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# On Covering Euclidean Spaces with Q-arrangements of Cones

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## Abstract

This paper is concerned with a covering problem of  $\mathbb{R}^n$  by stitching  $2^n$  cones along their common facets. The cones could be degenerate and are allowed to overlap. This covering problem provides a geometric equivalent reformulation of a relatively old, yet unsolved, problem that originated in the optimization community: under which conditions on the  $n \times n$  matrix  $M$ , does the so called linear complementarity problem given by  $w - Mz = q$ ,  $w, z \geq 0$ , and  $w \cdot z = 0$ , have a solution  $(w, z)$  for all vectors  $q \in \mathbb{R}^n$ ? If a solution exists for all  $q$ , the matrix  $M$  is called a Q-matrix. We investigate the regions where no solution exists (so called holes) and show that they only occur around specific vectors for  $n \leq 3$ . This property allows to fully characterize Q-matrices for  $n \leq 3$  as it shows that the covering reduces to studying locally the problem. In particular, Theorems 15 generalizes [1, Theorem 4.7] as it drops the strong non-degeneracy requirement. This in turn provides a symbolic characterization for Q-matrices for  $n \leq 3$ , that is an automatically generated list of constraints on the entries of the matrix that are satisfied if and only if the matrix is a Q-matrix.

**Keywords:** linear complementarity problem, Q-matrix, covering of Euclidean space, convex geometry, symbolic computation

## 1 Introduction

Given a vector  $q \in \mathbb{R}^n$  and an  $n \times n$  matrix  $M$  over the reals, the *linear complementarity problem* [2],  $\text{LCP}(q, M)$ , asks whether there exists a pair  $w, z \in \mathbb{R}^n$  satisfying  $w - Mz = q$ ,  $w, z \geq 0$ , and  $w \cdot z = 0$ , where  $x \geq 0$  means that  $x$  belongs to  $\mathbb{R}_+^n$ , the non-negative orthant of  $\mathbb{R}^n$ , and  $w \cdot z$  is the scalar product of  $w$  and  $z$ . When  $\text{LCP}(q, M)$  admits

a solution,  $\text{LCP}(q, M)$  is said to be *solvable*. When a partial solution satisfying only  $w, z \geq 0$  exists (i.e. when dropping the scalar product requirement),  $\text{LCP}(q, M)$  is said to be *feasible*. Related to the solvability and feasibility concepts, several classes of matrices are subsequently defined. When  $\text{LCP}(q, M)$  is feasible for all  $q$ ,  $M$  is called an *S-matrix*. When  $\text{LCP}(q, M)$  is solvable for all  $q$ ,  $M$  is called a *Q-matrix*. If furthermore such solution is unique for all  $q$ ,  $M$  is called a *P-matrix*. While a satisfying algebraic characterization for P-matrices has been known since the late fifties [3], the same cannot be said for Q-matrices.

Thinking of Q-matrixity as a property of the linear application  $M$  proved to be useful [4]. In this work, however, we rely on the geometric reformulation of Q-matrixity as a covering problem of the Euclidean space. Instead of fixing  $q \in \mathbb{R}^n$  and solving for  $w$  and  $z$ , one could rather fix the pair  $(w, z)$  and solve for the vectors  $q$ . The constraints  $w, z \geq 0$  and  $w \cdot z = 0$  imply that, for each  $i$ , either  $w_i$  or  $z_i$  has to vanish and the remaining component has to be non-negative. Let  $I$  denote the identity matrix of dimension  $n$ . One forms a piecewise linear map  $[-M, I] : b \mapsto [-M, I]_b$  with respect to a string of bits  $b$  of length  $n$  where  $[-M, I]_b$  denote the  $n \times n$  matrix having  $-M_j$  as its  $j$ th column if  $b_j = 1$  and  $I_j$  otherwise. Then,  $\text{LCP}(q, M)$  is solvable if there exists a  $b$  such that  $q$  belongs to the image set of  $\mathbb{R}_+^n$  through  $[-M, I]_b$ . Asking for a solution for each  $q \in \mathbb{R}^n$  becomes then equivalent to asking whether  $\mathbb{R}^n$  is included in the union of the  $2^n$  images of  $\mathbb{R}_+^n$  through  $[-M, I]_b$  for each valuation of  $b$  (see for instance [1] for more details about the correspondence between LCP and piecewise linear maps).

**Contributions.** For  $n = 3$ , we show in Proposition 14 that, when  $M$  is an S-matrix, the regions where no solution exists (holes) are located precisely around non-degenerate convex cones generated by the pair  $I_i, -M_i$ . This property is exploited in Theorems 15 and 24 to derive two characterizations of the problem by a local reduction around the columns of the matrices  $I$  and  $-M$ . In particular, Theorems 15 generalizes [1, Theorem 4.7] as it drops the strong non-degeneracy requirement. Unlike [1], we do not appeal to degree theory of piecewise linear maps to establish our results. We only require basic knowledge of convex geometry. Theorem 24 is amenable to an algebraic characterization leading to a decision procedure for  $3 \times 3$  Q-matrices. Our implementation proved useful to find interesting examples of Q-matrices.

## 2 Q-covering

Let  $g_1, \dots, g_m \in \mathbb{R}^n$ . The convex cone or simply *cone* (resp. linear subspace) spanned by the vectors  $g_i$  will be denoted by  $\langle g_1, \dots, g_m \rangle$  (resp.  $(g_1, \dots, g_m)$ ). A cone is said to be *non-degenerate* if its generators are linearly independent (as vectors in  $\mathbb{R}^n$ ). It is *degenerate* otherwise (and its topological interior is thus empty). The relative interior of a cone  $C$  will be denoted by  $C^\circ$ .<sup>1</sup> One proves that  $\langle g_1, \dots, g_m \rangle^\circ$  is the set spanned by the positive linear combination of  $g_1, \dots, g_m$ . When  $C = \langle g_1, \dots, g_m \rangle = (g_1, \dots, g_m)$ ,  $C$  is said to be *flat*. Flatness and degeneracy should not be confused. A flat cone is not necessarily degenerate (e.g.  $\mathbb{R}^n$  is flat and non-degenerate) and a degenerate cone is not necessarily flat (e.g. the half line  $\langle g_1, g_1 \rangle$ , with  $g_1 \neq 0$ ). Let

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<sup>1</sup>In classical textbooks, like [5],  $\langle g_1, \dots, g_m \rangle$  is denoted by  $\text{cone}\{g_1, \dots, g_m\}$  and the relative interior of  $C$  is denoted by  $\text{ri } C$ .

$\{a_1, a'_1\}, \dots, \{a_n, a'_n\}$  denote  $n$  pairs of vectors in  $\mathbb{R}^n$ . Let the matrices  $A$  and  $A'$  denote respectively  $(a_1 \cdots a_n)$  and  $(a'_1 \cdots a'_n)$ . Consider the mapping

$$\begin{aligned} [A, A'] : \quad & \{0, 1\}^n \rightarrow \mathbb{R}^{n \times n} \\ & b \mapsto [A, A']_b \end{aligned}$$

where the  $i$ th column vector of the matrix  $[A, A']_b$  is  $a_i$  if  $b_i = 1$  and  $a'_i$  otherwise. A *complementary cone*, or *c-cone*,  $C$  is the cone generated by the column vectors of  $[A, A']_b$  for some valuation of  $b$ , that is  $C = [A, A']_b(\mathbb{R}_+^n)$ , the image set of  $\mathbb{R}_+^n$  through the linear application represented by  $[A, A']_b$ . We call the column vectors of  $[A, A']_b$ , *the generators of the c-cone  $C$* . A *complementary facet*, or a *c-facet*, is a facet of a c-cone. Thus an element of a c-facet is a non-negative combination of  $m$ ,  $1 \leq m \leq n$ , column vectors of  $[A, A']_b$ . We define similarly a *complementary subspace*, or *c-subspace*, as the linear combination of  $m$ ,  $1 \leq m \leq n$ , columns of  $[A, A']_b$  for some  $b$ .

**Definition 1** (Covered and surrounded sets). *A vector is said to be covered if it belongs to a c-cone. A subset of  $\mathbb{R}^n$  is said to be covered if all its vectors are covered. A vector is said to be surrounded if it has a covered neighborhood. A subset of  $\mathbb{R}^n$  is said to be surrounded if all its vectors are surrounded.*

We are interested in understanding what conditions the vectors  $a_i$  and  $a'_i$  have to satisfy for the Euclidean space to be covered. Formally:

**Definition 2** (Q-covering). *Let  $\{a_i, a'_i\}$ ,  $i = 1, \dots, n$  denote a list of  $n$  pairs of vectors in  $\mathbb{R}^n$ . The Q-covering decision problem asks whether  $\mathbb{R}^n$  is covered, that is whether  $\mathbb{R}^n$  is included in  $\Sigma$  where*

$$\Sigma = \{a_1, a'_1\} \oplus \cdots \oplus \{a_n, a'_n\} := \bigcup_{b \in \{0, 1\}^n} [A, A']_b(\mathbb{R}_+^n) .$$

We adopt the sum notation for  $\Sigma$  in the sequel. We observe that  $\Sigma$  is invariant under any permutation of the indices ( $\oplus$  is commutative). We think of the vector  $a'_i$  as *symmetric* to  $a_i$  in the sense that  $\Sigma$  remains invariant when swapping  $a_i$  and  $a'_i$  for any  $i$ . In fact, the prime ‘ ’ can be thought of as an operator providing the symmetric of  $a_i$ . In particular,  $(a'_i)'$  is the same as  $a_i$  (involution). For instance, the opposite is a very special ‘ ’ operator: when  $a'_i = -a_i$ , for all  $i$ ,  $\Sigma$  is the partition of  $\mathbb{R}^n$  into the  $2^n$  standard orthants.

If  $L$  is a non-singular matrix, then  $q \in \mathbb{R}^n$  is covered if and only if  $q$  belongs to a c-cone  $\langle a_1, \dots, a_n \rangle$  say, which is equivalent to  $Lq \in \langle La_1, \dots, La_n \rangle$ . Therefore, the Q-covering problem is invariant under non-singular linear transformations of the involved vectors. If all c-cones are degenerate, then by Baire’s theorem,  $\mathbb{R}^n$  cannot be covered. Thus, at least one c-cone must be non-degenerate and its generators form therefore a basis for the Euclidean space. There is thus no loss of generality in considering the standard basis  $e_1, \dots, e_n$  of  $\mathbb{R}^n$  as required in the definition of the linear complementarity problem. The Q-covering problem doesn’t require the standard basis in its definition and this slight difference proves useful for symbolic computation purposes.

**Remark 3.** *Regardless of the exact geometric intersection between a pair of c-cones, each c-cone has  $n$  (abstract) neighbors where two c-cones respectively generated by the column vectors of  $[A, A']_b$  and  $[A, A']_{b'}$  are neighbors if and only if the Hamming distance between  $b$  and  $b'$  is exactly one. Therefore, the c-cones can be put in correspondence with the vertices of an  $n$ -dimensional hypercube graph  $Q_n$  where the neighboring relationship is represented by the adjacency of the vertices in  $Q_n$ . Stitching together all c-cones, one at a time, following their neighborhood relationship, amounts to following the longest Hamiltonian cycle of  $Q_n$  (of length  $2^n$ ). In general, a family of convex sets for which the aforementioned neighboring relation makes sense is said to form a Q-arrangement [6, Definition 3].*

By introducing the equivalence relation  $\simeq$  over  $\mathbb{R}^n \setminus \{0\}$  defined by  $u \simeq v$  if and only if  $u = \lambda v$  for some positive scalar  $\lambda$ , one observes that if a non-zero vector  $q$  is covered then so is  $q' \simeq q$ . Thus one can equivalently study the covering problem of the quotient space  $(\mathbb{R}^n \setminus \{0\}) / \simeq$  instead of  $\mathbb{R}^n$  (observing that  $q = 0$  is trivially covered as it belongs to all c-cones). The quotient space  $(\mathbb{R}^n \setminus \{0\}) / \simeq$  is homeomorphic to the unit sphere  $S_{n-1}$  where each half-line is represented by its unit generator. Similarly, each c-cone is represented by the (possibly degenerate) spherical  $(n - 1)$ -simplex formed by the representatives of the  $n$  generators of the cone. The spherical covering was for instance instrumental in [6, 7]. (Notice that the collection of the so obtained c-simplices needs not form a simplicial complex [8] since degeneracy and full dimensional intersections are allowed.)

In the sequel, we will find it useful to fix one or several coordinates of the Boolean vector  $b$ . For instance we define

$$\Sigma(a_i) := \bigcup_{\substack{b \in \{0,1\}^n \\ b_i=1}} [A, A']_b(\mathbb{R}_+^n), \quad \Sigma(a'_i) := \bigcup_{\substack{b \in \{0,1\}^n \\ b_i=0}} [A, A']_b(\mathbb{R}_+^n) .$$

Clearly, for all  $i$ ,  $\Sigma = \Sigma(a_i) \cup \Sigma(a'_i)$ . We say that  $\Sigma(a_i)$  is the set of c-cones *rooted* at  $a_i$  to make the syntactic requirement of the definition salient. (Indeed, a c-cone can be generated by  $a_i$  without necessarily having  $b_i = 1$ . For instance when  $a_2 \simeq a_1$ , the cone  $\langle a'_1, a_2 \rangle$  is generated by  $a_1$  while  $b_1 = 0$ .) In the sequel, the set of c-cones will be denoted by  $\text{cones}(\Sigma)$ . Similarly,  $\text{cones}(\Sigma(a_i))$  will denote the set of c-cones rooted at  $a_i$ .

A necessary condition for  $\mathbb{R}^n$  to be covered is that, for all  $i$ ,  $a_i, a'_i \neq 0$  and  $a'_i \not\approx a_i$  (geometrically, this means that  $\langle a_i, a'_i \rangle$  is not reduced to the origin nor it is a half-line). **Lemma 4.** *Assume there exists at least one non-degenerate c-cone. If all non-degenerate c-cones meet at a non-zero vector then  $\Sigma \subset \mathbb{R}^n$ . Moreover, if there exists an index  $i$  such that  $a_i = 0$  or  $a'_i \simeq a_i$  (with  $a_i \neq 0$ ) then  $\Sigma \subset \mathbb{R}^n$ .*

*Proof.* If  $a_i = a'_i = 0$  for some index  $i$ , then all c-cones are degenerate contradicting the assumption. Assume next that, there exists at least one non-degenerate c-cone and that for all  $i$ , either  $a_i \neq 0$  or  $a'_i \neq 0$ . Suppose there exists a point  $p \neq 0$  that belongs to all non-degenerate c-cones. The non-degenerate c-cones cannot cover  $-p$  as this would contradict the non-degeneracy assumption. This means that an open neighborhood  $U$  of  $-p$  is left to be covered by a finite number of degenerate cones,

which is impossible (by Baire's theorem). In particular, if  $a_i = 0$  and  $a'_i \neq 0$ , then all non-degenerate c-cones meet at  $p \simeq a'_i$  and the covering is impossible. Moreover, when  $a_i \neq 0$  and  $a'_i \simeq a_i$  then all c-cones meet at  $p \simeq a_i$  (in particular the non-degenerate ones), and the covering is also impossible.  $\square$

**Definition 5** (Hole). *A hole is a non-empty open connected region in  $\Sigma^c := \mathbb{R}^n \setminus \Sigma$ , the complement of  $\Sigma$  with respect to  $\mathbb{R}^n$ . A maximal hole is a hole  $K$  such that if  $K'$  is another hole,  $K \subseteq K'$  implies  $K' = K$ .*

Since  $\Sigma$  is a union of finitely many closed sets,  $\Sigma^c$  is an open set. In general,  $\Sigma^c$  is a union of disconnected maximal holes (in the sequel, we will consider only maximal holes and refer to them as holes). For instance, for  $n = 2$ ,  $\Sigma = \{e_1, -e_2\} \oplus \{e_2, -e_1\}$ ,  $\Sigma^c$  has two disconnected holes, namely  $\langle e_1, -e_2 \rangle^\diamond$  and  $\langle -e_1, e_2 \rangle^\diamond$ . Notice that a hole is not necessarily convex. For instance, when  $\Sigma = \{e_1, e_1\} \oplus \{e_2, e_2\} = \langle e_1, e_2 \rangle$ ,  $\Sigma^c = \mathbb{R}^2 \setminus \langle e_1, e_2 \rangle$  is a non-convex hole.

**Definition 6** (Separation). *Let  $H$  denote a hyperplane and let  $h$  denote a non-zero vector orthogonal to  $H$ . Two vectors  $u, v$  are said to be separated by  $H$  if and only if the scalar products  $u \cdot h$  and  $v \cdot h$  are not zero and have opposite signs. (Geometrically,  $u$  and  $v$  belong each to a distinct open half-space bounded by  $H$ .)*

Separation provides a simple, yet interesting, necessary condition on the pairs  $\{a_i, a'_i\}$  for the space to be covered.

