^k manifolds for some $k \geq r$). An obvious but important property of $J_s^r(X, Y)$ is that its set of source points $X^{(s)}$ is not compact, even if X is; this will be the cause of some distress later on. As for $J^r(X, Y)$, we call charts of $J_s^r(X, Y)$ induced by charts of X and Y *adapted charts*.

In line with our previous notations, we usually write an s -fold r -jet $a \in J_s^r(X, Y)$ as

$$a = (x_1, \dots, x_s, y_1, \dots, y_s, [f_1], \dots, [f_s]),$$

where $[f_i]$ is a class of equivalence for the relation “ \sim_r at x_i ”. Just as before, a map $f \in C^k(X, Y)$ induces a multijet C^{k-r} map

$$\begin{aligned} j_s^r(f) : X^{(s)} &\longrightarrow J_s^r(X, Y) \\ (x_1, \dots, x_s) &\longmapsto (x_1, \dots, x_s, f(x_1), \dots, f(x_s), [f], \dots, [f]) \end{aligned}$$

(which is simply the map $j^r(f) \times \dots \times j^r(f) : X^{(s)} \rightarrow J_s^r(X, Y)$ up to our choice of notations).

8.2 Two transversality theorems

We are now ready to quote the two statements that will be our main tools in the remainder of the section. The first one is a particularly complete variant of Mather’s adaptation of Thom’s transversality theorem to multijets (in [Mat70]), which we get from Damon’s work. Before that, a few definitions are required.

Remember that a closed subset F of a manifold X is *Whitney stratified* if it is a locally finite disjoint union of smooth submanifolds F_α called *strata*, such that if $F_\alpha \cap \overline{F_\beta} \neq \emptyset$ (where $\overline{F_\beta}$ is the closure of F_β in X), then $F_\alpha \subset \overline{F_\beta}$, and such that each pair of strata satisfies two technical conditions on their tangent spaces and secant lines, called *Whitney’s conditions a) and b)*. As such a stratification is usually not unique, one often says that F is *Whitney stratifiable* to say that it admits at least one such stratification. The dimension of a Whitney stratified set is the

maximum of the dimensions of its strata (it can be shown that it does not depend on the choice of stratification).

If $f : X \rightarrow Y$ is a map between two C^k manifolds and $F \subset X$, $G \subset Y$ are closed Whitney stratified sets with strata $\{F_\alpha\}$ and $\{G_\beta\}$, we say that f is *transverse to G on F* if for each pair of strata F_α and G_β , the map $f|_{F_\alpha} : F_\alpha \rightarrow Y$ is transverse to G_β in the usual sense.

In what follows, we (and the reader) will be shielded from the full complexity of Whitney stratifications; we only need to know that semialgebraic subsets of the Euclidean space (whose definition we give later in this paper) are Whitney stratifiable, as was shown by Thom in [Tho66]. To learn more about Whitney stratifications, read e.g. [Mat76] or [Gib76].

Finally, remember that a subset of a topological space B is *residual* (in B) if it is a countable intersection of subsets of B whose interiors are dense in B . If B is a *Baire space*, then its residual subsets are dense.

The theorem below is a special case of Corollary 1.11 from [Dam97].

Theorem 8.1 (Damon). *Let $s \geq 2$ and $r \in \mathbb{N}$ and let X, Y be C^k manifolds for $k \in \mathbb{N}_{\geq r+1} \cup \{\infty\}$. Assume that X is compact, and let $A \subset X^{(s)}$ and $W \subset J_s^r(X, Y)$ be closed Whitney stratified sets in their respective ambient space. Then*

$$F_{W,A} := \{f \in C^k(X, Y) : j_s^r(f) \text{ is transverse to } W \text{ on } A \text{ in } J_s^r(X, Y)\}$$

is a residual subset of $C^k(X, Y)$ for the Whitney C^k -topology. Furthermore, if A is compact, then $F_{W,A}$ is also open in the Whitney C^{r+1} -topology.

Remark 8.2. *Note that being dense in the Whitney C^k -topology implies being dense in the Whitney C^l -topology for all $l \leq k$, and being open in the Whitney C^{r+1} -topology implies being open in the Whitney C^l -topology for all $r+1 \leq l \leq k$.*

Proof. This statement is an altered, simplified and less general version of [Dam97, Corollary 1.11] for increased readability. Nonetheless, should a distrustful reader want to precisely compare the two to check that Theorem 8.1 is indeed a direct consequence of Damon’s theorem, let us give a few remarks (using his notations).

Firstly, the distinction that he makes between the weak C^{r+1} (which he calls “regular”) and the Whitney C^{r+1} topology is moot as they coincide in our case (because we assume X to be compact). We also let the set \mathcal{H} from Damon’s statement be the set of all mappings $C^k(X, Y)$ (and the map Φ is simply the identity). As observed in the *Note* below Damon’s definition 1.8 in the same article, it is a standard fact that $C^k(X, Y)$ satisfies what he calls “having smooth image in $J_s^r(X, Y)$ off $\Delta^{(s)}$ ”; furthermore (still using his notations), ${}_s\mathcal{H}^{(r)} = {}_sC^k(X, Y)^{(r)}$ is simply equal to $\{j_s^r(f)(x_1, \dots, x_s) : f \in C^k(X, Y), (x_1, \dots, x_s) \in X^{(s)}\} = J_s^r(X, Y)$, hence the condition that “ Φ be transverse to W in $J_s^r(X, Y)$ ” is trivially satisfied.

Another difference is that Damon’s theorem is for C^∞ maps from X to Y , but as he points out in the remark before his Definition 1.1, the same statement works for C^k maps for $k \geq r+1$.

Finally, Damon’s original result only states that $F_{W,A}$ is dense in C^∞ for the Whitney C^{r+1} -topology (rather than for the Whitney C^k -topology). However, the proof of his theorem shows that we also have $F_{W,A}$ dense in the Whitney C^m -topology for any $m \in \mathbb{N}$ such that $r+1 \leq m \leq k$. This is our statement if k is finite, and if $k = \infty$, the set $F_{W,A}$ being dense in the Whitney C^m -topology for all $m \in \mathbb{N}$ implies that $F_{W,A}$ is dense in the Whitney C^∞ -topology by its definition. \square

We also need another transversality statement that specifically deals with the case of distance functions in Euclidean spaces. For M a manifold and $f \in C^k(M, \mathbb{R}^D)$, let us define

$$\begin{aligned} \rho_f : M \times \mathbb{R}^D &\longrightarrow \mathbb{R} \\ (m, z) &\longmapsto \|f(m) - z\|^2. \end{aligned}$$

If we see $z \in \mathbb{R}^D$ as a parameter and write $\rho_{f,z} := \rho_f(-, z) \in C^k(M, \mathbb{R}^D)$, we can consider the multijet map of ρ_f with respect to $m \in M$:

$$\begin{aligned} (j_s^r)_M \rho_f : M^{(s)} \times \mathbb{R}^D &\longrightarrow J_s^r(M, \mathbb{R}) \\ (m_1, \dots, m_s, z) &\longmapsto (j^r(\rho_{f,z})(m_1), \dots, j^r(\rho_{f,z})(m_s)). \end{aligned}$$

In what follows, we will write an s -fold r -jet $a \in J_s^r(X, Y)$ in the special case where $X = M$ is a compact manifold and $Y = \mathbb{R}$ as

$$a = (m_1, \dots, m_s, r_1, \dots, r_s, [f_1], \dots, [f_s]),$$

with $m_i \in M$ and $r_i \in \mathbb{R}$. We will say that a subset $F \subset J_s^r(M, \mathbb{R})$ is invariant under addition if it is invariant under the action of \mathbb{R} defined as

$$t + (m_1, \dots, m_s, r_1, \dots, r_s, [f_1], \dots, [f_s]) := (m_1, \dots, m_s, r_1 + t, \dots, r_s + t, [f_1 + t], \dots, [f_s + t])$$

for any $t \in \mathbb{R}$. Then the following statement is true, which is a variant of a theorem by Looijenga [Loo74]; this version in particular comes (modulo some alterations and simplifications) from Damon's and Gasparovic's [DG17].

Theorem 8.3 (Damon, Gasparovic). *Let $s \geq 2$ and $r \in \mathbb{N}$ and let M be a compact C^k manifold for $k \in \mathbb{N}_{\geq r+1} \cup \{\infty\}$. Let $A \subset M^{(s)} \times \mathbb{R}^D$ and $W \subset J_s^r(M, \mathbb{R}^D)$ be closed Whitney stratified sets in their respective ambient space. Furthermore, assume that each stratum of W is invariant under addition. Then*

$$D_{W,A} := \{f \in C^k(M, \mathbb{R}^D) : (j_s^r)_M \rho_f \text{ is transverse to } W \text{ on } A \text{ in } J_s^r(M, \mathbb{R})\}$$

is a residual subset of $C^k(M, \mathbb{R}^D)$ for the Whitney C^k -topology. Furthermore, if A is compact, then $D_{W,A}$ is also open in the Whitney C^{r+1} -topology.

Proof. As for Theorem 8.1, we briefly explain how the statement is a direct adaptation of Theorem 16.5 in [DG17].

The remarks we made in the proof of Theorem 8.1 regarding the weak and Whitney C^{r+1} topologies, the compactness of M , the topologies in which $F_{W,A}$ (here $D_{W,A}$) is dense and the distinctions between C^∞ and C^k maps remain valid.

Two new elements are to be commented. First, we let $E = \emptyset$ and $Y = M^{(s)}$ in Damon and Gasparovic's statement. Secondly, similarly to the "proof" of Theorem 8.1, we let \mathcal{H} be $C^k(M, \mathbb{R}^D)$. Then it remains to show that, in the words of Damon and Gasparovic, the map

$$\begin{aligned} \rho : C^k(M, \mathbb{R}^D) &\longrightarrow C^k(M \times \mathbb{R}^D, \mathbb{R}^D) \\ f &\longmapsto \rho_f : (m, z) \mapsto \|f(m) - z\|^2 \end{aligned}$$

(which corresponds to Ψ in their statement) is "transverse to W " (on X) (this is defined in Definition 16.4 of [DG17]) for any closed Whitney stratified set $W \subset J_s^r(M, \mathbb{R}^D)$ whose strata are invariant under addition. This is shown in the same way as in Looijenga's proof of the original statement (in [Loo74]; one can also read C. Wall's account [Wal77], which is easier to find online). \square

8.3 Geometric properties (P1-4) as transversality conditions

Our plan in this subsection is to translate the geometric properties (P1-4) that we introduced earlier in terms of transversality of the multijet map $j_s^r(i)$ of the embedding $i : M \rightarrow \mathbb{R}^D$ to some well-chosen set $W \subset J_s^r(M, \mathbb{R}^D)$ (respectively the transversality of the multijet map $(j_s^r)_{M\rho_i}$ to $W \subset J_s^r(M, \mathbb{R})$); later, in Subsection 8.4, we use Theorems 8.1 and 8.3 to show that the transverse embeddings are a dense set among all embeddings. There will be two subcases: either W represents configurations to be absolutely avoided and the intersection of the image of $j_s^r(i)$ (respectively $(j_s^r)_{M\rho_i}$) with W will be empty due to a dimension argument, or the intersection will be non-empty but the transversality will constrain the geometry of the configurations in the intersection to be well-behaved. As in the previous subsections, we write $a \in J_s^r(M, \mathbb{R})$ as $a = (m_1, \dots, m_s, r_1, \dots, r_s, [f_1], \dots, [f_s])$ and $b \in J_s^r(M, \mathbb{R}^D)$ as $b = (m_1, \dots, m_s, x_1, \dots, x_s, [g_1], \dots, [g_s])$ with $m_i \in M$, $r_i \in \mathbb{R}$ and $x_i \in \mathbb{R}^D$. As usual, we let M be a compact C^k manifold of dimension m , but unlike in the remainder of this article, we have to assume in this subsection that $k \geq 3$ rather than $k \geq 2$. The reason for this is that if $i \in \text{Emb}^k(M, \mathbb{R}^D)$, then $j_s^r(i)$ is of regularity C^{k-r} , hence the map $j_s^r(i)$ being transverse to some set $W \subset J_s^r(M, \mathbb{R}^D)$ only makes sense if $r + 1 \leq k$. In what follows, some of the sets $W \subset J_s^r(M, \mathbb{R}^D)$ which we will consider can only be defined if $r \geq 2$, hence we need $k \geq r + 1 = 3$ to ask that $j_s^r(i)$ be transverse to them. We will see in the next subsection that this subtlety matters not, and that the density part of Theorem 1.1 remains true when M is only C^2 (and in fact even only C^1 , as suggested by Remark 1.3).

8.3.1 Condition (P1), part 1

The definition of the first set W and its properties come from Yomdin's [Yom81]. It allows us to handle a weaker version of Condition (P1) that we call (P0). We take care of the second part of (P1) later in Subsubsection 8.3.3.

Definition 8.4. For any $s, r \in \mathbb{N}$ with $s \geq 2$ and $r \leq k - 1$, we define the following set:

$$W_0^{(s)} := \{a \in J_s^r(M, \mathbb{R}) : r_1 = \dots = r_s\}.$$

Remark 8.5. $W_0^{(s)}$ is clearly a submanifold of $J_s^r(M, \mathbb{R})$ of codimension $s-1$ (hence in particular a Whitney stratified set) and invariant under addition.