**Proposition 7.** *If  $\mathbb{R}^n \subseteq \Sigma$  then there must exist a complementary hyperplane (that is a c-subspace of dimension  $n - 1$ ) separating each pair  $\{a_i, a'_i\}$ .*

*Proof.* By Lemma 4, for all  $i$ ,  $a_i, a'_i \neq 0$  and  $a_i \not\sim a'_i$ . The proof is by contradiction. Fix  $i$  and let  $V_r$ ,  $r = 1, \dots, 2^{n-1}$ , denote the c-subspaces generated by the remaining  $(n-1)$  vectors  $a_j, a'_j$ ,  $j \neq i$ . If  $\dim(V_r) < n - 1$  for all  $r$ , then all c-cones are degenerate and  $\Sigma \subset \mathbb{R}^n$ . Thus, there must exist a non-empty subset of indices  $S \subseteq \{1, \dots, 2^n\}$  such that  $V_s$  is a hyperplane for all  $s \in S$ . Assume that the pair  $\{a_i, a'_i\}$  is not separated by any hyperplane  $V_s$ ,  $s \in S$ . If  $a_i, a'_i \in V_s$  for all  $s$ , then again, all c-cones are degenerate and  $\Sigma \subset \mathbb{R}^n$ . Thus, there must exist a set of indices  $K \subseteq S$  such that  $a_i \notin V_k$  or  $a'_i \notin V_k$  for all  $k \in K$ . Let  $V_k(i)$ ,  $k \in K$ , denote the closed half-space bounded by  $V_k$  and containing either  $a_i$  or  $a'_i$  in its interior. We thus have  $a_i, a'_i \in V := \bigcap_{k \in K} V_k(i)$ . The boundary of  $V$ , denoted  $\partial V$ , is a subset of the union of the boundaries of  $V_k(i)$ . Since  $V$  is a (closed) convex cone then  $\langle a_i, a'_i \rangle \subseteq V$ . We show by contradiction that the topological interior of  $V$  is non-empty. Assume it is empty. Then  $V$  is equal to its boundary  $\partial V$  and  $V = \partial V \subseteq \bigcup_{k \in K} V_k$ . Since  $a'_i \in V$ , there exists an index  $k_1 \in K$  such that  $a'_i \in V_{k_1}$  and  $a_i \notin V_{k_1}$ . Thus  $\dim(a_i, a'_i) = 2$  and we construct a sequence of vectors  $p_j \neq 0$ ,  $1 \leq j \leq |K|$ , such that  $p_j \in \langle a_i, p_{j-1} \rangle^\diamond$  with  $p_1 = a'_i$ . For each  $j$ ,  $p_j \in V$ , so there exists an index  $k_j \in K$  such that  $p_j \in V_{k_j}$ . If  $k_j = k_\ell$  for  $1 \leq j, \ell \leq |K|$ ,  $j \neq \ell$ , then  $p_j, p_\ell \in V_{k_j} = V_{k_\ell}$ , and  $(a_i, a'_i) = (p_j, p_\ell) \subseteq V_{k_j}$ . But then  $a_i, a'_i \in V_{k_j}$ , contradicting the fact that  $k_j \in K$ . Thus  $k_1, \dots, k_{|K|}$  are all distinct making  $K = \{k_1, \dots, k_{|K|}\}$ . Now consider a vector  $p \neq 0$  in  $\langle a_i, p_{|K|} \rangle^\diamond$ . Then there exists  $k_j \in K$  such that  $p \in V_{k_j}$  and therefore  $(a_i, a'_i) = (p, p_j) \subseteq V_{k_j}$ , a contradiction. So the topological interior of  $V$  is non empty. We denote it by  $V^\diamond$ . We show next that  $-V^\diamond$  is a hole. First, by definition of  $V^\diamond$ ,  $V^\diamond \subseteq V_k(i)$ , thus  $-V^\diamond \subseteq V_k(i)^c$ , for each

$k \in K$ . Therefore  $-V^\diamond \subseteq \bigcap_k V_k(i)^c = (\bigcup_k V_k(i))^c$ . Second, let  $C(a_i, V_r)$  denote the c-cone spanned by  $a_i$  and the generators of  $V_r$ . For  $k \in K$ , since  $V_k$  doesn't separate the pair  $\{a_i, a'_i\}$  then  $C(a_i, V_k) \cup C(a'_i, V_k) \subseteq V_k(i)$ . Thus

$$\Sigma_K := \bigcup_{k \in K} (C(a_i, V_k) \cup C(a'_i, V_k)) \subseteq \bigcup_{k \in K} V_k(i),$$

and  $(\bigcup_{k \in K} V_k(i))^c \subseteq (\Sigma_K)^c$ . Thus  $-V^\diamond \subseteq (\Sigma_K)^c$ . Finally, all is left to cover  $-V^\diamond$  is a finite set of degenerate c-cones, namely  $C(a_i, V_r)$  and  $C(a'_i, V_r)$ ,  $r \notin K$ . Covering an open set with a finite number of degenerate cones is impossible (by Baire's theorem). So there must exist an index  $s \in S$  such that  $V_s$  is a hyperplane separating the pair  $\{a_i, a'_i\}$ .  $\square$

With respect to the notations of the proof of Proposition 7, the seminal work of Samelson et al. [3] shows that  $\Sigma$  partitions  $\mathbb{R}^n$  if and only if, for each index  $i = 1, \dots, n$  and for each  $r = 1, \dots, 2^{n-1}$ , the c-subspace  $V_r$  is a hyperplane separating the pair  $\{a_i, a'_i\}$ . Proposition 7 shows that, for  $\Sigma$  to be (only) covering, it is necessary that at least one c-subspace  $V_r$  is a hyperplane separating the pair. We also know that the necessary condition stated in Proposition 7 is not sufficient in general to prove that  $\Sigma$  is covering. For instance, in the plane, let  $a'_1 = -e_1 + e_2$  and  $a'_2 = -e_1$  and consider  $\Sigma = \{e_1, a'_1\} \oplus \{e_2, a'_2\}$  which only covers the closed upper half-plane. The pair  $\{e_1, a'_1\}$  is separated by the line  $(e_2)$  and the pair  $\{e_2, a'_2\}$  is separated by the line  $(a'_1)$ .

### 3 Q-covering for $n = 3$

**Proposition 8.** *Assume that  $a_i \neq 0$  and suppose that  $\langle a_i, a'_i \rangle$  is flat. Let  $a_i^\perp$  denote the hyperplane orthogonal to  $a_i$ , and let  $\bar{v}$  denote the orthogonal projection of a vector  $v$  onto  $a_i^\perp$ . Then  $\mathbb{R}^n \subseteq \Sigma$  if and only if the sum  $\{\bar{a}_1, \bar{a}'_1\} \oplus \dots \oplus \{\bar{a}_{i-1}, \bar{a}'_{i-1}\} \oplus \{\bar{a}_{i+1}, \bar{a}'_{i+1}\} \oplus \dots \oplus \{\bar{a}_n, \bar{a}'_n\}$  covers  $a_i^\perp$ .*

*Proof.* For clarity we fix  $i$  to 1. Let  $h \in a_1^\perp$ . Since  $\mathbb{R}^n \subseteq \Sigma$ , the vector  $x_h := \alpha_1 a_1 + h$  with  $\alpha_1 \in \mathbb{R}$ , belongs to a c-cone  $\langle a_1, \dots, a_n \rangle$  say. Thus  $x_h = \sum_i \lambda_i a_i$  where  $\lambda_i \geq 0$ . Therefore  $h = x_h - \alpha_1 a_1 = (\lambda_1 - \alpha_1) a_1 + \sum_{i \geq 2} \lambda_i a_i$ , and  $h = \bar{h} = \sum_{i \geq 2} \lambda_i \bar{a}_i$  as stated. The same occurs if  $x_h \in \langle a'_1, a_2, \dots, a_n \rangle$  since  $a_1 \simeq -a'_1$ . To prove the converse, let  $x \in \mathbb{R}^n$  and decompose  $x = \alpha_1 a_1 + h$  with  $h \in a_1^\perp$ . Thus  $h$  belongs to some cone  $\langle \bar{a}_2, \dots, \bar{a}_n \rangle$ . Equivalently, there exist  $\lambda_i \geq 0$  such that  $h = \sum_{i \geq 2} \lambda_i \bar{a}_i$ . We have  $a_i = \gamma_i a_1 + \bar{a}_i$ ,  $\gamma_i \in \mathbb{R}$ , for each  $i$ . Therefore

$$\begin{aligned} x &= \alpha_1 a_1 + h = \alpha_1 a_1 + \sum_{i \geq 2} \lambda_i \bar{a}_i = \alpha_1 a_1 + \sum_{i \geq 2} \lambda_i (a_i - \gamma_i a_1) \\ &= \underbrace{\left( \alpha_1 - \sum_{i \geq 2} \lambda_i \gamma_i \right)}_{\alpha} a_1 + \sum_{i \geq 2} \lambda_i a_i \end{aligned}$$

If  $\alpha \geq 0$  then  $x \in \langle a_1, \dots, a_n \rangle \subseteq \Sigma$  as required. If  $\alpha < 0$  then there exists  $\alpha' \geq 0$  such that  $\alpha a_1 = \alpha' a'_1$  (using  $a_1 \simeq -a'_1$ ) and  $x \in \langle a'_1, \dots, a_n \rangle$  proving that  $x \in \Sigma$  as well.  $\square$

**Proposition 9.** *The cone  $\langle a_i, a'_i \rangle$  cannot be partially covered, i.e. either  $\langle a_i, a'_i \rangle \subseteq \Sigma$  or  $\langle a_i, a'_i \rangle^\diamond \subseteq \Sigma^c$ .*

*Proof.* We fix  $i$  to 1 for clarity. If  $\dim(a_1, a'_1) \leq 1$ , then  $\langle a_1, a'_1 \rangle \subseteq \Sigma$ . Assume next that  $\dim(a_1, a'_1) = 2$  (in particular  $a'_1 \neq -a_1$ ). Suppose there exists  $a \in \langle a_1, a'_1 \rangle^\diamond$  such that  $a \in \Sigma$ . So  $a$  belongs to a c-cone  $C = \langle a_1, \dots, a_n \rangle$  say (otherwise swap  $a_1$  and  $a'_1$  in what follows). If  $a'_1 \in C$  then, since  $C$  is a convex cone,  $\langle a_1, a'_1 \rangle \subseteq C \subseteq \Sigma$ . Assume  $a'_1 \notin C$ . Using Lemma 32 (cf. Appendix A),  $\langle a_1, a'_1 \rangle^\diamond$  intersects the facet  $\langle a_2, \dots, a_n \rangle$  of  $C$  in a non-zero vector. By Lemma 38 (cf. Appendix B),  $\langle a_1, a'_1, a_2, \dots, a_n \rangle \subseteq \langle a_1, a_2, \dots, a_n \rangle \cup \langle a'_1, a_2, \dots, a_n \rangle$  which implies  $\langle a_1, a'_1 \rangle \subseteq \Sigma$ . Finally, if such  $a$  doesn't exist then  $\langle a_1, a'_1 \rangle^\diamond \subseteq \Sigma^c$ , which ends the proof.  $\square$

**Proposition 10.**  *$\langle a_i, a'_i \rangle \subseteq \Sigma$  if and only if either  $\dim(a_i, a'_i) \leq 1$  or  $\dim(a_i, a'_i) = 2$  and there exists a c-facet  $F_i := \langle a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n \rangle$  and a vector  $p \neq 0$  such that one of the following conditions occurs:*

1.  $p \in \langle a_i, a'_i \rangle \cap F_i$ ,    2.  $p \in \langle -a_i, a'_i \rangle^\diamond \cap F_i$ ,    3.  $p \in \langle a_i, -a'_i \rangle^\diamond \cap F_i$  .

*Proof.* We fix  $i$  to 1 for clarity. Suppose  $\langle a_1, a'_1 \rangle \subseteq \Sigma$  and that  $\dim(a_1, a'_1) = 2$ . Let  $a \in \langle a_1, a'_1 \rangle^\diamond$ , that is  $a = \alpha_1 a_1 + \alpha'_1 a'_1$ ,  $\alpha_1, \alpha'_1 > 0$ . Since  $a \in \Sigma$ , then either (i)  $a \in \langle a_1, \dots, a_n \rangle$  or (ii)  $a \in \langle a'_1, \dots, a_n \rangle$  for some c-facet  $F_1 = \langle a_2, \dots, a_n \rangle$ . For (i), one gets  $\alpha_1 a_1 + \alpha'_1 a'_1 = \sum_i \beta_i a_i$  that is  $(\alpha_1 - \beta_1) a_1 + \alpha'_1 a'_1 = \sum_{i \geq 2} \beta_i a_i$ . Let  $p = (\alpha_1 - \beta_1) a_1 + \alpha'_1 a'_1$ .

- If  $\alpha_1 - \beta_1 \geq 0$  then  $p \in \langle a_1, a'_1 \rangle \cap F_1$ . If  $p = 0$  then  $\dim(a_1, a'_1) < 2$ . Thus  $p \neq 0$  (Case 1 of the lemma).
- If  $\alpha_1 - \beta_1 < 0$  then  $p \in \langle -a_1, a'_1 \rangle^\diamond \cap F_1$  (Case 2 of the lemma).

The same discussion holds for by swapping  $a_1$  and  $a'_1$  for (ii), adding Case 3 of the lemma. For the converse, if  $\dim(a_1, a'_1) \leq 1$  then it is immediate that  $\langle a_1, a'_1 \rangle \subseteq \Sigma$ . Otherwise, one of the stated conditions holds. If the first condition holds, then by Lemma 38 (cf. Appendix B)  $\langle a_1, a'_1, a_2, \dots, a_n \rangle \subseteq \langle a_1, a_2, \dots, a_n \rangle \cup \langle a'_1, a_2, \dots, a_n \rangle \subseteq \Sigma$ . If the second condition holds, then by Lemma 39 (cf. Appendix B),  $\langle a_1, a'_1, a_2, \dots, a_n \rangle \subseteq \langle a_1, \dots, a_n \rangle \subseteq \Sigma$ . If the third condition holds, then by Lemma 39  $\langle a_1, a'_1, a_2, \dots, a_n \rangle \subseteq \langle a'_1, \dots, a_n \rangle \subseteq \Sigma$ . Thus  $\langle a_1, a'_1 \rangle \subseteq \langle a_1, a'_1, a_2, \dots, a_n \rangle \subseteq \Sigma$ , as stated.  $\square$

Let  $\Sigma_k$ ,  $0 \leq k \leq n$ , denote the  $k$ -skeleton of  $\Sigma$ , that is the union of all c-facets of dimensions up to  $k$  (for instance  $\Sigma_0$  is the set of vectors  $a_1, a'_1, \dots, a_n, a'_n$  and  $\Sigma_1$  is the set of half lines generated by these vectors). Let  $\Gamma = \langle a_1, a'_1, \dots, a_n, a'_n \rangle$ . Since we are working in dimension  $n$ ,  $\Gamma$  can be seen as the union of  $\binom{2n}{n}$  cones generated by any  $n$  vectors in  $\Sigma_0$ . The set of such cones will be denoted by  $\text{cones}(\Gamma)$ .

**Lemma 11.** *Assume  $\Gamma = \mathbb{R}^n$ . Then for any  $x \in \mathbb{R}^n$ , there exists a minimal cone  $G \in \text{cones}(\Gamma)$  containing  $x$ , that is  $G$  is non-degenerate for any other non-degenerate cone  $G' \in \text{cones}(\Gamma)$ ,  $G' \subseteq G$  implies  $G' = G$ .*

*Proof.* Since  $\Gamma = \mathbb{R}^n$ , by the Caratheodory theorem, there exists a non-degenerate cone  $G_1$  such that  $x \in G_1$  ( $G_1$  needs not be unique). Let  $g_1, \dots, g_n \in \Sigma_0$  denote the  $n$



generators of  $G_1$  and suppose there exists a non-zero vector  $g_{n+1} \in \Sigma_0 \setminus \{g_1, \dots, g_n\}$  such that  $g_{n+1} \in G_1$  and  $g_{n+1} \neq g_i$  for all  $1 \leq i \leq n$ . By Lemma 34 (cf. Appendix A), there exists a non-degenerate cone  $G_2 \subset G_1$  containing  $x$ . Suppose without loss of generality that  $g_1 \notin G_2$  (such vector must exist by construction of  $G_2$ ), so  $G_2 = \langle g_2, \dots, g_{n+1} \rangle$ . If  $G_2$  itself contains a vector  $g_{n+2} \in \Sigma_0 \setminus \{g_1, \dots, g_{n+1}\}$  such that  $g_{n+2} \neq g_i$  for all  $2 \leq i \leq n+1$ , then again by Lemma 34, there exists a non-degenerate cone  $G_3 \subset G_2$  containing  $x$ . At step  $k$ , we ask if  $G_k$  contains a vector  $g_{n+k} \in \Sigma_0 \setminus \{g_1, \dots, g_{n+k-1}\}$  such that  $g_{n+k} \neq g_i$  for all  $k \leq i \leq n+k-1$ . The process must terminate after at most  $n$  steps since  $\Sigma_0 \setminus \{g_1, \dots, g_{n+n}\}$  is empty. We let  $G = G_k$ ,  $1 \leq k \leq n$ , where  $G_k$  doesn't contain any vector from  $\Sigma_0$  (apart from those in the equivalence classes of its generators with respect to  $\simeq$ ). In particular there is no other cone  $G' \in \text{cones}(\Gamma)$  such that  $G' \subset G$ .  $\square$

Minimal cones play an important role in locating holes.

**Proposition 12.** *Assume  $\mathbb{R}^n = \Gamma$ . If  $\Sigma \subset \mathbb{R}^n$  then there exists a minimal cone  $G \in \text{cones}(\Gamma) \setminus \text{cones}(\Sigma)$  such that  $G \cap \Sigma^c$  is non-empty.*

*Proof.* Let  $x \in \Sigma^c$ . By Lemma 11, there exists a minimal cone  $G \in \text{cones}(\Gamma)$  containing  $x$ . The cone  $G$  cannot be a c-cone since it contains  $x$ . So  $G \notin \text{cones}(\Sigma)$ . Clearly we have  $x \in G \cap \Sigma^c$  and the intersection is non-empty.  $\square$

**Remark 13.** *One can tighten the statement of Proposition 12 by saying that  $G^\diamond \cap \Sigma^c$  is a non-empty open set. If  $x$  belongs to the interior of  $G$  then  $G^\diamond \cap \Sigma^c$  is non-empty. Otherwise, since  $\Sigma^c$  is an open set, a small perturbation of  $x$  remains in  $\Sigma^c$  while avoiding the boundaries of  $G$ . Thus  $G^\diamond \cap \Sigma^c$  is also non-empty.*

Observe that a cone in  $\text{cones}(\Gamma)$  which is not a c-cone must be rooted at both  $a_i$  and  $a'_i$  for some indices  $i$ . Proposition 9 showed that the cone  $\langle a_i, a'_i \rangle$  enjoys a special property: it cannot be partially covered. For  $n = 3$ , the next proposition shows that minimal cones of the form  $\langle a_i, a'_i, a_j \rangle$ ,  $j \neq i$ , enjoy themselves a special covering property.

**Proposition 14.** *Let  $G = \langle a_i, a'_i, a_j \rangle$ ,  $j \neq i$ , denote a minimal cone. If  $\langle a_i, a'_i \rangle \subseteq \Sigma$  then either  $G \subseteq \Sigma$  or  $a_i$  is not surrounded or  $a'_i$  is not surrounded.*

*Proof.* For clarity, we consider the cone  $\langle a_1, a'_1, a_2 \rangle$ . It suffices to permute the indices to match any other configuration. The proof is by case distinction using the characterization provided in proposition 10. Since  $G$  is minimal then  $\dim(a_1, a'_1) = 2$  and  $a'_2, a_3, a'_3 \notin G$ . Suppose that the c-facet  $\langle a_2, a_3 \rangle$  is responsible for  $\langle a_1, a'_1 \rangle \subseteq \Sigma$ , then:

- $\exists p \neq 0, p \in \langle a_1, a'_1 \rangle \cap \langle a_2, a_3 \rangle$ . Then, by Lemma 38 (cf. Appendix B),  $\langle a_1, a'_1, a_2, a_3 \rangle \subseteq \langle a_1, a_2, a_3 \rangle \cup \langle a'_1, a_2, a_3 \rangle$  and therefore  $G \subseteq \Sigma$ .
- $\exists p \neq 0, p \in \langle -a_1, a'_1 \rangle^\diamond \cap \langle a_2, a_3 \rangle$ . Then, by Lemma 39 (cf. Appendix B),  $\langle a_1, a'_1, a_2, a_3 \rangle \subseteq \langle a_1, a_2, a_3 \rangle$  and therefore  $G \subseteq \Sigma$ .
- $\exists p \neq 0, p \in \langle a_1, -a'_1 \rangle^\diamond \cap \langle a_2, a_3 \rangle$ . Then, by Lemma 39,  $\langle a_1, a'_1, a_2, a_3 \rangle \subseteq \langle a'_1, a_2, a_3 \rangle$  and therefore  $G \subseteq \Sigma$ .

Suppose that the c-facet  $\langle a'_2, a_3 \rangle$  is responsible for  $\langle a_1, a'_1 \rangle \subseteq \Sigma$ , then:

- $\exists p \neq 0, p \in \langle a_1, a'_1 \rangle \cap \langle a'_2, a_3 \rangle$ . By Proposition 45 (cf. Appendix B),  $G \subseteq \Sigma$  or  $G^\diamond \subseteq \Sigma(a_3)^c$ .

- $\exists p \neq 0, p \in \langle -a_1, a'_1 \rangle^\circ \cap \langle a'_2, a_3 \rangle$ . Then, by Proposition 49 (cf. Appendix B) either  $G \subseteq \Sigma$  or  $G^\circ \subseteq \Sigma(a_3)^c$  or there exists a non zero vector  $q$  such that  $\langle a_1, a_2, q \rangle^\circ \subseteq \Sigma(a_3)^c$  and  $q \in \langle a_1, a'_2 \rangle^\circ \cap \langle a'_1, a_2 \rangle^\circ$ .
- $\exists p \neq 0, p \in \langle a_1, -a'_1 \rangle^\circ \cap \langle a'_2, a_3 \rangle$ . Then, by Proposition 49 either  $G \subseteq \Sigma$  or  $G^\circ \subseteq \Sigma(a_3)^c$  or there exists a non zero vector  $q'$  such that  $\langle a'_1, a_2, q' \rangle^\circ \subseteq \Sigma(a_3)^c$  and  $q' \in \langle a'_1, a'_2 \rangle^\circ \cap \langle a_1, a_2 \rangle^\circ$ .

Notice that  $\langle a_1, a_2, q \rangle^\circ, \langle a'_1, a_2, q' \rangle^\circ \subseteq G^\circ$  and that  $q$  and  $q'$  cannot exist simultaneously. The only requirement for  $a_3$  is  $a_3 \notin G$ . Since  $a'_3 \notin G$ , the exact same statements hold for the c-facets  $\langle a_2, a'_3 \rangle$  and  $\langle a'_2, a'_3 \rangle$ . Moreover, the existence of  $q$  and  $q'$  is independent from  $a_3, a'_3$ . Several cases can therefore occur. We enumerate them all below:

- $G \subseteq \Sigma$ .
- $G^\circ \subseteq \Sigma(a_3)^c \cap \Sigma(a'_3)^c = \Sigma^c$ , so  $a_1, a'_1, a_2$  are not surrounded.
- $G^\circ \subseteq \Sigma(a_3)^c$  and  $\langle a_1, a_2, q \rangle^\circ \subseteq \Sigma(a'_3)^c$  making  $\langle a_1, a_2, q \rangle^\circ \subseteq \Sigma(a_3)^c \cap \Sigma(a'_3)^c = \Sigma^c$  and the same conclusion holds when swapping  $a_3$  and  $a'_3$ . Therefore  $a_1, a_2$  are not surrounded.
- $G^\circ \subseteq \Sigma(a_3)^c$  and  $\langle a'_1, a_2, q' \rangle^\circ \subseteq \Sigma(a'_3)^c$  making  $\langle a'_1, a_2, q' \rangle^\circ \subseteq \Sigma(a_3)^c \cap \Sigma(a'_3)^c = \Sigma^c$  and the same conclusion holds when swapping  $a_3$  and  $a'_3$ . Therefore  $a'_1, a_2$  are not surrounded.

In all case, either  $G \subset \Sigma$  or  $a_1$  is not surrounded or  $a'_1$  is not surrounded.  $\square$

**Theorem 15.**  $\Sigma = \mathbb{R}^3$  if and only if, for all  $i$ , both  $a_i$  and  $a'_i$  are surrounded.

*Proof.* Necessity is immediate. For sufficiency, we prove the contra-opposite, that is if  $\Sigma^c$  is non-empty then there exists an index  $i$  for which either  $a_i$  or  $a'_i$  is not surrounded. If  $\Gamma \subset \mathbb{R}^3$  then there exists a vector  $a \in \Sigma_0$  such that  $a$  is in the boundary of  $\Gamma$  and therefore  $a$  cannot be surrounded (if  $U$  is a neighborhood of  $a$ , then  $U \not\subseteq \Gamma$  and since  $\Sigma \subseteq \Gamma$ ,  $U \not\subseteq \Sigma$ ). By definition of  $\Sigma_0$ , there exists an index  $i$  such that either  $a = a_i$  or  $a = a'_i$ . Next, assume that  $\Gamma = \mathbb{R}^3$ . Since  $\Sigma \subset \mathbb{R}^3$ , by Proposition 12, there must exist a minimal cone  $G = \langle a_i, a'_i, a_j \rangle$ ,  $j \neq i$ , such that  $G \cap \Sigma^c$  is non-empty. By Proposition 9, either  $\langle a_i, a'_i \rangle^\circ \subseteq \Sigma^c$  or  $\langle a_i, a'_i \rangle \subseteq \Sigma$ . If the former holds then both  $a_i$  and  $a'_i$  are not surrounded. If the latter holds, then by Proposition 14, either  $a_i$  or  $a'_i$  is not surrounded. ( $G \subseteq \Sigma$  is impossible since  $G \cap \Sigma^c$  is non-empty.)  $\square$

Theorem 15 generalizes [1, Theorem 4.7] in that it doesn't require the so called *strong non degenerate* assumption, which translates for  $n = 3$  to saying that all cones (not only c-cones) in  $\text{cones}(\Gamma)$  are non-degenerate.<sup>2</sup> In [7, Theorem 3], Morris gives an example in  $n = 4$  where both  $a_i$  and  $a'_i$  are surrounded for all  $i$  without having a Q-covering. Therefore Theorem 15 cannot be generalized as is in higher dimension. In fact the (counter)example in  $n = 4$  shows that there exists a minimal cone  $G$  having all its generators surrounded while still having a hole in it formed by the intersection of c-facets. This observation isn't a surprise: the astute reader will observe that while the vectors  $q, q'$  in the proof of Proposition 14 cannot exist simultaneously, this isn't

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<sup>2</sup>The exact definition of strong non-degeneracy in [1] requires the non-degeneracy of *almost* c-cones which are cones rooted at exactly one pair  $\{a_i, a'_i\}$ . When  $n = 3$ , a cone in  $\text{cones}(\Gamma)$  is either a c-cone or an almost c-cone.

necessarily the case when  $n \geq 4$  where facets can intersect simultaneously surrounding the original vectors without preventing the existence of a hole. To better appreciate how one could eventually generalize Theorem 15 to higher dimensions, one needs to first understand the different kinds of surrounding. We do this in the next section in an attempt to make Theorem 15 amenable to a symbolic characterization of Q-matrices for  $n \leq 3$ .

## 4 Characterizing Q-matrices for $n \leq 3$

The following result gives a necessary condition for  $\mathbb{R}^n \subseteq \Gamma$ .

**Proposition 16.** *Let  $k \geq n + 1$  and  $g_1, \dots, g_k$  denote  $k$  non-zero vectors of  $\mathbb{R}^n$ . If  $\mathbb{R}^n = \langle g_1, \dots, g_k \rangle$  then there exists  $1 \leq m \leq n$  such that  $\langle g_{i_1}, \dots, g_{i_{m+1}} \rangle$  is a flat of dimension  $m$ .*

*Proof.* The proof is by induction on  $n$ . For  $n = 1$ , suppose  $\mathbb{R} = \langle g_1, \dots, g_k \rangle$  with  $k \geq 2$ . Then  $0 \in \langle g_1, \dots, g_k \rangle^\circ$  and there exist  $\lambda_1, \dots, \lambda_k > 0$  such that  $\sum_i \lambda_i g_i = 0$ . If  $g_i < 0$  for all  $i$  we get a contradiction, so there exists  $g_j > 0$ . Similarly, if  $g_i > 0$  for all  $i$  we also get a contradiction, so there exists  $g_\ell < 0$  and we have  $\mathbb{R} = \langle g_j, g_\ell \rangle$  and  $m = n = 1$ . Let  $\pi$  denote the orthogonal projection onto the hyperplane  $g_1^\perp$  ( $g_1 \neq 0$  by hypothesis). Let  $x \in g_1^\perp \subseteq \mathbb{R}^n = \langle g_1, \dots, g_k \rangle$ . Then  $x = \sum_{i=1}^k \lambda_i g_i$  with  $\lambda_i \geq 0$ . Thus  $x = \pi(x) = \sum_{i=2}^k \lambda_i \pi(g_i)$  and therefore  $g_1^\perp \subseteq \langle \pi(g_2), \dots, \pi(g_k) \rangle$ . The converse inclusion is immediate by definition of  $\pi$ . Thus  $g_1^\perp = \langle \pi(g_2), \dots, \pi(g_k) \rangle$ . By the induction hypothesis, there exists  $1 \leq m \leq n - 1$  such that  $\langle \pi(g_{i_2}), \dots, \pi(g_{i_{m+2}}) \rangle$  is a flat of dimension  $m$ . In particular  $\pi(g_{i_2}), \dots, \pi(g_{i_{m+1}})$  are linearly independent. Assume without loss of generality that  $i_j = j$ . We have  $0 = \sum_{j=2}^{m+2} \alpha_j \pi(g_j)$ , with  $\alpha_j > 0$ . Moreover, for each  $g_i$ , there exists  $\theta_i \in \mathbb{R}$  such that  $g_i = \theta_i g_1 + \pi(g_i)$ . Thus  $0 = \sum_{j=2}^{m+2} \alpha_j (g_j - \theta_j g_1) = (-\sum_{j=2}^{m+2} \alpha_j \theta_j) g_1 + \sum_{j=2}^{m+2} \alpha_j g_j$ . Let  $\gamma = -\sum_{j=2}^{m+2} \alpha_j \theta_j$ . If  $\gamma > 0$ , then by Lemma 35 (cf. Appendix B),  $(g_1, \dots, g_{m+2}) = (g_1, \dots, g_{m+1}) = \langle g_1, \dots, g_{m+2} \rangle$ . We further prove that  $g_1, \dots, g_{m+1}$  are linearly independent. Suppose  $0 = \sum_{i=1}^{m+1} \sigma_i g_i$  then  $0 = \sum_{i=2}^{m+1} \sigma_i \pi(g_i)$  and since  $\pi(g_2), \dots, \pi(g_{m+1})$  are linearly independent,  $\sigma_i = 0$  for all  $2 \leq i \leq m+1$ . Thus  $0 = \sigma_1 g_1$  and therefore  $\sigma_1 = 0$ . Thus  $\langle g_1, \dots, g_{m+2} \rangle$  is a flat of dimension  $m + 1$ . If  $\gamma = 0$ , then by Lemma 35,  $(g_2, \dots, g_{m+1}) = (g_2, \dots, g_{m+2}) = \langle g_2, \dots, g_{m+2} \rangle$ . Thus  $\langle g_2, \dots, g_{m+2} \rangle$  is a flat of dimension  $m$ . If  $\gamma < 0$ , then  $g_1 \in \langle g_2, \dots, g_{m+2} \rangle$  and  $\mathbb{R}^n = \langle g_2, \dots, g_k \rangle$ . We repeat the same reasoning with  $g_2$  and either we find a flat of dimension  $m$  or  $m + 1$  or  $g_2$  can be also removed from the list of generators till eventually reaching  $n + 1$  generators for  $\mathbb{R}^n$  at which point  $m = n$  and we are done.  $\square$

**Corollary 17.**  $\mathbb{R}^3 = \Gamma := \langle g_1, \dots, g_6 \rangle$  if and only if

- either there are 4 vectors such that  $\mathbb{R}^3 = \langle g_{i_1}, \dots, g_{i_4} \rangle$ ,
- or there are 3 vectors such that  $\langle g_{i_1}, g_{i_2}, g_{i_3} \rangle$  is a plane that separates two other vectors  $g_{i_4}$  and  $g_{i_5}$ ,
- or there are 2 vectors such that  $\langle g_{i_1}, g_{i_2} \rangle$  is a line and the plane  $g_{i_1}^\perp$  is spanned by  $\langle \pi(g_{i_3}), \dots, \pi(g_{i_6}) \rangle$ .

*Proof.* This is an immediate corollary of Proposition 16 where we distinguish all the cases for  $m$ . When  $n = 3$ ,  $m$  can be either 3, 2 or 1.  $\square$

**Definition 18** (Self surrounding). *The vector  $a_i$  is said to be self surrounded if it has a neighborhood covered by  $\Sigma(a_i)$ .*

Self surrounding can be seen as a Q-covering problem in lower dimension.

**Lemma 19.** *Suppose  $a_i \neq 0$ . Let  $\bar{v}$  denote the orthogonal projection of the vector  $v$  on  $a_i^\perp$ , the hyperplane orthogonal to  $a_i$ . Then  $a_i$  is self-surrounded if and only if  $\{\bar{a}_1, \bar{a}'_1\} \oplus \cdots \oplus \{\bar{a}_{i-1}, \bar{a}'_{i-1}\} \oplus \{\bar{a}_{i+1}, \bar{a}'_{i+1}\} \oplus \cdots \oplus \{\bar{a}_n, \bar{a}'_n\}$  defines a Q-covering of  $a_i^\perp$  (which is isomorphic to  $\mathbb{R}^{n-1}$ ).*

*Proof.* Let's fix  $i$  to 1. If there exists an neighborhood  $U \subseteq \Sigma(a_1)$ , then its projection  $\bar{U}$  onto  $a_1^\perp$  contains 0 in its relative interior and must be included by definition of  $\Sigma(a_1)$  in  $\{\bar{a}_2, \bar{a}'_2\} \oplus \cdots \oplus \{\bar{a}_n, \bar{a}'_n\}$ . Any non-zero vector in  $a_1^\perp$  has a representative (with respect to  $\simeq$ ) in  $\bar{U}$  and is therefore covered. Thus  $a_1^\perp \subseteq \{\bar{a}_2, \bar{a}'_2\} \oplus \cdots \oplus \{\bar{a}_n, \bar{a}'_n\}$ . For the converse, consider  $\Sigma' = \{a_1, -a_1\} \oplus \{a_2, a'_2\} \oplus \cdots \oplus \{a_n, a'_n\}$ . By Proposition 8,  $\mathbb{R}^n \subseteq \Sigma'$  and therefore  $a_1$  is surrounded with respect to  $\Sigma'$ . Moreover,  $a_1$  can only be self-surrounded (with respect to  $\Sigma'$ ) since if it belongs to any c-cone in  $\text{cones}(\Sigma'(-a_1))$  then such a c-cone must be degenerate and therefore it doesn't effectively contribute in surrounding  $a_1$ . Thus there exists a neighborhood  $U$  of  $a_1$  such that  $U \subseteq \Sigma'(a_1)$ . By definition of  $\Sigma'$ ,  $\Sigma'(a_1) = \Sigma(a_1)$ . Thus  $U \subseteq \Sigma(a_1)$  and  $a_1$  is self-surrounded with respect to  $\Sigma$  as desired.  $\square$

**Remark 20.** *In light of Definition 18 and Lemma 19, Proposition 8 can be reformulated as follows. Suppose that there exists an index  $i$  such that  $a_i \neq 0$  and that  $\langle a_i, a'_i \rangle$  is flat. Then  $\mathbb{R}^n$  is covered if and only if  $a_i$  is self surrounded.*

**Definition 21** (Lazy surrounding). *We say that  $a_i$  is lazily surrounded if it belongs to the (topological) interior of a non-degenerate c-cone rooted at  $a'_i$ .*

The vector  $a_i$  can be surrounded even if it is not self nor lazily surrounded. Surrounding however isn't easy to characterize in all cases even for  $n = 3$ . Therefore Theorem 15 has little practical interest when it comes to effectively checking Q-covering. Under the assumption  $\mathbb{R}^3 \subseteq \Gamma$ , we show below that we can alleviate the need for checking surrounding in general and that self surrounding is enough to characterize Q-covering for  $n = 3$ . We start by proving some special cases before stating the main theorem of this section (cf. Theorem 24 below).

**Lemma 22.** *Assume  $\Gamma = \mathbb{R}^3$ ,  $a_i, a'_i \neq 0$ , and  $a'_i \not\parallel a_i$ . If  $a_i \in \langle a'_i, a_j \rangle$ ,  $j \neq i$ , and  $a'_i$  is self surrounded or  $a'_i \in \Sigma(a_i)$  then all minimal cones rooted at  $a_i, a'_i$  are covered.*

*Proof.* For clarity we fix  $(i, j)$  to  $(1, 2)$ . (It suffices to permute the indices accordingly for any other configuration.) Since  $a_1 \in \langle a'_1, a_2 \rangle$ , then  $\langle a_1, a'_1, a_2 \rangle \subseteq \langle a'_1, a_2 \rangle \subseteq \Sigma$ ,  $\langle a_1, a'_1, a_3 \rangle \subseteq \langle a'_1, a_2, a_3 \rangle \subseteq \Sigma$ , and  $\langle a_1, a'_1, a'_3 \rangle \subseteq \langle a'_1, a_2, a'_3 \rangle \subseteq \Sigma$ . Only  $G = \langle a_1, a'_1, a'_2 \rangle$  is left out. Suppose  $G$  is minimal. Then  $a_2 \notin G$  and  $a_1 \in \langle a'_1, a_2 \rangle$  becomes equivalent to  $a_2 \in \langle -a'_1, a_1 \rangle^\diamond$ . Applying Lemma 47 (cf. Appendix B) for  $a_3$  and  $a'_3$ , either  $G \subseteq \Sigma$  or  $G^\diamond \subseteq \Sigma^c(a_3) \cap \Sigma^c(a'_3) = \Sigma^c$ . We show next that  $G^\diamond \subseteq \Sigma^c$  contradicts the hypothesis on  $a'_1$ . Suppose  $G^\diamond \subseteq \Sigma^c$ . Then  $a'_1$  is not surrounded. Moreover, by Lemma 47, there exist non zero vectors in  $\langle a'_2, a_3 \rangle \cap \langle -a_1, -a'_1 \rangle$  and  $\langle a'_2, a'_3 \rangle \cap \langle -a_1, -a'_1 \rangle$ . This implies that (i)  $-a'_1 \in \Sigma(a_1, a'_2)$  and (ii)  $a'_1 \notin \Sigma(a_1, a'_2)$  (otherwise,  $\langle a_1, a'_1 \rangle \subseteq \Sigma(a_1, a'_2)$

contradicting the intersections of  $\langle a'_2, a_3 \rangle$  and  $\langle a'_2, a'_3 \rangle$  with  $\langle -a_1, -a'_1 \rangle$ —cf. the proof of Proposition 10). Using (i) one gets  $a_2 \in \Sigma(a_1, a'_2)$  (because  $a_2 \in \langle -a'_1, a_1 \rangle^\circ$ ) leading to  $\Sigma(a_1) \subseteq \Sigma(a_1, a'_2)$ . Using (ii)  $a'_1 \notin \Sigma(a_1)$ .  $\square$

**Lemma 23.** *Assume  $\Gamma = \mathbb{R}^3$ ,  $a_i, a'_i \neq 0$ , and  $a'_i \neq a_i$ . Suppose  $a_i \in \langle a_j, a_k \rangle$  with  $i, j, k$  distinct. Then all minimal cones rooted at  $a_i, a'_i$  are covered.*