Lemma 8.6 (Yomdin, [Yom81]). For any $0 \leq r \leq k - 1$, if $i \in \text{Emb}^k(M, \mathbb{R}^D)$ is such that $(j_s^r)_{M\rho_i} : M^{(s)} \times \mathbb{R}^D \rightarrow J_s^r(M, \mathbb{R})$ is transverse to $W_0^{(s)}$ for $s = 2, \dots, D + 2$, then $\mathbf{M} = i(M)$ satisfies condition

(P0) For any critical point $z \in Z(\mathbf{M})$, the projections $\pi_{\mathbf{M}}(z)$ are the vertices of a non-degenerate simplex of \mathbb{R}^D . In particular, z has at most $D + 1$ projections.

Proof. This is proved by Yomdin in [Yom81]. In fact, he proves it for the points z in the cut locus of $i(M)$, which is in general larger than its medial axis. \square

Remark 8.7. It is very easy to find manifolds that do not verify condition (P0); e.g. the center z of a (Euclidean) sphere S of dimension greater than 1 is such that $\pi_S(z)$ has infinitely many points. Examples where the points of $\pi_{i(M)}(z)$ are in finite numbers but are not affinely independent are equally easy to construct.

The rest of the subsection is modelled after these first definition, remark and lemma: we will define sets W , show that they are Whitney stratified and of the correct dimension, and show that embeddings that are transverse to them have nice geometric properties (proving that such embeddings are dense will come later).

8.3.2 Conditions (P2) and (P3)

Definition 8.8. For any $s, r \in \mathbb{N}$ with $s \geq 2$ and $1 \leq r \leq k$, we let $W_2^{(s)}$ be the set of multijets $(m_1, \dots, m_s, x_1, \dots, x_s, [f_1], \dots, [f_s]) \in J_s^r(M, \mathbb{R}^D)$ such that there exists $z \in \text{Conv}(x_1, \dots, x_s)$ with $d(x_1, z) = \dots = d(x_s, z)$ and such that for all $i \in [s]$, $\text{Im}((df_i)_{m_i} : T_{m_i}M \rightarrow T_{x_i}\mathbb{R}^D) \perp (z - x_i)$.

Remark 8.9. As for $(m_1, \dots, m_s, x_1, \dots, x_s, [f_1], \dots, [f_s]) \in J_s^r(M, \mathbb{R}^D)$ the image of $(df_i)_{m_i}$ in $T_{x_i}\mathbb{R}^D$ does not depend on the choice of representative f_i for $[f_i] \subset C^k(M, \mathbb{R}^D)_{m_i, x_i}$, the set $W_2^{(s)}$ is well-defined.

The definition of $W_2^{(s)}$ should be understood as follows: let $i : M \rightarrow \mathbb{R}^D$ be an embedding. If z_0 is a critical point of $i(M)$ with $\pi_{i(M)}(z_0) = \{x_1, \dots, x_s\}$, then

$$(m_1, \dots, m_s, x_1, \dots, x_s, [i], \dots, [i]) \in W_2^{(s)} \cap \text{Im}(j_s^r(i) : M^{(s)} \rightarrow J_s^r(M, \mathbb{R}^D))$$

for some $m_1, \dots, m_s \in M$ with z_0 satisfying the constraints for $(m_1, \dots, m_s, x_1, \dots, x_s, [i], \dots, [i])$ in the definition of $W_2^{(s)}$. This is proved further below with Lemma 8.15. Note that the set of $W_2^{(s)}$ also depends on r , though we hide that dependency in our notations.

Lemma 8.10. For any $s, r \in \mathbb{N}$ with $2 \leq s \leq D + 1$ and $1 \leq r \leq k - 1$, the set $W_2^{(s)}$ is a finite union of closed Whitney stratifiable subsets $W_{2,\alpha}^{(s)}$ of $J_s^r(M, \mathbb{R}^D)$ of codimension at least sm .

To prove Lemma 8.10, rather than to identify possible strata and to manually show that they verify Whitney's conditions, we will use the fact that semialgebraic subsets of an Euclidean space are Whitney stratifiable (see [Tho66]). Remember that a set $A \subset \mathbb{R}^n$ is called *semialgebraic set* if it can be written as

$$A = \bigcup_{i=1}^l \bigcap_{j=1}^{k_i} \{x_1, \dots, x_n \in \mathbb{R}^n : P_{i,j}(x_1, \dots, x_n) \text{ op}_{i,j} 0\},$$

where $P_{i,j}$ is a polynomial in n variables and $\text{op}_{i,j} \in \{<, =, >\}$ for all $i = 1, \dots, l$ and $j = 1, \dots, k_i$. This definition is deceptively simple, as there are some semialgebraic sets that one would not naively expect to be. To learn more on semialgebraic sets, read e.g. [BR90].

Proof. We fix $s \in \{2, \dots, D + 1\}$ in the proof, and write $W_2 = W_2^{(s)}$ to alleviate notation. First of all, let us define the set

$$\begin{aligned} \tilde{W}_2 := & \{(m_1, \dots, m_s, x_1, \dots, x_s, [f_1], \dots, [f_s]) \in J^r(M, \mathbb{R}^D)^s : \exists z \in \mathbb{R}^D \text{ s.t.} \\ & \|x_1 - z\|^2 = \dots = \|x_s - z\|^2, \forall i \in [s], \text{Im}((df_i)_{m_i} : T_{m_i}M \rightarrow T_{x_i}\mathbb{R}^D) \perp (z - x_i) \\ & \text{and } \exists \lambda_1, \dots, \lambda_s \in [0, 1] \text{ s.t. } \sum_{i=1}^s \lambda_i = 1 \text{ and } z = \sum_{i=1}^s \lambda_i x_i\}. \end{aligned}$$

(26)

There are two differences between W_2 and \tilde{W}_2 : the first is that it's defined as a subset of $J^r(M, \mathbb{R}^D)^s$ rather than of $J_s^r(M, \mathbb{R}^D) \subset J^r(M, \mathbb{R}^D)^s$, the second is that the constraints in its definitions are the same, but have been rewritten in a more visibly algebraic fashion. As a result, we have that $W_2 = \tilde{W}_2 \cap J_s^r(M, \mathbb{R}^D)$. We will use \tilde{W}_2 to circumvent the lack of compactness of $M^{(s)}$.

Closedness. Let us first show that \tilde{W}_2 is closed. Let $(a_n)_{n \in \mathbb{N}} \subset \tilde{W}_2$ be such that $\lim_{n \rightarrow \infty} a_n = a \in J^r(M, \mathbb{R}^D)^s$, and let us write $a_n = (m_{1,n}, \dots, m_{s,n}, x_{1,n}, \dots, x_{s,n}, [f_{1,n}], \dots, [f_{s,n}])$. We want to show that $a \in \tilde{W}_2$.

For each $n \in \mathbb{N}$, let $z_n \in \mathbb{R}^D$ be a point such that $d(x_{1,n}, z_n) = \dots = d(x_{s,n}, z_n)$, $\text{Im}((df_{i,n})_{m_{i,n}} : T_{m_{i,n}}M \rightarrow T_{x_{i,n}}\mathbb{R}^D) \perp (z_n - x_{i,n})$ for $i \in [s]$ and $z_n \in \text{Conv}(x_{1,n}, \dots, x_{s,n})$ (in fact, such a point z_n is unique). Write $z_n = \sum_{i=1}^s \lambda_{i,n} x_{i,n}$ for some $\lambda_{i,n} \in [0, 1]$.

We can find a subsequence for which $(\lambda_{1,n}, \dots, \lambda_{s,n})$ converges to $(\lambda_1, \dots, \lambda_s)$, hence for which z_n converges to $z := \sum_{i=1}^s \lambda_i x_i \in \mathbb{R}^D$. Then by definition $z \in \text{Conv}(x_1, \dots, x_s)$ and by continuity $d(x_1, z) = \dots = d(x_s, z)$. Furthermore, choose charts of M around m_1, \dots, m_s , and consider the induced adapted chart of $J^r(M, \mathbb{R}^D)^s$ around a . In the coordinates given by this chart, the derivative $(df_{i,n})_{m_{i,n}} : T_{m_{i,n}}M \rightarrow T_{x_{i,n}}\mathbb{R}^D$ (for each $i = 1, \dots, s$) can be represented by D degree 1 homogeneous polynomials in m variables, or equivalently by a $D \times m$ matrix $A_{i,n}$. The condition that $\text{Im}((df_{i,n})_{m_{i,n}}) \perp (z_n - x_{i,n})$ can be written as $(A_{i,n})^\top (z_n - x_{i,n}) = 0$ (as soon as n is large enough for $m_{i,n}$ to belong to the domain of the chart around m_i), which is a closed constraint, hence $\text{Im}((df_i)_{m_i} : T_{m_i}M \rightarrow T_{x_i}\mathbb{R}^D) \perp (z - x_i)$ and z satisfies all three conditions in the definition of \tilde{W}_2 . This shows that $a \in \tilde{W}_2$, and thus that \tilde{W}_2 is closed in $J^r(M, \mathbb{R}^D)^s$. As $W_2 = \tilde{W}_2 \cap J_s^r(M, \mathbb{R}^D)$, we get that W_2 is also closed in $J_s^r(M, \mathbb{R}^D)$.

Finite union of closed Whitney stratified sets. A preliminary remark: as was mentioned earlier, it was shown by Thom in [Tho66] that semialgebraic subsets of Euclidean space are Whitney stratifiable. As \tilde{W}_2 is semialgebraic in the coordinates of any adapted chart of $J^r(M, \mathbb{R}^D)^s$ (see below), its intersection with the domain of any such chart is Whitney stratifiable. We are somewhat convinced that this should easily imply that \tilde{W}_2 is globally Whitney stratifiable (on the other hand, we cannot in general expect \tilde{W}_2 to be globally semialgebraic), but we could not find any such statement in the literature. Consequently, we have to consider each chart separately,⁶ as follows.

For any $(m_1, \dots, m_s, x_1, \dots, x_s, [f_1], \dots, [f_s]) \in J^r(M, \mathbb{R}^D)^s$, we can find charts $\phi_i : U_i \rightarrow \mathbb{R}^m$ of M with domains $U_i \ni m_i$ for $i = 1, \dots, s$, and from them get an adapted chart

$$\phi : U \longrightarrow \phi_1(U_1) \times \dots \times \phi_s(U_s) \times (\mathbb{R}^D)^s \times (B_{m,D}^r)^s$$

with domain $U := \{(m_1, \dots, m_s, x_1, \dots, x_s, [f_1], \dots, [f_s]) \in J^r(M, \mathbb{R}^D)^s : m_i \in U_i \text{ for } i \in [s]\}$ (we do not need charts around the targets $x_i \in \mathbb{R}^D$, as \mathbb{R}^D is already an Euclidean space). For any such chart U , let us choose $\varepsilon_U > 0$ small enough that for $i \in [s]$ we have

$$\overline{B}(\phi_i(m_i), \varepsilon_U) \subset \phi_i(U_i) \subset \mathbb{R}^m.$$

Using the compactness of M^s , we can find a finite collection of such charts $(\phi_\alpha, U_\alpha)_{\alpha \in A}$ (where

⁶In fact, we think that a variant of the argument exposed below could be used to show that \tilde{W}_2 as a whole is Whitney stratifiable, though it is not required for our purposes.

A is a finite collection of indices) such that

$$\bigcup_{\alpha \in A} \phi_{1,\alpha}^{-1}(\overline{B}(\phi_{1,\alpha}(m_{1,\alpha}), \varepsilon_{U_\alpha})) \times \dots \times \phi_{s,\alpha}^{-1}(\overline{B}(\phi_{s,\alpha}(m_{s,\alpha}), \varepsilon_{U_\alpha})) = M^s,$$

which is equivalent to the charts $(\phi_\alpha, U_\alpha)_{\alpha \in A}$ corestricted to

$$\phi_{1,\alpha}^{-1}(\overline{B}(\phi_{1,\alpha}(m_{1,\alpha}), \varepsilon_{U_\alpha})) \times \dots \times \phi_{s,\alpha}^{-1}(\overline{B}(\phi_{s,\alpha}(m_{s,\alpha}), \varepsilon_{U_\alpha})) \times (\mathbb{R}^D)^s \times (B_{m,D}^r)^s$$

covering $J^r(M, \mathbb{R}^D)^s$.

In the coordinates of any such chart, the set \tilde{W}_2 (intersected with the domain of the chart) is semialgebraic. Indeed, a consequence of the Tarski-Seidenberg theorem (see e.g. [BR90, Theorem 2.3.4]) is that any set defined as

$$\{X \in \mathbb{R}^p : \exists Z \in \mathbb{R}^m \text{ s.t. } (P_1(X, Z) \text{ op}_1 0) \wedge \dots \wedge (P_q(X, Z) \text{ op}_q 0)\}$$

is semialgebraic, where P_i is a polynomial and $\text{op}_i \in \{>, \geq, =\}$ for $i = 1, \dots, q$ and \wedge is the AND logical operator (in fact, more general formula featuring OR operators, negations and parentheses are allowed, but this is enough for our purposes). In the coordinates of a chart (ϕ_α, U_α) , the set \tilde{W}_2 is of this form. Indeed, its definition (see (26)) is a combination of:

- A \exists operation for $z, \lambda_1, \dots, \lambda_s$.
- Algebraic constraints on z and the targets x_i ,
- A constraint that the image of the derivatives $(df_i)_{m_i}$ must be perpendicular to $z - x_i$. As noted earlier, this is equivalent to the transpose $(A_i)^\top$ of the matrix representing $(df_i)_{m_i}$ in the coordinates of the charts verifying $(A_i)^\top(z - x_i) = 0$, which is an algebraic constraint on the coordinates representing $(df_i)_{m_i}$, z and x_i .
- Some more algebraic constraints on the λ_i, x_i and z .

Furthermore, the closed ball $\overline{B}(\phi_{s,\alpha}(m_{s,\alpha}), \varepsilon_{U_\alpha})$ is also semialgebraic in the coordinate of the chart (ϕ_α, U_α) , as it is defined using a polynomial inequality. Consequently, the intersection

$$\tilde{W}_{2,\alpha} := \tilde{W}_2 \cap (\phi_{1,\alpha}^{-1}(\overline{B}(\phi_{1,\alpha}(m_{1,\alpha}), \varepsilon_{U_\alpha})) \times \dots \times \phi_{s,\alpha}^{-1}(\overline{B}(\phi_{s,\alpha}(m_{s,\alpha}), \varepsilon_{U_\alpha})) \times (\mathbb{R}^D)^s \times (B_{m,D}^r)^s)$$

is semialgebraic in the coordinates of (ϕ_α, U_α) , and as a result it is Whitney stratifiable in $J^r(M, \mathbb{R}^D)^s$ (the image of a Whitney stratifiable set by a diffeomorphism is also Whitney stratifiable). It is also closed.

As Whitney's conditions a) and b) are purely local, it is trivial to check that the intersection of a Whitney stratified set with an open set of its ambient manifold is still Whitney stratified. As $J_s^r(M, \mathbb{R}^D)$ is an open set of $J^r(M, \mathbb{R}^D)^s$, the intersection

$$W_{2,\alpha} := \tilde{W}_{2,\alpha} \cap J_s^r(M, \mathbb{R}^D)$$

is therefore still Whitney stratifiable (and closed) in $J_s^r(M, \mathbb{R}^D)$. We also have

$$W_2 = \tilde{W}_2 \cap J_s^r(M, \mathbb{R}^D) = \left(\bigcup_{\alpha \in A} \tilde{W}_{2,\alpha} \right) \cap J_s^r(M, \mathbb{R}^D) = \bigcup_{\alpha \in A} W_{2,\alpha}.$$

Dimension. We only have to show that the Whitney stratifiable sets $W_{2,\alpha}$ are of codimension sm in $J_s^r(M, \mathbb{R}^D)$ to conclude. To do so, consider the following auxiliary set:

$$\begin{aligned}
W_2' := & \{((m_1, \dots, m_s, x_1, \dots, x_s, [f_1], \dots, [f_s]), z) \in J^r(M, \mathbb{R}^D)^s \times \mathbb{R}^D : \\
& \|x_1 - z\|^2 = \dots = \|x_s - z\|^2, \forall i \in [s], \text{Im}((df_i)_{m_i} : T_{m_i}M \rightarrow T_{x_i}\mathbb{R}^D) \perp (z - x_i), \\
& \text{rank}((df_i)_{m_i}) = m, \exists \lambda_1, \dots, \lambda_s \in]0, 1[\text{ s.t. } \sum_{i=1}^s \lambda_i = 1, z = \sum_{i=1}^s \lambda_i x_i \\
& \text{and } \dim \text{Aff}(x_1, \dots, x_s) = s - 1\}.
\end{aligned} \tag{27}$$

Note the four differences compared to \tilde{W}_2 : the point z is among the coordinates rather than the object of a \exists statement, we ask that the points x_i be affinely independent and that the derivatives $(df_i)_{m_i}$ be of full rank, and the coefficients λ_i cannot be trivial. We claim that W_2' is a (non-closed) submanifold of $J^r(M, \mathbb{R}^D)^s \times \mathbb{R}^D$ of codimension $D + sm$, hence of dimension $\dim J^r(M, \mathbb{R}^D)^s - sm$.

Assume for now that the claim is true, and consider the projection

$$\pi : J^r(M, \mathbb{R}^D)^s \times \mathbb{R}^D \rightarrow J^r(M, \mathbb{R}^D)^s.$$

Restrict it to $U_\alpha \times \mathbb{R}^D$, where U_α is the domain of one of the charts defined in the previous section of the proof. In the coordinates induced by this chart, the set W_2' is semialgebraic⁷ (for the same reasons as \tilde{W}_2), and the map π is semialgebraic in the coordinates, as it is simply a projection from one vector space to another (see e.g. [BR90, Section 2.3]). Thus $\pi(W_2')$ is also semialgebraic in the coordinates of the chart,⁸ and so is its closure $\overline{\pi(W_2')}$ (see e.g. [Cos02, Proposition 1.12]). Furthermore, it is easy to see that the closure $\overline{\pi(W_2')}$ of $\pi(W_2')$ in $J^r(M, \mathbb{R}^D)^s$ is equal to \tilde{W}_2 . Hence (see [BR90, 2.5.4])

$$\dim \tilde{W}_2 = \dim(\overline{\pi(W_2')}) = \dim(\pi(W_2')) \leq \dim W_2' = \dim J^r(M, \mathbb{R}^D)^s - sm,$$

as the same inequalities are true when considering the intersections of those sets with the domain U_α (respectively $U_\alpha \times \mathbb{R}^D$) of any of the charts $(\phi_\alpha, U_\alpha)_\alpha$. As W_2 is the intersection of \tilde{W}_2 with the open set $J_s^r(M, \mathbb{R}^D)$ of $J^r(M, \mathbb{R}^D)^s$, we have $\dim W_2 \leq \dim \tilde{W}_2 \leq \dim J^r(M, \mathbb{R}^D)^s - sm = \dim J_s^r(M, \mathbb{R}^D) - sm$, and each of the closed subsets $W_{2,\alpha}$ of W_2 is of codimension at least sm in $J_s^r(M, \mathbb{R}^D)$.

The only thing left is to prove the claim that W_2' is a submanifold of $J^r(M, \mathbb{R}^D)^s \times \mathbb{R}^D$ of codimension $D + sm$. This is a local property; in what follows, we implicitly place ourselves in the coordinates of an adapted chart (ϕ, U) , without explicitly saying that we are considering the intersection of W_2' with $U \times \mathbb{R}^D$ or constantly specifying "in the coordinates of...".

Let (a, z) be a point of W_2' , where $a \in J^r(M, \mathbb{R}^D)^s$. The conditions that $\dim \text{Aff}(x_1, \dots, x_s) = s - 1$ and that $\text{rank}((df_i)_{m_i}) = m$ for $i = 1 \in [s]$ are open, and the conditions that $\|x_1 - z\|^2 = \dots = \|x_s - z\|^2$ and that there exist $\lambda_1, \dots, \lambda_s \in]0, 1[$ such that $\sum_{i=1}^s \lambda_i = 1$ and $z = \sum_{i=1}^s \lambda_i x_i$ mean that z must be the (unique) center of the circumsphere of x_1, \dots, x_s , and that this center

⁷In this paragraph, we are guilty of a small abuse of notations: when we say that W_2' is semialgebraic in the coordinates of the chart, we mean that its intersection with the domain of the chart is semialgebraic in the coordinates (and same for $\pi(W_2')$ and $\overline{\pi(W_2')}$).

⁸In fact, the projection π restricted to W_2' is a diffeomorphism onto its image, but it is not needed for the proof.

must belong to the relative interior of the convex hull of x_1, \dots, x_s , which is another open condition on x_1, \dots, x_s . Hence we can find a small neighborhood V of a such that these three open conditions are satisfied on the neighborhood $V \times \mathbb{R}^D$ of (a, z) . For any point $(b, z) = ((m_1, \dots, m_s, x_1, \dots, x_s, [f_1], \dots, [f_s]), z)$ in $V \times \mathbb{R}^D$, the point z must again be the center of the circumsphere of x_1, \dots, x_s , and for such a point $(b, z) \in V \times \mathbb{R}^D$, the coordinates of z are a (smooth) function $z(x_1, \dots, x_s)$ of the coordinates of x_1, \dots, x_s (see [Cox30]).

Identify the first order derivatives at m_i of the functions f_i with $D \times m$ matrices A_i and let $d^{\geq 2} f_i$ represent all derivatives at m_i of higher order, and consider the function $F : V \rightarrow (\mathbb{R}^m)^s$ given by

$$F(m_1, \dots, m_s, x_1, \dots, x_s, A_1, d^{\geq 2} f_1, \dots, A_s, d^{\geq 2} f_s) = (A_1^\top(z - x_1), \dots, A_s^\top(z - x_s)), \quad (28)$$

where $z = z(x_1, \dots, x_s)$. Then F is a submersion, as the derivative of $(A_i, x_1, \dots, x_s) \mapsto A_i^\top(z(x_1, \dots, x_s) - x_i)$ with respect to A_i is surjective (since $z - x_i \neq 0$). Consequently, the set

$$F^{-1}(\{0\}) = \{(m_1, \dots, m_s, x_1, \dots, x_s, A_1, d^{\geq 2} f_1, \dots, A_s, d^{\geq 2} f_s) \in V : \\ \forall i \in [s], A_i^\top(z(x_1, \dots, x_s) - x_i) = 0\}$$

is a submanifold of codimension sm in $J^r(M, \mathbb{R}^D)^s$, and $W_2' \cap (V \times \mathbb{R}^D)$ is simply the graph of the function

$$F^{-1}(\{0\}) \rightarrow \mathbb{R}^D \\ (m_1, \dots, m_s, x_1, \dots, x_s, [f_1], \dots, [f_s]) \mapsto z(x_1, \dots, x_s).$$

Thus it is a submanifold of codimension $sm + D$ in $J^r(M, \mathbb{R}^D)^s \times \mathbb{R}^D$, as claimed. \square

Remark 8.11. *We are quite convinced that $W_2^{(s)}$ is in fact a Whitney stratifiable subset of $J_s^r(M, \mathbb{R}^D)$ (rather than a union) - we know how to prove it in the case where M is smooth rather than just C^k using Nash-Tognoli's theorem, but it should be easy to prove in general using some theorem along the lines of "any subset of a manifold X that is semialgebraic in the coordinates of a well-chosen atlas of X is Whitney stratifiable"; however, we are no specialists of semialgebraic sets and could not find such a statement in the literature.*

Lemma 8.10 gives the existence of a decomposition $W_2^{(s)} = \cup_{\alpha \in A} W_{2,\alpha}^{(s)}$ of $W_2^{(s)}$ into a finite union of closed Whitney stratifiable sets of codimension at least sm , but this decomposition is not unique. We choose for a given M and for each $s, r \in \mathbb{N}$ with $2 \leq s \leq D+1$ and $1 \leq r \leq k-1$ such a decomposition $\{W_{2,\alpha}^{(s)}\}_{\alpha \in A}$, and we simply refer to it as "the decomposition from Lemma 8.10" or "the sets $\{W_{2,\alpha}^{(s)}\}_{\alpha \in A}$ from Lemma 8.10" in what follows.

We now need another set that is very similar to $W_2^{(s)}$, except that it will allow us to select embeddings $i \in \text{Emb}^k(M, \mathbb{R}^D)$ such that for each critical point $z_0 \in Z(i(M))$ and each $x \in \pi_{i(M)}(z_0)$, the sphere $S(z_0, d_{i(M)}(z_0))$ does not osculate $i(M)$ at x .

Definition 8.12. *For any $s, r \in \mathbb{N}$ with $s \geq 2$ and $2 \leq r \leq k$, we let $W_3^{(s)}$ be the set of multijets $(m_1, \dots, m_s, x_1, \dots, x_s, [f_1], \dots, [f_s]) \in J_s^r(M, \mathbb{R}^D)$ that belong to $W_2^{(s)}$ and such that there exists $i \in [s]$ for which the quadratic form $v \mapsto \|d(f_i \circ \phi)_0(v)\|^2 - \langle z - x_i, d^2(f_i \circ \phi)_0(v, v) \rangle$ is degenerate for any chart $\phi : (U, 0) \rightarrow (M, m_i)$.*

Remark 8.13. The set $W_3^{(s)}$ is well-defined. Indeed, the condition on the quadratic form only depends on the derivatives of order less than 2 of the functions f_i ; hence it is independent of the choice of representative $f_i \in [f_i] \subset C^k(M, \mathbb{R}^D)_{m_i, x_i}$. Furthermore, as $\langle z - x_i, d^2(f_i \circ \phi)_0(v, v) \rangle = \langle z - x_i, d^2(f_i)_{m_i}(d\phi_0(v), d\phi_0(v)) \rangle$ (because $\text{Im}((df_i)_{m_i}) \perp (z - x_i)$), it does not depend on the choice of the chart ϕ either - if it is true for one chart, it is true for all.

Lemma 8.14. For any $s, r \in \mathbb{N}$ with $2 \leq s \leq D + 1$ and $2 \leq r \leq k - 1$, the set $W_3^{(s)}$ is a finite union of closed Whitney stratifiable subsets $W_{3, \beta}^{(s)}$ of $J_s^r(M, \mathbb{R}^D)$ of codimension at least $sm + 1$.