*Proof.* For clarity we fix  $i, j, k$  to 1, 2, 3 respectively. (It suffices to permute the indices accordingly for any other configuration.) Since  $a_1 \in \langle a_2, a_3 \rangle$  then  $\langle a_1, a'_1, a_3 \rangle \subseteq \langle a'_1, a_2, a_3 \rangle \subseteq \Sigma$  and  $\langle a_1, a'_1, a_2 \rangle \subseteq \langle a'_1, a_2, a_3 \rangle \subseteq \Sigma$ . Moreover  $a_1 \in \langle a_2, a_3 \rangle$  implies  $a_1 \in \langle a_2, a_3 \rangle \cap \langle a_1, a'_1 \rangle$ . By Proposition 45 (cf. Appendix B), with  $p = a_1$  ( $a_1 \neq 0$ ), both  $\langle a_1, a'_1, a'_2 \rangle$  and  $\langle a_1, a'_1, a'_3 \rangle$  are covered.  $\square$

**Theorem 24.** *Assume  $\Gamma = \mathbb{R}^3$ . Then  $\Sigma = \mathbb{R}^3$  if and only if, for each  $i$ ,  $a_i, a'_i \neq 0$ ,  $a'_i \neq a_i$ ,  $a_i$  is self surrounded or  $a_i \in \Sigma(a'_i)$ , and  $a'_i$  is self surrounded or  $a'_i \in \Sigma(a_i)$ . (Observe that the conditions are invariant by swapping  $a_i$  and  $a'_i$ .)*

*Proof.* If  $\mathbb{R}^3 \subseteq \Sigma$ , then by Lemma 4,  $a_i, a'_i \neq 0$  and  $a'_i \neq a_i$ . Moreover  $a_i, a'_i$  must be surrounded for all  $i$ . If  $a_i$  is not self surrounded then necessarily  $a_i \in \Sigma(a'_i)$  (otherwise  $a_i$  cannot be surrounded contradicting  $\mathbb{R}^3 \subseteq \Sigma$ ). The same holds for  $a'_i$ . We prove the converse by contradiction. Suppose that there exists a hole  $K$ . Since  $\Gamma = \mathbb{R}^3$ , then by Proposition 12 there exists a minimal cone  $G = \langle a_i, a'_i, a_j \rangle$ ,  $j \neq i$ , such that  $G \cap \Sigma^c$  is non-empty. By hypothesis,  $\langle a_i, a'_i \rangle$  is covered. Thus by Proposition 14,  $a_i$ , say, is not surrounded (otherwise it suffices to swap  $a_i$  and  $a'_i$  in what follows). In particular  $a_i$  is not self nor lazily surrounded. Therefore  $a_i \in \Sigma_2(a'_i)$ . If  $a_i \in \langle a_j, a_k \rangle$ , with  $i, j, k$  distinct, then by Lemma 23,  $G$  is covered. If  $a_i \in \langle a'_i, a_j \rangle$ ,  $j \neq i$ , then by Lemma 22,  $G$  is also covered. Both cases lead to a contradiction since  $G \cap \Sigma^c$  is non-empty. Thus  $K$  doesn't exist and  $\mathbb{R}^3 \subseteq \Sigma$ .  $\square$

**Remark 25.** *In general, the  $Q$ -covering problem doesn't enjoy an induction property around the involved vectors, that is, if  $\mathbb{R}^n \subseteq \Sigma$ , it is not necessary that  $a_i$  is self surrounded for all  $i$ . However, under a strong non-degeneracy assumption, requiring that all cones (not only  $c$ -cones) are non-degenerate,  $a_i \notin \Sigma_{n-1}(a'_i)$  and surrounding reduces to either self or lazily surrounding. We have already seen in Lemma 19 that self surrounding reduces to  $Q$ -covering problem in dimension  $n - 1$ . One proves that lazy surrounding can be also encoded as a  $Q$ -covering problem in dimension  $n - 1$  (cf. Appendix C). It is worth mentioning that, in [9], Cottle introduced a subclass of  $Q$ -matrices, coined completely  $Q$ -matrices, in which the inductive nature is preserved.*

## 4.1 $Q$ -covering for $n \leq 2$

For  $n = 2$ , Theorem 15 holds and surrounding can be fully characterized.

**Theorem 26.**  $\Sigma = \mathbb{R}^2$  if and only if, for all  $i$ ,  $a_i$  and  $a'_i$  are surrounded.

*Proof.* Necessity is immediate. For sufficiency, we prove the contra-opposite. Suppose  $\Sigma \subset \mathbb{R}^2$ . If  $\Gamma \subset \mathbb{R}^2$  then there exists  $a \in \Sigma_0$  such that  $a$  is in the boundary of  $\Gamma$  (recall that  $\Gamma$  is a cone by definition). Since  $\Sigma \subseteq \Gamma$ ,  $a$  cannot be surrounded. Next, suppose that  $\Gamma = \mathbb{R}^2$ . By proposition 12, there must exist a minimal cone  $G = \langle a_i, a'_i \rangle$  such

that  $G \cap \Sigma^c$  is non-empty and, by proposition 9,  $G^\circ \subseteq \Sigma^c$  proving that both  $a_i$  and  $a'_i$  are not surrounded.  $\square$

**Proposition 27.** *Let  $n = 2$ . Assume that, for all  $i$ ,  $a_i, a'_i \neq 0$ ,  $a'_i \not\simeq a_i$ . Then  $a_i$  is surrounded if and only if it is either self or lazily surrounded or  $a_i \simeq a_j$ ,  $j \neq i$ , and the pair  $\{\bar{a}'_j, \bar{a}'_i\}$  form a 1-dimensional Q-covering.*

*Proof.* Let  $j \neq i$ . First,  $a_i$  can be self surrounded by the union  $\langle a_i, a_j \rangle \cup \langle a_i, a'_j \rangle$ . Otherwise,  $\Sigma(a'_i)$  is involved in surrounding  $a_i$ . If  $a_i$  belongs to the topological interior of  $\langle a'_i, a_j \rangle$  or  $\langle a'_i, a'_j \rangle$  then  $a_i$  is lazily surrounded. Assume that  $a_i$  is not self nor lazily surrounded. Then  $a_i \in \Sigma_1(a'_i)$  (otherwise  $a_i$  cannot be surrounded). If  $a_j \simeq a_i$ , then  $a_i \not\simeq a'_j$  (otherwise  $a_j \simeq a'_j$ ), and  $a_i \notin \langle a'_i, a'_j \rangle$ . Then  $a_i$  must be surrounded by  $\langle a_i, a_j \rangle \cup \langle a_i, a'_j \rangle \cup \langle a_i, a_j \rangle \cup \langle a'_i, a_j \rangle = \langle a_i, a'_j \rangle \cup \langle a'_i, a_j \rangle = \langle a_i, a'_j \rangle \cup \langle a_i, a'_i \rangle$  which is equivalent to checking that  $\{\bar{a}'_i, \bar{a}'_j\}$  form a 1-dimensional Q-covering.  $\square$

Proposition 27 is amenable to a set of symbolic constraints to mechanically check for Q-covering in  $n = 2$ . For lazy surrounding, checking if  $a_i \in \langle a'_i, a_j \rangle^\circ$ ,  $j \neq i$ , amounts to simply check that the determinants of the three matrices  $(a'_i \ a_j)$ ,  $(a_i \ a'_i)$ , and  $(a_i \ a_j)$  have the same sign. Likewise, checking if  $a_i$  is self surrounded amounts to verify that the determinants of  $(a_i \ a_j)$  and  $(a_i \ a'_j)$ ,  $j \neq i$ , have opposite signs. We observe that this condition should not be confused with  $a_i \in \langle a_j, a'_j \rangle^\circ$ , which is only a special case ( $a_i$  needs not be in the interior of  $\langle a_j, a'_j \rangle$  to be self surrounded). Notice that self surrounding can be equivalently checked by projecting on the orthogonal space of  $a_i$  (cf. Lemma 19) and appealing to the following simple fact.

**Theorem 28.** *Let  $a, a' \in \mathbb{R}$ . The pair  $\{a, a'\}$  defines a Q-covering of  $\mathbb{R}$  if and only if  $aa' < 0$ , providing thereby a partition for  $\mathbb{R}$ .*

*Proof.* The cones generated by  $a$  and  $a'$  cover, or more precisely partition,  $\mathbb{R}$  if and only if  $a, a'$  are both non zero and have opposite signs.  $\square$

**Remark 29.** *Observe that, for  $n = 1$ ,  $\Gamma$  and  $\Sigma$  coincide and that  $a$  is surrounded if and only if  $a \neq 0$ . Interestingly, Theorem 15 doesn't hold for  $n = 1$  since  $a$  and  $a'$  can be both (lazily) surrounded (for instance when  $a' \simeq a$ ) and  $\Sigma = \Gamma \subset \mathbb{R}$ . A similar problem holds for  $n = 4$  in the counter example provided by Morris [7] where  $\Sigma \subset \Gamma = \mathbb{R}^4$  with all vectors lazily surrounded. Intuitively, when  $a'_i$  is lazily surrounded, it is somehow 'redundant' with  $a_i$  (with  $a'_i \simeq a_i$  being the simplest—and perhaps strongest—form of redundancy). So when  $a'_i$  is redundant for all  $i$ ,  $\Sigma$  is unlikely to be covering. We conjecture that, for all  $n \geq 1$ , when  $a_i, a'_i$  are lazily surrounded for all  $i$ ,  $\Sigma \subset \mathbb{R}^n$  (without necessarily having  $\Gamma \subset \mathbb{R}^n$ ). We also know that lazy surrounding can be allowed for some indices without preventing  $\Sigma$  to be covering. A natural question arises: how many vectors are allowed to be lazily surrounded at most before breaking the covering of  $\Sigma$ ? (for  $n \leq 2$  the answer is 1.)*

## 4.2 Q-matrices for $n = 2$

Assuming  $a_1 = e_1, a_2 = e_2$ , and  $a'_1, a'_2$  are the columns of the matrix  $-\begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$ , the conditions of Proposition 27 translate to the following algebraic characterization.

**Theorem 30.** *The matrix  $\begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$  is a Q-matrix if and only if*

$$\begin{aligned} & (m_1 < 0 \wedge m_2 > 0 \wedge m_3 > 0 \wedge m_2 m_3 > m_1 m_4 > 0) \\ \vee & (m_1 < 0 \wedge m_2 > 0 \wedge m_3 < 0 \wedge 0 > m_1 m_4 > m_2 m_3) \\ \vee & (m_1 = 0 \wedge m_2 > 0 \wedge m_3 < 0 \wedge m_4 > 0) \\ \vee & (m_1 > 0 \wedge m_3 = 0 \wedge m_4 > 0) \\ \vee & (m_1 > 0 \wedge m_2 \geq 0 \wedge m_4 > 0) \\ \vee & (m_1 > 0 \wedge m_2 < 0 \wedge m_3 > 0 \wedge m_1 m_4 > m_2 m_3) \\ \vee & (m_1 > 0 \wedge m_2 < 0 \wedge m_3 < 0 \wedge m_1 m_4 > m_2 m_3) \end{aligned}$$

For the sake of comparison, we give below the relatively much simpler conditions for  $M$  to be a P-matrix

$$m_1 > 0 \wedge m_4 > 0 \wedge m_1 m_4 > m_2 m_3 .$$

For instance, we can readily use this characterization to prove that the Q-covering fails if there exists a flat c-cone ( $a_j \simeq -a_i, j \neq i$ ). This in particular means that, for  $n = 2$ , when  $\text{LCP}(0, M)$  admits several solutions,  $M$  is not a Q-matrix (cf. Section 5 for a more detailed account on regular and super-regular matrices). As we shall see in the next section, however, this statement cannot be generalized. Indeed, it is possible to have a Q-covering of  $\mathbb{R}^3$  with flat c-cones (cf. example 31 below).

### 4.3 Q-matrices for $n = 3$

Using the planar algebraic characterization of the Q-covering problem, we further derive a symbolic characterization for  $3 \times 3$  Q-matrices. To check whether a point  $p \in \langle a_1, a_2, a_3 \rangle^\diamond$ , we simply verify that  $\det \begin{pmatrix} a_1 & a_2 & a_3 \\ p & a_2 & a_3 \end{pmatrix}, \det \begin{pmatrix} a_1 & a_2 & a_3 \\ a_1 & p & a_3 \end{pmatrix}, \det \begin{pmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & p \end{pmatrix}$  are all positive or all negative. (We can alternatively regard lazy surrounding as a Q-covering problem as stated in Appendix C.) To project a generic vector on the orthogonal space of a vector  $(a, b, c) \neq 0$ , we use the following generic projector:

$$\pi := \begin{cases} \begin{pmatrix} 0 & c & 0 \\ c & 0 & 0 \end{pmatrix} & \text{if } a = b = 0 \\ \begin{pmatrix} b & -a & 0 \\ ac & bc & -a^2 - b^2 \end{pmatrix} & \text{otherwise} \end{cases}$$

Thus,  $\bar{v}$  is simply  $\pi v$  for a given vector  $v$ . The equivalence relation  $u \simeq v$  is encoded as  $u \times v = 0 \wedge u \cdot v > 0$  where  $u \times v$  denotes the cross product.

We implemented our characterization in Mathematica, and were able to generate an example in  $n = 3$  showcasing that, unlike the planar case, a covering with flat c-cones is actually possible starting from dimension 3.<sup>3</sup> To the best of our knowledge,

<sup>3</sup>Code available here [khalilghorbal.info/assets/QCovering\\_n3\\_MAPR.zip](http://khalilghorbal.info/assets/QCovering_n3_MAPR.zip).



the smallest known dimension for which such a covering is possible is  $n = 5$  (provided in [10]).

**Example 31.** Consider  $e'_1 = (-2, -4, -3)$ ,  $e'_2 = -e_1$ , and  $e'_3 = (1, 1, 1)$ . The  $c$ -cones  $\langle e_1, e'_2, e_3 \rangle$  and  $\langle e_1, e'_2, e'_3 \rangle$  are flat and therefore the following  $3 \times 3$  matrix  $M$  is not super-regular, that is  $LCP(0, M)$  admits several solutions. It is however a  $Q$ -matrix.<sup>4</sup>

$$M = - (e'_1 \ e'_2 \ e'_3) = \begin{pmatrix} 2 & 1 & -1 \\ 4 & 0 & -1 \\ 3 & 0 & -1 \end{pmatrix} .$$

## 5 Related work

The current work is part of an extensive and rich literature, initiated almost seventy years ago, about characterizing and/or efficiently recognizing a  $Q$ -matrix. Below we restrict our attention to topological and geometric approaches.

**Degree of Piecewise Linear Maps.** Informally, the topological degree of continuous map is a way to measure how many times its domain wraps around its range. This idea has been exploited in [1] using a known functional reformulation of LCP as a piecewise linear map and then defining its *conical degree* as the local degree of the center 0. Under mild extra conditions on  $A, A'$ , the  $Q$ -covering problem is formulated as the surjectivity of  $[A, A']_b$  (denoted as  $f_{b,A,A'}$  in [1]). The authors show that the  $Q$ -covering holds if and only if the conical degree is non-zero for strongly non-degenerate  $3 \times 3$  matrices [1, Theorem 4.6]. It is further shown that under the same strong non-degeneracy assumption, a local study around the original vectors defining the problem is necessary and sufficient to decide the  $Q$ -covering problem. Theorem 15 generalizes this result as it drops

**Spherical Geometry.** In the late seventies, Kelly and Watson applied spherical geometry in [6] providing a (non-constructive) characterization of non-degenerate  $Q$ -matrices. More precisely, the authors established that if  $M_i$  and  $-e_i$  are in different components of the complement of the complex formed by the (complementary)  $(n-1)$ -simplices obtained when removing the index  $i$ , then one gets a  $Q$ -matrix. While the approach is insightful, the characterization doesn't provide a uniform property that all the pairs (or points) need to satisfy. Moreover,  $Q$ -covering is specifically about understanding whether the complex formed by the complementary  $n$ -simplices is covering, and one is left with studying a different problem (namely two points being in different components) for (at most)  $n$  complexes involving each  $2^{n-1}$  simplices. In [11] Cottle, Rando and Stone also made use of spherical geometry to derive a necessary condition for  $Q$ -matrices. Specifically, a property of spherically convex sets that applies for spherical simplices led to a characterization of  $Q$ -matrices for  $n = 2$  and a counter example was provided to show that the condition is necessary but not sufficient for  $n > 2$ .

**Relaxations and Triangulation.** By dropping the (only) non-linear condition  $w.z = 0$ , one gets a standard linear program which is well understood and easier

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<sup>4</sup>The fact that the provided matrix is a  $Q$ -matrix can be checked independently using for instance a quantifier elimination procedure over the reals, like the Cylindrical Algebraic Decomposition or the spherical characterization of Kelly and Watson [6].



to solve. Geometrically, such relaxation amounts to consider all involved  $n$ -simplices, regardless of complementarity. (We called this set  $\Gamma$  in this work.) If the non-complementary simplices are included in the complementary ones, the matrix  $M$  is known as a  $K$ -matrix. If moreover all the simplices are covering, then one immediately gets a characterization for  $Q$ -matrices. It is easy to see that if the cone  $\langle M_1, \dots, M_n \rangle$  (with  $M_i$  being the  $i$ th column of  $M$ ) intersects the interior of  $\langle e_1, \dots, e_n \rangle$ , then the entire space is covered by the union of all cones ( $\Gamma = \mathbb{R}^n$ ). It thus remains to study under which conditions the non-complementary cones are included in the complementary ones. This was performed by Fredricksen, Watson and Murty in [12, Theorem 9] for  $n = 3$ . The main theorem states that if some specific set of finite points (precisely  $3 \times 5! = 360$ ) are covered, then  $M$  is a  $Q$ -matrix. A similar generic approach consists in using all the points  $(-M_i$  and  $e_i)$  at once to build a regular triangulation then check for inclusions in complementarity simplices [13, Theorem 1].

**Simplicial Homology.** Given the equivalent spherical reformulation of the covering problem, it is very reasonable to investigate the usefulness and insights of standard homology groups to understand  $Q$ -covering. As the considered set of simplices need not define a simplicial complex (degeneracy and overlaps are allowed), in [14], Naiman and Stone exploited a related abstract simplicial complex called the *nerve* of the covering. Let  $V = \{1, \dots, m\}$  denote a set of vertices (0-simplices) where  $m \leq 2^n$  denotes the number of the non-degenerate  $n$ -simplices involved in the covering. An (abstract) simplex of the nerve is a subset  $W \subseteq V$  such that  $\bigcap_{j \in W} C_j$  is non-empty. The simplicial homology group  $H_{n-1}$  of the nerve is isomorphic to  $\mathbb{Z}$  whenever the  $S_{n-1}$  is covered and is isomorphic to the trivial group 0 otherwise ([14, Corollary 13]). Apart from the fact that the effective computation of the simplicial homology group has a non-elementary complexity (given the exponential number of the  $n$ -simplices involved), such approach doesn't exploit the underlying structure of  $Q$ -covering. The nerve can be defined for any collection of subsets which makes it a powerful theoretical tool, but less insightful to capture the topological properties of the  $Q$ -covering.