Proof. The proof is almost identical to that of Lemma 8.10; we omit most details and only comment on the few meaningful differences. As in the previous proof, we fix $s \in \{2, \dots, D + 1\}$ and write $W_3 = W_3^{(s)}$ and $W_2 = W_2^{(s)}$. The set W_3 can be written as $W_3 = \bigcup_{i=1}^s W_3^i$, where W_3^i is the set of multijets $(m_1, \dots, m_s, x_1, \dots, x_s, [f_1], \dots, [f_s]) \in W_2$ for which the quadratic form $v \mapsto \|d(f_i \circ \phi)_0(v)\|^2 - \langle z - x_i, d^2(f_i \circ \phi)_0(v, v) \rangle$ is degenerate for any chart $\phi : (U, 0) \rightarrow (M, m_i)$. Our goal is to conclude using the same arguments as in the proof of Lemma 8.10 that each W_3^i is a finite union of closed Whitney stratifiable subsets of $J_s^r(M, \mathbb{R}^D)$ of codimension at least $sm + 1$, hence that W_3 can also be written as such a union.

The condition for a point $(m_1, \dots, m_s, x_1, \dots, x_s, [f_1], \dots, [f_s]) \in J_s^r(M, \mathbb{R}^D)$ that $v \mapsto \|d(f_i \circ \phi)_0(v)\|^2 - \langle z - x_i, d^2(f_i \circ \phi)_0(v, v) \rangle$ be degenerate is equivalent to a closed condition on the first and second derivative of the function $[f_i]$ (that does not depend on the choice of the representative f_i), and this condition is algebraic in adapted charts of $J_s^r(M, \mathbb{R}^D)$; indeed, in such a system of coordinates, it is equivalent to asking that the matrix $A_i^\top A_i - B_i$ has zero determinant, where A_i represents the first differential of f_i and B_i represents the bilinear form that applies the second differential of f_i to a pair of vectors and returns the scalar product of the result with $z - x_i$ (both being represented in the chosen system of coordinates).

It remains to show that the added condition does increase the codimension by one. To that end, we would like the condition on the determinant of the matrix to be transverse to our other conditions, but sadly it is not even transverse to the ambient space: the determinant is not a submersion from the space of $m \times m$ matrices to \mathbb{R} (in particular the set of all singular matrices is semialgebraic, but is not a manifold). However, this minor difficulty can be avoided. As the differential of the determinant at a matrix K is equal to $d(\det)_K : H \mapsto \text{tr}(\text{adj}(K)H)$, where $\text{adj}(K)$ is the adjugate of K and tr is the trace, the determinant is indeed a submersion at any matrix K of rank at least $m - 1$.

In the proof of Lemma 8.10, we defined an auxiliary set $W_2' \subset J^r(M, \mathbb{R}^D)^s \times \mathbb{R}^D$ such that $\pi(W_2') \cap J_s^r(M, \mathbb{R}^D) = W_2$ and such that $\dim W_2 \leq \dim W_2'$, see (27). Similarly, we define the set $(W_3^i)'$ of pairs $((m_1, \dots, m_s, x_1, \dots, x_s, [f_1], \dots, [f_s]), z) \in W_2'$ such that the form $v \mapsto \|d(f_i \circ \phi)_0(v)\|^2 - \langle z - x_i, d^2(f_i \circ \phi)_0(v, v) \rangle$ is of rank $m - 1$. We claim that the set $(W_3^i)'$ is a submanifold of codimension $sm + D + 1$ in $J^r(M, \mathbb{R}^D)^s \times \mathbb{R}^D$ (compared to the codimension $sm + D$ of W_2'). If this claim is true, the rest of the proof is the same as that of Lemma 8.10 (with the additional fact that the matrices of rank $m - 1$ are dense in the set of all degenerate matrices).

To prove the claim, note that the only difference with W_2' is the condition that the form $v \mapsto \|d(f_i \circ \phi)_0(v)\|^2 - \langle z - x_i, d^2(f_i \circ \phi)_0(v, v) \rangle$ be of rank $m - 1$. This is the intersection of the condition that it be degenerate (as in the definition of W_3), or equivalently that the determinant of the associated matrix be 0, and the condition that it be of rank at least $m - 1$, which is an open condition. Similarly to the proof of Lemma 8.10, any point in $(W_3^i)'$ admits an open neighborhood $V \times \mathbb{R}^D \subset J^r(M, \mathbb{R}^D)^s \times \mathbb{R}^D$ such that the intersection of the set $(W_3^i)'$ with this

neighborhood is the graph by the function $(m_1, \dots, m_s, x_1, \dots, x_s, [f_1], \dots, [f_s]) \mapsto z(x_1, \dots, x_s)$ of the intersection $F^{-1}(\{0\}) \cap G_i^{-1}(\{0\})$ of the zero sets of two maps. Here, the map F is defined in (28) and the map $G_i : V \rightarrow \mathbb{R}$ is given by

$$G_i(m_1, \dots, m_s, x_1, \dots, x_s, A_1, d^2 f_1, \dots, A_s, d^2 f_s, d^{\geq 3} f_1, \dots, d^{\geq 3} f_s) = \det(A_j^\top A_j - B_j) \quad (29)$$

where the $D \times m$ matrix A_i represents the first differential of f_i at m_i and the $m \times m$ matrix B_i represents the bilinear form that applies the second differential $(d^2 f_i)_{m_i}$ of f_i (which we simply write $d^2 f_i$ here) to a pair of vectors and returns the scalar product of the result with $z - x_i$. We already know that F is a submersion; so is G_j , because V is chosen (among other open conditions) so that the matrix $A_i^\top A_i - B_i$ be of rank at least $m - 1$ for any point in V . Furthermore, it is easy to show that the conditions “ $F = 0$ ” and “ $G_j = 0$ ” are transverse to each other, essentially because “ $F = 0$ ” does not constrain $(d^2 f_i)_{m_i}$ in any way. Hence $F^{-1}(\{0\}) \cap G_j^{-1}(\{0\}) \subset V$ is a submanifold of codimension $sm + 1$ in $J^r(M, \mathbb{R}^D)^s$, and $(W_3^i)' \cap (V \times \mathbb{R}^D)$ is a submanifold of codimension $sm + D + 1$ in $J^r(M, \mathbb{R}^D)^s \times \mathbb{R}^D$. This proves the claim, and thence the lemma. \square

As for Lemma 8.10, we choose for a given M and for each $s, r \in \mathbb{N}$ with $2 \leq s \leq D + 1$ and $2 \leq r \leq k - 1$ a decomposition $W_3^{(s)} = \cup_{\beta \in B} W_{3,\beta}^{(s)}$ of $W_3^{(s)}$ into a finite union of closed Whitney stratifiable sets of codimension at least $sm + 1$ whose existence is guaranteed by Lemma 8.14, and we simply refer to it as “the decomposition from Lemma 8.14” or “the sets $\{W_{3,\beta}^{(s)}\}_{\beta \in B}$ from Lemma 8.14” in what follows.

Lemma 8.15. *For any $2 \leq r \leq k - 1$, if $i \in \text{Emb}^k(M, \mathbb{R}^D)$ is such that $\mathbf{M} = i(M)$ satisfies condition (P0) and that for all $s = 2, \dots, D + 1$, $j_s^r(i) : M^{(s)} \rightarrow J_s^r(M, \mathbb{R}^D)$ is transverse to the sets $\{W_{2,\alpha}^{(s)}\}_{\alpha \in A}$ from Lemma 8.10 and to the sets $\{W_{3,\beta}^{(s)}\}_{\beta \in B}$ from Lemma 8.14, then $\mathbf{M} = i(M)$ satisfies conditions*

(P2) *The set $Z(\mathbf{M})$ is finite.*

(P3) *For every $z_0 \in Z(\mathbf{M})$ and every $x_0 \in \pi_{\mathbf{M}}(z_0)$, the sphere $S(z_0, d_{\mathbf{M}}(z_0))$ is non-osculating \mathbf{M} at x_0 .*

As will become apparent in the proof of Lemma 8.15, condition (P3) is a consequence of the transversality to the sets $\{W_{3,\beta}^{(s)}\}_{\beta \in B}$ only. On the other hand, we need transversality to both collections of sets to get condition (P2): transversality to the sets $\{W_{2,\alpha}^{(s)}\}_{\alpha \in A}$ is required so that the dimension of the set of critical points $Z(i(M))$ be 0, and transversality to the sets $\{W_{3,\beta}^{(s)}\}_{\beta \in B}$ so that $Z(i(M))$ have no accumulation point.

Proof. Fix $2 \leq r \leq k - 1$ and $i \in \text{Emb}^k(M, \mathbb{R}^D)$ as in the statement. As we assume that $i(M)$ satisfies condition (P0), we know that any critical point z of $i(M)$ is such that the set $\pi_{i(M)}(z)$ of its projections on $i(M)$ is of cardinality at most $D + 1$. Let us write as $Z(i(M))_s$ the points $z \in Z(i(M))$ such that $\pi_{i(M)}(z)$ is of cardinality exactly $2 \leq s \leq D + 1$.

Let $z \in Z(i(M))_s$ and let us write $\pi_{i(M)}(z) = \{x_1, \dots, x_s\}$. We also call $m_1, \dots, m_s \in M$ the points such that $i(m_j) = x_j$. By definition $d(x_1, z) = \dots = d(x_s, z)$, and $T_{x_j} i(M) = \text{Im}(di_{m_j} : T_{m_j} M \rightarrow T_{x_j} \mathbb{R}^D)$ must be perpendicular to $z - x_j$ for $j \in [s]$ for x_j to belong to $\pi_{i(M)}(z)$. Furthermore, the point z must belong to the convex hull of x_1, \dots, x_s , as it is equal

to $m(\{x_1, \dots, x_s\})$, the center of the smallest enclosing ball of $\{x_1, \dots, x_s\}$ (see Section 2.1 of [CCSL06]).

This shows that $j_s^r(i)(m_1, \dots, m_s) = (m_1, \dots, m_s, x_1, \dots, x_s, [i], \dots, [i])$ and z satisfy the conditions in the definition of $W_2^{(s)}$, hence that $j_s^r(i)(m_1, \dots, m_s)$ belongs to $W_2^{(s)} \cap \text{Im}(j_s^r(i)) \subset J_s^r(M, \mathbb{R}^D)$. In other words, each point z of $Z(i(M))_s$ corresponds to a point $j_s^r(i)(m_1, \dots, m_s)$ in the intersection $W_2^{(s)} \cap \text{Im}(j_s^r(i))$. Note that the converse is not true: there can be points $(m_1, \dots, m_s, x_1, \dots, x_s, [i], \dots, [i]) \in \text{Im}(j_s^r(i))$ and $z \in \mathbb{R}^D$ that satisfy the conditions of $W_2^{(s)}$, yet such that z is not a critical point of $i(M)$ (simply because the points x_j are not the closest points to z in $i(M)$).

Furthermore, for any point $(m_1, \dots, m_s, x_1, \dots, x_s, [f_1], \dots, [f_s]) \in J_s^r(M, \mathbb{R}^D)$, there can be at most a single point z for which the conditions of $W_2^{(s)}$ are satisfied, this point z being equal to the center of the smallest enclosing ball of $\{x_1, \dots, x_s\}$ (see Section 2.1 of [CCSL06]). Consequently, there is an injection $\psi : Z(i(M))_s \hookrightarrow W_2^{(s)} \cap \text{Im}(j_s^r(i))$.

Consider z a point of $Z(i(M))_s$ with $\pi_{i(M)}(z) = \{x_1, \dots, x_s\}$ and

$$\psi(z) = (m_1, \dots, m_s, x_1, \dots, x_s, [i], \dots, [i])$$

the associated point of $W_2^{(s)} \cap \text{Im}(j_s^r(i))$. Then by definition of $W_3^{(s)} \subset W_2^{(s)}$, $\psi(z) \in W_3^{(s)}$ if and only if there exists $j \in [s]$ such that for any chart $\phi : (U, 0) \rightarrow (M, m_j)$ the form $v \mapsto \|d(i \circ \phi)_0(v)\|^2 - \langle z - x_j, d^2(i \circ \phi)_0(v, v) \rangle$ is degenerate. For a fixed $j \in [s]$, this condition is equivalent to the sphere $S(z, d_{i(M)}(z))$ osculating $i(M)$ at x_j , as shown in the Osculation Characterization Lemma 3.1. In other words, we only need $W_3^{(s)} \cap \text{Im}(j_s^r(i))$ to be empty for $s = 2, \dots, D + 1$ for the embedding i to satisfy condition (P3).

But the strata of the Whitney stratified sets $W_{3,\beta}^{(s)}$ have codimension at least $sm + 1$ (see Lemma 8.14), and the map $j_s^r(i) : M^{(s)} \rightarrow J_s^r(M, \mathbb{R}^D)$ intersects them transversally by hypothesis. As $M^{(s)}$ is of dimension sm , the image $\text{Im}(j_s^r(i))$ can only have empty intersections with the strata of each $W_{3,\beta}^{(s)}$, hence with $W_3^{(s)}$ as a whole. We have shown that i verifies condition (P3).

Similarly, the strata of the Whitney stratified sets $W_{2,\alpha}^{(s)}$ have codimension at least sm , hence the image $\text{Im}(j_s^r(i))$ can only have a non-empty intersection with the strata of codimension sm of $W_{2,\alpha}^{(s)}$, and that intersection must be of dimension 0 and be discrete. The same holds for $W_2^{(s)} \cap \text{Im}(j_s^r(i))$.

Let us assume by contradiction that i does not satisfy condition (P2); there must be an index $2 \leq s \leq D + 1$ such that $Z(i(M))_s$ contains infinitely many points. As each critical point z and its projections $\pi_{i(M)}(z)$ belong to the compact set $\text{Conv}(i(M))$, we can find a sequence of critical points $(z_n)_{n \in \mathbb{N}}$ with projections $\pi_{i(M)}(z_n) = \{x_{n,1}, \dots, x_{n,s}\}$, whose preimages by i we write $\{m_{n,1}, \dots, m_{n,s}\} \subset M$, such that $\lim_{n \rightarrow \infty} z_n = z$, $\lim_{n \rightarrow \infty} m_{n,j} = m_j$ and $\lim_{n \rightarrow \infty} x_{n,j} = x_j$ for some points $z \in \mathbb{R}^D$, $m_1, \dots, m_s \in M$ and $x_1, \dots, x_s \in i(M)$ (with of course $i(m_j) = x_j$).

If the points m_j (or equivalently the points x_j) are pairwise distinct, the sequence of points $\psi(z_n) = (m_{n,1}, \dots, m_{n,s}, x_{n,1}, \dots, x_{n,s}, [i], \dots, [i]) \in W_2^{(s)} \cap \text{Im}(j_s^r(i))$ converges to

$$(m_1, \dots, m_s, x_1, \dots, x_s, [i], \dots, [i]) \in J_s^r(M, \mathbb{R}^D),$$

which does belong to $W_2^{(s)} \cap \text{Im}(j_s^r(i))$ as well (as $W_2^{(s)} \cap \text{Im}(j_s^r(i))$ is closed in $J_s^r(M, \mathbb{R}^D)$). Hence we have found an infinite sequence with an accumulation point in $W_2^{(s)} \cap \text{Im}(j_s^r(i))$, contradicting its discrete nature - this is impossible.

Let us now consider the case where the points m_j are not pairwise distinct. As $i(M)$ has positive reach, there cannot be critical points arbitrarily close to it; consequently z does not belong to $i(M)$. Furthermore, as the norm $\|\nabla d_{i(M)}\|$ of the generalized gradient is lower semicontinuous, we have $\nabla d_{i(M)}(z) = 0$, hence z is a critical point. By continuity, the limit points x_j are also such that $d(x_j, z) = d(i(M), z)$, hence $\{x_1, \dots, x_s\} \subset \pi_{i(M)}(z)$. As the points m_j are not pairwise distinct, we can assume (up to permuting some indices) that $m_1 = m_2$, hence that $\lim_{n \rightarrow \infty} x_{n,1} = \lim_{n \rightarrow \infty} x_{n,2} = x_1$. This means that for n large enough, there are points z_n arbitrarily close to a critical point z such that they have two distinct projections $x_{n,1}$ and $x_{n,2}$ themselves arbitrarily close to a projection x_1 of z . As we have shown earlier that i satisfies condition (P3), we know that the sphere $S(z, d_{i(M)}(z))$ does not osculate $i(M)$ at any of the x_j : this yields a contradiction with the local unicity of the projection from Lemma 4.1. Hence i satisfies condition (P2) as well, as claimed. \square

It might have been possible to handle all cases $s \in \{2, \dots, D+1\}$ at once, and in particular the case of the sequences of projections $x_{n,1}$ and $x_{n,2}$ having the same limit, using some compactification of $J_{D+1}^r(M, \mathbb{R}^D)$ such as the one recently defined in [AL23]. As there is no version of Thom and Mather's transversality theorem adapted to such a compactification yet, we favoured a more elementary approach.

Remark 8.16. *A tube in \mathbb{R}^3 is a simple example (that can easily be compactified) of a manifold that satisfies neither (P2) nor (P3).*

8.3.3 Condition (P1), part 2

We will now handle the second part of Condition (P1), i.e. the fact that each critical point $z \in Z(i(M))$ must belong to the relative interior of the convex hull of its projections on $i(M)$, in order to avoid the situation illustrated earlier in Example 7.2.

Definition 8.17. *For any $s, r \in \mathbb{N}$ with $s \geq 3$ and $1 \leq r \leq k$, we let $W_1^{(s)}$ be the set of multijets $(m_1, \dots, m_s, x_1, \dots, x_s, [f_1], \dots, [f_s]) \in J_s^r(M, \mathbb{R}^D)$ that belong to $W_2^{(s)}$ and such that the (unique) point $z \in \mathbb{R}^D$ satisfying the conditions in the definition of $W_2^{(s)}$ belongs to $\text{Conv}(x_1, \dots, x_{s-1})$.*

Lemma 8.18. *For any $s, r \in \mathbb{N}$ with $3 \leq s \leq D+1$ and $1 \leq r \leq k-1$, the set $W_1^{(s)}$ is a finite union of closed Whitney stratifiable subsets $W_{1,\gamma}^{(s)}$ of $J_s^r(M, \mathbb{R}^D)$ of codimension at least $sm+1$.*

Proof. The proof is yet again almost identical to that of Lemma 8.10; we omit most details and only comment on the few meaningful differences. As in the previous proofs, we fix $s \in \{3, \dots, D+1\}$ and write $W_1 = W_1^{(s)}$. The definition of W_1 is the same as that of W_2 , except that we ask that $z \in \text{Conv}(x_1, \dots, x_{s-1})$ rather than $z \in \text{Conv}(x_1, \dots, x_{s-1}, x_s)$. This simply replaces one closed semialgebraic condition by another; the only non-trivial point is to show that this change does increase the codimension by one compared to W_2 .

In the proof of Lemma 8.10, we defined an auxiliary set $W_2' \subset J^r(M, \mathbb{R}^D)^s \times \mathbb{R}^D$ such that $\pi(W_2') \cap J_s^r(M, \mathbb{R}^D) = W_2$ and such that $\dim W_2 \leq \dim W_2'$, see (27). Similarly, we can define

the set

$$\begin{aligned}
W'_1 := & \{((m_1, \dots, m_s, x_1, \dots, x_s, [f_1], \dots, [f_s]), z) \in J^r(M, \mathbb{R}^D)^s \times \mathbb{R}^D : \\
& \|x_1 - z\|^2 = \dots = \|x_s - z\|^2, \forall i \in [s], \text{Im}((df_i)_{m_i} : T_{m_i}M \rightarrow T_{x_i}\mathbb{R}^D) \perp (z - x_i), \\
& \text{rank}((df_i)_{m_i}) = m, \exists \lambda_1, \dots, \lambda_{s-1} \in]0, 1[\text{ s.t. } \sum_{i=1}^{s-1} \lambda_i = 1, z = \sum_{i=1}^{s-1} \lambda_i x_i \\
& \text{and } \dim \text{Aff}(x_1, \dots, x_s) = s - 1\}.
\end{aligned} \tag{30}$$

We claim that the set W'_1 is a submanifold of codimension $sm + D + 1$ in $J^r(M, \mathbb{R}^D)^s \times \mathbb{R}^D$ (compared to the codimension $sm + D$ of W'_2). Once this has been shown, the rest of the proof is the same as that of Lemma 8.10. This is a local property; in what follows, we implicitly place ourselves in the coordinates of an adapted chart (ϕ, U) of $J^r_s(M, \mathbb{R}^D)$, without explicitly saying that we are considering the intersection of W'_1 with $U \times \mathbb{R}^D$ or constantly specifying "in the coordinates of...".

Let (a, z) be a point of W'_1 , with $a \in J^r(M, \mathbb{R}^D)^s$. As in the proof of Lemma 8.10, the conditions that $\dim \text{Aff}(x_1, \dots, x_s) = s - 1$, that $\text{rank}((df_i)_{m_i}) = m$ for $i = 1, \dots, s$ and that the center of the circumsphere of x_1, \dots, x_{s-1} belongs to the relative interior of $\text{Conv}(x_1, \dots, x_{s-1})$ are open, and remain true in a small neighborhood V of a , hence in the neighborhood $V \times \mathbb{R}^D$ around (a, z) . Likewise, for $(b, z) = ((m_1, \dots, m_s, x_1, \dots, x_s, [f_1], \dots, [f_s]), z)$ in $V \times \mathbb{R}^D$, the conditions that $\|x_1 - z\|^2 = \dots = \|x_{s-1} - z\|^2$ and that there exist $\lambda_1, \dots, \lambda_{s-1} \in]0, 1[$ such that $\sum_{i=1}^{s-1} \lambda_i = 1$ and $z = \sum_{i=1}^{s-1} \lambda_i x_i$ means that the coordinates of z are a (smooth) function $z(x_1, \dots, x_{s-1})$ of the coordinates of x_1, \dots, x_{s-1} , as z must be the center of the circumsphere of x_1, \dots, x_{s-1} .

Similarly to the proofs of Lemmas 8.10 and 8.10, the set $W'_1 \cap (V \times \mathbb{R}^D) \subset J^r(M, \mathbb{R}^D)^s \times \mathbb{R}^D$ is the graph by the function $(m_1, \dots, m_s, x_1, \dots, x_s, [f_1], \dots, [f_s]) \mapsto z(x_1, \dots, x_{s-1})$ of the intersection $F^{-1}(\{0\}) \cap H^{-1}(\{0\})$ of the zero sets of two maps. Here, the map F is defined in (28) and the map $H : V \rightarrow \mathbb{R}$ is given by

$$H(m_1, \dots, m_s, x_1, \dots, x_s, d^{\geq 1}f_1, \dots, d^{\geq 1}f_s) = \|z - x_s\|^2 - \|z - x_1\|^2, \tag{31}$$

where $z = z(x_1, \dots, x_{s-1})$ and $d^{\geq 1}f_i$ represents the differentials of f_i of order greater than or equal to 1 at m_i .

We already know that F is a submersion and so is G (in particular because $z \neq x_s$). The conditions " $F = 0$ " and " $H = 0$ " are transverse to each other essentially because " $H = 0$ " does not constrain the differentials $(df_i)_{m_i}$ (represented by the matrices A_i) in any way. Hence $F^{-1}(\{0\}) \cap H^{-1}(\{0\}) \subset V$ is a submanifold of codimension $sm + 1$ in $J^r(M, \mathbb{R}^D)^s$, and $W'_1 \cap (V \times \mathbb{R}^D)$ is a submanifold of codimension $sm + D + 1$ in $J^r(M, \mathbb{R}^D)^s \times \mathbb{R}^D$. This proves the claim, and thence the lemma. \square

As for Lemmas 8.10 and 8.14, we choose for a given M and for each $s, r \in \mathbb{N}$ with $3 \leq s \leq D + 1$ and $1 \leq r \leq k - 1$ a decomposition $W_1^{(s)} = \cup_{\gamma \in C} W_{1,\gamma}^{(s)}$ of $W_1^{(s)}$ into a finite union of closed Whitney stratifiable sets of codimension at least $sm + 1$ whose existence is guaranteed by Lemma 8.18, and we simply refer to it as "the decomposition from Lemma 8.18" or "the sets $\{W_{1,\gamma}^{(s)}\}_{\gamma \in C}$ from Lemma 8.18" in what follows.

Lemma 8.19. *For any $1 \leq r \leq k - 1$, if $i \in \text{Emb}^k(M, \mathbb{R}^D)$ is such that $\mathbf{M} = i(M)$ satisfies condition (P0) and that for all $s = 3, \dots, D + 1$, $j_s^r(i) : M^{(s)} \rightarrow J_s^r(M, \mathbb{R}^D)$ is transverse to the sets $\{W_{1,\gamma}^{(s)}\}_{\gamma \in C}$ from Lemma 8.18, then \mathbf{M} satisfies condition*

(P1) *For every $z_0 \in Z(\mathbf{M})$, the projections $\pi_{\mathbf{M}}(z_0)$ are the vertices of a non-degenerate simplex of \mathbb{R}^D . In particular, z_0 has at most $D + 1$ projections. Moreover, the point z_0 belongs to the relative interior of the convex hull of $\pi_{\mathbf{M}}(z_0)$*

Proof. Fix $1 \leq r \leq k - 1$ and $i \in \text{Emb}^k(M, \mathbb{R}^D)$ as in the statement. As we assume that i satisfies condition (P0), we know that any critical point z of $i(M)$ is such that its projections $\pi_{i(M)}(z)$ are the vertices of a non-degenerate simplex of \mathbb{R}^D , and that in particular there can be no more than $D + 1$ of them. We only have to show that each such z must belong to the relative interior of this simplex for (P1) to be true.

Let $z \in Z(i(M))$ with $\pi_{i(M)}(z) = \{x_1, \dots, x_s\}$ for some $s = 2, \dots, D + 1$. Let us assume, by way of contradiction, that z does not belong to the relative interior of $\text{Conv}(x_1, \dots, x_s)$. As we have shown in the proof of Lemma 8.15 that z does belong to $\text{Conv}(x_1, \dots, x_s)$, it must then belong to a strict face of the simplex, hence to the convex hull of a strict subset of its vertices. If $s = 2$, we already find a contradiction (as z cannot be equal to any of the x_i). Otherwise, up to reordering the vertices, we can assume that $z \in \text{Conv}(x_1, \dots, x_{s-1})$. As in the proof of Lemma 8.15, we also have that the point $j_s^r(i)(m_1, \dots, m_s) = (m_1, \dots, m_s, x_1, \dots, x_s, [i], \dots, [i])$ and z are such that $d(x_1, z) = \dots = d(x_s, z)$ and that $\text{Im}((df_i)_{m_i}) \perp (z - x_i)$ for $i = 1, \dots, s$. Thus, the point $j_s^r(i)(m_1, \dots, m_s)$ must belong to $W_1^{(s)} \cap \text{Im}(j_s^r(i))$.