**Monotonicity.** Several attempts tackled successfully either sub-classes or larger ones with computationally practical properties hoping it would lead to a better understanding of  $Q$ -matrices (for an overview on the class of matrices found in complementarity theory one can check [15]). Some were characterized by signs of sub-determinants of  $M$  like  $P$ -matrices [3] and completely- $Q$ -matrices [9]. In [16], it has been shown that when  $M \geq 0$ ,  $M \in Q$  if and only if all the diagonal entries of  $M$  are positive. However in [17], Watson proved that a characterization of  $Q$ -matrices cannot only rely on the sign of principal minors of  $M$ . In [18], the characterizations of  $P$ -matrices and completely- $Q$ -matrices were extended to a more general setting, namely that of oriented matroids, which the author believed was the natural setting of the problem. Following an attempt to prove a similar result for  $Q$ -matrices in [19], Morris showed in [7] that  $Q$ -matrices cannot be characterized by signs of sub-determinants.

**Multiplicities and related classes.** In [20], Karamardian introduced the class of regular matrices, the first class of matrices, in connection with complementarity theory, that may have negative elements in its diagonal. He proved that if  $M$  is a regular matrix then  $M$  is a  $Q$ -matrix. Moreover, he was one of the first to draw a specific attention to  $R_0$  matrices (also known as super-regular matrices) for which the

LCP has a unique solution at  $q = 0$ . In [21], he proved that if in addition, for some strictly positive  $q$  the LCP has also a unique solution, then  $M$  is a Q-matrix. Shortly after, Saigal strengthened Karamardian's result in [22] by proving that if the LCP has a unique solution for  $q = 0$  and an odd number of non-degenerate solutions for some other vector  $q$ , then  $M$  is necessarily a Q-matrix. Later in the same decade, in their paper [23], Aganagic and Cottle proved that among  $P_0$  matrices (i.e. matrices for which all principal minors are non-negative), the subclasses of Q-matrices, regular matrices and  $R_0$ -matrices are equivalent. Pang then showed in [24] that a similar characterization to that of  $P_0$  matrices, holds for L-matrices. And conjectured the same to be true for semi-monotone matrices ( $L_1$ -matrices). In [10] Jeter and Pye provided a counter-example showcasing a semi-monotone  $5 \times 5$ -matrix that is Q but not  $R_0$  and thus falsifying Pang's conjecture. Then Gowda proved in [25] that the conjecture is true for symmetric matrices. In addition, Pye proved in [26] that for the class of square matrices that are not  $P_0$ -matrices but  $P_0$ -matrices of order  $n - 1$  and possess an inverse containing at least one positive entry, the classes of Q-matrices and regular matrices coincide. Also, he provided necessary and sufficient conditions for non-singular  $N_0$  matrices to be Q-matrices.

## Conclusion

The main contribution of this work is the particular covering property of minimal cones of the form  $\langle a_i, a'_i, a_j \rangle$ ,  $j \neq i$ , whenever  $\langle a_i, a'_i \rangle$  is covered (Proposition 14). This property offers a better understanding of the shape and location of holes when  $\Gamma$  is covering (Theorem 15). We exploited these insights to derive a symbolic characterization for Q-matrices in full generality (regardless of degeneracy) in the three dimensional case which, up to our knowledge, did not exist before (Theorem 24). We found such a characterizations useful in order to generate automatically counter-examples for serious conjectures out of reach with our current understanding of the Q-covering problem. For instance, while we knew that flat c-cones are not allowed in dimension 2 and possible in dimension 5, it wasn't clear whether they can occur in dimensions 3 or 4. Example 31 provides an example in dimension 3.

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## A Technical Lemmas

**Lemma 32.** *Let  $G = \langle g_1, \dots, g_m \rangle$ ,  $m \geq 2$ , denote a cone in  $\mathbb{R}^n$  and let  $q \notin G$  such that  $\langle g_1, q \rangle$  is non-flat. Suppose there exists a non-zero vector in  $\langle g_1, q \rangle^\circ \cap G$  then there exists a non-zero vector in  $\langle g_2, \dots, g_m \rangle \cap \langle g_1, q \rangle^\circ$ .*

*Proof.* Let  $p' \neq 0$  be in  $\langle g_1, q \rangle^\circ \cap G$ . Then there exists  $\lambda_1, \lambda > 0$  and  $\alpha_1, \dots, \alpha_m \geq 0$  such that  $p' = \lambda_1 g_1 + \lambda q = \sum_{i=1}^m \alpha_i g_i$ . Since  $q \notin G$  then  $0 < \lambda_1 - \alpha_1$ . Let  $p = (\lambda_1 - \alpha_1)g_1 + \lambda q$ . Then,  $p \in \langle g_2, \dots, g_m \rangle \cap \langle g_1, q \rangle^\circ$ . If  $p = 0$  then  $-g_1 \simeq q$  ( $q \neq 0$  because  $q \notin G$ ), making  $\langle g_1, q \rangle$  flat. Thus  $p \neq 0$ .<sup>5</sup>  $\square$

**Lemma 33.** *Let  $u, v, g_1, \dots, g_m \in \mathbb{R}^n$ ,  $1 \leq m \leq n - 1$ , such that  $u, g_1, \dots, g_m$  or  $v, g_1, \dots, g_m$  are linearly independent. Then  $u \in \langle v, g_1, \dots, g_m \rangle$  and  $v \in \langle u, g_1, \dots, g_m \rangle$  if and only if  $u \simeq v$ . When  $m = 1$ , the linear independence condition can be dropped.*

*Proof.* There exist  $\alpha, \lambda, \alpha_i, \lambda_i \geq 0$  such that  $u = \alpha v + \sum_{i=1}^m \alpha_i g_i$  and  $v = \lambda u + \sum_{i=1}^m \lambda_i g_i$ . Assume  $v, g_1, \dots, g_m$  are linearly independent (otherwise, swap  $u$  and  $v$  in what follows). We get  $(1 - \lambda\alpha)v = \sum_{i=1}^m (\lambda\alpha_i + \lambda_i)g_i$ . The linear independence implies  $1 - \lambda\alpha = 0$  and  $\alpha\lambda_i + \lambda_i = 0$  for all  $i$ . Thus  $\alpha, \lambda \neq 0$  and  $\alpha_i = \lambda_i = 0$  for all  $i$  proving the statement. When  $m = 1$ , if  $\lambda\alpha_1 + \lambda_1 = 0$  then  $\alpha, \lambda > 0$  and  $\alpha_1 = \lambda_1 = 0$  and therefore  $v \simeq u$ . Likewise, if  $\alpha\lambda_1 + \lambda_1 = 0$ ,  $u \simeq v$ . Suppose  $\lambda\alpha_1 + \lambda_1 > 0$  and  $\alpha\lambda_1 + \lambda_1 > 0$ . If  $1 - \lambda\alpha < 0$  then  $u \simeq -g_1$  and  $v \simeq -g_1$  so  $u \simeq v$ . Otherwise  $u \simeq v \simeq g_1$ .  $\square$

**Lemma 34.** *Let  $G = \langle g_1, \dots, g_n \rangle$  denote a cone in  $\mathbb{R}^n$ . Let  $g$  be a non-zero vector in  $G$  and let  $G_i$  denote the cone generated by  $\{g, g_1, \dots, g_n\} \setminus \{g_i\}$ ,  $1 \leq i \leq n$ . Then  $G = \cup_i G_i$ . If moreover  $G$  is non-degenerate then there exists a non-empty set of indices  $J \subseteq \{1, \dots, n\}$  such that  $G = \cup_{j \in J} G_j$ ,  $G_j$  is non-degenerate, and  $G_j \subseteq G$ , for all  $j \in J$ . Moreover  $G_j = G$  if and only if  $g \simeq g_j$ .*

*Proof.* We have  $g = \sum_{i=1}^n \alpha_i g_i$  for some non-negative coefficients  $\alpha_i$  and, since  $g \neq 0$ , there exists a non-empty set of indices  $J \subseteq \{1, \dots, n\}$  such that  $\alpha_j > 0$  for all  $j \in J$ . Let  $x \in G$ , i.e.  $x = \sum_{i=1}^n \beta_i g_i$  for some non-negative coefficients  $\beta_i$ . Let  $\lambda = \min_{j \in J} \{\frac{\beta_j}{\alpha_j}\} = \frac{\beta_k}{\alpha_k}$  for some  $k \in J$ . We then have, for all  $i = 1, \dots, n$ ,  $\beta_i - \lambda\alpha_i \geq 0$  and

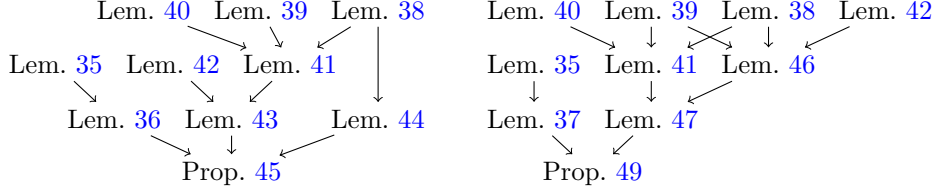
$$x = \sum_{i=1}^n \beta_i g_i = \lambda g + \sum_{i=1}^n (\beta_i - \lambda\alpha_i) g_i = \lambda g + \sum_{i \neq k} (\beta_i - \lambda\alpha_i) g_i .$$

Thus  $x \in G_k$  and  $G \subseteq \cup_{j \in J} G_j \subseteq \cup_i G_i$ . This proves the first statement. Suppose  $G$  is non-degenerate and  $G_j$  is degenerate for an index  $j \in J$ , then  $g$  belongs to the hyperplane  $H_j$  generated by  $g_i$ ,  $i \neq j$ . But then  $g_j \in H_j$  (because  $\alpha_j > 0$ ) and  $G$  is itself degenerate, a contradiction. Thus  $G_j$  is non-degenerate for all  $j \in J$ . Since  $g \in G$ ,  $G_j \subseteq G$  for all  $j$ . If  $G \subseteq G_j$  for some index  $j$ , then by Lemma 33,  $g \simeq g_j$ .  $\square$

<sup>5</sup>The original proof was by induction on  $m$ . The provided shorter constructive proof was suggested by Jean-Charles Gilbert in a private communication with the first author.

## B Proof of proposition 14

We start by stating several technical lemmas before proving the two main cases required in Proposition 14, namely when  $\langle a_1, a'_1 \rangle$  intersects  $\langle a'_2, a_3 \rangle$  in a non-zero vector (Proposition 45 in Section B.2) and when  $\langle -a_1, a'_1 \rangle^\diamond$  intersects  $\langle a'_2, a_3 \rangle$  in a non-zero vector (Proposition 49 in Section B.3). For convenience, the proof structures of both propositions are depicted in (the transitive closure of) the following graphs. (An arrow from  $X$  to  $Y$  means that  $X$  is required to prove  $Y$ .)



### B.1 Preliminaries

**Lemma 35.** *Suppose that  $0 \in \langle g_1, \dots, g_{m+1} \rangle^\diamond$ ,  $1 \leq m \leq n$ , then  $\langle g_1, \dots, g_{m+1} \rangle$  is flat of dimension at most  $m$ . Moreover,  $\langle g_1, \dots, g_{m+1} \rangle = \cup_i G_i$  where  $G_i$ ,  $1 \leq i \leq m+1$ , is the cone generated by  $\{g_1, \dots, g_{m+1}\} \setminus \{g_i\}$ .*

*Proof.* There exists  $\alpha_1, \dots, \alpha_{m+1} > 0$  such that  $0 = \sum_{i=1}^{m+1} \alpha_i g_i$ . Let  $x \in \langle g_1, \dots, g_m \rangle$ . Then  $x = \sum_{i=1}^m \lambda_i g_i$  where  $\lambda_i \in \mathbb{R}$ . If  $\lambda_i \geq 0$  for all  $1 \leq i \leq m$ , then  $x \in \langle g_1, \dots, g_m \rangle \subseteq \langle g_1, \dots, g_{m+1} \rangle$ . Otherwise, there exists a non-empty set of indices  $J$  such that  $\lambda_j < 0$  for all  $j \in J$ . Let  $\lambda = \max_{j \in J} \{-\frac{\lambda_j}{\alpha_j}\} > 0$ . So, for all  $i$ ,  $\lambda_i + \lambda \alpha_i \geq 0$ . Then

$$x = x + \lambda 0 = \sum_{i=1}^m \lambda_i g_i + \lambda \sum_{i=1}^{m+1} \alpha_i g_i = \lambda \alpha_{m+1} g_{m+1} + \sum_{i=1}^m (\lambda_i + \lambda \alpha_i) g_i$$

and  $x \in \langle g_1, \dots, g_{m+1} \rangle$ . Therefore  $\langle g_1, \dots, g_{m+1} \rangle = \langle g_1, \dots, g_m \rangle \subseteq \langle g_1, \dots, g_{m+1} \rangle \subseteq \langle g_1, \dots, g_{m+1} \rangle$ . This proves the first part of the lemma. For the second part, if  $J$  is empty, then  $x \in G_{m+1}$ , otherwise, if we let  $k \in J$  denotes the index for which  $\lambda = -\frac{\lambda_k}{\alpha_k}$  and one sees that  $x \in G_k$ . Thus  $\langle g_1, \dots, g_{m+1} \rangle = G_{m+1} \cup (\cup_{j \in J} G_j) \subseteq \cup_{i=1}^{m+1} G_i$ .  $\square$

**Lemma 36.** *Let  $G = \langle g_1, \dots, g_m \rangle$ ,  $2 \leq m \leq n$ , denote a non-degenerate cone of  $\mathbb{R}^n$  and let  $H = \langle h_1, h_2 \rangle$  with  $h_1, h_2 \in \langle g_1, \dots, g_m \rangle \cap G^c$  and  $H$  non-flat. If  $H$  intersects one facet of  $G$  in a non-zero vector then  $H$  intersects another facet of  $G$  in a non-zero vector.*

*Proof.* Without loss of generality, suppose that  $H$  intersects the facet  $\langle g_1, \dots, g_{m-1} \rangle$  in a non-zero vector  $p$ . There exists  $\lambda_1, \lambda_2, \alpha_1, \dots, \alpha_{m-1} \geq 0$  and  $(\lambda_1, \lambda_2) \neq (0, 0)$  such that  $p = \sum_{i=1}^{m-1} \alpha_i g_i = \lambda_1 h_1 + \lambda_2 h_2$ . If  $\lambda_1 = 0$  then  $\lambda_2 > 0$  and  $h_2 \in G$ , a contradiction. The same holds if  $\lambda_2 = 0$ . Thus  $\lambda_1, \lambda_2 > 0$ . If  $\alpha_i = 0$  for any index  $i = 1, \dots, m-1$ , then  $p$  belongs to another facet of  $G$ , namely the one generated by  $\{g_1, \dots, g_m\} \setminus \{g_i\}$ . Next, suppose  $\alpha_i > 0$  for all  $i$ . Since  $\lambda_1, \lambda_2 > 0$ , then  $0 \in \langle -h_1, -h_2, g_1, \dots, g_{m-1} \rangle^\diamond$ . By Lemma 35 (applied for  $-h_1$  as  $g_{m+1}$  in the lemma),  $\langle -h_1, -h_2, g_1, \dots, g_{m-1} \rangle =$

$\langle -h_2, g_1, \dots, g_m \rangle = \langle g_1, \dots, g_m \rangle$  and  $\langle g_1, \dots, g_m \rangle$  is the union of  $m + 1$  cones  $C_k$ ,  $k = 1, \dots, m + 1$ , spanned by any  $m$  vectors in the list  $\{-h_1, -h_2, g_1, \dots, g_{m-1}\}$ . Suppose  $-g_m \in \langle -h_1, g_1, \dots, g_{m-1} \rangle$  or equivalently  $-g_m = \beta_1(-h_1) + \sum_{i=1}^{m-1} \gamma_i g_i$  with  $\beta_1, \gamma_1, \dots, \gamma_i \geq 0$ . If  $\beta_1 > 0$  then  $h_1 \in G$ , a contradiction. So  $\beta_1 = 0$  and  $G$  is degenerate which is also a contradiction. Thus  $-g_m \notin \langle -h_1, g_1, \dots, g_{m-1} \rangle$ . Likewise,  $-g_m$  cannot be in a  $\langle -h_2, g_1, \dots, g_{m-1} \rangle$ . Thus  $-g_m$  belongs to a cone  $C_k$  having both  $-h_1$  and  $-h_2$  as generators. Suppose, without loss of generality, that  $-g_m \in \langle -h_1, -h_2, g_2, \dots, g_{m-1} \rangle$ . That is  $-g_m = \sigma_1(-h_1) + \sigma_2(-h_2) + \sum_{i=2}^{m-1} \theta_i g_i$  where  $\sigma_1, \sigma_2, \theta_2, \dots, \theta_{m-1} \geq 0$ . If  $\sigma_1 = \sigma_2 = 0$  then  $G$  is degenerate. If  $\sigma_1 = 0$  and  $\sigma_2 > 0$  then  $h_2 \in G$ . If  $\sigma_1 > 0$  and  $\sigma_2 = 0$  then  $h_1 \in G$ . Thus,  $\sigma_1, \sigma_2 > 0$  and  $H$  intersects the facets  $\langle g_2, \dots, g_m \rangle$  in  $h = \sigma_1 h_1 + \sigma_2 h_2$ . Moreover, since  $\langle h_1, h_2 \rangle$  is non-flat then  $h \neq 0$ . (Observe that  $h_1, h_2 \in G^c$  implies that  $h_1, h_2 \neq 0$ .)  $\square$

**Lemma 37.** *Let  $G = \langle g_1, \dots, g_m \rangle$  denote a cone of  $\mathbb{R}^n$ . Let  $p \in G$ ,  $p \neq 0$ , and  $q \in \langle g_1, \dots, g_m \rangle \cap G^c$ . If  $\langle p, q \rangle$  is non-flat then  $\langle p, q \rangle$  intersects one of the facets of  $G$  in a non-zero vector.*

*Proof.* The boundary of  $G$  is the union of its facets. If  $p$  belongs to the boundary of  $G$  then  $p$  belongs to one of its facets and the statement holds (for  $p$ ). Suppose next that  $p \in G^\circ$ . Thus  $0 = \langle -p, g_1, \dots, g_m \rangle^\circ$  and by Lemma 35,  $\langle g_1, \dots, g_m \rangle = \langle -p, g_1, \dots, g_m \rangle$  and  $\langle g_1, \dots, g_m \rangle$  is partitioned into  $m + 1$  cones  $C_k$  generated each by a combination of  $m$  vectors among  $-p, g_1, \dots, g_m$ . Since  $q \notin G$ ,  $q$  belongs to a cone  $C_k$  having  $-p$  as a generator. Suppose without loss of generality that  $q \in \langle -p, g_1, \dots, g_{m-1} \rangle$ . Then there exist  $\alpha, \alpha_1, \dots, \alpha_{m-1} \geq 0$  such that  $q = \alpha(-p) + \sum_{i=1}^{m-1} \alpha_i g_i$  and  $q + \alpha p \in \langle p, q \rangle \cap \langle g_1, \dots, g_{m-1} \rangle$ . If  $q + \alpha p = 0$  then  $\langle p, q \rangle$  is flat, contradicting the hypothesis. Thus  $q + \alpha p \neq 0$  proving the statement.  $\square$