But the strata of the Whitney stratified sets $W_{1,\gamma}^{(s)}$ have codimension at least $sm + 1$ (see Lemma 8.14), and the map $j_s^r(i) : M^{(s)} \rightarrow J_s^r(M, \mathbb{R}^D)$ intersects them transversally by hypothesis. As $M^{(s)}$ is of dimension sm , the image $\text{Im}(j_s^r(i))$ can only have empty intersections with the strata of each $W_{1,\gamma}^{(s)}$, hence with $W_1^{(s)}$ as a whole. This yields a contradiction, thus we have shown that $i(M)$ verifies condition (P1). \square

8.3.4 Condition (P4)

The somewhat delicate case of condition (P4) is all that remains; to tackle it, we will need the results from Section 6. Similarly to how we proceeded earlier, we want to define a family of sets $W_4^{(s)} \subset J_s^r(M, \mathbb{R}^D)$ such that transversality to them will yield the Big Simplex Property (BSP), and consequently condition (P4). To do so, we will use the expression from Lemma 4.3 of the differential of the projection in terms of charts of our C^k compact manifold M , as it is compatible with the formalism of jets.

Let $m_0 \in M$, $x_0, z_0 \in \mathbb{R}^D$ and $\phi : (V, u_0) \rightarrow (M, m_0)$ be any chart of M around m_0 . Let also $f \in C^2(M, \mathbb{R}^D)$ be some function that maps m_0 to x_0 such that df_{m_0} has full rank, that $\text{Im}(df_{m_0}) \perp z_0 - x_0$, and that the quadratic form $v \mapsto \|d(f \circ \phi)_{u_0}(v)\|^2 - \langle z_0 - x_0, d^2(f \circ \phi)_{u_0}(v, v) \rangle$ is positive definite. Note that these conditions depend only of the value and the first and second differentials of f at m_0 . Then, Lemma 4.3 guarantees the existence of $\delta > 0$ and of a neighborhood $U \subset M$ of m_0 such that $f|_U : U \rightarrow \mathbb{R}^D$ is an embedding and that each $z \in B(z_0, \delta)$ admits a unique projection $p(z)$ on $f(U)$ with $p(z) \in f(U)$. The lemma also gives an expression for the differential $dp_{z_0} : \mathbb{R}^D \rightarrow \text{Im}(df_{m_0})$ at z_0 of $p : B(z_0, \delta) \rightarrow f(U)$, and this expression also only depends on the first two differentials of f at m_0 . Hence, given $r \geq 2$, a jet $(m, x, [f]) \in J^r(M, \mathbb{R}^D)$ such that df_m has full rank and a point $z \in \mathbb{R}^D$ such that $\text{Im}(df_m) \perp z - x$, and assuming that $v \mapsto \|d(f \circ \phi)_0(v)\|^2 - \langle z - x, d^2(f \circ \phi)_0(v, v) \rangle$ is positive

definite (for some chart $\phi : (V, 0) \rightarrow (M, m)$), we can unambiguously define the differential at z of the projection p on the image of f , independently from the choice of representative $f \in [f]$.

Definition 8.20. *When properly defined, we write as $P_{[f],x,z}$ the differential at z of the projection p on the image of f around x .*

Now we can formulate an analogue in the language of jets of the quadratic form B from Lemma 6.7.

Definition 8.21. *Let $r, s \geq 2$ and let $a = (m_1, \dots, m_s, x_1, \dots, x_s, [f_1], \dots, [f_s]) \in J_s^r(M, \mathbb{R}^D)$ and $z \in \mathbb{R}^D$. Assume that for $i \in [s]$ the map $(df_i)_{m_i}$ is of rank m , the quadratic form $v \mapsto \|d(f_i \circ \phi_i)_0(v)\|^2 - \langle z - x_i, d^2(f_i \circ \phi_i)_0(v, v) \rangle$ is positive definite (for some chart $\phi_i : (V, 0) \rightarrow (M, m_i)$) and $\text{Im}((df_i)_{m_i}) \perp (z - x_i)$.*

We define the quadratic form

$$B_{a,z}(h) := \sum_{i,j=1}^s \det(A_{a,z,i}(h)^\top A_{a,z,j}(h))$$

on \mathbb{R}^D , where $A_{a,z,i}(h)$ is the $D \times s$ matrix whose j -th column is equal to $x_j - z$ if $j \neq i$ and whose i -th column is $P_{[f_i],x_i,z}(h) - h$.

We will typically consider the case where $z \in \text{Conv}(x_1, \dots, x_s)$ and $h \in \text{Vec}(x_1 - z, \dots, x_s - z)^\perp$, but this is not needed for the definition to make sense.

We can now finally formulate (the negation of) the Big Simplex Property (BSP) in a way that is compatible with jets and thence define $W_4^{(s)}$.

Definition 8.22. *For any $s, r \in \mathbb{N}$ with $s \geq 2$ and $2 \leq r \leq k$, we let $(W_4^\circ)^{(s)}$ be the set of multijets $(m_1, \dots, m_s, x_1, \dots, x_s, [f_1], \dots, [f_s]) \in J_s^r(M, \mathbb{R}^D)$ that belong to $W_2^{(s)}$ and such that*

- for all $i \in [s]$, $\text{rank}((df_i)_{m_i}) = m$;
- for all $i \in [s]$ and for any chart $\phi : (U, 0) \rightarrow (M, m_i)$, the form $v \mapsto \|d(f_i \circ \phi)_0(v)\|^2 - \langle z - x_i, d^2(f_i \circ \phi)_0(v, v) \rangle$ is positive definite;
- and the quadratic form $B_{a,z}$ is degenerate on $\text{Vec}(x_1 - z, \dots, x_s - z)^\perp$.

Furthermore, we let $W_4^{(s)}$ be the closure of $(W_4^\circ)^{(s)}$ in $J_s^r(M, \mathbb{R}^D)$.

Lemma 8.23. *For any $s, r \in \mathbb{N}$ with $2 \leq s \leq D + 1$ and $2 \leq r \leq k - 1$, the set $W_4^{(s)}$ is a finite union of closed Whitney stratifiable subsets $W_{4,\nu}^{(s)}$ of $J_s^r(M, \mathbb{R}^D)$ of codimension at least $sm + 1$.*

Proof. The proof is yet again similar to that of Lemma 8.10; as for Lemmas 8.14 and 8.18, we omit details and focus on the differences of import. As in the previous proofs, we fix $s \in \{2, \dots, D + 1\}$ and write $W_4 = W_4^{(s)}$.

The definition of W_4° is similar to that of W_2 , but three conditions have been added: the first is that for $(m_1, \dots, m_s, x_1, \dots, x_s, [f_1], \dots, [f_s]) \in J_s^r(M, \mathbb{R}^D)$ the rank of $(df_i)_{m_i}$ must be m for $i \in [s]$. This is an algebraic constraint in any adapted chart, and an open condition, and does not add any difficulty to the proof. The second is that for any $i \in [s]$ and any chart $\phi : (U, 0) \rightarrow (M, m_i)$, the form $v \mapsto \|d(f_i \circ \phi)_0(v)\|^2 - \langle z - x_i, d^2(f_i \circ \phi)_0(v, v) \rangle$ must be positive definite (where z , if it exists, is the only point in $\text{Conv}(x_1, \dots, x_s)$ such that $d(x_1, z) = \dots = d(x_s, z)$).

We know from the proof of Lemma 8.14 that this condition is algebraic in adapted charts, and it is also open. The third and last addition is that we ask that the form $B_{a,z}$ be degenerate on $\text{Vec}(x_1 - z, \dots, x_s - z)^\perp$. If we prove that this condition is semialgebraic in adapted charts and that it does increase the codimension by one, the same arguments as in Lemmas 8.10, 8.14 and 8.18 show that W_4° is locally semialgebraic of codimension $sm + 1$, hence that so is its closure W_4 , which can then be decomposed as a union of finitely many closed Whitney stratified sets $\{W_{4,\nu}\}_{\nu \in N}$ of codimension at least $sm + 1$ (for some finite set of indices N), as claimed.

Let $(m, x, [f]) \in J^r(M, \mathbb{R}^D)$ and $z \in \mathbb{R}^D$ be such that the conditions of Lemma 4.3 are satisfied by x, f, z . Let $\phi : (V, 0) \rightarrow (M, m)$ be a chart of M around m , and let us place ourselves in the adapted chart of $J^r(M, \mathbb{R}^D)$ to which ϕ gives rise. In Lemma 4.3, we defined the linear endomorphism $\Lambda : \text{Im}(df_m) \rightarrow \text{Im}(df_m)$ associated to x, z, f by stating that $\Lambda(v)$ must be the only vector in $\text{Im}(df_m)$ such that

$$\langle z - x, d^2(f \circ \phi)_0(d(f \circ \phi)_0^{-1}(v), d(f \circ \phi)_0^{-1}(w)) \rangle = \langle \Lambda(v), w \rangle$$

for any $w \in \text{Im}(df_{m_0})$. Let A be the $D \times m$ matrix that represents $d(f \circ \phi)_0 : \mathbb{R}^m \rightarrow \mathbb{R}^D$, and let B be the $m \times m$ matrix that represents the bilinear form $\mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}, (v, w) \mapsto \langle z - x, d^2(f \circ \phi)_0(v, w) \rangle$. Then $\tilde{\Lambda} := d(f \circ \phi)_0^{-1} \circ \Lambda \circ d(f \circ \phi)_0 : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is such that

$$\langle z - x, d^2(f \circ \phi)_0(v, w) \rangle = \langle d(f \circ \phi)_0(\tilde{\Lambda}(v)), d(f \circ \phi)_0(w) \rangle$$

for any $v, w \in \mathbb{R}^m$, which in terms of matrices means that

$$w^\top Bv = w^\top A^\top A \tilde{\Lambda}(v)$$

for all $v, w \in \mathbb{R}^m$, or equivalently

$$(A^\top A)^{-1} Bv = \tilde{\Lambda}(v)$$

(where we use the same notation for the map $\tilde{\Lambda}$ and the matrix that represents it in the chosen system of coordinates). We also see that the map $d(f \circ \phi)_0^{-1} \circ \pi_{\text{Im}(df_m)} : \mathbb{R}^D \rightarrow \mathbb{R}^m$ is represented by the pseudo-inverse matrix

$$(A^\top A)^{-1} A^\top.$$

This expression is well-defined because A is of full rank, and the coefficients of the resulting matrix are polynomial fractions in the coefficients of A .

Finally, remember that we defined $P_{[f],x,z}$ as the differential at z on the image of f , which was shown in Lemma 4.3 to be equal to

$$P_{[f],x,z} = (\text{id} - \Lambda)^{-1} \circ \pi_{\text{Im}(df_m)} : \mathbb{R}^D \rightarrow \text{Im}(df_m).$$

This can be rewritten as

$$\begin{aligned} P_{[f],x,z} &= (\text{id} - \Lambda)^{-1} \circ \pi_{\text{Im}(df_m)} = d(f \circ \phi)_0 d(f \circ \phi)_0^{-1} (\text{id} - \Lambda)^{-1} d(f \circ \phi)_0 d(f \circ \phi)_0^{-1} \pi_{\text{Im}(df_m)} \\ &= d(f \circ \phi)_0 (\text{id} - d(f \circ \phi)_0^{-1} \Lambda d(f \circ \phi)_0)^{-1} d(f \circ \phi)_0^{-1} \pi_{\text{Im}(df_m)} \\ &= d(f \circ \phi)_0 (\text{id} - \tilde{\Lambda})^{-1} d(f \circ \phi)_0^{-1} \pi_{\text{Im}(df_m)} \end{aligned}$$

which is thus represented by the $D \times D$ matrix

$$A(I_m - (A^\top A)^{-1} B)^{-1} (A^\top A)^{-1} A^\top$$

whose coefficients are polynomial fractions in the coordinates of the adapted chart.

For $a = (m_1, \dots, m_s, x_1, \dots, x_s, [f_1], \dots, [f_s]) \in J_s^r(M, \mathbb{R}^D)$ and $z \in \mathbb{R}^D$, we defined $B_{a,z}$ as

$$B_{a,z}(h) := \sum_{i,j=1}^s \det(A_{a,z,i}(h)^\top A_{a,z,j}(h))$$

for $h \in \mathbb{R}^D$, where $A_{a,z,i}(h)$ is the $D \times s$ matrix whose j -th column is equal to $x_j - z$ if $j \neq i$ and whose i -th column is $P_{[f_i],x_i,z}(h) - h$. Sums, scalar products and determinants are all polynomial operations, and we have shown that the coefficients of the matrices representing the maps $P_{[f_i],x_i,z}$ are polynomial fractions in the coordinates, hence so are the coefficients of the $D \times D$ matrix O representing $B_{a,z}$. One can locally define a linear map $\iota : \mathbb{R}^{D-m} \rightarrow \mathbb{R}^D$ such that the image of ι is $E^\perp = \text{Vec}(x_2 - x_1, \dots, x_s - x_2)^\perp$ and such that it is represented by a matrix J whose coefficients are polynomial fractions in the coordinates of x_1, \dots, x_s ; then $B_{a,z}$ is degenerate when restricted to E^\perp if and only if the determinant of $J^\top O J$ is zero, and this determinant is a polynomial fraction in the coordinates of the adapted chart (possibly restricted to a smaller domain). This proves that asking that $B_{a,z}$ be degenerate is a semialgebraic constraint in the adapted charts.

We now turn to the codimension of W_4° . In what follows, we restrict ourselves to an adapted chart of sufficiently small domain U of $J_s^r(M, \mathbb{R}^D)$ that there exists a polynomial P in the coordinates of the chart and in those of $z \in \mathbb{R}^D$ such that wherever $B_{a,z}$ is well-defined, it is degenerate on E^\perp if and only if $P(a, z) = 0$ (to simplify notations, we write $P(a, z)$ for the value of P evaluated in the coordinates corresponding to a and z). Such a polynomial exists because we have shown that the determinant of $B_{a,z}$ can be locally expressed as a polynomial fraction, which is 0 if and only if its numerator is 0.

We only need to show that the intersection of W_4° with the domain of such a chart is of codimension at least $sm + 1$: then the same arguments as in the proofs of Lemmas 8.10, 8.14 and 8.18 are enough to conclude. In what follows, we do not always specify "in the coordinates of the chart" or "restricted to the domain of the chart" and write f_i for the precomposition of f_i with the coordinates of the chart.

We face two challenges: the first is geometric in nature. Indeed, it is not trivial that asking that the form $B_{a,z}$ be degenerate on E^\perp truly increases the codimension by one; after all, the form $B_{a,z}$ is degenerate on \mathbb{R}^D . Nonetheless, we will show that it is true. The second difficulty is that as in the proof of Lemma 8.10, the polynomial constraint $P(a, z) = 0$ is not a priori everywhere transverse to the other constraints, as P might have singularities. A few tricks are enough to circumvent this mostly technical problem.

Write $a \in U$ as $a = (a_{\leq 1}, a_{\geq 2})$, where

$$\begin{aligned} a_{\leq 1} &:= (m_1, \dots, m_s, x_1, \dots, x_s, (df_1)_{m_1}, \dots, (df_s)_{m_s}) \\ a_{\geq 2} &:= ((d^2 f_1)_{m_1}, \dots, (d^2 f_s)_{m_s}, (d^{\geq 3} f_1)_{m_1}, \dots, (d^{\geq 3} f_s)_{m_s}), \end{aligned}$$

(up to some reordering of the coordinates). Let a be such that for some (unique) $z \in \mathbb{R}^D$, we have $\dim \text{Aff}(x_1, \dots, x_s) = s - 1$, $\|x_1 - z\|^2 = \dots = \|x_s - z\|^2$, $\text{Im}((df_i)_{m_i}) \perp (z - x_i)$ and $\text{rank}((df_i)_{m_i}) = m$ for $i = 1, \dots, s$. Let us consider all $a' \in U$ such that the sources m'_i , the targets x'_i and the first differential $(df'_i)_{m'_i}$ of a' are equal to those of a , i.e. such that $a'_{\leq 1} = a_{\leq 1}$ (we let the $a'_{\geq 2}$ vary freely). All the conditions listed above are true for all such a' .

The key observation is the following: we can make the eigenvalues associated to the form $v \mapsto \langle z - x_i, d^2(f'_i)_{m'_i}(v, v) \rangle$ arbitrarily smaller than 0 (and doing so will ensure, as a side effect,

that $v \mapsto \|d(f'_i)_{m_i}(v)\|^2 - \langle z - x_i, d^2(f'_i)_{m_i}(v, v) \rangle$ be positive definite). To summarize what follows, this makes the differential $P_{[f'_i], x_i, z}$ at z of the projection on the image of f'_i tend to 0, which in turn makes $B_{a', z}(h)$ approximately equal to the square of the s -volume of the simplex $\text{Conv}(x_1, \dots, x_s, z + h)$ for $h \in E^\perp$, which is non-zero. This shows that the polynomial that associates the differentials of degree 2 of a' to $P(a', z)$ is non-trivial, thus that this polynomial is non-zero for most a' and defines a codimension 1 constraint. Now let us formalize this argument.

Remember that we defined the linear endomorphism $\Lambda : \text{Im}(d(f'_i)_{m_i}) \rightarrow \text{Im}(d(f'_i)_{m_i})$ associated to x_i, z, f'_i by stating that $\Lambda(v)$ must be the only vector in $\text{Im}(d(f'_i)_{m_i})$ such that (in the coordinates)

$$\langle z - x, d^2(f'_i)_{m_i}((d(f'_i)_{m_i})^{-1}(v), (d(f'_i)_{m_i})^{-1}(w)) \rangle = \langle \Lambda(v), w \rangle$$

for any $w \in \text{Im}(d(f'_i)_{m_i})$. As the eigenvalues associated to the form $v \mapsto \langle z - x_i, d^2(f'_i)_{m_i}(v, v) \rangle$ go to minus infinity, so must the eigenvalues of Λ , hence the linear map

$$P_{[f'_i], x_i, z} = (\text{id} - \Lambda)^{-1} \circ \pi_{\text{Im}(d(f'_i)_{m_i})} : \mathbb{R}^D \rightarrow \text{Im}(d(f'_i)_{m_i})$$

tends to 0. Then $B_{a', z}$ tends to

$$h \mapsto \sum_{i, j=1}^s \det(\tilde{A}_{a, z, i}(h)^\top \tilde{A}_{a, z, j}(h)),$$

where $\tilde{A}_{a, z, i}(h)$ is the $D \times s$ matrix whose j -th column is equal to $x_j - z$ if $j \neq i$ and whose i -th column is $-h$. Now the same arguments as in the proof of Lemma 6.7 show that

$$\sum_{i, j=1}^s \det(\tilde{A}_{a, z, i}(h)^\top \tilde{A}_{a, z, j}(h)) = (s!)^2 \text{Vol}_s(\text{Conv}(x_1, \dots, x_s, z + h))^2 + o(\|h\|^2).$$

But as $\dim \text{Aff}(x_1, \dots, x_s) = s-1$, if $h \in E^\perp = \text{Vec}(x_1 - z, \dots, x_s - z)^\perp$ then $\text{Vol}_s(\text{Conv}(x_1, \dots, x_s, z + h)) \geq C\|h\|$ for some constant $C > 0$ that depends on the positions of $x_1, \dots, x_s \in \mathbb{R}^D$. This shows that $B_{a', z}$ is non-degenerate if the eigenvalues associated to $v \mapsto \langle z - x_i, d^2(f'_i)_{m_i}(v, v) \rangle$ are negative enough.

For the same fixed $a = (a_{\leq 1}, a_{\geq 2}) \in U$ and associated $z \in \mathbb{R}^D$, the map

$$P_{a_{\leq 1}} : a'_{\geq 2} \mapsto P((a_{\leq 1}, a'_{\geq 2}), z)$$

is polynomial in the coordinates. As $B_{a', z}$ is non-degenerate for an appropriate choice of $a'_{\geq 2}$, the polynomial $P_{a_{\leq 1}}$ is non-trivially 0, hence

$$S_{a_{\leq 1}} := \{a'_{\geq 2} : B_{(a_{\leq 1}, a'_{\geq 2}), z} \text{ is well-defined and } P_{a_{\leq 1}}(a'_{\geq 2}) = 0\}$$

is an algebraic hypersurface (possibly with singularities) in the open set

$$\{a'_{\geq 2} : B_{(a_{\leq 1}, a'_{\geq 2}), z} \text{ is well-defined}\},$$

and in particular a semialgebraic set of codimension (at least) 1.

Consider the projection

$$\pi_{\leq 1} : (a_{\leq 1}, a_{\geq 2}) \mapsto a_{\leq 1}$$

defined on the domain U of the adapted chart. The image of the restriction of $\pi_{\leq 1}$ to $W_4^\circ \cap U$ is included in the set $W_2 = W_2^{(s)}$ of Definition 8.8 for $r = 1$. We know from Lemma 8.10 that W_2

is a finite union of Whitney stratified sets of codimension sm in $J_s^1(M, \mathbb{R}^D)$, and we can in fact assume that the image of $\pi_{\leq 1}|_{W_4^\circ \cap U}$ is included in one of those Whitney stratified sets by making U small enough. An easy consequence of Theorem 2.7.1 from [BR90] is that if $f : A \rightarrow B$ is a semialgebraic map between A and B two semialgebraic sets, the dimension of A is less than or equal to the dimension of B plus the maximum of the dimensions of the fibers $f^{-1}(b)$ (for $b \in B$). Now $\pi_{\leq 1} : W_4^\circ \cap U \rightarrow W_2$ is a semialgebraic map, and we have just shown that the fibers $(\pi_{\leq 1}|_{W_4^\circ \cap U})^{-1}(a_{\leq 1})$ for $a_{\leq 1} \in \text{Im}(\pi_{\leq 1}|_{W_4^\circ \cap U})$ have codimension at least 1. Hence $W_4^\circ \cap U$ has codimension at least $sm + 1$, which is enough to conclude. \square

As for Lemmas 8.10, 8.14 and 8.18, we choose for a given M and for each $s, r \in \mathbb{N}$ with $2 \leq s \leq D + 1$ and $2 \leq r \leq k - 1$ a decomposition $W_4^{(s)} = \cup_{\nu \in N} W_{4,\nu}^{(s)}$ of $W_4^{(s)}$ into a finite union of closed Whitney stratifiable sets of codimension at least $sm + 1$ whose existence is guaranteed by Lemma 8.23, and we simply refer to it as “the decomposition from Lemma 8.23” or “the sets $\{W_{4,\nu}\}_{\nu \in N}$ from Lemma 8.23” in what follows.

Lemma 8.24. *For any $2 \leq r \leq k - 1$, if $i \in \text{Emb}^k(M, \mathbb{R}^D)$ is such that $\mathbf{M} = i(M)$ satisfies condition (P1), (P2) and (P3) and is such that for all $s = 2, \dots, D + 1$, $j_s^r(i) : M^{(s)} \rightarrow J_s^r(M, \mathbb{R}^D)$ is transverse to the sets $\{W_{4,\nu}^{(s)}\}_{\nu \in N}$ from Lemma 8.23, then \mathbf{M} satisfies condition*

(P4) *There exist constants $C > 0$ and $\mu_0 \in (0, 1)$ such that for every $\mu \in [0, \mu_0)$, the set $Z_\mu(\mathbf{M})$ is included in a tubular neighborhood of size $C\mu$ of $Z(\mathbf{M})$, that is every point of $Z_\mu(\mathbf{M})$ is at distance less than $C\mu$ from $Z(\mathbf{M})$.*

Proof. Fix $2 \leq r \leq k - 1$ and $i \in \text{Emb}^k(M, \mathbb{R}^D)$ as in the statement. Our goal is to show that each critical point of $i(M)$ satisfies the criterion in Corollary 6.8; then the corollary will guarantee that $i(M)$ satisfies Condition (P4).

Let $z_0 \in Z(\mathbf{M})$ with $\pi_{\mathbf{M}}(z_0) = \{x_1, \dots, x_s\}$ for some $s \in \{2, \dots, D + 1\}$, and $E = \text{Vec}(x_j - x_l : j, l \in [s])$. Let $\delta > 0$ be small enough that there exist maps $p_j : B(z_0, \delta) \rightarrow B(x_j, \delta') \cap \mathbf{M}$ that satisfy the conclusions of Lemma 4.8 for $j = 1, \dots, s$ (and for some $\delta' > 0$). We have to prove that there exist constants $\delta > \tilde{\delta} > 0$ and $L > 0$ such that for any $h \in E^\perp$ with $\|h\| < \tilde{\delta}$, the s -volume of the s -simplex $\Delta(h) := \text{Conv}(\{z_0 + h\} \cup \{p_j(z_0 + h) : j \in [s]\})$ satisfies

$$\text{Vol}_s(\Delta(h)) \geq L\|h\|. \quad (32)$$

As in the proof of Lemma 8.15, the point $a = (m_1, \dots, m_s, x_1, \dots, x_s, [i], \dots, [i]) \in J_s^r(M, \mathbb{R}^D)$ (where $i(m_j) = x_j$) is such that $d(x_1, z_0) = \dots = d(x_s, z_0)$, $z_0 \in \text{Conv}(x_1, \dots, x_s)$, the rank of di_{m_j} is m , and for any chart $\phi : (U, 0) \rightarrow (M, m_j)$ the quadratic form $\mathbb{R}^m \rightarrow \mathbb{R}, v \mapsto \|d(i \circ \phi)_0(v)\|^2 - \langle z_0 - x_j, d^2(i \circ \phi)_0(v, v) \rangle$ is positive definite (for $j = 1, \dots, s$). The last point is true because we have assumed that i satisfies Condition (P3), which is equivalent to this property (as was shown in Lemma 3.1). If the form B_{a,z_0} from Definition 8.21 was degenerate on E^\perp , the point a would satisfy all the conditions of the definition of $(W_4^\circ)^{(s)}$ and hence would belong to $W_4^{(s)} \cap \text{Im}(j_s^r(i)) \subset J_s^r(M, \mathbb{R}^D)$. But by hypothesis $j_s^r(i) : M^{(s)} \rightarrow J_s^r(M, \mathbb{R}^D)$ is transverse to the Whitney stratifiable sets $\{W_{4,\nu}^{(s)}\}_{\nu \in N}$ and those are of codimension at least $sm + 1$; hence the intersection of $\text{Im}(j_s^r(i))$ with $W_{4,\nu}^{(s)}$ is trivial for any $\nu \in N$, and so is its intersection with $W_4^{(s)}$. Consequently, the form B_{a,z_0} is non-degenerate on E^\perp .