**Lemma 38.** *Fix  $m \geq 2$  and let  $C = \langle v_1, v_2, \dots, v_m \rangle$  and  $C' = \langle v'_1, v_2, \dots, v_m \rangle$ . If there exists a non zero vector in  $\langle v'_1, v_1 \rangle \cap \langle v_2, \dots, v_m \rangle$  then  $\langle v'_1, v_1, v_2, \dots, v_m \rangle = C \cup C'$ .*

*Proof.* Let  $c \in \langle v'_1, v_1, v_2, \dots, v_m \rangle$ . There exist  $\alpha'_1, \alpha_1, \dots, \alpha_m \geq 0$  such that  $c = \alpha_1 v_1 + \alpha'_1 v'_1 + \sum_{i \geq 2} \alpha_i v_i$ . By assumption, there exist  $\lambda'_1, \lambda_1, \dots, \lambda_m \geq 0$ ,  $(\lambda_1, \lambda'_1) \neq (0, 0)$ , such that  $\lambda_1 v_1 + \lambda'_1 v'_1 = \sum_{i \geq 2} \lambda_i v_i$ . Suppose  $\lambda_1, \lambda'_1 > 0$  and let  $\beta = \min\{\frac{\alpha_1}{\lambda_1}, \frac{\alpha'_1}{\lambda'_1}\} \geq 0$ . Then

$$\begin{aligned} c &= c + \beta \times 0 = \alpha_1 v_1 + \alpha'_1 v'_1 + \sum_{i \geq 2} \alpha_i v_i + \beta(-\lambda_1 v_1 - \lambda'_1 v'_1 + \sum_{i \geq 2} \lambda_i v_i) \\ &= (\alpha_1 - \beta \lambda_1) v_1 + (\alpha'_1 - \beta \lambda'_1) v'_1 + \sum_{i \geq 2} (\alpha_i + \beta \lambda_i) v_i. \end{aligned}$$

By definition of  $\beta$ , the coefficients of  $v_1$  and  $v'_1$  are both non-negative and one of them has to vanish. Thus  $c \in C \cup C'$ . The same holds when  $\lambda_1 = 0$  (resp.  $\lambda'_1 = 0$ ) by taking  $\beta$  as  $\frac{\alpha'_1}{\lambda'_1}$  (resp.  $\frac{\alpha_1}{\lambda_1}$ ). The converse inclusion is immediate.  $\square$

**Lemma 39.** *Fix  $m \geq 2$  and let  $C = \langle v_1, v_2, \dots, v_m \rangle$  and  $C' = \langle v'_1, v_2, \dots, v_m \rangle$ . If  $\langle v_1, -v'_1 \rangle^\circ \cap \langle v_2, \dots, v_m \rangle$  is non-empty then  $\langle v_1, v'_1, v_2, \dots, v_m \rangle \subseteq C'$ . Similarly, if  $\langle -v_1, v'_1 \rangle^\circ \cap \langle v_2, \dots, v_m \rangle$  is non-empty then  $\langle v_1, v'_1, v_2, \dots, v_m \rangle \subseteq C$ .*

*Proof.* By assumption, there exist  $\alpha'_1, \alpha_1, \dots, \alpha_m \geq 0$  such that  $\alpha_1 v_1 + \alpha'_1(-v'_1) = \sum_{i \geq 2} \alpha_i v_i$  with  $\alpha_1, \alpha'_1 > 0$ . Thus  $v_1 \in C'$  and  $\langle v_1, v'_1, v_2, \dots, v_m \rangle \subseteq C'$ . It suffices to swap  $v_1$  and  $v'_1$  to prove the second part of the lemma.  $\square$

**Lemma 40.** *Let  $G = \langle v_1, v'_1, v_2, \dots, v_{n-1} \rangle$  denote a non-degenerate cone in  $\mathbb{R}^n$ . Suppose there exists a vector  $p \neq 0$  such that  $p \in \langle -v_1, -v'_1 \rangle \cap \langle v_2, \dots, v_n \rangle$  then  $G^\circ \subseteq (\langle v_1, v_2, \dots, v_n \rangle \cup \langle v'_1, v_2, \dots, v_n \rangle)^c$ .*

*Proof.* There exist  $\lambda_1, \lambda'_1, \lambda_2, \dots, \lambda_n \geq 0, (\lambda_1, \lambda'_1) \neq (0, 0)$  such that

$$p = \lambda_1(-v_1) + \lambda'_1(-v'_1) = \sum_{i=2}^n \lambda_i v_i . \quad (1)$$

If  $\lambda_n = 0$  then  $G$  is degenerate. Thus  $\lambda_n > 0$ . Let  $g \in G^\circ$ . Then  $g = \alpha_1 v_1 + \alpha'_1 v'_1 + \sum_{i=2}^{n-1} \alpha_i v_i$  with  $\alpha_1, \alpha'_1, \alpha_2, \dots, \alpha_{n-1} > 0$ . Suppose that  $g \in \langle v_1, \dots, v_n \rangle$ , then there exist  $\beta_i \geq 0$  such that  $g = \sum_{i=1}^n \beta_i v_i$ . Thus  $\alpha_1 v_1 + \alpha'_1 v'_1 + \sum_{i=2}^{n-1} \alpha_i v_i = \sum_{i=1}^n \beta_i v_i$  and

$$\alpha'_1 v'_1 + \sum_{i=1}^{n-1} (\alpha_i - \beta_i) v_i = \beta_n v_n . \quad (2)$$

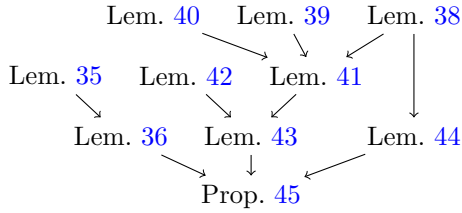
If  $\beta_n = 0$  then  $\alpha'_1 = 0$  (because  $v_1, v'_1, v_2, \dots, v_{n-1}$  are linearly independent), which contradicts  $\alpha'_1 > 0$ . Thus  $\beta_n > 0$ . Since  $\lambda_n, \beta_n > 0$  we can eliminate  $v_n$  in Eq. 1 and Eq. 2. We get

$$(\beta_n \lambda'_1 + \lambda_n \alpha'_1)(-v'_1) + \sum_{i=1}^{n-1} (\beta_n \lambda_i + \lambda_n (\alpha_i - \beta_i))(-v_i) = 0$$

since  $v_1, v'_1, v_2, \dots, v_{n-1}$  are linearly independent, all the coefficients must vanish. In particular,  $\beta_n \lambda'_1 + \lambda_n \alpha'_1 = 0$  which implies  $\lambda_n \alpha'_1 = 0$ , a contradiction. Thus  $g \notin \langle v_1, \dots, v_n \rangle$ . Swapping  $v_1$  and  $v'_1$  in the discussion above, we prove that  $g \notin \langle v'_1, v_2, \dots, v_n \rangle$  and therefore  $G^\circ \subseteq (\langle v_1, \dots, v_n \rangle \cup \langle v'_1, v_2, \dots, v_n \rangle)^c$  as stated.  $\square$

## B.2 Front intersection

The main result of this section is Proposition 45. It states that when the c-facet  $\langle a'_2, a_3 \rangle$  intersects  $\langle a_1, a'_1 \rangle$  in a non-zero vector and  $G = \langle a_1, a'_1, a_2 \rangle$  is a minimal cone then  $G \subseteq \Sigma$ . For convenience, we recall the proof structure of this section.





**Lemma 41.** *Let  $G = \langle a_1, a'_1, a_2 \rangle$  denote a non-degenerate cone in  $\mathbb{R}^3$  and let  $H(a_2)$  denote the open half-space bounded by the plane  $(a_1, a'_1)$  and containing  $a_2$ . If  $a'_2, a_3 \in G^c \cap H(a_2)^c$  then either  $G \subseteq \Sigma$  or  $G^\circ \subseteq \Sigma(a_3)^c$ . (More precisely,  $G^\circ \subseteq \Sigma(a_3)^c$  occurs whenever there exists a non-zero vector in  $\langle -a_1, -a'_1 \rangle \cap \langle a_2, a_3 \rangle$ .)*

*Proof.* Since  $a_1, a'_1, a'_2, a_3 \in H(a_2)^c$ ,  $\langle a_1, a'_2, a_3 \rangle \cup \langle a'_1, a'_2, a_3 \rangle \subseteq H(a_2)^c$ . But  $G^\circ \subseteq H(a_2)$ . Thus (i)  $G^\circ \subseteq (\langle a_1, a'_2, a_3 \rangle \cup \langle a'_1, a'_2, a_3 \rangle)^c$ . If  $\langle a_2, a_3 \rangle$  is flat then  $a_3 \simeq -a_2$  and  $a_3 \in (a_1, a_2)$  which gives  $\langle a_1, a_2, a_3 \rangle \subset (a_1, a_2)$ . Likewise  $a_3 \in (a'_1, a_2)$  and  $\langle a'_1, a_2, a_3 \rangle \subseteq (a'_1, a_2)$ . The planes  $(a_1, a_2)$  and  $(a'_1, a_2)$  support  $G$ , thus  $G^\circ \subseteq (a_1, a_2)^c \cap (a'_1, a_2)^c$ . Therefore (ii)  $G^\circ \subseteq (\langle a_1, a_2, a_3 \rangle \cup \langle a'_1, a_2, a_3 \rangle)^c$ . Combining (i) and (ii), we get  $G^\circ \subseteq \Sigma(a_3)^c$ . The only case remaining is when  $\langle a_2, a_3 \rangle$  is non-flat and must then intersect the plane  $(a_1, a'_1)$  in a non-zero vector  $p$ . Since  $G$  is non-degenerate, the cone  $\langle a_1, a'_1 \rangle$  is non-degenerate and the plane  $(a_1, a'_1)$  is partitioned into four cones, namely  $\langle a_1, a'_1 \rangle$ ,  $\langle -a_1, a'_1 \rangle$ ,  $\langle a_1, -a'_1 \rangle$ , and  $\langle -a_1, -a'_1 \rangle$ .

- If  $p \in \langle a_2, a_3 \rangle \cap \langle a_1, a'_1 \rangle$  then by Lemma 38,  $\langle a_1, a'_1, a_2, a_3 \rangle \subseteq \langle a_1, a_2, a_3 \rangle \cup \langle a'_1, a_2, a_3 \rangle$  and therefore  $G \subseteq \langle a_1, a_2, a_3 \rangle \cup \langle a'_1, a_2, a_3 \rangle \subseteq \Sigma$ .
- If  $p \in \langle a_2, a_3 \rangle \cap \langle -a_1, a'_1 \rangle^\circ$  then by Lemma 39  $\langle a_1, a'_1, a_2, a_3 \rangle \subseteq \langle a_1, a_2, a_3 \rangle$  and therefore  $G \subseteq \langle a_1, a_2, a_3 \rangle \subseteq \Sigma$ . We get the same result when  $a_1$  and  $a'_1$  are swapped.
- If  $p \in \langle a_2, a_3 \rangle \cap \langle -a_1, -a'_1 \rangle$  then by Lemma 40, we get (iii)  $G^\circ \subseteq (\langle a_1, a_2, a_3 \rangle \cup \langle a'_1, a_2, a_3 \rangle)^c$ . Combining (i) and (iii) we obtain  $G^\circ \subseteq \Sigma(a_3)^c$ .

□

**Lemma 42.** *Let  $G = \langle a_1, a'_1, a_2 \rangle$  denote a non-degenerate cone and let  $H(a_2)$  denote the open half-space bounded by  $(a_1, a'_1)$  and containing  $a_2$ . Suppose  $u \in H(a_2)$ . Then either  $u \in G$  or  $a_2 \in \langle a_1, a'_1, u \rangle$  or there exists a vector  $p \neq 0$  such that  $p \in \langle a_1, u \rangle \cap \langle a'_1, a_2 \rangle$  or  $p \in \langle a'_1, u \rangle \cap \langle a_1, a_2 \rangle$ .*

*Proof.* Since  $u \in H(a_2)$  and  $\langle a_1, a'_1 \rangle$  is non-degenerate,  $\langle a_1, a'_1, u \rangle$  is non-degenerate. We partition the closer of  $H(a_2)$  into four cones, namely  $\langle a_1, a'_1, a_2 \rangle$ ,  $\langle -a_1, a'_1, a_2 \rangle$ ,  $\langle a_1, -a'_1, a_2 \rangle$ , and  $\langle -a_1, -a'_1, a_2 \rangle$ .

- If  $u \in \langle a_1, a'_1, a_2 \rangle$  then  $u \in G$ .
- If  $u \in \langle -a_1, -a'_1, a_2 \rangle$  then there exist  $\lambda_1, \lambda'_1, \lambda_2 \geq 0$  such that  $u = \lambda_1(-a_1) + \lambda'_1(-a'_1) + \lambda_2 a_2$ . If  $\lambda_2 = 0$  then  $\langle a_1, a'_1, u \rangle$  is degenerate, a contradiction. Thus  $\lambda_2 > 0$  and  $a_2 \in \langle a_1, a'_1, u \rangle$ .
- If  $u \in \langle -a_1, a'_1, a_2 \rangle$  then there exist  $\lambda_1, \lambda'_1, \lambda_2 \geq 0$  such that  $u = \lambda_1(-a_1) + \lambda'_1 a'_1 + \lambda_2 a_2$ . Let  $p = u + \lambda_1 a_1$ , then  $p \in \langle u, a_1 \rangle \cap \langle a'_1, a_2 \rangle$ . If  $p = 0$  then  $u = -\lambda_1 a_1$  and  $u \in (a_1, a'_1)$ , contradicting  $u \in H(a_2)$ . Therefore  $p \neq 0$ .
- If  $u \in \langle a_1, -a'_1, a_2 \rangle$  then it suffices to swap  $a_1$  and  $a'_1$  in the previous case to arrive at the stated result.

□

**Lemma 43.** *Let  $G = \langle a_1, a'_1, a_2 \rangle$  denote a non-degenerate cone in  $\mathbb{R}^3$ . Suppose that  $a'_2, a_3 \in G^c$  and  $a'_1 \in \langle a'_2, a_3 \rangle$ . Then  $G \subseteq \Sigma$ .*

*Proof.* Let  $H(a_2)$  denote the open half-space bounded by  $(a_1, a'_1)$  and containing  $a_2$ .

- If  $a'_2, a_3 \in H(a_2)^c$  then  $a'_2, a_3 \in G^c \cap H(a_2)^c$  and by Lemma 41, either  $G \subseteq \Sigma$  or  $G^\circ \subseteq \Sigma(a_3)^c$ . The latter occurs precisely when there exists  $p \neq 0$  such that  $p \in \langle a_2, a_3 \rangle \cap \langle -a_1, -a'_1 \rangle$  and  $p = a_3$  (indeed, since  $a'_2, a_3 \in H(a_2)^c$  and  $a'_1 \in \langle a'_2, a_3 \rangle$  then  $a_3, a'_2 \in \langle a_1, a'_1 \rangle$ ). Thus  $a_3 \in \langle -a_1, -a'_1 \rangle$  forcing  $a'_2 \in \langle a_1, a'_1 \rangle$  which is impossible. Thus  $G \subseteq \Sigma$ .
- If  $a'_2, a_3 \in H(a_2)$  then  $\langle a'_2, a_3 \rangle \cap \langle a_1, a'_1 \rangle$  is reduced to zero and therefore  $a'_1 = 0$  contradicting the non-degeneracy of  $G$ .
- If  $a_3 \in H(a_2)$  and  $a'_2 \in H(a_2)^c$ . Then by Lemma 42 (with  $u = a_3$ ) one of the following occurs.
  - $a_3 \in G$  which is impossible by hypothesis on  $a_3$ .
  - $a_2 \in \langle a_1, a'_1, a_3 \rangle$ . Then  $G \subseteq \langle a_1, a'_1, a_3 \rangle$ . Since  $a'_1 \in \langle a'_2, a_3 \rangle$  then  $\langle a_1, a'_1, a_3 \rangle \subseteq \langle a_1, a'_2, a_3 \rangle$ . Thus  $G \subseteq \langle a_1, a'_2, a_3 \rangle \subseteq \Sigma$ .
  - $\exists p \neq 0, p \in \langle a_1, a_3 \rangle \cap \langle a'_1, a_2 \rangle$ . By Lemma 38  $\langle a_1, a'_1, a_2, a_3 \rangle \subseteq \langle a_1, a'_1, a_3 \rangle \cup \langle a_1, a_2, a_3 \rangle$ . Since  $a'_1 \in \langle a'_2, a_3 \rangle$  then  $\langle a_1, a'_1, a_3 \rangle \subseteq \langle a_1, a'_2, a_3 \rangle$  and therefore  $G \subseteq \langle a_1, a'_1, a_2, a_3 \rangle \subseteq \langle a_1, a'_1, a_3 \rangle \cup \langle a_1, a_2, a_3 \rangle \subseteq \langle a_1, a'_2, a_3 \rangle \cup \langle a_1, a_2, a_3 \rangle \subseteq \Sigma$ .
  - $\exists p \neq 0, p \in \langle a'_1, a_3 \rangle \cap \langle a_1, a_2 \rangle$ . (Observe that one cannot conclude by simply swapping  $a_1$  and  $a'_1$  here since  $a'_1$  enjoys a special property, namely  $a'_1 \in \langle a'_2, a_3 \rangle$ .) By Lemma 38  $\langle a_1, a'_1, a_2, a_3 \rangle \subseteq \langle a_1, a'_1, a_3 \rangle \cup \langle a'_1, a_2, a_3 \rangle$ . Since  $a'_1 \in \langle a'_2, a_3 \rangle$  then  $\langle a_1, a'_1, a_3 \rangle \subseteq \langle a_1, a'_2, a_3 \rangle$  and therefore  $G \subseteq \langle a_1, a'_1, a_2, a_3 \rangle \subseteq \langle a_1, a'_1, a_3 \rangle \cup \langle a'_1, a_2, a_3 \rangle \subseteq \langle a_1, a'_2, a_3 \rangle \cup \langle a'_1, a_2, a_3 \rangle \subseteq \Sigma$ .
- If  $a_3 \in H(a_2)^c$  and  $a'_2 \in H(a_2)$ . Then by Lemma 42 (with  $u = a'_2$ ) one of the following occurs.
  - $a'_2 \in G$  which is impossible by hypothesis on  $a'_2$ .
  - $a_2 \in \langle a_1, a'_1, a'_2 \rangle$ . Then  $G \subseteq \langle a_1, a'_1, a'_2 \rangle$  and since  $a'_1 \in \langle a'_2, a_3 \rangle$ ,  $\langle a_1, a'_1, a'_2 \rangle \subseteq \langle a_1, a'_2, a_3 \rangle$  and therefore  $G \subseteq \langle a_1, a'_2, a_3 \rangle \subseteq \Sigma$ .
  - $\exists p \neq 0, p \in \langle a_1, a'_2 \rangle \cap \langle a'_1, a_2 \rangle$ . There exist non-negative coefficients such that (i)  $\alpha_1 a_1 + \alpha'_2 a'_2 = \alpha'_1 a'_1 + \alpha_2 a_2$  with  $(\alpha_1, \alpha'_2), (\alpha'_1, \alpha_2) \neq (0, 0)$ . In addition  $\alpha_1 > 0$  because otherwise  $\alpha'_2 > 0$  and  $a'_2 \in G$ . Moreover there exist  $\lambda'_2, \lambda_3 \geq 0$  such that (ii)  $a'_1 = \lambda'_2 a'_2 + \lambda_3 a_3$ . If  $\lambda'_2 = 0$  then either  $\lambda_3 = 0$  and  $a'_1 = 0$  contradicting the non-degeneracy of  $G$ , or  $\lambda_3 > 0$  and  $a'_1 \simeq a_3$  contradicting  $a_3 \in H(a_2)^c$ . Thus  $\lambda'_2 > 0$  and by eliminating  $a'_2$  from (i) and (ii) we get  $\lambda'_2 \alpha_1 a_1 + \alpha'_2 (a'_1 - \lambda_3 a_3) = \lambda'_2 \alpha'_1 a'_1 + \lambda'_2 \alpha_2 a_2$  or equivalently  $\lambda'_2 \alpha_1 a_1 + (\alpha'_2 - \lambda'_2 \alpha'_1) a'_1 = \alpha'_2 \lambda_3 a_3 + \lambda'_2 \alpha_2 a_2$ . If  $\alpha'_2 - \lambda'_2 \alpha'_1 < 0$  then  $a_1 \in \langle a'_1, a_2, a_3 \rangle$  (recall that  $\alpha_1 > 0$ ). and therefore  $G \subseteq \langle a'_1, a_2, a_3 \rangle \subseteq \Sigma$ . If  $\alpha'_2 - \lambda'_2 \alpha'_1 \geq 0$  then there exists a non zero vector in  $\langle a_1, a'_1 \rangle \cap \langle a_2, a_3 \rangle$ . By Lemma 38,  $\langle a_1, a'_1, a_2, a_3 \rangle \subseteq \langle a_1, a_2, a_3 \rangle \cup \langle a'_1, a_2, a_3 \rangle$  making  $G \subseteq \Sigma$ .
  - $\exists p \neq 0, p \in \langle a'_1, a'_2 \rangle \cap \langle a_1, a_2 \rangle$ . Then  $p = \alpha'_1 a'_1 + \alpha'_2 a'_2$  with  $\alpha'_1, \alpha'_2 \geq 0$  and  $(\alpha'_1, \alpha'_2) \neq (0, 0)$ . If  $\alpha'_2 = 0$  then  $p \simeq a'_1$  and  $a'_1 \in \langle a_1, a_2 \rangle$  contradicting the non-degeneracy of  $G$ . Thus  $\alpha'_2 > 0$ . Since  $a'_1 \in \langle a'_2, a_3 \rangle$  then  $a'_1 = \lambda'_2 a'_2 + \lambda_3 a_3$  with  $\lambda'_2, \lambda_3 \geq 0$ . Thus  $\alpha'_2 a'_1 = \alpha'_2 \lambda'_2 a'_2 + \alpha'_2 \lambda_3 a_3 = \lambda'_2 (p - \alpha'_1 a'_1) + \alpha'_2 \lambda_3 a_3$  and  $(\alpha'_2 + \lambda'_2 \alpha'_1) a'_1 = \lambda'_2 p + \lambda_3 \alpha'_2 a_3$  with  $\alpha'_2 + \lambda'_2 \alpha'_1 > 0$ . Therefore  $a'_1 \in \langle p, a_3 \rangle \subseteq \langle a_1, a_2, a_3 \rangle$ . But then  $G \subseteq \langle a_1, a_2, a_3 \rangle \subseteq \Sigma$ .

□

**Lemma 44.** Let  $G = \langle a_1, a'_1, a_2 \rangle$  denote a non-degenerate cone in  $\mathbb{R}^3$ . Suppose there exist two vectors  $p, q \neq 0$  such that  $p \in \langle a_1, a'_1 \rangle \cap \langle a'_2, a_3 \rangle$  and  $q \in \langle a_1, a_3 \rangle \cap \langle a'_1, a_2 \rangle$  then  $G \subseteq \Sigma$ .

*Proof.* Since  $q \in \langle a_1, a_3 \rangle \cap \langle a'_1, a_2 \rangle$ , applying Lemma 38, we get  $\langle a_1, a'_1, a_2, a_3 \rangle \subseteq \langle a_1, a_2, a_3 \rangle \cup \langle a_1, a'_1, a_3 \rangle$  and therefore (i)  $G \subseteq \langle a_1, a'_1, a_2, a_3 \rangle \subseteq \langle a_1, a_2, a_3 \rangle \cup \langle a_1, a'_1, a_3 \rangle \subseteq \Sigma$ . In addition, using  $p \in \langle a_1, a'_1 \rangle \cap \langle a'_2, a_3 \rangle$  and applying Lemma 38, we get  $\langle a_1, a'_1, a'_2, a_3 \rangle \subseteq \langle a_1, a'_2, a_3 \rangle \cup \langle a'_1, a'_2, a_3 \rangle$  and therefore (ii)  $\langle a_1, a'_1, a_3 \rangle \subseteq \langle a_1, a'_2, a_3 \rangle \cup \langle a'_1, a'_2, a_3 \rangle$ . Combining (i) and (ii). We get  $G \subseteq \langle a_1, a_2, a_3 \rangle \cup (\langle a_1, a'_2, a_3 \rangle \cup \langle a'_1, a'_2, a_3 \rangle) \subseteq \Sigma$ .  $\square$

**Proposition 45.** Let  $G = \langle a_1, a'_1, a_2 \rangle$  denote a non-degenerate cone in  $\mathbb{R}^3$ . Suppose that  $a'_2, a_3 \in G^c$  and that there exists a vector  $p \neq 0$  such that  $p \in \langle a_1, a'_1 \rangle \cap \langle a'_2, a_3 \rangle$  then  $G \subseteq \Sigma$ .

*Proof.* By Lemma 36, there exists a vector  $q \neq 0$  such that  $q \in \langle a_1, a_2 \rangle \cap \langle a'_2, a_3 \rangle$  or  $q \in \langle a'_1, a_2 \rangle \cap \langle a'_2, a_3 \rangle$ . Assuming the latter (otherwise it suffices to swap  $a_1$  and  $a'_1$ ). The hypothesis  $p \in \langle a_1, a'_1 \rangle \cap \langle a'_2, a_3 \rangle$  implies that there exist  $\lambda_1, \lambda'_1, \beta'_2, \beta_3 \geq 0$ ,  $(\lambda_1, \lambda'_1), (\beta'_2, \beta_3) \neq (0, 0)$ , such that (i)  $\lambda_1 a_1 + \lambda'_1 a'_1 = \beta'_2 a'_2 + \beta_3 a_3$ . Similarly,  $q \in \langle a'_1, a_2 \rangle \cap \langle a'_2, a_3 \rangle$  implies that there exist  $\gamma'_1, \gamma_2, \phi'_2, \phi_3 \geq 0$ ,  $(\gamma_2, \gamma'_1), (\phi'_2, \phi_3) \neq (0, 0)$ , such that (ii)  $\gamma'_1 a'_1 + \gamma_2 a_2 = \phi'_2 a'_2 + \phi_3 a_3$ . If  $\beta'_2 = 0$  and  $\beta_3 > 0$  then  $a_3 \in G$  (contradicting  $a_3 \notin G$ ). If  $\beta'_2 > 0$  and  $\beta_3 = 0$  then  $a'_2 \in G$  (contradicting  $a'_2 \notin G$ ). Thus  $\beta'_2, \beta_3 > 0$ . Similarly, one proves that  $\phi'_2, \phi_3 > 0$ . We can therefore eliminate  $a'_2$  from equations (i) and (ii) above to get

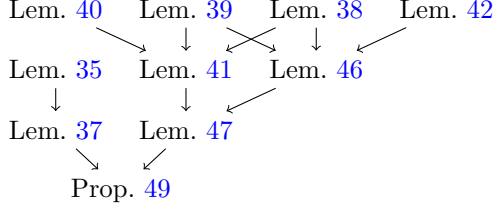
$$\phi'_2 \lambda_1 a_1 + \underbrace{(\phi'_2 \lambda'_1 - \beta'_2 \gamma'_1)}_{\rho_1} a'_1 = \beta'_2 \gamma_2 a_2 + \underbrace{(\phi'_2 \beta_3 - \beta'_2 \phi_3)}_{\rho_2} a_3$$

- If  $\rho_1, \rho_2 < 0$  then there exists a non-zero vector in  $\langle a_1, a_3 \rangle \cap \langle a'_1, a_2 \rangle$ . By hypothesis  $p \in \langle a_1, a'_1 \rangle \cap \langle a'_2, a_3 \rangle$ . Thus Lemma 44 applies proving  $G \subseteq \Sigma$ .
- If  $\rho_1, \rho_2 > 0$  then there exists a non-zero vector in  $\langle a_1, a'_1 \rangle \cap \langle a_2, a_3 \rangle$  and, using Lemma 38,  $G \subseteq \langle a_1, a'_1, a_2, a_3 \rangle \subseteq \langle a_1, a_2, a_3 \rangle \cup \langle a'_1, a_2, a_3 \rangle \subseteq \Sigma$ .
- If  $\rho_1 \leq 0, \rho_2 \geq 0$  and  $\lambda_1 > 0$  then  $a_1 \in \langle a'_1, a_2, a_3 \rangle$  and  $G \subseteq \langle a'_1, a_2, a_3 \rangle \subseteq \Sigma$ .
- If  $\rho_1 \geq 0, \rho_2 \leq 0$  and  $\gamma_2 > 0$  then  $a_2 \in \langle a_1, a'_1, a_3 \rangle$  and  $G \subseteq \langle a_1, a'_1, a_3 \rangle$ . Since  $p \in \langle a_1, a'_1 \rangle \cap \langle a'_2, a_3 \rangle$ , by Lemma 38,  $\langle a_1, a'_1, a'_2, a_3 \rangle \subseteq \langle a_1, a'_2, a_3 \rangle \cup \langle a'_1, a'_2, a_3 \rangle \subseteq \Sigma$ . Therefore  $G \subseteq \Sigma$ .

Finally, if  $\lambda_1 = 0$  (resp.  $\gamma_2 = 0$ ) then by (i) (resp. (ii))  $a'_1 \in \langle a'_2, a_3 \rangle$  and by Lemma 43  $G \subseteq \Sigma$ .  $\square$

### B.3 Side intersection

The main result of this section is Proposition 49. It states that when the c-facet  $\langle a'_2, a_3 \rangle$  intersects  $\langle -a_1, a'_1 \rangle$  in a non-zero vector and  $G = \langle a_1, a'_1, a_2 \rangle$  is a minimal cone then either  $G \subseteq \Sigma$  or  $a_1$  is not surrounded or  $a'_1$  is not surrounded. We recall the proof structure of this section.



**Lemma 46.** *Let  $G = \langle a_1, a'_1, a_2 \rangle$  denote a non-degenerate cone in  $\mathbb{R}^3$  and let  $H(a_2)$  denote the open half-space bounded by the plane  $(a_1, a'_1)$  and containing  $a_2$ . If  $a'_2, a_3 \in G^c$ ,  $a_3 \in H(a_2)$ , and there exists  $p \neq 0$  such that  $p \in \langle -a_1, a'_1 \rangle^\circ \cap \langle a_3, a'_2 \rangle$  then  $G \subseteq \Sigma$ .*

*Proof.* Since  $a_3 \in H(a_2)$ , then by Lemma 42 (with  $u = a_3$ ), either  $a_3 \in G$  or  $a_2 \in \langle a_1, a'_1, a_3 \rangle$  or there exists  $q \neq 0$  such that  $q \in \langle a_1, a_3 \rangle \cap \langle a'_1, a_2 \rangle$  or  $q \in \langle a_1, a_2 \rangle \cap \langle a'_1, a_3 \rangle$ .

- $a_3 \in G$  is impossible by hypothesis.
- If  $a_2 \in \langle a_1, a'_1, a_3 \rangle$  then  $G \subseteq \langle a_1, a'_1, a_3 \rangle$ . Since  $p \in \langle -a_1, a'_1 \rangle^\circ \cap \langle a'_2, a_3 \rangle$ , by Lemma 39,  $\langle a_1, a'_1, a'_2, a_3 \rangle \subseteq \langle a_1, a'_2, a_3 \rangle$  and therefore  $G \subseteq \langle a_1, a'_1, a_3 \rangle \subseteq \langle a_1, a'_2, a_3 \rangle \subseteq \Sigma$ .
- If  $q \in \langle a_1, a_3 \rangle \cap \langle a'_1, a_2 \rangle$  then Lemma 38 applies and  $\langle a_1, a'_1, a_2, a_3 \rangle \subseteq \langle a_1, a_2, a_3 \rangle \cup \langle a_1, a'_1, a_3 \rangle$ . Thus  $G \subseteq \langle a_1, a_2, a_3 \rangle \cup \langle a_1, a'_1, a_3 \rangle$ . Moreover since  $p \in \langle -a_1, a'_1 \rangle^\circ \cap \langle a_3, a'_2 \rangle$ , by Lemma 39,  $\langle a_1, a'_1, a'_2, a_3 \rangle \subseteq \langle a_1, a'_2, a_3 \rangle$  and  $\langle a_1, a'_1, a_3 \rangle \subseteq \langle a_1, a'_2, a_3 \rangle$ . Therefore  $G \subseteq \langle a_1, a_2, a_3 \rangle \cup \langle a_1, a'_1, a_3 \rangle \subseteq \langle a_1, a_2, a_3 \rangle \cup \langle a_1, a'_2, a_3 \rangle \subseteq \Sigma$ .
- If  $q \in \langle a_1, a_2 \rangle \cap \langle a'_1, a_3 \rangle$  then Lemma 38 applies and  $\langle a_1, a'_1, a_2, a_3 \rangle \subseteq \langle a'_1, a_2, a_3 \rangle \cup \langle a_1, a'_1, a_3 \rangle$ . Thus  $G \subseteq \langle a'_1, a_2, a_3 \rangle \cup \langle a_1, a'_1, a_3 \rangle$ . Moreover  $p \in \langle -a_1, a'_1 \rangle^\circ \cap \langle a'_2, a_3 \rangle$ . Thus by Lemma 39,  $\langle a_1, a'_1, a'_2, a_3 \rangle \subseteq \langle a_1, a'_2, a_3 \rangle$  and  $\langle a_1, a'_1, a_3 \rangle \subseteq \langle a_1, a'_2, a_3 \rangle$ . Therefore  $G \subseteq \langle a'_1, a_2, a_3 \rangle \cup \langle a_1, a'_1, a_3 \rangle \subseteq \langle a'_1, a_2, a_3 \rangle \cup \langle a_1, a'_2, a_3 \rangle \subseteq \Sigma$ . (Observe that, while the proof of this case is similar to the previous case, we cannot simply swap  $a_1$  and  $a'_1$  to conclude as the hypothesis of the lemma is not necessarily satisfied by such a swap.)

□

**Lemma 47.** *Let  $G = \langle a_1, a'_1, a_2 \rangle$  denote a non-degenerate cone in  $\mathbb{R}^3$  and suppose that  $a_3 \in G^c$  and  $a'_2 \in \langle -a_1, a'_1 \rangle^\circ$ . Then either  $G \subseteq \Sigma$  or  $G^\circ \subseteq \Sigma(a_3)^c$ . (More precisely,  $G^\circ \subseteq \Sigma(a_3)^c$  occurs whenever there exists a non-zero vector  $p \in \langle -a_1, -a'_1 \rangle \cap \langle a_2, a_3 \rangle$ .)*

*Proof.* Let  $H(a_2)$  denote the open half-space bounded by  $(a_1, a'_1)$  and containing  $a_2$ . We have  $a'_2 \in G^c \cap H(a_2)^c$ . If  $a_3 \in H(a_2)^c$  then  $a_3 \in G^c \cap H(a_2)^c$  and by Lemma 41, either  $G \subseteq \Sigma$  or  $G^\circ \subseteq \Sigma(a_3)^c$ . Next, suppose  $a_3 \in H(a_2)$ . By hypothesis  $a'_2 \in \langle -a_1, a'_1 \rangle^\circ \cap \langle a'_2, a_3 \rangle$ . Thus Lemma 46 applies and  $G \subseteq \Sigma$ . □

The following lemma is akin to the results of [6], in particular the concepts of *visibility* and *starlike* sets introduced therein. We give below a direct proof inline with our purposes.

**Lemma 48.** *Let  $G = \langle a_1, a'_1, a_2 \rangle$  denote a non-degenerate cone in  $\mathbb{R}^3$  and suppose that there exists a non zero vector  $q$  in  $\langle a_1, a'_2 \rangle^\circ \cap \langle a'_1, a_2 \rangle^\circ$ . If  $-a_3 \in \langle a_1, a_2, q \rangle$  then  $\langle a_1, a_2, q \rangle^\circ \subseteq \Sigma(a_3)^c$ . (The reader familiar with [6] can think of  $G$  as starlike from  $-a_3$  and, dually,  $\Sigma(a_3)$  as the closure of the visibility set of  $a_3$  with respect to  $G$ .)*

*Proof.* We first observe that  $\langle a_1, a_2, q \rangle$  is non degenerate. By hypothesis,  $-a_3 = \lambda_1 a_1 + \lambda_2 a_2 + \lambda q$  with  $\lambda_1, \lambda_2, \lambda \geq 0$ . Let  $x \in \langle a_1, a_2, q \rangle^\circ$ , that is  $x = \alpha_1 a_1 + \alpha_2 a_2 + \alpha q$ , with  $\alpha_1, \alpha_2, \alpha > 0$ . The proof is by contradiction discussing all possible cases.