Remember that by definition, the form B_{a,z_0} is defined as

$$B_{a,z_0}(h) := \sum_{j_1, j_2=1}^s \det(A_{a,z_0,j_1}(h)^\top A_{a,z_0,j_2}(h))$$

for $h \in \mathbb{R}^D$, where $A_{a,z_0,j}(h)$ is the $D \times s$ matrix whose l -th column is equal to $x_l - z_0$ if $l \neq j$ and whose j -th column is $P_{[i],x_j,z_0}(h) - h$, with $P_{[i],x_j,z_0}$ being the differential at z_0 of the projection on a small neighborhood of x_j in M (as in the statement of Lemma 4.3). This projection coincides with $p_j : B(z_0, \delta) \rightarrow B(x_j, \delta') \cap M$, hence by definition of the form B from Equation 25 in Lemma 6.7 we have $B(h) = B_{a,z_0}(h)$ for all $h \in E^\perp$ with $\|h\| < \bar{\delta}$ for some $0 < \bar{\delta} < \delta$. By applying Corollary 6.8 and using the same reasoning for each critical point of M , this is enough to conclude that M satisfies Condition (P4). \square

8.4 Density of the transversality conditions

In the previous subsection, we defined five sets in the space of multijets that can be written as finite unions of closed Whitney stratified sets, and shown that the transversality to those sets of the map $j_s^r(i) : M^{(s)} \rightarrow J_s^r(M, \mathbb{R}^D)$ (respectively the map $(j_s^r)_{M\rho_i} : M^{(s)} \times \mathbb{R}^D \rightarrow J_s^r(M, \mathbb{R})$) induced by an embedding $i : M \rightarrow \mathbb{R}^D$ implies that $i(M)$ satisfies the conditions (P1)-(P4) that are of interest to us.⁹ It remains to show that the set of embeddings for which these properties of transversality are satisfied is dense in the Whitney topology; with this, we will have proved the second half of the Genericity Theorem 1.1, as well as Remark 1.3. To that end, the transversality theorems presented in Subsection 8.2 are the key ingredients.

Theorem 8.25 (The Density Theorem). *Let M be a compact C^k manifold for some $k \in \mathbb{N} \cup \{\infty\}$ with $k \geq 1$. Then, the set of C^k embeddings $i : M \rightarrow \mathbb{R}^D$ such that $M = i(M)$ satisfies conditions (P1), (P2), (P3) and (P4) is dense in $Emb^k(M, \mathbb{R}^D)$ for the Whitney C^k -topology.*

Proof. The only difficulty here is that the transversality theorems 8.1 and 8.3 only state results regarding the density of C^{r+1} functions whose multijets are transverse to sets in $J_s^r(M, \mathbb{R})$ and $J_s^r(M, \mathbb{R}^D)$, as opposed to C^r functions, and relatedly, we had to assume throughout Subsection 8.3 that M was a C^k manifold for $k \geq 3$. Hence we have to be somewhat cautious to account for the cases $k = 1, 2$, as proceeding naively would force us to ask that M be at least C^3 . The key ingredient here will be that C^{r_1} maps are dense in C^{r_2} maps for $r_1 \geq r_2$ and for the Whitney C^{r_2} -topology.

Throughout the proof, the sets $W_0^{(s)}, W_1^{(s)}, W_2^{(s)}, W_3^{(s)}$ and $W_4^{(s)}$ are as defined in Subsection 8.3. Let us start by assuming that $k \geq 3$. For $r = 2$ and a given $s = 2, \dots, D + 2$, the set $W_0^{(s)} \subset J_s^2(M)$ is a closed submanifold (hence in particular a closed Whitney stratifiable set). As $k \geq r + 1 = 3$, we can apply the transversality theorem adapted to the case of distance functions (Theorem 8.3) to $W_0^{(s)}$ (simply with $A = M^{(s)} \times \mathbb{R}^D$, in the notations of the theorem) to get that the set

$$F_{W_0^{(s)}} := \{f \in C^k(M, \mathbb{R}^D) : (j_s^2)_{M\rho_f} \text{ is transverse to } W_0^{(s)} \text{ in } J_s^2(M, \mathbb{R})\}$$

is residual in $C^k(M, \mathbb{R}^D)$ (note that we have not yet restricted ourselves to embeddings). Similarly, we have shown in Lemmas 8.10, 8.14, 8.18 and 8.23 that $W_1^{(s)}, W_2^{(s)}, W_3^{(s)}, W_4^{(s)} \subset J_s^2(M, \mathbb{R}^D)$ are finite unions of closed Whitney stratifiable subsets $\{W_{2,\alpha}^{(s)}\}_{\alpha \in A}$, $\{W_{3,\beta}^{(s)}\}_{\beta \in B}$, $\{W_{1,\gamma}^{(s)}\}_{\gamma \in C}$ and $\{W_{4,\nu}^{(s)}\}_{\nu \in N}$ respectively. The transversality theorem 8.1 (with $A = M^{(s)}$) then yields that for any $s = 2, \dots, D + 1$ and $\alpha \in A$, the set

$$F_{W_{2,\alpha}^{(s)}} := \{f \in C^k(M, \mathbb{R}^D) : j_s^2(f) \text{ is transverse to } W_{2,\alpha}^{(s)} \text{ in } J_s^2(M, \mathbb{R})\}$$

⁹The property of transversality must be verified for either $s = 2, \dots, D + 2$, $s = 2, \dots, D + 1$ or $s = 3, \dots, D + 1$ depending on the set considered.

is residual in $C^k(M, \mathbb{R}^D)$, and similarly for the sets

$$F_{W_{3,\beta}^{(s)}} := \{f \in C^k(M, \mathbb{R}^D) : j_s^2(f) \text{ is transverse to } W_{3,\beta}^{(s)} \text{ in } J_s^2(M, \mathbb{R})\}$$

and

$$F_{W_{4,\nu}^{(s)}} := \{f \in C^k(M, \mathbb{R}^D) : j_s^2(f) \text{ is transverse to } W_{4,\nu}^{(s)} \text{ in } J_s^2(M, \mathbb{R})\}$$

for $\beta \in B$, $\nu \in N$ and $s = 2, \dots, D+1$ and the sets

$$F_{W_{1,\gamma}^{(s)}} := \{f \in C^k(M, \mathbb{R}^D) : j_s^2(f) \text{ is transverse to } W_{1,\gamma}^{(s)} \text{ in } J_s^2(M, \mathbb{R})\}$$

for $\gamma \in C$ and $s = 3, \dots, D+1$. As $C^k(M, \mathbb{R}^D)$ is a Baire space (see e.g. [Hir94, Thm 4.4]), the (finite) intersection

$$F := \bigcap_{s=2}^{D+2} F_{W_0^{(s)}} \cap \bigcap_{s=3}^{D+1} \left(\bigcap_{\gamma \in C} F_{W_{1,\gamma}^{(s)}} \right) \cap \bigcap_{s=2}^{D+1} \left(\bigcap_{\alpha \in A} F_{W_{2,\alpha}^{(s)}} \cap \bigcap_{\beta \in B} F_{W_{3,\beta}^{(s)}} \cap \bigcap_{\nu \in N} F_{W_{4,\nu}^{(s)}} \right)$$

of residual sets is yet again residual, and in particular dense in $C^k(M, \mathbb{R}^D)$. As the set of embeddings $\text{Emb}^k(M, \mathbb{R}^D)$ is open in $C^k(M, \mathbb{R}^D)$ (see [Mic80, Prop 5.3]), the set $E := F \cap \text{Emb}^k(M, \mathbb{R}^D)$ is dense in $\text{Emb}^k(M, \mathbb{R}^D)$.

Let $i \in E$; then $(j_s^2)_M \rho_i$ is transverse to $W_0^{(s)}$ for $s = 2, \dots, D+2$, hence we know from Yomdin's Lemma 8.19 that $i(M)$ satisfies Condition (P0). From Lemma 8.15 (which we can apply, as we have assumed that i is C^k with $k \geq 3 = r+1$), and as $j_s^2(i)$ is transverse to the sets $\{W_{2,\alpha}^{(s)}\}_{\alpha \in A}$ and $\{W_{3,\beta}^{(s)}\}_{\beta \in B}$ for $s = 2, \dots, D+1$, we get that $i(M)$ satisfies conditions (P2) and (P3). This, in addition to the transversality of $j_s^2(i)$ to $\{W_{1,\gamma}^{(s)}\}_{\gamma \in C}$ for $s = 3, \dots, D+1$ and to $\{W_{4,\nu}^{(s)}\}_{\nu \in N}$ for $s = 2, \dots, D+1$, allows us to apply Lemmas 8.19 and 8.24 (again, i is at least C^3) to get that $i(M)$ also fulfills Conditions (P1) and (P4). Hence we have shown that the C^k embeddings that satisfy conditions (P1)-(P4) are dense in $\text{Emb}^k(M, \mathbb{R}^D)$ for the Whitney C^k -topology, which completes the proof when $k \geq 3$.

Assume now that $k \in \{1, 2\}$. Using Theorem 2.9 from [Hir94, Chapter 2], we can equip M with a C^3 differential structure that is compatible with its given C^k differential structure. To make a clear distinction, we write M_3 to refer to M equipped with this structure. Let $U \subset \text{Emb}^k(M, \mathbb{R}^D) = \text{Emb}^k(M_3, \mathbb{R}^D)$ be any non-empty open set of the Whitney C^k -topology. As the C^3 embeddings $\text{Emb}^3(M_3, \mathbb{R}^D)$ seen as a subset of $\text{Emb}^k(M_3, \mathbb{R}^D)$ are dense in $\text{Emb}^k(M_3, \mathbb{R}^D)$ for the Whitney C^k -topology (see e.g. [Hir94, Section 2, Thm 2.7]), there is a C^3 embedding $i \in U \cap \text{Emb}^3(M_3, \mathbb{R}^D)$. As we have shown above that the C^3 embeddings i' such that $i'(M)$ satisfies Conditions (P1-4) are dense in $\text{Emb}^3(M_3, \mathbb{R}^D)$ for the Whitney C^3 -topology, we can find such embeddings arbitrarily close to i for the C^3 -norm, hence also for the coarser C^k -norm. Such an embedding i' is in particular a C^k embedding for the initial structure on M , and if $\|i - i'\|_{C^k}$ is small enough, then i' must belong to U . Hence we can find $i' \in U$ such that $i'(M)$ satisfies Conditions (P1-4), which is enough to conclude. \square

Remark 8.26. *The observant reader might wonder why we did not use the ‘‘openness’’ part of the transversality theorems 8.1 and 8.3 to directly get that the set of C^k embeddings $i : M \rightarrow \mathbb{R}^D$ such that $i(M)$ satisfies conditions (P1), (P2), (P3) and (P4) is open in $\text{Emb}^k(M, \mathbb{R}^D)$ for the Whitney C^k -topology. The even more observant reader remembers that Theorem 8.1 guarantees*

that if $k \geq r + 1$, the set of C^k embeddings i such that $j_s^r(i)$ is transverse on $A \subset M^{(s)}$ to some closed Whitney stratified set $W \subset J_s^r(M, \mathbb{R}^D)$ is open in the Whitney C^{r+1} -topology under the condition that A be compact (and similarly for Theorem 8.3). Here we face two problems: the first is that we need transversality on $A = M^{(s)}$, which is not compact. This hurdle can in fact be avoided by controlling the distance between the projections of critical points, though some work is required. The second problem is that for our sets $W \subset J_s^r(M, \mathbb{R}^D)$ to be defined, we need $r \geq 2$, hence $k \geq 3$, and we could not find a way around this limitation; hence, though we could have shown that the set of suitable embeddings is open in the Whitney C^3 -topology using the two transversality theorems, we needed the more concrete and geometric proof of the C^2 Stability Theorem 1.4 to show that this set is also open in the Whitney C^2 -topology.

As a corollary of the Density Theorem and the C^2 Stability Theorem 1.4, we finally get the Genericity Theorem 1.1.

Proof of the Genericity Theorem 1.1. The density part of the statement comes directly from the Density Theorem 8.25. From the C^2 Stability Theorem 1.4, we get that the set of embeddings $i \in \text{Emb}^2(M, \mathbb{R}^D)$ such that $i(M)$ satisfies (P1), (P2), (P3) and (P4) is open in $\text{Emb}^2(M, \mathbb{R}^D)$ for the Whitney C^2 -topology. As the Whitney C^2 -topology on $\text{Emb}^k(M, \mathbb{R}^D) \subset \text{Emb}^2(M, \mathbb{R}^D)$ is the subspace topology inherited from the Whitney C^2 -topology on $\text{Emb}^2(M, \mathbb{R}^D)$, the set of C^k embeddings such that $i(M)$ satisfies (P1), (P2), (P3) and (P4) is also open in $\text{Emb}^k(M, \mathbb{R}^D)$. \square

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