- If  $x \in \langle a_1, a_2, a_3 \rangle$  then  $x = \beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3$  with  $\beta_1, \beta_2, \beta_3 \geq 0$ . Thus  $x = \alpha_1 a_1 + \alpha_2 a_2 + \alpha q = \beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3$ . If  $\beta_3 = 0$  then  $q \in \langle a_1, a_2 \rangle$  contradicting the non degeneracy of  $\langle a_1, a_2, q \rangle$ . Thus  $\beta_3 > 0$  and  $-\beta_3 a_3 = (\beta_1 - \alpha_1) a_1 + (\beta_2 - \alpha_2) a_2 - \alpha q$ . Since  $a_1, a_2, q$  form a basis,  $\beta_3 \lambda = -\alpha$  which is impossible.
- If  $x \in \langle a_1, q, a_3 \rangle$  then  $x = \beta_1 a_1 + \beta q + \beta_3 a_3 = \alpha_1 a_1 + \alpha_2 a_2 + \alpha q$  with  $\beta_1, \beta, \beta_3 \geq 0$ . Thus  $(\alpha_1 - \beta_1) a_1 + \alpha_2 a_2 = (\beta - \alpha) q + \beta_3 a_3$ . If  $\beta_3 = 0$  then  $\alpha_2 = 0$  because  $a_1, a_2, q$  are independent. Thus  $\beta_3 > 0$  and  $-\beta_3 a_3 = -(\alpha_1 - \beta_1) a_1 - \alpha_2 a_2 + (\beta - \alpha) q = \beta_3 \lambda_1 a_1 + \beta_3 \lambda_2 a_2 + \beta_3 \lambda q$  which leads to  $-\alpha_2 = \beta_3 \lambda_2$ , a contradiction.
- If  $x \in \langle a'_1, q, a_3 \rangle$  then  $x = \beta'_1 a'_1 + \beta q + \beta_3 a_3 = \alpha_1 a_1 + \alpha_2 a_2 + \alpha q$  with  $\beta'_1, \beta, \beta_3 \geq 0$ . Thus  $\alpha_1 a_1 + \alpha_2 a_2 - \beta'_1 a'_1 = (\beta - \alpha) q + \beta_3 a_3$ . Suppose  $\beta_3 = 0$ . Then  $\beta - \alpha$  cannot be zero since  $a_1, a'_1, a_2$  are independent. Moreover, if  $\beta - \alpha \neq 0$ ,  $q \in \langle a'_1, a_2 \rangle$  forces  $\alpha_1$  to be zero contradicting  $\alpha_1 > 0$ . So  $\beta_3 > 0$  and  $-\beta_3 a_3 = -\alpha_1 a_1 - \alpha_2 a_2 + \beta'_1 a'_1 + (\beta - \alpha) q = \beta_3 \lambda_1 a_1 + \beta_3 \lambda_2 a_2 + \beta_3 \lambda q$ . Since  $q = \gamma_1 a_1 + \gamma'_2 a'_2 = \gamma'_1 a'_1 + \gamma_2 a_2$ , one can substitute  $a'_1$  for a linear combination of  $a_2$  and  $q$  which leads to  $-\alpha_1 = \beta_3 \lambda_1$ , a contradiction.
- One gets similar contradictions when  $x \in \langle a_2, q, a_3 \rangle$  or when  $x \in \langle a'_2, q, a_3 \rangle$  since the used arguments remain valid when swapping the indices 1 and 2.

To summarize, we just proved that  $x$  cannot be in  $\langle a_1, a_2, a_3 \rangle \cup \langle a_1, q, a_3 \rangle \cup \langle a'_1, q, a_3 \rangle \cup \langle q, a_2, a_3 \rangle \cup \langle q, a'_2, a_3 \rangle$ . Since  $q \in \langle a_1, a'_2 \rangle^\circ \cap \langle a'_1, a_2 \rangle^\circ$ , one has  $\langle a'_1, a_2, a_3 \rangle = \langle a'_1, q, a_3 \rangle \cup \langle q, a_2, a_3 \rangle$  and  $\langle a_1, a'_2, a_3 \rangle = \langle a_1, q, a_3 \rangle \cup \langle q, a'_2, a_3 \rangle$ . Therefore  $x \notin \langle a'_1, a_2, a_3 \rangle \cup \langle a_1, a'_2, a_3 \rangle$ . Finally, if  $x \in \langle a'_1, a'_2, a_3 \rangle$  then  $x = \beta'_1 a'_1 + \beta'_2 a'_2 + \beta_3 a_3 = \alpha_1 a_1 + \alpha_2 a_2 + \alpha q$  with  $\beta'_1, \beta'_2, \beta_3 \geq 0$ . Using  $q = \gamma_1 a_1 + \gamma'_2 a'_2 = \gamma'_1 a'_1 + \gamma_2 a_2$ , one gets  $\alpha_1 a_1 + \alpha_2 a_2 + \alpha q = \frac{\beta'_1}{\gamma'_1} (q - \gamma_2 a_2) + \frac{\beta'_2}{\gamma'_2} (q - \gamma_1 a_1) + \beta_3 a_3$ . If  $\beta_3 = 0$  then  $\alpha_1 + \frac{\beta'_2}{\gamma'_2} \gamma_1 = 0$  contradicting  $\alpha_1 > 0$ . Thus  $\beta_3 > 0$  and  $-(\alpha_1 + \frac{\beta'_2}{\gamma'_2} \gamma_1) = \beta_3 \lambda_1$  which is impossible. Thus  $x \notin \langle a'_1, a'_2, a_3 \rangle$ . Thus  $\langle a_1, a_2, q \rangle^\circ \subseteq \Sigma(a_3)^c$  as stated.  $\square$

**Proposition 49.** *Let  $G = \langle a_1, a'_1, a_2 \rangle$  denote a non-degenerate cone in  $\mathbb{R}^3$  and suppose that  $a'_2, a_3 \in G^c$ . If there exists a vector  $p \neq 0$  such that  $p \in \langle -a_1, a'_1 \rangle^\circ \cap \langle a'_2, a_3 \rangle$  then either  $G \subseteq \Sigma$  or  $G^\circ \subseteq \Sigma(a_3)^c$  or there exists a non zero vector  $q$  such that  $\langle a_1, a_2, q \rangle^\circ \subseteq \Sigma(a_3)^c$  and  $q \in \langle a_1, a'_2 \rangle^\circ \cap \langle a'_1, a_2 \rangle^\circ$ . (More Precisely,  $G^\circ \subseteq \Sigma(a_3)^c$  occurs whenever  $a'_2 \in \langle -a_1, a'_1 \rangle^\circ$  and there exists a non-zero vector in  $\langle a_2, a_3 \rangle \cap \langle -a_1, -a'_1 \rangle$ ; and  $\langle a_1, a_2, q \rangle^\circ \subseteq \Sigma(a_3)^c$  occurs whenever  $-a_3 \in \langle a_1, a_2, q \rangle$ .)*

*Proof.* Since  $p \in \langle -a_1, a'_1 \rangle^\circ \cap \langle a'_2, a_3 \rangle$ , there exist  $\lambda_1, \lambda'_1 > 0$  and  $\lambda'_2, \lambda_3 \geq 0$ ,  $(\lambda'_2, \lambda_3) \neq (0, 0)$  such that (i)  $p = \lambda_1(-a_1) + \lambda'_1 a'_1 = \lambda'_2 a'_2 + \lambda_3 a_3$ . If  $\lambda'_2 = 0$  and  $\lambda_3 > 0$  then  $a_3 \in \langle -a_1, a'_1 \rangle^\circ$ . So  $a_3 \in \langle -a_1, a'_1 \rangle^\circ \cap \langle a_2, a_3 \rangle$ . By Lemma 39 ( $a_3 \neq 0$  because  $a_3 \notin G$ ),  $\langle a_1, a'_1, a_2, a_3 \rangle \subseteq \langle a_1, a_2, a_3 \rangle$ . Thus  $G \subseteq \langle a_1, a_2, a_3 \rangle \subseteq \Sigma$ . If  $\lambda_3 = 0$  then  $a'_2 \in \langle -a_1, a'_1 \rangle^\circ$ . By Lemma 47, either  $G \subseteq \Sigma$  or  $G^\circ \subseteq \Sigma(a_3)^c$ . We are thus left with the case where  $\lambda'_2, \lambda_3 > 0$ . Equation (i) implies that  $a'_1 \in C := \langle a_1, a'_2, a_3 \rangle$ . If  $a_2 \in C$ , then  $G \subseteq C \subseteq \Sigma$ . Suppose next that  $a_2 \notin C$ .  $\langle a'_1, a_2 \rangle$  is non-flat (because  $G$  is non-degenerate), Lemma 37 implies that  $\langle a'_1, a_2 \rangle$  must intersect one of the three facets of the cone  $C$  in a non-zero vector. We detail the three cases below.

Case 1.  $\langle a'_1, a_2 \rangle$  intersects the facet  $\langle a_1, a'_2 \rangle$  in a non-zero vector  $q$ . Thus, there exist  $\gamma'_1, \gamma_2, \gamma_1, \gamma'_2 \geq 0$ ,  $(\gamma'_1, \gamma_2), (\gamma_1, \gamma'_2) \neq (0, 0)$ , such that (ii)  $q = \gamma'_1 a'_1 + \gamma_2 a_2 = \gamma_1 a_1 + \gamma'_2 a'_2$ . If  $\gamma'_2 = 0$  then  $a_1, a'_1, a_2$  are linearly dependent contradicting the non-degeneracy of  $G$ . We can thus assume that  $\gamma'_2 > 0$ . If  $\gamma_1 = 0$  then  $a'_2 \in \langle a'_1, a_2 \rangle$  contradicting  $a'_2 \notin G$ . Thus  $\gamma_1 > 0$ . Since  $\lambda'_2, \gamma'_2 > 0$ , we can eliminate  $a'_2$  from (i) and (ii) to get

$$\underbrace{(\lambda'_2 \gamma_1 - \gamma'_2 \lambda_1)}_{\theta_1} a_1 + \underbrace{(\gamma'_2 \lambda'_1 - \lambda'_2 \gamma'_1)}_{\theta'_1} a'_1 = \lambda'_2 \gamma_2 a_2 + \gamma'_2 \lambda_3 a_3 \quad (3)$$

- If  $\theta_1 > 0$  and  $\theta'_1 < 0$  then  $\langle a_1, a'_1, a_2, a_3 \rangle \subseteq \langle a'_1, a_2, a_3 \rangle$  by Lemma 39 and therefore  $G \subseteq \langle a'_1, a_2, a_3 \rangle \subseteq \Sigma$ .
- If  $\theta_1 < 0$  and  $\theta'_1 > 0$  then  $\langle a_1, a'_1, a_2, a_3 \rangle \subseteq \langle a_1, a_2, a_3 \rangle$  by Lemma 39 and therefore  $G \subseteq \langle a_1, a_2, a_3 \rangle \subseteq \Sigma$ .
- If  $\theta_1, \theta'_1 \geq 0$  with  $(\theta_1, \theta'_1) \neq (0, 0)$ , then  $\langle a_1, a'_1, a_2, a_3 \rangle \subseteq \langle a_1, a_2, a_3 \rangle \cup \langle a'_1, a_2, a_3 \rangle$  by Lemma 38. Thus  $G \subseteq \langle a_1, a_2, a_3 \rangle \cup \langle a'_1, a_2, a_3 \rangle \subseteq \Sigma$ .
- If  $\theta_1, \theta'_1 \leq 0$  then  $\frac{\gamma'_1}{\lambda'_1} \geq \frac{\gamma'_2}{\lambda'_2} \geq \frac{\gamma_1}{\lambda_1}$  and  $\gamma'_1 > 0$ . Since  $\lambda'_1, \gamma'_1 > 0$  we can eliminate  $a'_1$  from (i) and (ii) to get (with  $\rho_1 \geq 0$ )

$$\underbrace{(\gamma'_1 \lambda_1 - \lambda'_1 \gamma_1)}_{\rho_1} a_1 + \lambda'_1 \gamma_2 a_2 + (-\theta'_1) a'_2 = -\gamma'_1 \lambda_3 a_3 \quad (4)$$

If  $\gamma_2 = 0$  we get from (ii)  $a'_2 \in \langle -a_1, a'_1 \rangle^\circ$ . By Lemma 47, either  $G \subseteq \Sigma$  or  $G^\circ \subseteq \Sigma(a_3)^c$ . If  $\gamma_2 > 0$ , we have  $q \in \langle a_1, a'_2 \rangle^\circ \cap \langle a'_1, a_2 \rangle^\circ$  and  $\langle a_1, a_2, a'_2 \rangle \cap \langle a_1, a'_1, a_2 \rangle = \langle a_1, a_2, q \rangle$ . From Equations (3) and (4), one gets  $-a_3 \in \langle a_1, a_2, q \rangle$  and therefore  $\langle a_1, a_2, q \rangle^\circ \subseteq \Sigma(a_3)^c$  (cf. [6], or see Lemma 48 for a detailed proof).

Case 2.  $\langle a'_1, a_2 \rangle$  intersects the facet  $\langle a_1, a_3 \rangle$  in a non-zero vector. Using Lemma 38, we get  $\langle a_1, a'_1, a_2, a_3 \rangle \subseteq \langle a_1, a'_1, a_3 \rangle \cup \langle a_1, a_2, a_3 \rangle$ . Moreover  $a'_1 \in C$  so  $\langle a_1, a'_1, a_3 \rangle \subseteq C$ . Thus  $G \subseteq C \cup \langle a_1, a_2, a_3 \rangle \subseteq \Sigma$ .

Case 3.  $\langle a'_1, a_2 \rangle$  intersects the facet  $\langle a'_2, a_3 \rangle$  in a non-zero vector. There exist  $\gamma'_1, \gamma_2, \gamma'_2, \gamma_3 \geq 0$ ,  $(\gamma'_1, \gamma_2), (\gamma'_2, \gamma_3) \neq (0, 0)$ , such that (iii)  $\gamma'_1 a'_1 + \gamma_2 a_2 = \gamma'_2 a'_2 + \gamma_3 a_3$ . If  $\gamma'_2 = 0$  and  $\gamma_3 > 0$  then  $a_3 \in \langle a'_1, a_2 \rangle$  contradicting  $a_3 \notin G$ . Thus  $\gamma'_2 > 0$ . Since  $\lambda'_2, \gamma'_2 > 0$  we can eliminate  $a'_2$  using (i) and (iii). We get

$$\gamma'_2 \lambda_1 a_1 + \lambda'_2 \gamma_2 a_2 + \underbrace{(\gamma'_2 \lambda_3 - \lambda'_2 \gamma_3)}_{\theta_3} a_3 = \underbrace{(\gamma'_2 \lambda'_1 - \lambda'_2 \gamma'_1)}_{\theta'_1} a'_1$$

- If  $\theta_3 \leq 0$  and  $\theta'_1 \geq 0$  then  $\langle a_1, a_2 \rangle$  intersects  $\langle a'_1, a_3 \rangle$  in  $\gamma'_2 \lambda_1 a_1 + \lambda'_2 \gamma_2 a_2$  which is non-zero because  $\gamma'_2 \lambda_1 > 0$  and  $G$  is non-degenerate. Using Lemma 38, we get  $G \subseteq \langle a_1, a'_1, a_2, a_3 \rangle \subseteq \langle a_1, a'_1, a_3 \rangle \cup \langle a'_1, a_2, a_3 \rangle$ . Moreover  $a'_1 \in C$  so  $\langle a_1, a'_1, a_3 \rangle \subseteq C$ . Thus  $G \subseteq C \cup \langle a'_1, a_2, a_3 \rangle \subseteq \Sigma$ .
- If  $\theta_3 < 0$  and  $\theta'_1 \leq 0$  then  $a_3 \in G$  contradicting  $a_3 \notin G$ .
- If  $\theta_3 \geq 0$  and  $\theta'_1 > 0$  then  $a'_1 \in \langle a_1, a_2, a_3 \rangle$  and  $G \subseteq \langle a_1, a_2, a_3 \rangle \subseteq \Sigma$ .
- If  $\theta_3 > 0$  and  $\theta'_1 < 0$ . If  $\gamma_3 = 0$  then by (iii)  $a'_2 \in \langle a'_1, a_2 \rangle$  contradicting  $a'_2 \notin G$ . Thus  $\gamma_3 > 0$ . The condition  $\theta_3 > 0$  implies that  $\lambda_3 > 0$  and  $\frac{\gamma_3}{\lambda_3} < \frac{\gamma'_2}{\lambda'_2} < \frac{\gamma'_1}{\lambda'_1}$ . Since

$\lambda_3, \gamma_3 > 0$ , we can eliminate  $a_3$  from (i) and (iii) to get

$$\gamma_3 \lambda_1 a_1 + \underbrace{(\lambda_3 \gamma'_1 - \gamma_3 \lambda'_1)}_{\rho'_1} a'_1 + \lambda_3 \gamma_2 a_2 = \theta_3 a'_2$$

with  $\rho'_1 > 0$  which implies that  $a'_2 \in G$ , a contradiction.  $\square$

## C Lazy Surrounding

Similar to self surrounding, lazy surrounding can be regarded as a Q-covering problem in lower dimension.

**Lemma 50.** *Suppose  $a_1$  belongs to the topological interior of a c-cone  $C = \langle a'_1, a_2, \dots, a_n \rangle$ . Then  $\{\bar{a}'_1, \bar{a}_2\} \oplus \dots \oplus \{\bar{a}'_1, \bar{a}_n\}$  defines a Q-covering of  $a_1^\perp$ . The converse holds only when  $\det(a_1 \ a_2 \ \dots \ a_n)$  and  $\det(a'_1 \ a_2 \ \dots \ a_n)$  have both the same sign.<sup>6</sup>*

*Proof.* Suppose  $a_1$  belongs to the topological interior of a c-cone  $C = \langle a'_1, a_2, \dots, a_n \rangle$ , then  $a_1 \neq 0$  and  $0 \in \langle -a_1, a'_1, a_2, \dots, a_n \rangle^\diamond$ . By Lemma 35,  $\langle -a_1, a'_1, a_2, \dots, a_n \rangle = (-a_1, a'_1, a_2, \dots, a_n) = (a'_1, a_2, \dots, a_n) = \mathbb{R}^n$  (because  $C$  is non-degenerate) and  $\langle -a_1, a'_1, a_2, \dots, a_n \rangle = C \cup \langle -a_1, a_2, \dots, a_n \rangle \cup \bigcup_{i=2}^n \langle -a_1, a'_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_n \rangle = \mathbb{R}^n$ . Moreover, by Lemma 34,  $C = \langle a_1, \dots, a_n \rangle \cup \bigcup_{i=2}^n \langle a'_1, a_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_n \rangle$ . Let  $\Sigma' = \{a_1, -a_1\} \oplus \{a'_1, a_2\} \oplus \dots \oplus \{a'_1, a_n\}$ . Then by definition of  $\Sigma'$  one has

$$\begin{aligned} \Sigma'(a_1) &= \langle a_1, \dots, a_n \rangle \cup \bigcup_{i=2}^n \langle a_1, a'_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_n \rangle = C \\ \Sigma'(-a_1) &= \langle -a_1, a_2, \dots, a_n \rangle \cup \bigcup_{i=2}^n \langle -a_1, a'_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_n \rangle \end{aligned}$$

Therefore,  $\Sigma'(a_1) \cup \Sigma'(-a_1) = \mathbb{R}^n$ . By Proposition 8,  $\{\bar{a}'_1, \bar{a}_2\} \oplus \dots \oplus \{\bar{a}'_1, \bar{a}_n\}$  defines a Q-covering of  $a_1^\perp$ . The converse implies that either  $a_1$  or  $-a_1$  are in the topological interior of  $C$ . To ensure the former,  $a_1$  and  $a'_1$  must not be separated by the hyperplane  $(a_2, \dots, a_n)$  which is equivalent to saying that  $\det(a_1 \ a_2 \ \dots \ a_n)$  and  $\det(a'_1 \ a_2 \ \dots \ a_n)$  have both the same sign (if  $a_1$  and  $a'_1$  are separated by the hyperplane  $(a_2, \dots, a_n)$  then surely  $a_1 \notin \langle a'_1, a_2, \dots, a_n \rangle$ ).  $\square$

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<sup>6</sup>In this case, the hyperplane  $(a_2, \dots, a_n)$  is sometimes said to be *reflective*.